Detection, Estimation Inference for Signal Processing

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Example: Power Spectrum of Sine Wave

- The power spectrum of the random phase sine-wave is obtained as:

$$\mathcal{S}_X\left(e^{j\Omega}\right) = \sum_{m=-\infty}^{\infty} r_{XX}[m]e^{-jm\Omega} = \sum_{m=-\infty}^{\infty} 0.5a^2\cos\left[m\omega_0\right]e^{-jm\Omega}$$

$$= 0.25a^2 \times \sum_{m=-\infty}^{\infty}\left(\exp\left(jm\omega_0\right) + \exp\left(-jm\omega_0\right)\right)e^{-jm\Omega}$$

$$= 0.5\pi a^2 \times \sum_{m=-\infty}^{\infty}\delta\left(\Omega - \omega_0 - 2m\pi\right) + \delta\left(\Omega + \omega_0 - 2m\pi\right)$$

- The last line above is derived via the Fourier series of a periodic train of δ functions[1]:

$$\sum_{m=-\infty}^{\infty}\delta(\Omega - \omega_0 - 2m\pi) = \sum_{m=-\infty}^{\infty} c_m e^{-jm(\Omega-\omega_0)}$$

- The coefficient c_m can be solve by take the inverse DTFT of the delta train:

$$\Rightarrow c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty}\delta(\Omega - \omega_0 - 2m\pi)e^{jm(\Omega-\omega_0)}\mathrm{d}\Omega$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi}(\ldots)\mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(\Omega - \omega_0)\mathrm{d}\Omega = \frac{1}{2\pi}$$

[1] aka Dirac comb, impulse train and sampling function

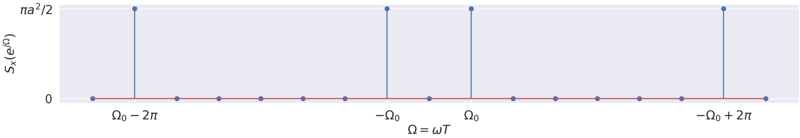
Example: Power Spectrum of Sine Wave

- '$\delta(\cdot)\exp(\cdot)$' is uniform convergent, thus we can permute the integral and summation
- since $\delta(\Omega - \omega_0 - 2m\pi)$ is zero over the interval $[-\pi, \pi]$ for $m \neq 0$, thus there is only one term in the summation that contributes to the integral
- thus

$$2\pi \sum_{m=-\infty}^{\infty} \delta\left(\Omega - \omega_0 - 2m\pi\right) = \sum_{m=-\infty}^{\infty} e^{-jm(\Omega-\omega_0)}$$

as desired

- we can sketch power spectrum as the combination of two Dirac combs:



General Framework of Using Filters

- In practical, usually we do not know the statistic characteristics such as mean and variance(aka variables of interest, VoI) of the signal $\{X_n\}$, thus we need measurements
- If the signal generation process is wide-sense stationary yet ergodic[2] as well, then easy estimation of VoI methods exist:
    - if we measure a realization $\{x_n\}$, of the process $\{X_n\}$(e.g., one waveform from the ensemble of possible waveforms/ one(or a few) battery performance test to infer the general performance of this battery type, sort of to monitor/measure x_n for a period of time, such x_n is just one possible waveform of the ground truth)
- Wiener filter - linear estimator for stationary signals
- Kalman filter - non stationary signals estimator(not covered in 3F3)

[2] such as mean and variance ergodic

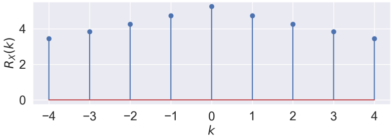
Extension of Wiener Filter

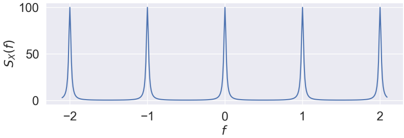
Wiener Filter can readily be extended to deal with cases outside the regular noise reduction:

- Prediction of a noisy signal $\{u_n\}$
- Smoothing of a noisy signal
- Deconvolution

$$R_X(k) = a^{|k|}\sigma_X^2, \quad k \in \mathbb{Z}$$

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-j2\pi fk} = \sigma_X^2 \sum_{k=-\infty}^{\infty} a^{|k|}e^{-j2\pi fk} = \frac{\sigma^2}{1 + a^2 - 2a\cos(2\pi f)}$$





What is 'ergodic'???

TBD

recall the definition of $J(y)$(same as in the lecture notes):

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} h_1\left(Y_1, \ldots, Y_n\right) \\ \vdots \\ h_n\left(Y_1, \ldots, Y_n\right) \end{pmatrix}, \quad J(y) = \begin{pmatrix} \frac{\partial}{\partial y_1} h_1, \ldots, \frac{\partial}{\partial y_n} h_1 \\ \vdots \\ \frac{\partial}{\partial y_1} h_n, \cdots, \frac{\partial}{\partial y_n} h_n \end{pmatrix}$$

where $h_j = \sum_{i=1}^{n} s_{ij} y_i, \;\; s_{ij}$ is the (i, j) entry of S^{-1}

the (j, i) entry of $J(y)$ is: $\dfrac{\partial}{\partial y_i} h_j = s_{ij} \Rightarrow J(y) = (S^{-1})^T \Rightarrow |\det J(y)| = |\det(S^{-1})|$

- **mean vector**: $\mu_Y = \mu_X S^{-1} = (\mu_0, 0, \cdots, 0)\, S^{-1}$
- **covariance matrix**:[3]
  recall the exponential term of multivariate Guassian:

$$X \Sigma_X^{-1} X^T = (Y S^{-1}) \Sigma_X^{-1} (Y S^{-1})^T = Y S^{-1} \Sigma_X^{-1} (S^{-1})^T Y^T = Y \Sigma_Y^{-1} Y^T$$
$$\Rightarrow \Sigma_Y = S^T \Sigma_X S, \text{ where } \Sigma_X = \mathrm{diag}(\sigma_0^2, \sigma^2, \cdots, \sigma^2)$$

[3] usually we use the column vector $(X_0, X_1, \ldots, X_k)^T$ to describe multivariate Gaussian, here we use row vectors, but they are identical. Also it is clear to know that: $(A^{-1})^T = (A^{-T})^1$ since $(A^{-1}A)^T = A^T (A^{-1})^T = \mathrm{I} = A^T (A^T)^{-1}$