1、利用传播波法，求解波动方程的古尔萨（Goursat）问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u|_{x-at=0} = \varphi(x) \\ u|_{x+at=0} = \psi(x)\ ,\ \varphi(0) = \psi(0) \end{cases}$$

解：设 $u(x,t) = F(x-at) + G(x+at)$

令 $x-at=0$，则 $\varphi(x) = F(0) + G(x+at)$，而 $x+at = x+x = 2x$，

则 $\varphi(x) = F(0) + G(2x)$ ①

令 $x+at=0$，则 $\psi(x) = G(0) + F(x-at)$，而 $x-at = x+x = 2x$，

则 $\psi(x) = G(0) + F(2x)$ ②

由①②得 $\begin{cases} F(2x) = \psi(x) - G(0) \\ G(2x) = \varphi(x) - F(0) \end{cases}$，则 $\begin{cases} F(x) = \psi(\frac{x}{2}) - G(0) \\ G(x) = \varphi(\frac{x}{2}) - F(0) \end{cases}$，则 $\begin{cases} F(0) = \psi(0) - G(0) \\ G(0) = \varphi(0) - F(0) \end{cases}$

则 $u(x,t) = F(x-at) + G(x+at)$，而 $F(0) + G(0) = \varphi(0) = \psi(0)$

$= \psi(\frac{x-at}{2}) - G(0) + \varphi(\frac{x+at}{2}) - F(0)$

$= \psi(\frac{x-at}{2}) + \varphi(\frac{x+at}{2}) - \psi(0)$

2、求解下列半无界弦自由振动问题

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad (0<x<\infty,\ t>0), \\ u|_{t=0} = 0,\quad u_t|_{t=0} = 0, \\ u|_{x=0} = \sin 2t. \end{cases}$$

解：延拓：

$$\begin{cases} U_{tt} - U_{xx} = 0 \quad (-\infty<x<+\infty,\ t>0), \\ U|_{t=0} = U(x,0) = \Phi(x) = \begin{cases} 0 & (x \geq 0), \text{①} \\ \varphi(x) & (x<0), \text{②} \end{cases} \\ U_t|_{t=0} = U_t(x,0) = \psi(x) = 0 \quad (-\infty<x<+\infty), \\ U(x,t) = \dfrac{\Phi(x-at) + \Phi(x+at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(y)\,dy\ ,\ 而\ a=1, \end{cases}$$

则 $U(x,t) = \dfrac{\Phi(x-t) + \Phi(x+t)}{2} \quad (-\infty<x<+\infty,\ t>0)$

当 $x-t>0$ 时，即当 $x>t$ 时，$u(x,t) = U(x,t) = \dfrac{\Phi(x-t)+\Phi(x+t)}{2} = 0$（由①得）

当 $x-t\leq 0$ 时，即当 $x\leq t$ 时，$u(x,t) = U(x,t) = \dfrac{\Phi(x-t)+\Phi(x+t)}{2} = \dfrac{\varphi(x-t)}{2}$（由①②得）

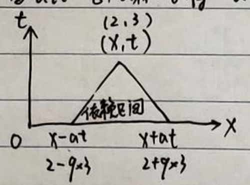
$U(0,t) = \dfrac{\varphi(-t)}{2} = u(0,t) = u|_{x=0} = \sin 2t$

$\varphi(-t) = 2\sin 2t$

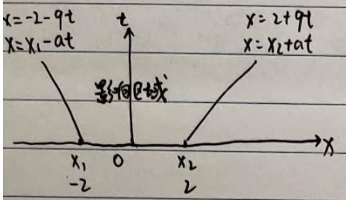
$\varphi(t) = -2\sin 2t$，则 $u(x,t) = \begin{cases} 0 & (x>t), \\ \dfrac{-2\sin 2(x-t)}{2} & (x\leq t), \end{cases}$ 则 $u(x,t) = \begin{cases} 0 & (t<x<\infty,\ t>0), \\ -\sin 2(x-t) & (0<x\leq t,\ t>0) \end{cases}$

3、写出方程 $u_{tt}-81u_{xx}=0$ 点 $(x_0,t_0)=(2,3)$ 的依赖区间和 x 轴上区间 $[-2,2]$ 的影响区域。

解：由 $u_{tt}-81u_{xx}=0$ 得 $a=9$



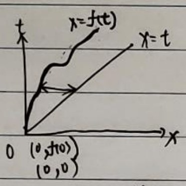
$\Rightarrow$ 点 $(x_0,t_0)=(2,3)$ 的依赖区间：$[-25,29]$



$\Rightarrow$ x 轴上区间 $[-2,2]$ 的影响区域：$\begin{cases} -2-9t\le x\le 2+9t \\ t>0 \end{cases}$

4. 求方程 $\begin{cases} u_{tt}-u_{xx}=0, & f(t)<x<t, \\ u|_{x=t}=\varphi(t), \\ u|_{x=f(t)}=\psi(t) \end{cases}$ 的解，其中 $\varphi(0)=\psi(0)$，$x=f(t)$ 为由原点出发的，介于 $x=t$ 与 $x=-t$ 之间的光滑曲线（即一阶连续可导），且 $|f'(t)|\ne 1$ 对一切 t 成立。

解：



设 $u(x,t)=F(x-t)+G(x+t)$

令 $x=t$，则 $u(t,t)=F(t-t)+G(t+t)=F(0)+G(2t)=\varphi(t)$，

则 $\varphi(t)=F(0)+G(2t)$ ①

令 $x=f(t)$，则 $u(f(t),t)=F(f(t)-t)+G(f(t)+t)=\psi(t)$，

令 $\overset{H(t)}{\cancel{H(t)}}=f(t)-t=m$，则 $t=H^{-1}(m)$ 则 $\psi(t)=F(f(t)-t)+G(f(t)+t)$ ②

由①②得 $\begin{cases} F(f(t)-t)=\psi(t)-G(f(t)+t) \\ G(2t)=\varphi(t)-F(0) \end{cases}$，则 $\begin{cases} F(m)=\psi[H^{-1}(m)]-G[m+2H^{-1}(m)] \quad \cancel{F(t)=\psi(f(t)-t)-G(t)} \\ G(t)=\varphi(\frac{t}{2})-F(0) \end{cases}$，则 $\begin{cases} F(0)=\psi(0)-G(0) \\ G(0)=\varphi(0)-F(0) \end{cases}$

则 $u(x,t)=F(x-t)+G(x+t)$，即 $F(0)+G(0)=\varphi(0)=\psi(0)$

$=\psi[\overset{H^{-1}(x-t)}{\cancel{f(x-t)-(x-t)}}]-G[\overset{(x-t)+2H^{-1}(x-t)}{\cancel{(x-t)}}]+\varphi(\frac{x+t}{2})-F(0)=\psi[H^{-1}(x-t)]-\varphi[\frac{x-t}{2}+H^{-1}(x-t)]+F(0)+\varphi(\frac{x+t}{2})-F(0)$

$\cancel{=\psi[\overset{H^{-1}(x-t)}{f(x-t)-(x-t)}]-\varphi(\frac{t-x}{2})+F(0)+\varphi(\frac{x+t}{2})-F(0)}$

$\cancel{=\psi[f(x-t)-(x-t)]+\varphi(\frac{x+t}{2})-\varphi(\frac{t-x}{2})}$

则 $u(x,t)=\psi[H^{-1}(x-t)]-\varphi[\frac{x-t}{2}+H^{-1}(x-t)]+\varphi(\frac{x+t}{2})$，其中 $H(x-t)=f(x-t)-(x-t)$

且 $H^{-1}(x-t)$ 为 $H(x-t)$ 的反函数