1、把下列方程化成标准型.

$$6U_{xx} - U_{xy} + U = y^2$$

解: $\Delta = \left(-\frac{1}{2}\right)^2 - 0 > 0 \Rightarrow$ 双曲

$6\,dy^2 + dx\,dy = 0 \Rightarrow dy\,(6dy + dx) = 0$

特征线 $y = C_1$, $6y + x = C_2$

取 $\begin{cases} \xi = y \\ \eta = 6y + x \end{cases}$ $\begin{cases} U_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi 0 + U_\eta 1 = U_\eta \\ U_y = U_\xi \xi_y + U_\eta \eta_y = U_\xi 1 + U_\eta 6 = U_\xi + 6U_\eta \end{cases}$

$$\begin{cases} U_{xx} = U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x = U_{\eta\xi}0 + U_{\eta\eta}1 = U_{\eta\eta} \\ U_{yy} = U_{\xi\xi}\xi_y + U_{\xi\eta}\eta_y + 6U_{\eta\xi}\xi_y + 6U_{\eta\eta}\eta_y = U_{\xi\xi}1 + U_{\xi\eta}6 + 6U_{\eta\xi}1 + 6U_{\eta\eta}6 \Rightarrow \\ \quad = U_{\xi\xi} + 12U_{\xi\eta} + 36U_{\eta\eta} \text{ (针对本题，可以不算)} \\ U_{xy} = U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y = U_{\eta\xi}1 + U_{\eta\eta}6 = U_{\xi\eta} + 6U_{\eta\eta} \end{cases}$$

$6U_{xx} - U_{xy} + U = y^2 \Rightarrow 6U_{\eta\eta} - U_{\xi\eta} - 6U_{\eta\eta} + U = \xi^2 \Rightarrow U_{\xi\eta} = U - \xi^2$

2、把下列方程化成标准型.

$$u_{xx}+2u_{xy}+3u_{yy}+4u_x+5u_y+u=e^x$$

解: $\Delta=\left(\frac{2}{2}\right)^2-3<0 \Rightarrow$ 椭圆

$dy^2-2dxdy+3dx^2=0 \Rightarrow \lambda_{1,2}=1\pm i\sqrt{2} \Rightarrow dy=(1\pm i\sqrt{2})dx \Rightarrow y=(1\pm i\sqrt{2})x+C$

$\begin{cases} y-(1+i\sqrt{2})x=C_1 \\ y-(1-i\sqrt{2})x=C_2 \end{cases}$ ▲

取 $\begin{cases} \xi=y-x \\ \eta=-\sqrt{2}x \end{cases}$ $\begin{cases} u_x=u_\xi \xi_x+u_\eta \eta_x=u_\xi(-1)+u_\eta(-\sqrt{2})=-u_\xi-\sqrt{2}u_\eta \\ u_y=u_\xi \xi_y+u_\eta \eta_y=u_\xi 1+u_\eta 0=u_\xi \end{cases}$

$u_{xx}=-u_{\xi\xi}\xi_x-u_{\xi\eta}\eta_x-\sqrt{2}u_{\eta\xi}\xi_x-\sqrt{2}u_{\eta\eta}\eta_x=-u_{\xi\xi}(-1)-u_{\xi\eta}(-\sqrt{2})-\sqrt{2}u_{\eta\xi}(-1)$

$\quad -\sqrt{2}u_{\eta\eta}(-\sqrt{2})=u_{\xi\xi}+2\sqrt{2}u_{\xi\eta}+2u_{\eta\eta}$

$u_{yy}=u_{\xi\xi}\xi_y+u_{\xi\eta}\eta_y=u_{\xi\xi}1+u_{\xi\eta}0=u_{\xi\xi}$

$u_{xy}=u_{yx}=u_{\xi\xi}\xi_x+u_{\xi\eta}\eta_x=u_{\xi\xi}(-1)+u_{\xi\eta}(-\sqrt{2})=-u_{\xi\xi}-\sqrt{2}u_{\xi\eta}$

$u_{xx}+2u_{xy}+3u_{yy}+4u_x+5u_y+u=e^x \Rightarrow$

$u_{\xi\xi}+\cancel{2\sqrt{2}u_{\xi\eta}}+2u_{\eta\eta}-2u_{\xi\xi}-\cancel{2\sqrt{2}u_{\xi\eta}}+3u_{\xi\xi}-4u_\xi-4\sqrt{2}u_\eta+5u_\xi+u=e^{-\frac{\eta}{\sqrt{2}}}$

$2u_{\xi\xi}+2u_{\eta\eta}+u_\xi-4\sqrt{2}u_\eta+u=e^{-\frac{\eta}{\sqrt{2}}}$

$u_{\xi\xi}+u_{\eta\eta}=-\frac{1}{2}u_\xi+2\sqrt{2}u_\eta-\frac{1}{2}u+\frac{1}{2}e^{-\frac{\sqrt{2}}{2}\eta}$

3. 把下列方程化成标准型.

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0 \quad (xy \neq 0)$$

解: $\Delta = \left(\frac{2xy}{2}\right)^2 - x^2y^2 = 0 \Rightarrow$ 抛物

$x^2 dy^2 - 2xy\, dx\, dy + y^2 dx^2 = 0 \Rightarrow (x\,dy - y\,dx)^2 = 0$

特征线 $\frac{y}{x} = C$

取 $\begin{cases} \xi = \frac{y}{x} \\ \eta = y \end{cases}$ $\quad \begin{cases} u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi\left(-\frac{y}{x^2}\right) + u_\eta 0 = -\frac{y}{x^2} u_\xi = -\frac{y}{x^2} u_\xi \\ u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi\left(\frac{1}{x}\right) + u_\eta 1 = \frac{1}{x} u_\xi + u_\eta \end{cases}$

$u_{xx} = \frac{2y}{x^3} u_\xi - \frac{y}{x^2}\left(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x\right) = \frac{2y}{x^3} u_\xi - \frac{y}{x^2}\left[u_{\xi\xi}\left(-\frac{y}{x^2}\right) + u_{\xi\eta} 0\right] = \frac{2y}{x^3} u_\xi + \frac{y^2}{x^4} u_{\xi\xi}$

$u_{xy} = u_{yx} = -\frac{1}{x^2} u_\xi + \frac{1}{x}\left[u_{\xi\xi}\left(-\frac{y}{x^2}\right) + u_{\xi\eta} 0\right] + u_{\eta\xi}\left(-\frac{y}{x^2}\right) + u_{\eta\eta} 0$

$= -\frac{1}{x^2} u_\xi - \frac{y}{x^3} u_{\xi\xi} - \frac{y}{x^2} u_{\xi\eta}$

$u_{yy} = \frac{1}{x}\left[u_{\xi\xi}\left(\frac{1}{x}\right) + u_{\xi\eta} 1\right] + u_{\eta\xi}\left(\frac{1}{x}\right) + u_{\eta\eta} 1 = \frac{1}{x^2} u_{\xi\xi} + \frac{2}{x} u_{\xi\eta} + u_{\eta\eta}$

$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0 \Rightarrow \cancel{\frac{2y}{x}} u_\xi + \cancel{\frac{y^2}{x^2}} u_{\xi\xi} - \cancel{\frac{2y}{x}} u_\xi - \cancel{\frac{2y^2}{x^2}} u_{\xi\xi} - \cancel{\frac{2y^2}{x}} u_{\xi\eta}$

$+ \cancel{\frac{y^2}{x^2}} u_{\xi\xi} + \cancel{\frac{2y^2}{x}} u_{\xi\eta} + y^2 u_{\eta\eta} = 0$

$y^2 u_{\eta\eta} = 0 \Rightarrow \eta^2 u_{\eta\eta} = 0 \qquad (xy \neq 0 \Rightarrow x \neq 0$ 且 $y \neq 0 \Rightarrow \eta \neq 0)$

$u_{\eta\eta} = 0$

4、证明：两个自变量的二阶线性方程经过自变量的可逆变换后，其类型不会改变，即变换后

$\Delta = a_{12}^2 - a_{11}a_{12}$ 的符号不变。

证明：两个自变量的二阶线性方程：$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu = 0$

取自变量的可逆变换 $\begin{cases} \xi = \varphi(x,y) \\ \eta = \Phi(x,y) \end{cases} \Rightarrow J = \dfrac{\partial(\varphi,\Phi)}{\partial(x,y)} = \begin{vmatrix} \varphi_x & \varphi_y \\ \Phi_x & \Phi_y \end{vmatrix} \neq 0 \Rightarrow \varphi_x\Phi_y - \varphi_y\Phi_x = J \neq 0$

新的两个自变量的二阶线性方程：$A_{11}u_{\xi\xi} + 2A_{12}u_{\xi\eta} + A_{22}u_{\eta\eta} + B_1u_\xi + B_2u_\eta + Cu = 0$

$A_{11} = a_{11}\varphi_x^2 + 2a_{12}\varphi_x\varphi_y + a_{22}\varphi_y^2$

$A_{12} = a_{11}\varphi_x\Phi_x + a_{12}(\varphi_x\Phi_y + \varphi_y\Phi_x) + a_{22}\varphi_y\Phi_y$

$A_{22} = a_{11}\Phi_x^2 + 2a_{12}\Phi_x\Phi_y + a_{22}\Phi_y^2$

B_1、B_2、C 针对本题，不会改变其类型，可以不算

$$\Delta' = A_{12}^2 - A_{11}A_{22} = a_{11}^2\varphi_x^2\Phi_x^2 + a_{12}^2(\varphi_x\Phi_y + \varphi_y\Phi_x)^2 + a_{22}^2\varphi_y^2\Phi_y^2 + 2a_{11}a_{22}\varphi_x\Phi_x\varphi_y\Phi_y$$

$$+ 2a_{11}a_{12}\varphi_x\Phi_x(\varphi_x\Phi_y + \varphi_y\Phi_x) + 2a_{12}a_{22}\varphi_y\Phi_y(\varphi_x\Phi_y + \varphi_y\Phi_x) - a_{11}^2\varphi_x^2\Phi_x^2 - 2a_{11}a_{12}\varphi_x^2\Phi_x\Phi_y$$

$$- a_{11}a_{22}\varphi_x^2\Phi_y^2 - 2a_{11}a_{12}\Phi_x^2\varphi_x\varphi_y - 4a_{12}^2\varphi_x\varphi_y\Phi_x\Phi_y - 2a_{12}a_{22}\Phi_y^2\varphi_x\varphi_y - a_{11}a_{22}\varphi_y^2\Phi_x^2 - 2a_{12}a_{22}\varphi_y^2\Phi_x\Phi_y$$

$$- a_{22}^2\varphi_y^2\Phi_y^2 = a_{12}^2\varphi_x^2\Phi_y^2 + a_{12}^2\varphi_y^2\Phi_x^2 + 2a_{12}^2\varphi_x\varphi_y\Phi_x\Phi_y + 2a_{11}a_{22}\varphi_x\varphi_y\Phi_x\Phi_y - a_{11}a_{22}\varphi_x^2\Phi_y^2$$

$$- 4a_{12}^2\varphi_x\varphi_y\Phi_x\Phi_y - a_{11}a_{22}\varphi_y^2\Phi_x^2 = a_{12}^2\varphi_x^2\Phi_y^2 + a_{12}^2\varphi_y^2\Phi_x^2 + 2a_{11}a_{22}\varphi_x\varphi_y\Phi_x\Phi_y - a_{11}a_{22}\varphi_x^2\Phi_y^2$$

$$- 2a_{12}^2\varphi_x\varphi_y\Phi_x\Phi_y - a_{11}a_{22}\varphi_y^2\Phi_x^2 = (a_{12}^2 - a_{11}a_{22})(\varphi_x^2\Phi_y^2 + \varphi_y^2\Phi_x^2 - 2\varphi_x\varphi_y\Phi_x\Phi_y)$$

$$= \Delta(\varphi_x\Phi_y - \varphi_y\Phi_x)^2 \Rightarrow \Delta' = \Delta J^2 \ (J^2 > 0) \Rightarrow \Delta' \text{ 与 } \Delta \text{ 同号}$$

即变换后 $\Delta = a_{12}^2 - a_{11}a_{12}$ 的符号不变