18.103 Lecture Notes

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This class is really about Fourier Series/ Fourier Integrals, that is harmonic analysis dealing with things like

$$f(x) = \sum_{k=-\infty}^{\infty} c_n \exp(i\pi kn), \qquad \int_{-\infty}^{\infty} f(x)e^{ixt}dx$$

the thing is we first have to do some measure theory, because we want to know about integration (Lebesgue integration). Measure theory itself is boring, because its a stripped down compact theory, so we learn probability mixed in with measure theory.

1 Simple Probability and Definition of Measure

First consider *the Coin Toss*. Toss a fair coin and record the outcomes of the unbiased coin. We notice that there are not an equal number of heads and tails (even though we would expect as such "in the long run"). What is the relationship between this and measure theory?

Start by saying that $H = 1$, $T = 0$, so the sequence of heads and tails can be encoded as a sequence of 1 and 0. Our **state space** is the space of all possible outcomes

$$\begin{aligned} \mathcal{B} &= \{\text{all possible sequences of } 0,\ 1\} \\ &= \{a_n, n \in \{1 \dots\}, a_n = 0, 1\} \end{aligned}$$

Bernoulli (hence the $\mathcal{B}$) thought of these as binary expansions of real numbers, that is as a map

$$\{a_n\}_{n=1}^{\infty} \in \mathcal{B} \mapsto \omega = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

and $\omega \in [0, 1]$, obviously.

The problem with this map is that it is not one-to-one and onto. For example

$$.1000\bar{0} = .0111\bar{1} = \frac{1}{2}$$

map to the same number.

So this mapping is not quite 1-1, but it is "close enough". The problem is recurring 1's.

For $\omega \in (0,1]$ consider the binary expansion which ends in infinite $1's$. If we have such an expansion, we do the following

$$a_1 \dots a_k 0111 \dots \to a_1 \dots a_k 1$$

we "kill" all sequences that end in infinite number of 1's. This defines unique maps

$$e : (0,1] \curvearrowright \mathcal{B} \backslash \mathcal{B}_{\text{deg}}, \qquad b : \mathcal{B} \backslash \mathcal{B}_{\text{deg}} \curvearrowright (0,1]$$

where $\mathcal{B}_{\text{deg}}$ sequences that end in an infinite number of 1's, e is the binary expansion, b is the binary sum, b and e are inverses (so in particular they are both 1-1 and onto).

Observe two things

1. $\mathcal{B}_{\text{deg}}$ is countable
2. $\mathcal{B}$ is not countable

We can see that $\mathcal{B}_{\text{deg}}$ is countable by the following argument. Define

$$\mathcal{B}_{\text{deg}}^N \{\{a_n\}_{n=1}^{\infty} : a_n = 0, n \geq N\}$$

this is countable, and now

$$\mathcal{B}_{\text{deg}} = \bigcup_N \mathcal{B}_{\text{deg}}^N$$

this is a countable union of countable sets, and so it is countable. So we can throw $\mathcal{B}_{\text{deg}}$ away.

Question. What is the probability of a certain "outcome"?

The outcome being identified with a subset of the state space, $S \subset \mathcal{B}$. Now we would like to map $S \subset \mathcal{B} \leftrightarrow S' \subset (0,1]$, that is we want to say the probability of the outcome is $\mu(S')$, where μ denotes measure, where measure, for now, is length.

Example. $a_1 = H$ is an outcome

$$S = 1 \dots, \text{ then } S' = \left\{ \omega = \sum_{n=1}^{\infty} \frac{a_n}{2^n} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{a_n}{2^n} \right\}$$

this set is simply $(1/2, 1]$.

Example. $a_2 = H$. Then

$$S = \begin{cases} 11 \dots \\ 01 \dots \end{cases} \to S' = \begin{cases} (\frac{3}{4}, 1] \\ (\frac{1}{4}, \frac{1}{2}] \end{cases}$$

so the measure is the sum of those $\mu(S) = 1/2$.

In general $a_N = H$, then the interval $(0,1)$ is split up into parts of size $1/2^N$.

Now we can ask complicated questions, like
"On average" how many heads occur in the first N tosses as $N \to \infty$
If we say that

$$\text{number of heads} = \sum_{i=1}^{N} a_i = S_N$$

we would expect that

$$\lim_{N\to\infty} \frac{S_N}{N} = \frac{1}{2}$$

this is the **weak law of large numbers**. However when we are trying to take the measure of something we can see that S can be nasty.

Problem. Can we measure the "length" of any subset of $(0,1]$? Can we do the same with volume in $\mathbb{R}^n$?

Our first job is to construct Lebesgue measure.
We know lengths of "simple" subsets of $\mathbb{R}$. We can agree that $\mu((a,b)) = b - a$ where $b \geq a$. We can agree on the lengths of finite unions of disjoint intervals. That is

$$\mu\left(\bigcup_{i=1}^{N} I_i\right) = \sum_{i=1}^{N}(b_i - a_i), \qquad I_i = (a_i, b_i)$$

Idea We abstract this definition and just work in the abstract picture. Our abstract setting is a set, instead of $(0,1]$. Let X be our base set $(\mathbb{R}^n)$. We have the set theoretic operations $A \cap B$, $A \cup B$, $A - B = A \backslash B = \{x \in A, x \notin B\}$ and sometimes we have $A \ominus B$ the symmetric difference $A\backslash B \cup B \backslash A$. And also $A^c = X \backslash A$.

Definition. Ring
A ring of subsets is a collection of subsets of X closed under unions and symmetric difference, that is

$$A, B \in \mathcal{R} \to A \cup B \in \mathcal{R}, A \ominus B \in \mathcal{R}$$

Question Is the intersection in the ring $\mathcal{R}$. The answer is yes since

$$(A \cup B) \ominus (A \ominus B) = A \cap B$$

so it contains intersection.

Example. $2^X = \mathcal{P}$, the collection of all subsets of X

Example. Our finite unions of disjoint intervals in $\mathbb{R}$ is a ring.

$$I_1, \ldots, I_N, \qquad \text{and} \qquad I'_1, \ldots, I'_m$$

and $A = \bigcup I_i$, $B = \bigcup I'_i$, then $A \cup B \in \mathcal{R}$, since we can always split up $A \cup B$ into non-overlapping chunks.