Proofs of Identities

Reading Material
Discrete Mathematics Course
on Coursera

In the previous text we have introduced operation on sets, introduced the notion of equivalence on sets and considered a simple example. Now we will consider a more advanced example.

As our example we will show that we can actually express symmetric difference through other operations:

$$A \triangle B = (A \cup B) \setminus (A \cap B)$$

Let's try to prove this. For convenience let us denote $F = (A \cup B) \setminus (A \cap B)$. For the lefthand side we have $x \in A \triangle B$ if and only if x is in exactly one of the two sets A and B.

Now, for the righthand side we have $x \in F$ if and only if $x \in A \cup B$, but $x \notin A \cap B$. Here we used the definition of set difference. How we have to decompose this further. We have $x \in A \cup B$ if and only if $x \in A$ or $x \in B$. We have $x \in A \cap B$ if and only if $x \in A$ and $x \in B$ Substituting this into the condition for F overall we have $x \in F$ if and only if x is in at least one of the sets A and B, but not in both sets A and B.

So, after decomposing both sides of the equation we have the following. On one hand, $x \in A \triangle B$ if and only if x is in exactly in one of the two sets A and B. On the other hand, $x \in F$ if and only if x is in at least one of the sets A and B, but not in both sets A and B. Note that this is the same condition! Thus on both sides of the equation we have the same set and this is equivalence.

Overall, the proof scheme is straightforward. First, we decompose both sides of equivalence by the definitions of set operations. Next, we check that both decompositions are the same.

However, this might become complicated. Decompositions of sides of the equation might become long and complex. It might get hard to keep everything in mind.

To overcome this problem we will consider another somewhat simpler approach. This approach works better for complicated formulas.

Suppose we want to check equivalence $F = G$. To show that $F = G$ it is enough to show that $F \subseteq G$ and $G \subseteq F$ The idea is to show these two inclusions separately. To show that $F \subseteq G$ we can consider $x \in F$, decompose what it means by definitions and show that $x \in G$ To check that $G \subseteq F$ we can do a symmetric argument.

To illustrate this let us consider out previous example:

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Suppose $x \in A \triangle B$. This means that exactly one of the following is true: $x \in A, x \in B$. Clearly, such an x is in $A \cup B$. Clearly, such an x is not in $A \cap B$. Thus, x is in the difference of these two sets, that is, x is in the righthand side.

In the other direction, suppose $x \in (A \cup B) \setminus (A \cap B)$. Then, $x \in A \cup B$ and $x \notin A \cap B$ This means that x is in at least one of the sets A and B, and x is not in both sets A and B. Thus x is in exactly one of the sets A and B. That is, $x \in A \triangle B$.

We basically repeated the same argument as before, but now it is somewhat easier to go through the argument. We do not have to keep track of having 'if and only if' statements each time. We also do not have to parse complicated statements. Instead we can process one operation at a time. In the next segment we will see how to visualize these arguments.