CPSC 320: NP-Completeness Practice Solutions*

1 NP True or False

Let X and X' be decision problems, where both problems have Yes instances and No instances. Indicate whether each of the following statements must be true, might be false, or is an open question.

1. Suppose that P $\neq$ NP. If $X \leq_p X'$ and X' is NP-complete, then X is NP-complete.

   SOLUTION:

   ○ True
   ● False
   ○ Open Question

2. If X is in P and X' is NP-complete, then $X \leq_p X'$.

   SOLUTION:

   ● True
   ○ False
   ○ Open Question

3. If X is in NP and X' is NP-complete, then $X' \leq_p X$.

   SOLUTION:

   ○ True
   ○ False
   ● Open Question

4. Your friend has come to you with a fascinating and important computational problem, the SnigSnag decision problem. Suppose that you prove that the SnigSnag problem belongs to the class NP, and give a (correct) polynomial-time reduction from SnigSnag to 3-SAT. Have you then proved that the SnigSnag problem is NP-Complete?

   ○ Yes ● No

5. Let P_1 and P_2 be two NP-complete problems. Is it the case that there is a polynomial-time reduction from P_1 to P_2, and *also* a polynomial-time reduction from P_2 to P_1?

   ● Yes ○ No

2 Exam Breakout Rooms

A new university rule during COVID-19 stipulates that students who have been collaborating on classwork must be placed in different breakout rooms for the final exam, and each breakout room must get a different version of the exam. Anne and the TAs would like to minimize the number of breakout rooms (and different exams that need to be prepared).

An instance of the Exam Assignment optimization problem is n lists, where n is the number of students in the class and the ith list provides the classmates that have collaborated with the ith student in the class. The problem is to determine the minimum number of breakout rooms needed, so as to satisfy the university rule?

1. Describe a decision version of the Exam Assignment optimization problem.

   SOLUTION: Given n lists as above, and a nonnegative integer k, is there a way to assign students to k breakout rooms while satisfying the university rule.

2. Assuming that P $\neq$ NP, is your decision problem in P or is it NP-complete?

   SOLUTION: The Graph Colouring problem can be reduced to the Exam Assignment decision problem, and so the problem is NP-complete.

3 More P vs NP-complete

Consider these well-known problems (mostly, but not all, NP-complete, assuming that P $\neq$ NP).

A. Vertex Cover
B. Subset Sum
C. Stable Matching
D. Graph Colouring
E. Traveling Salesperson
F. Minimum Spanning Tree
G. Bipartite Matching
H. 3-SAT
I. 3-Dimensional Matching

Each of the problems below is a "disguised" version of one of the problems above. Assuming that P $\neq$ NP, indicate whether you think that each problem is in P or is NP-complete.

1. Radio frequencies must be assigned to cell towers, but two towers whose signals could interfere with each other cannot be assigned the same frequency. A problem instance is a set of towers and a list of pairs of towers that interfere with each other, plus a positive integer k. The question is whether the number of radio frequencies needed is at most k.

   ● NP-complete

   ○ In P

   (There is an easy reduction from graph colouring: nodes are mapped to cell towers, with two towers interfering iff there is an edge between the corresponding nodes. The graph can be coloured with at most k colours iff the cell towers need at most k frequencies.)

2. Given an instance of stable matching, i.e., lists of the rankings of n employers and n applicants, plus a positive integer k, you want to know if there is a stable matching in which at least k applicants get their top-ranked (best) choice. Assuming that P $\neq$ NP, is this problem in P or is it NP-complete?

   ○ NP-complete

   ● In P

   (To determine the answer, use the "applicant-optimal" version of the Gale-Shapley algorithm where applicants make choices, and see whether at least k applicants get their top-ranked choice.)

3. With schools reopening in the fall, the Vancouver School Board urgently needs to assign each incoming kindergartner to a school. The number of kindergartners that can attend school i is capped at $C[i]$. The family of each child has provided an unranked set of preferred schools. The Kindergarten Assignment problem is to determine whether all children can be assigned to one of their preferred schools, while ensuring that each school i is assigned at most $C[i]$ kindergartners.

   ○ NP-complete

   ● In P

   (This can be reduced to the bipartite (2D) matching problem, which is in P.)

4. An instance of the Tug-O-War problem is an array $w[1..n]$ that provides the weights of $n > 1$ people. The problem is to determine if the people can be partitioned into two teams that are perfectly balanced, that is, the total weight of team 1 equals the total weight of team 2.

   ● NP-complete

   ○ In P

   (This is the subset sum problem in disguise.)