A Practical Approach: Exercise 1

- $\lim_{x\to 3} \frac{1}{1+e^{x-3}}$
- $\lim_{x\to -1} \sqrt{x^2+4}$
- $\lim_{x\to 1} \left(\frac{x-2}{x^2-5}\right)^{\frac{x}{2}}$
- $\lim_{x\to 3} \ln\left(\frac{x-2}{2x-5}\right)$
- $\lim_{x\to -\infty}(x^2+e^{-x})$
- $\lim_{x\to +\infty} x^2 \ln x$
- $\lim_{x\to 2^+} \frac{1}{(x-2)(e^2-e^x)}$
- $\lim_{x\to 0^+} \frac{\ln x}{x}$
- $\lim_{x\to 0^+} x \ln x$

Simple Rules for Limits [SHS 6.5]

If A and B are real numbers:

RULES FOR LIMITS

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, then

(a) $\lim_{x \to a} \big(f(x) \pm g(x)\big) = A \pm B$

(b) $\lim_{x \to a} \big(f(x) \cdot g(x)\big) = A \cdot B$

(c) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$ (if $B \neq 0$)

(d) $\lim_{x \to a} \big(f(x)\big)^r = A^r$ (if A^r is defined and r is any real number)

Further Rules (see also SHS 7.9)

(e) $\lim_{x \to a} f(x)^{g(x)} = A^B$, with $A, B \neq 0, A, B \neq 1, \ , A, B \neq \pm\infty$.

(f) if f continuous (*) at B, then $\lim_{x \to a} f(g(x)) = f(B)$.

(*) See later the definition of continuous function in the book.

Rules for Limits, extended to $0^+, 0^-, +\infty, -\infty$

- 0^+ (0^-) is a strictly positive (negative) number closer and closer to 0
- $+\infty$ is a quantity bigger than any finite threshold M, with $M > 0$
- $-\infty$ is a quantity smaller than any finite threshold $-M$, with $M > 0$

Then the following rules hold:

- $+\infty + \infty = +\infty$, $-\infty - \infty = -\infty$
- $(+\infty)(+\infty) = +\infty,\ (-\infty)(-\infty) = +\infty,\ (+\infty)(-\infty) = -\infty$
- $0^+ * 0^+ = 0^+, \quad 0^- * 0^- = 0^+, \quad 0^+ * 0^- = 0^-,$
- $\frac{1}{0^+} = +\infty, \quad \frac{1}{0^-} = -\infty, \quad \frac{1}{+\infty} = 0^+, \quad \frac{1}{-\infty} = 0^-$

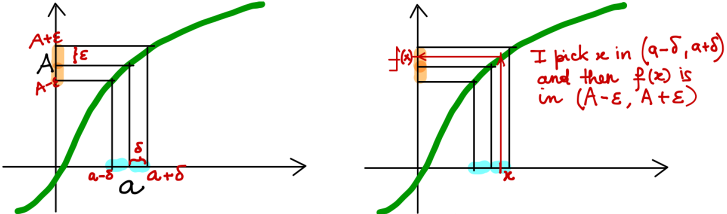
while the following expressions are indefinite (Indefinite Forms)

$$+\infty - \infty = ?, \quad \frac{0}{0} = ?, \quad \frac{\infty}{\infty} = ?, \quad 0 \cdot \infty = ?, \quad 1^\infty = ?, \quad 0^0 = ?$$

Exercise 2

- Spot all of the rules of limits that are used to compute the limits in Exercise 1.
- Solve all exercises for section 6.5 in the textbook.

(A glance to the) Formal Definition of Limit [SHS 7.9]



- $|x-a|<\delta \iff x\in(a-\delta,a+\delta)$
  $(a-\delta,a+\delta)$ is called a (symmetric) neighborhood of a;
- $|f(x)-A|<\varepsilon \iff f(x)\in(A-\varepsilon,A+\varepsilon)$
  $(A-\varepsilon,A+\varepsilon)$ is called a (symmetric) neighborhood of A.

Definition

By $\lim_{x\to a} f(x)=A$ we mean: for all $\varepsilon>0$, there exists $\delta>0$ such that $|x-a|<\delta$ $(x\neq a)$ implies $|f(x)-A|<\varepsilon$.

That is: if we set one threshold of error ε, then we may find a corresponding threshold δ so that x closer to a than δ, then $f(x)$ is closer to A than ε.

Exercise 3 - First tricks when $x \to \pm\infty$

(Hint: put in evidence the strongest power)

- $\lim_{x \to +\infty} (x^2 - x^3)$
- $\lim_{x \to +\infty} \frac{\ln x + 1}{2 + 3 \ln^2 x}$
- $\lim_{x \to -\infty} \frac{x^2 + 1}{1 - 3x^3}$

Exercise 3 - continued

(derationalize)

- $\lim_{x \to +\infty} (\sqrt{x+1} - \sqrt{x})$

- $\lim_{x \to +\infty} (\sqrt{x^2+1} - \sqrt{x})$

Exercise 4 - Change of Variables

Change of Variables

$$\lim_{x\to a} g(f(x)) = \lim_{t\to f(a)} g(t)$$

- $\lim_{x\to 0^+} \frac{e^x}{e^x+1} =$

- $\lim_{x\to +\infty} \frac{\sqrt{e^{2x}+1}}{8e^x-1}$

Exercise 4 - Change of Variables

- $\lim_{x \to -\infty} \sqrt{1 - x}$
- $\lim_{y \to -\infty} \frac{1}{y} \sqrt{\frac{1+y^3}{y}}$

Comparing infinities of different strength 1/3

Definition

Assume $\lim_{x\to+\infty} f(x) = +\infty$, and $\lim_{x\to+\infty} g(x) = +\infty$. Then if

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 0,$$

we say that *$g(x)$ overtakes $f(x)$*, for $x \to +\infty$, ora that *g is an infinity of greater strength than f*.

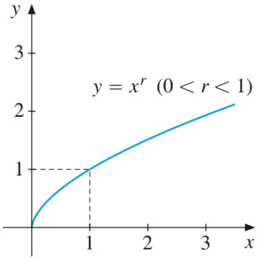
Example 1: Powers. The highest the strongest.

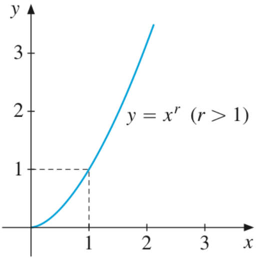
$$\lim_{x\to+\infty} \frac{x^a}{x^b} = \lim_{x\to+\infty} x^{a-b} = \begin{cases} 0, & a < b \\ +\infty, & a > b \\ 1, & a = b. \end{cases}$$

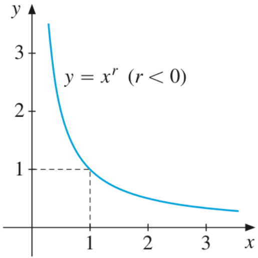
Example 2: Polynomials. Behaviour determined by the highest power.

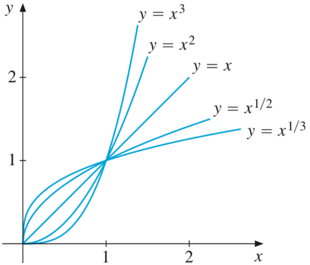
$$\lim_{x\to+\infty} \frac{5x+2}{x-1} = 5, \quad \lim_{x\to+\infty} \frac{x^2+2}{x-1} = +\infty, \quad \lim_{x\to-\infty} \frac{x^2+2}{x^3-1} = 0$$

Graph of Power Functions

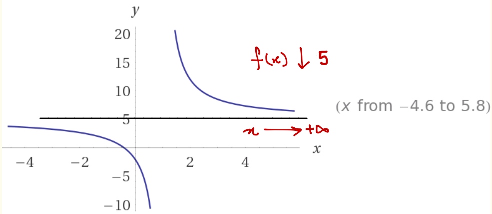




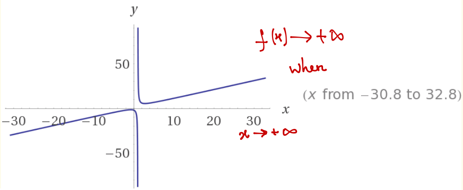




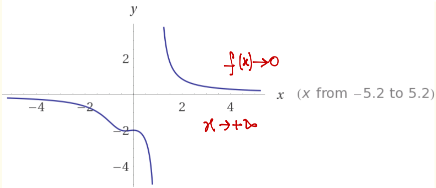
Graphs



$$\frac{5x+2}{x-1} \longrightarrow 5, \text{ as } x \to +\infty$$



$$\frac{x^2+2}{x-1} \longrightarrow +\infty, \text{ as } x \to +\infty$$



$$\frac{x^2+2}{x^3-1} \longrightarrow 0, \text{ as } x \to +\infty$$

Comparing infinities of different strength 2/3

Logarithms an Powers

For $a > 1$, $b > 0$, $c > 0$

$$\lim_{x \to +\infty} \frac{\log_a^c x}{x^b} = 0$$

Exponentials and Powers

For $a > 1$, for $b > 0$

$$\lim_{x \to +\infty} \frac{x^b}{a^x} = 0$$

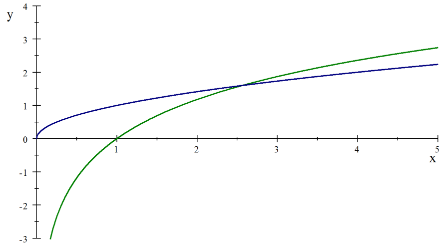
Summing up:

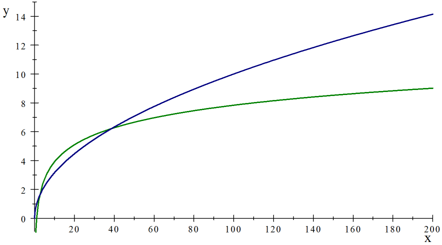
$$\log_a x << x^b << a^x,$$

$$\text{for any } \; a > 1, b > 0, \; \text{when } x \to +\infty,$$

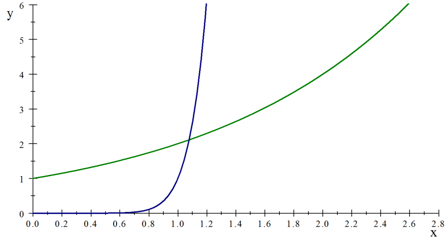
Graphs

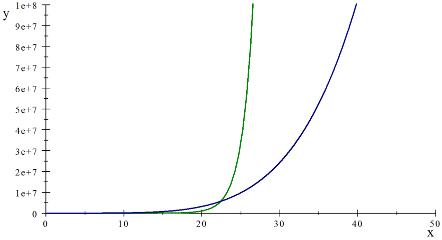
Comparison between $\log_{1.8} x$ in green, $x^{0.5}$ in blue.





Comparison between 2^x in green, x^5 in blue.





Comparing infinities of different strength 3/3

Idea to resolve indefinite forms of type $\frac{\infty}{\infty}$:

- factor the strongest infinity at numerator and denominator
- the limit is that of the ratio of such strongest infinities, consistently with the previous rules

Exercise 5 I

Show, justifying accurately all steps, that:

1. $\lim_{x \to +\infty} \frac{(\ln x)^{4000}}{x^{0.000001}} = 0$
2. $\lim_{x \to +\infty} \frac{(\ln x)^{4000}}{1.001^x} = 0$
3. $\lim_{x \to +\infty} \frac{\ln^4 x + 2x^4 - x^2 + 1}{14x^4 + 2\ln^5 x - 2^{-x}} = \frac{1}{7}$
4. $\lim_{x \to +\infty} \frac{\ln^4 x + 1}{\log_{34} x - 2^{-x}} = +\infty$

Exercise 5 II

5. $\lim_{x \to +\infty} \frac{x}{e^{\ln x}} = 1$

6. $\lim_{x \to +\infty} \frac{2^{x+1} - x^5}{e^x - x^2} = 0$

7. $\lim_{x \to +\infty} \frac{e^{x+1} - x^5}{e^{x^2} - x^2} = 0$

Exercise 6

Consider the function $f(x) = \frac{4-2x}{3x-1}$. Find: (a) natural domain; (b) sign; (c) limits at the boundary of the domain; (d) first derivative and monotonicity; (e) plot the graph.

Indefinite Forms of Type $\frac{0}{0}$

Exercise 7

Compute the following limits of ratios of polynomials:

(a) $\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2}$
(Hint: factor $(x-2)$ both at numerator and denominator, simplify.)

(b) $\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3}$
(Hint: factor $(x-1)$ twice, both at num/den, simplify.)

(c) $\lim_{x\to -1} \frac{x^2-1}{x^2+3x+2}$

(d) $\lim_{x\to 1} \left(\frac{1}{1-x} - \frac{3}{1-x^3}\right)$

Some notable $\frac{0}{0}$ and 1^∞ limits

The following limits are given as known

Theorem: Notable limits

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1; \qquad \lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right)^x = e$$

Corollary/Exercise 8

Prove that

(a) $\displaystyle\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$

(b) $\displaystyle\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

(c) $\displaystyle\lim_{x\to\pm\infty} \left(1 + \frac{a}{x}\right)^x = e^a$

Exercise 9

Show that:

- $\lim_{x\to 0} \frac{e^x-1}{e^{4x}-1} = \frac{1}{4}$

- $\lim_{x\to 1} \frac{\ln(x)}{x-1} = 1$

- $\lim_{x\to +\infty} \left(1+\frac{2}{x+1}\right)^x$.

L'Hopital's Rule [SHS 7.12]

Theorem

Assume f and g are differentiable functions in (α, β), and $a \in (\alpha, \beta)$:

- (SHS Theorem 7.12.1) if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, $g'(x) \neq 0$ at all $x \in (\alpha, \beta)$, and there exists $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then:
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$
- if $\lim_{x\to a} f(x) = \pm\infty$, and $\lim_{x\to a} g(x) = \pm\infty$, and there exists $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then:
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$
- The same results hold for $a = +\infty$ (respectively, $a = -\infty$) and f and g differentiable on a positive (respectively, negative) half line, or for $x \to \alpha^+$, and $x \to \beta^-$ (see one-sided limits).

Exercise 10

- Solve all exercises of section 7.9 in textbook.
- Confirm the results of the previous exercises using L'Hopital's Rule.
- Solve all exercises of section 7.12 in textbook.

One-sided limits [SHS 7.9]

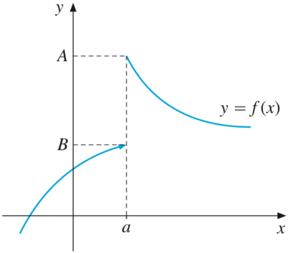
Definition

Left limit as x tends to a (or "limit as x tends to a from the left")

$$\lim_{x \to a^-} f(x) \equiv \lim_{\substack{x \to a \\ x < a}} f(x)$$

Right limit as x tends to a (or "limit as x tends to a from the right")

$$\lim_{x \to a^+} f(x) \equiv \lim_{\substack{x \to a \\ x > a}} f(x)$$



Examples

Left and right limit may or may not exist separately, and may or may not be equal when they exist.

- $\lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^+} \frac{1}{x} = +\infty$
- $\lim_{x\to 0^-} xe^{\frac{1}{x}} =$

  $\lim_{x\to 0^+} xe^{\frac{1}{x}} =$

In such case the (overall) limit DOES NOT exist.
Conversely, if they do exist, they are finite and they are equal, the limit exists and coincide with their common value

Theorem

$$\exists \lim_{x\to a} f(x) = A \Leftrightarrow \begin{cases} \exists \lim_{x\to a^-} f(x) = A \\ \exists \lim_{x\to a^+} f(x) = A \end{cases}$$

Example - optional

Examples of a function not having left and right limits $f(x) = sin\frac{1}{x}$