Exercise 5

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May 30, 2012

5.1.1

$$
\begin{aligned}
\boldsymbol{x} &= (x_1, x_2)^T \\
k(\boldsymbol{x}, \boldsymbol{z}) &= (\boldsymbol{x}^T \boldsymbol{z})^3 \\
&= x_1^3 z_1^3 + 3x_1^3 z_1^2 x_2 z_2 + 3x_1 z_1 x_2^2 z_2^2 + x_2^3 z_2^3 \\
\Rightarrow \boldsymbol{\Phi}(\boldsymbol{x}) &= (x_1^3, \sqrt{3} x_1^2 x_2, \sqrt{3} x_1 x_2^2, x_2^3)^T \\
\Phi : & \quad \mathbb{R}^2 \mapsto \mathbb{R}^4
\end{aligned}
$$

5.1.2

$$
\begin{aligned}
k(\boldsymbol{x}, \boldsymbol{z}) &= c * k_1(\boldsymbol{x}, \boldsymbol{z}) && c = const && (1) \\
k(\boldsymbol{x}, \boldsymbol{z}) &= \exp\left(k_1(\boldsymbol{x}, \boldsymbol{z})\right) && && (2) \\
k(\boldsymbol{x}, \boldsymbol{z}) &= k_1(\boldsymbol{x}, \boldsymbol{z}) + k_2(\boldsymbol{x}, \boldsymbol{z}) && && (3) \\
k(\boldsymbol{x}, \boldsymbol{z}) &= k_1(\boldsymbol{x}, \boldsymbol{z}) \cdot k_2(\boldsymbol{x}, \boldsymbol{z}) && && (4) \\
& && && (5)
\end{aligned}
$$

$$
\begin{aligned}
k(\boldsymbol{x}, \boldsymbol{z}) &= \exp\left(-\frac{(\boldsymbol{x} - \boldsymbol{z})^2}{2\sigma^2}\right) && && (6) \\
&= \prod_{i=1}^{3} \exp\left(-\frac{k_i}{2\sigma^2}\right) && && (7) \\
k'(\boldsymbol{x}, \boldsymbol{z}) &= \boldsymbol{x}^T \boldsymbol{z} && = \boldsymbol{\Phi}^T \boldsymbol{\Phi} && (8) \\
k_1 &= k'(\boldsymbol{x}, \boldsymbol{x}) && && (9) \\
k_2 &= k'(\boldsymbol{z}, \boldsymbol{z}) && && (10) \\
k_3 &= k'(\boldsymbol{x}, \boldsymbol{z}) && && (11) \\
\Phi : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n) & && && (12) \\
& && && (13)
\end{aligned}
$$

k_i are valid kernels as k' can be expressed by a scalar product of a mapping of $\boldsymbol{x}$ and $\boldsymbol{z}$. According to equation 1 $\tilde{k}_i = -\frac{k_i}{2\sigma^2}$ are valid kernels as well with $c = -\frac{1}{2\sigma^2}$. From equation 2 we can see that $\exp\left(\tilde{k}_i\right)$ are valid kernels as well. Finally a product of valid kernels results in a valid kernel (equation 4).

Listing 1: Gaussian kernel response in Matlab

```
function k = gaussian_kernel(x, z, sigma)
% x are matrices of size pxn, containing n (number of samples) feature
% vectors of size p. z is a vector of size p. sigma is a scalar.

n = size(x, 2);
k = zeros(1, n);
SGM = 2*sigma^2;
z = z';

for i=1:n
    arg = x(:,i) - z;
    k(i) = exp(-(arg'*arg)/(SGM));
end
end
```

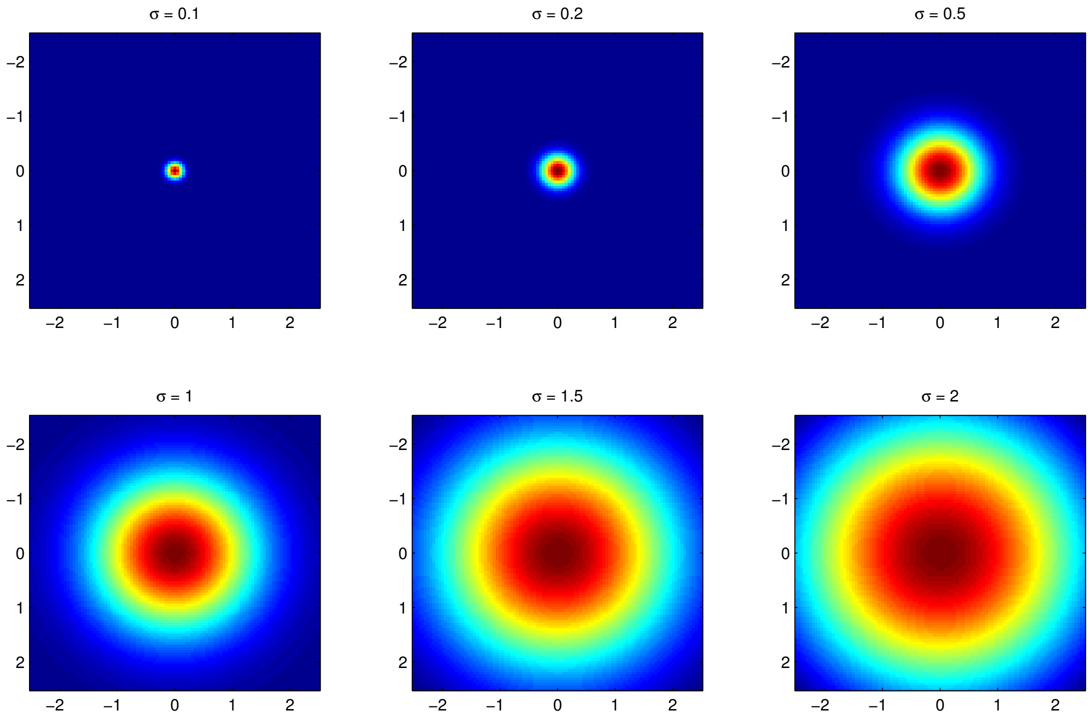

Figure 1: Responses for various values of σ (see plot titles).

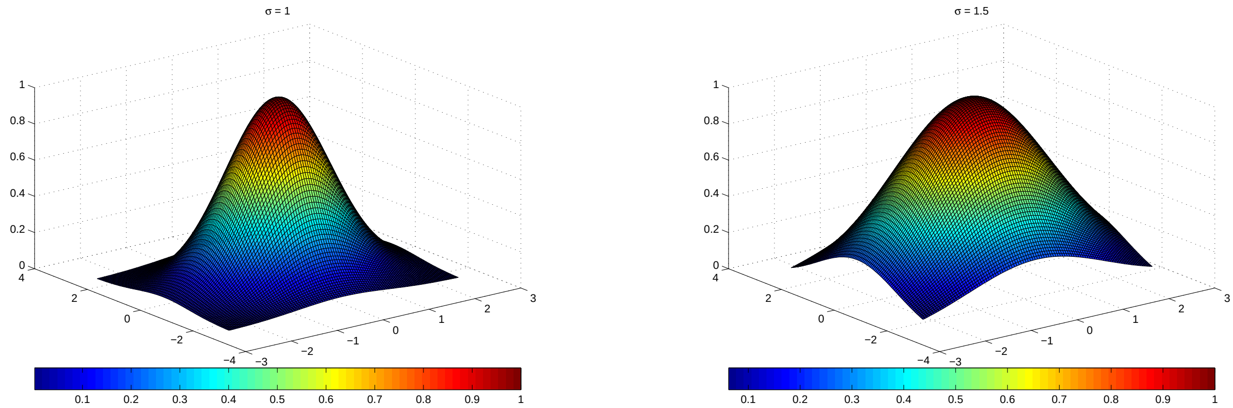

Figure 2: surface plots for $\sigma = 1$ and $\sigma = 1.5$