Identities on Sets

Reading Material
Discrete Mathematics Course
on Coursera

We have seen that the language of sets is a convenient language to talk about Combinatorics. On the other hand, it is one of the simplest math structures: sets are one of the most basic mathematical objects and operations on them are simple and natural[1] Due to their simplicity operations on sets provide a good setting to practice mathematical reasoning. This is exactly what we will do in this lesson.

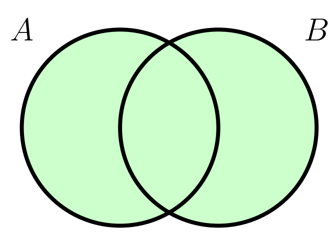
We have already seen some basic operations on sets.

One of them was a union operation. *Union* $A \cup B$ consists of all elements belonging to at least one of the sets A and B. Let us also introduce a standard mathematical notation for this:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Here the righthand side denotes the set. Before the bar we introduce the notation for elements of the set. In this case we denote elements by xs. After the bar we specify the condition that x should satisfy to be in the set. This gives us a way to introduce sets of elements by specifying the condition they should satisfy.

We have already introduce Venn diagrams and here is a Venn diagram for the union of two sets.



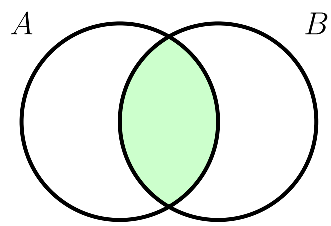
[1]We should mention here, however, that there are deep mathematical complications related to sets. We will not encounter them in this course and we will not discuss them.

Another operation we have seen was intersection. *Intersection* $A \cap B$ consists of all elements belonging to both sets. Here is again mathematical notation for this set:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

It is the same as before, except the condition had changed.

Here is a Venn diagram for intersection.

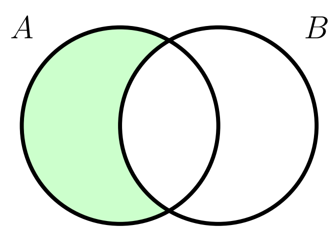


There are some other standard basic operations on sets we have not see before. We will introduce two of them.

Difference $A \smallsetminus B$ consists of all elements belonging to A, but not to B:

$$A \smallsetminus B = \{x \mid x \in A, \text{ but } x \notin B\}$$

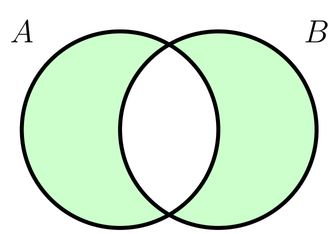
Here is a Venn diagram for difference.



Symmetric difference $A \triangle B$ is the set of all elements belonging to exactly one of the sets A and B:

$$A \triangle B = \{x \mid x \in A \text{ or } x \in B, \text{ but not both}\}$$

Here is a Venn diagram for symmetric difference.



We have also seen the notion of inclusion between sets. The set A is a *subset* of the set B if for any $x \in A$ we have $x \in B$. This is denoted by $A \subseteq B$. We can introduce equality between sets in the standard way: $A = B$ if A and B consist of the same elements.

We can make a simple, but important observation: $A = B$ iff $A \subseteq B$ and $B \subseteq A$. Indeed, $A = B$ means that A and B consist of exactly the same elements. In other words, all elements of A should be elements of B and all elements of B should be elements of A.

Using operations on sets we can construct complicated expressions from basic sets. For example, we can consider $(A \setminus B) \cap C$ or $(A \cup B) \setminus (A \cap B)$.

This is actually can be useful. For example, suppose we have a dataset and A is the set of objects in it. Denote by B the subset of objects we use for training the model (and the rest we will use for testing the model). Denote by C the set of all objects on which our model is correct. Then the set of all testing data on which the model is correct can be written as $(A \setminus B) \cap C$.

Suppose now there are arbitrary sets A, B and C. Suppose there are two formulas F and G constructed from A, B and C by set operations. We can consider equality of these expressions $F = G$. Sometimes this equality holds no matter what the sets A, B and C are. In this case wee say that the equation $F = G$ is *equivalence*. That is, the equivalence in the equation that is true for all possible sets A, B and C.

Sets A, B, C are actually variables in the expression $F = G$. That is, the equation $F = G$ is equivalence if it is true for all possible values of its variables.

Note that there actually might be more than three variable sets in the equation. The notion of equivalence is analogous for any number of variables. We used three variables A, B and C just to simplify the definition.

Let us consider a simple example of equivalence. Consider the equation $A \cap B = B \cap A$. In the previous notation here we have two variable sets A and B, and we have $F = A \cap B$ and $G = B \cap A$. This equality is called *commutativity of intersection* and it is actually an equivalence. This is pretty easy to see: both the left and the right side consists of all elements lying in both A and B.

This example is very simple, but still let us break down an explanation for it into basic steps. To show that two sets are equal we need to show that they consist of exactly the same elements. We have $x \in A \cap B$ if and only if $x \in A$ and $x \in B$. On the other hand, we have $x \in B \cap A$ if and only if $x \in B$ and $x \in A$ Obviously, these are the same conditions on x.

In this text we have introduced (and recalled) operation on sets, we have introduced the notion of equivalence on sets and we have considered a simple example. Next we will proceed to more advanced examples.