1. 求解 $\begin{cases} u_{tt} = a^2(u_{xx}+u_{yy}+u_{zz}) \\ u|_{t=0} = x^3+y^2z, \quad u_t|_{t=0} = \sin x. \end{cases}$

解: $u = u_1 + u_2$

$$\begin{cases} u^1_{tt} = a^2 u^1_{xx} \\ u^1|_{t=0} = x^3, \quad u^1_t|_{t=0} = \sin x \end{cases} \xRightarrow{\text{达朗贝尔公式}} u^1(x,t) = \frac{(x-at)^3+(x+at)^3}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \sin\xi \, d\xi$$

$$u^1(x,t) = \frac{(x-at)^3+(x+at)^3}{2} + \frac{\cos(x-at)-\cos(x+at)}{2a}$$

$$\begin{cases} u^2_{tt} = a^2(u^2_{yy}+u^2_{zz}) \\ u^2|_{t=0} = y^2z, \quad u^2_t|_{t=0} = 0 \end{cases} \xRightarrow{\text{二维泊松公式}} u^2(y,z,t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at}\int_0^{2\pi} \frac{(y+r\cos\theta)^2(z+r\sin\theta)}{\sqrt{(at)^2-r^2}} r\,d\theta\,dr\right]$$

$$u^2(y,z,t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at} \frac{2\pi(y^2z+\frac{z}{2}r^2)}{\sqrt{(at)^2-r^2}} r\,dr\right]$$

$$u^2(y,z,t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[a y^2 z t \; \frac{z}{6}(2+a^2t^2)\sqrt{1-a^2t^2} \; \frac{2\pi z}{3}\right]$$

$$u^2(y,z,t) = y^2z + \frac{z}{2}\frac{a^3t^3}{\sqrt{1-a^2t^2}}$$

$u = u_1 + u_2$

$u(x,y,z,t) = u^1(x,t) + u^2(y,z,t)$

$$u(x,y,z,t) = \frac{(x-at)^3+(x+at)^3}{2} + \frac{\cos(x-at)-\cos(x+at)}{2a} + y^2z + \frac{z}{2}\frac{a^3t^3}{\sqrt{1-a^2t^2}}$$

、写出方程 $u_{tt} = 16\Delta_3 u$ 点 $(x_1,x_2,x_3,t) = (1,1,1,2)$ 的依赖区域。

解: 锥面 $K = \{(x,y,z,t) \mid (x-1)^2+(y-1)^2+(z-1)^2 = 16(t-2)^2\} \; (0 \le t \le 2)$

、求解 $\begin{cases} u_{tt} - \Delta_3 u = 0 \quad (t>0) \\ t=0: u = \sin(x+y+z), \; u_t = 0 \end{cases}$

解: $u(x,y,z,t) = \cancel{\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at}\int_0^{2\pi}\right.} \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin(x+at\sin\theta\cos\varphi+y+at\sin\theta\sin\varphi+z+at\cos\theta)\sin\theta\,d\theta\,d\varphi\right)$

$\cancel{u_{tt}-3u_{mm}=0 \; (t>0)}$

$\cancel{t=0: u=\sin m, \; u_t=0}$

$$u(x,y,z,t) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin(x+y+z+t\sin\theta\cos\varphi+t\sin\theta\sin\varphi+t\cos\theta)\sin\theta\,d\theta\,d\varphi\right)$$

$$= \sin(x+y+z)\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\cos(t\sin\theta\cos\varphi+t\sin\theta\sin\varphi+t\cos\theta)\sin\theta\,d\theta\,d\varphi\right) +$$

$$\cos(x+y+z)\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin(t\sin\theta\cos\varphi+t\sin\theta\sin\varphi+t\cos\theta)\sin\theta\,d\theta\,d\varphi\right) \quad (\times)$$

$$x+y+z = m \Rightarrow \Delta_3 u = u_{xx}+u_{yy}+u_{zz} = \frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z} = \frac{\partial \frac{\partial u}{\partial m}\frac{\partial m}{\partial x}}{\partial x} + \frac{\partial \frac{\partial u}{\partial m}\frac{\partial m}{\partial y}}{\partial y} + \frac{\partial \frac{\partial u}{\partial m}\frac{\partial m}{\partial z}}{\partial z} = 3u_{mm}$$

$$\begin{cases} u_{tt} - 3u_{mm} = 0 \quad (t>0) \\ t=0: u = \sin m, \; u_t = 0 \end{cases} \xRightarrow{\text{达朗贝尔公式}} u(m,t) = \frac{\sin(m-\sqrt{3}t)+\sin(m+\sqrt{3}t)}{2}$$

$$u(x+y+z,t) = \frac{\sin(x+y+z-\sqrt{3}t)+\sin(x+y+z+\sqrt{3}t)}{2}$$

$$u(x,y,z,t) = \sin(x+y+z) - \frac{3}{2}t^2\sin(x+y+z) = \sin(x+y+z)\left(1-\frac{3}{2}t^2\right)$$

4. 求解 $\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz} + 2(y-t) \\ u|_{t=0} = 0,\ u_t|_{t=0} = x^2 + yz \end{cases}$

解：$u = u_1 + u_2 + u_3$

$$\begin{cases} u^1_{tt} = u^1_{xx} \\ u^1|_{t=0} = 0,\ u^1_t|_{t=0} = x^2 \end{cases} \xRightarrow{\text{达朗贝尔公式}} u^1(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} \xi^2 d\xi = \frac{1}{2}\int_{x-t}^{x+t} \xi^2 d\xi = \frac{1}{6}\xi^3\Big|_{x-t}^{x+t}$$

$$= \frac{(x+t)^3-(x-t)^3}{6} = x^2 t + \frac{t^3}{3}$$

$$\begin{cases} u^2_{tt} = u^2_{zz} \\ u^2|_{t=0} = 0,\ u^2_t|_{t=0} = yz \end{cases} \xRightarrow{\text{达朗贝尔公式}} u^2(z,t) = \frac{1}{2a}\int_{z-at}^{z+at} y\xi\, d\xi = \frac{y}{2}\int_{z-t}^{z+t} \xi\, d\xi = \frac{y}{4}\xi^2\Big|_{z-t}^{z+t}$$

$$= \frac{y}{4}\left[(z+t)^2-(z-t)^2\right] = yzt$$

$$\begin{cases} u^3_{tt} = u^3_{yy} + 2(y-t) \\ u^3|_{t=0} = 0,\ u^3_t|_{t=0} = 0 \end{cases}$$

$\Downarrow$ 齐次化原理

$$\begin{cases} u^3_{tt} = u^3_{yy} \\ u^3|_{t=\tau} = 0,\ u^3_t|_{t=\tau} = 2(y-\tau) \end{cases}$$

$$u^3(y,t) = \frac{1}{2a}\int_{y-a(t-\tau)}^{y+a(t-\tau)} 2(\xi-\tau)\, d\xi = \int_{y-(t-\tau)}^{y+(t-\tau)} (\xi-\tau)\, d\xi = \frac{(y+t-\tau)^2-(y-t+\tau)^2}{2} + 2\tau^2 - 2t\tau$$

$$= 2yt - 2y\tau + 2\tau^2 - 2t\tau$$

$$u^3(y,t) = \int_0^t (2yt - 2y\tau + 2\tau^2 - 2t\tau)\, d\tau = 2yt^2 - yt^2 + \frac{2}{3}t^3 - t^3 = yt^2 - \frac{1}{3}t^3$$

$$u(x,y,z,t) = u_1 + u_2 + u_3$$

$$= x^2 t + \frac{t^3}{3} + yzt + yt^2 - \frac{1}{3}t^3$$

$$= x^2 t + yzt + yt^2$$