CPSC 320 2024W1: Midterm 1 Solutions, 15:00-15:45

1 SMP, Asymptotic Analysis

No justification is needed for these questions.

1. [2 points] Complete the preference lists for three employers and three applicants, such that the Gale-Shapley algorithm (provided on page 7) matches employer e_1 with applicant a_1.

e_1:	a_2	a_3	a_1
e_2:	a_2	a_3	a_1
e_3:	a_2	a_3	a_1

a_1:	e_2	e_3	e_1
a_2:	e_2	e_3	e_1
a_3:	e_2	e_3	e_1

2. [2 points] Let $f(n)$ be the number of different SMP instances of size n such that e_i is the top choice of a_i, and a_i is the top choice of e_i, for all i from 1 to n, We saw in Assignment 1 that $f(n) = ((n-1)!)^{2n}$. Which of the following bounds are valid for $f(n)$? Choose all that apply.

   ○ $f(n) = O(n)$ ○ $f(n) = O(((n-1)!)^n)$ ○ $f(n) = O((n)^{2n})$

   ● $f(n) = \Omega(n)$ ● $f(n) = \Omega(((n-1)!)^n)$ ● $f(n) = \Omega((n)^{2n})$

3. [2 points] Let $f(n)$ and $g(n)$ be functions that map nonnegative integers to positive real numbers. Indicate whether the following statements must be true (for *all* such functions f and g), or could be false (for *some* such functions f and g).

   - If $f(n) = o(n \cdot g(n))$ then $f(n) = O(g(n))$.

     ○ True ● False

   - If $f(n) = \Omega(g(n))$ then $g(n) = o(f(n))$.

     ○ True ● False

2 True/False++

For each of the following claims below, select whether the claim is True or False. If the claim is True, justify your answer by proving the claim is always True. If the claim is False, justify your answer by providing and briefly explaining a counterexample in which the claim is not True.

1. [5 points] Given: An SMP instance with n employers and applicants in which all employers have the same first choice applicant. **Claim**: The Gale-Shapley algorithm (provided on page 7) will take at least $2n - 1$ while-loop iterations to converge.

   Choose one: ● **TRUE** ○ **FALSE**

   Justification (proof for True or counterexample for False):

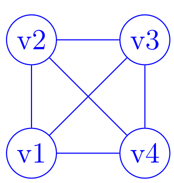
   The claim is True. Since all employers have the same first choice applicant, they will all make an offer to that applicant, who will only ultimately accept one of them. So after this we will have n offers made (and thus n while loop iterations), and only one matched pair. Thus, the other $n - 1$ employers will each need to make at least one additional offer, leading to at least $2n - 1$ loop iterations.

2. [5 points] Given: A connected graph G and a spanning tree T of G. **Claim**: Every diametric node[1] in G is also a diametric node in T.

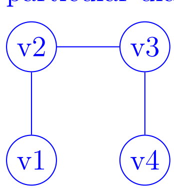
   Choose one: ○ **TRUE** ● **FALSE**

   Justification (proof for True or counterexample for False):

   The claim is False. Consider the following graph:

   This graph has diameter 1 and every node is a diametric node (distance 1 away from at least one other node). But for the sake of our counterexample, we'll identify two particular diametric nodes v_2, v_3. However, if we consider the spanning tree:

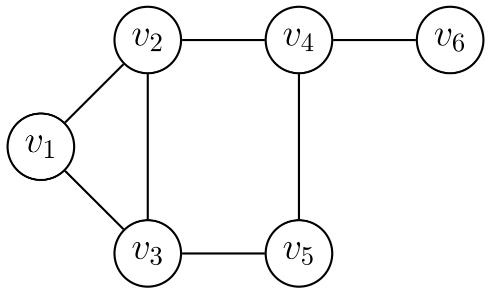
   

   The only diametric nodes are v_1 and v_4. Thus, the nodes v_2 and v_3 are diametric in G but not in T.

[1] Recall that a "diametric node" is a node whose distance from another node defines the diameter of a graph.

3 Reduction

Given a graph $G = (V, E)$, a **dominating set** is a subset D of vertices in V with the property that every vertex in V either belongs to D, or has a neighbour in D. The graph below has a dominating set of size 2 given by $\{v_3, v_4\}$, but no dominating set of size 1.



In this question, we consider the *Dominating Set* problem (DSP):

> Given a graph $G = (V, E)$ with n nodes, plus an integer $k \geq 1$, does G have a dominating set of size k? That is, is there a subset D of vertices in V such that $|D| = k$, and all vertices in V are either in D or are adjacent to a vertex in D?

Recall also the **Boolean satisfiability** (SAT) problem, defined as follows:

> The input is a collection of m clauses over r boolean variables X_1, X_2, $\ldots X_r$. Each clause is a disjunction of some of the variables or their complements.
>
> The problem consists of answering the question "Is there a way to assign truth values to each variable that makes **every** clause of the instance TRUE?

In the following questions you will complete a reduction from DSP to SAT.

1. [1 point] Our reduction defines nk variables $X_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq k$. What should the variable $X_{i,j}$ represent? Choose the **best** answer.

   ○ Whether vertex v_i is in the potential dominating set
   ○ Whether vertex v_j is in the potential dominating set
   ● Whether vertex v_i is the jth element of the potential dominating set
   ○ Whether vertex v_j is the ith element of the potential dominating set

2. [2 points] Consider the following **partially complete** reduction from DSP to SAT. Define nk variables $X_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq k$.

   (A) For each a from 1 to k, add the clause:

$$X_{1,a} \vee X_{2,a} \vee \ldots \vee X_{n,a}.$$

(B) For every p from 1 to n and every distinct a, b from 1 to k, add the clause:

$$\overline{X}_{p,a} \vee \overline{X}_{p,b}$$

(C) For every distinct p, q from 1 to n and all a from 1 to k, add the clause:

$$\overline{X}_{p,a} \vee \overline{X}_{q,a}$$

For this reduction, Which of the following best describes the purpose of the clauses in (A)-(C)? Select all that apply.

○ They ensure that G contains a dominating set of size k

○ They ensure that every vertex in V is either in the dominating set, or is adjacent to a vertex in the dominating set

● They ensure that the potential dominating set of G set contains exactly k vertices

○ None of the above

3. [6 points] Complete the reduction by adding the necessary clauses in (D) below:

(D) Let the neighbours of vertex v_p be denoted by $v_{p1}, v_{p2}, \ldots, v_{pq}$. For each vertex v_p, add the clause(s):

$X_{p,1} \vee X_{p,2} \ldots \vee X_{p,k} \vee X_{p1,1} \vee X_{p1,2} \ldots \vee X_{p1,k} \vee X_{p2,1}$
$\vee X_{p2,2} \ldots \vee X_{p2,k} \ldots \vee X_{pq,1} \vee X_{pq,2} \ldots \vee X_{pq,k}$

In English (this was not required for your answer): one of $v_p, v_{p1}, \ldots, v_{pq}$ is in some position $(1, \ldots, k)$ of the dominating set.

Appendix: Gale-Shapley Algorithm

```
set all s ∈ S and e ∈ E to free
while some free employer e hasn't made an offer to every student do
    s ← the highest-ranking student e hasn't made an offer to
    if s is free then
        hire(e, s)
    else
        e′ ← s's current employer
        if s prefers e to e′ then
            set e′ to free
            hire(e, s)
        endif
    endif
endwhile
return the set of pairs
```

Extra Space

If you write any part of your answers here, you must CLEARLY indicate it in the space allocated for that question.