CPSC 320 2024W1: NP-Completeness Tutorial Problems

1 Scheduling classes

A college wishes to offer evening courses and has n possible courses that can be offered. There are k students interested in taking a course and each student is interested in some subset of the n possible courses. To manage costs, the college has a bound b on the number of courses it is willing to actually offer. Is there a way to choose b courses out of the n possibilities so that every student can take a course that interests them? Formally, an instance of the CS problem consists of

- a set C of n possible courses, numbered 1 through n,
- subsets $S_1, S_2, \ldots, S_k$ of $\{1, 2, \ldots, n\}$ (the courses of interest to each of the students), and
- a bound b (the maximum number of classes actually offered).

The problem is to determine whether there is a subset B of $\{1, 2, \ldots, n\}$, where the size of B is $\leq b$, such that $B \cap S_i$ is not empty for all i, $1 \leq i \leq k$.

1. Show that the CS problem is in NP.

2. Describe an efficient reduction from the Vertex Cover problem to the Class Scheduling (CS) problem. The Vertex Cover problem is: Given an undirected graph $G = (V, E)$ and an integer K, does G have a vertex cover with size at most K? A vertex cover is a subset W of V such that every edge in E has at least one endpoint in W.

3. Your reduction maps instances (G, K) of Vertex Cover to instances $(C, S_1, \ldots, S_k, b)$ of CS. Show that G has a vertex cover of size at most K if and only if in the mapped instance $(C, S_1, \ldots, S_k, b)$, there is a set of courses of size at most b such that every student can take a course that interests them. (Remember that you need to two parts here, one for "if" and one for "only if".)

4. Explain why your reduction runs in polynomial time.

5. Put the previous parts together to conclude that CS is NP-complete.

2 NP True or False

Let X and X' be decision problems, where both problems have Yes instances and No instances. State whether you think each of the following statements must be true, must be false, or is an open question. Justify your answer.

1. **Statement:** If $X \leq_p X'$ and X is *not* in NP, then X' is not in NP.
2. **Statement:** If $X \leq_p X'$, X is in P and X' is in NP, then X' must be in P.