Probability excercise

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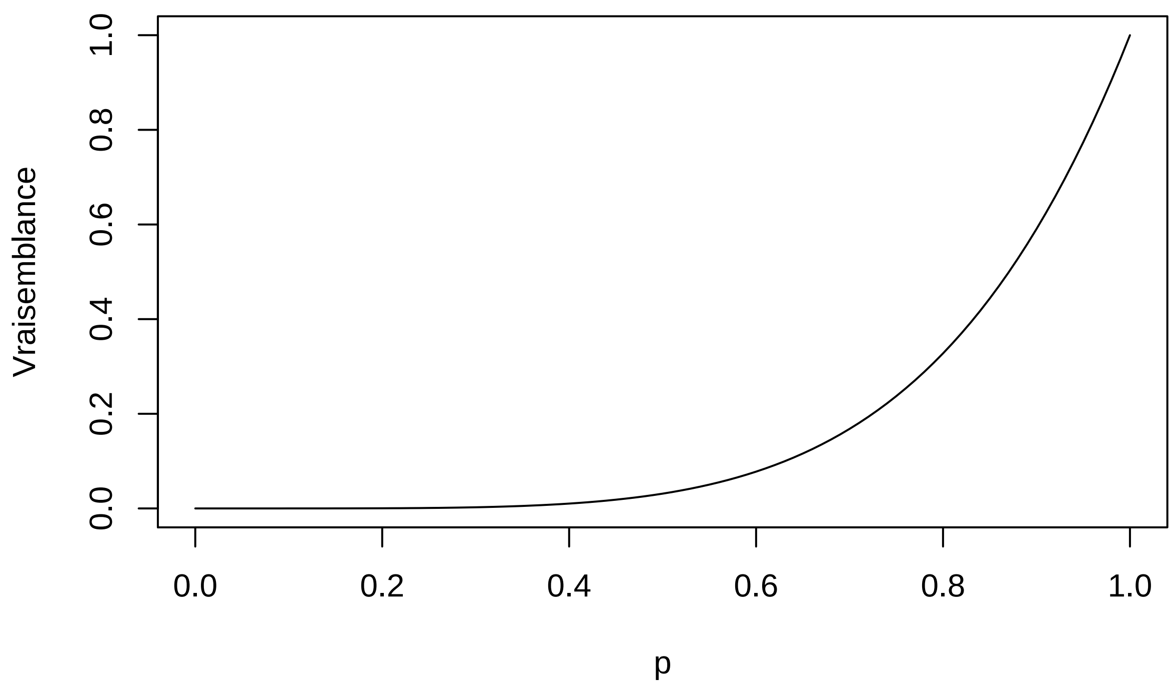
Problem

You throw a coin 5 times. Each time we get *HEAD*. What is the probability p of getting *HEAD* on the 6th throw?

Solution

Likelihood of the event according to p

The main idea is to see p as a measure of the bias of the used coin. If p=0.5 then the coin is unbiased since it has as much chance to fall on *HEAD* as on *TAIL*. We therefore start by computing the likelihood of the event *H-H-H-H-H* as a function of p. The graph below represents the likelihood of the *H-H-H-H-H* event as a function of all possible p values.

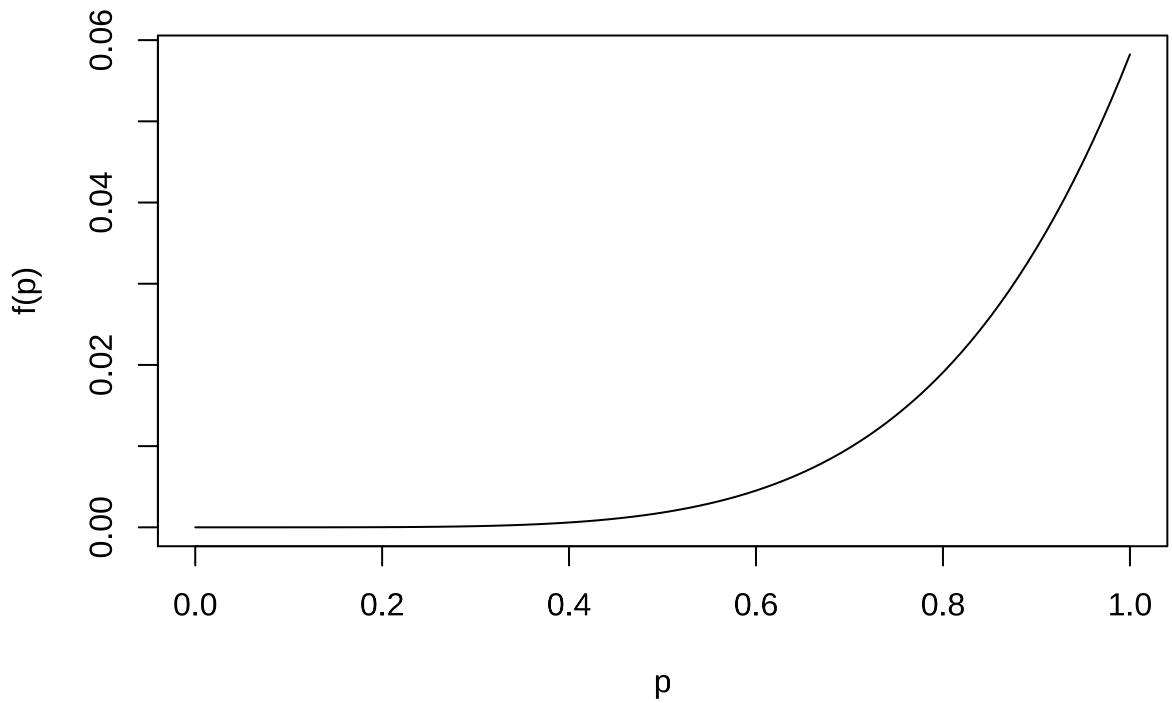


As one might expect, the likelihood is higher as we get to 1. Indeed, getting *HEAD* 5 times in a row is extremely unlikely if the coin is not unbiased (p=0.5). On the other hand, it is very likely to happen if p=0.8.

Probability density function of the variable p

We can see p as the variable representing the bias of the coin we used. In this case, its probability density *f(p)* is equal to one constant to the function *V(p)* which gives the likelihood of the event *H-H-H-H* as a function of p. The integral of *f(p)* is 1. The graph below shows the probability density function of the random variable p.

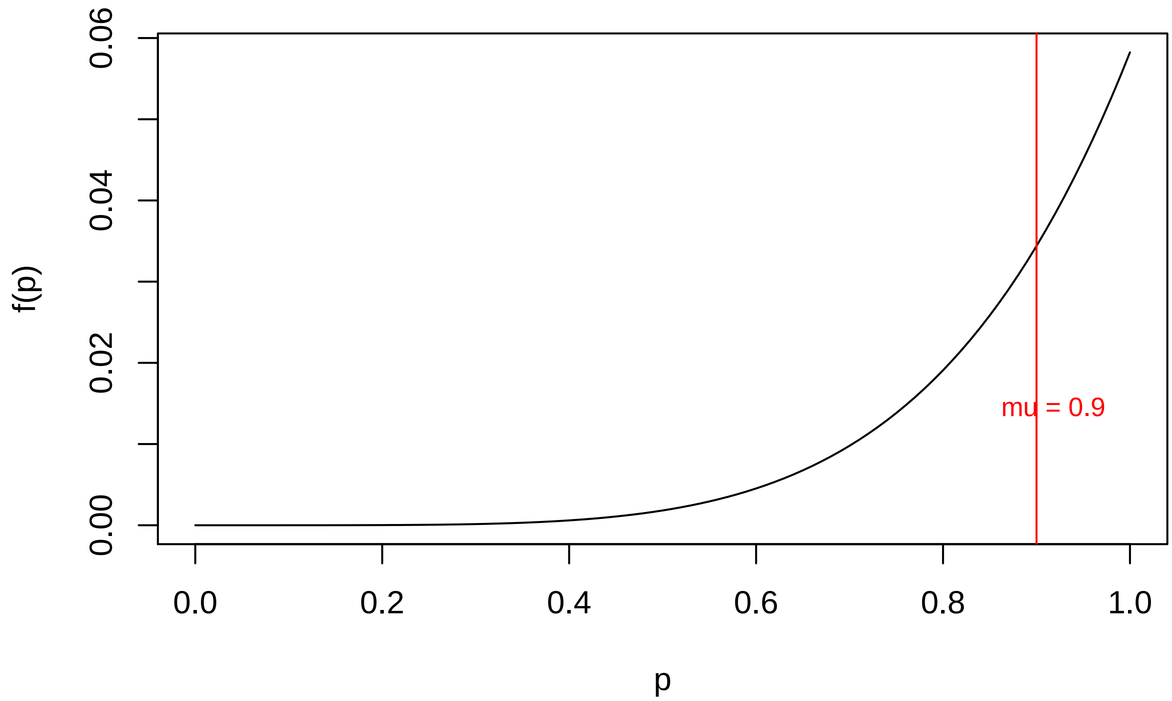
$$f(p) = \frac{V(p)}{\int_0^1 V(p)dp} = \frac{V(p)}{Q}$$



Estimation of p

Now that we have the probability density function of p, it is easy to determine its average. This corresponds to the value of p for which the area under the curve on the left and right are equal. Be careful not to confuse the mean and the mode which are often the same in symmetrical probability densities. We are looking for mu so that :

$$\int_0^\mu f(p)dp = \int_\mu^1 f(p)dp = 0.5$$



Conclusion

In our case, it turns out that the mean mu of the random variable p is 0.9. In other words, the most probable value of p representing the bias of the coin used is 0.9. **In the 6th trial, we have 90% chance of getting *HEAD* again**.