Assignment 8 (PDEs)
Phys 512 Computational Physics

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Q1:
Plugging in the solution and doing a bit of algebra:

$$\begin{aligned}
\frac{f(t+dt,x)-f(t-dt,x)}{2dt} &= -v\frac{f(t,x+dx)-f(t,x-dx)}{2dx} \\
\frac{\xi^{t+dt}e^{ik(x)}-\xi^{t-dt}e^{ik(x)}}{2dt} &= -v\frac{\xi^{t}e^{ik(x+dx)}-\xi^{t}e^{ik(x-dx)}}{2dx} \\
\frac{\xi^{dt}-\xi^{-dt}}{2dt} &= -v\frac{e^{ik(dx)}-e^{-ik(dx)}}{2dx} \\
\frac{\xi^{dt}-\xi^{-dt}}{2dt} &= -i\frac{v}{dx}\sin(k(dx)) \\
\xi^{dt}-\xi^{-dt} &= -i\frac{2v(dt)}{dx}\sin(k(dx)) \\
\frac{\xi^{2dt}-1}{\xi^{dt}} &= -i\frac{2v(dt)}{dx}\sin(k(dx)) \\
\xi^{2dt}-1 &= -i\xi^{dt}\frac{2v(dt)}{dx}\sin(k(dx))
\end{aligned}$$

The dt in the exponents on ξ actually just represent a time step of 1 unit, so we can really write this as

$$\xi^2-1=-i\xi\frac{2v(dt)}{dx}\sin(k(dx))$$

Solving the quadratic equation:

$$\xi=-i\frac{vdt}{dx}\sin(kdx)\pm\sqrt{1-\left(\frac{vdt}{dx}\sin(kdx)\right)^2}$$

We want to find when $|\xi|=1$, since that's when the solution won't blow up.

$$\begin{aligned}
|\xi|^2 &= \xi\cdot\mathrm{conj}(\xi) \\
&= \frac{vdt}{dx}\sin(kdx)+1-\left(\frac{vdt}{dx}\sin(kdx)\right)^2
\end{aligned}$$

Let $y=\frac{vdt}{dx}\sin(kdx)$

$$\begin{aligned}
|\xi|^2=1 &= -y^2+y+1 \\
0 &= y(1-y)
\end{aligned}$$

So either $\frac{vdt}{dx}=0$ (trivial solution) or

$$\begin{aligned}
1-y &= 0 \\
1 &= \frac{vdt}{dx}\sin(kdx)
\end{aligned}$$

Since we can't control $\sin(kdx)$, we have

$$\begin{aligned} 1 &= \frac{vdt}{dx} \\ \implies dt &= \frac{dx}{v} \end{aligned}$$

But it also works if y is larger, since that would make $|\xi|$ smaller, so the condition is

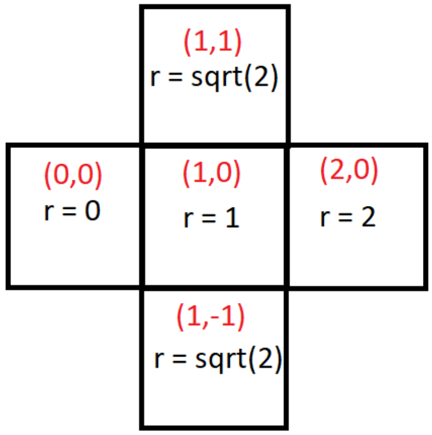
$$\frac{vdt}{dx} \geq 1$$

which is exactly the Courant condition.

Q2:

a) To work out the scale of the potential, we'll use the rule $V(r) = a\ln(r) + b$ where a is some a priori unknown scale factor, and b is an offset.

This is the diagram of the cells next to (1,0):



I'm pretty sure the following is wrong somehow, but here's what I get. First I use the equation that relates the potential to the average of its neighbors when there's ρ present.

$$\begin{aligned}
V(0,0) - \frac{V(1,0) + V(0,1) + V(-1,0) + V(0,-1)}{4} &= \rho \\
1 - \frac{a\ln(1) + b + a\ln(1) + b + a\ln(1) + b + a\ln(1) + b}{4} &= 1 \\
\frac{4b}{4} &= 0 \\
\implies b &= 0
\end{aligned}$$

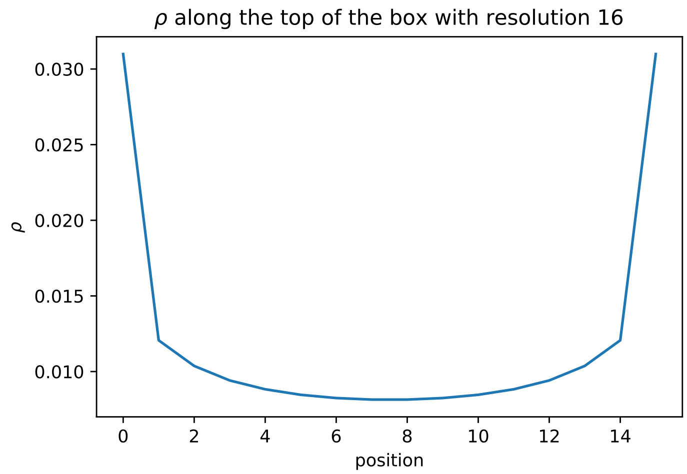
Then I use the case without ρ at (0,1)

$$\begin{aligned}
V(1,0) &= \frac{V(0,0) + V(1,1) + V(1,-1) + V(2,0)}{4} \\
a\ln(1) &= \frac{1 + a\ln\left(\sqrt{2}\right) + a\ln\left(\sqrt{2}\right) + a\ln(2)}{4} \\
0 &= \frac{1 + 2a\ln\left(\sqrt{2}\right) + a\ln(2)}{4} \\
-1 &= 2a\ln(2) \\
\implies a &\approx -0.7213475204444817
\end{aligned}$$

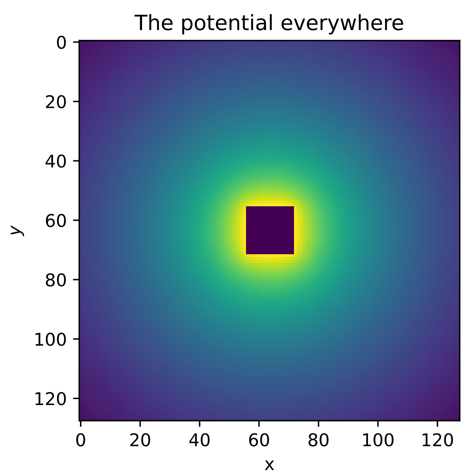
This gives the final rule as $V(r) = -0.7213475204444817\log(r)$.

The potential at (0,1) is 0 and the potential at (0,2) is -0.5. But I also get that the potential at (5,0) is -1.16 instead of -1.05 so I must be wrong somewhere.

b)
Here's the plot of the charge on the side of the square:

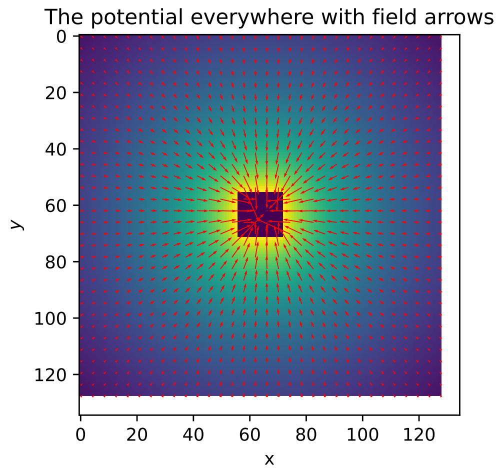


c)
Here's the plot of the potential everywhere:



I calculated the difference between the highest voltage in the square and the average, and it gave me 0. This is expected though, since we manually set it to 0 every time we do apply_laplace.

Here's the plot with the arrows of the field:



The arrows at the surface seem to be pointing more or less perpendicular to the surface, at least those that are near the middle of the square.

(Also the arrows might be slightly misplaced. pyplot.quiver is really confusing.)