Basic Counting Techniques

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Outline

Why counting

Rule of Sum

Convenient Language: Sets

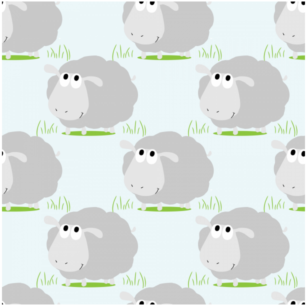
Generalizing Rule of Sum

Recursive Counting: Number of Paths

Rule of Product

Why Counting?

- Counting is one of the basic tasks related to mathematics



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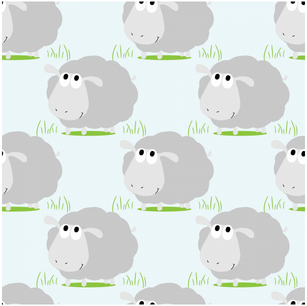
Why Counting?

- Counting is one of the basic tasks related to mathematics
- Goal: to tell how many objects are there without actually counting them one by one



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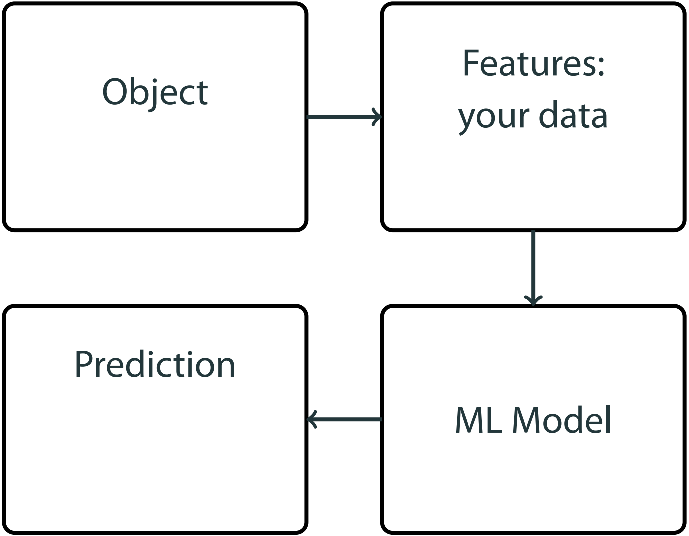
- Important application: computing probabilities
- Important application: estimating the size of data

Counting in Data Analysis?



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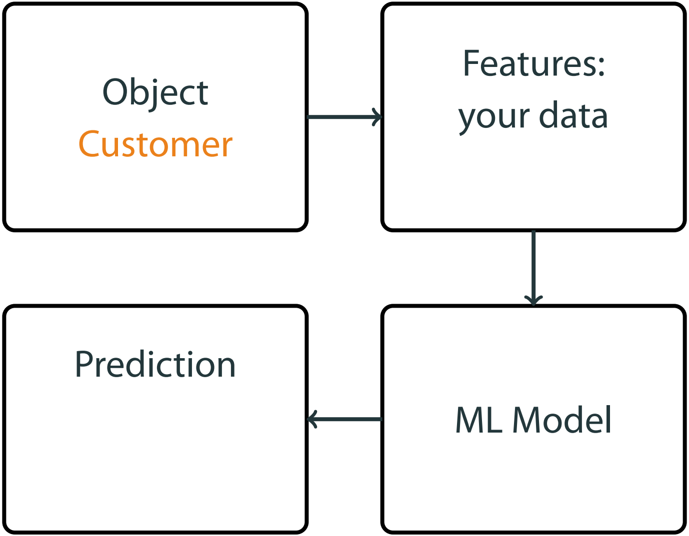

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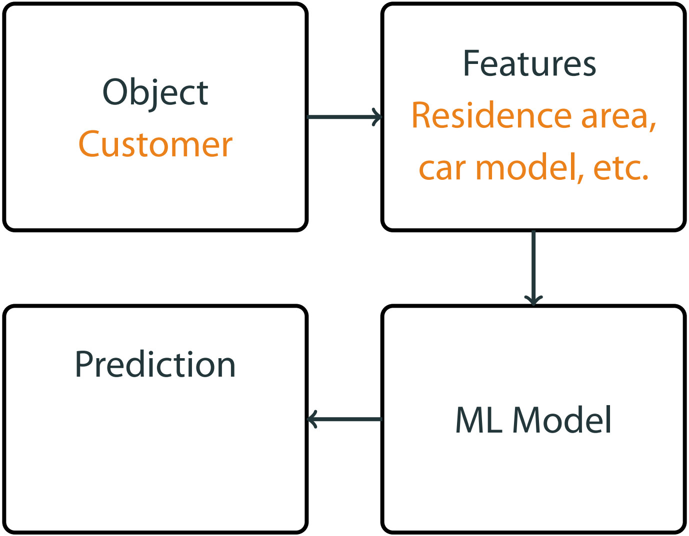

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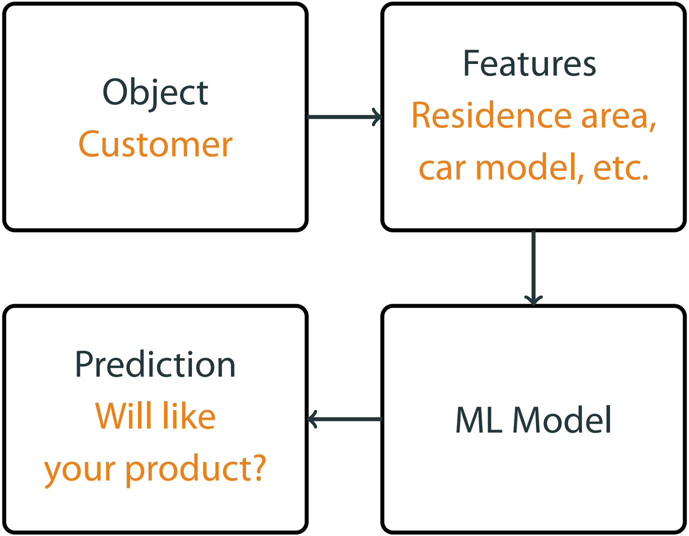

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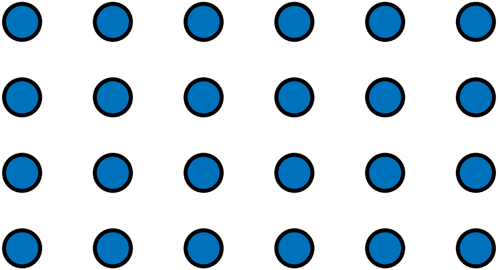
- In many models the amount of data needed is comparable to the number of all possible different objects in your feature setting
- It is useful to estimate the number of all possible collections of feature values

Why Study Counting?

- On the other hand, there are important ideas on how to count



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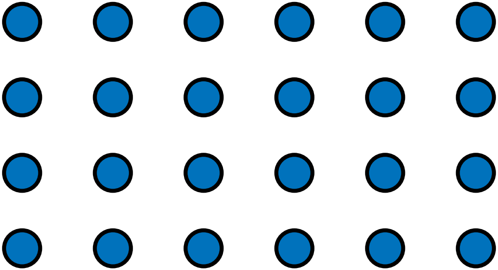
- Toy example: can you tell how many blue circles are in the picture without counting them by hand?



Real Life Example, Preview

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- Suppose a country, a state or a region introduces a new format of a license plate

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- Will we have enough plates for everyone?

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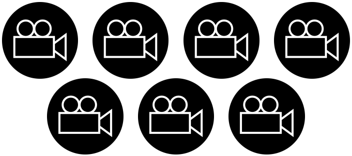

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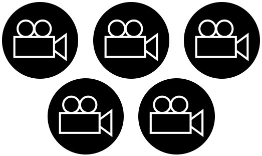

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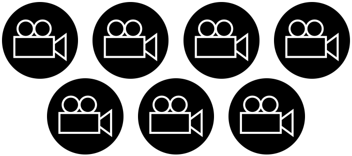

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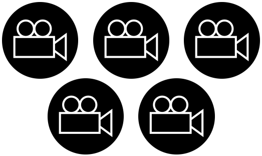
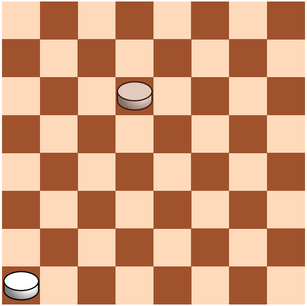


7+5=12 videos in total

Piece on a Chessboard

Piece on a chessboard

A piece stays in the bottom left corner of a chessboard. In one move it can move one step to the right or one step up. How many moves are needed to get to the position on the picture?



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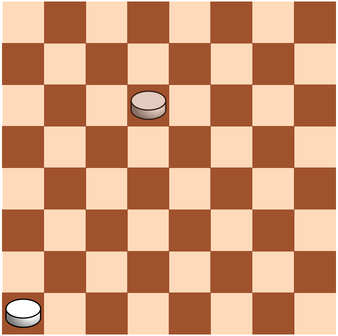

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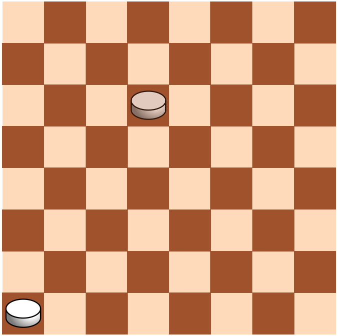

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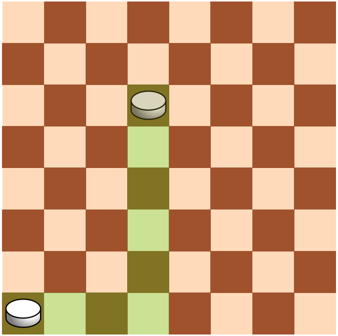

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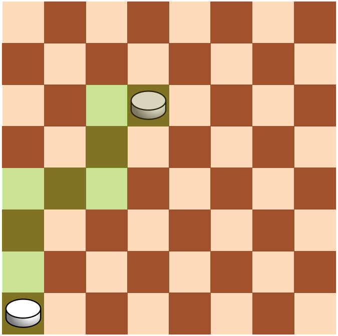

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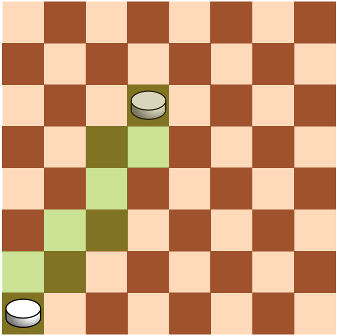

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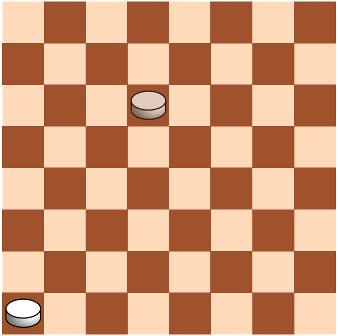

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- In all cases we need 8 moves

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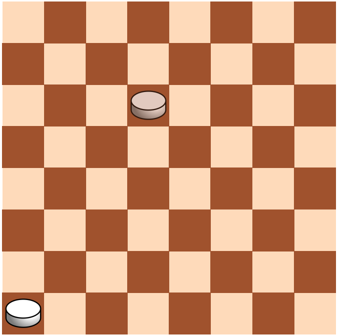

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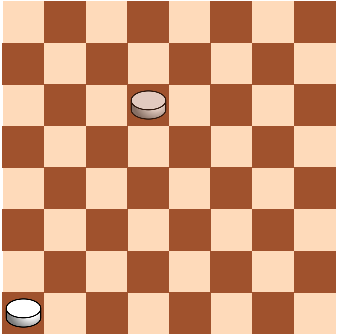

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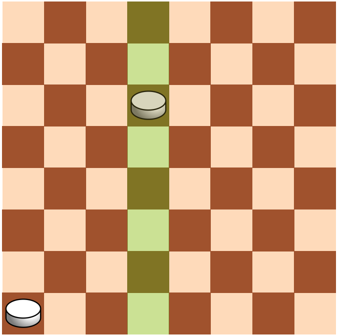

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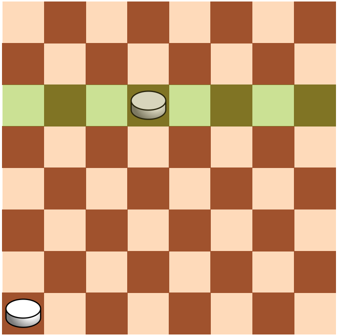

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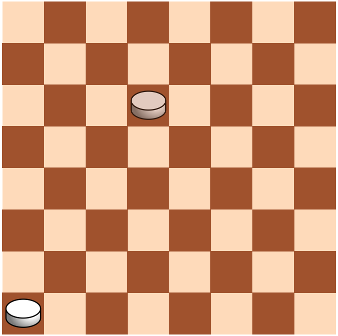

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4. In total we need 3+5=8 moves

One More Example

- Suppose in our data we have 7 videos of length ≥ 4 minutes and 5 music videos

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- What if some of our music videos are longer than 4 minutes?

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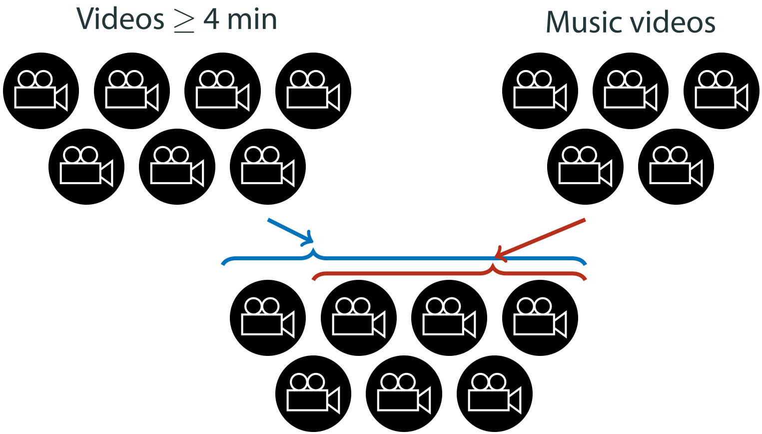

Rule of sum does not work here!

More Subtle Example

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- Suppose in your data there are 7 videos of one category and 5 videos of another category
- Do you have 12 videos that fall in one of the two categories?
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- For example, there might be funny videos with cats

Rule of Sum Revisited

Rule of Sum

If there are k objects of the first type and there are n objects of the second type, then there are $n + k$ objects of one of two types

- Important lesson: in the rule of sum no object should belong to both classes!

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- Repetitions in the list of elements are not important: $\{0, 1, 2, 3\} = \{1, 0, 1, 3, 2, 3\}$

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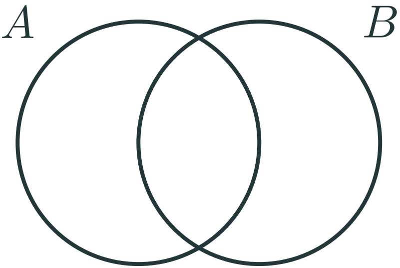
- However, there are pitfalls
- “Set consisting of all sets” is a dangerous construction
- We will not encounter these difficulties in our program and will not discuss them

Venn Diagrams

- A convenient way to view sets is by the means of Venn Diagrams



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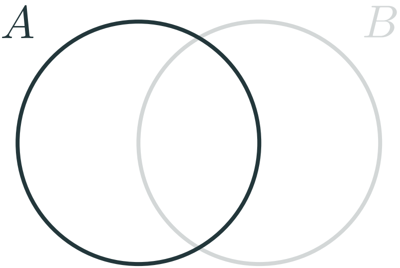

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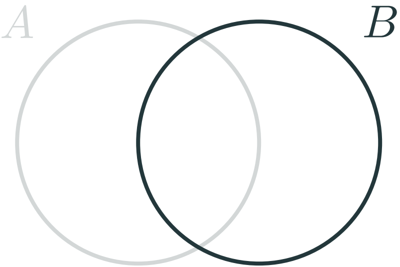

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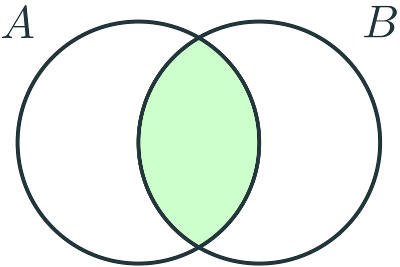
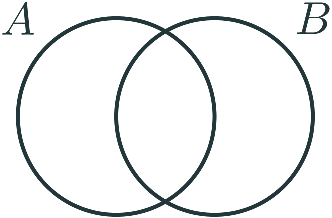
- Intersection corresponds to elements belonging to both sets



Useful Notations



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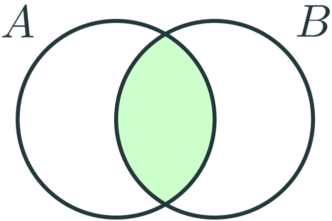

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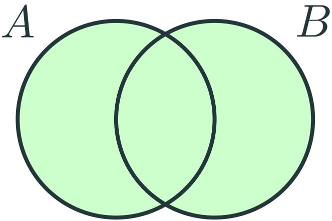

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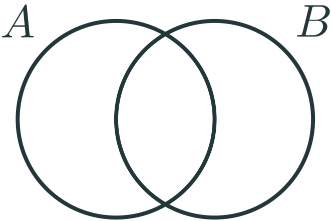

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- The set $A \cap B$ is an intersection of these sets: it consists of elements belonging to both sets
- The set $A \cup B$ is a union of these sets: it consists of elements belonging to at least one of the sets
- If every element of A is also an element of B, then A is a subset of B; we write $A \subseteq B$

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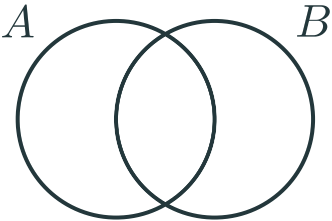

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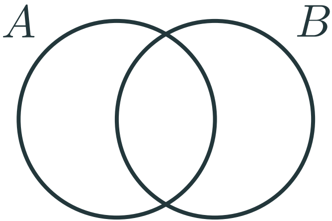

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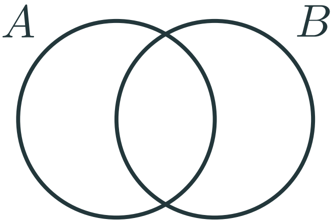

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- A set without elements is denoted by $\emptyset$ and is called empty set

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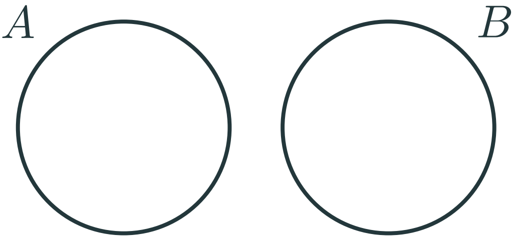

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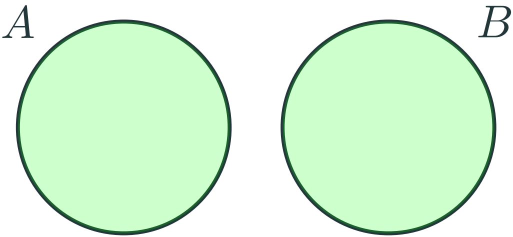

Generalized Rule of Sum

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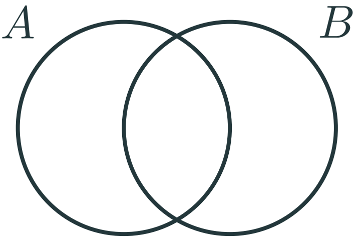

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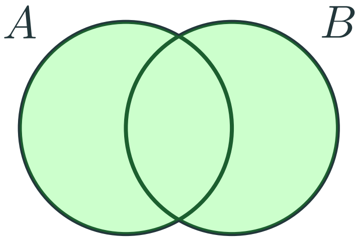

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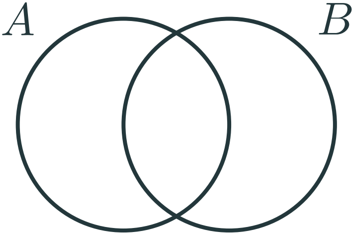

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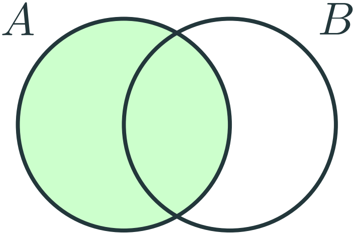

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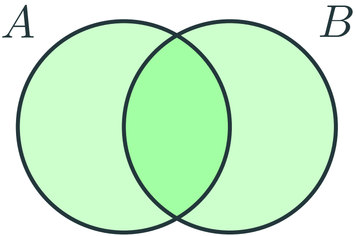

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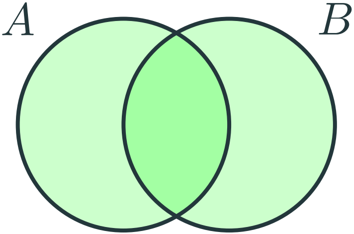

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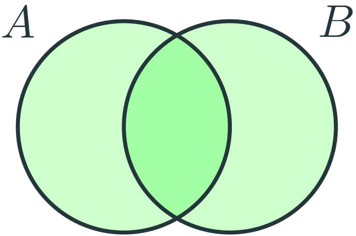

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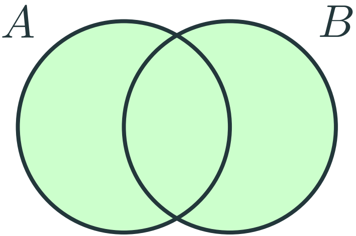

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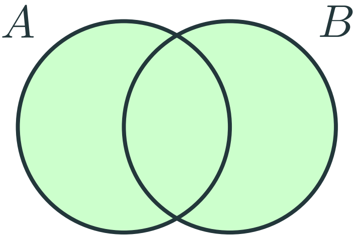

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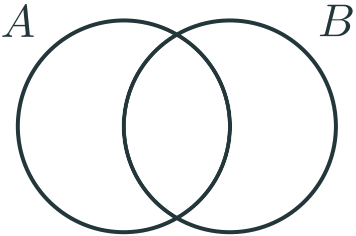
- We count elements that belong to both A and B twice
- So let's subtract them now!
- This gives the right result: $|A \cup B| = |A| + |B| - |A \cap B|$

Generalized Rule of Sum

Rule of Sum

If there are finite sets A and B, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$



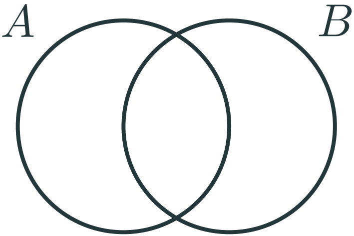
Generalized Rule of Sum

Rule of Sum

If there are finite sets A and B, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Covers the original rule: $|A \cap B| = 0$

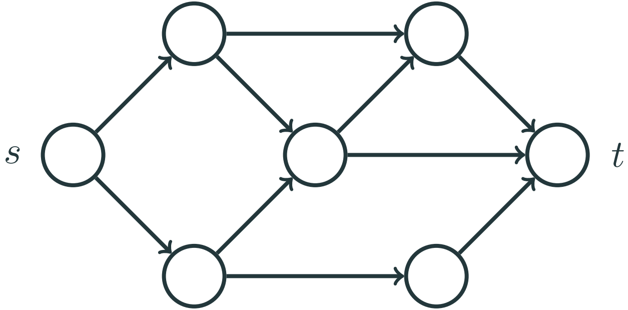


Outline

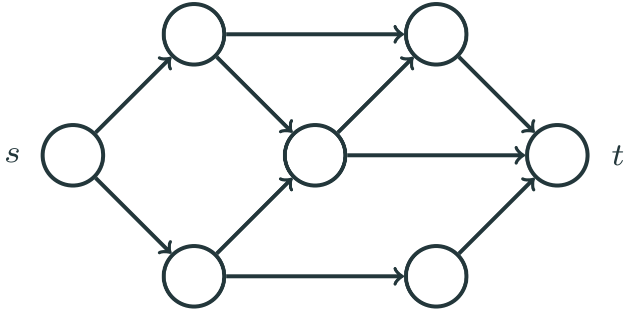
Recursive Counting: Number of Paths

Problem

Suppose there are several points connected by arrows. There is a starting point s (called source) and a final point t (called sink). How many different ways are there to get from s to t?

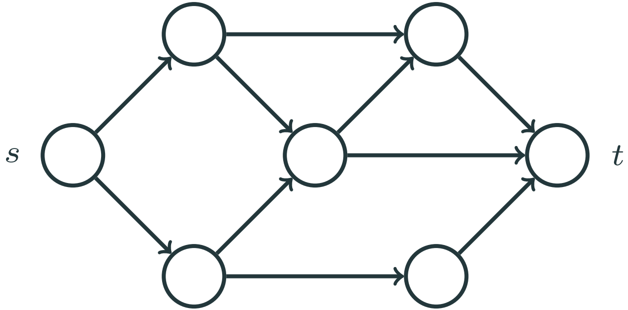


Number of Paths



- There are several various paths; how not to miss anything when counting?

Number of Paths



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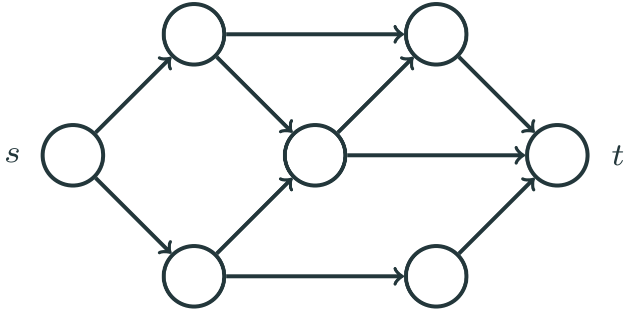

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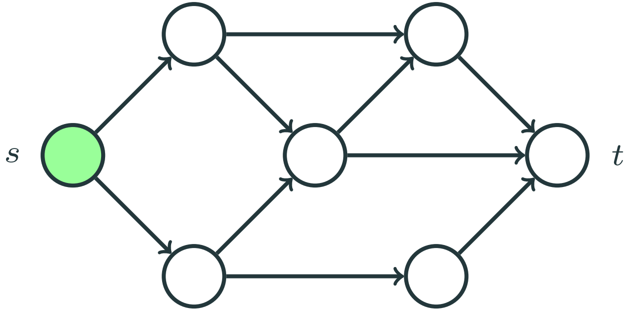

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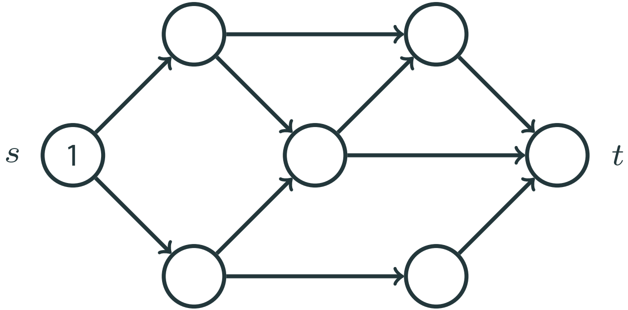

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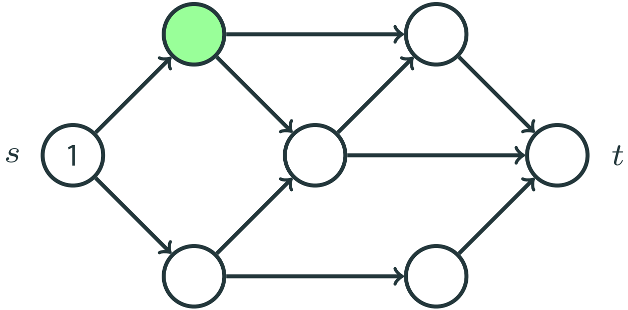

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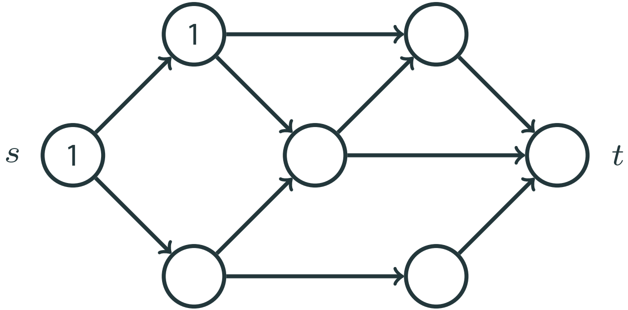

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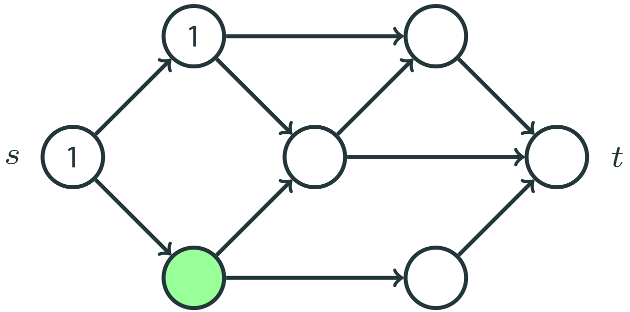

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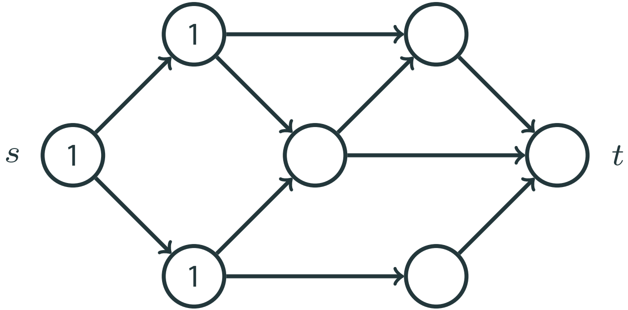

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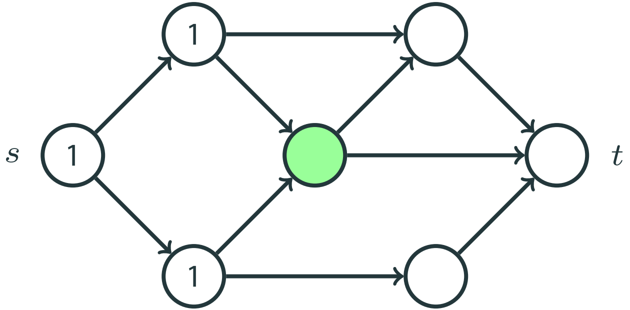

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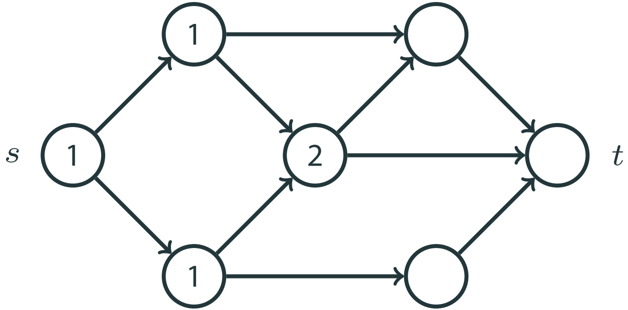

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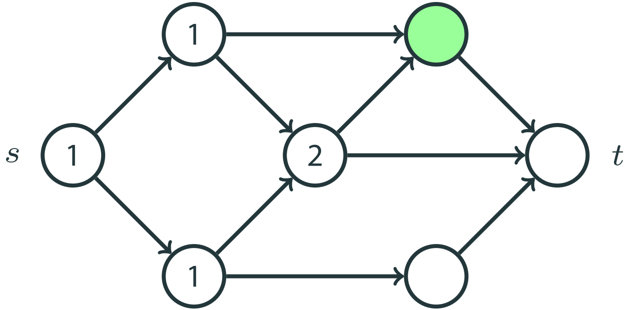

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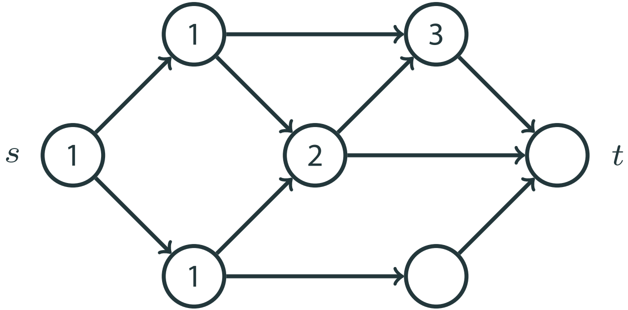

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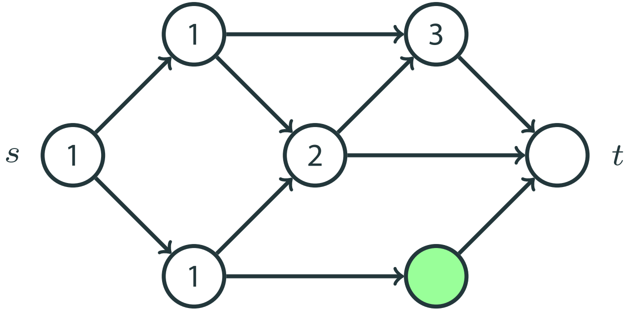

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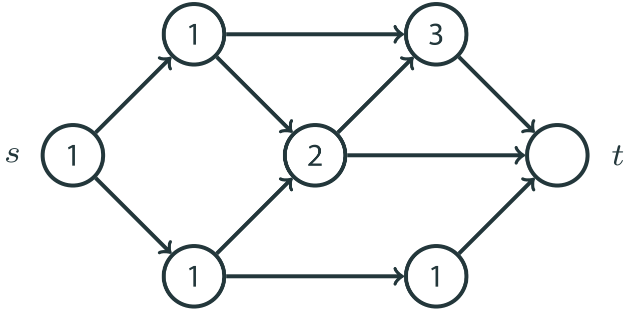

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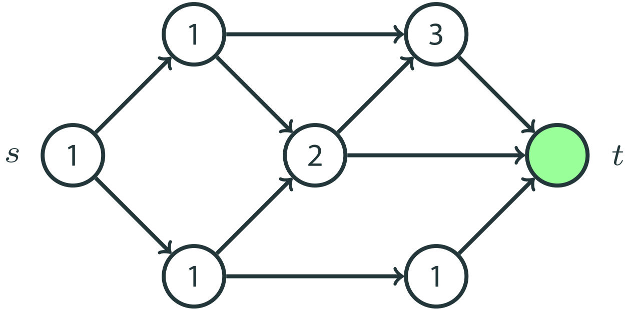

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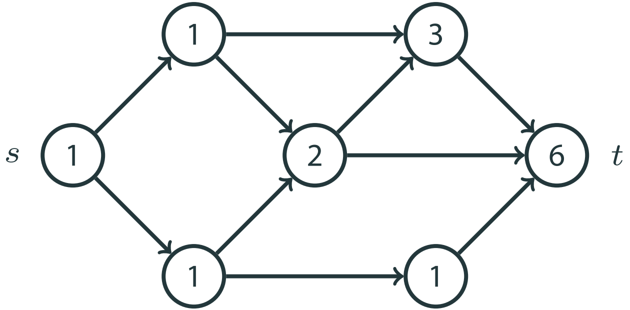

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Outline

Why counting

Rule of Sum

Convenient Language: Sets

Generalizing Rule of Sum

Recursive Counting: Number of Paths

Rule of Product

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If there are k object of the first type and there are n object of the second type, then there are $k \times n$ pairs of objects, the first of the first type and the second of the second type

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Videos Persons

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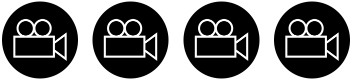


Persons

$4 \times 3 = 12$ evaluations

List of All Evaluations

	5	3	2	7
	8	7	5	1
	1	7	8	10

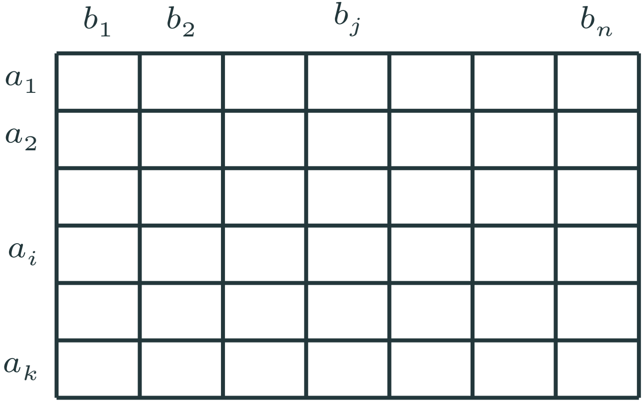
Rule of Product in the Set Language

Rule of Product

If there is a finite set A and a finite set B, then there are $|A| \times |B|$ pairs of objects, the first from A and the second from B

Why the Rule of Product is True?

$$A = \{a_1, \dots, a_k\}$$
$$B = \{b_1, \dots, b_n\}$$



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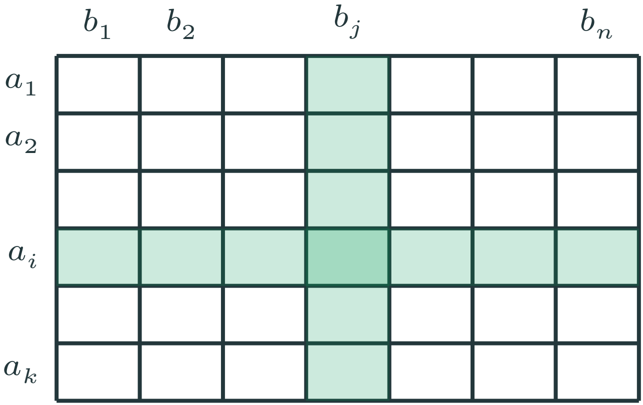

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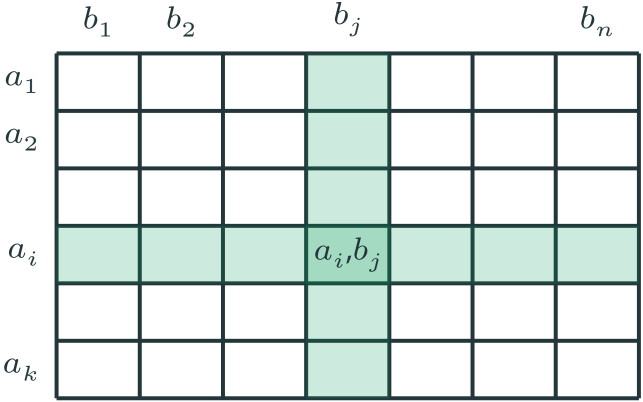

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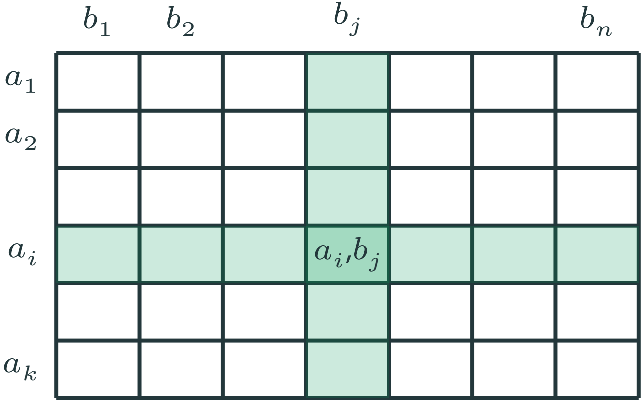

There are as many pairs as cells in this table

Rule of Product as the Number of Paths

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If there is a finite set A and a finite set B, then there are $|A| \times |B|$ pairs of objects, the first from A and the second from B

Rule of Product as the Number of Paths

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Can we express this counting rule in terms of counting the number of paths?

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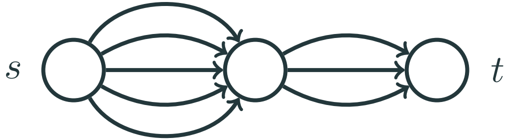

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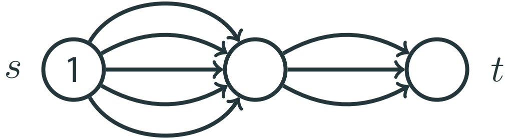

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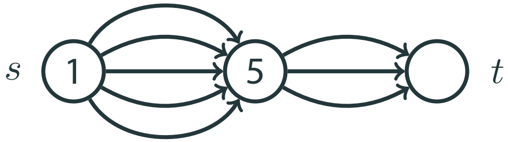

$$1 + 1 + 1 + 1 + 1 = 5$$

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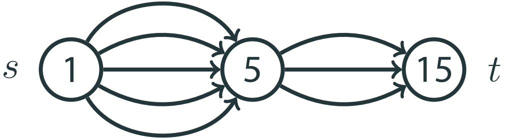

$$5 + 5 + 5 = 3 \times 5 = 15$$

Summary

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- Counting starts with simple things
- But even rule of sum can be tricky
- Some simple basic ideas can already be helpful
- Next we will see how to build something more involved from the basic building blocks