Assignment 5
Phys 512 Computational Physics

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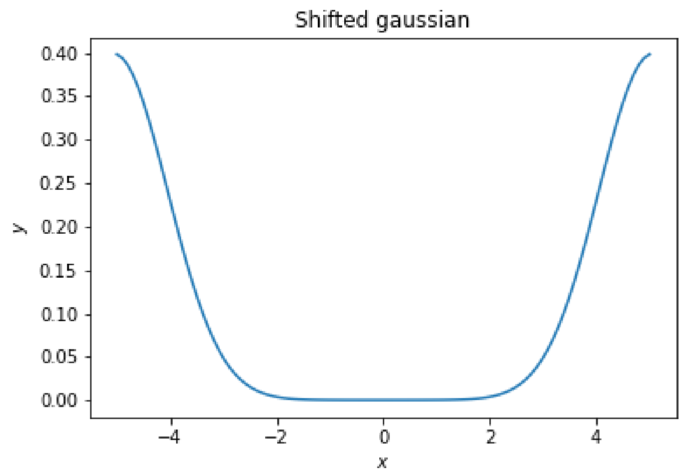
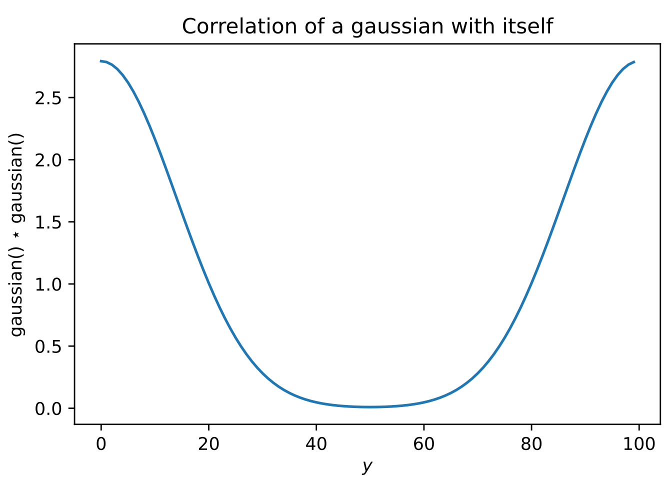
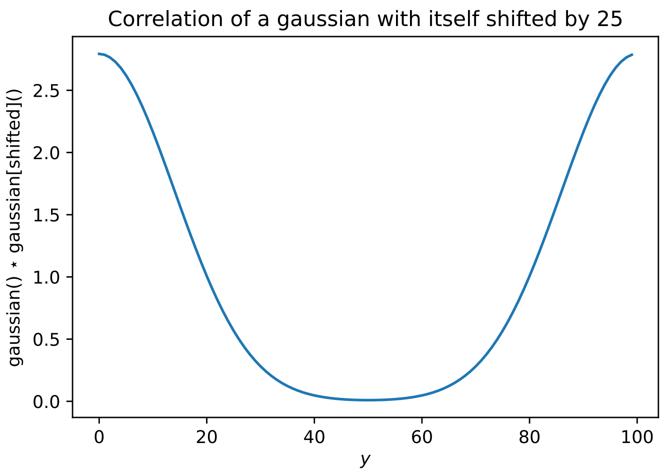
Q1:
The resulting plot is this:



Q2:
The resulting plot is this:



Q3:
I tried shifting by different amounts and always got the same output:



To explain, let's say we're shifting array f by m amount. Let g be the zero array but with 1 in the mth place. The equation we used for shifting is:

$$\begin{aligned}
\text{shift}(f) &= f * g \\
&= \text{idft}(\text{dft}(f)\text{dft}(g)) \\
&= \text{idft}(\text{dft}(f)\exp(-2\pi imk/N)) \\
&= \text{idft}\left(\sum_{n=0}^{N-1} f[n]\exp(-2\pi ink/N)\exp(-2\pi imk/N)\right) \\
&= \text{idft}\left(\sum_{n=0}^{N-1} f[n]\exp(-2\pi i(n+m)k/N)\right)
\end{aligned}$$

(I didn't actually use it; I just did a direct convolution, but it's equivalent.)

Meanwhile for correlation we use

$$f \star h = \text{idft}(\text{dft}(f)\ \text{conj}(\text{dft}(h)))$$

If we let $h = \text{shift}(f)$ we get:

$$\begin{aligned}
f \star \text{shift}(f) &= \text{idft}(\text{dft}(f)\ \text{conj}(\text{dft}(\text{shift}(f)))) \\
&= \text{idft}\left(\text{dft}(f)\ \text{conj}\left(\sum_{n=0}^{N-1} f[n]\exp(-2\pi i(n+m)k/N)\right)\right) \\
&= \text{idft}\left(\text{dft}(f)\ \left(\sum_{n=0}^{N-1} \text{conj}(f[n])\exp(2\pi i(n+m)k/N)\right)\right) \\
&= \text{idft}\left(\left(\sum_{n=0}^{N-1} f[n]\exp(-2\pi ink/N)\right)\ \left(\sum_{n=0}^{N-1} \text{conj}(f[n])\exp(2\pi i(n+m)k/N)\right)\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(\left(\sum_{n=0}^{N-1} f[n]\exp(-2\pi ink/N)\right)\ \left(\sum_{n=0}^{N-1} \text{conj}(f[n])\exp(2\pi i(n+m)k/N)\right)\right)\exp(2\pi iky/N)
\end{aligned}$$

The part that the shift amount m affects is in the exponent in the second sum. $(n+m)k$. But notice that every value of n and k are getting summed over anyway, so the m is just changing the order of terms in the sum. (This is also because everything is mod N, so mk will wrap around.) That's why we don't expect the shift amount to change anything.

Q4:

The way I'll do it is to start the g array shifted almost entirely to the left, so that only the last element of g and the first element of f multiply, then gradually shift over g until only the first element of g and the last element of f multiply.

If N is the length of the f array and M is the length of the g array, my output array will be $N + M - 1$ long.

The code is in A5_Q4.py

Q5:

a)

Let $\alpha = \exp(-2\pi ik/N)$. This gives

$$\begin{aligned}
\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) &= \sum_{x=0}^{N} \alpha^x \\
&= \frac{1-\alpha^N}{1-\alpha} && \text{geometric sum} \\
&= \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)}
\end{aligned}$$

b) At $k = 0$, this is $\frac{0}{0}$, so we can use l'Hopital (treating k as nondiscrete variable):

$$\begin{aligned}
\lim_{k\to 0} \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} &= \lim_{k\to 0} \frac{2\pi i\exp(-2\pi ik)}{\frac{2\pi i}{N}\exp(-2\pi ik/N)} \\
&= N \lim_{k\to 0} \exp(-2\pi ik(1+1/N)) \\
&= N
\end{aligned}$$

This is true when k is continuous, so it must also be true when k is discrete by limit properties.

Proving it's always equal to 0 if k isn't a multiple of N:

Since k is an integer, the numerator $1-\exp(-2\pi ik) = 1-1$ is always 0. The denominator $1-\exp(-2\pi ik/N)$ is always nonzero since k/N isn't an integer. Dividing 0 by nonzero always gives 0.

c)

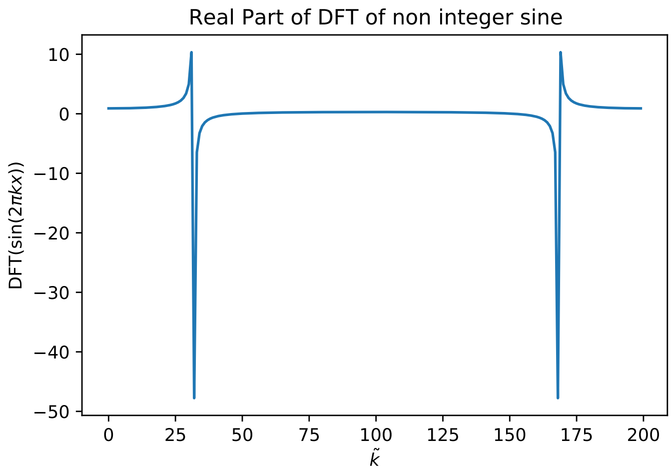
Choose k to be a non integer. Then the non integer sine wave is

$$\sin(2\pi kx) = \frac{\exp(2\pi ikx) - \exp(-2\pi ikx)}{2i}$$

Let $F(\tilde{k})$ be the dft, where $\tilde{k}$ is an integer. We write that out as:

$$\begin{aligned}
F(\tilde{k}) &= \sum_{x=0}^{N-1} \left(\frac{\exp(2\pi ikx) - \exp(-2\pi ikx)}{2i}\right) \exp\Big(-2\pi i\tilde{k}x/N\Big) \\
&= \frac{1}{2i}\sum_{x=0}^{N-1} \exp\Big(2\pi i(k-\tilde{k}/N)x\Big) - \frac{1}{2i}\sum_{x=0}^{N-1} \exp\Big(2\pi i(-k-\tilde{k}/N)x\Big) \\
&= \frac{1}{2i}\frac{1-\exp\Big(2\pi i(kN-\tilde{k})\Big)}{1-\exp\Big(2\pi i(k-\tilde{k}/N)\Big)} - \frac{1}{2i}\frac{1-\exp\Big(2\pi i(-kN-\tilde{k})\Big)}{1-\exp\Big(2\pi i(-k-\tilde{k}/N)\Big)}
\end{aligned}$$

This is the resulting plot: (k is chosen to be $\frac{1}{2\pi}$ and it's plotting 200 points.)

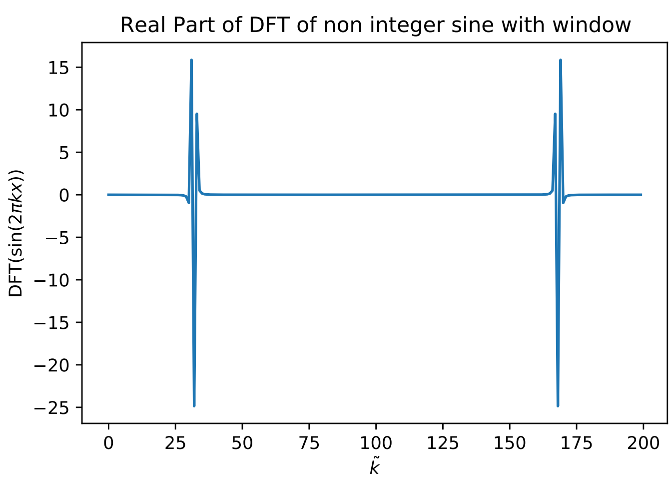


The maximum absolute value of the difference between the analytic points and the numpy fft points is about 10^{-12}, but this goes down when testing fewer points. It's probably just caused by the ringing. With 100 points, it's down to 10^{-13}. With 10 points it's at 10^{-14}.

It's not really a delta function. It's a negative delta function and then a tiny positive delta function beside it. Also the delta spike has a wide base instead of a narrow base.

d)

This is the resulting plot (for same k and N values)



Well the base of the spike is a lot narrower, which is good, but it's still not just one spike.

But maybe this makes sense. The spike is centered around 30, and $\frac{200}{2\pi} \approx 31.83$. So maybe those spikes are like $\tilde{k} = 31$ and $\tilde{k} = 32$, and it needs a linear combination for the two to produce the original sine function.

e) The Fourier transform of the window is

$$\begin{aligned}
W(k) &= \sum_{x=0}^{N-1} w(x)\exp(-2\pi ikx/N) \\
&= \sum_{x=0}^{N-1} \left(\frac{1}{2} - \frac{1}{2}\left(\frac{\exp(2\pi ix/N) + \exp(-2\pi ix/N)}{2}\right)\right)\exp(-2\pi ikx/N) \\
&= \sum_{x=0}^{N-1} \left(\frac{1}{2}\exp(-2\pi ikx/N) - \frac{1}{4}\left(\exp(2\pi i(1-k)x/N) + \exp(-2\pi i(1+k)x/N)\right)\right) \\
&= \frac{1}{2}\left(\frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ikx/N)}\right) - \frac{1}{4}\left(\frac{1-\exp(2\pi i(1-k))}{1-\exp(2\pi i(1-k)x/N)}\right) - \frac{1}{4}\left(\frac{1-\exp(-2\pi i(1+k))}{1-\exp(-2\pi i(1+k)x/N)}\right)
\end{aligned}$$

The first term goes to $\frac{N}{2}$ (proved earlier) only when $k=0$, and it's 0 otherwise. The second term is 0 unless $k=1$, in which case it takes the value $-\frac{N}{4}$. The final term is 0 unless $k=-1$, which is the same as saying $k=N-1$. It also takes the value of $-\frac{N}{4}$.

The resulting array looks like $[\frac{N}{2}, -\frac{N}{4}, 0, ..., 0, -\frac{N}{4}]$.

Let the unwindowed Fourier transform array look like $[F[0], F[1], ..., F[N-1]]$. We're taking a convolution of this and the above array, which we'll call $G = [\frac{N}{2}, -\frac{N}{4}, 0, ..., 0, -\frac{N}{4}]$.

Let D be the convolution array. Then $D[i]$ is:

$$D[i] = \sum_{j=0}^{N-1} F[j]G[i-j]$$

The only nonzero terms are $j=i$, $j=i+1$, and $j=i-1$.

$$\begin{aligned}
D[i] &= F[i-1]G[-1] + F[i]G[0] + F[i+1]G[1] \\
&= -\frac{N}{4}F[i-1] + \frac{N}{2}F[i] - \frac{N}{4}F[i+1]
\end{aligned}$$

And this $D[i]$ should be the exact same thing we get by multiplying in real space and then taking the Fourier transform.

Q6:

a) From the Wiener-Khinchin Theorem, we get that the power spectrum is the Fourier transform of the correlation function.

The correlation function we found is $g(\delta) = c - |\delta|$. Then the Fourier transform is

$$\begin{aligned} G(k) &= \sum_{\delta=0}^{N-1} (c-\delta) e^{-2\pi i k \delta/N} \\ &= c \sum_{\delta=0}^{N-1} e^{-2\pi i k \delta/N} - \sum_{\delta=0}^{N-1} \delta e^{-2\pi i k \delta/N} \end{aligned}$$

Using partial sum identities, we can write

$$\begin{aligned} \sum_{\delta=0}^{N-1} e^{-2\pi i k \delta/N} &= \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)} \\ \sum_{\delta=0}^{N-1} \delta e^{-2\pi i k \delta/N} &= \frac{\exp(2\pi i k/N) - N\exp(-2\pi i k) + (N-1)\exp\left(-2\pi i k \frac{N+1}{N}\right)}{(1-\exp(-2\pi i k/N))^2} \end{aligned}$$

So we subtract the two fractions:

$$G(k) = \frac{c - c\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)} - \frac{\exp(2\pi i k/N) - N\exp(-2\pi i k) + (N-1)\exp\left(-2\pi i k \frac{N+1}{N}\right)}{(1-\exp(-2\pi i k/N))^2}$$

Putting them over the same common denominator means we need to multiply out the numerator of the first fraction:

$$(c - c\exp(-2\pi i k))(1-\exp(-2\pi i k/N)) = c - c\exp(-2\pi i k) - c\exp(-2\pi i k/N) + c\exp\left(-2\pi i k \frac{N+1}{N}\right)$$

Adding those terms into the rest gives:

$$G(k) = \frac{c - (c+1)\exp(2\pi i k/N) + (N-c)\exp(-2\pi i k) - (N-1-c)\exp\left(-2\pi i k \frac{N+1}{N}\right)}{(1-\exp(-2\pi i k/N))^2}$$

Now we're going into approximation land. We'll take N to be large, so $\frac{N+1}{N} \to 1$.

$$G(k) = \frac{c - (c+1)\exp(2\pi i k/N) + (N-c)\exp(-2\pi i k) - (N-1-c)\exp(-2\pi i k)}{(1-\exp(-2\pi i k/N))^2}$$

So we can subtract off the $(N-c)$ multiple.

$$G(k) = \frac{c - (c+1)\exp(2\pi i k/N) + \exp(-2\pi i k)}{(1-\exp(-2\pi i k/N))^2}$$

And actually $\exp(-2\pi i k)$ is always 1 since k is an integer.

$$\begin{aligned} G(k) &= \frac{c - (c+1)\exp(2\pi i k/N) + 1}{(1-\exp(-2\pi i k/N))^2} \\ &= (c+1)\frac{1-\exp(2\pi i k/N)}{(1-\exp(-2\pi i k/N))^2} \\ &= (c+1)\frac{1}{1-\exp(-2\pi i k/N)} \\ &= \frac{c}{1-\exp(-2\pi i k/N)} \end{aligned}$$

In the last line, we redefined c as $c+1$ since it's nothing but a constant multiple anyway.

Now we use the fact that $g(\delta) = c - |\delta| = c - |-\delta| = g(-\delta)$ to conclude that the Fourier transform $G(k)$ is real. The easiest way to find the real part of this is to find the absolute value squared and then square root.

$$
\begin{aligned}
|G(\delta)|^2 &= \left(\frac{c}{1 - \exp(-2\pi ik/N)}\right)\left(\frac{c}{1 - \exp(2\pi ik/N)}\right) \\
&= \frac{c^2}{1 - \exp(2\pi ik/N) - \exp(-2\pi ik/N) + 1} \\
&= \frac{c^2}{2 - 2\cos(2\pi k/N)}
\end{aligned}
$$

This concludes the part of the math where I know what's happening.
If we assume k is small relative to N, then we can Taylor expand cos around k/N.

$$\cos(2\pi ik/N) \approx 1 - 2\pi^2 \frac{k^2}{N^2}$$

Plugging this in gives:

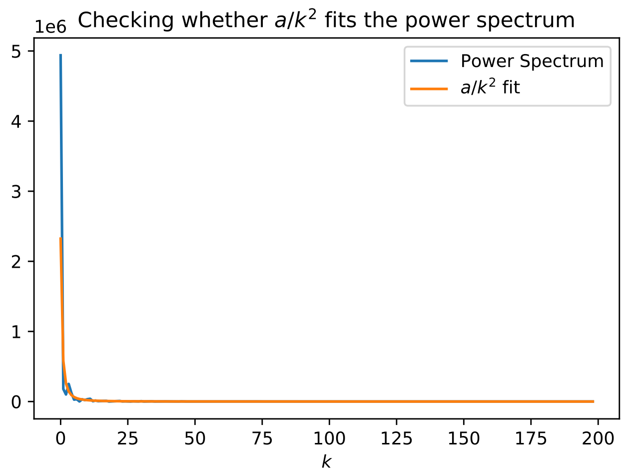
$$
\begin{aligned}
|G(\delta)|^2 &= \frac{c^2}{2 - 2(1 - 2\pi^2 \frac{k^2}{N^2})} \\
&= \frac{c^2 N^2}{4\pi^2 k^2}
\end{aligned}
$$

This *does* go like k^{-2}, but the power spectrum is supposed to be the square root of this.

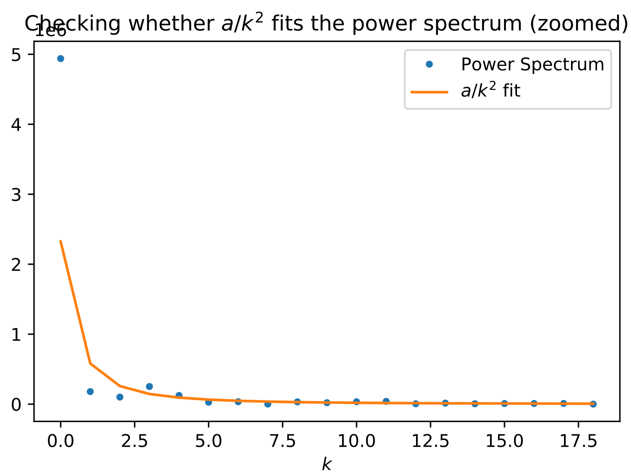
$$G(\delta) = \frac{cN}{2\pi k}$$

Which goes like k^{-1}. Apparently I dropped a power of 2 somewhere.

b) Using an example random walk, we can plot the power spectrum and compare it to an $\frac{a}{k^2}$ function:



And zoomed-in on the interesting part:



We can see that the fit is pretty good.