(两端固定)

1. 求弦振动方程 $u_{tt}-a^2u_{xx}=0$，$0<x<l$，$t>0$ 满足以下定解条件的解：$\begin{cases} u|_{x=0}=u_x|_{x=l}=0 \\ u|_{t=0}=\underbrace{\sin\frac{3}{2l}\pi x}_{\varphi(x)},\ u_t|_{t=0}=\underbrace{\sin\frac{5}{2l}\pi x}_{\psi(x)} \end{cases}$

解：令 $u(x,t)=X(x)T(t)$，则 $X(x)T''(t)-a^2X''(x)T(t)=0$

令 $X''(x)=-\lambda X(x)$，则 $X''(x)+\lambda X(x)=0$ (特征方程)，$\cancel{则\ T''(t)+a^2\lambda T(t)=0}$

$\because u|_{x=0}=u_x|_{x=l}=0$

$\therefore X(0)=X'(l)=0$

$\therefore \begin{cases} \lambda_n=\left[\frac{(2n+1)}{2l}\pi\right]^2,\ n\geq 0\ (特征值) \\ X_n(x)=\sin\left[\frac{(2n+1)}{2l}\pi x\right],\ n\geq 0\ (特征函数) \end{cases}$

令 $\varphi(x)=\sum_{n=0}^{\infty}\varphi_n X_n(x)$，则 $\varphi_n=\frac{2}{l}\int_0^l\varphi(s)X_n(s)ds=\frac{2}{l}\int_0^l\sin\frac{3}{2l}\pi s\sin\left[\frac{(2n+1)}{2l}\pi s\right]ds=\begin{cases}1 & n=1\\ 0 & n\neq 1\end{cases}$

令 $\psi(x)=\sum_{n=0}^{\infty}\psi_n X_n(x)$，则 $\psi_n=\frac{2}{l}\int_0^l\psi(s)X_n(s)ds=\frac{2}{l}\int_0^l\sin\frac{5}{2l}\pi s\sin\left[\frac{(2n+1)}{2l}\pi s\right]ds=\begin{cases}1 & n=2\\ 0 & n\neq 2\end{cases}$

令 $u(x,t)=\sum_{n=0}^{\infty}\cancel{T_n(t)}X_n(x)T_n(t)$，则 $\sum_{n=0}^{\infty}\cancel{T}X_n(x)T_n''(t)-a^2\sum_{n=0}^{\infty}\overset{(-\lambda_n)}{\cancel{(-\lambda)}}X_n(x)T_n(t)=0$

$\therefore \begin{cases}\cancel{T_n''(t)+a^2\lambda_nT_n(t)=0}\\ \cancel{T_n(0)=}\end{cases}$ $\because u(x,0)=\sum_{n=0}^{\infty}X_n(x)T_n(0)$，$u_t(x,0)=\sum_{n=0}^{\infty}X_n(x)T_n'(0)$

$\therefore T_n(0)=\varphi_n$，$T_n'(0)=\psi_n$

$\therefore \begin{cases}T_n''(t)+a^2\lambda_nT_n(t)=0\\ T_n(0)=\varphi_n,\ T_n'(0)=\psi_n\end{cases}$

令 $T_n(t)=C_n\cos\frac{n\pi a}{l}t+d_n\sin\frac{n\pi a}{l}t$，则 $T_n(0)=C_n=\varphi_n$，$T_n'(0)=\psi_n=d_n\frac{n\pi a}{l}$，则 $d_n=\frac{\psi_n l}{n\pi a}$

$\therefore T_n(t)=\varphi_n\cos\frac{n\pi a}{l}t+\frac{\psi_n l}{n\pi a}\sin\frac{n\pi a}{l}t$

$\therefore u(x,t)=\sum_{n=0}^{\infty}\varphi_n\cos\frac{n\pi a}{l}t\sin\left[\frac{(2n+1)}{2l}\pi x\right]+\sum_{n=0}^{\infty}\frac{\psi_n l}{n\pi a}\sin\frac{n\pi a}{l}t\sin\left[\frac{(2n+1)}{2l}\pi x\right]$

$\therefore u(x,t)=\cos\frac{\pi a}{l}t\sin\frac{3\pi}{2l}x+\frac{l}{2\pi a}\sin\frac{2\pi a}{l}t\sin\frac{3\pi}{2l}x$

$\therefore u(x,t)=\sin\frac{3\pi x}{2l}\cos\frac{\pi a t}{l}+\frac{l}{2\pi a}\sin\frac{5\pi x}{2l}\sin\frac{2\pi a t}{l}$

2. 求解 $\begin{cases} u_{tt} = a^2 u_{xx},\ 0 \le x \le l & (\text{两端固定}) \\ u_x(t,0)=0,\ u_x(t,l)=0 \\ u(0,x)=0,\ u_t(0,x)=Ae^{-\alpha x} \end{cases}$

$\qquad\qquad\qquad\qquad\qquad \varphi(x) \qquad\qquad \psi(x)$

解：令 $u(x,t)=X(x)T(t)$，则 $X(x)T''(t)=a^2X''(x)T(t)$

令 $X''(x)=-\lambda X(x)$，则 $X''(x)+\lambda X(x)=0$（特征方程）

$\because u_x(t,0)=u_x(t,l)=0$

$\therefore X'(0)=X'(l)=0$

$\therefore \begin{cases} \lambda_n=\left(\frac{n\pi}{l}\right)^2,\ n\ge 0 & (\text{特征值}) \\ X_n(x)=\cancel{\sin}\cos\frac{n\pi}{l}x,\ n\ge 0 & (\text{特征函数}) \end{cases}$

令 $\varphi(x)=\sum_{n=0}^{\infty}\varphi_n X_n(x)$，则 $\varphi_n=\frac{2}{l}\int_0^l \varphi(s)X_n(s)ds=0$

令 $\psi(x)=\sum_{n=0}^{\infty}\psi_n X_n(x)$，则 $\psi_n=\frac{2}{l}\int_0^l \cancel{Ae^{-\alpha s}}\psi(s)X_n(s)ds=\frac{2A}{l}\int_0^l \cancel{A}e^{-\alpha s}\cos\frac{n\pi}{l}s\,ds$

$\int_0^l e^{-\alpha s}\cos\frac{n\pi}{l}s\,ds=\int_0^l\left(-\frac{1}{\alpha}\right)\cos\frac{n\pi}{l}s\,de^{-\alpha s}=\left(-\frac{1}{\alpha}\right)\cos\frac{n\pi}{l}s\,e^{-\alpha s}\Big|_0^l-\int_0^l e^{-\alpha s}d\left(-\frac{1}{\alpha}\right)\cos\frac{n\pi}{l}s=\frac{1-\cos n\pi\, e^{-\alpha l}}{\alpha}-\frac{n\pi}{\alpha l}\int_0^l e^{-\alpha s}\sin$

$\qquad=\frac{1-\cos n\pi\, e^{-\alpha l}}{\alpha}-\left(\frac{n\pi}{\alpha l}\right)^2\int_0^l e^{-\alpha s}\cos\frac{n\pi}{l}s\,ds$

$\int_0^l e^{-\alpha s}\cos\frac{n\pi}{l}s\,ds=\frac{\frac{1-\cos n\pi\, e^{-\alpha l}}{\alpha}}{1+\left(\frac{n\pi}{\alpha l}\right)^2}=\frac{1-(-1)^n e^{-\alpha l}}{\alpha\,\frac{\alpha^2l^2+n^2\pi^2}{\alpha^2l^2}}=\frac{\alpha l^2\left[1-(-1)^n e^{-\alpha l}\right]}{\alpha^2l^2+n^2\pi^2}$

$\therefore \psi_n=\frac{2A\alpha l}{\alpha^2l^2+n^2\pi^2}\left[1-(-1)^n e^{-\alpha l}\right]$

由第1题得：$T_n(t)=\varphi_n\cos\frac{n\pi a}{l}t+\frac{\psi_n l}{n\pi a}\sin\frac{n\pi a}{l}t$，$\psi=\frac{2A\alpha l^2\left[1-(-1)^n e^{-\alpha l}\right]}{\alpha^2l^2 n\pi a+n^3\pi^3 a}\sin\frac{n\pi a}{l}t$

$u(x,t)=\sum_{n=0}^{\infty}\frac{2A\alpha l^2\left[1-(-1)^n e^{-\alpha l}\right]}{\alpha^2l^2 n\pi a+n^3\pi^3 a}\sin\frac{n\pi a}{l}t\cos\frac{n\pi}{l}x$