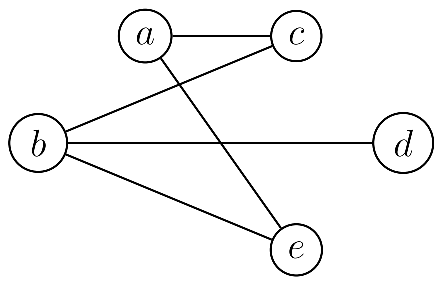
CPSC 320 2024W1: A Reduction Tutorial Problem

A SAT Reduction

A graph $G = (V, E)$ is **bipartite** if we can partition the vertices V into two disjoint sets U and W such that no two vertices in U are connected, and no two vertices in W are connected. For instance, the graph below:



is bipartite if we define $U = \{a, b\}$ and $W = \{c, d, e\}$. In the **Bipartite Graph Problem** (BGP), we want to determine if a given input graph is bipartite. In this problem, you will reduce BGP to Boolean Satisfiability (SAT), defined below.

> **SAT:** The input is a collection of m *clauses* over n boolean variables X_1, X_2, $\ldots X_n$. Each clause is a disjunction of some of the variables or their complements.
>
> The problem consists in answering the question "Is there a way to assign truth values to each variable that makes **every** clause of the instance TRUE?

For example, the SAT instance given by:

$$(X_1 \vee \overline{X}_2) \wedge (X_2) \wedge (\overline{X}_1 \vee X_3 \vee X_4)$$

is satisfiable by setting all variables to True. (This is not the only truth assignment that works for this instance.)

1. Given a BGP instance, we need to figure out how to express it as a SAT instance. The first step is to figure out what the **variables** in our SAT instance should represent. Is there any aspect of the BGP problem that we can encode as a choice between two options (since this behaves like a variable in SAT)?

   Give your variables a name, and describe what each variable represents. Hint: my reduction introduces one variable for each vertex in V.

   Our obvious choice between two options in this problem is that, for each vertex v_i, it can belong to either the set U or the set W. So we'll let the variable X_i represent whether a vertex belongs to U. We'll say that X_i is True if vertex v_i is in set U and False if v_i is in W.

2. Consider a pair of vertices v_i, v_j. What, if anything, can we say about their corresponding variables in the SAT instances if v_i and v_j share an edge? What about if they don't share an edge?

   If v_i and v_j share an edge, they can't both be in the same set U/W. That is, we can't have X_i and X_j both be True, nor can they both be False. If they don't share an edge, there's nothing useful we can say (in terms of our variables in the SAT reduction): v_i and v_j might both be in U, or both be in W, or one in U and one in W (that is, any combination of truth assignments to X_i and X_j might be possible, depending on the structure of the rest of the graph).

3. Combine your answers to questions 1 and 2 to give a complete reduction from BGP to SAT.

   We define n variables X_i, where X_i represents whether vertex v_i is in the set U. The constraint we need to encode in our solution is that if two nodes are connected by an edge, they must be in different subsets U or W. Thus, for each edge (v_i, v_j), we need at least one of the vertices v_i and v_j to be in U, and at least one of them to be in W. In terms of the reduction, the clauses to add are:

   For each edge $(v_i, v_j) \in E$, add the clauses

   $$(X_i \vee X_j) \wedge (\bar{X}_i \vee \bar{X}_j).$$

   The first clause says that one of the vertices i and j must be in U, and the second says that one of them must be in W.

   The graph G is bipartite if and only if the reduced SAT instance is satisfiable.

4. In the next two questions, we'll prove the correctness of your reduction from BGP to SAT – that is, we'll show that the reduced SAT instance is satisfiable if and only if the input to BGP is a bipartite graph.

   For the first direction: Prove that, if the input graph to BGP is bipartite, your reduced SAT instance is satisfiable. Hint: if G is bipartite, you know there's a way to assign vertices to be in V or W, such that there are no edges between any vertices in U or between any vertices in W. Try to use this assignment of vertices to construct a truth assignment to the variables in SAT.

   Suppose that G is bipartite. This means we can assign the n vertices v_i to be in either U or W, with no vertices in U being adjacent to each other and no vertices in W being adjacent to each other.

   We construct a truth assignment to our SAT variables as follows. For each vertex v_i, we set X_i to be True if v_i belongs to the set U, and False if v_i belong to the set V.

   We now prove that this is a satisfying truth assignment to our reduced SAT instance. Because we have a valid partitioning of our vertices into the sets U and W, we know that any two adjacent vertices (i.e., vertices that are connected by an edge) must belong to different sets. Therefore, for each edge (v_i, v_j), exactly one of X_i, X_j is True and exactly one of them is False. This means that the clauses $(X_i \vee X_j) \wedge (\bar{X}_i \vee \bar{X}_j)$ will both evaluate to True. Since this accounts for all clauses in our reduced SAT instance, this is a satisfying truth assignment – and, therefore, the reduced SAT instance is satisfiable.

5. Now, for the opposite direction: prove that, if the reduced SAT instance is satisfiable, the input graph to BGP is bipartite. Hint: if the reduced SAT instance is satisfiable, you know there is a truth assignment such that every clause is True. Try to use this truth assignment to partition the vertices in V into the sets U and W.

   Suppose that our reduced SAT instance has a satisfying truth assignment – that is, an assignment of True/False values to our variables X_i such that all our clauses evaluate to True. We construct an assignment of vertices as follows: for each i, if X_i is True, we assign vertex i to be in U; if X_i is False, we say vertex i is in W.

We now prove that this is a valid bipartite graph partitioning – that is, there are no edges between any two vertices in U, or between any two vertices in W. Our clauses are only defined for vertex pairs v_i, v_j that share an edge. The corresponding clauses $(X_i \vee X_j) \wedge (\bar{X}_i \vee \bar{X}_j)$ will only both evaluate to True if one of X_i, X_j is True and one of X_i, X_j is False. This means that any two adjacent vertices will be in different sets. Therefore, no two vertices in the same set will share an edge; this means that we have shown G is bipartite, because we can partition the vertices into two sets that don't contain any edges within the set.