3F3 Example Paper3

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December 2, 2019

Q2

$$\mathbb{E}[Y_n] = \mathbb{E}\left[\sum_{p=-\infty}^{+\infty} h_p X_{n-p}\right] = \sum_{p=-\infty}^{+\infty} h_p \mathbb{E}\left[X_{n-p}\right] = \sum_{p=-\infty}^{+\infty} h_p \mathbb{E}\left[X_n\right] (\text{since}\{X_n\} \text{ is WSS})$$

$$r_{YY}[k] = \mathbb{E}\left[(\sum_{l=-\infty}^{\infty} h_l X_{n-l})(\sum_{i=-\infty}^{\infty} h_i X_{n-i-k})\right] = \sum_l \sum_i h_l h_i \mathbb{E}\left[X_{n-l} X_{n-i-k}\right]$$

- b) by using mathematical induction

Q3

- the probability of X_m (for any m) in state i_m is:

$$p_{X_m}(i_m) = (\lambda P^n)_{i_m} = (\lambda P)_{i_m} = \lambda_{i_m}$$

$$\text{since } P^2 = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix}^2 = \begin{pmatrix} \lambda_1 \sum_{i=1}^n \lambda_i & \cdots & \lambda_n \sum_{i=1}^n \lambda_i \\ \lambda_1 \sum_{i=1}^n \lambda_i & \cdots & \lambda_n \sum_{i=1}^n \lambda_i \\ \vdots & & \vdots \\ \lambda_1 \sum_{i=1}^n \lambda_i & \cdots & \lambda_n \sum_{i=1}^n \lambda_i \end{pmatrix} = P$$

thus $P^n = P$

$$\text{and } \lambda P = (\lambda_1, \lambda_2, \cdots, \lambda_n) \cdot \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} = (\lambda_1, \lambda_2, \cdots, \lambda_n) = \lambda$$

thus no matter at which point(X_m), the probability of being in state i_m is always λ_{i_m}

- thus the joint PMF

$$p(X_m, X_{m+1}, \cdots, X_{m+k}) = p(i_m P_{i_m, i_{m+1}} P_{i_m+1, i_{m+2}} \ldots P_{i_{m+k-1}, i_{m+k}}) = \lambda_{i_m} \lambda_{i_{m+1}} \ldots \lambda_{i_{m+k}}$$

is irrelavant to either m or k, which shows that this is strictly stationary

Q5

- b)

$$
\begin{aligned}
\mathbb{E}(X_n) &= \mathbb{E}(A)\cos(nf_0) + \mathbb{E}(B)\sin(nf_0) = 0 \quad \text{since } \mathbb{E}(A) = \mathbb{E}(B) = 0 \\
\mathbb{E}(X_{n1}X_{n2}) &= \mathbb{E}(((A\cos(n_1 f_0) + B\sin(n_1 f_0))((A\cos(n_2 f_0) + B\sin(n_2 f_0))) \\
&= \mathbb{E}(A^2)\cos(n_1 f_0)\cos(n_2 f_0) + \mathbb{E}(B^2)\sin(n_1 f_0)\sin(n_2 f_0) \quad \text{since } \mathbb{E}(AB) = 0 \\
&= (\sigma_A^2 + \mathbb{E}(A)^2)\cos(n_1 f_0)\cos(n_2 f_0) + (\sigma_B^2 + \mathbb{E}(B)^2)\sin(n_1 f_0)\sin(n_2 f_0) \\
&= 2(\cos(n_1 f_0)\cos(n_2 f_0) + \sin(n_1 f_0)\sin(n_2 f_0)) \\
&= \cos((n_1 - n_2)f_0) \Rightarrow \text{thus it is WSS}
\end{aligned}
$$

- c)

$$
\begin{aligned}
\mathbb{E}(Y_n) &= \mathbb{E}(X_n) - \mathbb{E}(X_{n-1}) = 0 \quad \text{since } \mathbb{E}(X_n) = \mathbb{E}(X_{n-1}) = 2q - 1 \\
\mathbb{E}(Y_{n1}Y_{n2}) &= \mathbb{E}((X_{n_1} - X_{n_1-1})(X_{n_2} - X_{n_2-1})) \\
&= \mathbb{E}(X_{n_1}X_{n_2}) - \mathbb{E}(X_{n_1-1}X_{n_2}) - \mathbb{E}(X_{n_1}X_{n_2-1}) + \mathbb{E}(X_{n_1-1}X_{n_2-1}) \\
&= (2q-1)^2 - \mathbb{E}(X_{n_1-1}X_{n_2}) - \mathbb{E}(X_{n_1}X_{n_2-1}) + (2q-1)^2 \\
&= \begin{cases} (2q-1)^2 - (2q-1)^2 - (2q-1)^2 + (2q-1)^2 = 0 & \text{if } |n_1 - n_2| = 1 \\ (2q-1)^2 - 0 - (2q-1)^2 + (2q-1)^2 = (2q-1)^2 & \text{elsewhere} \end{cases} \\
&\Rightarrow \text{it is WSS, we can denote the correlation function as:} \\
&\quad R_Y(k) = 1 \text{ if } k = 1, R_Y(k) = 0 \text{ elsewhere}
\end{aligned}
$$

Q6

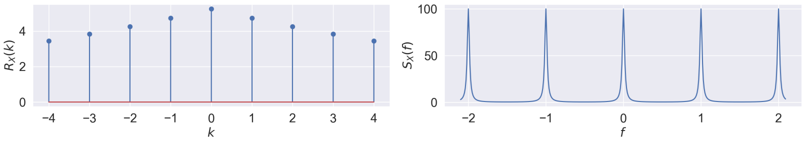
- c) from a) and b), we know that(it is also included in the lecture notes):

$$R_X(k) = a^{|k|}\sigma_X^2, \quad k \in \mathbb{Z}$$
$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi fk} = \sigma_X^2 \sum_{k=-\infty}^{\infty} a^{|k|} e^{-j2\pi fk} = \frac{\sigma^2}{1 + a^2 - 2a\cos(2\pi f)}$$

thus we can derive a and σ^2 directly:

$$a = R_X(k)/R_X(k-1) = R_X(1)/R_X(0) = 4.74/5.26 \approx 0.9$$
$$\sigma^2 = (1-a^2)\sigma_X^2 = (1-a^2)R_X(0) = 0.19 * 5.26 \approx 1.0$$

thus we can sketch the $R_X(k)$ and $S_X(f)$:



Q7

- **joint pdf:** it is not straightforward to write down the joint pdf of $(X_0, X_1, ..., X_k)$, thus we first write down the joint pdf of $(X_0, W_1, ..., W_k)$, then apply the rules of change of variables for random vectors to get our goal:

$$f\left(x_0, w_1, \ldots, w_k\right) = f_{X_0}\left(x_0\right) \prod_{i=1}^{k} f_W\left(w_i\right), \text{ where } f_{X_0} \sim \mathcal{N}(\mu_0, \sigma_0^2),\ f_W \sim \mathcal{N}(0, \sigma^2)$$

which is a multivariate Gaussian, plus we know that:

$$(X_0, \ldots, X_k) = (X_0, W_1 \ldots, W_k) \begin{pmatrix} 1 & a & \cdots & a^k \\ & 1 & \cdots & a^{k-1} \\ & & \ddots & \vdots \\ & & 1 & a \\ & & & 1 \end{pmatrix} \text{ denoted as: } \mathrm{Y} = \mathrm{XS}$$

$$\Rightarrow X = YS^{-1} \text{ since } \det(S) = 1 \text{ thus it has an inverse}$$

recall the rules of change of variables, now we have:

$$f_Y(y) = f_X(H(y))|\det J(y)| = f_X(yS^{-1})|\det(S^{-1})| = f_X(yS^{-1}) \quad \text{same notations as lecture notes}$$

here $f_X(\cdot) = f_{X_0, W_1, \ldots, W_k}(x_0, w_1, ..., w_k)$ is the multivariate Gaussian mentioned above

now is to show that: $|\det J(y)| = |\det(S^{-1})|$

Q7

recall the definition of $J(y)$(same as in the lecture notes):

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} h_1\left(Y_1, \ldots, Y_n\right) \\ \vdots \\ h_n\left(Y_1, \ldots, Y_n\right) \end{pmatrix}, \quad J(y) = \begin{pmatrix} \frac{\partial}{\partial y_1} h_1, \ldots, \frac{\partial}{\partial y_n} h_1 \\ \vdots \\ \frac{\partial}{\partial y_1} h_n, \cdots, \frac{\partial}{\partial y_n} h_n \end{pmatrix}$$

where $h_j = \sum_{i=1}^{n} s_{ij} y_i,\ s_{ij}$ is the (i, j) entry of S^{-1}

the (j, i) entry of $J(y)$ is: $\frac{\partial}{\partial y_i} h_j = s_{ij} \Rightarrow J(y) = (S^{-1})^T \Rightarrow |\det J(y)| = |\det(S^{-1})|$

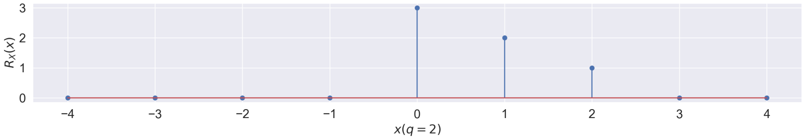
- **mean vector**: $\mu_Y = \mu_X S^{-1} = (\mu_0, 0, \cdots, 0)\, S^{-1}$
- **covariance matrix**:[1]
  recall the exponential term of multivariate Guassian:

$$X \Sigma_X^{-1} X^T = (Y S^{-1}) \Sigma_X^{-1} (Y S^{-1})^T = Y S^{-1} \Sigma_X^{-1} (S^{-1})^T Y^T = Y \Sigma_Y^{-1} Y^T$$
$$\Rightarrow \Sigma_Y = S^T \Sigma_X S, \text{ where } \Sigma_X = \mathrm{diag}(\sigma_0^2, \sigma^2, \cdots, \sigma^2)$$

[1] usually we use the column vector $(X_0, X_1, \ldots, X_k)^T$ to describe multivariate Gaussian, here we use row vectors, but they are identical. Also it is clear to know that: $(A^{-1})^T = (A^{-T})^1$ since $(A^{-1}A)^T = A^T (A^{-1})^T = \mathrm{I} = A^T (A^T)^{-1}$

Q8

- a) $X_n = \sum_{i=-\infty}^{\infty} h_i W_{n-i}$ where $\{h_i\}$ is the impulse response of a causal LTI system with input $\{W_i\}$. Indeed $h_0 = 1, h_i = b_i$ for $0 < i \leq q$ and $h_i = 0$ otherwise.
- b)
$$R_X(n, n+k) = \mathbb{E}\{X_n X_{n+k}\} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j \mathbb{E}\{W_{n-i} W_{n+k-j}\} = \sigma^2 \sum_{i=-\infty}^{\infty} h_i h_{i+k}$$
since $\mathbb{E}(X_n X_n) = \sigma^2, \quad \mathbb{E}(X_n X_{n+k}) = 0 (k \neq 0), R_X(n, n+k)$ is irrelavant to $n \Rightarrow$ it is WSS
- c) when $b_i = 1, R_X(0) = q+1, R_X(1) = q, \ldots, R_X(q) = 1$ and $R_X(k) = 0$ for $k > q$. As a remark, we see that even without the restriction that $b_i = 1, R_X(k) = 0$ for $k > q$.



Q10

- a) cross-correlation function between $\{b_n\}$ and $\{x_n\}$:

$$r_{bx}[k] = \mathbb{E}[b_n x_{n+m}] = \mathbb{E}\left[b_n \sum_{i=0}^{1} c_i b_{n+m-i}\right] = \sum_{i=0}^{1} c_i \mathbb{E}\left[b_n b_{n+m-i}\right] = \begin{cases} c_0, & m=0 \\ c_1, & m=1 \\ 0, & \text{otherwise} \end{cases}$$

the autocorrelation function of $\{x_n\}$ is:

$$r_{xx}[k] = \mathbb{E}[x_n x_{n+m}] = \mathbb{E}\left[\sum_{i=0}^{1} c_i b_{n-i} \sum_{j=0}^{1} c_j b_{n+m-j}\right] = \sum_{i=0}^{1}\sum_{j=0}^{1} c_i c_j \mathbb{E}\left[b_{n-i} b_{n+m-j}\right]$$

$$= \sum_{i=0}^{1} c_i c_0 \mathbb{E}\left[b_{n-i} b_{n+m}\right] + \sum_{i=0}^{1} c_i c_1 \mathbb{E}\left[b_{n-i} b_{n+m-1}\right]$$

$$= c_0^2 \mathbb{E}\left[b_n b_{n+m}\right] + c_0 c_1 (\mathbb{E}\left[b_{n-1} b_{n+m}\right] + \mathbb{E}\left[b_n b_{n+m-1}\right]) + c_1^2 \mathbb{E}\left[b_{n-1} b_{n+m-1}\right]$$

$$= \begin{cases} c_0 c_1, & m=-1,1 \\ c_1^2 + c_0^2, & m=0 \\ 0, & \text{otherwise} \end{cases}$$

Q10

- b) require to minimise:

$$\mathbb{E}[(b_n - \hat{b}_n)^2] = \mathbb{E}[(b_n - \sum_{i=0}^{1} h_i x_{n-i})^2] = \mathbb{E}[(b_n - \mathbf{h}^T \mathbf{x}_n)^2] = \mathbb{E}[b_n^2] + \mathbf{h}^T \mathbb{E}[\mathbf{x}_n \mathbf{x}_n^T]\mathbf{h} - 2\mathbb{E}[b_n \mathbf{h}^T \mathbf{x}_n]$$

differtiate w.r.t **h** and equate to zero to get:

$$2\mathbb{E}\left[\mathbf{x}_n \mathbf{x}_n^T\right]\mathbf{h} = 2\mathbb{E}\left[b_n \mathbf{x}_n\right] \Rightarrow \mathbf{h} = \mathbf{R}_x^{-1}\mathbf{r}_{bx} = \begin{bmatrix} 1.01 & 0.1 \\ 0.1 & 1.01 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$$