Connectedness

Stepan Kuznetsov

Computer Science Department, Higher School of Economics

Outline

Connected Graphs and Connected Components

Inequations on the Number of Connected Components

Circuit Rank

Connected Vertices

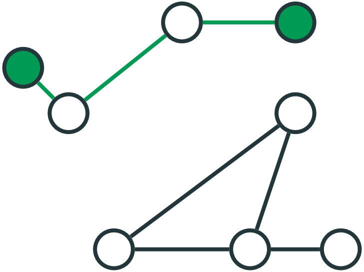
- Two vertices in a graph are connected, if there exists a path between them.

Connected Vertices

- Two vertices in a graph are connected, if there exists a path between them.
- In other words, u and v are connected, if $d(u, v) < \infty$.

Connected Vertices

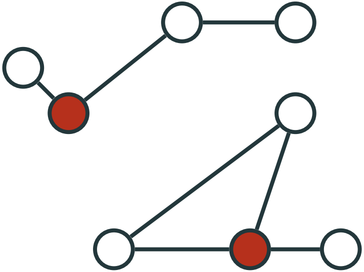
- Two vertices in a graph are connected, if there exists a path between them.
- In other words, u and v are connected, if $d(u, v) < \infty$.



connected

Connected Vertices

- Two vertices in a graph are connected, if there exists a path between them.
- In other words, u and v are connected, if $d(u, v) < \infty$.



not connected

Airlines

Jackexu10 @ Wikipedia

Airlines

- Vertices = airports (cities).

Airlines

- Vertices = airports (cities).
- Edges = direct flights.

Airlines

- Vertices = airports (cities).
- Edges = direct flights.
- Usually if there is a flight form A to B, there is also a flight from B to A. So our graph is undirected.

Airlines

- Vertices = airports (cities).
- Edges = direct flights.
- Usually if there is a flight form A to B, there is also a flight from B to A. So our graph is undirected.
- Cities A and B are connected if you can get from A to B by air (possibly changing airplanes).

Airlines

- The whole graph (airline network) is connected, if any two vertices are connected.

Airlines

- The whole graph (airline network) is connected, if any two vertices are connected.
- For example, one can get from St. Petersburg, Russia (LED) to Lima, Peru (LIM) via the following route:

LED — CDG — ATL — LIM

Connected Components

- Vertex v belongs to the connected component of vertex u, if v and u are connected.

Connected Components

- Vertex v belongs to the connected component of vertex u, if v and u are connected.
- Denote the connected component of u by $[u]$.

Connected Components

- Vertex v belongs to the connected component of vertex u, if v and u are connected.
- Denote the connected component of u by $[u]$.
- It is easy to see that if $v \in [u]$, then $[v] = [u]$...

Connected Components

- Vertex v belongs to the connected component of vertex u, if v and u are connected.
- Denote the connected component of u by $[u]$.
- It is easy to see that if $v \in [u]$, then $[v] = [u]$...
- ... and if $v \notin [u]$, then $[v] \cap [u] = \emptyset$.

Connected Components

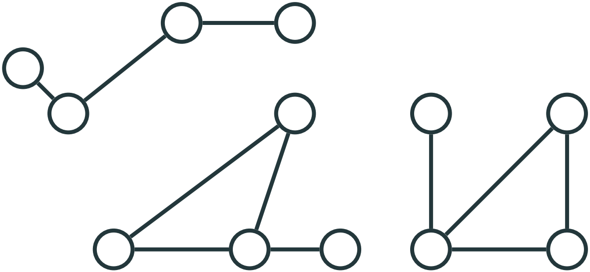
- Thus, each graph gets split into several disjoint connected components.

Connected Components

- Thus, each graph gets split into several disjoint connected components.
- The number of connected components is a graph invariant.

Connected Components

- Thus, each graph gets split into several disjoint connected components.
- The number of connected components is a graph invariant.
- Example (3 connected components):



Connected Components

- In airlines, the major airline network forms a big connected component.

Connected Components

- In airlines, the major airline network forms a big connected component.
- However, there could be small local airline networks not connected to the major one.

Connected Components

- In airlines, the major airline network forms a big connected component.
- However, there could be small local airline networks not connected to the major one.
- Each connected component can be seen as an independent graph.

Connected Components

- In airlines, the major airline network forms a big connected component.
- However, there could be small local airline networks not connected to the major one.
- Each connected component can be seen as an independent graph.
- In particular, each component obeys all graph-theoretic principles, for example, handshaking lemma.

Componentwise Handshaking

- **Question.** Suppose there is one city with 15 direct airline connections, one city with only 3, and several cities with 10. Prove that the first two cities are connected.

Componentwise Handshaking

- **Question.** Suppose there is one city with 15 direct airline connections, one city with only 3, and several cities with 10. Prove that the first two cities are connected.
- **Answer.** If not, they are in different connected components. But then each of these components violates the handshaking lemma, having exactly one odd vertex.

Outline

Connected Graphs and Connected Components

Inequations on the Number of Connected Components

Circuit Rank

The Minimum

- Let us estimate the minimal number of connected components in a graph with n vertices and m edges.

The Minimum

- Let us estimate the minimal number of connected components in a graph with n vertices and m edges.
- A connected graph on n vertices should have at least $n - 1$ edges (minimal example: tree).

The Minimum

- Let us estimate the minimal number of connected components in a graph with n vertices and m edges.
- A connected graph on n vertices should have at least $n-1$ edges (minimal example: tree).
- Proof: pick one vertex as the starting point and add other vertices one by one. Each of these vertices should have at least one new edge in order to get connected.

The Minimum

- Let us estimate the minimal number of connected components in a graph with n vertices and m edges.
- A connected graph on n vertices should have at least $n-1$ edges (minimal example: tree).
- Proof: pick one vertex as the starting point and add other vertices one by one. Each of these vertices should have at least one new edge in order to get connected.
- Thus, if $m < n-1$, then the graph has at least two connected components.

The Minimum

- More generally, if a graph on n vertices has k connected components, it should have at least $n - k$ edges.

The Minimum

- More generally, if a graph on n vertices has k connected components, it should have at least $n - k$ edges.
- Proof: take a starting vertex in each connected component (there will be k of them). Each other vertex brings at least one new edge.

The Minimum

- More generally, if a graph on n vertices has k connected components, it should have at least $n - k$ edges.
- Proof: take a starting vertex in each connected component (there will be k of them). Each other vertex brings at least one new edge.
- Thus, $m \geq n - k$.

The Minimum

- More generally, if a graph on n vertices has k connected components, it should have at least $n - k$ edges.
- Proof: take a starting vertex in each connected component (there will be k of them). Each other vertex brings at least one new edge.
- Thus, $m \geq n - k$.
- In other words, $k \geq n - m$.

The Maximum

- However, even if $m \geq n - 1$, the graph could be disconnected.

The Maximum

- However, even if $m \geq n - 1$, the graph could be disconnected.
- How could we guarantee connectedness?

The Maximum

- However, even if $m \geq n - 1$, the graph could be disconnected.
- How could we guarantee connectedness?
- If a connected component includes n_1 vertices, then the number of edges is less or equal than $n_1 \cdot (n_1 - 1)/2$.

The Maximum

- However, even if $m \geq n-1$, the graph could be disconnected.
- How could we guarantee connectedness?
- If a connected component includes n_1 vertices, then the number of edges is less or equal than $n_1 \cdot (n_1 - 1)/2$.
- Thus, if a graph has two (or more) connected components, $n_1 + n_2 = n$, then $m \leq n_1 \cdot (n_1 - 1)/2 + n_2 \cdot (n_2 - 1)/2$.

The Maximum

- However, even if $m \geq n-1$, the graph could be disconnected.
- How could we guarantee connectedness?
- If a connected component includes n_1 vertices, then the number of edges is less or equal than $n_1 \cdot (n_1 - 1)/2$.
- Thus, if a graph has two (or more) connected components, $n_1 + n_2 = n$, then $m \leq n_1 \cdot (n_1 - 1)/2 + n_2 \cdot (n_2 - 1)/2$.
- If m is greater, the graph should be connected.

The Maximum

- The maximal value of $n_1 \cdot (n_1 - 1)/2 + n_2 \cdot (n_2 - 1)/2$ is the one with $n_1 = n - 1$ and $n_2 = 1$ (or symmetrically, $n_1 = 1$, $n_2 = n - 1$).

The Maximum

- The maximal value of $n_1 \cdot (n_1 - 1)/2 + n_2 \cdot (n_2 - 1)/2$ is the one with $n_1 = n - 1$ and $n_2 = 1$ (or symmetrically, $n_1 = 1, n_2 = n - 1$).
- Thus, if $m \geq (n - 1) \cdot (n - 2)/2$, then the graph should be connected.

The Maximum

- The maximal value of $n_1 \cdot (n_1 - 1)/2 + n_2 \cdot (n_2 - 1)/2$ is the one with $n_1 = n - 1$ and $n_2 = 1$ (or symmetrically, $n_1 = 1$, $n_2 = n - 1$).
- Thus, if $m \geq (n-1) \cdot (n-2)/2$, then the graph should be connected.
- We leave it as an exercise to find out the inequation for m sufficient for a graph to have no more than k connected component.

Outline

Connected Graphs and Connected Components

Inequations on the Number of Connected Components

Circuit Rank

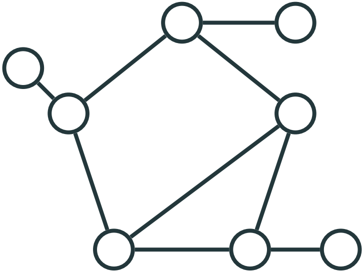
Circuit Rank

- While connectedness ensures the *possibility* of getting from one vertex to another, the circuit rank measures *reliability* of the system.

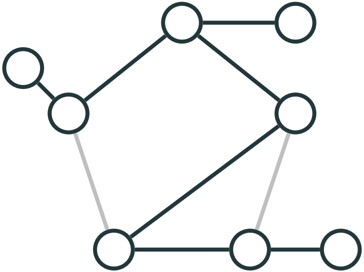
Circuit Rank

- While connectedness ensures the *possibility* of getting from one vertex to another, the circuit rank measures *reliability* of the system.
- The circuit rank is the graph invariant counted as follows: it is the minimum number of edges which should be removed from the graph to break all its cycles (i.e., to make it a tree or forest).

Circuit Rank: Example



Circuit Rank: Example



circuit rank = 2

History and Applications

- The notion of circuit rank was introduced by G. Kirchhof for electric circuits.

History and Applications

- The notion of circuit rank was introduced by G. Kirchhof for electric circuits.
- In chemistry (for graphs of molecular structure) it is known as the Frèrejacque number.

History and Applications

- The notion of circuit rank was introduced by G. Kirchhof for electric circuits.
- In chemistry (for graphs of molecular structure) it is known as the Frèrejacque number.
- Circuit rank is used in computer science (software metrics) for estimating structural complexity of program code (so-called cyclomatic complexity).

Computing the Circuit Rank

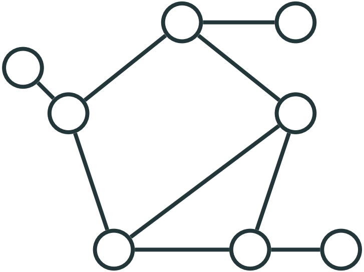
- Notice that the circuit rank of a graph is *not* merely the number of cycles.

Computing the Circuit Rank

- Notice that the circuit rank of a graph is *not* merely the number of cycles.
- Cycles could be dependent, and removing one edge can destroy more than one cycle.

Computing the Circuit Rank

- Notice that the circuit rank of a graph is *not* merely the number of cycles.
- Cycles could be dependent, and removing one edge can destroy more than one cycle.
- Example:



circuit rank = 2; # cycles = 3

Computing the Circuit Rank

- Fortunately, it is easy to compute the circuit rank, since there is an explicit formula.

Computing the Circuit Rank

- Fortunately, it is easy to compute the circuit rank, since there is an explicit formula.
- $r = m - n + c$,
  where m is the number of edges, n is the number of vertices, and c is the number of connected components.

Formula for Circuit Rank: Proof

- $r = m - n + c$

Formula for Circuit Rank: Proof

- $r = m - n + c$
- It is sufficient to consider one connected component (i.e., a connected graph), since they are independent.

Formula for Circuit Rank: Proof

- $r = m - n + c$
- It is sufficient to consider one connected component (i.e., a connected graph), since they are independent.
- For a connected graph, we have to prove $r = m - n + 1$.

Formula for Circuit Rank: Proof

- $r = m - n + c$
- It is sufficient to consider one connected component (i.e., a connected graph), since they are independent.
- For a connected graph, we have to prove $r = m - n + 1$.
- If we remove $m - n + 1$ edges, then the number of edges becomes $n - 1$, which is the minimal possible for a connected graph.

Formula for Circuit Rank: Proof

- $r = m - n + c$
- It is sufficient to consider one connected component (i.e., a connected graph), since they are independent.
- For a connected graph, we have to prove $r = m - n + 1$.
- If we remove $m - n + 1$ edges, then the number of edges becomes $n - 1$, which is the minimal possible for a connected graph.
- Thus, this graph is a tree (otherwise we could remove one more edge), so $r \leq m - n + 1$.

Formula for Circuit Rank: Proof

- On the other hand, if we remove less than $m - n + 1$ edges, the graph could not become a tree or forest: in a forest, the number of edges is $\leq n - 1$.

Formula for Circuit Rank: Proof

- On the other hand, if we remove less than $m - n + 1$ edges, the graph could not become a tree or forest: in a forest, the number of edges is $\leq n - 1$.
- Thus, the graph would still contain cycles.

Formula for Circuit Rank: Proof

- On the other hand, if we remove less than $m - n + 1$ edges, the graph could not become a tree or forest: in a forest, the number of edges is $\leq n - 1$.
- Thus, the graph would still contain cycles.
- Therefore, $r \geq m - n + 1$.