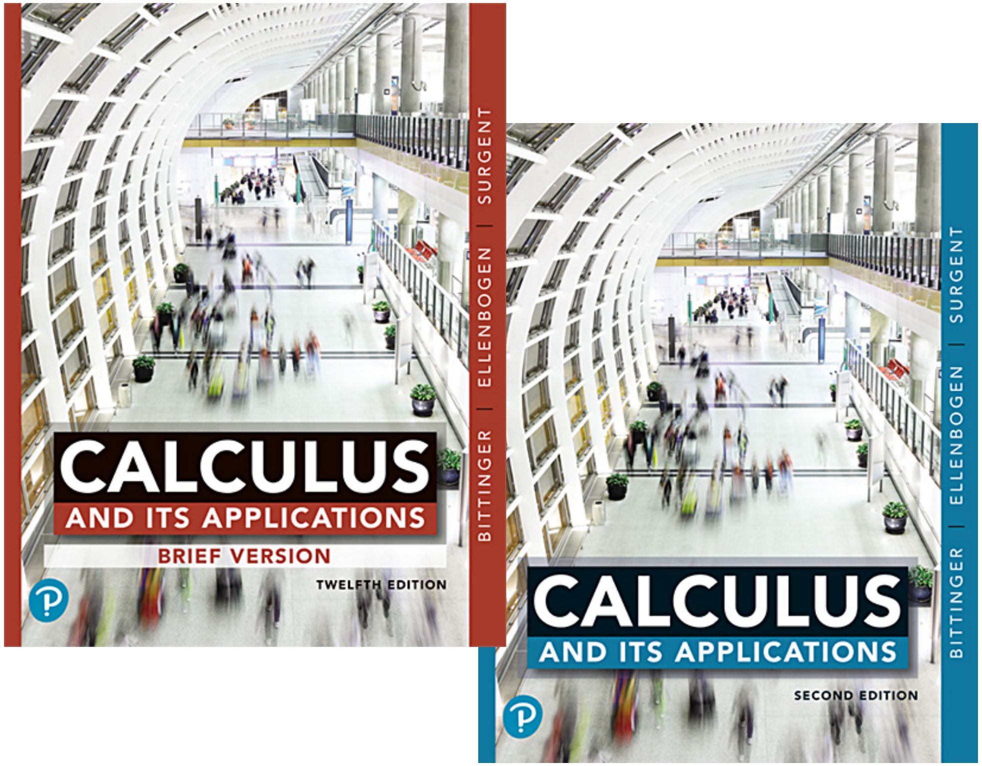
Chapter 3

Applications of Differentiation



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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

OBJECTIVE

- Find relative extrema of a continuous function using the First-Derivative Test.
- Sketch graphs of continuous functions.

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

DEFINITIONS:

A function f is **increasing** over I if, for every a and b in I, if $a < b$, then $f(a) < f(b)$.

(If the input a is less than the input b, then the output for a is less than the output for b.

A function f is **decreasing** over I if, for every a and b in I, if $a < b$, then $f(a) > f(b)$.

(If the input a is less than the input b, then the output for a is greater than the output for b.)

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 1

Let f be differentiable over an open interval I.

If $f'(x) > 0$ for all x in an interval I, then f is increasing over I.

If $f'(x) < 0$ for all x in an interval I, then f is decreasing over I.

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

DEFINITION:

A **critical value** of a function f is any number c in the domain of f for which the tangent line at $(c, f(c))$ is horizontal or for which the derivative does not exist.

That is, c is a critical value if $f(c)$ exists and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

If c is a critical value of a function f, then $(c, f(\mathrm{c}))$ is a **critical point**.

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

DEFINITIONS:

Let I be the domain of f:

$f(c)$ is a **relative minimum** if there exists within I an open interval I_1 containing c such that $f(c) \leq f(x)$ for all x in I_1;

and

$f(c)$ is a **relative maximum** if there exists within I an open interval I_2 containing c such that $f(c) \geq f(x)$ for all x in I_2.

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 2

If a function f has a relative extreme value $f(c)$ on an open interval; then c is a critical value, and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 3: The First-Derivative Test for Relative Extrema

For any continuous function f that has exactly one critical value c in an open interval (a, b);

F1. f has a relative minimum at c if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b). That is, f is decreasing to the left of c and increasing to the right of c.

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 3: The First-Derivative Test for Relative Extrema (continued)

F2. f has a relative maximum at c if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b). That is, f is increasing to the left of c and decreasing to the right of c.

F3. f has neither a relative maximum nor a relative minimum at c if $f'(x)$ has the same sign on (a, c) and (c, b).

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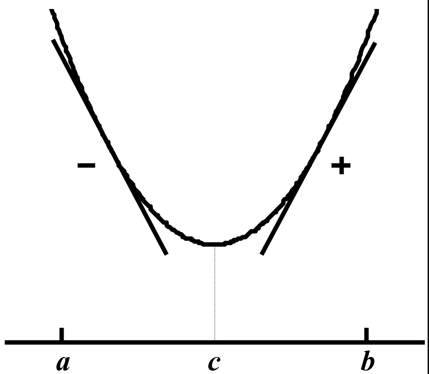
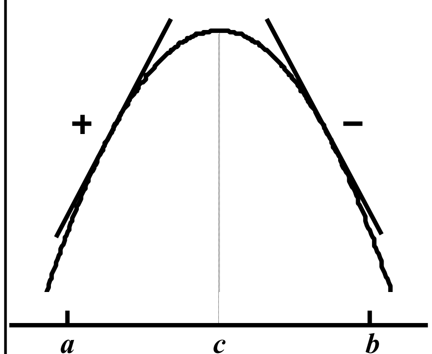
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	Relative minimum	–	+	Decreasing on (a, c); increasing on (c, b)
	Relative maximum	+	–	Increasing on (a, c); decreasing on (c, b)

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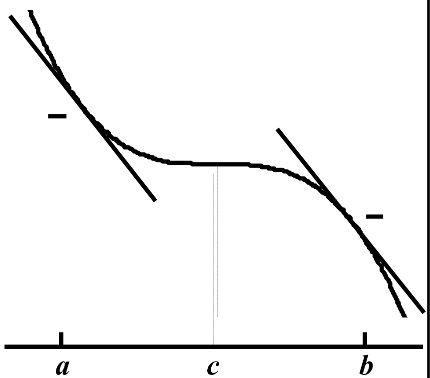
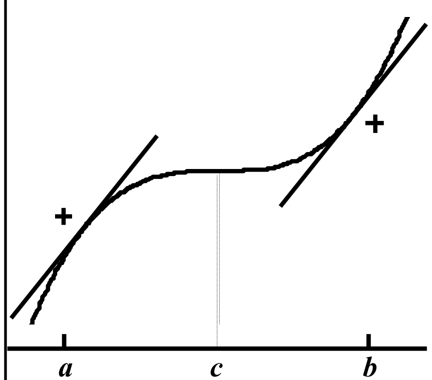
Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	No relative maxima or minima	–	–	Decreasing on (a, b)
	No relative maxima or minima	+	+	Increasing on (a, b)

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1: Graph the function f given by

$$f(x) = 2x^3 - 3x^2 - 12x + 12$$

and find the relative extrema.

Suppose that we are trying to graph this function but do not know any calculus. What can we do? We can plot a few points to determine in which direction the graph seems to be turning. Let's pick some x-values and see what happens.

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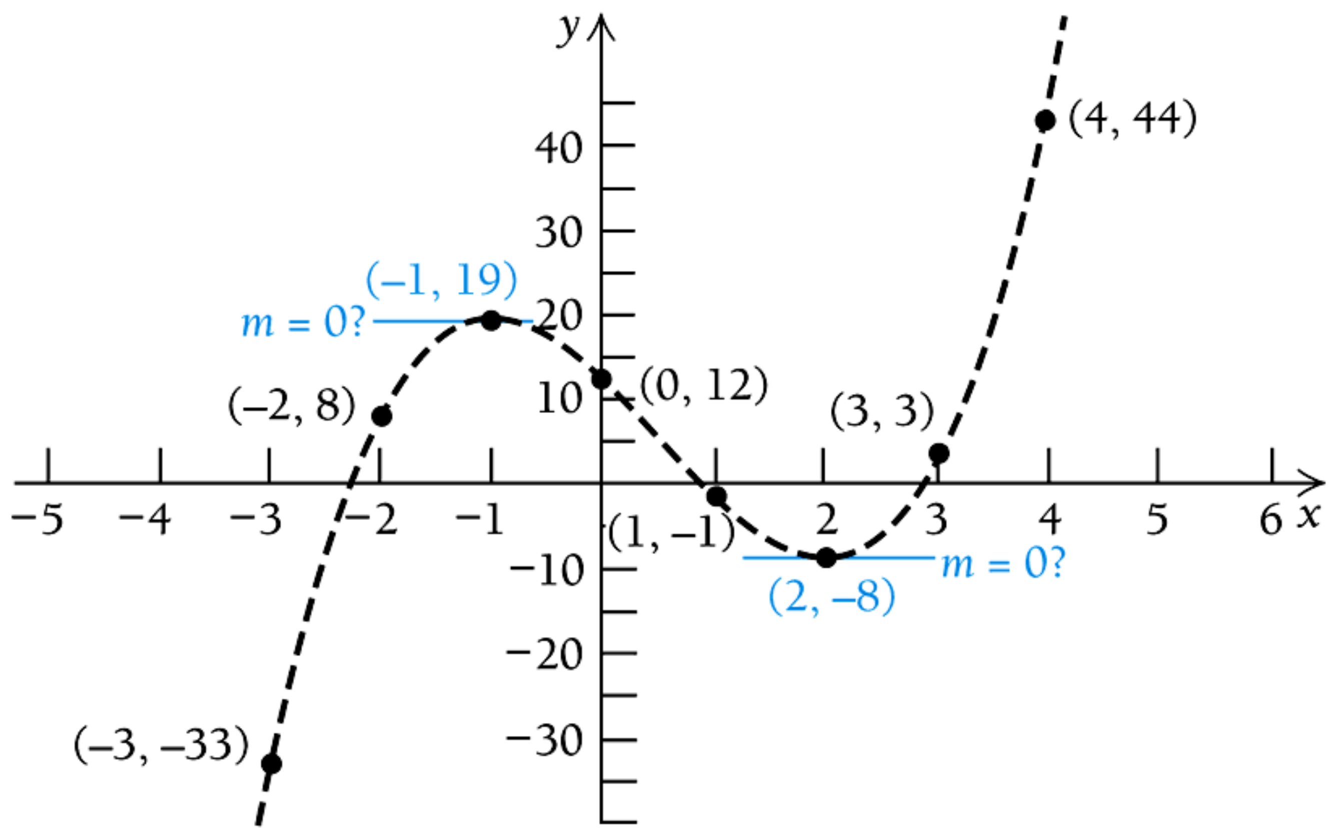
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

x	$f(x)$
-3	-33
-2	8
-1	19
0	12
1	-1
2	-8
3	3
4	44

Example 1 (continued):



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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

We can see some features of the graph from the sketch. Now we will calculate the coordinates of these features precisely.

1. Find a general expression for the derivative.

$$f'(x) = 6x^2 - 6x - 12$$

2. Determine where $f'(x)$ does not exist or where $f'(x) = 0$.

Since $f'(x)$ is a polynomial, there is no value where $f'(x)$ does not exist. So, the only possibilities for critical values are where $f'(x) = 0$.

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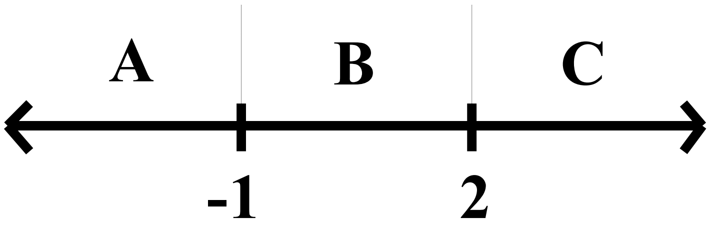
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

$$
\begin{aligned}
6x^2 - 6x - 12 &= 0 \\
x^2 - x - 2 &= 0 \\
(x-2)(x+1) &= 0
\end{aligned}
$$

$$x = 2 \quad \text{or} \quad x = -1$$

These two critical values partition the number line into 3 intervals: A $(-\infty, -1)$, B $(-1, 2)$, and C $(2, \infty)$.



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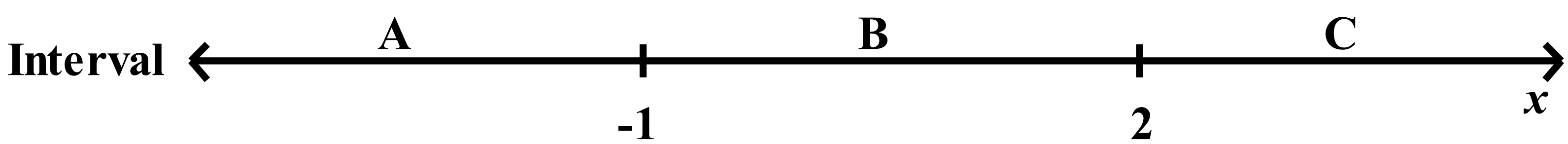
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

3rd analyze the sign of $f'(x)$ in each interval.



Test Value	$x = -2$	$x = 0$	$x = 4$
Sign of $f'(x)$	$+$	$-$	$+$
Result	f is increasing on $(-\infty, -1)$	f is decreasing on $(-1, 2)$	f is increasing on $(2, \infty)$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (concluded):

Therefore, by the First-Derivative Test,

f has a relative maximum at $x = -1$ given by

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 12 = 19$$

Thus, (–1, 19) is a relative maximum.

And f has a relative minimum at $x = 2$ given by

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 12 = -8.$$

Thus, (2, –8) is a relative minimum.

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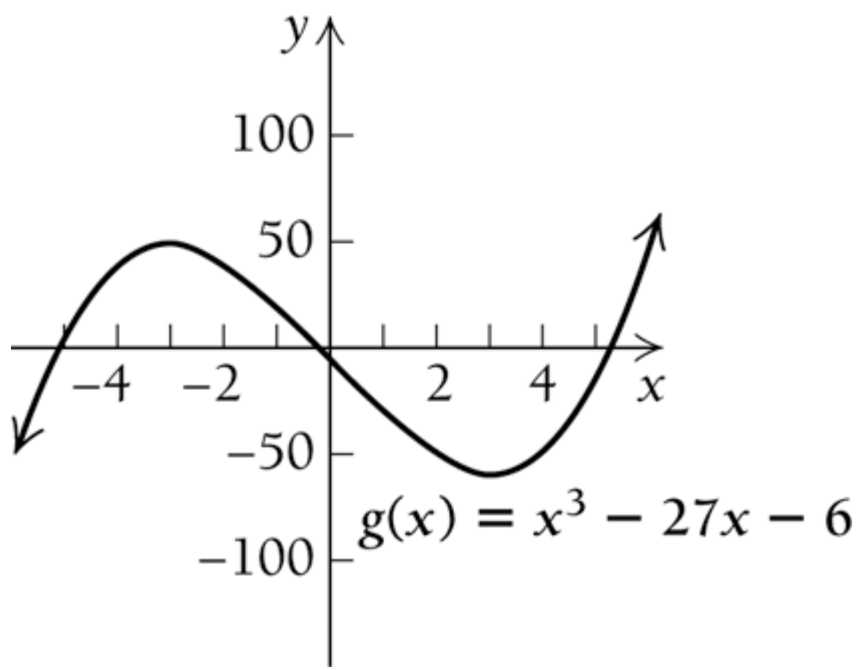
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 1

Graph the function g given by $g(x) = x^3 - 27x - 6$, and find the relative extrema.



Relative Maximum at: $(-3, 48)$

Relative Minimum at: $(3, -60)$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2: Find the relative extrema for the Function $f(x)$ given by

$$f(x) = (x-2)^{2/3} + 1.$$

Then sketch the graph.

First find $f'(x)$.

$$f'(x) = \frac{2}{3}(x-2)^{-1/3}$$

$$f'(x) = \frac{2}{3\sqrt[3]{x-2}}$$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

Then, find where $f'(x)$ does not exist or where $f'(x) = 0$.

Note that $f'(x)$ does not exist where the denominator equals 0. Since the denominator equals 0 when $x = 2$, $x = 2$ is a critical value.

$f'(x) = 0$ where the numerator equals 0. Since $\dfrac{2}{3\sqrt[3]{x-2}} \neq 0$, there is no solution for $f'(x) = 0$.

Thus, $x = 2$ is the only critical value.

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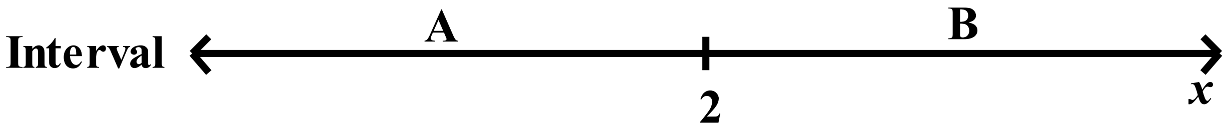
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

Next, $x = 2$ partitions the number line into 2 intervals: A $(-\infty, 2)$ and B $(2, \infty)$. So, analyze the signs of $f'(x)$ in both intervals.



Test Value	$x = 0$	$x = 3$
Sign of $f'(x)$	$-$	$+$
Result	f is decreasing on $(-\infty, 2)$	f is increasing on $(2, \infty)$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

Therefore, by the First-Derivative Test,

f has a relative minimum at $x = 2$ given by

$$f(2) = (2-2)^{2/3} + 1 = 1$$

Thus, (2, 1) is a relative minimum.

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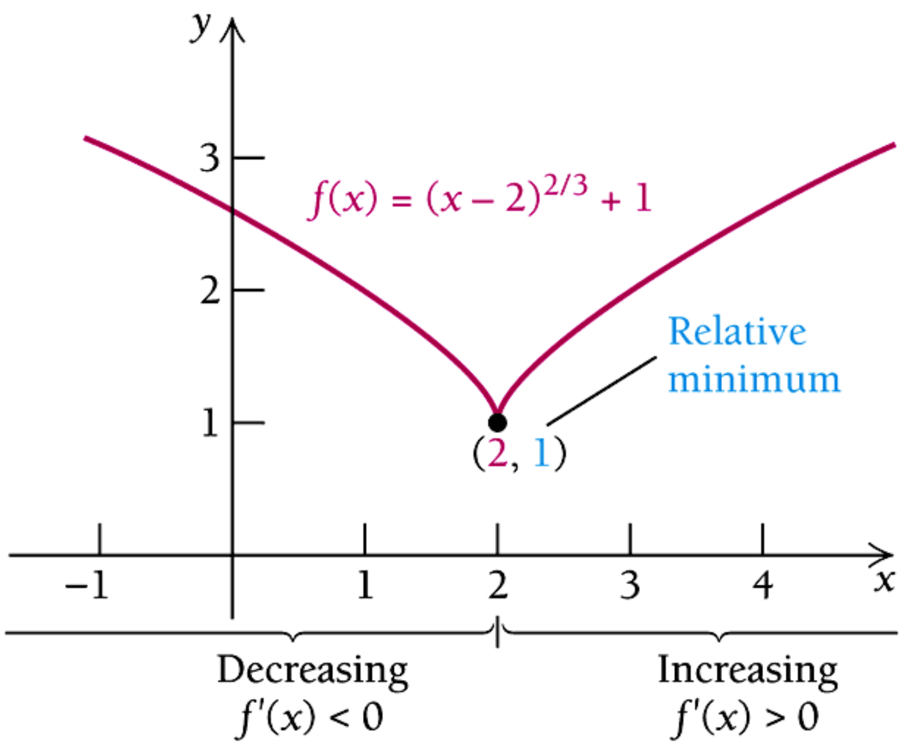
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (concluded):

We use the information obtained to sketch the graph below, plotting other function values as needed.



x	$f(x)$, approximately
−1	3.08
−0.5	2.84
0	2.59
0.5	2.31
1	2
1.5	1.63
2	1
2.5	1.63
3	2
3.5	2.31
4	2.59

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2

Find the relative extrema of the function h given by

$h(x) = x^4 - \frac{8}{3}x^3$. Then sketch the graph.

First find $h'(x)$: $h'(x) = 4 \cdot x^3 - 3 \cdot \frac{8}{3}x^2$

$$h'(x) = 4x^3 - 8x^2$$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2 Continued

Next find where $h'(x)$ does not exist or where $h'(x)=0$.

$$h'(x) = 4x^3 - 8x^2 = 0$$

$$4h^2(h-2) = 0$$

So $h'(x)=0$ when $h=0$ and $h=2$.

Thus $h=0$ and $h=2$ are the critical values.

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2 Continued

Third, $h = 0$ and $h = 2$ partitions the number line into three intervals: A$(-\infty, 0)$, B$(0, 2)$, and C$(2, \infty)$. So analyze the signs of $h'(x)$ in all three intervals.

Interval	A	B	C
Test Value	$h = -2$	$h = 1$	$h = 3$
Sign of $h'(x)$	–	–	+
Result	h is decreasing	h is decreasing	h is increasing

Thus, there is a minimum at $h = 2$. Therefore, $\left(2, \frac{-16}{3}\right)$ is a minimum.

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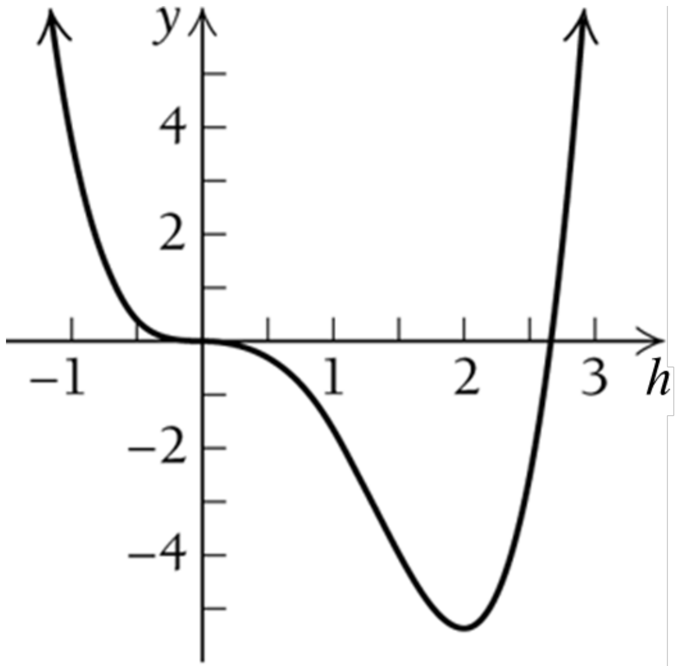
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2 Concluded

From the information we have gathered, the graph of $h(x)$ looks like:



$h(x) = x^4 - \frac{8}{3}x^3$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 3

Find the relative extrema of the function g given by $g(x) = 3 - x^{1/3}$. Then sketch the graph.

First find $g'(x)$: $\quad g'(x) = -\frac{1}{3}x^{\frac{1}{3}-1}$

$$g(x) = -\frac{1}{3x^{\frac{2}{3}}}$$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 3 Continued

Second, find where $g'(x)$ does not exist or where $g'(x) = 0.$

Note: $g'(x)$ does not exist when the denominator = 0. So $g'(x)$ does not exist when $x = 0$. Also, there is no value of x that makes $g'(x) = 0.$

Thus there is a critical value at $x = 0$.

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 3 Continued

Third, $x = 0$ partitions the number line into two intervals: $A(-\infty, 0)$ and $B(0, \infty)$. So analyze the signs of $g'(x)$ for both intervals.

Interval	$A(-\infty, 0)$	$B(0, \infty)$
Test Value	$x = -1$	$x = 1$
Sign of $g'(x)$	+	+
Results	g is increasing	g is increasing

Thus there is no extrema for $g(x)$.

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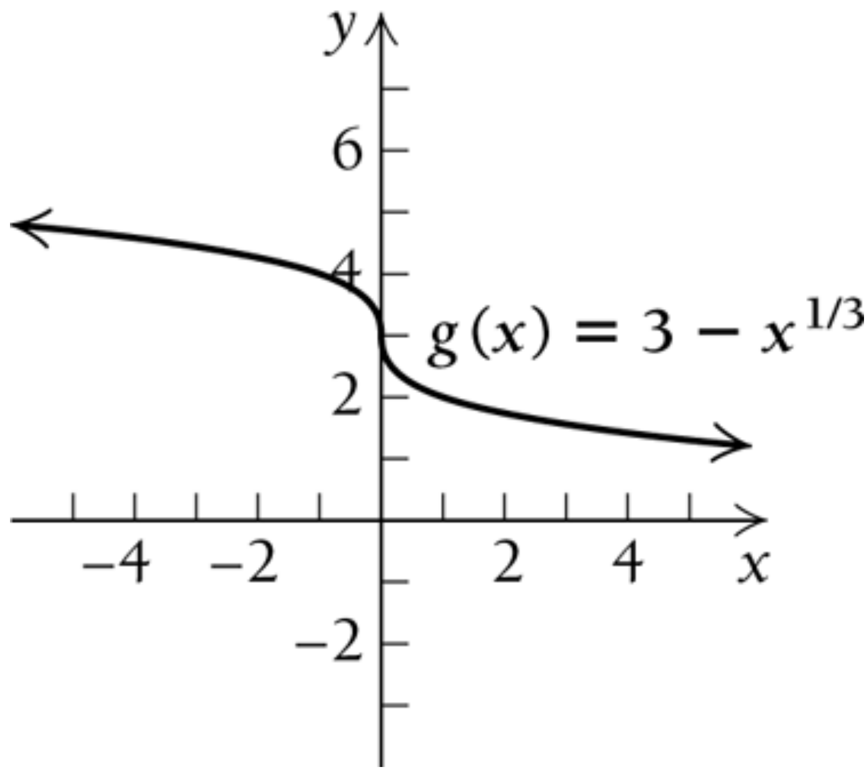
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 3 Concluded

Using the information gathered, the graph of $g\left(x\right)$ looks like:



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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3: Find any relative extrema and sketch the graph of the function g given by $g(x) = x^2 e^{-x}$.

Solution: To find the critical values, we must find $g'(x)$.

$$g'(x) = x^2(-e^{-x}) + 2xe^{-x} \quad \text{By the Product Rule}$$
$$= xe^{-x}(2 - x) \quad \text{Factoring}$$

Next, we need to find where $g'(x) = 0$ or where g' does not exist. Since $g'(x)$ can be evaluated for any real number x, the only candidates for critical values are those where $g'(x) = 0$: $xe^{-x}(2 - x) = 0$ Since $e^{-x} > 0$ for all x, we get:

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3 continued:

$$xe^{-x}(2-x)=0$$

$$x=0 \quad \text{or} \quad 2-x=0$$
$$x=2$$

The critical values are 0 and 2. We use these values to divide the *x*-axis into three intervals: $(-\infty,0),(0,2),\text{and}\,(2,\infty)$.

Interval	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
Test Value	$x=-1$	$x=1$	$x=3$
Sign of $g'(x)$	$g'(-1)<0$	$g'(1)>0$	$g'(3)<0$
Results	g is decreasing	g is increasing	g is decreasing

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3 continued:

By the First Derivative Test, the change in the sign of $g'(x)$ from negative to positive indicates a relative minimum at x = 0 while the change in sign of $g'(x)$ from positive to negative indicates a relative maximum at x = 2.

Evaluating $g(x)$ at these x-values, we have:

$g(0) = 0^2 e^{-0} = 0 \cdot 1 = 0.$ So, (0,0) is a relative minimum point.

$g(2) = 2^2 e^{-2} = 4e^{-2}.$ So, $(2, 4e^{-2})$ is a relative maximum point.

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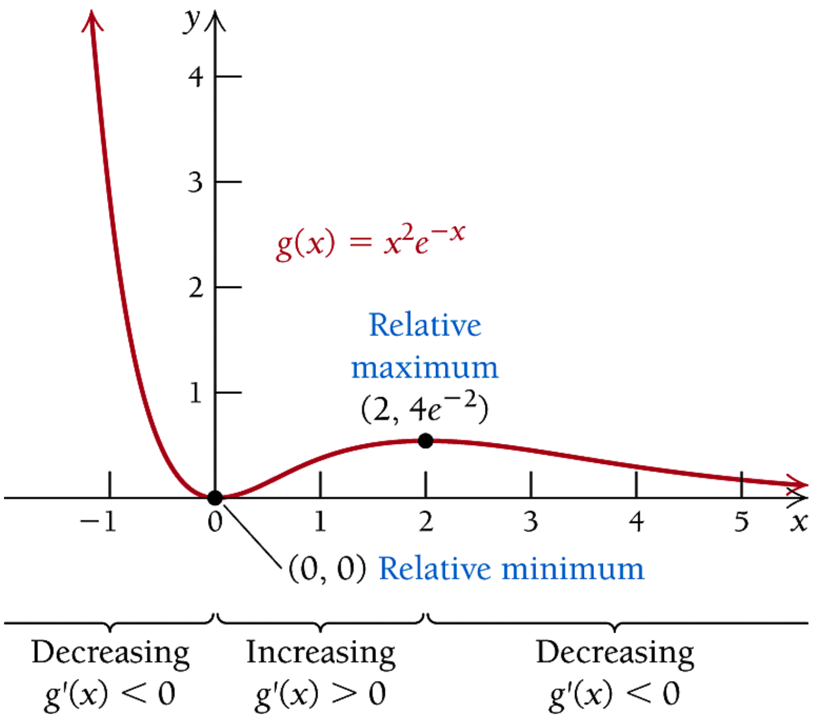
3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3 concluded:

We use this information, with other points found using a calculator, to sketch the graph of g below:

x	$g(x)$, approximately	
-2	29.556	
-1	2.718	
0	0	Relative minimum
1	0.368	
2	0.541	Relative maximum
3	0.448	
4	0.293	



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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Section Summary

- A function f is *increasing* over an interval I if, for all a and b such that $a < b, f(a) < f(b)$. Equivalently, the slope of the secant line connecting a and b is positive:

$$\frac{f(b)-f(a)}{b-a} > 0$$

- A function f is *decreasing* over an interval I if, for all a and b in such that $a < b, f(a) > f(b)$. Equivalently, the slope of the secant line connecting a and b is negative:

$$\frac{f(b)-f(a)}{b-a} < 0$$

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Section Summary Continued

- Using the first derivative, a function is *increasing* over an open interval I if, for all x in I, the slope of the tangent line at x is positive; that is, $f'(x) > 0$. Similarly, a function is *decreasing* over an open interval I if, for all x in I, the slope of the tangent line is negative; that is $f'(x) < 0$.

- A *critical value* is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist. The point $(c, f(c))$ is called a *critical point.*

- A relative maximum point is higher than all other points in some interval containing it. Similarly, a relative minimum point is lower than all other points in some interval containing it. They y-value of such a point is called a relative maximum (or minimum) *value* of the function.

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3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Section Summary Concluded

- Minimum and maximum points are collectively called *extrema*.
- Critical values are candidates for possible relative extrema. The *First-Derivative Test* is used to classify a critical value as a relative minimum, a relative maximum, or neither.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

OBJECTIVE

- Classify the relative extrema of a function using the Second-Derivative Test.
- Graph a continuous function in a manner that shows concavity.

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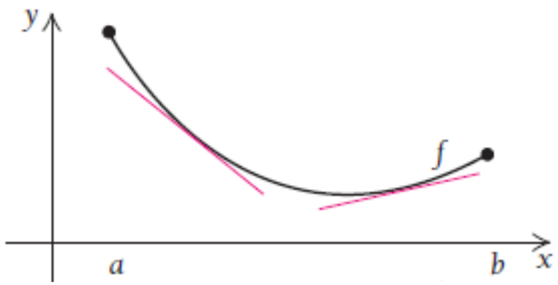
3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

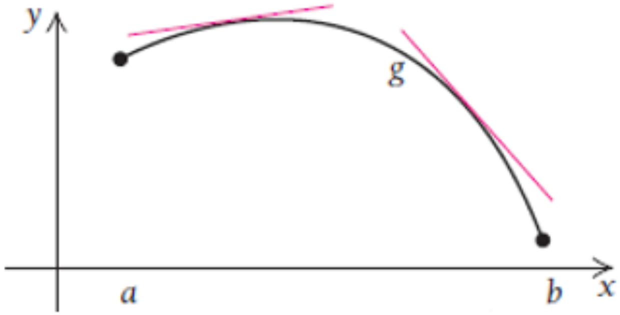
DEFINITION:

Suppose that f is a function whose derivative f' exists at every point in an open interval I. Then

f is **concave up** on I if f' is increasing over I.



f is **concave down** on I if f' is decreasing over I.



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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 4: A Test for Concavity

1. If $f''(x) > 0$ on an interval I, then the graph of f is concave up. (f' is increasing, so f is turning up on I.)

2. If $f''(x) < 0$ on an interval I, then the graph of f is concave down. (f' is decreasing, so f is turning down on I.)

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 5: The Second Derivative Test for Relative Extrema

Suppose that f is differentiable for every x in an open interval (a, b) and that there is a critical value c in (a, b) for which $f'(c) = 0$. Then:

1. $f(c)$ is a relative minimum if $f''(c) > 0$.
2. $f(c)$ is a relative maximum if $f''(c) < 0$.

For $f''(c) = 0$, the First-Derivative Test can be used to determine whether $f(c)$ is a relative extremum.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1: Graph the function f given by

$$f(x) = x^3 + 3x^2 - 9x - 13,$$

and find the relative extrema.

1st find $f'(x)$ and $f''(x)$.

$$f'(x) = 3x^2 + 6x - 9,$$

$$f''(x) = 6x + 6.$$

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

2nd solve $f'(x) = 0$.

$$\begin{aligned} 3x^2 + 6x - 9 &= 0 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \end{aligned}$$

$$\begin{aligned} x + 3 &= 0 \\ x &= -3 \end{aligned} \quad \text{or} \quad \begin{aligned} x - 1 &= 0 \\ x &= 1 \end{aligned}$$

Thus, $x = -3$ and $x = 1$ are critical values.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

3rd use the Second Derivative Test with –3 and 1.

$$f''(-3) = 6(-3) + 6 = -18 + 6 = -12 < 0 : \text{Relative maximum}$$

$$f''(1) = 6(1) + 6 = 6 + 6 = 12 > 0 : \text{Relative minimum}$$

Lastly, find the values of $f(x)$ at –3 and 1.

$$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 13 = 14$$

$$f(1) = (1)^3 + 3(1)^2 - 9(1) - 13 = -18$$

So, (–3, 14) is a relative maximum and (1, –18) is a relative minimum.

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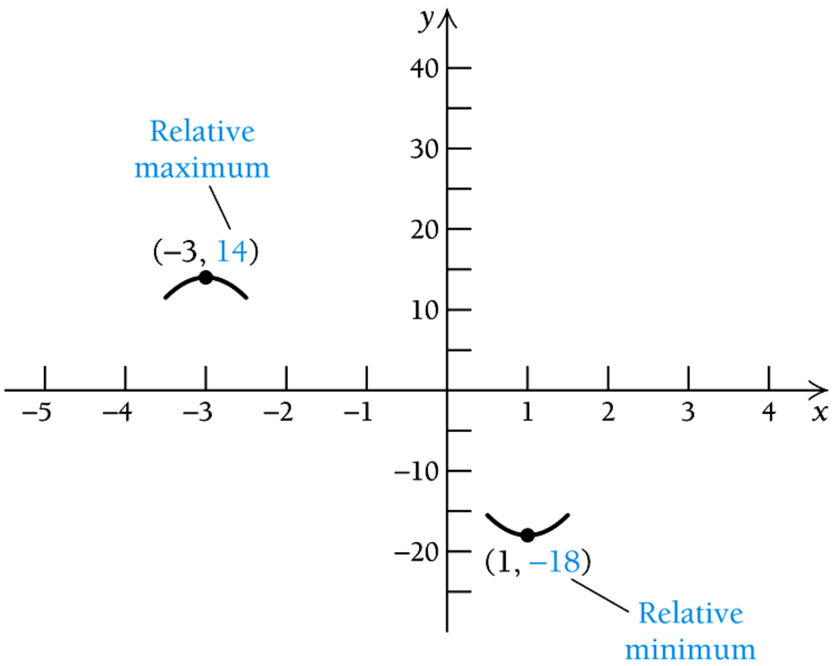
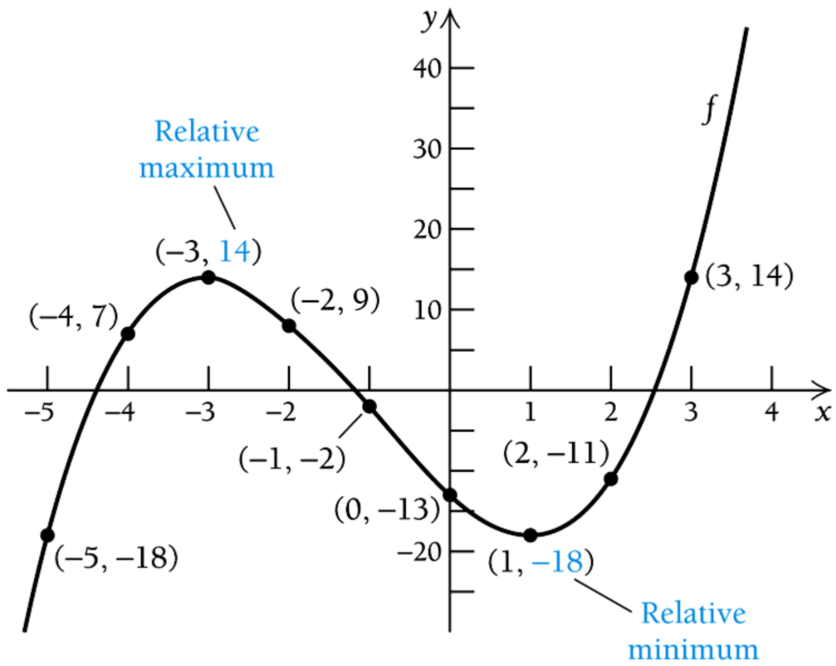
3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (concluded):

Then, by calculating and plotting a few more points, we can make a sketch of $f(x)$, as shown below.





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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2: Graph the function given by $k(x) = e^{2x} - e^{x}$, and find the relative extrema.

1st find $k'(x)$ and $k''(x)$.

$$k'(x) = 2e^{2x} - e^{x},$$

$$k''(x) = 4e^{2x} - e^{x}.$$

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

2nd solve $k'(x) = 0$.

$$\begin{aligned} 2e^{2x} - e^{x} &= 0 \\ e^{x}(2e^{x} - 1) &= 0 \end{aligned}$$

Since $e^{x} > 0$, for all x, only $(2e^{x} - 1)$ will yield solutions.

$$\begin{aligned} (2e^{x} - 1) &= 0 \\ 2e^{x} &= 1 \\ e^{x} &= 0.5 \\ x &= \ln(0.5) \end{aligned}$$

Thus, $x = \ln(0.5)$ is a critical value.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

3rd use the Second Derivative Test with ln(0.5).

$$k''(\ln 0.5) = 4e^{2(\ln 0.5)} - e^{\ln 0.5}$$
$$= 4e^{\ln 0.5^2} - e^{\ln 0.5}$$
$$= 4 \cdot 0.5^2 - 0.5 = 0.5 > 0: \text{ Relative minimum}$$

Lastly, find the values of $k(x)$ at ln(0.5).

$$k(\ln 0.5) = e^{2(\ln 0.5)} - e^{\ln 0.5}$$
$$= e^{\ln 0.5^2} - e^{\ln 0.5}$$
$$= 0.5^2 - 0.5 = -0.25$$

So, (ln0.5, -0.25) is a relative minimum.

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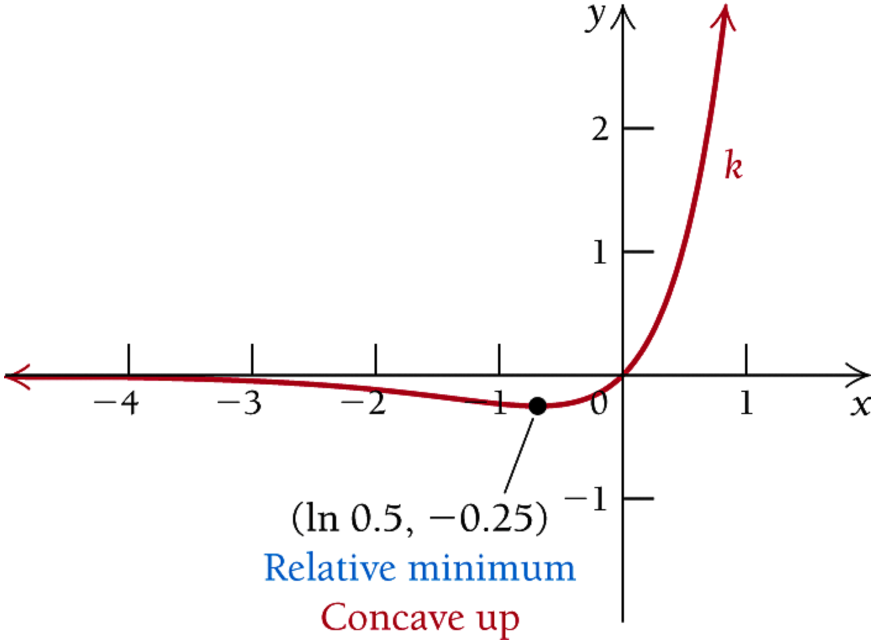
3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (concluded):

Then, by calculating and plotting a few more points, we can make a sketch of $k(x)$, as shown below.



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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 1

Find the relative extrema of the function g given by $g(x) = 10x^3 - 6x^5$, and sketch the graph.

First find $g'(x)$ and $g''(x)$:

$$g'(x) = 30x^2 - 30x^4$$

$$g''(x) = 60x - 120x^3$$

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 1 Continued

Second, solve $g'(x) = 0$:

$$g'(x) = 30x^2 - 30x^4 = 0$$
$$30x^2(1 - x^2) = 0$$

Thus $x = 0$, $x = 1$, and $x = -1$.

Therefore, there are critical values at $(0,0)$, $(1,4)$, and $(-1,-4)$.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 1 Continued

Third, use the Second Derivative Test with $x = 0$, $x = 1$, and $x = -1$:

$g''(0) = 60(0) - 120(0)^3 = 0$, Second Derivative test tells us nothing about this point, so we use the First Derivative test, which will show there is no extrema at this point.

$g''(1) = 60(1) - 120(1)^3 < 0$, relative maximum.

$g''(-1) = 60(-1) - 120(-1)^3 > 0$, relative minimum.

Therefore there is a relative maximum at $(1, 4)$ and a relative minimum at $(-1, -4)$.

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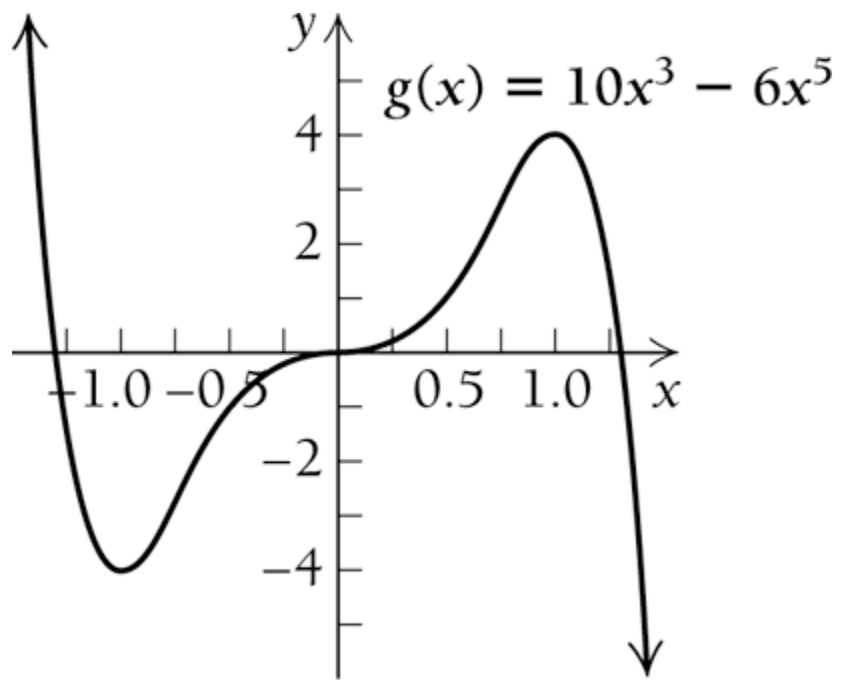
3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 1 Concluded

Using the information we gathered, we can plot $g(x)$:



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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 6: Finding Points of Inflection

If a function f has a point of inflection, it must occur at a point x_0, where.

$$f''(x_0) = 0 \quad \text{or} \quad f''(x_0) \text{ does not exist.}$$

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2

Determine the points of inflection for the function given by

$$g(x) = 10x^3 - 6x^5$$

We will use Theorem 6 (Finding Points of Inflection):

If a function has a point of inflection, it must occur at a point x_0, where:

$$f''(x_0) = 0 \quad \text{or} \quad f''(x_0) \text{ does not exist.}$$

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2 Concluded

So, to find the point of inflection, we need to find where $g''(x) = 0$, and then show that there is a change in the sign of $g''(x)$ on either side of the point of inflection.

First, we need to find what $g''(x)$ is, then find where it equals 0.

If $g(x) = 10x^3 - 6x^5$ then, $g''(x) = (g'(x))'$

$$= (30x^2 - 30x^4)' = 60x - 120x^3 = 60x\left(1 - 2x^2\right)$$

Now we know that the points of inflection are when $60x = 0$ and $1 - 2x^2 = 0$. We can now solve for both, which gives us

$(0,0)$, $\left(\frac{\sqrt{2}}{2}, 2.475\right)$, and $\left(-\frac{\sqrt{2}}{2}, -2.475\right)$ as the points of inflection.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 3

Determine the points of inflection for the function given by

$$k(x) = e^{2x} - e^{x}$$

We will use Theorem 6 (Finding Points of Inflection):

If a function has a point of inflection, it must occur at a point x_0 , where:

$$f''(x_0) = 0 \quad \text{or } f''(x_0) \text{ does not exist.}$$

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 3 Continued

To find the point of inflection, we need to find where $k''(x) = 0$ and then show that there is a change in the sign of $k''(x)$ on either side of the point of inflection.

First, Let's find where $k''(x) = 0$. If $k(x) = e^{2x} - e^{x}$ then,

$$k''(x) = (k'(x))' = (2e^{2x} - e^{x})' = 4e^{2x} - e^{x} = e^{x}\left(4e^{x} - 1\right)$$

The inflection points occur where $4e^{x} - 1 = 0$. Solving we get:

$$4e^{x} = 1$$

$$e^{x} = 0.25$$

$$x = \ln 0.25 \approx -1.386$$

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 3 Concluded

The y-coordinate of this potential inflection point is:

$$k(\ln 0.25) = e^{2\ln 0.25} - e^{\ln 0.25} = 0.25^2 - 0.25 = -\frac{3}{16}$$

Now, determine if there is a change in the sign of $k''(x)$ on either side of this potential inflection point: $x \approx -1.386$

Test $x = -2$: $k''(-2) = 4e^{2(-2)} - e^{(-2)} \approx -0.06$

Test $x = 0$: $k''(0) = 4e^{2(0)} - e^{(0)} = 4 - 1 = 3$

Since there is a change of sign, $\left(\ln 0.25, -\frac{3}{16}\right)$ is an inflection point.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Strategy for Sketching Graphs:

a) *Derivatives and Domain*. Find $f'(x)$ and $f''(x)$. Note the domain of f.

b) *Critical values of f*. Find the critical values by solving $f'(x) = 0$ and finding where $f'(x)$ does not exist. Find the function values at these points.

c) *Increasing and/or decreasing; relative extrema*. Substitute each critical value, x_0, from step (b) into $f''(x)$ and apply the Second Derivative Test.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Strategy for Sketching Graphs (continued):

d) *Inflection Points*. Determine candidates for inflection points by finding where $f''(x) = 0$ or where $f''(x)$ does not exist. Find the function values at these points.

e) *Concavity*. Use the candidates for inflection points from step (d) to define intervals. Use the relative extrema from step (b) to determine where the graph is concave up and where it is concave down.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Strategy for Sketching Graphs (concluded):

f) *Sketch the graph*. Sketch the graph using the information from steps (a) – (e), calculating and plotting extra points as needed.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3: Find the relative extrema of the function f given by

$$f(x) = x^3 - 3x + 2,$$

and sketch the graph.

a) *Derivatives and Domain*.

$$f'(x) = 3x^2 - 3,$$

$$f''(x) = 6x.$$

The domain of f is all real numbers.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3 (continued):

b) Critical values of f.

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

And we have $f(-1) = 4$ and $f(1) = 0$.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3 (continued):

c) *Increasing and/or Decreasing; relative extrema.*

$$f''(-1) = 6(-1) = -6 < 0$$

So (–1, 4) is a relative maximum, and $f(x)$ is increasing on (–∞, –1) and decreasing on (–1, 1). The graph is also concave down at the point (–1, 4).

$$f''(1) = 6(1) = 6 > 0$$

So (1, 0) is a relative minimum, and $f(x)$ is decreasing on (–1, 1) and increasing on (1, ∞). The graph is also concave up at the point (1, 0).

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3 (continued):

d) *Inflection Points.*

$$\begin{aligned} 6x &= 0 \\ x &= 0 \end{aligned}$$

And we have $f(0) = 2$.

e) *Concavity.* From step (c), we can conclude that f is concave down on the interval $(-\infty, 0)$ and concave up on $(0, \infty)$.

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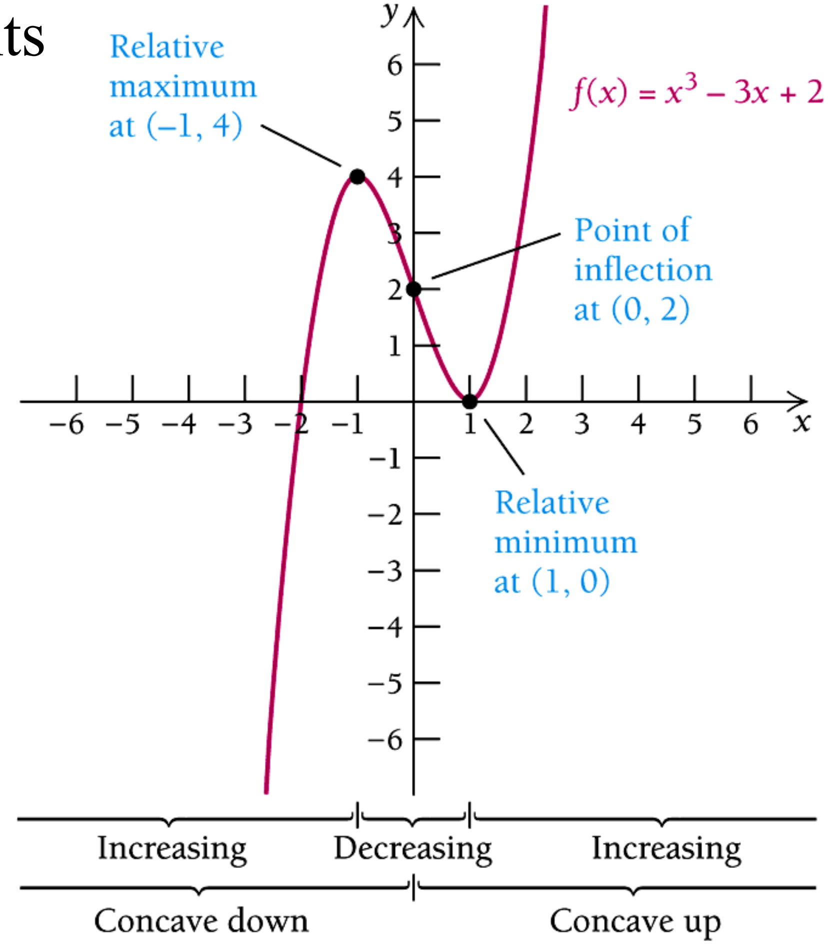
3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 3 (concluded)

f) *Sketch the graph*. Using the points from steps (a) – (e), the graph follows.

x	$f(x)$
-3	-16
-2	0
-1	4
0	2
1	0
2	4
3	20



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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 4

Find the relative maxima and minima of the function f given by $f(x) = 1 + 8x^2 - x^4$ and sketch the graph.

a.) *Derivatives and Domain*

$$f'(x) = 16x - 4x^3$$

$$f''(x) = 16 - 12x^2$$

The domain of f is all real numbers.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 4 Continued

b.) Find the critical values:

$$16x - 4x^3 = 0$$

$$4x(4 - x^2) = 0$$

$$4 - x^2 = 0$$

$$-x^2 = -4$$

$$x^2 = 4$$

$$x = \pm 2$$

And we have $f(0) = 1$, $f(2) = 17$, and $f(-2) = 17$.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 4 Continued

c.) *Increasing and/or decreasing; relative extrema.*

$$f''(0) = 16 > 0$$

So $(0,1)$ is a relative minimum, and $f(x)$ is decreasing on $(-2,0)$ and increasing on $[0,2]$. Also, the graph is concave up at $(0,1)$.

$$f''(2) = 16 - 12(2)^2 = 16 - 48 < 0$$

So $(2,17)$ is a relative maximum, and $f(x)$ is increasing on $(0,2)$ and decreasing on $[2,\infty)$. Also, the graph is concave down at $(2,17)$.

$$f''(-2) = 16 - 12(2)^2 = 16 - 48 < 0$$

So $(-2,17)$ is a relative maximum, and $f(x)$ is increasing on $(-\infty,-2)$ and decreasing on $(-2,0)$. Also, the graph is concave down at $(-2,17)$.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 4 Continued

d.) *Inflection Points.*

$$16-12x^2=0$$
$$-12x^2=-16$$
$$x^2=\frac{4}{3}$$
$$x=\pm\frac{2\sqrt{3}}{3}$$

e.) *Concavity:* From step c.) we can conclude that f is concave down on the intervals $(-\infty,-2)$ and $(2,\infty)$, and is concave up on the interval $(-2,-2)$.

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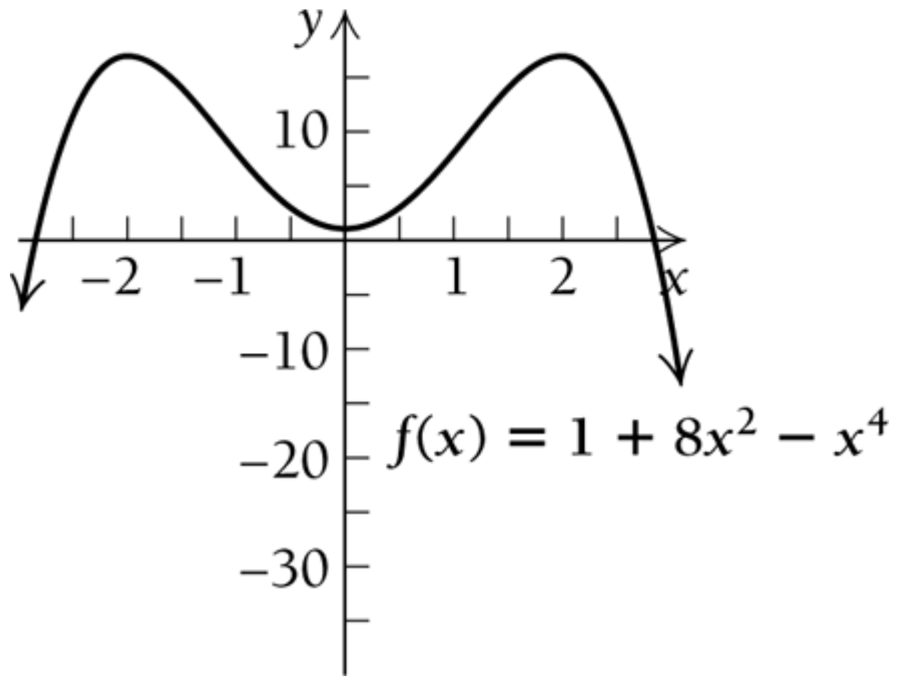
3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 4 Concluded

We can graph $f(x)$ from the points we have gathered:



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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 4: Graph the function f given by

$$f(x) = (2x - 5)^{1/3} + 1.$$

List the coordinates of any extreme points and points of inflection. State where the function is increasing or decreasing, as well as where it is concave up or concave down.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 4 (continued)

a) *Derivatives and Domain.*

$$f'(x) = \frac{1}{3}\left(2x-5\right)^{-2/3} \cdot 2 = \frac{2}{3}(2x-5)^{-2/3} = \frac{2}{3(2x-5)^{2/3}}$$

$$f''(x) = -\frac{4}{9}\left(2x-5\right)^{-5/3} \cdot 2 = -\frac{8}{9}(2x-5)^{-5/3} = \frac{-8}{9(2x-5)^{5/3}}$$

The domain of f is all real numbers.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 4 (continued)

b) *Critical values*. Since $f'(x)$ is never 0, the only critical value is where $f'(x)$ does not exist. Thus, we set its denominator equal to zero.

$$\begin{aligned} 3(2x-5)^{2/3} &= 0 \\ (2x-5)^{2/3} &= 0 \\ 2x-5 &= 0 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

And, we have

$$f\left(\frac{5}{2}\right) = \left(2\cdot\frac{5}{2}-5\right)^{1/3}+1$$

$$f\left(\frac{5}{2}\right) = 0+1$$

$$f\left(\frac{5}{2}\right) = 1$$

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 4 (continued)

c) *Increasing and/or decreasing; relative extrema.*

$$f''\left(\frac{5}{2}\right) = \frac{-8}{9\left(2 \cdot \frac{5}{2} - 5\right)^{5/3}}$$

$$f''\left(\frac{5}{2}\right) = \frac{8}{9 \cdot 0}$$

$$f''\left(\frac{5}{2}\right) = \frac{8}{0}$$ Cannot divide by zero!

Since $f''(x)$ does not exist, the Second Derivative Test fails. Instead, we use the First Derivative Test.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 4 (continued)

c) *Increasing and/or decreasing; relative extrema (continued).* Selecting 2 and 3 as test values on either side of $\frac{5}{2}$,

$$f'(2) = \frac{2}{3(2\cdot 2-5)^{2/3}} = \frac{2}{3(-1)^{2/3}} = \frac{2}{3\cdot 1} = \frac{2}{3} > 0$$

$$f'(3) = \frac{2}{3(2\cdot 3-5)^{2/3}} = \frac{2}{3(1)^{2/3}} = \frac{2}{3\cdot 1} = \frac{2}{3} > 0$$

Since $f'(x)$ is positive on both sides of $\frac{5}{2}$, $\frac{5}{2}$ is not an extremum.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 4 (continued)

d) *Inflection points*. Since $f''(x)$ is never 0, we only need to find where $f''(x)$ does not exist. And, since $f''(x)$ cannot exist where $f'(x)$ does not exist, we know from step (b) that a possible inflection point is $(\frac{5}{2}, 1)$.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 4 (continued)

e) *Concavity*. Again, using 2 and 3 as test points on either side of $\frac{5}{2}$,

$$f''(2) = \frac{-8}{9(2\cdot 2-5)^{\frac{5}{3}}} = \frac{-8}{9\cdot -1} = \frac{8}{9} > 0$$

$$f''(3) = \frac{-8}{9(2\cdot 3-5)^{\frac{5}{3}}} = \frac{-8}{9\cdot 1} = -\frac{8}{9} < 0$$

Thus, $\left(\frac{5}{2}, 1\right)$ is a point of inflection.

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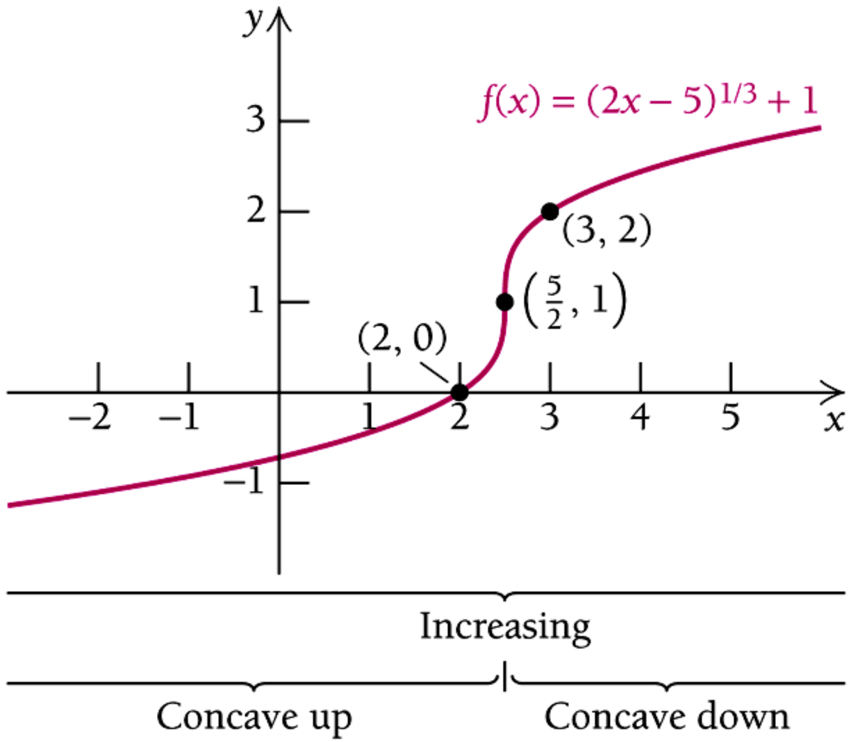
3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 4 (concluded)

f) *Sketch the graph*. Using the information in steps (a) – (e), the graph follows.



x	$f(x)$, approximately
0	−0.71
1	−0.44
2	0
$\frac{5}{2}$	1
3	2
4	2.44
5	2.71

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Section Summary

- The second derivative f'' determines the ***concavity*** of the graph of function f.
- If $f''(x) > 0$ for all x in an open interval I, then the graph of f is ***concave up*** over I.
- If $f''(x) < 0$ for all x in an open interval I, then the graph of f is ***concave down*** over I.
- If c is a critical value and $f''(c) > 0$, then $f(c)$ is a relative minimum.
- If c is a critical value and $f''(c) > 0$, then $f(c)$ is a relative maximum.

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3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Section Summary Concluded

- If c is a critical value and $f''(c) = 0$, the First-Derivative Test must be used to classify $f(c)$.

- If $f''(x_0) = 0$ or $f''(x_0)$ does not exist, and there is a change in concavity to the left and to the right of x_0, then the point $(x_0, f(x_0))$ is called a ***point of inflection***.

- Finding the extrema, intervals over which a function is increasing or decreasing, intervals of upward or downward concavity, and points of inflection is all part of a strategy for accurate curve sketching.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

OBJECTIVE

- Find absolute extrema using Maximum-Minimum Principle 1.
- Find absolute extrema using Maximum-Minimum Principle 2.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

DEFINITION:

Suppose that f is a function with domain I.

$f(c)$ is an **absolute minimum** if $f(c) \leq f(x)$ for all x in I.

$f(c)$ is an **absolute maximum** if $f(c) \geq f(x)$ for all x in I.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

THEOREM 7: The Extreme Value Theorem

A continuous function f defined over a closed interval $[a, b]$ must have an absolute maximum value and an absolute minimum value over $[a, b]$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

THEOREM 8: Maximum-Minimum Principle 1

Suppose that f is a continuous function defined over a closed interval $[a, b]$. To find the absolute maximum and minimum values over $[a, b]$:

a) First find $f'(x)$.

b) Then determine all critical values in $[a, b]$. That is, find all c in $[a, b]$ for which

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

THEOREM 8: Maximum-Minimum Principle 1 (continued)

c) List the values from step (b) and the endpoints of the interval:

$$a, c_1, c_2, \ldots., c_n, b.$$

d) Evaluate $f(x)$ for each value in step (c):

$$f(a), f(c_1), f(c_2), \ldots., f(c_n), f(b).$$

The largest of these is the **absolute maximum of f over [a, b]**. The smallest of these is the **absolute minimum of f over [a, b]**.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 1: Find the absolute maximum and minimum values of $f(x) = x^3 - 3x + 2$ over the interval $[-2, \frac{3}{2}]$.

a) $f'(x) = 3x^2 - 3$

b) Note that $f'(x)$ exists for all real numbers.

$$3x^2 - 3 = 0$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 1 (continued):

c) $-2, -1, 1, \frac{3}{2}$

d)
$$
\begin{aligned}
f(-2) &= (-2)^3 - 3(-2) + 2 &&= 0 \\
f(-1) &= (-1)^3 - 3(-1) + 2 &&= 4 \\
f(1) &= (1)^3 - 3(1) + 2 &&= 0 \\
f\left(\frac{3}{2}\right) &= \left(\frac{3}{2}\right)^3 - 3\left(\frac{3}{2}\right) + 2 &&= \frac{7}{8}
\end{aligned}
$$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 1 (concluded):

Thus, the absolute maximum value of $f(x)$ on $[-2, \frac{3}{2}]$ is 4, which occurs at $x = 1$. The absolute minimum value of $f(x)$ on $[-2, \frac{3}{2}]$ is 0, which occurs at $x = 2$ and $x = 1$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 1

Find the absolute maximum and minimum values of the function given in Example 1 over the interval $[0,3]$.

Recall that the function in Example 1 was $f(x)=x^3-3x+2$.

So, $f'(x)=3x^2-3$. Set this equal to zero and you get:

$$\begin{aligned}3x^2-3&=0\\3x^2&=3\\x^2&=1\\x&=\pm 1\end{aligned}$$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 1 Concluded

Next we check the equation at $x = -1, 0, 1, 3$ to find the minimum and maximum:

$$f(-1) = (-1)^3 - 3(-1) + 2 \;= -1 + 3 + 2 \;= 4$$

$$f(0) = (0)^3 - 3(0) + 2 \;= 0 + 0 + 2 \;= 2$$

$$f(1) = (1)^3 - 3(1) + 2 \;= 1 - 3 + 2 \;= 0$$

$$f(3) = (3)^3 - 3(3) + 2 \;= 27 - 9 + 2 \;= 20$$

The absolute maximum is 20 at $x = 3$.

The absolute minimum is 0 at $x = 1$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

THEOREM 9: Maximum-Minimum Principle 2

Suppose that f is a function such that $f'(x)$ exists for every x in an interval I, and that there is *exactly one* (critical) value c in I, for which $f'(c) = 0$. Then

$f(c)$ is the absolute maximum value over I if $f''(c) < 0$

or

$f(c)$ is the absolute minimum value over I if $f''(c) > 0$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 2: Find the absolute maximum and minimum values of $f\left(x\right)=2e^{3x}$ over $\left[-2,\frac{1}{2}\right]$.

Solution: $f'(x)=6e^{3x}$ is defined for all real numbers.

Note that $6e^{3x}>0$ for all x.

Thus, there are no critical values and the absolute extrema must occur at the endpoints of the interval:

$$f(-2)=2e^{3(-2)}=2e^{-6}\approx 0.005$$

$$f(0.5)=2e^{3(0.5)}=2e^{1.5}\approx 8.963$$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 2 concluded:

The absolute maximum of $f(x) = 2e^{3x}$ over $\left[-2, \frac{1}{2}\right]$ is:

$2e^{1.5} \approx 8.963$ and occurs at $x = \frac{1}{2}$.

The absolute minimum of $f(x) = 2e^{3x}$ over $\left[-2, \frac{1}{2}\right]$ is:
$2e^{-6} \approx 0.005$ and occurs at $x = -2$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 3: Find the absolute maximum and minimum values of $f(x) = 4x - x^2$.

When no interval is specified, we consider the entire domain of the function. In this case, the domain is the set of all real numbers.

a) $f'(x) = 4 - 2x$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 3 (continued):

b) $4 - 2x = 0$

$$-2x = -4$$

$$x = 2$$

c) Since there is only one critical value, we can apply Maximum-Minimum Principle 2 using the second derivative.

$$f''(x) = -2$$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 3 (concluded):

Since $f''(x) < 0$ and $f(2) = 4(2) - (2)^2 = 4$, $f(x)$ has an absolute maximum value of 4 at $x = 2$.

$f(x)$ has no absolute minimum value.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 2

Find the absolute maximum and minimum values of $f(x) = x^2 - 10x$ over each interval:

a.) $[0,6]$

b.) $[4,10]$

Looking at the function, we can see that the absolute minimum over $(-\infty, \infty)$ is -25 located at $x = 5$. Since $x = 5$ is in both intervals, the absolute minimum for both intervals will be -25 located at $x = 5$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 2 Concluded

Now we need to check both endpoints for both intervals to find the absolute maximum:

a.) $f(0) = 0^2 - 10(0) = 0 - 0 = 0$

$f(6) = 6^2 - 10(6) = 36 - 60 = -24$

So the absolute maximum over $[0, 6]$ is 0 located at $x = 0$.

b.) $f(4) = 4^2 - 10(4) = 16 - 40 = -24$

$f(10) = 10^2 - 10(10) = 100 - 100 = 0$

So the absolute maximum over $[4, 10]$ is 0 located at $x = 10$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

A Strategy for Finding Absolute Maximum and Minimum Values:

To find absolute maximum and minimum values of a continuous function over an interval:

a) Find $f'(x)$.

b) Find the critical values.

c) If the interval is closed and there is more than one critical value, use Maximum-Minimum Principle 1.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

A Strategy for Finding Absolute Maximum and Minimum Values (continued):

d) If the interval is closed and there is exactly one critical value, use either Maximum-Minimum Principle 1 or Maximum-Minimum Principle 2. If it is easy to find $f''(x)$, use Maximum-Minimum Principle 2.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

A Strategy for Finding Absolute Maximum and Minimum Values (concluded):

e) If the interval is not closed, such as $(-\infty, \infty)$, $(0, \infty)$, or (a, b), and the function has only one critical value, use Maximum-Minimum Principle 2. In such a case, if the function has a maximum, it will have no minimum; and if it has a minimum, it was have no maximum.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 4: Find the absolute maximum and minimum values of $f(x) = 5x + \frac{35}{x}$ over the interval $(0, \infty)$.

a) $f'(x) = 5 - \frac{35}{x^2}$

b) Since $f'(x)$ exists for all values of x in $(0, \infty)$, the only critical values are those for which $f'(x) = 0$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 4 (continued):

$$5 - \frac{35}{x^2} = 0$$

$$5 = \frac{35}{x^2}$$

$$5x^2 = 35$$

$$x^2 = 7$$

$$x = \sqrt{7}$$

c) The interval $(0, \infty)$ is not closed, and the only critical value is $\sqrt{7}$. Thus, we can use Maximum-Minimum Principle 2 using the second derivative.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 4 (continued):

$$f''(x) = \frac{70}{x^3}$$

$$f''\left(\sqrt{7}\right) = \frac{70}{\left(\sqrt{7}\right)^3} > 0$$

Thus $f(x)$ has an absolute minimum at $x = \sqrt{7}$.

$$f\left(\sqrt{7}\right) = 5 \cdot \sqrt{7} + \frac{35}{\sqrt{7}}$$

$$f\left(\sqrt{7}\right) = 5\sqrt{7} + \frac{35\sqrt{7}}{7}$$

$$f\left(\sqrt{7}\right) = 5\sqrt{7} + 5\sqrt{7} = 10\sqrt{7}$$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 4 (concluded):

Thus, the absolute minimum of $f(x)$ is $10\sqrt{7}$, which occurs at $x = \sqrt{7}$.

$f(x)$ has no absolute maximum value.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 3

Find the absolute maximum and minimum values of $g(x) = \dfrac{2x^2 + 18}{x}$ over the interval $(0, \infty)$.

First we want to find $g'(x)$:

$$g'(x) = \frac{x(4x) - (2x^2 + 18)}{x^2} = \frac{4x^2 - 2x^2 - 18}{x^2} = \frac{2x^2 - 18}{x^2}$$

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 3 Continued

Next we find out where $g'(x) = 0$.

$$\frac{2x^2 - 18}{x^2} = \frac{2x^2}{x^2} - \frac{18}{x^2} = 2 - \frac{18}{x^2} = 0$$

$$2 = \frac{18}{x^2}$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

Since $x = -3$ is not in the interval, the only critical value is at $x = 3$.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 3 Concluded

We can now apply the Maximum-Minimum Principle 2:

$$f''(x) = \frac{36}{x^3}$$

$$f''(3) = \frac{36}{3^3} > 0$$

So an absolute minimum occurs at $x = 3$, and is 12.

Since this is on an open interval, there is no absolute maximum.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Section Summary

- An *absolute minimum* of a function f is a value $f(c)$ such that $f(c) \leq f(x)$ for all x in the domain of f.
- An *absolute maximum* of a function f is a value $f(c)$ such that $f(c) \geq f(x)$ for all x in the domain of f.
- If the domain of f is a closed interval and f is continuous over that domain, then the *Extreme-Value Theorem* guarantees the existence of both an absolute minimum and an absolute maximum.

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3.4 Optimization: Finding Absolute Maximum and Minimum Values

Section Summary Concluded

- Endpoints of a closed interval may be absolute extrema, but not relative extrema.

- If there is exactly one critical value c such that $f'(c) = 0$ in the domain of f, then *Maximum-Minimum Principle 2* may be used. Otherwise, *Maximum-Minimum Principle 1* has to be used.

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