I1. Let "NB" be the event the email doesn't contain the word "free".
"S" .... is a spam
"NS" ... is not a spam.

$\Omega = \{ (NB, NS), (NB, S), (B, NS), (B, S) \}$.

note that

$p(B,S) = p(B|S) \cdot p(S) = 0.8 \times 0.25 = 0.2$

$p(NB,S) = (1 - p(B|S)) \cdot p(S) = 0.05$

$p(NB,NS) = (1-p(B|NS)) \cdot (1-p(S)) = 0.9 \times 0.75 = 0.675$

$p(B,NS) = p(B|NS) \cdot (1-p(S)) = 0.1 \times 0.75 = 0.075$

I2. "AB are independent" $\Leftrightarrow$ $p(AB) = p(A) \cdot p(B)$.

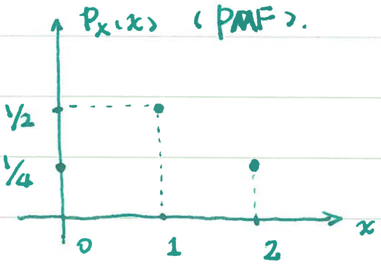
$AB = A \cap B = \{2, 4\}$.

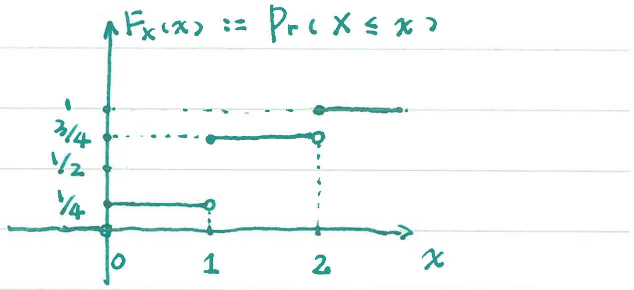
$p(AB) = 2/6 = 1/3$ $\quad p(A) = 1/2$ $\quad p(B) = 2/3$

$\Rightarrow p(AB) = p(A) \cdot p(B)$

$\Rightarrow$ A and B are independent.

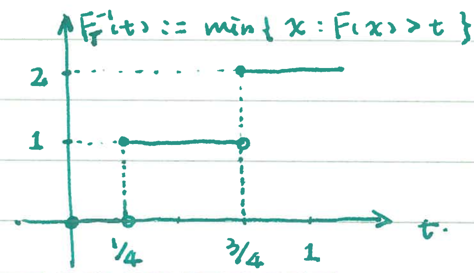
I3. (a).





right continuous: $\lim_{x \to x_0^+} f(x) = f(x_0)$

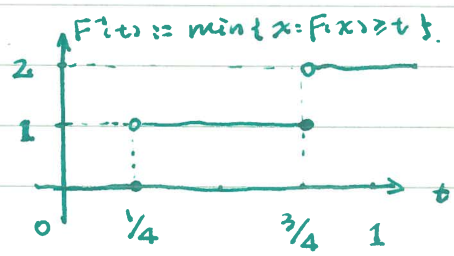
(b).



$F^{-1}(1/2) = 1$

$F^{-1}(3/4) = 2$

(c)



$F^{-1}(1/2) = 1$

$F^{-1}(3/4) = 1$. (left continuous)

Are both definition of the inverse correct ?

$\Rightarrow$ Can they both represent identical CMF ?

$\Rightarrow$ if so, Both are correct.

$\Rightarrow$ Find PMF of $F^{-1}(U)$ where U is uniform random variable in $[0,1]$.

This method is also called "The Inverse Transform Method".

$\Rightarrow$ Proposition: (

the inverse transform method", define $X = F^{-1}(U)$, $u \in \text{Uniform}([0,1])$

$\Rightarrow P(X \le x) = F(x)$

Proof:

$F(F^{-1}(y)) = y$. $\Rightarrow$ if $F^{-1}(U) \le x$

$\Rightarrow F(F^{-1}(U)) = U \le F(x)$ (F is monotonically increase)

similarly $F^{-1}(F(y)) = y$ $\Rightarrow$ if $U \le F(x)$

$\Rightarrow F^{-1}(U) \le F^{-1}(F(x)) = x$ (F^{-1} is monotonically increasing).

$\Rightarrow \{F^{-1}(U) \le x\} \subseteq \{U \le F(x)\} \subseteq \{F^{-1}(U) \le x\}$.

$\Rightarrow \{F^{-1}(U) \le x\} = \{U \le F(x)\}$.

$\Rightarrow P(\{F^{-1}(U) \le x\}) = P(\{U \le F(x)\}) = F(x)$.

$\Rightarrow$ both definitions of F^{-1}, correspond to the same CDF, thus CMF, thus both are correct

14. (a).

Recall the axioms of probability:

1) $P(\Omega) = 1$

2) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \phi$

3) $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ if $A_i \cap A_j = \phi$ when $i \neq j$.

$F \cup G = (F - FG) \cup (G)$, and $(F - FG) \cap G = \phi$

$\Rightarrow P(F \cup G) = P(F - FG) + P(G)$. then $FG \cap F - FG = \phi$

$\Rightarrow P(FG \cup F - FG) = P(FG) + P(F - FG) = P(F)$

$\Rightarrow P(F - FG) = P(F) - P(FG)$

$\Rightarrow P(F \cup G) = P(F) + P(G) - P(FG)$

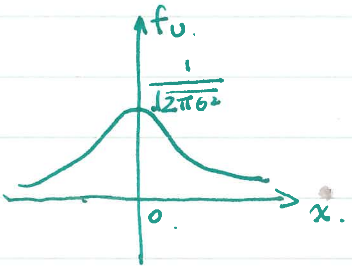
b). $p(G|F_i)\,p(F_i) = p(G, F_i)$.

$\{G, F_i\} \cup \{G, F_j\} = \{G, F_i \cup F_j\}$ since $F_i \cap F_j = \phi$

$\Rightarrow \sum_{i=1}^{n} p(G|F_i)\,p(F_i) = \sum_{i=1}^{n} p(G, F_i) = p(G, \bigcup_{i=1}^{n} F_i) = p(G, \Omega) = p(G)$.

15. (a) $1 - F_X(0.5)$ (b) $F_X(0.25)$ (c) $F_X(0.7) - F_X(0.3)$.

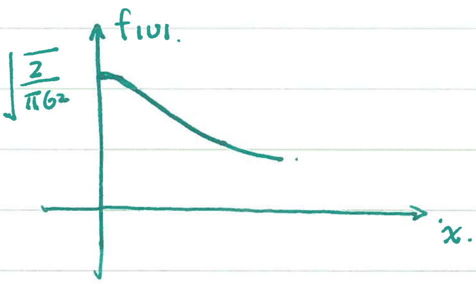
16. (a) $f_U = N(0, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2})$.



(b). $Pr(|U| \le r) = F(r) - F(-r)$ $(r \ge 0)$.

$\frac{d}{dr}(F(r) - F(-r)) = \frac{d}{dr}F(r) - \frac{d}{dr}F(-r) \cdot \frac{d}{dr}(-r) = 2 \cdot \frac{d}{dr}F(r) = 2f(r)$

$\Rightarrow f_{|U|} = 2f(u)$ $(u \ge 0)$.



(c). Method 1. $g(U) = U^2 = |U|^2$ $\Rightarrow$ change of variables.

$f_{U^2} = f_{|U|}(\sqrt{u}) \cdot \frac{d\sqrt{u}}{du}$

$= f_{|U|}(\sqrt{u}) \cdot \frac{1}{2\sqrt{u}}$ $(u \ge 0)$.

$= \cancel{2} \cdot f_U(\sqrt{u}) \cdot \frac{1}{\cancel{2}\sqrt{u}}$ $(u \ge 0)$.

$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{u}{2\sigma^2}) \cdot \frac{1}{\sqrt{u}}$

method 2.

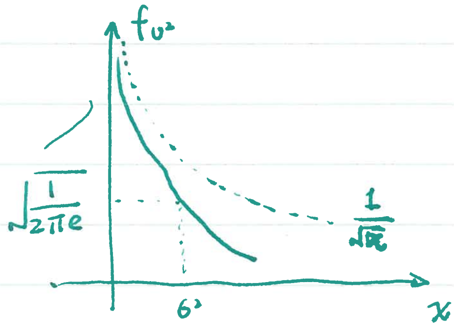
$Pr \; N(0, \sigma^2) \le x$

$$Pr(U^2 \le r) = F(\sqrt{r}) - F(-\sqrt{r}) \qquad (r \ge 0)$$

$$\Rightarrow f_{U^2} = \frac{d}{dr}\left(F(\sqrt{r}) - F(-\sqrt{r})\right).$$

$$= f(\sqrt{r}) \cdot 2 \cdot \frac{d\sqrt{r}}{dr}$$

$$= f(\sqrt{u}) \cdot \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{u}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{u}}$$



(d). $U < -a \Rightarrow g(u) = u + a \Rightarrow u = g(u) - a. \Rightarrow g(u) \in (-\infty, 0)$

$U > a \Rightarrow g(u) = u - a \Rightarrow u = g(u) + a. \Rightarrow g(u) \in (0, +\infty)$.

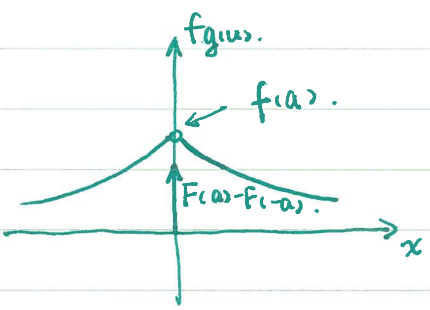
$-a \le U \le a \Rightarrow g(u) = 0$

$$\Rightarrow. \; f_{g(u)} = f_U(u-a)\left(\frac{d(u+a)}{du}\right) = f_U(u-a) \qquad (g(u) > 0).$$

similarly.

$$f_{g(u)} = f_U(u+a) \quad \text{for } g(u) < 0.$$

$$f_{g(0)} = \int_{-a}^{a} f_{(U)}(u) \cdot du = F(a) - F(-a). \rightarrow \text{mass}.$$



I7. $Cov(X,Y) = \mathbb{E}(XY) - \mu_X \mu_Y$

$= \mathbb{E}(aX^2+bX) - \mathbb{E}(aX+b)\cdot\mathbb{E}(X).$

$= a\mathbb{E}(X^2) + b\mathbb{E}(X) - (a\mathbb{E}(x)+b)\cdot\mathbb{E}(x).$

$= a\cdot(\mathbb{E}(x^2) - \mathbb{E}^2(x))$

$\sigma_X^2\sigma_Y^2 = (\mathbb{E}(x^2) - \mathbb{E}^2(x))(\mathbb{E}((ax+b)^2) - \mathbb{E}^2(ax+b)).$

$= (\mathbb{E}(x^2) - \mathbb{E}^2(x))\cdot(a^2\mathbb{E}(x^2) - a^2\mathbb{E}^2(x)).$

$\Rightarrow \rho(X,Y) = sign(a)$

II8. $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}\exp(-\frac{x^2}{2\sigma^2})\cdot\exp(-\frac{y^2}{2\sigma^2}).$

$Pr(a\le X\le b,\ c\le Y\le d) = \int_c^d\int_a^b \frac{1}{2\pi\sigma^2}\exp(-\frac{(x^2+y^2)}{2\sigma^2})\,dx\,dy.$

Change to Polar Coordinates:

first is Jacobian: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$$

$\Rightarrow f_{R,\theta}(r,\theta) = f_{XY}(r\cos\theta, r\sin\theta)\cdot r$

$= \frac{1}{2\pi\sigma^2}\exp(-\frac{r^2}{2\sigma^2})\cdot r$

$f_R(r) = \int_0^{2\pi}\frac{r}{2\pi\sigma^2}\exp(-\frac{r^2}{2\sigma^2})\cdot d\theta = \frac{r}{\sigma^2}\exp(-\frac{r^2}{2\sigma^2}) \sim$ Rayleigh PDF.

I 11.

a).

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}.$$ by definition.

$$f_{XY}(x,y) = g(x)\cdot h(y).$$

$$= \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-m_1, y-m_2)^T \Sigma^{-1} (x-m_1, y-m_2)\right).$$

$$= \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left(-\frac{1}{2|\Sigma|} [x-m_1, y-m_2] \begin{bmatrix} \sigma_2^2 & -\rho \\ -\rho & \sigma_1^2 \end{bmatrix} \begin{bmatrix} x-m_1 \\ y-m_2 \end{bmatrix}\right).$$ (cross term cancelled)

$$\sim \frac{-1}{2|\Sigma|}\left[(x-m_1)^2\sigma_2^2 + (y-m_2)^2\sigma_1^2 - 2(x-m_1)(y-m_2)\rho\right].$$

Complete the squares. →

$$= \frac{-1}{2|\Sigma|\sigma_2^{-2}}\left[(x-m_1)^2 - 2(x-m_1)\frac{(y-m_2)\rho}{\sigma_2^2} + \left(\frac{(y-m_2)\rho}{\sigma_2^2}\right)^2 - \left(\frac{(y-m_2)\rho}{\sigma_2^2}\right)^2 + (y-m_2)^2\frac{\sigma_1^2}{\sigma_2^2}\right].$$

$$= \frac{1}{2|\Sigma|\sigma_2^{-2}}\Big[\underbrace{\left(x-m_1-\frac{\rho}{\sigma_2^2}(y-m_2)\right)^2}_{g(x)} + \underbrace{(y-m_2)^2\frac{\sigma_1^2}{\sigma_2^2} - \left(\frac{(y-m_2)\rho}{\sigma_2^2}\right)^2}_{h(y).}\Big].$$

$$\Rightarrow f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{g(x)\cancel{h(y)}}{\int_{-\infty}^{\infty} g(x)\cancel{h(y)}\,dx} = \frac{\exp\left(\frac{\sigma_2^2}{2|\Sigma|}[(x-m_1-\frac{\rho}{\sigma_2^2}(y-m_2)]^2\right)}{\int_{-\infty}^{\infty}(\quad)\,dx}.$$

$$= \frac{\mathcal{N}\left(x;\ m_1+\frac{\rho}{\sigma_2^2}(y-m_2),\ \frac{|\Sigma|}{\sigma_2^2}\right)}{\int \mathcal{N}\left(x;\ m_1+\frac{\rho}{\sigma_2^2}(y-m_2),\ \frac{|\Sigma|}{\sigma_2^2}\right)dx = 1.}$$

$$\frac{1}{\sqrt{2\pi}\sigma}$$

$$= \mathcal{N}\left(x;\ m_1+\frac{\rho}{\sigma_2^2}(y-m_2),\ \sigma_1^2-\frac{\rho^2}{\sigma_2^2}\right) \cdot \cancel{\text{const}}$$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2$$

b).

$$f_Y(y) = \frac{f_{XY}(x,y)}{f_{X|Y}(x|y)}$$

$$= \frac{\sqrt{2\pi}|\Sigma|^{1/2}/\sigma_2}{2\pi|\Sigma|^{1/2}} \times \frac{\exp\left(-\frac{1}{2|\Sigma|}\left((x-m_1)^2\sigma_2^2 + (y-m_2)^2\sigma_1^2 - 2(x-m_1)(y-m_2)\rho\right)\right)}{\exp\left(-\frac{\sigma_2^2}{2|\Sigma|}\left(x-m_1-\frac{\rho}{\sigma_2^2}(y-m_2)\right)^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(\frac{1}{2\sigma_2^2}(y-m_2)^2\right)$$

$$\frac{\exp\left(-\frac{1}{2|\Sigma|}[(y-m_2)\sigma_1^2]\right)}{\exp\left(-\frac{\cancel{\sigma_2}}{2|\Sigma|}\frac{\rho^2}{\sigma_2^4}(y-m_2)^2\right)} = \frac{\exp\left(-\frac{1}{2|\Sigma|}[(y-m_2)^2\sigma_1^2]\right)}{\exp\left(-\frac{1}{2|\Sigma|}\frac{\sigma_1^2\sigma_2^2-|\Sigma|}{\sigma_2^2}(y-m_2)^2\right)}$$

$$= \exp\left(-\frac{1}{2\sigma_2^2}(y-m_2)^2\right).$$

finished !