Detection, Estimation Inference for Signal Processing

Hanchen Wang

Ph.D Candidate in
Engineering, University of Cambridge

January 19, 2020

Example: Power Spectrum of Sine Wave

- The power spectrum of the random phase sine-wave is obtained as:

$$\begin{aligned}
\mathcal{S}_X\left(e^{j\Omega}\right) &= \sum_{m=-\infty}^{\infty} r_{XX}[m]e^{-jm\Omega} = \sum_{m=-\infty}^{\infty} 0.5a^2\cos\left[m\omega_0\right]e^{-jm\Omega} \\
&= 0.25a^2 \times \sum_{m=-\infty}^{\infty} \left(\exp\left(jm\omega_0\right) + \exp\left(-jm\omega_0\right)\right)e^{-jm\Omega} \\
&= 0.5\pi a^2 \times \sum_{m=-\infty}^{\infty} \delta\left(\Omega-\omega_0-2m\pi\right) + \delta\left(\Omega+\omega_0-2m\pi\right)
\end{aligned}$$

- The last line above is derived via the Fourier series of a periodic train of δ functions[1]:

$$\sum_{m=-\infty}^{\infty} \delta(\Omega-\omega_0-2m\pi) = \sum_{m=-\infty}^{\infty} c_m e^{-jm(\Omega-\omega_0)}$$

- The coefficient c_m can be solve by take the inverse DTFT of the delta train:

$$\begin{aligned}
\Rightarrow c_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty} \delta(\Omega-\omega_0-2m\pi)e^{jm(\Omega-\omega_0)}\mathrm{d}\Omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi}(\dots)\mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(\Omega-\omega_0)\mathrm{d}\Omega = \frac{1}{2\pi}
\end{aligned}$$

[1] aka Dirac comb, impulse train and sampling function

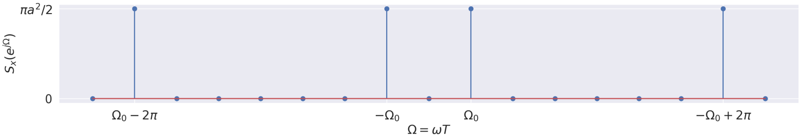
Example: Power Spectrum of Sine Wave

- '$\delta(\cdot)\exp(\cdot)$' is uniform convergent, thus we can permute the integral and summation
- since $\delta(\Omega - \omega_0 - 2m\pi)$ is zero over the interval $[-\pi, \pi]$ for $m \neq 0$, thus there is only one term in the summation that contributes to the integral
- thus

$$2\pi \sum_{m=-\infty}^{\infty} \delta\left(\Omega - \omega_0 - 2m\pi\right) = \sum_{m=-\infty}^{\infty} e^{-jm(\Omega - \omega_0)}$$

as desired

- we can sketch power spectrum as the combination of two Dirac combs:



General Framework of Using Filters

- In practical, usually we do not know the statistic characteristics such as mean and variance(aka variables of interest, VoI) of the signal $\{X_n\}$, thus we need measurements
- If the signal generation process is wide-sense stationary yet ergodic[2] as well, then easy estimation of VoI methods exist:
  - if we measure a realization $\{x_n\}$, of the process $\{X_n\}$(e.g., one waveform from the ensemble of possible waveforms/ one(or a few) battery performance test to infer the general performance of this battery type, sort of to monitor/measure x_n for a period of time, such x_n is just one possible waveform of the ground truth)
- Wiener filter - linear estimator for stationary signals
- Kalman filter - non stationary signals estimator(not covered in 3F3)

[2] such as mean and variance ergodic

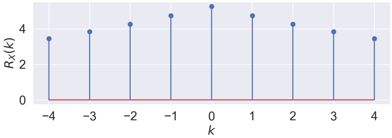
Extension of Wiener Filter

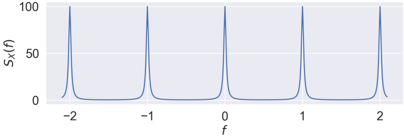
Wiener Filter can readily be extended to deal with cases outside the regular noise reduction:

- Prediction of a noisy signal $\{u_n\}$
- Smoothing of a noisy signal
- Deconvolution

$$R_X(k) = a^{|k|}\sigma_X^2, \quad k \in \mathbb{Z}$$

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi fk} = \sigma_X^2 \sum_{k=-\infty}^{\infty} a^{|k|} e^{-j2\pi fk} = \frac{\sigma^2}{1 + a^2 - 2a\cos(2\pi f)}$$





What is 'ergodic'???

TBD