1、证明下列定解问题解唯一

$$\begin{cases} u_{xx}+u_{yy}=x^2-y^2, & x^2+y^2<1, \\ u|_{x^2+y^2=1}=0. \end{cases} \Longrightarrow \text{相应的齐次问题只有零解} \begin{cases} u_{xx}+u_{yy}=0, & x^2+y^2<1, \\ u|_{x^2+y^2=1}=0. \end{cases}$$

解：设 $\Omega\subset R^n$ 为有界区域，边界 $\partial\Omega=\Gamma$ 分块光滑，$\bar{\Omega}=\Omega\cup\Gamma$

方程两端同乘以 u，然后积分

$$\int_\Omega u(u_{xx}+u_{yy})dx=0$$

根据 Green 第一公式，如果 $u\in C^2(\Omega)\cap C^1(\bar{\Omega})$，那么

$$\begin{aligned}\int_\Omega u(u_{xx}+u_{yy})dx &= \int_\Omega u\Delta u\,dx \\ &= \int_\Gamma (\nabla\cdot(u\nabla u)-\nabla u\cdot\nabla u)dx \\ &= \int_\Gamma u\frac{\partial u}{\partial n}ds-\int_\Omega \nabla u\cdot\nabla u\,dx\end{aligned}$$

$$\int_\Omega \nabla u\cdot\nabla u\,dx=\int_\Omega|\nabla u|^2dx=0$$

$u\equiv const$

又因为 $u|_{x^2+y^2=1}=0$，那么 $u\equiv 0$，只有零解

则该定解问题解唯一.

2. 求变分问题 u 满足的边值问题：

$$J[u(x,y)]=\int_{x^2+y^2\le 1}[|\nabla u|^2-2xyu]dxdy,\quad u|_{x^2+y^2=1}=xy.$$

解：$(V)\to(D)$：令 $u=\min v$

$$\frac{d}{d\varepsilon}J(u+\varepsilon v)|_{\varepsilon=0}=0$$

$$J(u+\varepsilon v)=\int_\Omega\left(\frac{|\nabla u|^2}{2}+\varepsilon\nabla u\cdot\nabla v+\frac{\varepsilon^2|\nabla v|^2}{2}-fu-\varepsilon fv\right)dx$$

$$\begin{aligned}\frac{d}{d\varepsilon}J(u+\varepsilon v)|_{\varepsilon=0}&=\int_\Omega(\nabla u\cdot\nabla v-\varepsilon fv)dx \\ &=-\int_\Omega v(\Delta u+f)dx\end{aligned}$$

$$J(u)=\min_{v\in D(J)}J(v)$$

$$D(J)=\{v\in C^2(\Omega)\cap C^1(\bar{\Omega}):v|_\Gamma=0\}$$

$$\Rightarrow J(v)=\int_\Omega\left(\frac{1}{2}|\nabla v|^2-fv\right)dx$$

$$\therefore 2J(v)=\int_\Omega[|\nabla v|^2-2fv]dx=\int_{x^2+y^2\le1}[|\nabla v|^2-2fv]dxdy,\quad f=xy$$

$\therefore$ 变分问题 u 满足的边值问题：

$$(D)\begin{cases}-u_{xx}-u_{yy}=xy, & x^2+y^2<1, \\ u|_{x^2+y^2=1}=0\end{cases}\Rightarrow(D)\begin{cases}u_{xx}+u_{yy}=-xy, & x^2+y^2<1, \\ u|_{x^2+y^2=1}=0\end{cases}$$

3. 设 $J[v]=\iiint_\Omega \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]dxdydz+\iint_\Gamma\left[\frac{1}{2}\sigma v^2-gv\right]ds$，考察变分问题：求 $u\in V=\{v\in C^2(\Omega)\cap C^1(\bar\Omega)\}$

使得 $J[u]=\min\limits_{v\in V}J[v]$. 试推导 u 满足的边值问题.

解：$\because J[v]=\iiint_\Omega \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]dxdydz+\iint_\Gamma\left[\frac{1}{2}\sigma v^2-gv\right]ds$ $\qquad \nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}$

$=\iiint_\Omega\frac{1}{2}\left[v_x^2+v_y^2+v_z^2\right]dxdydz+\iint_\Gamma\left[\frac{1}{2}\sigma v^2-gv\right]ds$

对于 $\iiint_\Omega\frac{1}{2}\left[v_x^2+v_y^2+v_z^2\right]dxdydz=\iiint_\Omega\frac{1}{2}\left[(v_x\cdot v_y\cdot v_z)^2\cancel{-2v_xv_y-2v_xv_z-2v_yv_z}\right]dV$

$=\iiint_\Omega\frac{1}{2}\left[|\nabla v|^2\cancel{-2v_xv_y-2v_xv_z-2v_yv_z}\right]dV$

$\therefore J[v]=\int_\Omega F(x,y,z,v(x,y,z),v_x(x,y,z),v_y(x,y,z),v_z(x,y,z))\,dxdydz$

$\therefore$ $\cancel{f=g \text{ 且 } \sigma u^2=(u_x+u_y+u_z)^2}$ $\quad D(J)=\{v\in C^2(\Omega):v|_\Omega=f\}$

$\therefore$ $\cancel{\sqrt{\sigma}u=u_x+u_y+u_z}$ 根据 Green 第二公式，如果 $u,v\in C^2(\Omega)\cap C^1(\bar\Omega)$，即有

$\therefore \int_\Omega(u\Delta v-v\Delta u)dx=\int_\Gamma\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds$

$\therefore J(v)=\int_\Omega\left(\frac{1}{2}|\nabla v|^2-fv\right)dx$

$=\int_\Omega\left(\frac{1}{2}|\nabla u|^2-\overset{xyz}{(6+g)}v\right)dx$

$\therefore$ u 满足的边值问题：

$$(D)\begin{cases}-u_{xx}-u_{yy}-u_{zz}=(6+g)xyz\\ u|_\Gamma=0\end{cases}\Rightarrow (D)\begin{cases}u_{xx}+u_{yy}+u_{zz}=-(6+g)xyz\\ u|_\Gamma=0\end{cases}$$