上海交通大學

SHANGHAI JIAO TONG UNIVERSITY

1、指出下面方程的类型、阶数和是否齐次，并求解.

(1) $u_{xy} = x+y$,

(2) $\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} = 0 \quad (u = u(x,y,z))$.

解：(1) 二阶线性非齐次方程

$\because u_{xy} = x+y$

$\therefore u_x = xy + \frac{1}{2}y^2 + f_0(x)$ 　$f_0(x)$为关于x的任意函数，$f_1(x)$为$f_0(x)$的原函数

$\therefore u = \frac{1}{2}yx^2 + \frac{1}{2}xy^2 + f_1(x) + f(y)$ 　$f_1(x)$为关于x的任意函数，$f(y)$为关于y的任意函数

$\Rightarrow u = \frac{1}{2}yx^2 + \frac{1}{2}xy^2 + g(x) + g(y)$ 　$g(x)$、$g(y)$为任意函数

(2) 二阶线性齐次方程

$\because \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} = 0 \quad (u = u(x,y,z))$

令 $p(x,y,z) = \frac{\partial u}{\partial x}$，则 $p + \frac{\partial p}{\partial y} = 0$，则 $\frac{\partial p}{p} = -\partial y$，则 $\ln|p| = -y + f_0(x,z)$ 　$f_0(x,z)$为关于x、z的任意函数

$\therefore |p| = e^{f_0(x,z)-y}$，则 $p = \pm e^{f_0(x,z)-y}$，则 $p = f_1(x,z)e^{-y}$ 　$f_1(x,z)$为关于x、z的任意函数

$\therefore \frac{\partial u}{\partial x} = f_1(x,z)e^{-y}$

$\therefore u = f_2(x,z)e^{-y} + f(y,z)$ 　$f_2(x,z)$为关于x、z的任意函数，$f(y,z)$为关于y、z的任意常数

$\Rightarrow u = g(x,z)e^{-y} + g(y,z)$ 　$g(x,z)$、$g(y,z)$为任意函数

2. 证明函数 $u=\frac{1}{4\pi r}$, $r=\sqrt{x^2+y^2+z^2}\neq 0$, 满足 $\Delta_3 u=(\partial_{xx}+\partial_{yy}+\partial_{zz})u=0$.

证明: $\because u=\frac{1}{4\pi r}$, $r=\sqrt{x^2+y^2+z^2}$

$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}=-\frac{1}{4\pi r^2}\frac{x}{\sqrt{x^2+y^2+z^2}}=-\frac{x}{4\pi r^3}$

同理 $\frac{\partial u}{\partial y}=-\frac{y}{4\pi r^3}$, $\frac{\partial u}{\partial z}=-\frac{z}{4\pi r^3}$

$\therefore \frac{\partial^2 u}{\partial x^2}=-\frac{1}{4\pi}\frac{\partial\left(\frac{x}{r^3}\right)}{\partial x}=-\frac{1}{4\pi}\frac{r^3-3rx^2}{r^6}=-\frac{1}{4\pi}\frac{r^2-3x^2}{r^5}$

同理 $\frac{\partial^2 u}{\partial y^2}=-\frac{1}{4\pi}\frac{r^2-3y^2}{r^5}$, $\frac{\partial^2 u}{\partial z^2}=-\frac{1}{4\pi}\frac{r^2-3z^2}{r^5}$

$\therefore \Delta_3 u=(\partial_{xx}+\partial_{yy}+\partial_{zz})u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$

$=-\frac{1}{4\pi}\frac{3r^2-3(x^2+y^2+z^2)}{r^5}=-\frac{1}{4\pi}\frac{3r^2-3r^2}{r^5}$

$=0$

得证