Graphs

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Outline

The Notion of Graph

Trees

Colorings. Bipartite Graphs

The Notion of Graph

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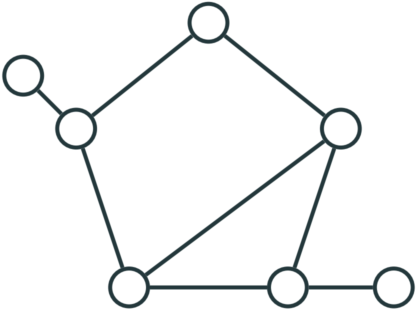
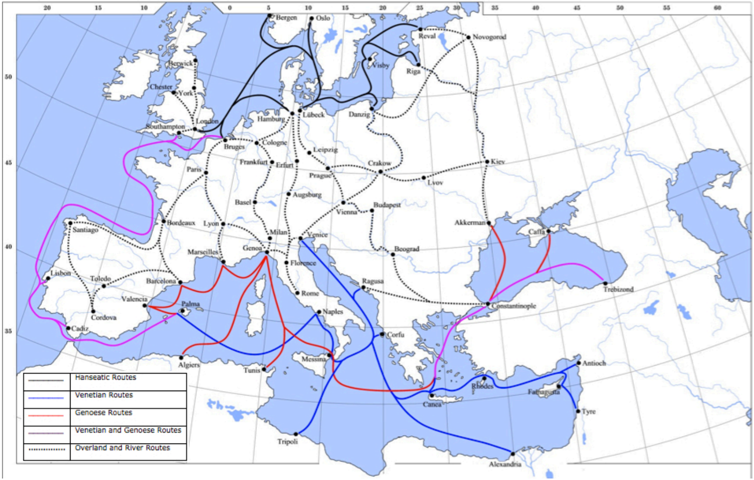


Applications of Graphs

— Graphs are everywhere!

Applications of Graphs

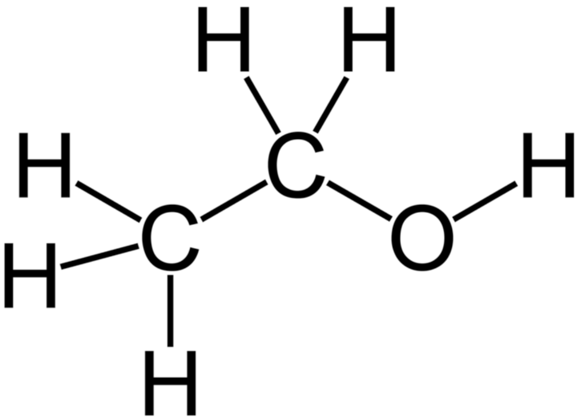
Maps (GIS): vertices = cities, edges = routes.



Wikipedia

Applications of Graphs

Chemistry: graphs of molecular structure.



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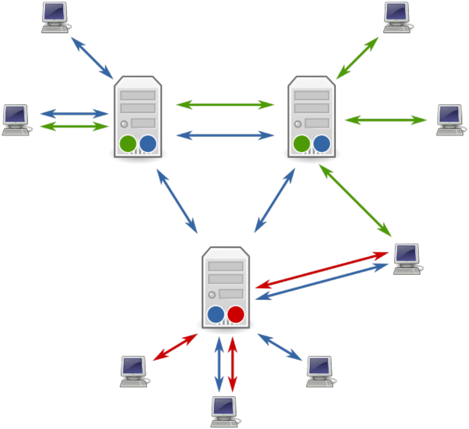
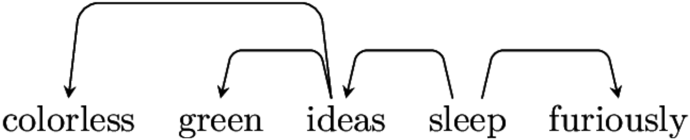
Internet: network topology.



Benjamin D. Esham @ Wikipedia

Applications of Graphs

Linguistics: syntactic dependencies.



Applications of Graphs

... and many, many more!

Directed and Undirected Graphs

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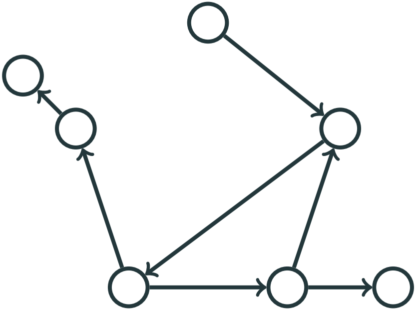

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- In contrast, the *subscription* graph is *directed.*
- Say, Carl is subscribed to Wall Street Journal, while the editorial of Wall Street Journal doesn't even know about Carl.
- In this course, most graphs will be undirected.

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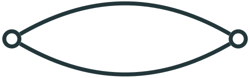
- *Parallel edges:* two vertices connected by more than one edge.



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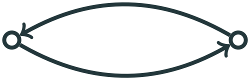
- Note that in a directed graph edges connecting two vertices in different directions are **not** considered parallel.



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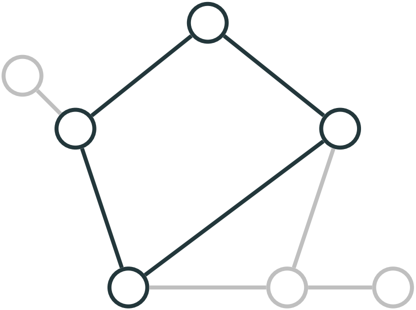

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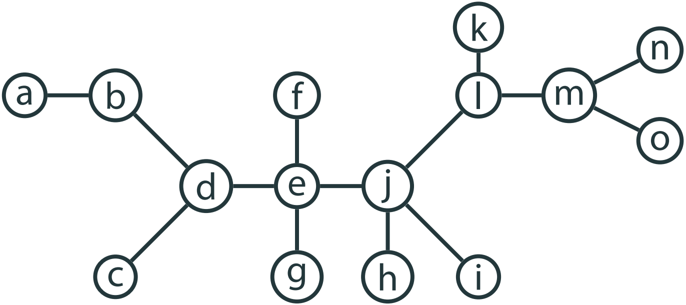

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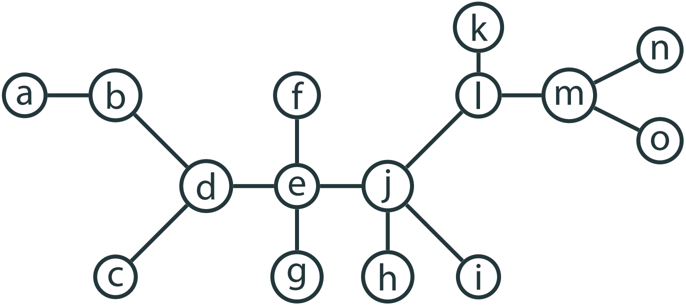

- Non-simple cycles could still exist, for example d–e–f–e–g–d on the graph above.

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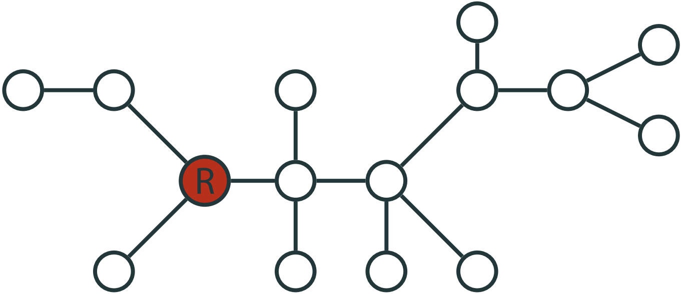

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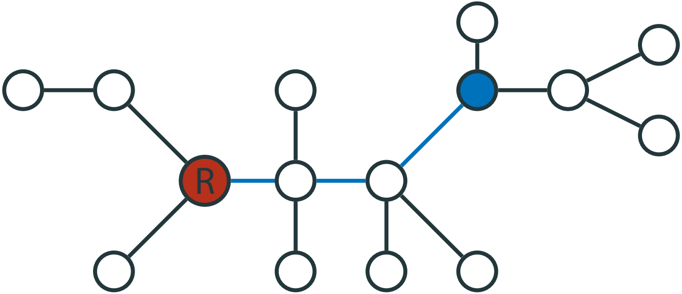

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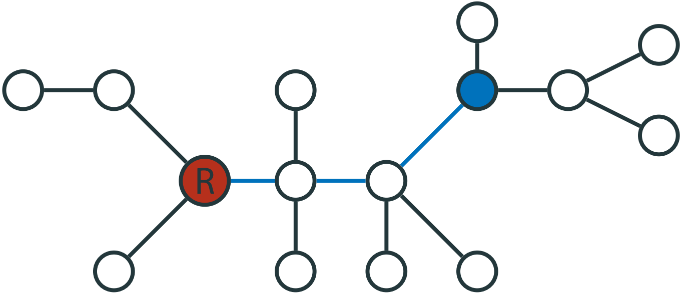

- For any vertex (except the root) there is a unique path from the root to this vertex.
- Otherwise, there would be a cycle.

Parents and Children

- Thus, for any vertex (except the root), there is exactly one vertex, which is next to it on the path towards the root.
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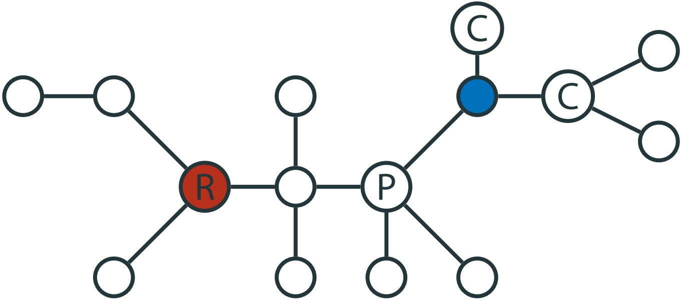
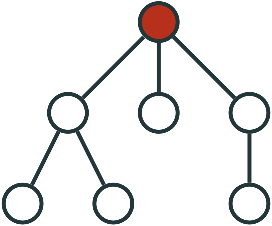


Drawing Rooted Trees

- A tree with a root is usually drawn in such a way that for each vertex its parent is located above and its children below the vertex.



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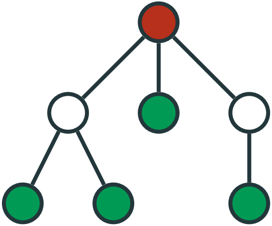

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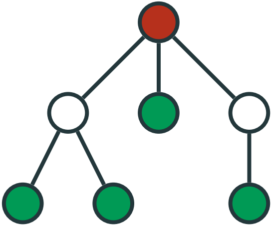

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- *Notice:* in computer science, trees usually grow down (root at the top, leaves at the bottom)!

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- Decision trees are also used as a computational model in complexity theory.
- In a decision tree, the inner nodes represent queries (usually binary) of input data, and leaves are answers.

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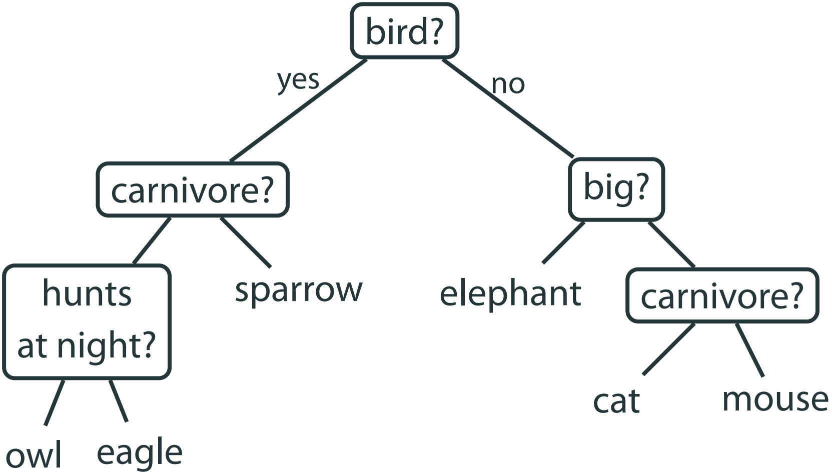

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- Thus, $m = n - 1$.

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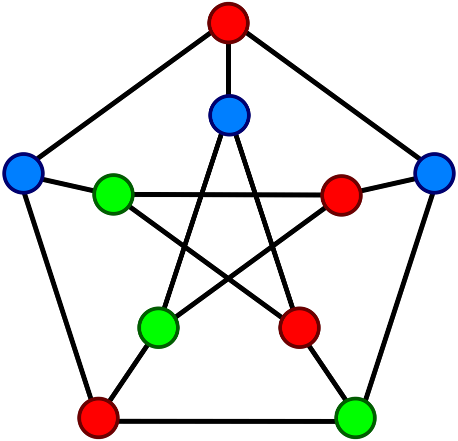
- A coloring is *proper,* if endpoints of any edge have different colors.

Vertex Colorings

- Example: Petersen graph properly 3-colored.



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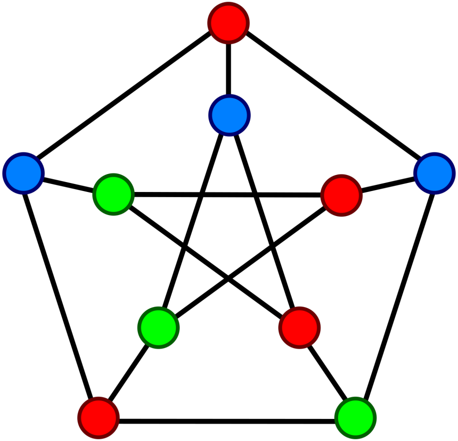

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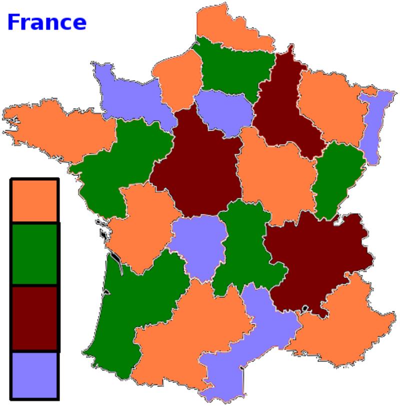

Proved by K. Appel and W. Haken in 1976.

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Vertices = areas; edges connect adjacent areas.

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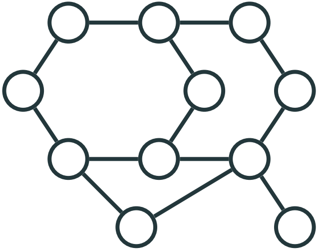

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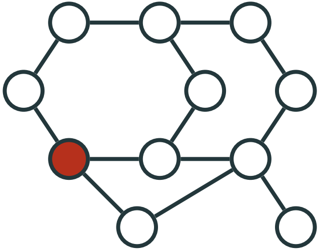

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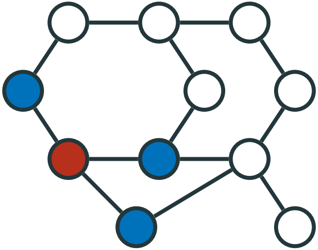

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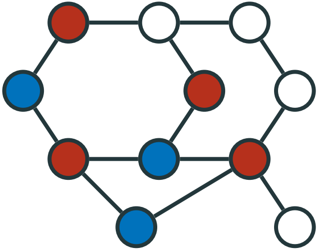

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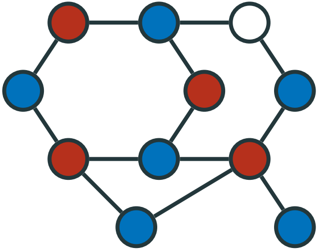

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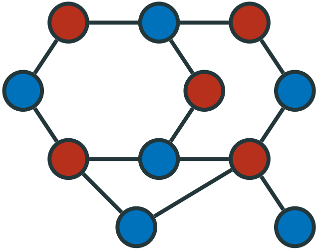

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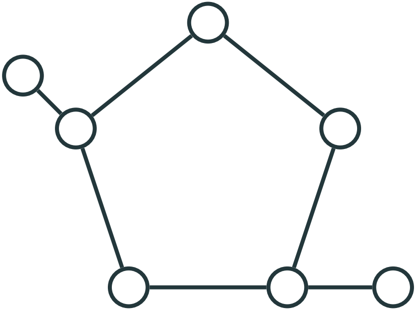

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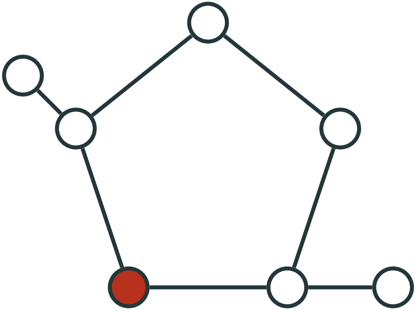

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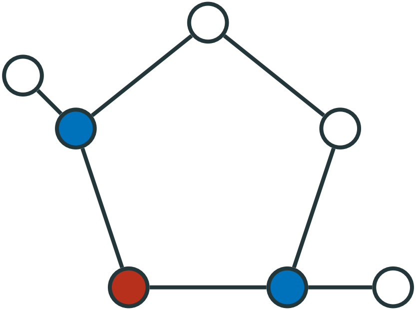

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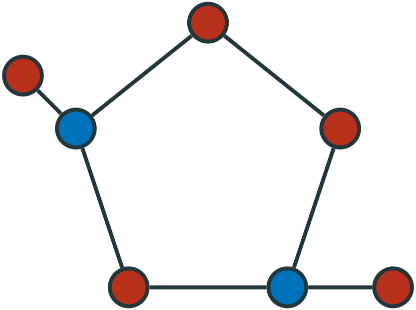

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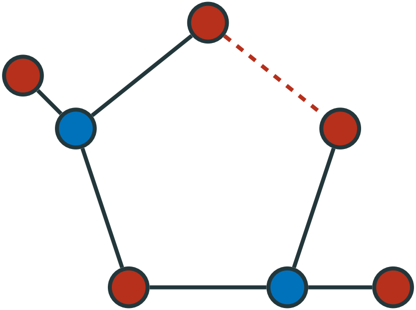

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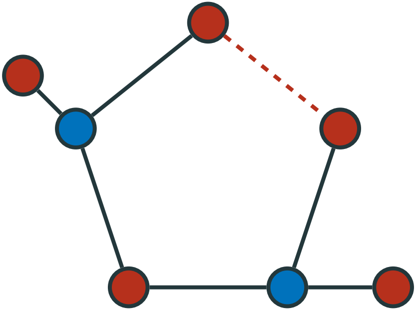

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- For such a graph, 2-coloring is impossible.

Bipartite Graphs

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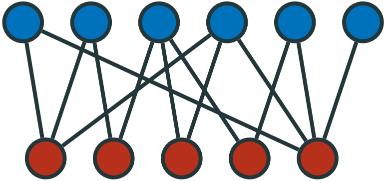

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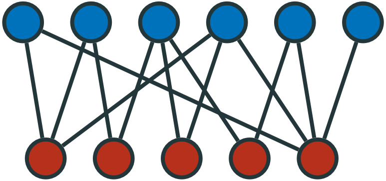

- Bipartite graphs can express *correspondences* between objects of different kind (say, students and courses they attend).