Engineering Tripos Part IIA THIRD YEAR

3F3 – Statistical Signal Processing

Examples Paper 2

Questions marked as '†' are straightforward questions testing fundamental concepts. The rest are Tripos style although not necessarily Tripos length.

1. Let $\{X_n\}_{n\geq 0}$ be a Markov chain taking values in $S = \{1, \ldots, L\}$ and let $P = (P_{i,j} : i, j \in S)$ be its transition probability matrix, i.e.
$$p\left(i_n | i_0, \ldots, i_{n-1}\right) = P_{i_{n-1}, i_n}.$$
   (a) †Show that the conditional probability mass function (cpmf) of X_2 given $X_1 = i_1$ satisfies
$$p\left(i_2 | i_1\right) = P_{i_1, i_2}.$$
   (b) By generalising your solution, show
$$p\left(i_n | i_{n-1}\right) = P_{i_{n-1}, i_n}.$$

2. A Bernoulli random process is a sequence of iid random variables X_n where X_n takes the values 1 or -1 with probabilities q and $1 - q$ respectively.
   (a) †Write down a Markov chain representation of this process, i.e. give the pmf of X_0 and its transition probability matrix P.
   (b) Determine if this Bernoulli process is wide sense stationary.

3. Let $\{X_n\}_{n\geq 0}$ be a Markov chain taking values in $S = \{1, \ldots, L\}$. Let $\lambda = (\lambda_i : i \in S)$ be the initial distribution of the chain (i.e. the pmf of X_0) and P be the transition probability matrix of the chain where
$$P = \begin{bmatrix} \lambda_1 & \ldots & \lambda_L \\ \vdots & & \vdots \\ \lambda_1 & \ldots & \lambda_L \end{bmatrix}.$$
   Show the Markov chain is *strictly stationary*, i.e. the joint pmf of $(X_0, \ldots, X_k)$ and $(X_m, \ldots, X_{m+k})$ are the same for any integers m and k.

4. Consider the AR(1) process
$$X_n = aX_{n-1} + W_n$$
   where $|a| < 1$ and $\{W_n\}_{n\in\mathbb{Z}}$ is a sequence of random variables satisfying $\mathbb{E}\left(W_n\right) = 0$, $\mathbb{E}\left(W_n^2\right) = \sigma^2$ and $\mathbb{E}\left(W_i W_j\right) = 0$ for $i \neq j$.

(a) Assuming X_n has constant mean μ, find μ.

(b) Assuming X_n has constant variance σ_X^2, find σ_X.

5. Determine if each of the following processes is wide sense stationary.

   (a) $X_n = U$ where U is a random variable.

   (b) $X_n = A\cos(nf_0) + B\sin(nf_0)$ where A and B are independent zero mean random variables with equal variances with value 2.

   (c) $Y_n = X_n - X_{n-1}$ where X_n is the Bernoulli process of Q2.

6. Consider the AR(1) process in Q4.

   (a) Find $\mathbb{E}\{X_n X_{n+k}\}$ for all integers n and k. Is the process wide sense stationary?

   (b) Find the power spectrum density $S_X(f)$.

   (c) Values of $R_X(k)$, estimated from data, are $R_X(0) = 5.26$ and $R_X(1) = 4.74$. Find parameters a and σ^2 and sketch $S_X(f)$.

7. For the AR(1) process in Q4, assume that $\{W_n\}_{n\in\mathbb{Z}}$ is a sequence of iid Gaussian random variables, $\mathbb{E}(W_n) = 0$ and $\mathbb{E}(W_n^2) = \sigma^2$. Assuming X_0 is $\mathcal{N}(\mu_0, \sigma_0^2)$, show that the random vector $(X_0, \ldots, X_k)$ is a Gaussian random vector and find its mean vector and covariance matrix.

8. The MA(q) process is

$$X_n = \left(\sum_{i=1}^{q} b_i W_{n-i}\right) + W_n$$

   where $b_1, \ldots, b_q$ are the coefficients of the MA process and $\{W_n\}_{\mathbb{Z}}$ is a sequence of random variables with mean zero, variance σ^2 and $\mathbb{E}(W_i W_j) = 0$ when $i \neq j$.

   (a) Write X_n as the output of an LTI filter with infinite impulse response $\{h_k\}$ and give h_k.

   (b) Find $\mathbb{E}\{X_n X_{n+k}\}$ for all integers n and k. Is the process wide sense stationary?

   (c) Assuming $b_i = 1$ for $1 \leq i \leq q$, sketch $R_X(k)$.

   (d) Find $S_X(f)$ for the case $q = 2$. Plot $S_X(f)$ in Matlab. (Be careful to capture its periodicity.)

Answers:

Q 2 (a) The state-space is $S = \{-1, 1\}$, transition probability matrix is $P = \begin{bmatrix} q & 1-q \\ q & 1-q \end{bmatrix}$ and initial distribution $\lambda = (q, 1-q)$. (b) Is WSS.

Q 4 (a) $\mu = 0$. (b) $\sigma_X^2 = \frac{\sigma^2}{1-a^2}$.

Q 5 Yes for all parts.

Q 6 (a) Yes. $R_X(k) = a^{|k|} R_X(0)$. (b) $S_X(f) = \sigma^2(1 + a^2 - 2a\cos(2\pi f))^{-1}$. (c) $a = 1$, $\sigma^2 = 1$.

Q 8 (a) $h_0 = 1$, $h_i = b_i$ for $0 < i \leq q$ and $h_i = 0$ otherwise. (b) Yes. $R_X(k) = \sigma^2 \sum_{i=-\infty}^{\infty} h_i h_{k+i}$.

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