Cycles in Graphs

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Outline

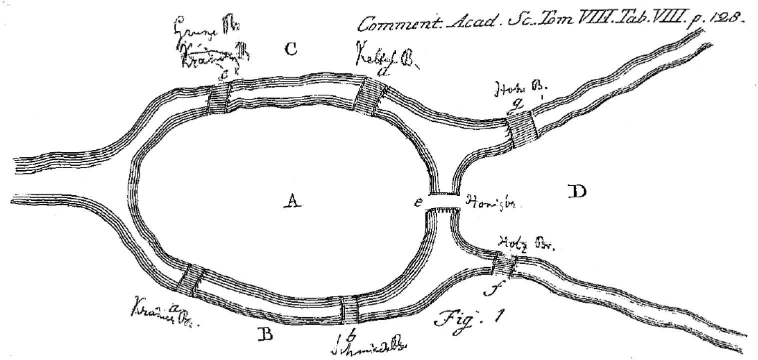
Königsberg Bridges. Euler Cycles

Constructing a Euler Cycle

Hamiltonian Paths

Acyclic Directed Graphs. Topological Sorting

The Seven Bridges of Königsberg



- A notable question by Leonhard Euler (1736).

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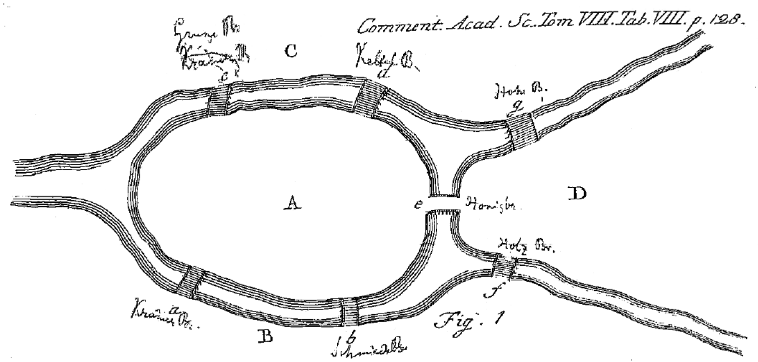

- A notable question by Leonhard Euler (1736).
- You see a map of Königsberg (nowadays Kaliningrad) from the times of Euler.

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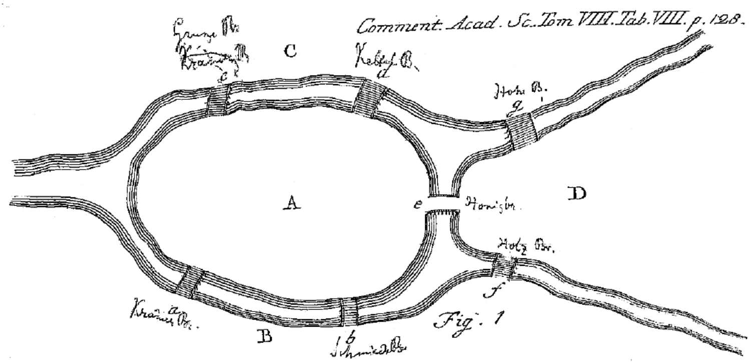

- Two islands (A and D) and two banks of the river (C and B) were connected by 7 bridges.

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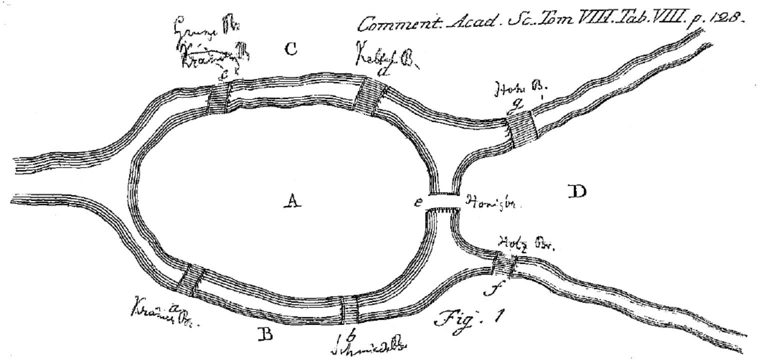
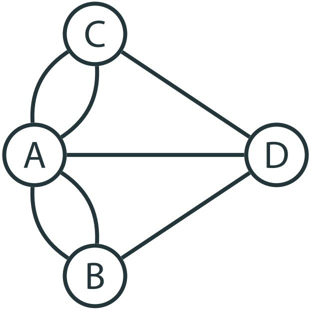


- Two islands (A and D) and two banks of the river (C and B) were connected by 7 bridges.
- Could one visit each bridge **exactly** once?

Königsberg Bridges as a Graph



- City areas = vertices; bridges = edges.

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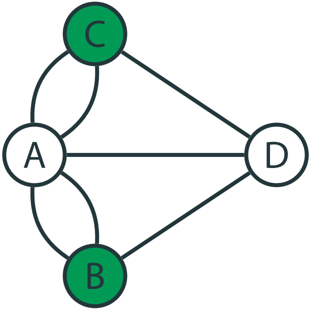

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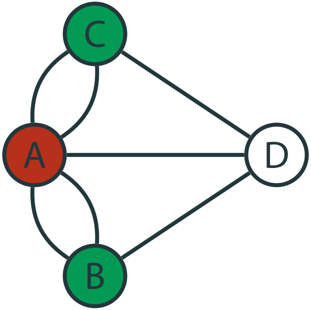

- City areas = vertices; bridges = edges.
- Suppose our tour starts at B, ends at C and visits each edge exactly once.
- Vertex A is visited **in the middle.**

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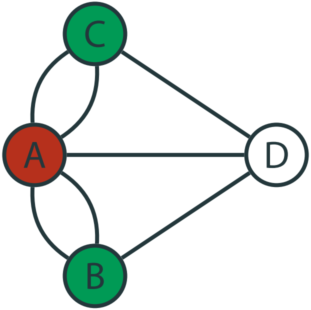

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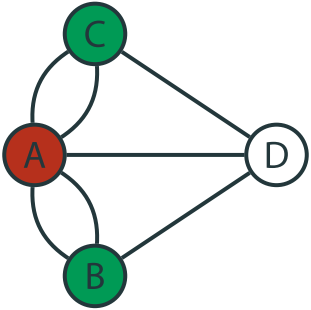

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- The number of edges adjacent to A is $2k$.

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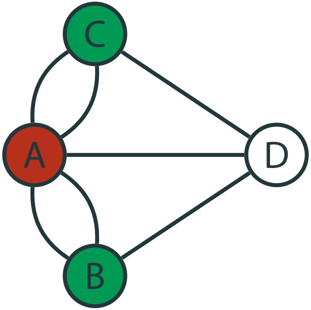

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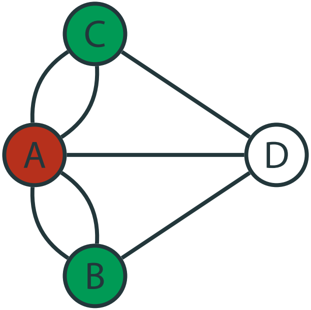

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- **Contradiction:** 5 is odd!
- All vertices are odd, thus, Euler's problem is unsolvable.

Euler Paths and Cycles

Formal definitions:

- A Euler path is a path in the graph which visits each edge exactly once.

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- A Euler path is a path in the graph which visits each edge exactly once.
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- An odd vertex is a vertex which has an odd number of edges adjacent to it.

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- If there exists a Euler cycle, then the graph has no odd vertices.
- **Question:** can a graph have exactly one odd vertex?

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- Suppose our graph has no odd vertices. Does this guarantee existence of a Euler cycle?

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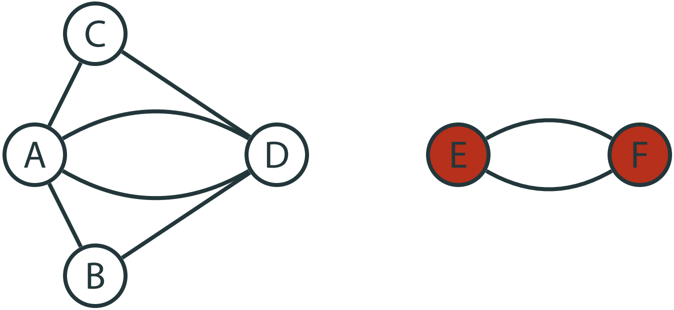
- E. g., the graph could be **disconnected:**



Greedy Algorithm

- OK, OK, suppose the graph is connected: there is a path between any two vertices...

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- OK, OK, suppose the graph is connected: there is a path between any two vertices...
- ... and there are no odd vertices ...
- ... then let's just start walking!

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Greedy Algorithm

- No odd vertices, so we never get stuck ...
- ... at some point, we return to the starting vertex, completing the cycle ...
- ... but what if there is still something left?

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- A Hamiltonian cycle is a Hamiltonian path which starts and ends at the same vertex (and this is counted as one visit).

Hamiltonian Cycle on a Dodecahedron

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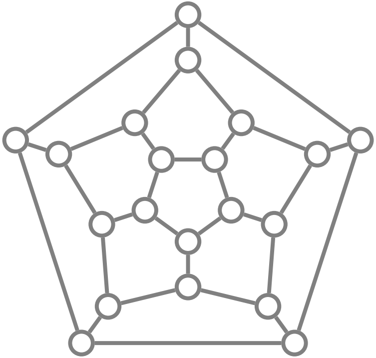
- Using the polar projection, the dodecahedron can be represented as a graph on the plane:



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- No “good” criterion whether a Hamiltonian cycle exists (in a given graph) is known.
- Unless P=NP, there is also no efficient algorithm for finding Hamiltonian cycles.

Knight's Tour

- A specific case of Hamiltonian cycle is the *knight tour question.*

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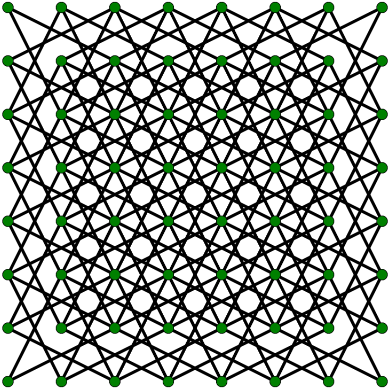
- The challenge is to find a walk of a chess knight over the board, in which each square is visited exactly once.

Knight's Tour

As one can easily see, a knight tour is exactly a Hamiltonian path in the following *knight's graph:*



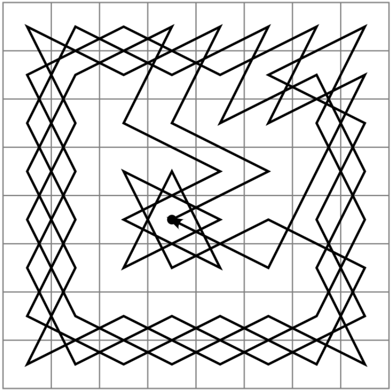
Wikipedia

Knight's Tour

- If it is a Hamiltonian cycle, then the knight tour is called *closed*.
- Otherwise, if the tour does not return to the initial square, it is *open*.

Knight's Tour

Example of a closed knight's tour on the standard 8×8 chessboard:



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- In contrast to the general case, there exist efficient algorithms for constructing knight's tours on an arbitrary $n \times n$ board.
- Good criteria whether such a tour exists are also available.

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- In computer graphics, Hamiltonian paths help in compact representation of data.

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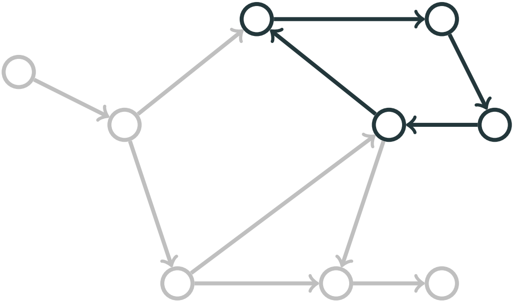

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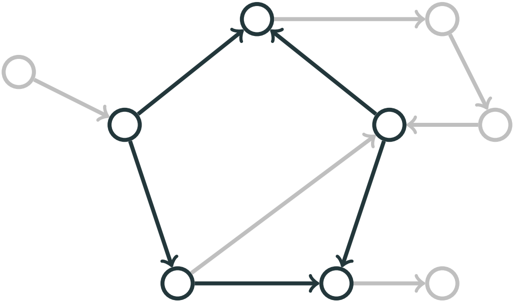

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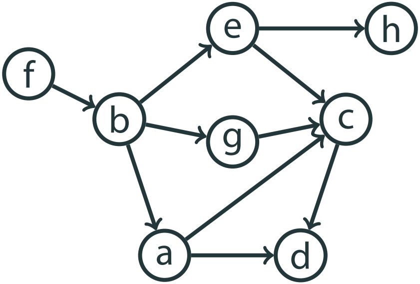

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- In dependency graphs, topological sorting represents *correct execution order* of actions.

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- This algorithm is based on depth-first search.

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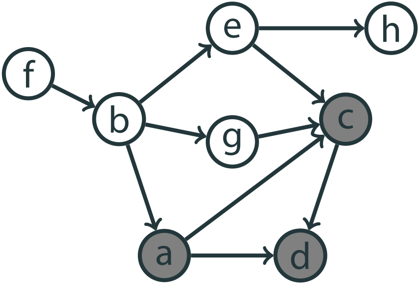
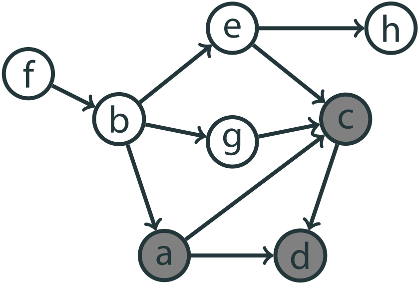


a — c — d

Suppose we have already correctly sorted some of the vertices...

Topological Sorting Using DFS



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...and wish to append a new one *to the beginning* of the list.

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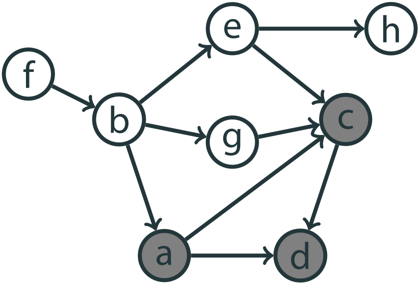

? — a — c — d

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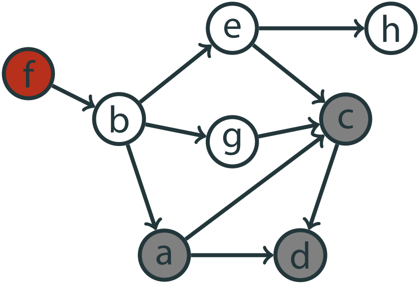

? — a — c — d

We pick a random vertex...

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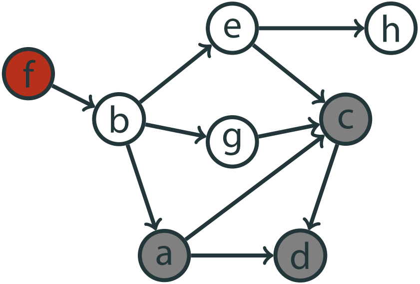

? — a — c — d

...and follow arrows until a vertex from which no edge leads to a vertex not previously sorted.

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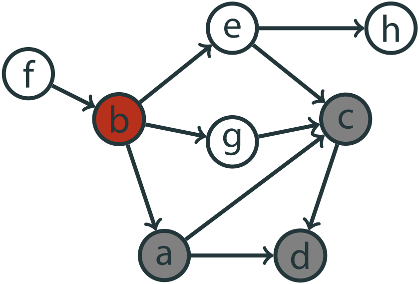

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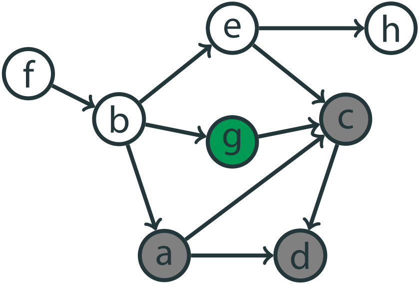

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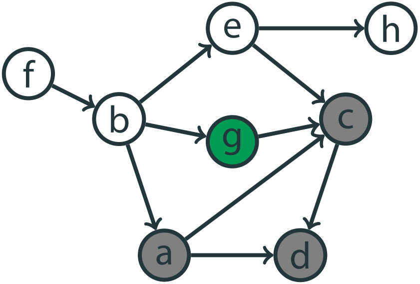

? — a — c — d

This vertex is safe to add!

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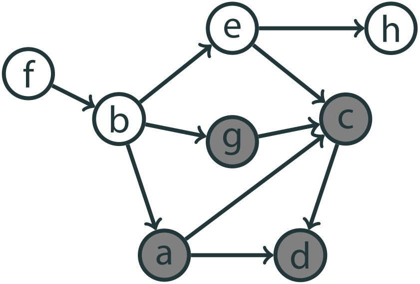

g — a — c — d

This vertex is safe to add!

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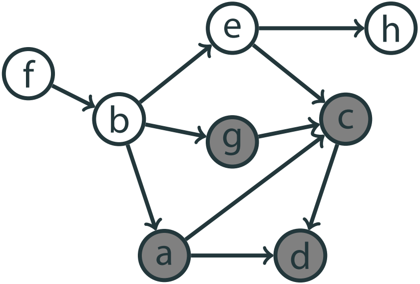

g — a — c — d

Vertices a, c, d were added before, in the same manner...

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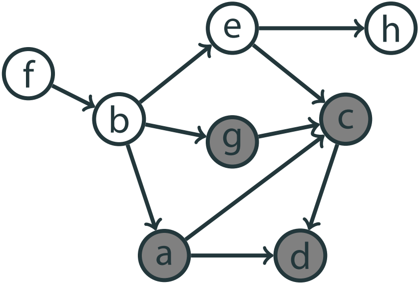

g — a — c — d

... and we can continue this process, until all vertices are listed.

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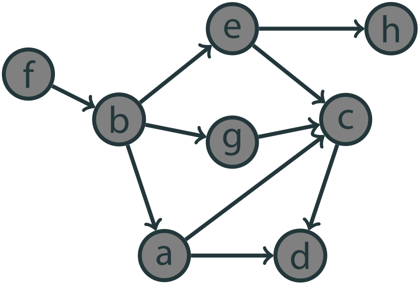

f — b — e — h — g — a — c — d

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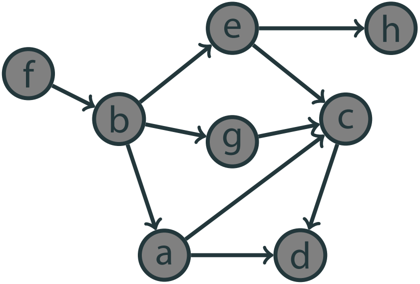

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This is the necessary topological sorting.