Subgraphs

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Outline

Cliques, Independent Sets

Vertex Covers

Approximating Optimal Vertex Cover

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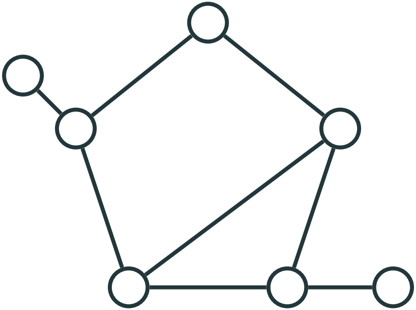

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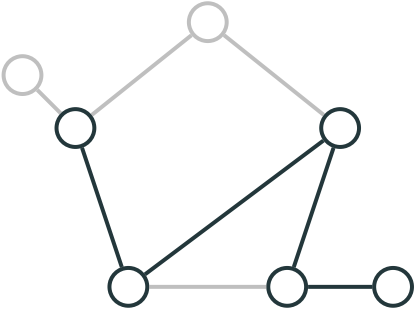

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- A *spanning* subgraph includes all vertices of the original graph (but maybe not all edges).

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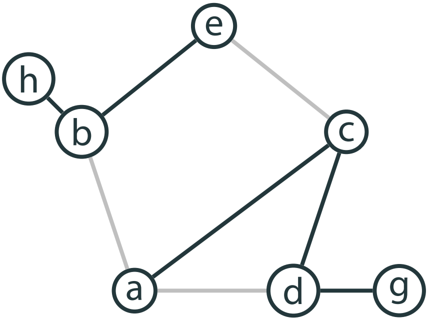

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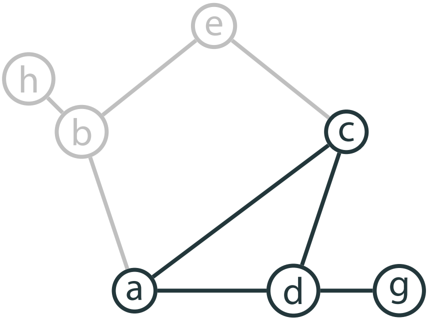

induced

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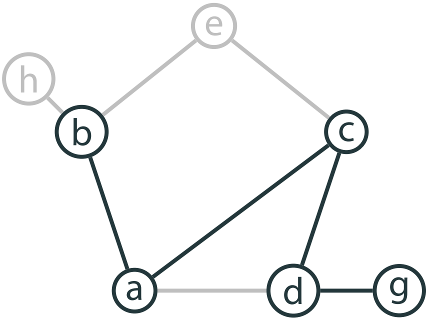

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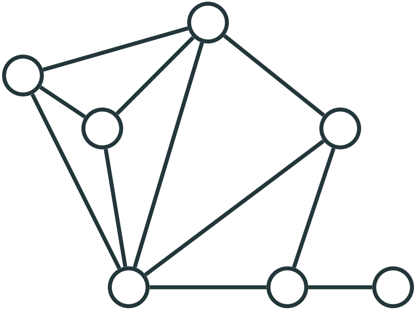

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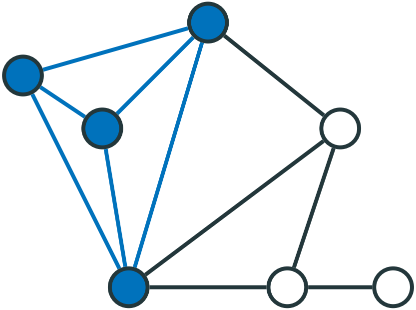

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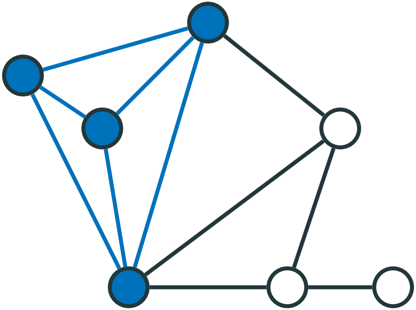

- Clique in social network graph = group of users who are friends with each other.

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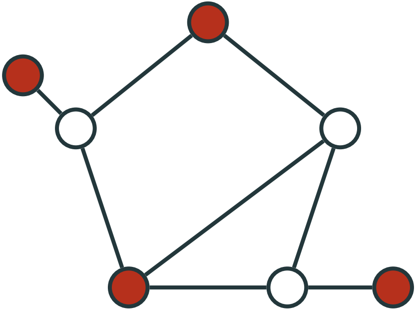

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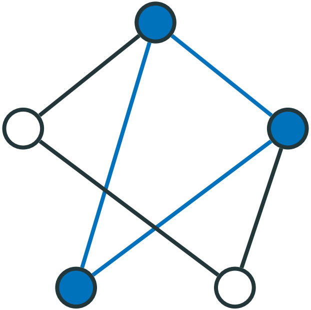

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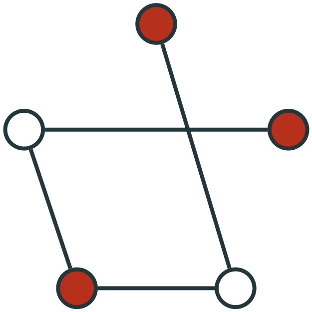

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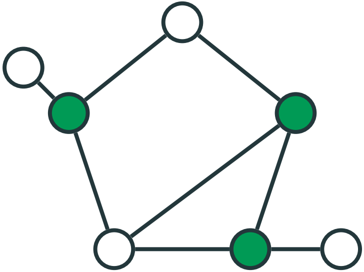

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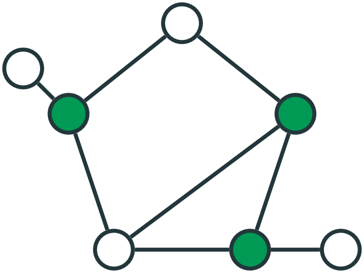

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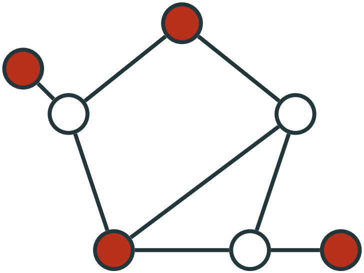

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- Indeed, no edge could connect two vertices outside C.
- Notice that this is a *different* duality than that of independent sets and cliques: here we take the complement of the set, not of the graph.

Computing the Optimal Vertex Cover

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  - If no, we do the same, but force the roots of these subtrees to belong to C.
  - Finally, we compare the results and yields the smaller one.

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- For arbitrary graphs, however, the task of finding an optimal vertex cover is hard (not solvable in polynomial time unless P=NP).
- In the next section we present an approximation: an algorithm which finds a vertex cover which is no more than twice bigger than the optimal one.

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- Suppose there is a set of m edges in a graph, such that none of these edges are adjacent.

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- Otherwise, some of the edges in the matching would be left uncovered.

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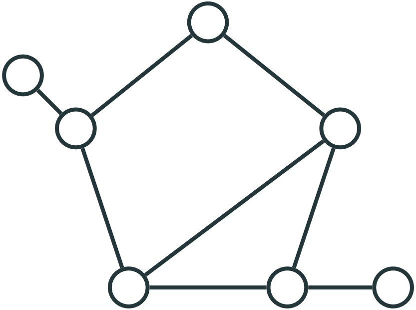

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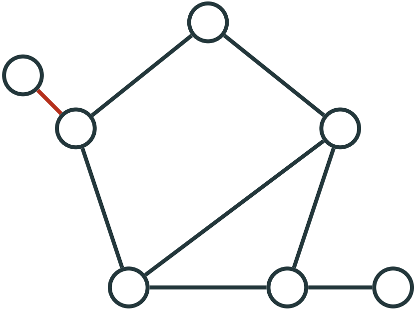

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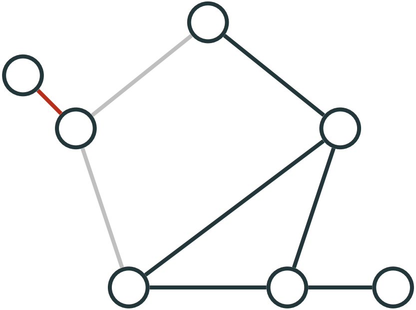

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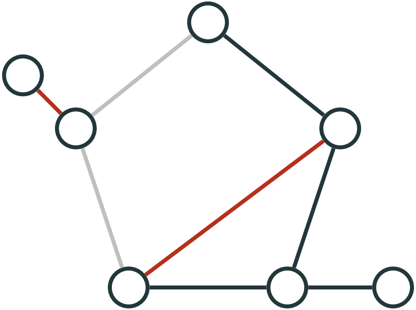

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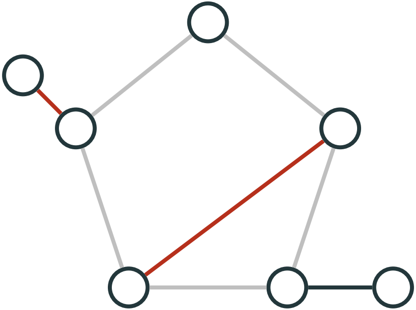

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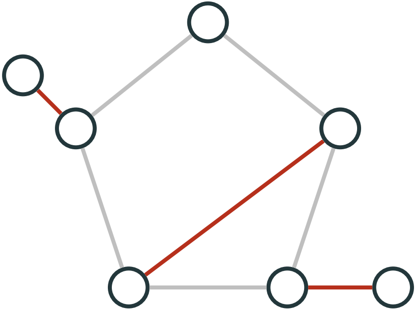

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- Thus, C is a vertex cover!

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- Thus, we have constructed a vertex cover which is no more than twice bigger than the optimal one: $|C| \leq 2 \cdot |C_{\text{opt}}|$.

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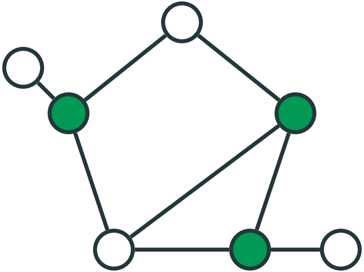

the optimal vertex cover

$$|C_{\text{opt}}| = 3$$

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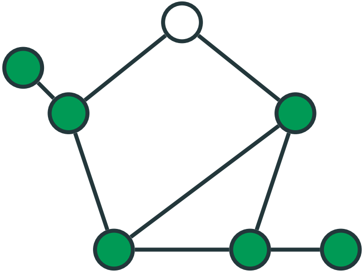

the approximation given by the algorithm

$$|C| = 6$$