Homework 1: Implied Volatility for EUA Options

Background. You work on a bank's EU Emissions (CO_2) derivatives desk. The desk quotes implied volatility (IV) for European-style options written on EUA futures. The desk needs IV to compare strikes.

Data:

- Valuation date: **23 Jan 2026**
- Underlying: **EUA Dec-26 futures** (quoted in EUR per EUA/tonne CO_2)
- Futures last/settlement price: $F_0 = 88.44$ **EUR**
- Option expiry: **15 Dec 2026** (European exercise)
- Discounting: use a **flat continuously-compounded annual rate** $r = 1.933\%$ (approx. STR around this date).
- Contract size (for PnL interpretation): **1 contract = 1,000 EUAs**.
  A 1 EUR move in the futures price changes the futures PnL by **1,000 EUR per contract**.

Observed option premiums (puts, settlement).
All options below are **puts** on the EUA Dec-26 futures with expiry 15 Dec 2026. Premiums are settlement prices in EUR per EUA.

Strike K (EUR)	65	70	75	80	85
Put settlement premium P^{mkt} (EUR)	1.55	2.19	3.08	4.43	6.21

Model: Black–76 on futures.
Let F_0 be the futures price, K the strike, T time to expiry in years, r the (flat) continuously-compounded rate. Define

$$d_1 = \frac{\ln(F_0/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}. \tag{1}$$

The Black–76 put price is

$$P(\sigma) = e^{-rT}\left(K\,N(-d_2) - F_0\,N(-d_1)\right), \tag{2}$$

where $N(\cdot)$ is the standard normal CDF.

Task 1.

- Compute T using ACT/365 from 23 Jan 2026 to 15 Dec 2026.
- Compute the discount factor e^{-rT}.
- Implement the Black–76 put pricing function $P(\sigma)$.

- Verify that $P(\sigma)$ is increasing in σ.

Task 3: Implied volatility via root finding. For each strike, compute implied volatility $\hat{\sigma}$ that solves:

$$f(\sigma) = P(\sigma) - P^{mkt} = 0. \tag{3}$$

- Use a volatility bracket such as $\sigma \in [10^{-6}, 5]$.
- Define a reasonable stopping rule.
- Record the number of iterations per option.
- If an input violates no-arbitrage bounds and no solution exists, flag it and explain why.

Task 4: Volatility smile.

- Plot implied volatility (y-axis) against strike (x-axis).
- Briefly describe the shape (skew/smile) and why it might arise in CO_2 markets.

Task 5: Client quote for an off-chain strike (bank setting). A client asks to **buy** a put with strike $K = 55$ (same expiry and underlying).

- (a) Propose an implied volatility for $K = 55$ by **extrapolating** your implied-vol curve. Justify your choice.
- (b) Using that volatility, compute a **model premium** for the $K = 55$ put.
- (c) As the bank, you will **sell** the put to the client. Propose a **quote** (ask price). Use, for example, a simple spread assumption.

Task 6: Hedging discussion (short write-up). Assume the bank sells the $K = 55$ put to the client.

- Compute the put **delta** under Black–76 at your quoted volatility.
- Explain how to construct a **delta hedge** using the EUA Dec-26 futures, and why the hedge must be adjusted over time.
- Mention at least one additional risk (gamma/vega) and one practical way the desk could reduce that risk (e.g., trading another option).

Deliverables.

- Code (Julia or R) that: (i) computes implied vols for all strikes, (ii) produces the vol-smile plot, (iii) computes the $K = 55$ quote and hedge quantities.
- A short PDF write-up (max 1 page) summarizing your results and answers to Tasks 4–6.