

# Engineering Pt IIa, Module 3F3 - Random Processes, Detection and Estimation Examples paper

1. In this question you prove a number of key properties about random processes. You need many of these properties to answer later questions (and exam questions!).

For jointly wide-sense stationary, discrete-time, real-valued random processes  $\{X_n\}$  and  $\{Y_n\}$ :

- (a) + Show that the autocorrelation function is even:

$$r_{XX}[k] = r_{XX}[-k]$$

- (b) + [Hence] show that power spectrum is always real-valued.

- (c) \* Show the maximum of the autocorrelation function:

$$\max |r_{XX}[k]| = r_{XX}[0]$$

Hint: start with the certain relationship, which holds for any value of  $a$ :

$$\mathbb{E}[(x_{n+k} - ax_n)^2] \geq 0$$

Then simplify and choose particular value(s) of  $a$  which give the required result.

- (d) Show for the Cross-correlation function:

$$r_{XY}[k] = r_{YX}[-k]$$

- (e) Show that power spectrum is even:

$$\mathcal{S}_X(e^{j\omega}) = \mathcal{S}_X(e^{-j\omega})$$

- (f) + Show that power spectrum is periodic:

$$\mathcal{S}_X(e^{j(\omega+2n\pi)}) = \mathcal{S}_X(e^{j\omega})$$

- (g) Show that the cross-power spectrum of two processes has the following symmetry:

$$\mathcal{S}_{XY}(e^{j\omega}) = \mathcal{S}_{YX}^*(e^{j\omega})$$

where \* denotes the complex conjugate.

2. \*  $\{X_n\}$  is a wide sense stationary random process with finite mean. If  $\{Y_n\}$  is the output of a linear time-invariant (LTI) system having impulse response  $\{h_n\}$  and input  $\{X_n\}$ , then show that:

$$\mathbb{E}[Y_n] = \mathbb{E}[X_n] \sum_{p=-\infty}^{+\infty} h_p$$

and

$$r_{YY}[k] = \sum_{l=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h_l h_i r_{XX}[l - i + k] = (\tilde{h} * h * r_{XX})[k]$$

where  $\tilde{h}$  is the time-reversed sequence  $h$ . Show that  $\{Y_n\}$  is guaranteed to be wide-sense stationary if:

$$\sum_{k=-\infty}^{+\infty} |h_k| < \infty$$

(i.e. the LTI system is strictly stable). To show this you will need to use the earlier property

$$\max_k (|r_{XX}[k]|) = r_{XX}[0]$$

3. The Bernoulli random process is defined such that each time point is independently assigned a value of +1 or -1 with probabilities  $p$  and  $1 - p$  respectively. A possible sample function from the process is, for example:

$$\{x_n\} = \{\dots, -1, +1, +1, -1, +1, +1, \dots\}$$

Is the Bernoulli process a white noise process?

4. A stationary random process has autocorrelation function:

$$r_{XX}[0] = 2.8, \quad r_{XX}[\pm 1] = 1$$

and zero values elsewhere. It is known that  $x_n$  is a noisy version of another signal  $d_n$ :

$$x_n = d_n + v_n$$

where  $v_n$  is a zero mean white noise signal with variance equal to 0.5 and uncorrelated with  $d_n$ . Determine the optimal first order (i.e. with two coefficients) FIR Wiener filter for estimation of  $d_n$  [Hint: first show that  $r_{dd}[k] = r_{xx}[k] - r_{vv}[k]$ ].

Calculate the minimum mean-squared error for this filter.

How much (if at all) does the Wiener filter improve the mean-squared error compared with simply setting  $\hat{d}_n = x_n$ ?

5. Determine the optimal frequency domain Wiener filter for the same signal as the previous question. Sketch the power spectrum of  $x_n$ ,  $d_n$ ,  $v_n$  and the frequency response of the filter. Comment on how reasonable the filter seems intuitively.

6. \* The error signal for a signal estimation problem for estimating a signal  $d_n$  from some measurements  $x_n$  is:

$$\epsilon_n = d_n - \sum_{p=-\infty}^{\infty} h_p x_{n-p}$$

- (a) Determine the autocorrelation function of  $\{\epsilon_n\}$ , i.e.  $r_{\epsilon\epsilon}[k] = \mathbb{E}[\epsilon_n \epsilon_{n+k}]$   
(b) Hence show that the power spectrum of the error is of the form

$$\mathcal{S}_\epsilon = \mathcal{S}_D - \mathcal{S}_{DX}H - \mathcal{S}_{XD}H^* + |H|^2\mathcal{S}_X$$

where the dependence of all  $\mathcal{S}$  and  $H$  terms on  $(e^{j\omega})$  has been omitted for clarity.  $H(e^{j\omega})$  is the frequency response of the LTI filter.

- (c) Use this result to show that the optimal Wiener Filter has a mean-squared error of

$$J_{\min} = \mathbb{E}[\epsilon_n^2] = r_{\epsilon\epsilon}[0] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathcal{S}_D(e^{j\omega}) - \mathcal{S}_{XD}^*(e^{j\omega})H^{\text{opt}}(e^{j\omega})d\omega$$

where  $H^{\text{opt}}$  is the optimal Wiener gain derived in the lecture notes.

- (d) Hence show that in the standard scenario where  $x_n = d_n + v_n$  and  $v_n$  is zero-mean and uncorrelated with  $d_n$ , we get

$$J_{\min} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathcal{S}_D(e^{j\omega})(1 - H^{\text{opt}}(e^{j\omega}))d\omega$$

Hint for whole question: you need several of the properties derived in q.1 to answer this question.

7. \* A stationary AR(1) process can be written as:

$$d_n = a_1 d_{n-1} + e_n$$

where  $e_n$  is zero mean white noise with variance  $\sigma_e^2$ .

- (a) Determine the power spectrum of the AR(1) process and its autocorrelation function. Sketch the power spectrum of the process for a few values of  $a_1$  between 0 and 1, over the range  $\omega = -4\pi, \dots, +4\pi$ .  
(b) The AR process is now observed in zero mean white noise  $v_n$  with variance  $\sigma_v^2$ :

$$x_n = d_n + v_n$$

Assuming that  $v_n$  is uncorrelated with  $d_n$ , determine the frequency response  $H(e^{j\omega})$  for the IIR Wiener filter which estimates  $d_n$  optimally from  $x_n$ .

- (c) Hence, with  $\sigma_v^2 = 0.25$ ,  $\sigma_e^2 = 0.25$  and  $a_1 = 0.5$ , determine the Wiener filter impulse response. Is this filter causal or non-causal? Sketch the frequency response of this filter and explain why it is intuitively reasonable.

You may use the following DTFT pair:

$$\sum_{n=-\infty}^{\infty} \alpha^{|n|} e^{-jn\omega} = \frac{1 - \alpha^2}{|1 - \alpha e^{-j\omega}|^2}$$

8. It is desired to predict a future value of a WSS signal  $x_n$   $m$  steps ahead of the current time, so that the desired signal is  $d_n = x_{n+m}$ . We are given the power spectrum of  $\{x_n\}$  as  $\mathcal{S}_x$ .

Show that the standard non-causal frequency domain Wiener filter formula is optimal for this problem:

$$H(e^{j\omega}) = \frac{\mathcal{S}_{xd}(e^{j\omega})}{\mathcal{S}_x(e^{j\omega})}$$

Show that in this case,

$$\mathcal{S}_{xd}(e^{j\omega}) = \mathcal{S}_x(e^{j\omega}) e^{jm\omega}$$

and hence that the Wiener filter gain is:

$$H(e^{j\omega}) = e^{jm\omega}.$$

Determine the impulse response of this filter and explain why this result could have been guessed without calculations.

9. \* In an aircraft cockpit noise cancellation application a noisy signal is measured at a primary microphone:

$$x_n = d_n + v_{1,n}$$

where  $d_n$  is the pilot's voice and  $v_{1,n}$  is background noise. At a second microphone in the cockpit a measurement of pure noise  $v_{2,n}$  is made.  $v_{1,n}$  and  $v_{2,n}$  are assumed jointly wide-sense stationary with cross-correlation function  $r_{v_1 v_2}[k]$ , and both are uncorrelated with  $d_n$ .

It is required to estimate the voice signal by FIR filtering the noise signal and subtracting from  $x_n$ :

$$\hat{d}_n = x_n - \sum_{p=0}^P h_p v_{2,n-p}$$

- (a) Using the mean-squared error criterion, show that the filter coefficients must satisfy

$$r_{v_2 v_1}[k] = \sum_{q=0}^P h_q r_{v_2 v_2}[k - q]$$

for  $k = 0, 1, 2, \dots, P$ . Hence show that the matrix form of the Wiener-Hopf equations for this case is:

$$\mathbf{R}_{v_2} \mathbf{h} = \mathbf{r}_{v_2 v_1}$$

where the matrices and vectors should be carefully defined.

- (b) Explain how, in a real environment,  $r_{v_2 v_2}[k]$  can be measured, assuming  $v_2$  to be ergodic.

(c) Show that when  $d_n$  is stationary,

$$r_{v_2 v_1}[k] = r_{v_2 x}[k]$$

and hence explain how  $r_{v_2 v_1}[k]$  could be estimated. Is the stationary assumption on  $d_n$  realistic in this application?

10. \* In a digital communication system random bits are transmitted as a Bernoulli random process, that is, each time point is independently assigned a value of +1 or -1 with equal probabilities of 0.5 .

In the communications system the channel can be modelled by the following formula,

$$x_n = \sum_{i=0}^1 c_i b_{n-i}$$

where  $c_0 = 1$  and  $c_1 = 0.1$ .

- (a) Determine the cross-correlation function between  $\{b_n\}$  and  $\{x_n\}$ , and also the autocorrelation function of  $\{x_n\}$ .
- (b) It is desired to optimally estimate the bit sequence  $\{b_n\}$  from the channel data  $\{x_n\}$ . Design the first order FIR Wiener filter for this task, i.e. form an estimate of the type:

$$\hat{b}_n = \sum_{i=0}^1 h_i x_{n-i}$$

where  $h_0$  and  $h_1$  are to be determined according to the Wiener criterion.

11. An FIR filter is applied to the problem of detecting a signal  $s_{1,n}$ :

$$s_{1,n} = \begin{cases} 1 & 0 \leq n < D/2 \\ -1 & D/2 \leq n < D \\ 0 & \text{Otherwise} \end{cases}$$

For simplicity,  $D$  is a multiple of 4.

$s_{1,n}$  is observed at an unknown time location  $n_0$ , and in additive, zero mean white noise of variance  $\sigma_v^2$ :

$$x_n = s_{1,n-n_0} + v_n, \quad n = 0, 1, \dots, N-1, \quad N \gg D$$

$x_n$  is filtered with a FIR filter  $h_{1,p}$  to give  $y_n$ :

$$y_n = \sum_{p=0}^P h_{1,p} x_{n-p}$$

- (a) Using such a filter, explain how to perform optimal detection of the time  $n_0$  where  $s_1$  occurs. What are the filter coefficients? Sketch  $s_1$ , the optimal filter  $h_1$  and the output of the optimal filter in response to  $x_n$ . This part is most easily done in Matlab. The following generates the signal  $s_1$  for  $D = 12$  with the usual one sample offset for Matlab vectors:

```

D=12
s_1=ones(D,1);
s_1(D/2+1:D)=-1;

```

- (b) Determine the maximum expected Signal-to-Noise Ratio (SNR) for the optimal filter at time  $n_0 + D - 1$  in terms of  $D$  and compare this with the maximum SNR obtainable from the unfiltered  $x_n$ . Suggest another significant advantage of the matched filter over this simple scheme for this choice of signal  $s_{1,n}$ .
- (c) Using the optimal filter, express the expected signal-to-noise ratio (SNR) at the output of the filter for times  $n \neq n_0 + D - 1$ . Hence show the output from the filter at times  $n \neq n_0 + D - 1$  always has SNR less than that at  $n_0$ .
- (d) True or false (and why?): this result guarantees correct detection of the time  $n_0$  at which the signal occurs.
- (e) A second signal  $s_{2,n}$  may also be present in the data:

$$s_{2,n} = \begin{cases} -1 & 0 \leq n < D/4 \\ +1 & D/4 \leq n < 3D/4 \\ -1 & 3D/4 \leq n < D \\ 0 & \text{Otherwise} \end{cases}$$

Determine and sketch the output of filter  $h_{1,p}$  applied to  $s_{2,n}$ . Again, most easily done in Matlab. The following generates  $s_2$ :

```

s_2=ones(D,1);
s_2(1:D/4)=-1;
s_2(3*D/4+1:D)=-1;

```

Hence suggest how simple matched filters can be used to detect the potential presence of *either or both*  $s_1$  and  $s_2$  in the data.

## Answers

1.

2.

3. Yes

4.

$$h_0 = 0.795 \quad h_1 = 0.073, \quad J_{min} = 0.398$$

5.

6. (a)

$$r_{dd}[k] - h_p * r_{dx}[k] - h_{-p} * r_{xd}[k] + h_p * h_{-p} * r_{xx}[k]$$

7. (a) Power spectrum is

$$\frac{\sigma_e^2}{|(1 - a_1 e^{-j\omega})|^2}$$

Autocorrelation function is:

$$\frac{\sigma_e^2}{1 - a_1^2} a_1^{|n|}$$

(b)

$$H(e^{j\omega}) = \frac{\sigma_e^2}{\sigma_e^2 + \sigma_v^2(1 - a_1 e^{-j\omega})(1 - a_1 e^{j\omega})}$$

(c)  $h_p = 0.496(0.234)^{|p|}$ , non-causal.

8.

9. Impulse response is:

$$h_p = \delta[p + m]$$

10. (a)

$$r_{bx}[m] \begin{cases} c_0, & m = 0 \\ c_1, & m = 1 \\ 0, & \text{otherwise} \end{cases}$$

$r_{xx}[0] = 1.01$ ,  $r_{xx}[\pm 1] = 0.1$ , and zero otherwise

(b)

$$\mathbf{h} = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix}$$

11. (d) False

Suitable past tripos questions from 3F3:

2010 Q2(c)

2011 Q3

2012 Q3

2014 Q3

2015 Q3

2016 Q3 (mainly for lectures 5-8)

2017 Q2 (d) and (e)

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