Example: Power Spectrum of Sine Wave

- The power spectrum of the random phase sine-wave is obtained as:

$$\mathcal{S}_X\left(e^{j\Omega}\right) = \sum_{m=-\infty}^{\infty} r_{XX}[m]e^{-jm\Omega} = \sum_{m=-\infty}^{\infty} 0.5a^2\cos\left[m\omega_0\right]e^{-jm\Omega}$$

$$= 0.25a^2 \times \sum_{m=-\infty}^{\infty}\left(\exp\left(jm\omega_0\right) + \exp\left(-jm\omega_0\right)\right)e^{-jm\Omega}$$

$$= 0.5\pi a^2 \times \sum_{m=-\infty}^{\infty}\delta\left(\Omega-\omega_0-2m\pi\right) + \delta\left(\Omega+\omega_0-2m\pi\right)$$

- The last line above is derived via the Fourier series of a periodic train of δ functions[1]:

$$\sum_{m=-\infty}^{\infty}\delta(\Omega-\omega_0-2m\pi) = \sum_{m=-\infty}^{\infty} c_m e^{-jm(\Omega-\omega_0)}$$

- The coefficient c_m can be solve by take the inverse DTFT of the delta train:

$$\Rightarrow c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty}\delta(\Omega-\omega_0-2m\pi)e^{jm(\Omega-\omega_0)}\mathrm{d}\Omega$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi}(\dots)\mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(\Omega-\omega_0)\mathrm{d}\Omega = \frac{1}{2\pi}$$

[1] aka Dirac comb, impulse train and sampling function

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- '$\delta(\cdot)\exp(\cdot)$' is uniform convergent, thus we can permute the integral and summation
- since $\delta(\Omega - \omega_0 - 2m\pi)$ is zero over the interval $[-\pi, \pi]$ for $m \neq 0$, thus there is only one term in the summation that contributes to the integral
- thus

$$2\pi \sum_{m=-\infty}^{\infty} \delta\left(\Omega - \omega_0 - 2m\pi\right) = \sum_{m=-\infty}^{\infty} e^{-jm(\Omega-\omega_0)}$$

as desired

- we can sketch power spectrum as the combination of two Dirac combs: