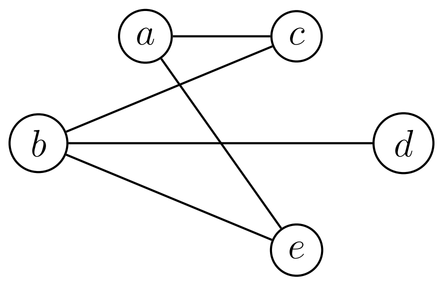
CPSC 320 2024W1: A Reduction Tutorial Problem

A SAT Reduction

A graph $G = (V, E)$ is **bipartite** if we can partition the vertices V into two disjoint sets U and W such that no two vertices in U are connected, and no two vertices in W are connected. For instance, the graph below:



is bipartite if we define $U = \{a, b\}$ and $W = \{c, d, e\}$. In the **Bipartite Graph Problem** (BGP), we want to determine if a given input graph is bipartite. In this problem, you will reduce BGP to Boolean Satisfiability (SAT), defined below.

> **SAT:** The input is a collection of m *clauses* over n boolean variables X_1, X_2, $\ldots X_n$. Each clause is a disjunction of some of the variables or their complements.
>
> The problem consists in answering the question "Is there a way to assign truth values to each variable that makes **every** clause of the instance TRUE?

For example, the SAT instance given by:

$$(X_1 \vee \overline{X}_2) \wedge (X_2) \wedge (\overline{X}_1 \vee X_3 \vee X_4)$$

is satisfiable by setting all variables to True. (This is not the only truth assignment that works for this instance.)

1. Given a BGP instance, we need to figure out how to express it as a SAT instance. The first step is to figure out what the **variables** in our SAT instance should represent. Is there any aspect of the BGP problem that we can encode as a choice between two options (since this behaves like a variable in SAT)?

   Give your variables a name, and describe what each variable represents. Hint: my reduction introduces one variable for each vertex in V.

2. Consider a pair of vertices v_i, v_j. What, if anything, can we say about their corresponding variables in the SAT instances if v_i and v_j share an edge? What about if they don't share an edge?

3. Combine your answers to questions 1 and 2 to give a complete reduction from BGP to SAT.

4. In the next two questions, we'll prove the correctness of your reduction from BGP to SAT – that is, we'll show that the reduced SAT instance is satisfiable if and only if the input to BGP is a bipartite graph.

   For the first direction: Prove that, if the input graph to BGP is bipartite, your reduced SAT instance is satisfiable. Hint: if G is bipartite, you know there's a way to assign vertices to be in V or W, such that there are no edges between any vertices in U or between any vertices in W. Try to use this assignment of vertices to construct a truth assignment to the variables in SAT.

5. Now, for the opposite direction: prove that, if the reduced SAT instance is satisfiable, the input graph to BGP is bipartite. Hint: if the reduced SAT instance is satisfiable, you know there is a truth assignment such that every clause is True. Try to use this truth assignment to partition the vertices in V into the sets U and W.