1、用Fourier变换求解Cauchy问题 $\begin{cases} u_t - t u_{xx} = 0 & (1) \\ t=0: \ u = \cos x = \varphi(x) & (2) \end{cases}$

解：① $F[u(x,t)] = \tilde{u}(\lambda,t)$　　(1) $\frac{d\tilde{u}(\lambda,t)}{dt} + t\lambda^2 \tilde{u}(\lambda,t) = 0$　　$\frac{d\tilde{u}(\lambda,t)}{dt} = -\lambda^2 t \tilde{u}(\lambda,t)$　　$\frac{d\tilde{u}(\lambda,t)}{\tilde{u}(\lambda,t)} = -\lambda^2 t\,dt$　　$\ln|\tilde{u}(\lambda,t)| = -\frac{\lambda^2}{2}t^2 + C_1$

$F[\varphi(x)] = \tilde{\varphi}(\lambda)$　　(2) $\tilde{u}(\lambda,0) = \tilde{\varphi}(\lambda)$　　$\tilde{u}(\lambda,t) = \pm e^{-\frac{\lambda^2}{2}t^2 + C_1} = C_2 e^{-\frac{\lambda^2}{2}t^2}$　　$\tilde{u}(\lambda,0) = C_2 = \tilde{\varphi}(\lambda)$

② $F[f'(x)] = i\lambda F[f(x)]$　　$\tilde{u}(\lambda,t) = \tilde{\varphi}(\lambda) e^{-\frac{\lambda^2}{2}t^2} = g_1(\lambda) g_2(\lambda) = F[f_1]F[f_2]$

$F[\tilde{u}(\lambda,t)_{tt}] = i\lambda F[\tilde{u}(\lambda,t)_t] = i\lambda i\lambda F[\tilde{u}(\lambda,t)] = -\lambda^2 F[\tilde{u}(\lambda,t)]$

③ $F[f] \Rightarrow g(\lambda) = \int_{-\infty}^{+\infty} f(\xi) e^{-i\lambda\xi} d\xi$　　$F^{-1}[e^{-\frac{\lambda^2}{2}t^2}] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{\lambda^2}{2}t^2} e^{i\lambda x} d\lambda = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}\lambda^2 + ix\lambda} d\lambda$

$F^{-1}[g] \Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\lambda) e^{i\lambda x} d\lambda$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2t^2}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}(\lambda - \frac{ix}{t^2})^2} d(\lambda - \frac{ix}{t^2})$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2t^2}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}\lambda_1^2} d\lambda_1 = \frac{1}{2\pi} e^{-\frac{x^2}{2t^2}} \underbrace{\int_{-\infty}^{+\infty} e^{-(\frac{t}{\sqrt{2}}\lambda_1)^2} d(\frac{t}{\sqrt{2}}\lambda_1)}_{\sqrt{\pi}} \frac{\sqrt{2}}{t}$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2t^2}} \frac{\sqrt{2\pi}}{t} = \frac{1}{t\sqrt{2\pi}} e^{-\frac{x^2}{2t^2}}$$

$$u(x,t) = \int_{-\infty}^{+\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{2t^2}} d\xi \frac{1}{t\sqrt{2\pi}} = \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos\xi \, e^{-\frac{(x-\xi)^2}{2t^2}} d\xi$$

$$= \frac{1}{t\sqrt{2\pi}} t\sqrt{2\pi} e^{-\frac{t^2}{2}} \cos x = e^{-\frac{t^2}{2}} \cos x$$

2. 用Fourier变换求解Cauchy问题 $\begin{cases} u_{tt} - u_{xx} = 0 & (1) \\ t=0: \ u = \overset{(2)}{\cos x} = \varphi(x), \ u_t = \overset{(3)}{\sin x} = \psi(x) \end{cases}$. 注意：计算需要给出必要的步骤，不要直接套公式.

解：① $F[u(x,t)] = \tilde{u}(\lambda,t)$　　(1) $\frac{d^2\tilde{u}(\lambda,t)}{dt^2} + \lambda^2 \tilde{u}(\lambda,t) = 0$　　$\tilde{u}(\lambda,t) = A(\lambda)e^{i\lambda t} + B(\lambda)e^{-i\lambda t}$　　$\tilde{u}_t(\lambda,t) = i\lambda A(\lambda)e^{i\lambda t} - i\lambda B(\lambda) e^{-i\lambda t}$

$F[\varphi(x)] = \tilde{\varphi}(\lambda)$　　(2) $\tilde{u}(\lambda,0) \cancel{= \cos x} = \tilde{\varphi}(\lambda)$　　$A(\lambda) = \frac{1}{2}[\tilde{\varphi}(\lambda) + \frac{1}{i\lambda}\tilde{\psi}(\lambda)]$

$F[\psi(x)] = \tilde{\psi}(\lambda)$　　(3) $\tilde{u}_t(\lambda,0) \cancel{= \sin x} = \tilde{\psi}(\lambda)$　　$B(\lambda) = \frac{1}{2}[\tilde{\varphi}(\lambda) - \frac{1}{i\lambda}\tilde{\psi}(\lambda)]$

② $F[f'(x)] = i\lambda F[f(x)]$　　$\tilde{u}(\lambda,t) = \frac{1}{2}[\tilde{\varphi}(\lambda)e^{i\lambda t} + \tilde{\varphi}(\lambda)e^{-i\lambda t}] + \frac{1}{2i\lambda}[\tilde{\psi}(\lambda)e^{i\lambda t} - \tilde{\psi}(\lambda)e^{-i\lambda t}]$

$F[\tilde{u}(\lambda,t)_{tt}] = i\lambda F[\tilde{u}(\lambda,t)_t] = i\lambda i\lambda F[\tilde{u}(\lambda,t)] = -\lambda^2 F[\tilde{u}(\lambda,t)]$

③ $F[f] \Rightarrow g(\lambda) = \int_{-\infty}^{+\infty} f(\xi) e^{-i\lambda\xi} d\xi$　　$u(x,t) = \int_{-\infty}^{+\infty} \frac{1}{2}\varphi(\xi) F^{-1}[e^{i\lambda t}] d\xi + \int_{-\infty}^{+\infty} \frac{1}{2}\varphi(\xi) F^{-1}[e^{-i\lambda t}] d\xi$

$F^{-1}[g] \Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\lambda) e^{i\lambda x} d\lambda$　　$+ \int_{-\infty}^{+\infty} \frac{1}{2}\psi(\xi) F^{-1}[\frac{e^{i\lambda t}}{i\lambda} - \frac{e^{-i\lambda t}}{i\lambda}] d\xi$

$$\int_{-\infty}^{+\infty} \frac{1}{2}\varphi(\xi) F^{-1}[e^{i\lambda t}] d\xi = \frac{1}{2}\varphi(x+t)$$

$$\int_{-\infty}^{+\infty} \frac{1}{2}\varphi(\xi) F^{-1}[e^{-i\lambda t}] d\xi = \frac{1}{2}\varphi(x-t)$$

$$\int_{-\infty}^{+\infty} \frac{1}{2}\psi(\xi) F^{-1}[\frac{e^{i\lambda t}}{i\lambda} - \frac{e^{-i\lambda t}}{i\lambda}] d\xi = \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi) d\xi$$

$$u(x,t) = \frac{1}{2}[\varphi(x+t) + \varphi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi) d\xi = \frac{1}{2}[\cos(x+t) + \cos(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \sin\xi \, d\xi$$

$$= \frac{\cos(x+t)}{2} + \frac{\cos(x-t)}{2} - \frac{\cos(x+t)}{2} + \frac{\cos(x-t)}{2} = \cos(x-t)$$

3、利用Laplace变换求解 $\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u_x(t,0)=0,\ u(t,l)=1,\ t>0,\ 0<x<l, \\ u(0,x) = 1 + \cos\dfrac{3\pi x}{2l}. \end{cases}$

解：设 $v(x,t) = A(t)x + B(t)$，代入非齐次边界条件 $v_x(t,0) = A(t) = 0$

$v(t,l) = A(t)l + B(t) = 1$

得 $v(x,t) = 1$

再令 $u(x,t) = w(x,t) + 1$

则 $w(x,t)$ 是定解问题 $\begin{cases} \dfrac{\partial w}{\partial t} = a^2 \dfrac{\partial^2 w}{\partial x^2}, \text{ 的解} \\ w_x(t,0)=0,\ w(t,l)=0,\ t>0,\ 0<x<l, \\ w(0,x) = \cos\dfrac{3\pi x}{2l}. \end{cases}$ $\tilde{w}(0,x) =$

$L[w(x,t)] = \tilde{w}(x,p)$

$e^{0x}(1\cos\frac{3\pi}{2l}x + 0\sin\frac{3\pi}{2l}x)$

$p\tilde{w}(x,p) = a^2 \dfrac{d^2\tilde{w}(x,p)}{dx^2} + \cos\dfrac{3\pi x}{2l}$，令 $\tilde{w}(x,p) = y$，则 $py = a^2y'' + \cos\frac{3\pi}{2l}x$，则特征方程 $py = a^2y''$

则通解 $\tilde{y} = Ce^{\frac{\sqrt{p}}{a}x} + De^{-\frac{\sqrt{p}}{a}x}$，特解 $\overset{*}{y} = (qx+b)\cos\frac{3\pi}{2l}x + (rx+d)\sin\frac{3\pi}{2l}x$

$\tilde{w}_x(t,0) = 0$，$\tilde{w}(t,l) = 0$

$p(qx+b)\cos\frac{3\pi}{2l}x + p(rx+d)\sin\frac{3\pi}{2l}x = a^2y'' + \cos\frac{3\pi}{2l}x$

$C=0$，$D=0$

$$\overset{*}{y} = \frac{p + a^2\frac{9\pi^2}{4l^2}}{(p + a^2\frac{9\pi^2}{4l^2})^2 + \frac{9\pi^2}{4l^2}}$$

$$\tilde{w}(x,p) = y = \tilde{y} + \overset{*}{y} = \overset{*}{y} = \frac{p + a^2\frac{9\pi^2}{4l^2}}{(p + a^2\frac{9\pi^2}{4l^2})^2 + \frac{9\pi^2}{4l^2}}$$

$$w(x,t) = e^{-a^2\frac{9\pi^2}{4l^2}t}\cos\frac{3\pi}{2l}x$$

$$u(x,t) = w(x,t) + 1 = 1 + e^{-\frac{9\pi^2}{4l^2}a^2t}\cos\frac{3\pi}{2l}x$$