1、求解半圆上的Laplace方程

$$\begin{cases} u_{xx}+u_{yy}=0,\ r<1,\ 0<\theta<\pi, \\ u|_{r=1}=\sin^3\theta, \\ u|_{\theta=0}=u|_{\theta=\pi}=0 \end{cases}$$

解: $$\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\ (r<1,\ 0<\theta<\pi), \\ u|_{r=1}=\sin^3\theta, \\ u|_{\theta=0}=u|_{\theta=\pi}=0 \end{cases}$$

$u(r,\theta)=R(r)\Theta(\theta)$

$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda$

$r^2R''+rR'-\lambda R=0$

$$\begin{cases} \Theta''+\lambda\Theta=0 \\ \Theta(\theta)=\Theta(\theta+2\pi) \end{cases}$$

$\lambda=0$, $\Theta(\theta)=A+B\theta$, $B=0$, $\Theta(\theta)=A$

$\lambda<0$, $\Theta(\theta)=Ae^{\sqrt{-\lambda}\theta}+Be^{-\sqrt{-\lambda}\theta}$, 无解

$\lambda>0$, $\Theta(\theta)=A\cos\sqrt{\lambda}\theta+B\sin\sqrt{\lambda}\theta$, $\sqrt{\lambda}=n$, $n=0,1,2,\cdots$, $\Theta(\theta)=A\cos n\theta+B\sin n\theta$, $n=0,1,2,\cdots$

$\lambda_n=n^2$, $\Theta_n(\theta)=A_n\cos n\theta+B_n\sin n\theta$, $n=0,1,2,\cdots$

$n=0$, $R_0(r)=C_0+D_0\ln r$, $R_0(r)=C_0$

$n>0$, $R_n(r)=C_n$ [redacted] $^n+D_nr^{-n}$, $R_n(r)=C_nr^n$

$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos n\theta+\beta_n\sin n\theta)r^n$, $r=1$ (边界条件)

$a_n=\frac{1}{\pi}\int_0^{2\pi}\sin^3t\cos nt\,dt$, $n=0,1,2,\cdots$

$\beta_n=\frac{1}{\pi}\int_0^{2\pi}\sin^3t\sin nt\,dt$, $n=1,2,\cdots$ [redacted] $\cancel{\frac{1}{\pi}}$ [redacted] $\cancel{\frac{3}{4}\int_0^{2\pi}\sin t\cos nt\,dt}$ [redacted]

$a_n=\frac{1}{\pi}\int_0^{2\pi}(\frac{3}{4}\sin t\cos nt-\frac{1}{4}\sin 3t\cos nt)\,dt$ [redacted] $=0$

$$\beta_n=\frac{1}{\pi}\int_0^{2\pi}(\frac{3}{4}\sin t\sin nt-\frac{1}{4}\sin 3t\sin nt)\,dt=\begin{cases} \frac{1}{\pi}\frac{3}{4}\int_0^{2\pi}\sin t\sin nt\,dt=\frac{3}{4} & (n=1) \\ \frac{1}{\pi}(-\frac{1}{4})\int_0^{2\pi}\sin 3t\sin nt\,dt=-\frac{1}{4} & (n=3) \end{cases}$$

$u(r,\theta)=\frac{3}{4}r\sin\theta$ [redacted] $-\frac{1}{4}r^3\sin 3\theta$

$\cancel{u(x,y)=\frac{3}{4}y}$ $u(r,\theta)=\frac{3r\sin\theta-r^3\sin 3\theta}{4}$

2. 记(r,θ)为极坐标，求解定解问题

$$\begin{cases} u_{xx}+u_{yy}=1 & 1<r<2 \\ u|_{r=1}=1+\cos^2\theta \\ u|_{r=2}=\sin^2\theta \end{cases}$$

解：

$$\begin{cases} u_{xx}+u_{yy}=1 \quad 1<r<2 & u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=1 \quad 1<r<2 \\ u|_{r=1}=1+\cos^2\theta \\ u|_{r=2}=1-\cos^2\theta \end{cases}$$

$u(r,\theta)=v(r)+w(r,\theta)$

$v(r)=\frac{1}{4}r^2$

$$\begin{cases} w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}=0 \quad 1<r<2 \\ u|_{r=1}=\frac{3}{4}+\cos^2\theta=\frac{5}{4}+\frac{\cos 2\theta}{2} \\ u|_{r=2}=-\cos^2\theta=-\frac{1}{2}-\frac{\cos 2\theta}{2} \end{cases}$$

$r=1$: $a_0+b_0\ln 1=a_0=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{5}{4}+\frac{\cos 2\theta}{2}\right)d\theta=\frac{5}{4}+\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos 2\theta}{2}d\theta=\frac{5}{4} \quad\Rightarrow\quad a_0=\frac{5}{4}$

$a_n+c_n=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{5}{4}+\frac{\cos 2\theta}{2}\right)\cos n\theta\, d\theta=\frac{1}{2\pi}\int_0^{2\pi}\cos 2\theta\cos n\theta\, d\theta=\begin{cases}0 & n\neq 2\\ \frac{1}{2} & n=2\end{cases}$

$b_n+d_n=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{5}{4}+\frac{\cos 2\theta}{2}\right)\sin n\theta\, d\theta=0$

$r=2$: $a_0+b_0\ln 2=\frac{1}{2\pi}\int_0^{2\pi}\left(-\frac{1}{2}-\frac{\cos 2\theta}{2}\right)d\theta=-\frac{1}{2} \quad\Rightarrow\quad b_0=-\frac{7}{4\ln 2}$

$a_n 2^n+c_n 2^{-n}=\frac{1}{\pi}\int_0^{2\pi}\left(-\frac{1}{2}-\frac{\cos 2\theta}{2}\right)\cos n\theta\, d\theta=-\frac{1}{2\pi}\int_0^{2\pi}\cos 2\theta\cos n\theta\, d\theta=\begin{cases}0 & n\neq 2\\ -\frac{1}{2} & n=2\end{cases} \Rightarrow c_2=\frac{2}{3}$ 且 $a_2=-\frac{1}{6}$

$b_n 2^n+d_n 2^{-n}=\frac{1}{\pi}\int_0^{2\pi}\left(-\frac{1}{2}-\frac{\cos 2\theta}{2}\right)\sin n\theta\, d\theta=0 \quad\Rightarrow\quad b_n=0$ 且 $d_n=0$

$w(r,\theta)=\frac{5}{4}-\frac{7}{4\ln 2}\ln r+\left(-\frac{1}{6}r^2+\frac{2}{3}r^{-2}\right)\cos 2\theta$

$u(r,\theta)=v(r)+w(r,\theta)$

$u(r,\theta)=\frac{9}{4}-\frac{7}{4\ln 2}\ln r+\left(-\frac{1}{6}r^2+\frac{2}{3}r^{-2}\right)\cos 2\theta$

3. 求解下列方程

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 & (0<\theta<\frac{\pi}{2},\ 1<r<2) \\ u|_{\theta=0,\frac{\pi}{2}}=0 \\ u|_{r=1}=0,\quad u|_{r=2}=\sin 2\theta \end{cases}$$

解: $u(r,\theta)=R(r)\Theta(\theta)$

$$\begin{cases} \Theta''+\lambda\Theta=0 \\ \Theta(0)=\Theta(\frac{\pi}{2})=0 \end{cases}$$

$\lambda_n=(\frac{n\pi}{\frac{\pi}{2}})^2=4n^2$, $\Theta_n(\theta)=\sin\frac{n\pi}{\frac{\pi}{2}}\theta=\sin 2n\theta\quad(n=1.2,\cdots)$

$r^2R''+rR'-\lambda R=0$, $\cancel{R_n(r)=a_n r^{\frac{n\pi}{\frac{\pi}{2}}}+b_n r^{-\frac{n\pi}{\frac{\pi}{2}}}=a_n r^{2n}+b_n r^{-2n},\ |R(0)|<+\infty,\ R_n(r)=a_n r^{2n}}$

$r=1:\quad a_0+b_0\ln 1=a_0=0 \implies a_0=0$

[redacted] $a_n+c_n=0$

$b_n+d_n=0$

$r=2:\quad a_0+b_0\ln 2=\frac{1}{2\pi}\int_0^{2\pi}\sin 2\theta\, d\theta=0 \Rightarrow b_0=0$

$a_n 2^n+c_n 2^{-n}=\frac{1}{\pi}\int_0^{2\pi}\sin 2\theta\cos n\theta\, d\theta=0 \implies a_n=0$ 且 $c_n=0$

$b_n 2^n+d_n 2^{-n}=\frac{1}{\pi}\int_0^{2\pi}\sin 2\theta\sin n\theta\, d\theta=\begin{cases}0 & n\neq 2\\ 1 & n=2\end{cases}$

$4b_n+\frac{d_n}{4}=1 \implies b_n=\frac{4}{15}$ 且 $d_n=-\frac{4}{15}$

$u(r,\theta)=(\frac{4}{15}r^{2}-\frac{4}{15}r^{-2})\sin 2\theta$

$u(r,\theta)=\frac{4}{15}(r^2-r^{-2})\sin 2\theta$