# CPSC 320 2024W1: NP-Completeness Tutorial Problems

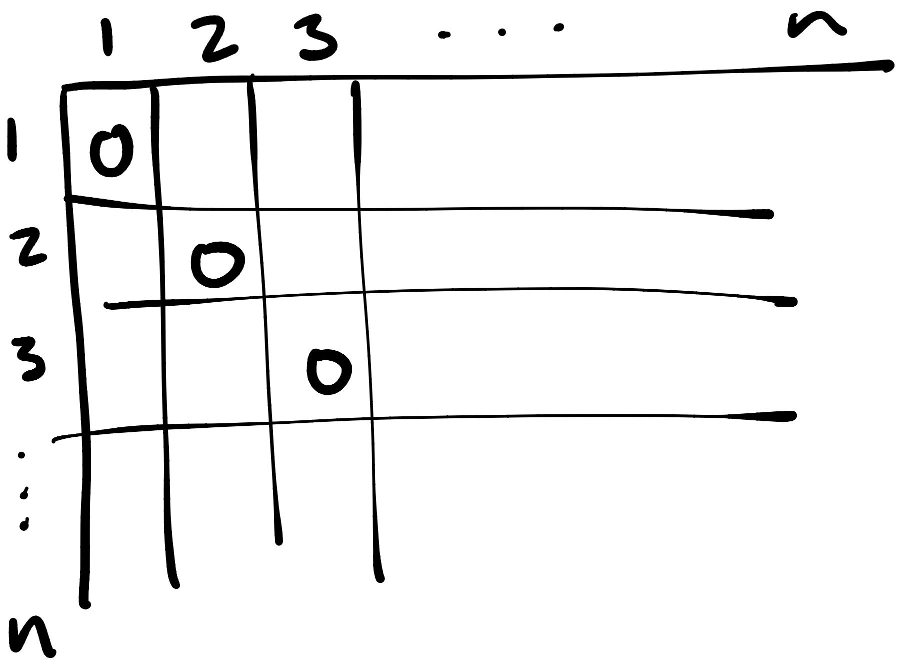
## 1 Strategically Placed Krispy Kremes

UBC Rec student leaders are planning their next fundraiser, and are seeking your help in identifying strategic locations to set up their stands of Krispy Kremes. They have a map showing $n$ locations of buildings and outdoor spots on campus. Their $k$ stands need to be set up in the outdoor spots. They want you to select $k$ spots such that the maximum distance from any of the $n$ locations to a stand is as small as possible. Formally an instance of the Strategically Placed Krispy Kremes optimization problem is a set $V$ of size $n$ (the locations), a subset $S$ of $V$ (the outdoor locations), and an integer $k$, $1 \leq k \leq |S|$, as well as a symmetric matrix $d[1..n][1..n]$ of distances between each pair of locations. The *optimization* problem is to find a subset $S' \subseteq S$ of size $k$, so as to minimize

$$\max_{v \in V} \min_{s \in S'} \{d(s, v) \mid s \in S'\}.$$

The corresponding Strategically Placed Krispy Kremes *decision* problem, which we'll call SPKK, is as follows. An instance of SPKK has a set $V$ of size $n$, a subset $S$ of $V$, an integer $k$, $1 \leq k \leq |S|$, and a symmetric matrix $d[1..n][1..n]$, plus an additional nonnegative integer $b$. The problem is to determine if there is a subset $S' \subseteq S$ of size $k$, such that

$$\max_{v \in V} \min_{s \in S'} \{d[v][s] \mid s \in S'\} \leq b. \tag{1}$$



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k=4

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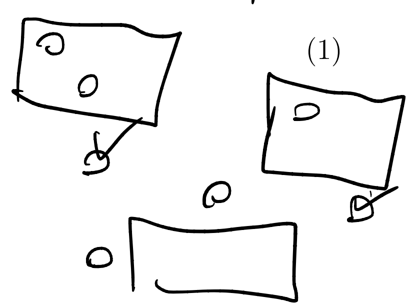


1. Show that the SPKK problem is in NP.

Let $I = (V, S, d[1..n][1..n], k, b)$

Let $S'$ as our solution

Verify that we use $=k$ locations $O(n \log n)$ $\Big\}\, O(n^2)$

" " our cost isn't exceeded $O(n^2)$

2. Suppose that, in specifying the decision problem, we replace the "$\leq$" in (2) with "$=$", i.e.,

$$\max_{v \in V} \min_{s \in S'} \{d[v][s] \mid s \in S'\} = b. \tag{2}$$

Call this new decision problem variant SPKK$'$. Can we easily adapt our certification algorithm from part 1 to SPKK$'$?

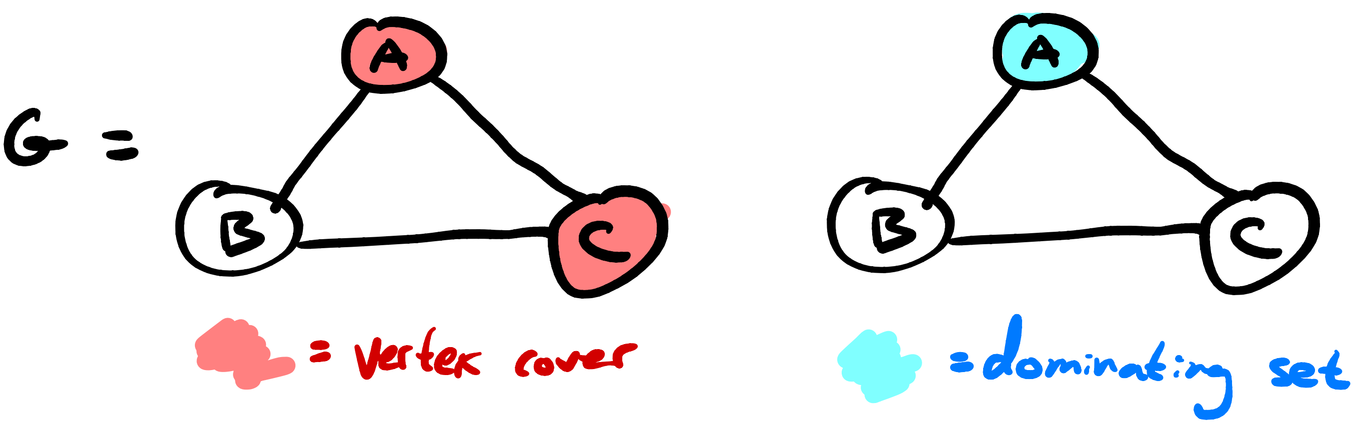
Yes

## 2 Reductions: Vertex Cover and Dominating Sets

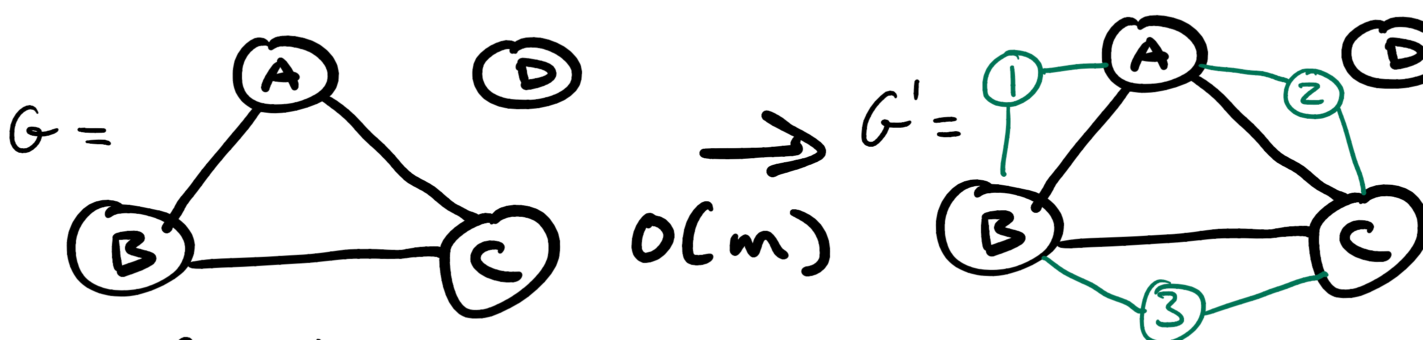
Your task is to find a reduction from Vertex Cover to Dominating Sets, two famous graph problems:

- *Vertex Cover:* An instance is a graph $G = (V, E)$ and an integer $K$. The problem asks: Is there a vertex cover with at most $K$ vertices in $G$? Here, a vertex cover is a subset $W$ of $V$ such that $|W| \leq K$, such that every edge in $E$ has at least one endpoint in $W$.
- *Dominating Set:* An instance is a graph $G = (V, E)$ and an integer $K$. The problem asks: Is there a dominating set with at most $K$ vertices in $G$? Here, a dominating set is a subset $W$ of $V$ such that $|W| \leq K$, such that every element of $V - W$ is joined by an edge to an element of $W$.

1. Give a reduction from Vertex Cover to Dominating Set. Explain why your reduction is correct, and runs in polynomial time.




Reducing VC -> DS $\quad m = |E|$




$I_{VC} = (G, K)$

$I_{DS} = (G', K' = K + \#$ isolated nodes$)$

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- *Dominating Set:* An instance is a graph $G = (V, E)$ and an integer $K$. The problem asks: Is there a dominating set with at most $K$ vertices in $G$? Here, a dominating set is a subset $W$ of $V$ such that $|W| \leq K$, such that every element of $V - W$ is joined by an edge to an element of $W$.

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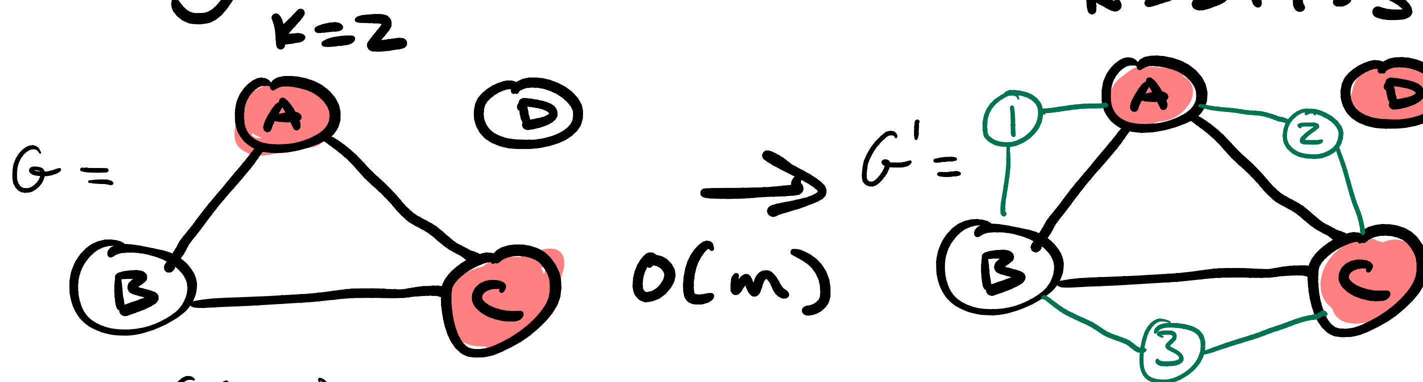


$I_{VC} = (G, K)$

$I_{DS} = (G', K' = K + \#$ isolated nodes$)$

If G has a vertex cover W w/ size K, all non-isolated nodes in V − W are joined by an edge to the nodes of W and all nodes added in the reduction are joined by an edge to a node in W.

If G' has a dominating set W' w/ size K', we can retrieve W by discarding isolated nodes, and for each node added in the reduction that is present in W', replace it with either of its endpoints.

* think of the nodes added in the reduction as ways to represent an edge in the original G.