Subgraphs

Stepan Kuznetsov

Computer Science Department, Higher School of Economics

Outline

Ramsey Numbers

Friends Among Six

We start with the following problem.

Problem

Show that among six people there are either three which are mutual friends, or three from which no two are friends.

Friends Among Six

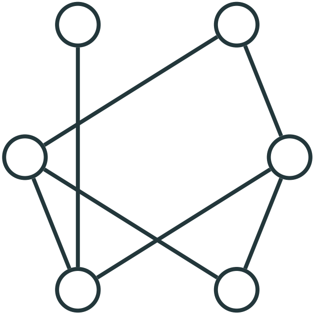
- In graph-theoretic terms: for any graph G on 6 vertices either G or its *complement* $\overline{G}$ includes a triangle.

Friends Among Six

- In graph-theoretic terms: for any graph G on 6 vertices either G or its *complement* $\overline{G}$ includes a triangle.
- We shall view G as an *edge coloring* of the complete graph on 6 vertices (K_6):

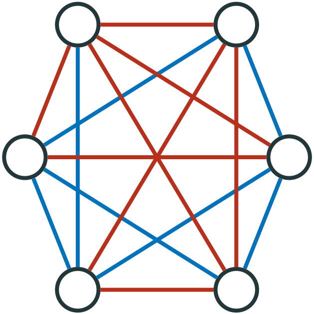
Friends Among Six

- In graph-theoretic terms: for any graph G on 6 vertices either G or its *complement* $\overline{G}$ includes a triangle.
- We shall view G as an *edge coloring* of the complete graph on 6 vertices (K_6):



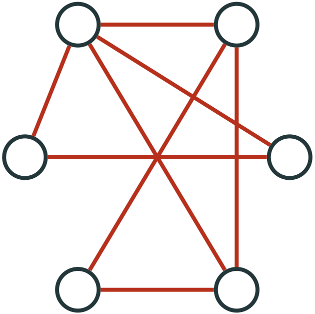
Friends Among Six

- In graph-theoretic terms: for any graph G on 6 vertices either G or its *complement* $\overline{G}$ includes a triangle.
- We shall view G as an *edge coloring* of the complete graph on 6 vertices (K_6):



Friends Among Six

- In graph-theoretic terms: for any graph G on 6 vertices either G or its *complement* $\overline{G}$ includes a triangle.
- We shall view G as an *edge coloring* of the complete graph on 6 vertices (K_6):



Proof

- Take an arbitrary vertex v.

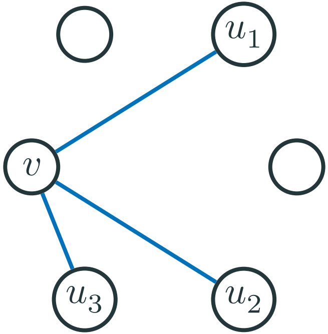
Proof

- Take an arbitrary vertex v.
- v has either at least 3 blue or at least 3 red adjacent edges: otherwise the total number is $\leq 2 + 2 = 4 < 5$.

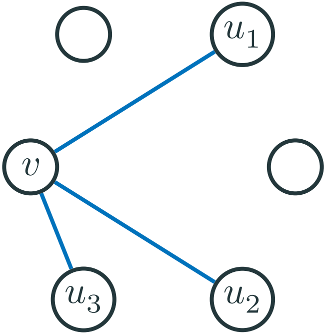
Proof

- Take an arbitrary vertex v.
- v has either at least 3 blue or at least 3 red adjacent edges: otherwise the total number is $\leq 2 + 2 = 4 < 5$.
- Let v have three blue adjacent edges.

Proof

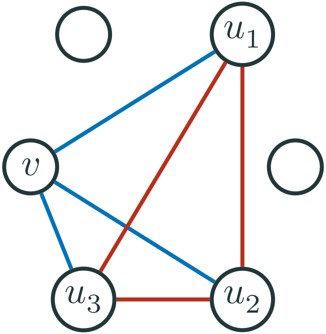


Proof



Consider two cases:

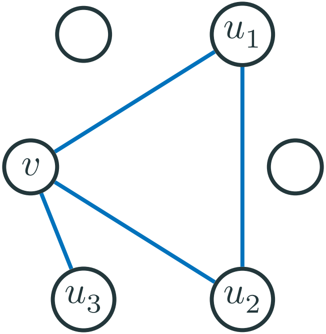
Proof



Consider two cases:

1. All edges between u_1, u_2, u_3 are red.

Proof



Consider two cases:

1. All edges between u_1, u_2, u_3 are red.
2. There is a blue edge between u_i and u_j.

Ramsey Number

Question

What is the *minimal* number n of vertices to guarantee that K_n, for any edge coloring in blue and red, includes either a blue triangle or a red triangle?

Ramsey Number

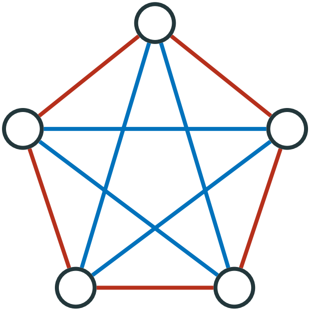
- The answer is denoted by $R(3,3)$.

Ramsey Number

- The answer is denoted by $R(3, 3)$.
- We have shown that $R(3, 3) \leq 6$.

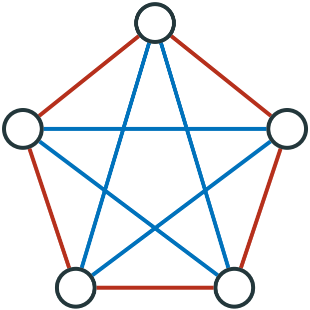
Ramsey Number

- The answer is denoted by $R(3,3)$.
- We have shown that $R(3,3) \leq 6$.
- $R(3,3) > 5$, as their is a counterexample:



Ramsey Number

- The answer is denoted by $R(3,3)$.
- We have shown that $R(3,3) \leq 6$.
- $R(3,3) > 5$, as their is a counterexample:



- Thus, $R(3,3) = 6$.

Ramsey Number

- In general, $R(m, k)$ is the minimal number n of vertices to guarantee that K_n with edges colored in blue and red includes a blue K_m or a red K_k.

Ramsey Number

- In general, $R(m, k)$ is the minimal number n of vertices to guarantee that K_n with edges colored in blue and red includes a blue K_m or a red K_k.
- R is called the *Ramsey number.*

Ramsey Number

- In general, $R(m, k)$ is the minimal number n of vertices to guarantee that K_n with edges colored in blue and red includes a blue K_m or a red K_k.
- R is called the *Ramsey number*.
- R can have more parameters, if the number of colors is bigger than two (blue, red, green, yellow, ...).

Properties of Ramsey Numbers

- Symmetry: $R(m,k) = R(k,m)$.
- Recursive inequation:
  $R(m,k) \leq R(m-1,k) + R(m,k-1)$.
- In particular, $R(m,k) < \infty$ for any m, k, that is, if a graph is big enough, then either it includes K_m or its complement includes K_k (m and k fixed). This is called **Ramsey's theorem.**
- Ramsey's theorem is proved by induction on $m + k$.

Proof of Recursive Inequation

- The proof of recursive inequation for $R(m, k)$ generalizes the solution of the “friends among six” problem.

Proof of Recursive Inequation

- The proof of recursive inequation for $R(m, k)$ generalizes the solution of the “friends among six” problem.
- Take an arbitrary v out of $R(m-1, k) + R(m, k-1)$ vertices.

Proof of Recursive Inequation

- The proof of recursive inequation for $R(m,k)$ generalizes the solution of the "friends among six" problem.
- Take an arbitrary v out of $R(m-1,k)+R(m,k-1)$ vertices.
- Let $M = \{u \mid \langle u,v\rangle \text{ is blue}\}$ and $N = \{u \mid \langle u,v\rangle \text{ is red}\}$.

Proof of Recursive Inequation

- The proof of recursive inequation for $R(m, k)$ generalizes the solution of the "friends among six" problem.
- Take an arbitrary v out of $R(m-1, k) + R(m, k-1)$ vertices.
- Let $M = \{u \mid \langle u, v\rangle \text{ is blue}\}$ and $N = \{u \mid \langle u, v\rangle \text{ is red}\}$.
- The total number of vertices is $|M| + |N| + 1$, thus either $|M| \geq R(m-1, k)$ or $|N| \geq R(m, k-1)$.

Proof of Recursive Inequation

- Let $|M| \geq R(m-1, k)$ (the second case is symmetric).

Proof of Recursive Inequation

- Let $|M| \geq R(m-1, k)$ (the second case is symmetric).
- Among vertices in M, there is either a red K_k or a blue K_{m-1}.

Proof of Recursive Inequation

- Let $|M| \geq R(m-1, k)$ (the second case is symmetric).
- Among vertices in M, there is either a red K_k or a blue K_{m-1}.
- In the first case, we get a red K_k in the original graph.

Proof of Recursive Inequation

- Let $|M| \geq R(m-1, k)$ (the second case is symmetric).
- Among vertices in M, there is either a red K_k or a blue K_{m-1}.
- In the first case, we get a red K_k in the original graph.
- In the second case, we get a blue K_m by adding v to the given K_{m-1} (all edges going from v to M are blue).

Properties of Ramsey Numbers

- Our recursive equation gives an exponential upper bound on $R(m, k)$.

Properties of Ramsey Numbers

- Our recursive equation gives an exponential upper bound on $R(m, k)$.
- Erdős (1947) proved an exponential lower bound for $R(k, k)$.

Properties of Ramsey Numbers

- Our recursive equation gives an exponential upper bound on $R(m, k)$.
- Erdős (1947) proved an exponential lower bound for $R(k, k)$.
- Calculating Ramsey's numbers for concrete m and k is hard: even $R(5, 5)$ is unknown (it is known that it is between 43 and 48).

Properties of Ramsey Numbers

- Our recursive equation gives an exponential upper bound on $R(m, k)$.
- Erdős (1947) proved an exponential lower bound for $R(k, k)$.
- Calculating Ramsey's numbers for concrete m and k is hard: even $R(5, 5)$ is unknown (it is known that it is between 43 and 48).
- Recursive inequation can be generalized to many colors:

$$R(k_1, \ldots, k_c) \leq R(k_1, \ldots, k_{c-2}, R(k_{c-1}, k_c)).$$