Degrees and Distances

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Outline

Graph Invariants and Graph Isomorphism

Handshaking Lemma

Clustering Coefficients

Distances. Diameter. Eccentricity

Graph Isomorphism

Sometimes graphs look different, but essentially are the same...

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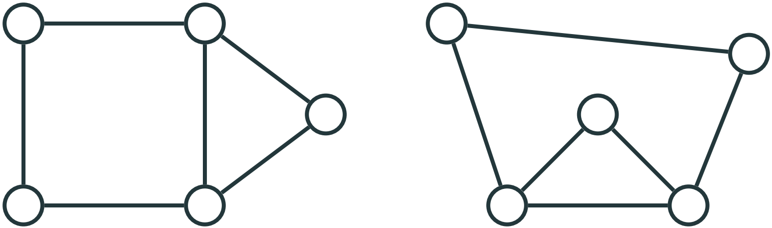

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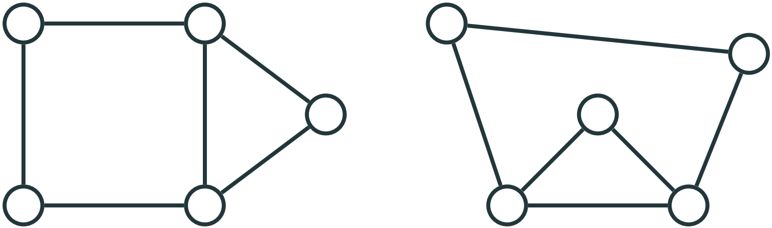

Here both graphs can be described as “a triangle and a quadrangle sharing a common edge.”

Graph Isomorphism

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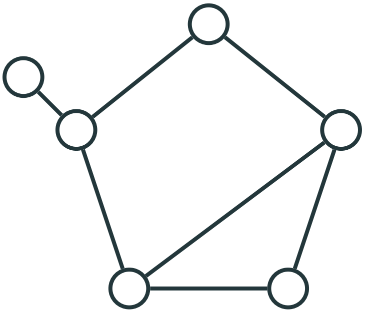

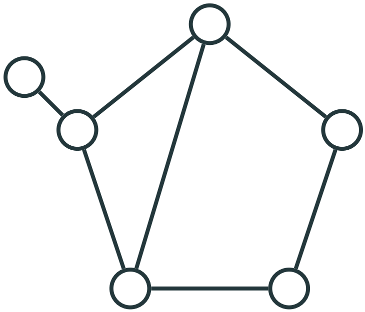

Graph Isomorphism

Isomorphic Graphs

Two graphs, $\mathcal{G}$ and $\mathcal{H}$, are *isomorphic,* if they have the same number n of vertices and vertices of each graph can be enumerated by numbers from 1 to n, so that vertices with numbers i and j are connected in $\mathcal{G}$ if and only if vertices with these numbers are connected in $\mathcal{H}$.

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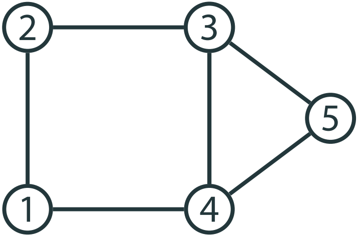
Isomorphic graphs can be seen as *different representations of the same graph*.

Isomorphic Graphs

$\mathcal{G}$



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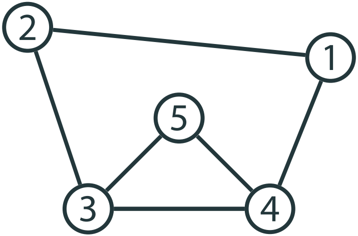

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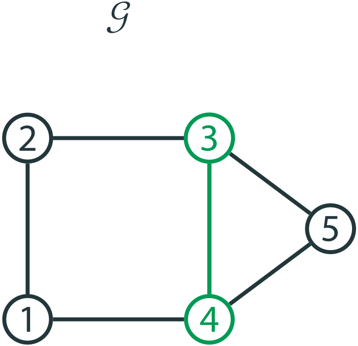

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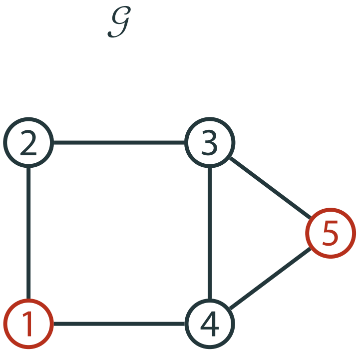

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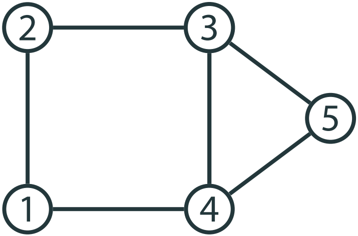

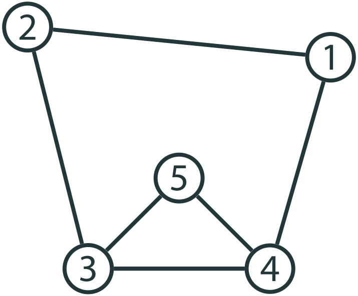

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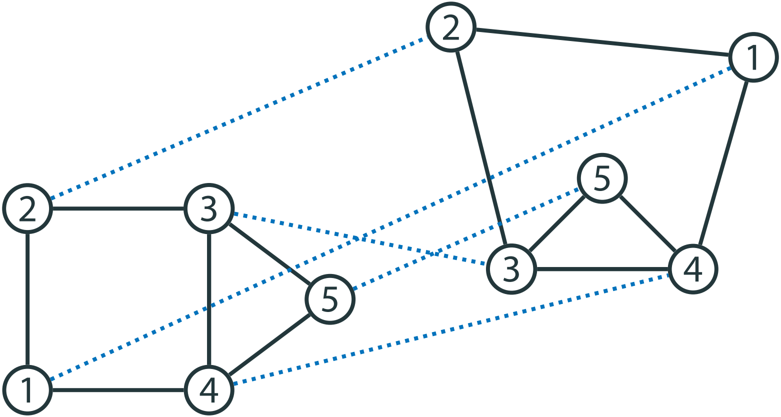

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- Graph invariants could capture interesting *properties* of graphs.

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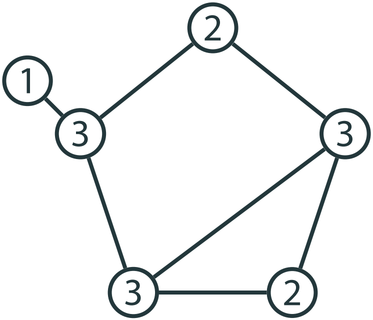

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- For example, if in $\mathcal{G}$ there are 14 vertices of degree 5 and in $\mathcal{H}$ there are 17 such vertices, then these graphs are not isomorphic.

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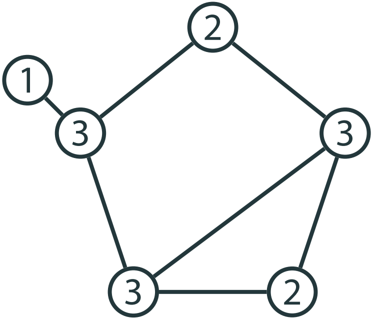

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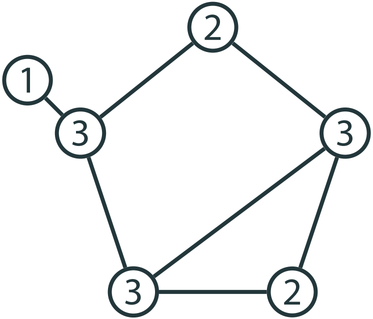

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number of edges $= 7$

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- Proof: in the sum of degrees, each edge is counted exactly twice.

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- This is a contradiction, since this sum is $2 \cdot (\text{number of edges})$.

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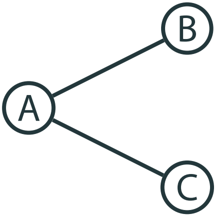

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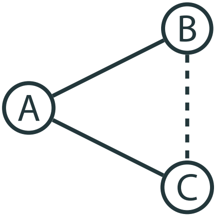

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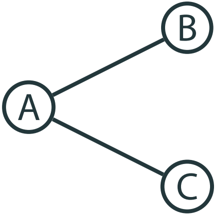

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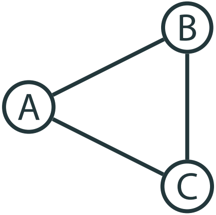

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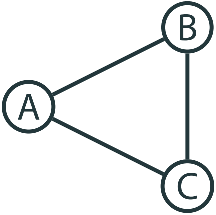

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- Thus, the GCC is the *probability* for a random triplet A, B, C in $\mathcal{G}$ to be closed (that is, B and C connected).

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- If A is an isolated vertex (degree = 0), then $LCC(A)$ is undefined (zero-by-zero division).

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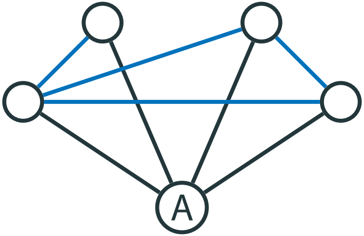

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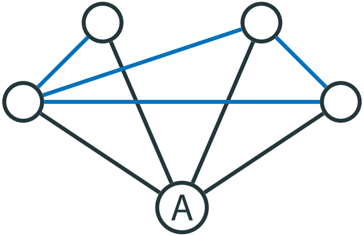

In this example, $LCC(A) = \frac{2 \cdot 4}{4 \cdot 3} = \frac{2}{3}$.

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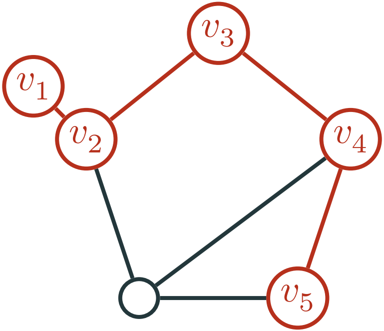

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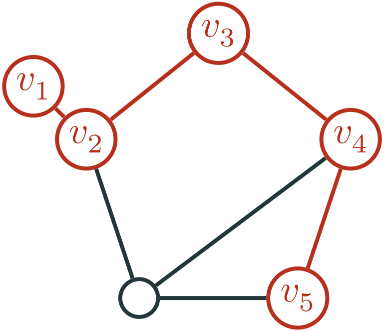

- The *length* of the path is the number of *edges* in it: $n - 1$ (here the length is 4).

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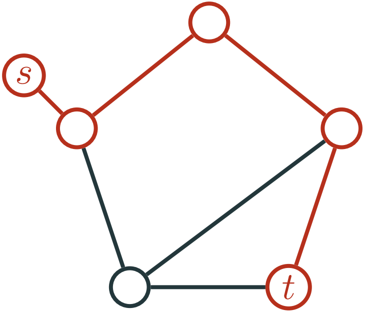

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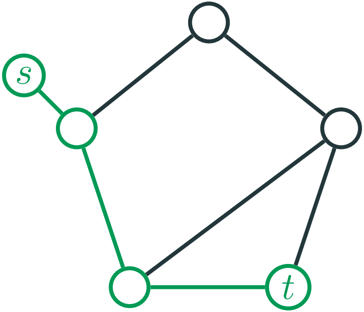

- Here the distance between s and t is 3:

$$d(s,t) = 3.$$

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- **Triangle inequality:**

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  for any vertices s, t, q.
- Proof: the *shortest* path between s and t is shorter, or at least not longer, then *any* path from s to t via q.

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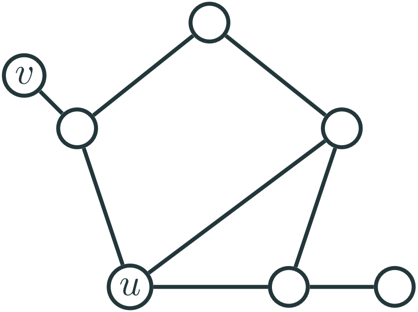
- In this example, the eccentricity of u is 2 and the eccentricity of v is 4:



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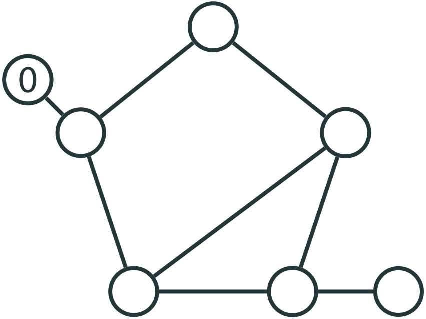

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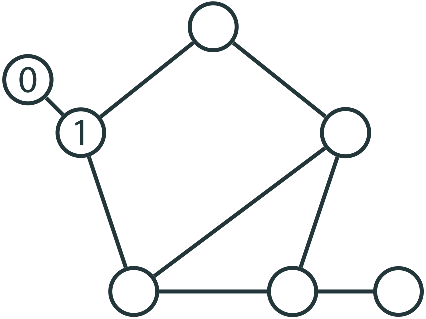

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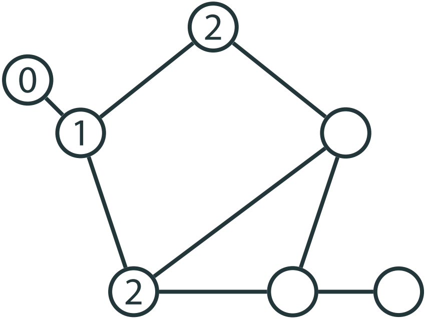

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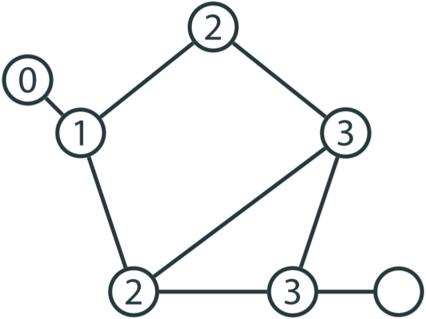

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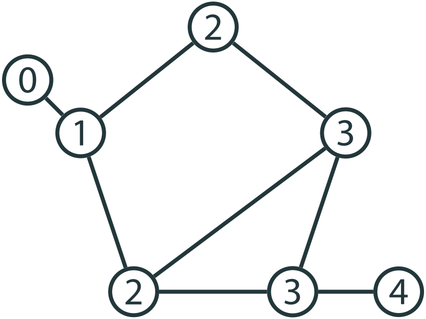

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- Otherwise, $ecc(s) = \infty$.

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- Equivalently, the diameter is the maximal eccentricity:

$$diam(\mathcal{G}) = \max_{u \in V} ecc(u).$$

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Diameter and Radius

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- It is easy to see (exercise!) that

$$r(\mathcal{G}) \leq diam(\mathcal{G}) \leq 2 \cdot r(\mathcal{G}).$$