3F3 Example Paper3

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Q1

- a) $r_{XX}[k] = \mathbb{E}[X_n X_{n+k}] = \mathbb{E}[X_{n+k} X_n] = r_{XX}[-k]$
- b)

$$\mathcal{S}_X\left(e^{j\Omega}\right) = \sum_{m=-\infty}^{\infty} r_{XX}[m] e^{-jm\Omega} = \sum_{m=1}^{\infty} r_{XX}[m](e^{-jm\Omega} + e^{jm\Omega}) + r_{XX}[0]$$
$$= \sum_{m=1}^{\infty} 2r_{XX}[m]\cos(m\Omega) + r_{XX}[0]$$

- c)

$$(x_{n+k} - ax_n)^2 \geq 0 \Rightarrow \mathbb{E}\left[(x_{n+k} - ax_n)^2\right] \geq 0$$
$$\Rightarrow \text{set } a = 1 : \mathbb{E}\left[(x_{n+k} - x_n)^2\right] = \mathbb{E}\left[(x_{n+k} - x_n)^2\right] = 2r_{XX}[0] - 2r_{XX}[k] \geq 0$$
$$\Rightarrow \text{this holds for } \forall k \text{ and } r_{XX}[k] = r_{XX}[-k] \Rightarrow \max|r_{XX}[k]| = r_{XX}[0]$$

- d) $r_{XY}[k] = \mathbb{E}[X_n Y_{n+k}] = \mathbb{E}[Y_{n+k} X_n] = r_{YX}[-k]$
- e,f) straightforward based on part b)
- g) straightforward based on part d)

Q2

$$\mathbb{E}[Y_n] = \mathbb{E}\left[\sum_{p=-\infty}^{+\infty} h_p X_{n-p}\right] = \sum_{p=-\infty}^{+\infty} h_p \mathbb{E}\left[X_{n-p}\right] = \sum_{p=-\infty}^{+\infty} h_p \mathbb{E}\left[X_n\right] \quad (\text{since} \{X_n\} \text{ is WSS})$$

$$\begin{aligned} r_{YY}[k] &= \mathbb{E}\left[\left(\sum_{p_1=-\infty}^{\infty} h_{p_1} X_{n-p_1}\right)\left(\sum_{p_2=-\infty}^{\infty} h_{p_2} X_{n-p_2+k}\right)\right] = \sum_{p_1}\sum_{p_2} h_{p_1} h_{p_2} \mathbb{E}\left[X_{n-p_1} X_{n-p_2+k}\right] \\ &= \sum_{p_1}\sum_{p_2} h_{p_1} h_{p_2} r_{XX}[p_1 - p_2 + k] \end{aligned}$$

recall the discrete convolution: $(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m]g[n-m]$

$$\begin{aligned} r_{YY}[k] &= \sum_{p_1}\sum_{p_2} h_{p_1} h_{p_2} r_{XX}\left[p_1 - p_2 + k\right] = \sum_{p_1} h_{p_1}\left(\sum_{p_2} h_{p_2} r_{XX}\left[p_1 - p_2 + k\right]\right) \\ &= \sum_{p_1} h_{p_1}\left(\left(h * r_{XX}\right)\left[p_1 + k\right]\right) \quad (\text{time reversal for } h_{p_1}) \\ &= \sum_{p_1'} h_{p_i'}\left(\left(h * r_{XX}\right)\left[k - p_1'\right]\right) \quad (p_1' = -p_1) \\ &= \left(\tilde{h} * h * r_{XX}\right)[k] \quad \text{where } \tilde{h} \text{ is the time-reversed sequence } h \end{aligned}$$

Q2

to show that $\{Y_n\}$ is wide-sense stationary, basically is to show the variance is finite:

$$\mu_Y = \mathbb{E}[Y_n] = \mathbb{E}[X_n] \sum_{p=-\infty}^{+\infty} h_p \leq \mathbb{E}[X_n] \sum_{p=-\infty}^{+\infty} |h_p| \quad \text{(constant and finite)}$$

$$r_{YY}[k] = \left(\tilde{h} * h * r_{XX}\right)[k] \quad \text{(only related to } k\text{)}$$

$$\mathbb{E}\left[(Y_n - \mu_Y)^2\right] = \mathbb{E}\left[Y_n^2\right] - \mathbb{E}[Y_n]^2 = r_{YY}[0] - \text{some constant}$$

$$r_{YY}[0] = \sum_{p_1}\sum_{p_2} h_{p_1} h_{p_2} r_{XX}[p_1 - p_2] \leq \sum_{p_1}\sum_{p_2} h_{p_1} h_{p_2} (r_{XX}[0])$$

$$\leq r_{XX}[0]\Big(\sum_{p_1} |h_{p_1}|\Big)\Big(\sum_{p_2} |h_{p_2}|\Big) < \infty \Rightarrow \text{variance is finite}$$

Q3

- $\mu_x = \mathbb{E}[x_n] = p \cdot (1) + (1-p) \cdot (-1) = 2p - 1 \Rightarrow$ constant
- $r_{xx}[0] = 1$, since x_n and $x_{n+i} (i \neq 0)$ are independent $\Rightarrow r_{XX}[k] = \mathbb{E}[x_n x_{n+k}] = 0 \ (k \neq 0)$
- $\mathbb{E}\left[(x_n - \mu_x)^2\right] = \mathbb{E}\left[x_n^2\right] - \mu_x^2 = r_{xx}[0]^2 - \mu_x^2 = 1 - (2p-1)^2 \Rightarrow$ finite $\Rightarrow$ thus WSS.
- $c_{xx}[m] = \mathbb{E}[(x_n - \mu_x)(x_{n+m} - \mu_x)] = r_{xx}[m] - \mu_x^2 = \delta[k]\left(1 - \mu_x^2\right) \Rightarrow$ thus White Noise.

Q4

- Wiener-Hopf equation: $\mathbf{R}_x\mathbf{h} = \mathbf{r}_{xd}$

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} r_{xd}[0] \\ r_{xd}[1] \end{bmatrix}$$

$$\begin{aligned} r_{xd}[k] &= \mathbb{E}\left[x_{n+k}d_n\right] = \mathbb{E}\left[x_{n+k}(x_n - v_n)\right] = r_{xx}[k] - r_{xv}[k] \\ r_{xv}[k] &= \mathbb{E}\left[v_n x_{n+k}\right] = \mathbb{E}\left[v_n(d_{n+k} + v_{n+k})\right] = r_{vd}[k] + r_{vv}[k] = r_{vv}[k] \\ &\Rightarrow r_{xd}[k] = r_{xx}[k] - r_{vv}[k] \end{aligned}$$

$$\begin{bmatrix} 2.8 & 1 \\ 1 & 2.8 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 2.3 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 0.795 \\ 0.073 \end{bmatrix}$$

- minimum mean-squared error for this filter is:

$$J = \mathbb{E}\left[v_n^2\right] = 0.5 \quad \text{(simply setting: } \hat{d}_n = x_n\text{)}$$

$$J_{\min} = \mathbb{E}\left[\epsilon_n d_n\right] = r_{dd}[0] - \sum_{p=-\infty}^{\infty} h_p r_{xd}[p] = r_{dd}[0] - h_0 r_{xd}[0] - h_1 r_{xd}[1] = 0.399$$

Q5

- for the frequency responces, we have (essentially in this case we have: $r_{xd}[m] = r_{dd}[m]$):

$$H\left(e^{j\Omega}\right) = \frac{\mathcal{S}_{xd}\left(e^{j\Omega}\right)}{\mathcal{S}_{x}\left(e^{j\Omega}\right)} = \frac{\sum_{m=-\infty}^{\infty} r_{xd}[m]e^{-jm\Omega}}{\sum_{m=-\infty}^{\infty} r_{xx}[m]e^{-jm\Omega}} = \frac{2.3 + 2\cos\Omega}{2.8 + 2\cos\Omega}$$

- Comments?

- a) $$\epsilon_n\epsilon_{n+k} = d_n d_{n+k} - d_n \sum_p h_p x_{n-p+k} - d_n \sum_p h_p x_{n-p+k} + \sum_{p_1}\sum_{p_2} h_{p_1} x_{n-p_1} h_{p_2} x_{n-p_2+k}$$
$$\begin{aligned} r_{\epsilon\epsilon}[k] &= r_{dd}[k] - \sum_p h_p r_{dx}[k-p] - \sum_p h_p r_{xd}[k+p] + \sum_{p_1}\sum_{p_2} h_{p_1} h_{p_2} r_{xx}[k-p_2+p_1] \\ &= r_{dd}[k] - (h * r_{dx})[k] - \left(\tilde{h} * r_{xd}\right)[k] + \left(\tilde{h} * h * r_{xx}\right)[k], \qquad \text{(from Q2)} \end{aligned}$$
- b) since $\mathcal{S}_X\left(e^{j\Omega}\right) = \sum_{m=-\infty}^{\infty} r_{XX}[m]e^{-jm\Omega}$, based on (a) we can directly have:
$$\mathcal{S}_\epsilon = \mathcal{S}_D - \mathcal{S}_{DX}H - S_{XD}H^* + |H|^2\mathcal{S}_X$$
- c) clearly $J_{\min} = \mathbb{E}\left[\epsilon_n^2\right] = r_{\epsilon\epsilon}[0]$, then:
$$\begin{aligned} J_{\min} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{S}_\epsilon d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathcal{S}_D - \mathcal{S}_{DX}H - \mathcal{S}_{XD}H^* + |H|^2\mathcal{S}_X\right) d\Omega\Big|_{H=H^{opt}=\frac{\mathcal{S}_{XD}}{\mathcal{S}_X}} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathcal{S}_D - \mathcal{S}_{DX}\mathcal{S}_{XD}/\mathcal{S}_X - \mathcal{S}_{XD}\mathcal{S}_{DX}/\mathcal{S}_X + \mathcal{S}_{XD}\mathcal{S}_{DX}/\mathcal{S}_X\right) d\Omega, \quad (\mathcal{S}_{DX} = \mathcal{S}_{XD}^*) \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \left(\mathcal{S}_D - \mathcal{S}_{XD}^* H^{\mathrm{opt}}\right) d\omega, \quad \text{as required} \end{aligned}$$
- d) in this case, $\mathcal{S}_{XD} = \mathcal{S}_D$ and $\mathcal{S}_D^* = \mathcal{S}_D$, it is one step from c)

Q7

- a) basically covered in lecture notes:

$$\mathcal{S}_d\left(e^{j\Omega}\right) = |H(e^{j\Omega})|^2\mathcal{S}_e\left(e^{j\Omega}\right) = \frac{\sigma_e^2}{\left(1-a_1e^{-j\Omega}\right)\left(1-a_1e^{j\Omega}\right)} = \frac{\sigma_e^2}{1+a_1^2-2a_1\cos\Omega}$$

$$= \sum_{m=-\infty}^{\infty} r_{dd}[m]e^{-jm\Omega} \Rightarrow r_{dd}[m] = \frac{1}{2\pi}\int_{2\pi}\mathcal{S}_d\cdot e^{jm\Omega}d\Omega = \frac{\sigma_e^2}{1-a_1^2}a_1^{|m|}, \quad \text{(DTFT)}$$

- sketch:
- b)

$$H\left(e^{j\Omega}\right) = \frac{\mathcal{S}_d\left(e^{j\Omega}\right)}{\mathcal{S}_d\left(e^{j\Omega}\right)+\mathcal{S}_v\left(e^{j\Omega}\right)} = \frac{\sigma_e^2}{\sigma_e^2+\left(1-a_1e^{-j\Omega}\right)\left(1-a_1e^{j\Omega}\right)\sigma_v^2}$$

- c) quite straightforward, non-causal

Q9

- a)

$$\epsilon_n = d_n - \hat{d}_n = d_n - x_n + \sum_{p=0}^{P} h_p v_{2,n-p}$$

$$\frac{\partial}{\partial h_p}\mathbb{E}[\epsilon_n^2] = \mathbb{E}[2\epsilon_n \frac{\partial \epsilon_n}{\partial h_p}] = \mathbb{E}[\epsilon_n v_{2,n-p}] = 0$$

$$\Rightarrow \mathbb{E}[\epsilon_n v_{2,n-p'}] = \mathbb{E}[(d_n - x_n + \sum_{p=0}^{P} h_p v_{2,n-p}) v_{2,n-p'}]$$

$$= r_{dv_2}[-p'] - r_{xv_2}[-p'] + \sum_{p=0}^{P} r_{v_2 v_2}[p-p']$$

$$\Rightarrow \quad r_{dv_2}[-p'] = 0 \quad \text{and} \quad r_{xv_2}[\cdot] = r_{(d+v_1)v_2}[\cdot] = r_{v_1 v_2}[\cdot]$$

$$\Rightarrow \quad r_{v_1 v_2}[-p'] = \sum_{p=0}^{p} h_p r_{v_2 v_2}[p-p'] = \sum_{p=0}^{p} h_p r_{v_2 v_2}[p'-p] = r_{v_2 v_1}[p'] \quad \text{as requried}$$

Hence:

$$\mathbf{R}_{v_2}\mathbf{h} = \mathbf{r}_{v_2 v_1}$$

- b) since v_2 is ergodic and is measured directly, we have:

$$\hat{r}_{v_2 v_2}[k] = \frac{1}{N} \sum_{n=1}^{N} v_{2,n} v_{2,n+k}$$

- c)

$$r_{v_2 v_1}[k] = \mathbb{E}\left[v_{2,n} v_{1,n+k}\right] = \mathbb{E}\left[v_{2,n}(x_{n+k} - d_{n+k})\right] = r_{v_2 x}[k] - r_{v_2 d}[k] = r_{v_2 x}[k]$$

$$\hat{r}_{v_2 v_1}[k] = \hat{r}_{v_2 x}[k] = \frac{1}{N} \sum_{n=1}^{N} v_{2,n} x_{n+k}$$

the stationary assumption on d (speech) is not realistic, maybe try out Kalman filter

- a)

$$r_{bx}[m] = \mathbb{E}\left[b_n x_{n+m}\right] = \mathbb{E}\left[b_n \sum_{i=0}^{1} c_i b_{n+m-i}\right] = \sum_{i=0}^{1} c_i \mathbb{E}\left[b_n b_{n+m-i}\right] = \begin{cases} c_0, & m=0 \\ c_1, & m=1 \\ 0, & \text{otherwise} \end{cases}$$

$$r_{xx}[m] = \mathbb{E}\left[x_n x_{n+m}\right] = \mathbb{E}\left[\sum_{i=0}^{1} c_i b_{n-i} \sum_{j=0}^{1} c_j b_{n+m-j}\right] = \sum_{i=0}^{1}\sum_{j=0}^{1} c_i c_j \mathbb{E}\left[b_{n-i} b_{n+m-j}\right]$$

$$= \sum_{i=0}^{1} c_i c_{m-i} = \begin{cases} 1.01, & m=0 \\ 0.1, & m=\pm 1 \\ 0, & \text{otherwise} \end{cases}$$

- b)

$$J = \mathbb{E}\left[\left(b_n - \hat{b}_n\right)^2\right] = \mathbb{E}\left[\left(b_n - \sum_{i=0}^{1} h_i x_{n-i}\right)^2\right] = \mathbb{E}\left[\left(b_n - \mathbf{h}^T \mathbf{x}_n\right)^2\right]$$

$$= \mathbb{E}\left[b_n^2\right] + \mathbf{h}^T \mathbb{E}\left[\mathbf{x}_n \mathbf{x}_n^T\right] \mathbf{h} - 2\mathbb{E}\left[b_n \mathbf{h}^T \mathbf{x}_n\right]$$

$$\partial J / \partial \mathbf{h} = 0 \Rightarrow \mathbb{E}\left[\mathbf{x}_n \mathbf{x}_n^T\right] \mathbf{h} = \mathbb{E}\left[b_n \mathbf{x}_n\right] \Rightarrow \mathbf{h} = \mathbf{R}_x^{-1} \mathbf{r}_{bx} = \begin{bmatrix} 1.01 & 0.1 \\ 0.1 & 1.01 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.99 \\ 0.001 \end{bmatrix}$$