1、证明下面定解问题解的唯一性：

$$\begin{cases} u_{tt}=a^2(u_{xx}+u_{yy})+f(t,x,y), & (t>0,\ (x,y)\in\Omega\subset R^2) \\ t=0:\ u=\varphi(x,y),\ u_t=\psi(x,y) \\ \left(\frac{\partial u}{\partial n}+\sigma u\right)\Big|_{\partial\Omega}=g(t,x,y) \end{cases}$$

证明：即证 $\begin{cases} u_{tt}=a^2(u_{xx}+u_{yy}), & (t>0,\ (x,y)\in\Omega\subset R^2) \\ t=0:\ u=0,\ u_t=0 \\ \left(\frac{\partial u}{\partial n}+\sigma u\right)\Big|_{\partial\Omega}=0 \end{cases}$

只有零解

$$\int_\Omega u_t\left[u_{tt}-a^2(u_{xx}+u_{yy})\right]dx=0$$

$$\int_\Omega u_t u_{tt}\,dx=\frac{d}{dt}\left(\frac{1}{2}\int_\Omega u_t^2\,dx\right)$$

$$\int_\Omega u_t(u_{xx}+u_{yy})\,dx=\int_\Omega u_t\Delta u\,dx=\int_\Omega u_t\nabla\cdot(\nabla u)=\int_\Omega\left[\nabla\cdot(u_t\nabla u)-\nabla u_t\cdot\nabla u\right]dx$$

$$=\int_{\partial\Omega}u_t\nabla u\cdot\vec{n}\,ds-\frac{d}{dt}\left(\frac{1}{2}\int_\Omega|\nabla u|^2dx\right)=\int_{\partial\Omega}u_t\frac{\partial u}{\partial n}ds-\frac{d}{dt}\left(\frac{1}{2}\int_\Omega|\nabla u|^2dx\right)$$

$\because \left(\frac{\partial u}{\partial n}+\sigma u\right)\Big|_{\partial\Omega}=0$

$\therefore \frac{\partial u}{\partial n}\Big|_\Gamma=-\sigma u\Big|_\Gamma$

$\therefore u_t\frac{\partial u}{\partial n}\Big|_\Gamma=-\sigma u u_t\Big|_\Gamma=-\frac{d}{dt}\left(\frac{1}{2}\sigma u^2\right)\Big|_\Gamma$

$\therefore \int_\Omega u_t\left[u_{tt}-a^2(u_{xx}+u_{yy})\right]dx=\frac{d}{dt}\left[\frac{1}{2}\int_\Omega u_t^2dx+\frac{1}{2}\int_\Omega a^2|\nabla u|^2dx+\frac{\sigma a^2}{2}u^2\Big|_\Gamma\right]$

$\because \frac{d}{dt}E(t)=0$

$\therefore E(t)=\frac{1}{2}\int_\Omega\left(u_t^2+a^2|\nabla u|^2\right)dx+\int_\Gamma\frac{\sigma a^2}{2}u^2dS$

$\therefore E(t)\equiv E(0)=\frac{1}{2}\int_\Omega\left[0^2+a^2\left(\sqrt{0^2+0^2}\right)^2\right]dx+\int_\Gamma\frac{\sigma a^2}{2}0^2dS=0$

得证：上面定解问题解的唯一性

2、求解 Cauchy 问题

$$\begin{cases} u_t-u_{xx}=0 \\ t=0:\ u=\sin x \end{cases}$$

解：$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(y)e^{-\frac{(y-x)^2}{4a^2t}}dy=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\sin(y-x+x)e^{-\frac{(y-x)^2}{4t}}d(y-x)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\left[\sin(y-x)\cos x+\cos(y-x)\sin x\right]e^{-\frac{(y-x)^2}{4t}}d(y-x)$

$=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\sin y\,e^{-\frac{(y-x)^2}{4t}}dy$

$=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\sin(y-x)\cos x\,e^{-\frac{(y-x)^2}{4t}}d(y-x)+\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\cos(y-x)\sin x\,e^{-\frac{(y-x)^2}{4t}}d(y-x)$

$=\frac{1}{2\sqrt{\pi t}}\cos x\int_{-\infty}^{+\infty}\sin m\,e^{-\frac{m^2}{4t}}dm+\frac{\sin x}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\cos(y-x)\,e^{-\frac{(y-x)^2}{4t}}d(y-x)$

$(m=y-x)$

$=\frac{\sin x}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\cos(y-x)e^{-\frac{(y-x)^2}{4t}}dy=\frac{\sin x}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(y-x)^2}{4t}}d\sin(y-x)=\frac{\sin x}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\sin(y-x)\frac{y-x}{2t}e^{-\frac{(y-x)^2}{4t}}dy$

$u(x,t)=\sin x\,e^{-t}$

3、设u为下列初边值问题的解

$$\begin{cases} u_t - a^2\Delta u = 0, & G = \Omega \times [0,T] \\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega\times[0,T]} = 0 \\ u|_{t=0} = \varphi(x,y,z) \end{cases}$$

其中Ω为有界域($\Omega \in R^3$)，试证明，对$\forall t\in(0,T)$：$\int_\Omega u(x,y,z,t)\,dxdydz = \int_\Omega \varphi(x,y,z)\,dxdydz$.

证明：即证 $\int_\Omega [u(x,y,z,t)-\varphi(x,y,z)]\,dxdydz = 0$

$\because \frac{\partial u}{\partial n}\Big|_{\partial\Omega\times[0,T]} = 0$

$\therefore \nabla u\cdot\vec{n}\Big|_{\partial\Omega\times[0,T]} = 0$

$\therefore \int_\Omega$ [redacted] $(u_{xx}+u_{yy}+u_{zz})dx = \int_\Omega$ [redacted] $\Delta u\,dx = \int_\Omega$ [redacted] $\nabla\cdot(\nabla u)$

$= \int_\Omega[\nabla\cdot($ [redacted] $\nabla u)$ [redacted] $]dx$

$= \int_{\partial\Omega}$ [redacted] $\nabla u\cdot\vec{n}\,ds$ [redacted]

$= 0$

$\therefore$ [redacted] $\int_\Omega \Delta u\,dx = 0 \Rightarrow \int_\Omega a^2\Delta u\,dx = 0$

$\because u_t - a^2\Delta u = 0 \Rightarrow u_t = a^2\Delta u$

$\therefore \int_\Omega u_t\,dx = 0$

$\therefore u(x,y,z,t) = \varphi(x,y,z)$

$\therefore \int_\Omega[u(x,y,z,t)-\varphi(x,y,z)]\,dxdydz = 0$

得证：对$\forall t\in(0,T)$：$\int_\Omega u(x,y,z,t)\,dxdydz = \int_\Omega \varphi(x,y,z)\,dxdydz$