1. 有一个半径为R的均匀球，初始温度分布为$\varphi(r)$（r为球内点到球心的距离）. 现把该球置于温度为$f(t)$的环境中，球的体密度为ρ，比热为c，热传导系数为k，球面与环境介质的热交换系数为h，试写出该球的温度分布$u(t,r)$所满足的定解问题.（只写出定解问题，不需要建模过程）

解：热传导方程：$u_t - a^2\Delta_3 u = 0 \quad (t>0 \text{且} 0<r<R) \quad (a^2=\frac{k}{c\rho})$

初始温度：$u(0,r)=\varphi(r)$

能量守恒定律：$dQ_1 \leftarrow \boxed{dS} \leftarrow dQ_2$，$dt$　　dt为单位时间、ds为单位面积、dQ为单位热量

$dQ_1 = k\,dS\,\frac{\partial u}{\partial r}\Big|_{r=R} dt + h u(t,R)\,dS\,dt$

$dQ_2 = hf(t)\,dS\,dt$

$dQ_1 = dQ_2 \Rightarrow k\,dS\,u_r\big|_{r=R} dt + hu(t,R)\,dS\,dt = hf(t)\,dS\,dt$

$k u_r(t,R) + h u(t,R) = hf(t)$

该热传导的定解问题：

$$\begin{cases} u_t - a^2\Delta_3 u = 0 \quad (t>0,\ 0<r<R) \quad (a^2=\frac{k}{c\rho}) \\ u(0,r)=\varphi(r) \\ ku_r(t,R)+hu(t,R)=hf(t) \end{cases}$$

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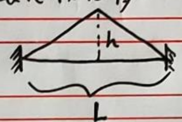
2. 设长度为L的弦两端固定，将弦的中点拉到高度h后放手，写出该弦振动的定解问题。

解：弦的纵振动方程：$u_{tt}-a^2u_{xx}=0 \quad (0<x<L \text{ 且 } t>0)$

两端初位移为0：$u(0,t)=0$ 且 $u(L,t)=0$

初速度为0：$u_t(x,0)=0$

其他初位移：



$$u(x,0)=\begin{cases}\frac{2h}{L}x & 0\le x<\frac{L}{2}\\ \frac{2h}{L}(L-x) & \frac{L}{2}\le x\le L\end{cases}$$

该弦振动的定解问题：

$$\begin{cases}u_{tt}-a^2u_{xx}=0 & (0<x<L,\ t>0)\\ u(0,t)=0 \\ u(L,t)=0 \\ u_t(x,0)=0 \\ u(x,0)=\begin{cases}\frac{2h}{L}x & 0\le x<\frac{L}{2}\\ \frac{2h}{L}(L-x) & \frac{L}{2}\le x\le L\end{cases}\end{cases}$$

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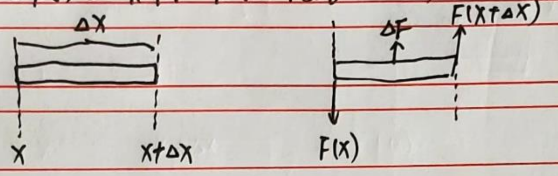
3. 推导均匀杆的纵振动的波动方程，其中 $u(x,t)$ 为杆上 x 点在 t 时刻点纵向位移，E 为杨氏模量，ρ 为密度，S 为横截面积。（需要写出过程）

解：



Δx 为单位长度，ΔV 为单位体积，Δm 为单位质量，ΔF 为惯性力，a_0 为加速度

$\Delta m = \rho\Delta V = \rho S\Delta x$　　$v(x,t)$ 为杆上 x 点在 t 时刻纵向速度

$$\Delta F = \Delta m\, a_0 = \Delta m \frac{\partial v(x,t)}{\partial t} = \Delta m \frac{\partial\left(\frac{\partial u(x,t)}{\partial t}\right)}{\partial t} = \Delta m \frac{\partial^2 u(x,t)}{\partial t^2} = \Delta m\, u_{tt} = \rho S\Delta x\, u_{tt}$$

$F(x+\Delta x)$ 为杆上 $(x+\Delta x)$ 点剪应力，$F(x)$ 为杆上 x 点剪应力，σ 为应力，ε 为应变

$$F(x+\Delta x) = \sigma S = E\varepsilon S = E\left.\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}\right|_{x+\Delta x} S = E\, u_x|_{x+\Delta x} S$$

$$F(x) = \sigma S = E\varepsilon S = E\left.\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}\right|_{x} S = E\, u_x|_x S$$

牛顿第二定律 $\Delta F = F(x+\Delta x) - F(x) = \rho S\Delta x\, u_{tt} = E\, u_x|_{x+\Delta x} S - E\, u_x|_x S$

令 $a^2 = \frac{E}{\rho}$，则 $\rho\Delta x\, u_{tt} = E\, u_x|_{x+\Delta x} - E\, u_x|_x$，则 $u_{tt} = \frac{E}{\rho}\frac{u_x|_{x+\Delta x} - u_x|_x}{\Delta x} = a^2 u_{xx}$

$u_{tt} - a^2 u_{xx} = 0$

4、分别写出2维和3维在球对称的情况下，拉普拉斯算子的表达式Δ_2和Δ_3.（需要具体过程）

解：①2维：$\begin{cases} x = r\cos\theta & r\geq 0,\ 0\leq\theta\leq 2\pi \\ y = r\sin\theta \end{cases}$

$\because r^2 = x^2+y^2,\quad 2r\frac{\partial r}{\partial x} = 2x,\quad r\frac{\partial r}{\partial x} = x,\quad \frac{\partial r}{\partial x} = \frac{x}{r}$

$r^2 = x^2+y^2,\quad 2r\frac{\partial r}{\partial y} = 2y,\quad r\frac{\partial r}{\partial y} = y,\quad \frac{\partial r}{\partial y} = \frac{y}{r}$

$\because \frac{x}{\cos\theta} = r = \frac{y}{\sin\theta},\quad x\sin\theta = y\cos\theta$

$\therefore \sin\theta + x\cos\theta\frac{\partial\theta}{\partial x} = -y\sin\theta\frac{\partial\theta}{\partial x},\quad \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{x\cos\theta+y\sin\theta} = -\frac{r\sin\theta}{xr\cos\theta+yr\sin\theta} = -\frac{y}{r^2}$

$x\cos\theta\frac{\partial\theta}{\partial y} = \cos\theta - y\sin\theta\frac{\partial\theta}{\partial y},\quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{x\cos\theta+y\sin\theta} = \frac{r\cos\theta}{xr\cos\theta+yr\sin\theta} = \frac{x}{r^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial\theta}\frac{y}{r^2}$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y} = \frac{\partial u}{\partial r}\frac{y}{r} + \frac{\partial u}{\partial\theta}\frac{x}{r^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial\theta}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \left(\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial\theta^2}\frac{\partial\theta}{\partial x}\right)\frac{\partial\theta}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial^2\theta}{\partial x^2}$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial\theta}{\partial y}\right)\frac{\partial r}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2} + \left(\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial\theta^2}\frac{\partial\theta}{\partial y}\right)\frac{\partial\theta}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial^2\theta}{\partial y^2}$

$\because \frac{\partial^2 r}{\partial x^2} = \frac{\partial\left(\frac{x}{r}\right)}{\partial x} = \frac{y^2}{r^3},\quad \frac{\partial^2\theta}{\partial x^2} = \frac{2xy}{r^4}$

$\frac{\partial^2 r}{\partial y^2} = \frac{\partial\left(\frac{y}{r}\right)}{\partial y} = \frac{x^2}{r^3},\quad \frac{\partial^2\theta}{\partial y^2} = -\frac{2xy}{r^4}$

$\therefore \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^4}\left(r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial\theta^2}\right)(x^2+y^2) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$

圆对称 $\Rightarrow \frac{\partial u}{\partial\theta} = 0$

$\therefore \Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$

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② 3维：

$$\begin{cases} x = r\sin\varphi\cos\theta & r\geq 0,\ 0\leq\varphi\leq\pi,\ 0\leq\theta\leq 2\pi \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$\begin{cases} \frac{\partial x}{\partial r} = \sin\varphi\cos\theta & \frac{\partial x}{\partial \varphi} = r\cos\theta\cos\varphi & \frac{\partial x}{\partial \theta} = -r\sin\varphi\sin\theta \\ \frac{\partial y}{\partial r} = \sin\varphi\sin\theta & \frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi & \frac{\partial y}{\partial \theta} = r\sin\varphi\cos\theta \\ \frac{\partial z}{\partial r} = \cos\varphi & \frac{\partial z}{\partial \varphi} = -r\sin\varphi & \frac{\partial z}{\partial \theta} = 0 \end{cases}$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = J \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \qquad J = \begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ r\cos\theta\cos\varphi & r\sin\theta\cos\varphi & -r\sin\varphi \\ -r\sin\varphi\sin\theta & r\sin\varphi\cos\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = J^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \qquad J^{-1} = \begin{pmatrix} \cos\theta\sin\varphi & \frac{\cos\theta\cos\varphi}{r} & -\frac{\sin\theta}{r\sin\varphi} \\ \sin\theta\sin\varphi & \frac{\sin\theta\cos\varphi}{r} & \frac{\cos\theta}{r\sin\varphi} \\ \cos\varphi & -\frac{\sin\varphi}{r} & 0 \end{pmatrix} \qquad \cancel{(AB)^T = B^T A^T}$$

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \cancel{\left(\frac{\partial}{\partial r}\ \frac{\partial}{\partial \varphi}\ \frac{\partial}{\partial \theta}\right)(J^{-1})^T J^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial \theta} \end{pmatrix}} \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}$$

$$= \left[\cos\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial \varphi} + \left(-\frac{\sin\theta}{r\sin\varphi}\right)\frac{\partial}{\partial \theta}\right]\frac{\partial}{\partial x} + \left[\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\sin\theta\cos\varphi}{r}\frac{\partial}{\partial \varphi} + \frac{\cos\theta}{r\sin\varphi}\frac{\partial}{\partial \theta}\right]\frac{\partial}{\partial y} +$$

$$\left[\cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}\right]\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial r}\frac{\partial}{\partial x} = \cos\theta\sin\varphi\frac{\partial}{\partial r}\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \frac{\sin\theta}{\sin\varphi}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial \varphi}\frac{\partial}{\partial x} = \cos\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{\partial}{\partial \varphi}\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi}\frac{\partial}{\partial \varphi} - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi} - \frac{\sin\theta}{r}\frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\varphi}\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial \theta}\frac{\partial}{\partial x} = -\sin\theta\left(\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi}\right) + \cos\theta\frac{\partial}{\partial \theta}\left(\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi}\right) - \frac{1}{r\sin\varphi}\left(\cos\theta\frac{\partial}{\partial \theta} + \sin\theta\frac{\partial^2}{\partial \theta^2}\right)$$

$$\frac{\partial}{\partial r}\frac{\partial}{\partial y} = \sin\theta\left(\sin\varphi\frac{\partial^2}{\partial r^2} + \cos\varphi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \varphi}\right)\right) + \frac{\cos\theta}{\sin\varphi}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial \varphi}\frac{\partial}{\partial y} = \sin\theta\left(\left(\cos\varphi\frac{\partial}{\partial r} + \sin\varphi\frac{\partial^2}{\partial \varphi\partial r}\right) + \frac{1}{r}\left(-\sin\varphi\frac{\partial}{\partial \varphi} + \cos\varphi\frac{\partial^2}{\partial \varphi^2}\right)\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\varphi}\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial \theta}\frac{\partial}{\partial y} = \cos\theta\left(\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi}\right) + \sin\theta\frac{\partial}{\partial \theta}\left(\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi}\right) + \frac{1}{r\sin\varphi}\left(-\sin\theta\frac{\partial}{\partial \theta} + \cos\theta\frac{\partial^2}{\partial \theta^2}\right)$$

$$\frac{\partial}{\partial r}\frac{\partial}{\partial z} = \cos\varphi\frac{\partial^2}{\partial r^2} - \sin\varphi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \varphi}\right)$$

$$\frac{\partial}{\partial \varphi}\frac{\partial}{\partial z} = -\sin\varphi\frac{\partial}{\partial r} + \cos\varphi\frac{\partial^2}{\partial\varphi\partial r} - \frac{1}{r}\left(\cos\varphi\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial^2}{\partial\varphi^2}\right)$$

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r\sin\varphi}\left(\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\theta^2}$$

球对称 $\Rightarrow$ $\begin{cases} \dfrac{\partial}{\partial\varphi} = 0 \\ \dfrac{\partial}{\partial\theta} = 0 \end{cases}$

$$\Delta_3 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}\right)$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$