1. 求解下列定解问题

$$\begin{cases} u_{xx} - u_{yy} + \sin x = 0, \quad y>0, -\infty < x < +\infty \\ u(x,0) = 0, \quad u_y(x,0) = 4x \end{cases}$$ (无穷长弦的强迫振动)

解：改写 $$\begin{cases} u_{yy} - u_{xx} = \sin x, \quad y>0, -\infty<x<+\infty \\ u|_{y=0} = 0, \quad u_y|_{y=0} = 4x \end{cases}$$

非齐次泛定方程解决方案 $u(x,y) = v(x,y) + w(x,y)$ （自由振动＋受迫振动）

$$\begin{cases} v_{yy} - \overset{a=1}{v_{xx}} = 0 & ① \\ v|_{y=0} = \underset{\varphi(x)}{0}, \quad v_y|_{y=0} = \underset{\psi(x)}{4x} & ② \end{cases}$$

①② 组成自由振动 ⇒ 达朗贝尔公式

$$v(x,y) = \frac{1}{2}[\varphi(x-ay) + \varphi(x+ay)] + \frac{1}{2a}\int_{x-ay}^{x+ay} \psi(\xi)d\xi$$
$$= \frac{1}{2}\int_{x-y}^{x+y} 4\xi d\xi = \int_{x-y}^{x+y} 2\xi d\xi = \xi^2 \Big|_{x-y}^{x+y} = (x+y)^2 - (x-y)^2 = 4xy$$

$$\begin{cases} w_{yy} - w_{xx} = \overset{f(x,\tau)}{\sin x} & ③ \\ w|_{y=0} = 0, \quad w_y|_{y=0} = 0 & ④ \end{cases}$$

③④ 组成受迫振动 ⇒ 冲量定理 $$\begin{cases} \bar{w}_{yy} - \bar{w}_{xx} = 0 \\ \bar{w}|_{y=\tau} = 0, \quad \bar{w}_y|_{y=\tau} = \overset{f(x,\tau)}{\sin x} \end{cases}$$ 其中 $\bar{w} = \frac{\partial w}{\partial \tau} =$

令 $y' = y - \tau$，则 $$\begin{cases} \bar{w}_{1yy} - \bar{w}_{1xx} = 0 \\ \bar{w}_1|_{y=0} = 0, \quad \bar{w}_{1y}|_{y=0} = \overset{f(x,\tau)}{\sin x} \end{cases}$$ ⇒ 达朗贝尔公式 $\bar{w}_1(x, \overset{\tau}{y'}) = \frac{1}{2}\int_{x-y'}^{x+y'} \sin\xi d\xi =$

$$\bar{w}(x,\overset{\tau}{y}) = \frac{1}{2}\int_{x-(y-\tau)}^{x+(y-\tau)} \sin\xi d\xi = \frac{1}{2}(-\cos\xi)\Big|_{x-(y-\tau)}^{x+(y-\tau)} = -\frac{1}{2}[\cos(x+y-\tau) - \cos(x-y+\tau)]$$

$$w(x,y) = \int_0^y \bar{w}(x,\tau)d\tau = \int_0^y -\frac{1}{2}[\cos(x+y-\tau) - \cos(x-y+\tau)]d\tau = \frac{1}{2}\int_0^y [\cos(x-y+\tau) - \cos(x+y-\tau)]d\tau$$
$$= \frac{1}{2}\{\sin(x-y+\tau) + \sin(x+y-\tau)\}\Big|_0^y$$
$$= \frac{1}{2}\{\sin x + \sin x - \sin(x-y) - \sin(x+y)\}$$
$$= \sin x - \frac{\sin(x-y) + \sin(x+y)}{2}$$

$$u(x,y) = v(x,y) + w(x,y)$$
$$= 4xy + \sin x - \frac{\sin(x-y) + \sin(x+y)}{2}$$

2、证明下面定解问题解的唯一性：

$$\begin{cases} u_{tt}-a^2u_{xx}+xu_x=f(x,t), & (0\le x\le l,\ t>0) \\ \overset{(\alpha_1 u_x+\beta_1 u)}{(u_x-\sigma_1 u)}\big|_{x=0}=\lambda(t),\quad \overset{(\alpha_2 u_x+\beta_2 u)}{u}\big|_{x=l}=\mu(t), & \sigma_1>0 \\ t=0:\ u=\varphi(x),\ u_t=\psi(x) & \end{cases} \qquad (P)$$

证明：假设问题 (P) 有两个解 u_1, u_2，则 $u=u_1-u_2$ 满足齐次问题

$$(P')\begin{cases} u_{tt}-a^2u_{xx}+xu_x=0, & (0\le x\le l,\ t>0) \\ (u_x-\sigma_1 u)\big|_{x=0}=0,\quad u\big|_{x=l}=0, & \sigma_1>0 \\ t=0:\ u=0,\ u_t=0 & \end{cases}$$

问题 (P) 有唯一解 $\Leftarrow$ 问题 (P') 只有零解

以边界条件 $(u_x-\sigma_1 u)\big|_{x=0}=u\big|_{x=l}=0$：方程两端同乘以 u_t，然后关于空间变量 x 积分：

$$\int_0^l u_t(u_{tt}-a^2u_{xx}+xu_x)\,dx=0 \Rightarrow \int_0^l u_tu_{tt}\,dx-\int_0^l a^2u_tu_{xx}\,dx+\int_0^l u_t x u_x\,dx=0$$

$$\int_0^l u_tu_{tt}\,dx=\int_0^l \frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right)dx=\frac{d}{dt}\int_0^l \frac{u_t^2}{2}\,dx$$

$$\cancel{u_t|_{x=0}=u_t|_{x=l}=0}$$

$$\int_0^l u_tu_{xx}\,dx=\int_0^l u_t\,du_x=u_tu_x\big|_0^l-\int_0^l u_x\,du_t=u_tu_x\big|_0^l-\int_0^l u_xu_{xt}\,dx=\underbrace{-}_{u_tu_x|_0^l}\int_0^l \frac{\partial}{\partial t}\left(\frac{u_x}{2}\right)^2 dx$$

$$\underbrace{=}_{u_tu_x|_0^l}-\frac{d}{dt}\int_0^l \left(\frac{u_x}{2}\right)^2 dx$$

$$\int_0^l u_t x u_x\,dx=\int_0^l u_t\left[\frac{\partial(xu)}{\partial x}-u\right]dx=\int_0^l u_t\,dxu-\int_0^l u_tu\,dx=\int_0^l u_t\,dxu-\frac{d}{dt}\int_0^l \frac{u^2}{2}\,d$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l (u_t^2+a^2u_x^2-u^2)\,dx-a^2u_tu_x\big|_0^l+\int_0^l u_t\,dxu=0 \Rightarrow \frac{d}{dt}E(t)=0$$

$E(t)=\cdots$ （不知道怎么化简了 😖）

$E(t)=E(0)=0 \Rightarrow u_t=0,\ u_x=0 \Rightarrow u=0$

3、利用能量积分方法，证明下述初边值问题解的唯一性： 其中常数 $\alpha, \beta > 0$.

$$
\begin{cases}
u_{tt} = c^2 u_{xx}, & 0<x<L,\ t>0 \\
u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & 0 \le x \le L \\
(-\alpha u_x + \beta u)|_{x=0} = p(t), & t \ge 0 \\
(\alpha u_x + \beta u)|_{x=L} = q(t), & t \ge 0
\end{cases}
\qquad (P)
$$

证明： 假设问题 (P) 有两个解 u_1, u_2，则 $u = u_1 - u_2$ 满足齐次问题

$$
(P')\begin{cases}
u_{tt} - c^2 u_{xx} = 0, & 0<x<L,\ t>0 \\
u(x,0) = 0,\ u_t(x,0) = 0, & 0 \le x \le L \\
(-\alpha u_x + \beta u)|_{x=0} = 0, & t \ge 0 \quad \Rightarrow -\alpha u_x(0,t) + \beta u(0,t) = 0 \\
(\alpha u_x + \beta u)|_{x=L} = 0, & t \ge 0 \quad \Rightarrow \alpha u_x(L,t) + \beta u(L,t) = 0
\end{cases}
$$

问题(P)有唯一解 $\Leftrightarrow$ 问题 (P') 只有零解

以边界条件 $(-\alpha u_x + \beta u)|_{x=0} = 0$ 和 $(\alpha u_x + \beta u)|_{x=L} = 0$：方程两端同乘以 u_t，然后关于空间变量 x 积分:

$$\int_0^L u_t(u_{tt} - c^2 u_{xx})dx = 0 \Rightarrow \int_0^L u_t u_{tt}\,dx - \int_0^L c^2 u_t u_{xx}\,dx = 0$$

$$\int_0^L u_t u_{tt} = \int_0^L \frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right)dx = \frac{d}{dt}\int_0^L \frac{u_t^2}{2}dx$$

$$\int_0^L u_t u_{xx}\,dx = \int_0^L u_t\,du_x = u_t u_x|_0^L - \int_0^L u_x\,du_t = u_t u_x|_0^L - \int_0^L u_x u_{xt}\,dx = u_t u_x|_0^L - \int_0^L \frac{\partial}{\partial t}\left(\frac{u_x^2}{2}\right)dx$$

$$= u_t u_x|_0^L - \frac{d}{dt}\int_0^L \left(\frac{u_x^2}{2}\right)dx$$

$$u_t u_x|_0^L = u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t) = u_t(L,t)\frac{\beta u(L,t)}{-\alpha} - u_t(0,t)\frac{\beta u(0,t)}{\alpha} = \frac{-\beta}{\alpha}\left[\frac{\alpha\,\partial\frac{u(x,t)^2}{2}}{\partial t}\Big|_{x=L} + \frac{\alpha\,\partial\left(\frac{u(x,t)}{2}\right)^2}{\partial t}\right|$$

$$= -\frac{\beta}{\alpha}\left(\frac{\partial u}{\partial t}\Big|_{x=L} + \frac{\partial u}{\partial t}\Big|_{x=0}\right)$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L u_t^2 + c^2 u_x^2 + c^2\frac{2\beta}{L\alpha}u|_{x=L} + c^2\frac{2\beta}{L\alpha}u|_{x=0}\,dx = 0 \Rightarrow \frac{d}{dt}E(t) = 0$$

$$E(t) = \frac{1}{2}\int_0^L u_t^2 + c^2\left(u_x^2 + \frac{2\beta}{L\alpha}u|_{x=L} + \frac{2\beta}{L\alpha}u|_{x=0}\right)dx$$

$$E(t) = E(0) = 0 \Rightarrow u_t = 0,\ u_x = 0 \Rightarrow u = 0$$