Connectedness

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Outline

Connectedness of Random Graphs

Random Graph

- Recall that $G(n, p)$ is the Erdős – Rényi random graph on n vertices, where each edge is planted with probability p, independently from others.

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- The probability p could depend on n.
- We start with $p = \frac{1}{2}$.
- Recall that the probability of $G(n, \frac{1}{2})$ to have an isolated vertex tends to 0 when $n \to \infty$.

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- The problem is that our cases overlap too much.

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- This implies connectedness, so connectedness will also have desired probability.

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- Indeed, v_i and v_j do not have a common neighbour if and only if for any w from the remaining $n-2$ vertices one of the edges $\langle w, v_i \rangle$ and $\langle w, v_j \rangle$ was not drawn. This happens with probability $\frac{3}{4}$.

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  - if $p > \frac{(1+\varepsilon)\ln n}{n}$, then $G(n, p)$ is almost surely connected.