Why Venn diagrams Provide Proofs?

Reading Material
Discrete Mathematics Course
on Coursera

In the previous segments we have discussed a rigorous way to prove equivalences on sets. But we have also discussed analysis through Venn diagrams that was much more convenient. But are these pictures actually provide us with proofs? Venn diagrams look convincing, but why drawing such a picture is a rigorous argument? And actually is it a rigorous argument? It turns out that Venn diagrams actually provide a complete proof! This is what we will discuss in this segment.

Consider an example:

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C).$$

To check that this is an equivalence we need to check that the left and the right side contain exactly the same set of elements. Observe that for each x we can tell whether it belong to each of the sets if we know which of the following statements are true and which are false: $x \in A$, $x \in B$ and $x \in C$.

Indeed, for example, consider x such that $x \in A$, $x \notin B$ and $x \in C$. Then $x \in A \cup B$. Then $x \notin (A \cup B) \setminus C$. For the other side, $x \notin (A \setminus C)$ and $x \notin (B \setminus C)$. That is, $x \notin (A \setminus C) \cup (B \setminus C)$.

There are 8 possible ways to pick, which of the statements are true and which are false. Here is the complete list:
$x \notin A$, $x \notin B$, $x \notin C$
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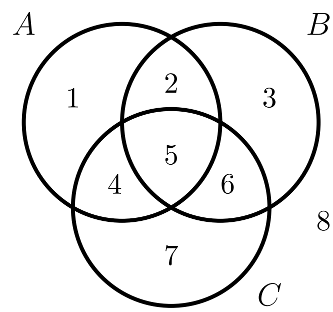
But we are studying Combinatorics, so let us calculate the number of cases using our knowledge. There are three statements: $x \in A$, $x \in B$ and $x \in C$. For each we have to pick one of the two options: True or False. Observe these are tuples! So, there are $2^3 = 8$ possibilities.

So there are 8 possible ways to pick, which of the statements are true and which are false. To check the equivalence it is enough to consider all 8 cases. For each case for both sides of equivalence we need to check, whether they contain corresponding x. If in all cases either both sides contain x, or both sides do not contain x, then they are equivalent. And if there is a case when one side contains such an x and the other side does not, they are not equivalent.

Now observe that on Venn diagram we consider exactly these 8 cases!



For example, area 1 corresponds to $x \in A$, $x \notin B$ and $x \notin C$. Area 4 corresponds to $x \in A$, $x \notin B$ and $x \in C$. Area 5 corresponds to $x \in A$, $x \in B$ and $x \in C$. And area 8 corresponds to $x \notin A$, $x \notin B$ and $x \notin C$.

Thus, Venn diagram actually corresponds to a complete case analysis. So, it provides a rigorous proof of equivalences!

Now, what happens in the general case, when we have more than three base sets in our equivalence? If there are n base sets, we cannot depict them on Venn diagram (or at least the picture becomes much more complicated). But the argument with case analysis still works. So, can we check arbitrarily large equivalences?

In fact, for n basic sets now we need to analyze 2^n cases. This is a lot, so this becomes impractical for large n. Is there another way to check equivalences effectively? This is actually a big open problem! It is one of numerous equivalent formulations of P vs. NP famous problem.

In this lesson we have studied sets and operations on them. This is one of the most basic mathematical structures capturing basic logical reasoning. For us it is a good way to practice rigorous reasoning. We have also seen that sometimes mathematical proofs enjoy graphic representation. This makes reasoning much easier.