

3F3 Example Paper2

Hanchen Wang

Ph.D Candidate in
Engineering, University of Cambridge

November 18, 2019

- a)

$$\begin{aligned}
 p(i_2|i_1) &= \frac{p(i_1, i_2)}{p(i_1)} = \frac{\sum_{i_0 \in S} p(i_0, i_1, i_2)}{p(i_1)} \\
 &= \frac{1}{p(i_1)} \sum_{i_0 \in S} \frac{p(i_0, i_1, i_2)}{p(i_0, i_1)} p(i_0, i_1) \\
 &= \frac{1}{p(i_1)} \sum_{i_0 \in S} p(i_2|i_0, i_1) p(i_0, i_1) \\
 &= \frac{1}{p(i_1)} \sum_{i_0 \in S} P_{i_1, i_2} \cdot p(i_0, i_1) \\
 &= P_{i_1, i_2}
 \end{aligned}$$

- b) by using mathematical induction

- the probability of X_m (for any m) in state i_m is:

$$p_{X_m}(i_m) = (\lambda P^n)_{i_m} = (\lambda P)_{i_m} = \lambda_{i_m}$$

$$\text{since } P^2 = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix}^2 = \begin{pmatrix} \lambda_1 \sum_{i=1}^n \lambda_i & \cdots & \lambda_n \sum_{i=1}^n \lambda_i \\ \lambda_1 \sum_{i=1}^n \lambda_i & \cdots & \lambda_n \sum_{i=1}^n \lambda_i \\ \vdots & & \vdots \\ \lambda_1 \sum_{i=1}^n \lambda_i & \cdots & \lambda_n \sum_{i=1}^n \lambda_i \end{pmatrix} = P$$

thus $P^n = P$

$$\text{and } \lambda P = (\lambda_1, \lambda_2, \dots, \lambda_n) \cdot \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} = (\lambda_1, \lambda_2, \dots, \lambda_n) = \lambda$$

thus no matter at which point(X_m), the probability of being in state i_m is always λ_{i_m}

- thus the joint PMF

$$p(X_m, X_{m+1}, \dots, X_{m+k}) = p(i_m P_{i_m, i_{m+1}} P_{i_{m+1}, i_{m+2}} \dots P_{i_{m+k-1}, i_{m+k}}) = \lambda_{i_m} \lambda_{i_{m+1}} \dots \lambda_{i_{m+k}}$$

is irrelevant to either m or k , which shows that this is strictly stationary

• b)

$$\begin{aligned}
 \mathbb{E}(X_n) &= \mathbb{E}(A) \cos(nf_0) + \mathbb{E}(B) \sin(nf_0) = 0 \quad \text{since } \mathbb{E}(A) = \mathbb{E}(B) = 0 \\
 \mathbb{E}(X_{n_1} X_{n_2}) &= \mathbb{E}(((A \cos(n_1 f_0) + B \sin(n_1 f_0))(A \cos(n_2 f_0) + B \sin(n_2 f_0))) \\
 &= \mathbb{E}(A^2) \cos(n_1 f_0) \cos(n_2 f_0) + \mathbb{E}(B^2) \sin(n_1 f_0) \sin(n_2 f_0) \quad \text{since } \mathbb{E}(AB) = 0 \\
 &= (\sigma_A^2 + \mathbb{E}(A)^2) \cos(n_1 f_0) \cos(n_2 f_0) + (\sigma_B^2 + \mathbb{E}(B)^2) \sin(n_1 f_0) \sin(n_2 f_0) \\
 &= 2(\cos(n_1 f_0) \cos(n_2 f_0) + \sin(n_1 f_0) \sin(n_2 f_0)) \\
 &= \cos((n_1 - n_2)f_0) \Rightarrow \text{thus it is WSS}
 \end{aligned}$$

• c)

$$\begin{aligned}
 \mathbb{E}(Y_n) &= \mathbb{E}(X_n) - \mathbb{E}(X_{n-1}) = 0 \quad \text{since } \mathbb{E}(X_n) = \mathbb{E}(X_{n-1}) = 2q - 1 \\
 \mathbb{E}(Y_{n_1} Y_{n_2}) &= \mathbb{E}((X_{n_1} - X_{n_1-1})(X_{n_2} - X_{n_2-1})) \\
 &= \mathbb{E}(X_{n_1} X_{n_2}) - \mathbb{E}(X_{n_1-1} X_{n_2}) - \mathbb{E}(X_{n_1} X_{n_2-1}) + \mathbb{E}(X_{n_1-1} X_{n_2-1}) \\
 &= (2q - 1)^2 - \mathbb{E}(X_{n_1-1} X_{n_2}) - \mathbb{E}(X_{n_1} X_{n_2-1}) + (2q - 1)^2 \\
 &= \begin{cases} (2q - 1)^2 - (2q - 1)^2 - (2q - 1)^2 + (2q - 1)^2 = 0 & \text{if } |n_1 - n_2| = 1 \\ (2q - 1)^2 - 0 - (2q - 1)^2 + (2q - 1)^2 = (2q - 1)^2 & \text{elsewhere} \end{cases} \\
 &\Rightarrow \text{it is WSS, we can denote the correlation function as:} \\
 &\quad R_Y(k) = 1 \text{ if } k = 1, R_Y(k) = 0 \text{ elsewhere}
 \end{aligned}$$

- c) from a) and b), we know that(it is also included in the lecture notes):

$$R_X(k) = a^{|k|} \sigma_X^2, \quad k \in \mathbb{Z}$$

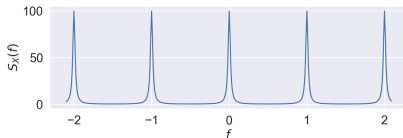
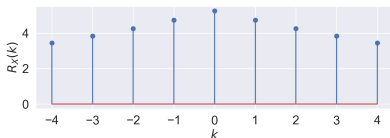
$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi f k} = \sigma_X^2 \sum_{k=-\infty}^{\infty} a^{|k|} e^{-j2\pi f k} = \frac{\sigma^2}{1 + a^2 - 2a \cos(2\pi f)}$$

thus we can derive a and σ^2 directly:

$$a = R_X(k)/R_X(k-1) = R_X(1)/R_X(0) = 4.74/5.26 \approx 0.9$$

$$\sigma^2 = (1 - a^2) \sigma_X^2 = (1 - a^2) R_X(0) = 0.19 * 5.26 \approx 1.0$$

thus we can sketch the $R_X(k)$ and $S_X(f)$:



- **joint pdf:** it is not straightforward to write down the joint pdf of (X_0, X_1, \dots, X_k) , thus we first write down the joint pdf of (X_0, W_1, \dots, W_k) , then apply the rules of change of variables for random vectors to get our goal:

$$f(x_0, w_1, \dots, w_k) = f_{X_0}(x_0) \prod_{i=1}^k f_W(w_i), \text{ where } f_{X_0} \sim \mathcal{N}(\mu_0, \sigma_0^2), f_W \sim \mathcal{N}(0, \sigma^2)$$

which is a multivariate Gaussian, plus we know that:

$$(X_0, \dots, X_k) = (X_0, W_1, \dots, W_k) \begin{pmatrix} 1 & a & \dots & a^k \\ & 1 & \dots & a^{k-1} \\ & & \ddots & \vdots \\ & & & 1 & a \\ & & & & 1 \end{pmatrix} \text{ denoted as: } Y = XS$$

$\Rightarrow X = YS^{-1}$ since $\det(S) = 1$ thus it has an inverse

recall the rules of change of variables, now we have:

$$f_Y(y) = f_X(H(y)) |\det J(y)| = f_X(yS^{-1}) |\det(S^{-1})| = f_X(yS^{-1}) \quad \text{same notations as lecture notes}$$

here $f_X(\cdot) = f_{X_0, W_1, \dots, W_k}(x_0, w_1, \dots, w_k)$ is the multivariate Gaussian mentioned above

now is to show that: $|\det J(y)| = |\det(S^{-1})|$

recall the definition of $J(y)$ (same as in the lecture notes):

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} h_1(Y_1, \dots, Y_n) \\ \vdots \\ h_n(Y_1, \dots, Y_n) \end{pmatrix}, \quad J(y) = \begin{pmatrix} \frac{\partial}{\partial y_1} h_1, \dots, \frac{\partial}{\partial y_n} h_1 \\ \vdots \\ \frac{\partial}{\partial y_1} h_n, \dots, \frac{\partial}{\partial y_n} h_n \end{pmatrix}$$

where $h_j = \sum_{i=1}^n s_{ij} y_i$, s_{ij} is the (i, j) entry of S^{-1}

the (j, i) entry of $J(y)$ is: $\frac{\partial}{\partial y_i} h_j = s_{ij} \Rightarrow J(y) = (S^{-1})^T \Rightarrow |\det J(y)| = |\det(S^{-1})|$

- **mean vector:** $\mu_Y = \mu_X S^{-1} = (\mu_0, 0, \dots, 0) S^{-1}$
- **covariance matrix:**¹

recall the exponential term of multivariate Gaussian:

$$\begin{aligned} X \Sigma_X^{-1} X^T &= (Y S^{-1}) \Sigma_X^{-1} (Y S^{-1})^T = Y S^{-1} \Sigma_X^{-1} (S^{-1})^T Y^T = Y \Sigma_Y^{-1} Y^T \\ &\Rightarrow \Sigma_Y = S^T \Sigma_X S, \text{ where } \Sigma_X = \text{diag}(\sigma_0^2, \sigma^2, \dots, \sigma^2) \end{aligned}$$

¹ usually we use the column vector $(X_0, X_1, \dots, X_k)^T$ to describe multivariate Gaussian, here we use row vectors, but they are identical. Also it is clear to know that: $(A^{-1})^T = (A^{-T})^1$ since $(A^{-1}A)^T = A^T(A^{-1})^T = I = A^T(A^T)^{-1} \Rightarrow (A^{-1})^T = (A^T)^{-1}$

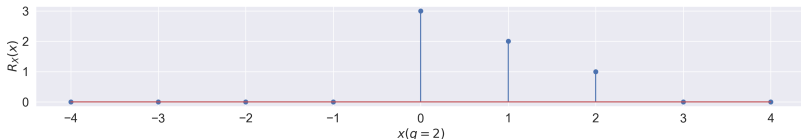
- a) $X_n = \sum_{i=-\infty}^{\infty} h_i W_{n-i}$ where $\{h_i\}$ is the impulse response of a causal LTI system with input $\{W_i\}$. Indeed $h_0 = 1$, $h_i = b_i$ for $0 < i \leq q$ and $h_i = 0$ otherwise.

- b)

$$R_X(n, n+k) = \mathbb{E}\{X_n X_{n+k}\} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j \mathbb{E}\{W_{n-i} W_{n+k-j}\} = \sigma^2 \sum_{i=-\infty}^{\infty} h_i h_{i+k}$$

since $\mathbb{E}(X_n X_n) = \sigma^2$, $\mathbb{E}(X_n X_{n+k}) = 0 (k \neq 0)$, $R_X(n, n+k)$ is irrelevant to $n \Rightarrow$ it is WSS

- c) when $b_i = 1$, $R_X(0) = q+1$, $R_X(1) = q, \dots, R_X(q) = 1$ and $R_X(k) = 0$ for $k > q$. As a remark, we see that even without the restriction that $b_i = 1$, $R_X(k) = 0$ for $k > q$.



- d) recall that:

$$H(f) = 1 + \sum_{k=1}^q b_k e^{-2\pi f k} \cdot S_X(f) = S_W(f) |H(f)|^2 = \sigma^2 |H(f)|^2$$

$$H(f) = 1 + b_1 e^{-j2\pi f} + b_2 e^{-j4\pi f}$$

$$|H(f)|^2 = H(f)H(f)^* = 1 + b_1^2 + b_2^2 + 2b_1(1 + b_2)\cos(2\pi f) + 2b_2\cos(4\pi f)$$

thus the sketch is:

