CPSC 320 Handouts: The Master Theorem

For a recurrence like $T(n) = aT(\frac{n}{b}) + f(n)$, where $a \geq 1$ and $b > 1$, the Master Theorem states three cases:

1. If $f(n) \in O(n^c)$ where $c < \log_b a$ then $T(n) \in \Theta(n^{\log_b a})$.
2. If for some constant $k \geq 0$, $f(n) \in \Theta(n^c (\log n)^k)$ where $c = \log_b a$, then $T(n) \in \Theta(n^c (\log n)^{k+1})$.
3. If $f(n) \in \Omega(n^c)$ where $c > \log_b a$ **and** $af(\frac{n}{b}) \leq kf(n)$ for some constant $k < 1$ and sufficiently large n, then $T(n) \in \Theta(f(n))$.

Master Theorem Corollary. Suppose that $T : \mathbb{N} \to \mathbb{R}_{\geq 0}$ satisfies

$$T(n) = \begin{cases} c, & \text{for } n < n_0, \\ aT(n/b) + cn^k, & \text{for } n \geq n_0, \end{cases}$$

where $a > 0$, $b > 1$, $c > 0$, and $k \geq 0$ are constants.

- If $a > b^k$, then $T(n) = \Theta(n^{\log_b a})$.
- If $a = b^k$, then $T(n) = \Theta(n^k \log n)$.
- If $a < b^k$, then $T(n) = \Theta(n^k)$.