Problems in Combinatorics

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Outline

Distributing Assignments Among People

Distributing Candies Among Kids

Numbers with Fixed Sum of Digits

Numbers with Non-increasing Digits

Splitting into Working Groups

Distributing Assignments Among People

Problem

Suppose there are 4 people and 9 different assignments. Each person should receive one assignment. Assignments for different people should be different. How many ways are there to do it?

- We have to count selections of assignments for 4 people

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- In this problem people are different, so the selection is ordered

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- No assignment can be given to two persons simultaneously, so no repetitions

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- We have to count selections of assignments for 4 people
- In this problem people are different, so the selection is ordered
- No assignment can be given to two persons simultaneously, so no repetitions
- So we are dealing with k-permutations

Distributing Assignments Among People

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Suppose there are 4 people and 9 different assignments. Each person should receive one assignment. Assignments for different people should be different. How many ways are there to do it?

Persons	1	2	3	4
Number of options				

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Number of options	9			

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Persons	1	2	3	4
Number of options	9	8		

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Number of options	9	8	7	

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Persons	1	2	3	4
Number of options	9	8	7	6

Distributing Assignments Among People

Problem

Suppose there are 4 people and 9 different assignments. Each person should receive one assignment. Assignments for different people should be different. How many ways are there to do it?

- The answer is $9 \times 8 \times 7 \times 6 = 3024$

Distributing Assignments Among People

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- The answer is $9 \times 8 \times 7 \times 6 = 3024$
- Needed to count permutations

Distributing Assignments Among People

Problem

There are 4 people and 9 different assignments. We need to distribute all assignments among people. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from 0 to 9. How many ways are there to do it?

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There are 4 people and 9 different assignments. We need to distribute all assignments among people. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from 0 to 9. How many ways are there to do it?

- Each person receives several assignments

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- Each person receives several assignments
- Can try to look at persons one by one

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- The first person is assigned arbitrary subset; we know how to count the number of subsets

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- Each person receives several assignments
- Can try to look at persons one by one
- The first person is assigned arbitrary subset; we know how to count the number of subsets
- The second person: the number of options left depends on what we choose for the first person

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There are 4 people and 9 different assignments. We need to distribute all assignments among people. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from 0 to 9. How many ways are there to do it?

- Seems tricky

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- Idea: look from the other point of view!

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There are 4 people and 9 different assignments. We need to distribute all assignments among people. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from 0 to 9. How many ways are there to do it?

- Seems tricky
- Idea: look from the other point of view!
- Don't give assignments to people, assign people to assignments instead

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There are 4 people and 9 different assignments. We need to distribute all assignments among people. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from 0 to 9. How many ways are there to do it?

Assignment	1	2	3	4	5	6	7	8	9
Number of options									

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Assignment	1	2	3	4	5	6	7	8	9
Number of options	4	4	4	4	4	4	4	4	4

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There are 4 people and 9 different assignments. We need to distribute all assignments among people. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from 0 to 9. How many ways are there to do it?

- So the answer is $4^9 = 262\,144$

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- Just needed to count tuples
- But needed also to look from the other side

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Numbers with Non-increasing Digits

Splitting into Working Groups

Distributing Candies Among Kids

Problem

There are 15 identical candies. How many ways are there to distribute them among 7 kids?

- Giving each candy we choose one of 7 kids

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- Giving each candy we choose one of 7 kids
- Repetitions are allowed, the same kid can receive a candy several times
- Candies are identical, so the order does not matter
- We are dealing with combinations with repetitions!

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- Number of candies is the size of a combination
- Number of kids is the number of options
- So, the answer is $\binom{15+(7-1)}{(7-1)} = \binom{21}{6} = 54\,264$

More Fair Distribution

Problem

There are 15 identical candies. How many ways are there to distribute them among 7 kids in such a way that each kid receives at least 1 candy?

- The previous solution does not work, we have additional restriction

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- But we can reduce the problem to the previous one

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- But we can reduce the problem to the previous one
- We have to give each kid at least one candy

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- But we can reduce the problem to the previous one
- We have to give each kid at least one candy
- So let's just do it!

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- Combinations with repetitions, the size of the combination is 8, the number of options is 7

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- Now we are left with $15 - 7 = 8$ candies
- And we can distribute them as in the previous problem!
- Combinations with repetitions, the size of the combination is 8, the number of options is 7
- So, the answer is $\binom{8+(7-1)}{(7-1)} = \binom{14}{6} = 3\,003$

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- Compare to the answer to the previous problem: $54\,264$

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- So, the answer is $\binom{8+(7-1)}{(7-1)} = \binom{14}{6} = 3\,003$
- Compare to the answer to the previous problem: $54\,264$
- Vast majority of ways to share candies will leave some kids without any

Outline

Distributing Assignments Among People

Distributing Candies Among Kids

Numbers with Fixed Sum of Digits

Numbers with Non-increasing Digits

Splitting into Working Groups

Numbers with Fixed Sum of Digits

Problem

How many non-negative integer numbers are there below 10 000 such that their sum of digits is equal to 9?

- Consider numbers as sequences of 4 digits

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How many non-negative integer numbers are there below 10 000 such that their sum of digits is equal to 9?

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- We can look at the problem from the side of numbers

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- Nine options for the first one; but already unclear for the second one

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- Consider numbers as sequences of 4 digits
- We can look at the problem from the side of numbers
- Nine options for the first one; but already unclear for the second one
- Idea: look from the other side

Numbers with Fixed Sum of Digits

$$* \quad * \quad * \quad *$$

- There are four positions

Numbers with Fixed Sum of Digits

$$* \quad * \quad * \quad *$$

- There are four positions
- We split the weight 9 among them

Numbers with Fixed Sum of Digits

0 0 0 0

- There are four positions
- We split the weight 9 among them
- Assume they are 0 in the beginning and add 1 nine times

Numbers with Fixed Sum of Digits

0 0 0 0

- There are four positions
- We split the weight 9 among them
- Assume they are 0 in the beginning and add 1 nine times
- Each time we pick one of the digits to increase

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- There are four positions
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Numbers with Fixed Sum of Digits

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- There are four positions
- We split the weight 9 among them
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- Order does not matter; there are repetitions
- We have combinations of size 9 among 4 options

Numbers with Fixed Sum of Digits

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- There are four positions
- We split the weight 9 among them
- Assume they are 0 in the beginning and add 1 nine times
- Each time we pick one of the digits to increase
- Order does not matter; there are repetitions
- We have combinations of size 9 among 4 options
- The answer is $\binom{9+(4-1)}{(4-1)} = \binom{12}{3} = 220$

Numbers with Fixed Sum of Digits

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- Looks very similar to the previous one

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- Distribute ten ones between four positions

Numbers with Fixed Sum of Digits

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How many non-negative integer numbers are there below 10 000 such that their sum of digits is equal to 10?

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- Distribute ten ones between four positions
- Combinations of size 10 among 4 options

Numbers with Fixed Sum of Digits

Problem

How many non-negative integer numbers are there below 10 000 such that their sum of digits is equal to 10?

- Looks very similar to the previous one
- Distribute ten ones between four positions
- Combinations of size 10 among 4 options
- The answer is $\binom{10+(4-1)}{(4-1)} = \binom{13}{3} = 286$

Numbers with Fixed Sum of Digits

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- Is everything right?

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- Distribute ten ones between four positions
- Combinations of size 10 among 4 options
- The answer is $\binom{10+(4-1)}{(4-1)} = \binom{13}{3} = 286$
- Is everything right? Let's check!

Numbers with Fixed Sum of Digits

The following code searches through all combinations:

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import itertools as it
count = 0
for d in it.product(range(10), repeat = 4):
    if sum(d) == 10:
        count += 1
print(count)
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Here is the output

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282
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The answer is off by 4!

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The answer is off by 4!

What went wrong?

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- With our approach we can assign all ten ones to the same position

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- What should we do now?

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- With our approach we can assign all ten ones to the same position
- But digits should be at most 9
- What should we do now?
- Just subtract the number of things that we should not have counted!

Numbers with Fixed Sum of Digits

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- What we should not have counted

Numbers with Fixed Sum of Digits

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- What we should not have counted
- Assignments of all ten ones to the same digit

Numbers with Fixed Sum of Digits

$$* \quad * \quad * \quad *$$

- What we should not have counted
- Assignments of all ten ones to the same digit
- But there are just 4 of them!

Numbers with Fixed Sum of Digits

10 0 0 0

- What we should not have counted
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Numbers with Fixed Sum of Digits

$$0 \quad 10 \quad 0 \quad 0$$

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Numbers with Fixed Sum of Digits

0 0 10 0

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- So our previous answer was of by just 4

Numbers with Fixed Sum of Digits

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- What we should not have counted
- Assignments of all ten ones to the same digit
- But there are just 4 of them!
- So our previous answer was of by just 4
- The correct answer is $286 - 4 = 282$

Outline

Distributing Assignments Among People

Distributing Candies Among Kids

Numbers with Fixed Sum of Digits

Numbers with Non-increasing Digits

Splitting into Working Groups

Numbers with Non-increasing Digits

Problem

How many four-digit numbers are there such that their digits are not increasing? Three-digit numbers are also four-digit, they just start with 0

Numbers with Non-increasing Digits

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How many four-digit numbers are there such that their digits are not increasing? Three-digit numbers are also four-digit, they just start with 0

- If we try to count options for each position and apply the product rule, there are problems

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Problem

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- If we try to count options for each position and apply the product rule, there are problems
- 10 options for the first position, but for the second the number of options depends on the first number

Numbers with Non-increasing Digits

Problem

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- If we try to count options for each position and apply the product rule, there are problems
- 10 options for the first position, but for the second the number of options depends on the first number
- Idea: look from the other side

Numbers with Non-increasing Digits

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- We pick digits from 0 to 9 to be in our number

Numbers with Non-increasing Digits

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- We pick digits from 0 to 9 to be in our number
- Once we picked four digits, our number is uniquely determined

Numbers with Non-increasing Digits

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Picked 3, 4 ,3, 7

- We pick digits from 0 to 9 to be in our number
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Numbers with Non-increasing Digits

7   4   3   3

Picked 3, 4 ,3, 7

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Numbers with Non-increasing Digits

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- We pick digits from 0 to 9 to be in our number
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- We pick digits from 0 to 9 to be in our number
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- We have combinations of size 4 from 10 options

Numbers with Non-increasing Digits

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- We pick digits from 0 to 9 to be in our number
- Once we picked four digits, our number is uniquely determined
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- We have combinations of size 4 from 10 options
- The answer is $\binom{4+(10-1)}{(10-1)} = \binom{13}{9} = 715$

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- Distributing Assignments Among People
- Distributing Candies Among Kids
- Numbers with Fixed Sum of Digits
- Numbers with Non-increasing Digits
- Splitting into Working Groups

Splitting into Working Groups

Problem

There are 12 students in the class. How many ways are there to split them into working groups of size 2 to work on the same assignment?

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- This problems is more tricky
- There are several ways to solve it
- But we need to combine several ideas

Splitting into Working Groups

- One solution goes by looking from the position of working groups

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- And so on

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- And so on
- Overall we have $\binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}$ options

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- Are we done?

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- Are we done? No!

Splitting into Working Groups

- Enumerate people by numbers from 1 to 12

Splitting into Working Groups

$$\{3, 7\}, \{1, 5\}, \{6, 9\}, \{11, 2\}, \{8, 12\}, \{4, 10\}$$

- Enumerate people by numbers from 1 to 12
- For example, we have counted these splittings in the groups

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- For example, we have counted these splittings in the groups
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- Order between groups also does not matter!
- So, what to do now?
- Apply the old idea! We have counted each splitting $6!$ times, for each permutation of 6 groups

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- In our first attempt the answer was $\binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}$

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- So the right answer is

$$\binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} \times \frac{1}{6!} =$$

$$\frac{12 \times 11}{2} \times \frac{10 \times 9}{2} \times \frac{8 \times 7}{2} \times \frac{6 \times 5}{2} \times \frac{4 \times 3}{2} \times \frac{2 \times 1}{2} \times \frac{1}{6!} =$$

$$\frac{12!}{6! \times 2^6} = 10\,395$$

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- There are situations that are so complicated, they stay unresolved
- Next week we will proceed to Probability