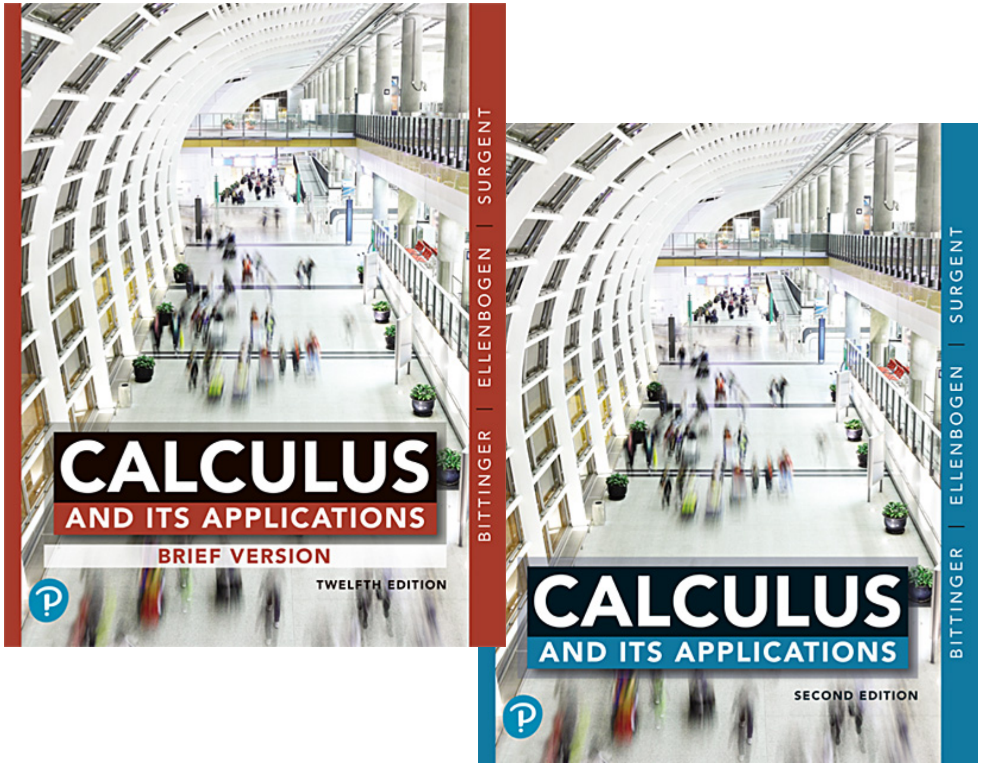
Chapter 4

Integration



4.1 Antidifferentiation

OBJECTIVE

- Find an antiderivative of a function.
- Evaluate indefinite integrals using basic rules of antidifferentiation.
- Use initial conditions to determine an antiderivative.

4.1 Antidifferentiation

THEOREM 1

The **antiderivative** of $f(x)$ is the set of functions $F(x)+C$ such that

$$\frac{d}{dx}\left[F(x)+C\right]=f(x).$$

The constant C is called the **constant of integration.**

4.1 Antidifferentiation

Integrals and Integration

Antidifferentiating is often called integration.

To indicate the antiderivative of x^2 is $x^3/3 + C$, we write $\int x^2 dx = \frac{x^3}{3} + C$, where the notation $\int f(x)dx$ is used to represent the antiderivative of $f(x)$.

More generally, $\int f(x)dx = F(x) + C$, where

$F(x) + C$ is the general form of the antiderivative of $f(x)$.

4.1 Antidifferentiation

Example 1: Determine these indefinite integrals. That is, find the antiderivative of each integrand:

a.) $\int 8dx = 8x + C$ $\quad$ $Check: \frac{d}{dx}(8x + C) = 8$

b.) $\int 3x^2 dx = x^3 + C$ $\quad$ Check: $\frac{d}{dx}\left(x^3 + C\right) = 3x^2$

c.) $\int e^x dx = e^x + C$ $\quad$ Check: $\frac{d}{dx}\left(e^x + C\right) = e^x$

d.) $\int \frac{1}{x} dx = \ln x + C$ $\quad$ Check: $\frac{d}{dx}(\ln x + C) = \frac{1}{x}$

4.1 Antidifferentiation

THEOREM 2: Basic Integration Formulas

1. $\int k \, dx = kx + C \quad (k \text{ is a constant})$

2. $\int x^r dx = \dfrac{x^{r+1}}{r+1} + C, \quad \text{provided } r \neq -1$

(To integrate a power of x other than -1, increase the power by 1 and divide by the increased power.)

4.1 Antidifferentiation

THEOREM 2: Basic Integration Formulas (continued)

3. $$\int x^{-1}dx = \int \frac{1}{x}dx = \int \frac{dx}{x} = \ln x + C, \qquad x > 0$$

$$\int x^{-1}dx = \ln|x| + C, \qquad x < 0$$

(We will generally assume that $x > 0$.)

4. $$\int be^{ax}dx = \frac{b}{a}e^{ax} + C$$

4.1 Antidifferentiation

Example 2: Use the Power Rule of Antidifferentiation to determine these indefinite integrals:

a) $\int x^7 dx$; b) $\int x^{99} dx$; c) $\int \sqrt{x}\,dx$; d) $\int \frac{1}{x^3} dx$

a.) $\int x^7 dx = \frac{x^{7+1}}{7+1} + C = \frac{1}{8}x^8 + C$

Check: $\frac{d}{dx}\left(\frac{1}{8}x^8 + C\right) = \frac{1}{8}\left(8x^7\right) = x^7$

b.) $\int x^{99} dx = \frac{x^{99+1}}{99+1} + C = \frac{1}{100}x^{100} + C$

Check: $\frac{d}{dx}\left(\frac{1}{100}x^{100} + C\right) = \frac{1}{100}\left(100x^{99}\right) = x^{99}$

4.1 Antidifferentiation

Example 2 (Continued)

c.) We note that $\sqrt{x} = x^{1/2}$. Therefore

$$\int \sqrt{x}\, dx = \int x^{1/2} = \frac{x^{1/2+1}}{\frac{1}{2}+1} + C = \frac{2}{3}x^{3/2} + C$$

Check: $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + C\right) = \frac{2}{3}\left(\frac{3}{2}x^{1/2}\right) = x^{1/2} = \sqrt{x}$

4.1 Antidifferentiation

Example 2 (Concluded)

d.) We note that $\frac{1}{x^3} = x^{-3}$. Therefore

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$$

Check: $\frac{d}{dx}\left(-\frac{1}{2}x^{-2} + C\right) = -\frac{1}{2}\left(-2x^{-3}\right) = x^{-3} = \frac{1}{x^3}$

4.1 Antidifferentiation

Quick Check 1

Determine these indefinite integrals:

a.) $\int x^{10}dx = \dfrac{x^{10+1}}{10+1} + C = \dfrac{1}{11}x^{11} + C$

b.) $\int x^{200}dx = \dfrac{x^{200+1}}{200+1} + C = \dfrac{1}{201}x^{201} + C$

c.) $\int \sqrt[6]{x} \cdot dx = \int x^{1/6}dx = \dfrac{x^{1/6+1}}{\frac{1}{6}+1} + C = \dfrac{6}{7}x^{7/6} + C = \dfrac{6}{7}\sqrt[6]{x^7} + C$

d.) $\int \dfrac{1}{x^4}dx = \int x^{-4}dx = \dfrac{x^{-4+1}}{-4+1} + C = \dfrac{1}{-3}x^{-3} + C = -\dfrac{1}{3x^3} + C$

4.1 Antidifferentiation

Example 3: Determine the indefinite integral $\int e^{4x}dx$.

Since we know that $\frac{d}{dx}e^{x} = e^{x}$, it is reasonable to make this initial guess:

$$\int e^{4x}dx = e^{4x} + C.$$

But this is (slightly) wrong, since

$$\frac{d}{dx}\left(e^{4x} + C\right) = 4e^{4x}$$

4.1 Antidifferentiation

Example 3 (Concluded): We modify our guess by inserting $\frac{1}{4}$ to obtain the correct antiderivative:

$$\int e^{4x}dx = \frac{1}{4}e^{4x} + C$$

This checks:

$$\frac{d}{dx}\left(\frac{1}{4}e^{4x} + C\right) = \frac{1}{4}\left(4e^{4x}\right) = e^{4x}$$

4.1 Antidifferentiation

Quick Check 2

Find each antiderivative:

a.) $\int e^{-3x}dx = \frac{1}{-3}e^{-3x} + C$

b.) $\int e^{(1/2)x}dx = 2e^{(1/2)x} + C$

4.1 Antidifferentiation

THEOREM 3

Properties of Antidifferentiation

P1. $$\int c f(x)dx = c\int f(x)dx$$

(The integral of a constant times a function is the constant times the integral of the function.)

P2. $$\int\left[f(x) \pm g(x)\right]dx = \int f(x)dx \pm \int g(x)dx$$

(The integral of a sum or difference is the sum or difference of the integrals.)

4.1 Antidifferentiation

Example 4: Determine these indefinite integrals. Assume $x > 0$.

a) $\int \left(3x^5 + 7x^2 + 8\right)dx$; b) $\int \frac{4 + 3x + 2x^4}{x}dx$

a.) We antidifferentiate each term separately:

$$\int \left(3x^5 + 7x^2 + 8\right)dx = \int 3x^5 dx + \int 7x^2 dx + \int 8dx$$

$$= 3\left(\frac{1}{6}x^6\right) + 7\left(\frac{1}{3}x^3\right) + 8x$$

$$= \frac{1}{2}x^6 + \frac{7}{3}x^3 + 8x + C$$

4.1 Antidifferentiation

Example 4 (Concluded):

b) We algebraically simplify the integrand by noting that x is a common denominator and then reducing each ratio as much as possible:

$$\frac{4+3x+2x^4}{x} = \frac{4}{x} + \frac{3x}{x} + \frac{2x^4}{x} = \frac{4}{x} + 3 + 2x^3$$

Therefore,

$$\int \frac{4+3x+2x^4}{x}\,dx = \int \left(\frac{4}{x} + 3 + 2x^3 \right) dx$$

$$= 4\ln x + 3x + \frac{1}{2}x^4 + C$$

4.1 Antidifferentiation

Quick Check 3

Determine these indefinite integrals:

a.) $\int\left(2x^4+3x^3-7x^2+x-5\right)dx = \frac{2}{5}x^5+\frac{3}{4}x^4-\frac{7}{3}x^3+\frac{1}{2}x^2-5x+C$

b.) $\int(x-5)^2\,dx = \int x^2-10x+25dx = \frac{1}{3}x^3-5x^2+25x+C$

c.) $\int\frac{x^2-7x+2}{x^2}\,dx = \int 1-\frac{7}{x}+\frac{2}{x^2}\,dx = x-7\ln x-\frac{2}{x}+C$

4.1 Antidifferentiation

Example 5: Find the function f such that

$$f'(x) = x^2 \text{ and } f(-1) = 2.$$

First find $f(x)$ by integrating.

$$f(x) = \int x^2 dx$$

$$f(x) = \frac{x^3}{3} + C$$

Example 5 (concluded):

Then, the initial condition allows us to find C.

$$\begin{aligned} f(-1) &= \frac{(-1)^3}{3} + C = 2 \\ -\frac{1}{3} + C &= 2 \\ C &= \frac{7}{3} \end{aligned}$$

Thus, $f(x) = \dfrac{x^3}{3} + \dfrac{7}{3}$.

4.1 Antidifferentiation

Section Summary

- The *antiderivative* of a function $f(x)$ is a set of functions
$$\frac{d}{dx}\left[F(x)+C\right]=f(x),$$
where the constant C is called the *constant of integration*.

- An antiderivative is denoted by an *indefinite integral* using the integral sign, $\int$. If $F(x)$ is an antiderivative of $f(x)$ we write
$$\int f(x)\,dx=F(x)+C.$$
We check the correctness of an antiderivative we have found by differentiating it.

4.1 Antidifferentiation

Section Summary Continued

- The *Constant Rule of Antidifferentiation* is $\int k \cdot dx = kx + C$.
- The *Power Rule of Antidifferentiation* is
$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C, \text{ for } n \neq -1.$$
- The *Natural Logarithm Rule of Antidifferentiation* is
$$\int \frac{1}{x} dx = \ln x + C, \text{ for } x > 0.$$
- The *Exponential Rule* (base *e*) *of Antidifferentiation* is
$$\int e^{ax} = \frac{1}{a} e^{ax} + C, \text{ for } a \neq 0.$$
- An *initial condition* is an ordered pair that is a solution of a particular antiderivative of an integrand.

4.2 Antiderivatives as Areas

OBJECTIVE

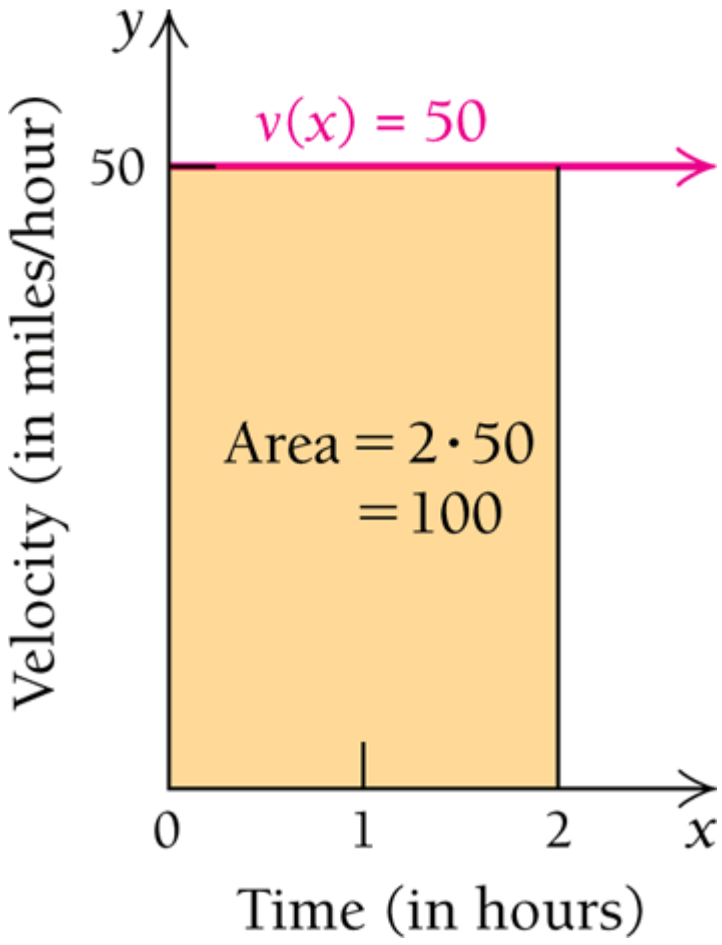
- Find the area under a graph and use it to solve real-world problems
- Use rectangles to approximate the area under a graph.

4.2 Antiderivatives as Areas

Example 1: A vehicle travels at 50 mi/hr for 2 hr. How far has the vehicle traveled?

The answer is 100 mi. We treat the vehicle's velocity as a function, $v(x) = 50.$ We graph this function, sketch a vertical line at $x = 2$, and obtain a rectangle. This rectangle measures 2 units horizontally and 50 units vertically. Its area is the distance the vehicle has traveled:

$$2\text{ hr} \cdot \frac{50\text{ mi}}{1\text{ hr}} = 100\text{ mi.}$$



4.2 Antiderivatives as Areas

Example 2: The velocity of a moving object is given by the function $v(x) = 3x$, where x is in hours and v is in miles per hour. Use geometry to find the area under the graph, which is the distance the object has traveled:

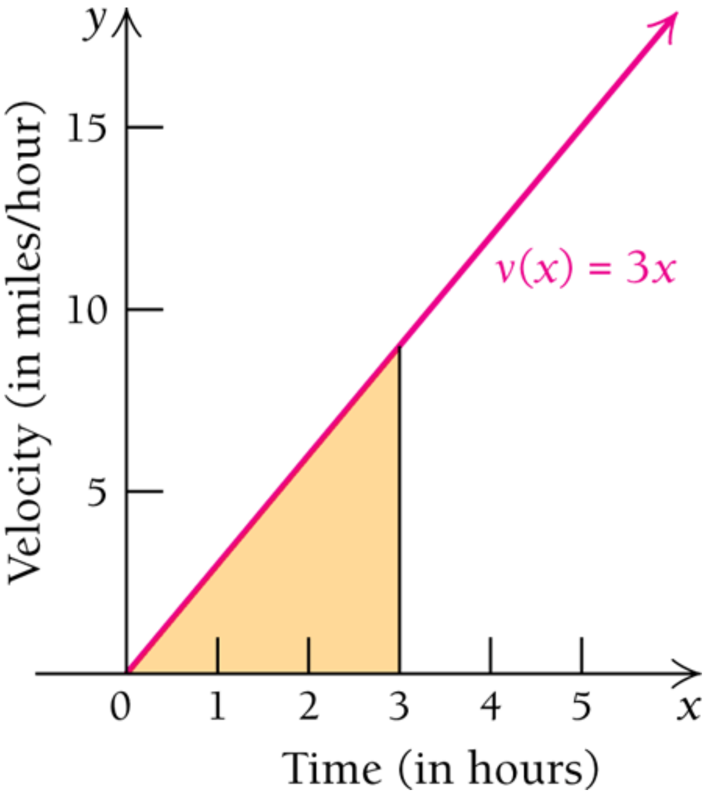
a.) during the first 3 hr $(0 \leq x \leq 3)$;

b.) between the third hour and the fifth hour $(3 \leq x \leq 5)$.

4.2 Antiderivatives as Areas

Example 2 (continued):

a.) The graph of the velocity function is shown at the right. We see the region corresponding to the time interval $0 \le x \le 3$ is a triangle with base 3 and height 8 (since $v(3) = 9$). Therefore, the area of this region is



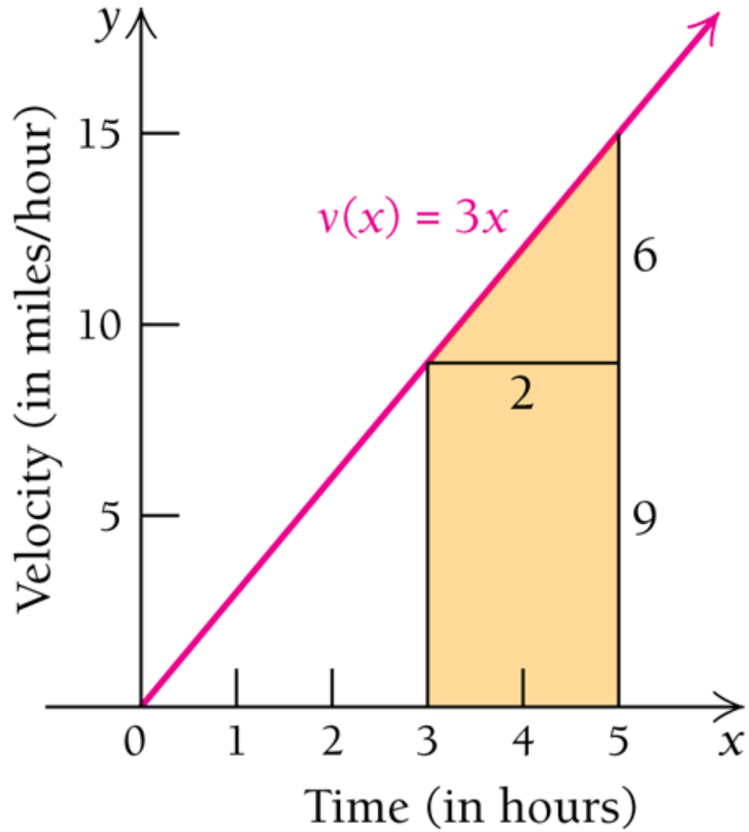
$$A = \frac{1}{2}(3)(9) = \frac{27}{2} = 13.5.$$

The object traveled 13.5 mi during the first 3 hr.

4.2 Antiderivatives as Areas

Example 2 (Continued):

b.) The region corresponding to the time interval $3 \le x \le 5$ is a trapezoid. It can be decomposed into a rectangle and a triangle as indicated in the figure to the right. The rectangle has a base 2 and height 9, and thus an area $A = (2)(9) = 18.$



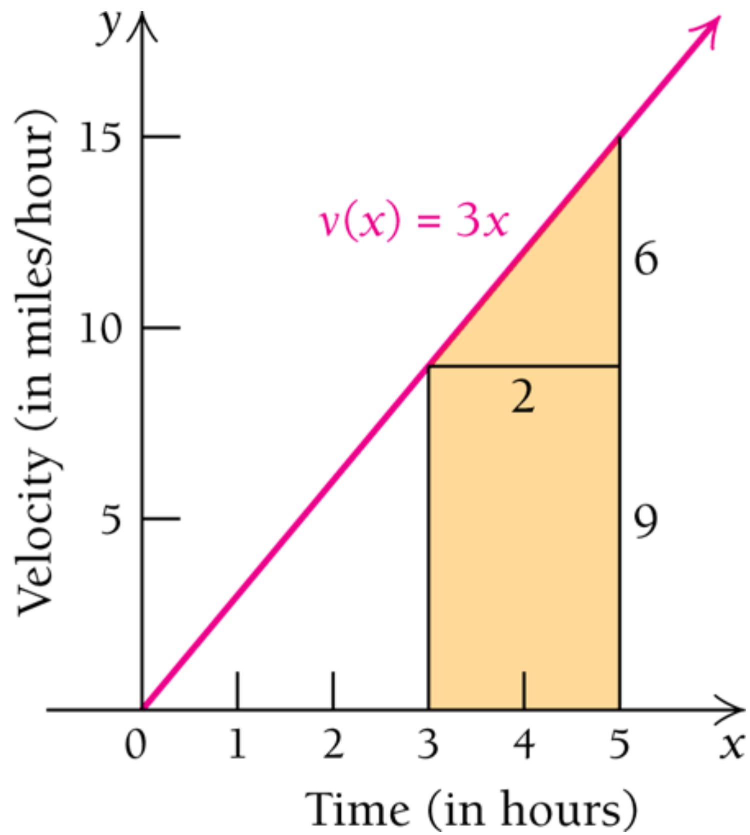
4.2 Antiderivatives as Areas

Example 2 (Concluded):

b.) The triangle has base 2 and height 6, for an area

$$A = \frac{1}{2}(2)(6) = 6.$$

Summing the two areas, we get 24. Therefore, the object traveled 24 mi between the third hour and the fifth hour.



4.2 Antiderivatives as Areas

Quick Check 1

An object moves with a velocity of $v(t) = \frac{1}{2}t$, where t is in minutes and v is in feet per minute.

a.) How far does the object travel during the first 30 min?

b.) How far does the object travel between the first hour and the second hour?

a.) We know that this is a linear function, so the region corresponding to the time interval $(0 \le t \le 30)$ is a triangle with base 30 and height 15 (since $v(30) = 15$). Therefore the area of this region is $A = \frac{1}{2}(30)(15) = 225$. The object traveled 225 feet in the first 30 minutes.

4.2 Antiderivatives as Areas

Quick Check 1 Concluded

b.) How far does the object travel between the first hour and the second hour?

The region corresponding to the time interval $60 \le t \le 120$ is a trapezoid. It can be decomposed into a rectangle and a triangle. The rectangle has base 60 and height 30 (since $v(60) = 30$). The triangle is has base 60 and height 30 (since $v(120) - v(60) = 60 - 30 = 30$). Therefore the area of this region is $A = (60)(30) + \frac{1}{2}(60)(30) = 2700$. Thus the object traveled 2,700 feet between the first hour and second hour.

4.2 Antiderivatives as Areas

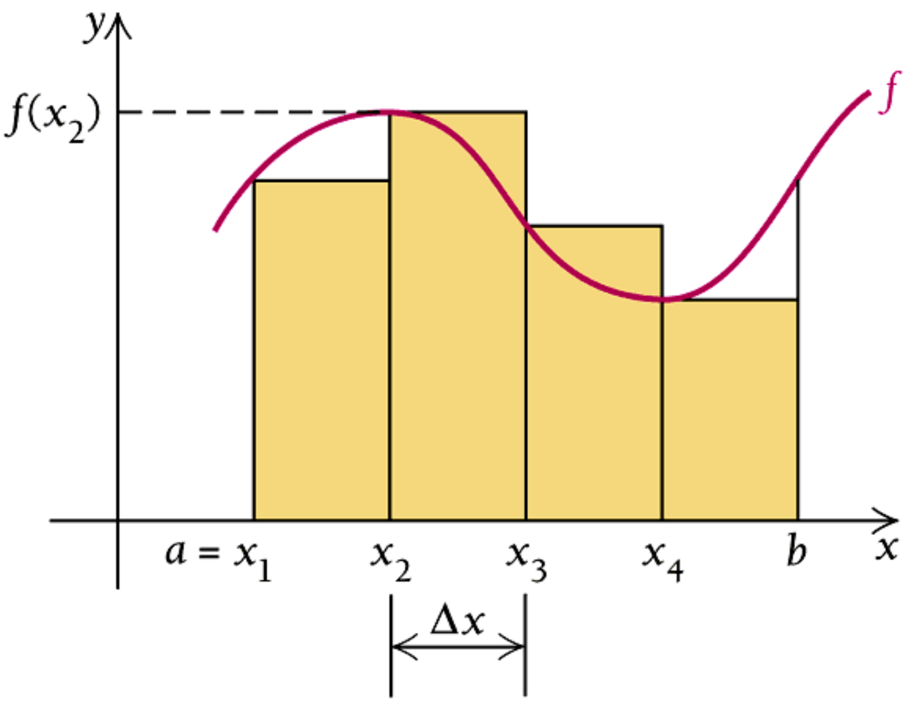
Riemann Sums:

The last two examples, the area function is an antiderivative of the function that generated the graph. Is this always true? Is the formula for the area under the graph of any function the antiderivative of that function? How do we handled curved graphs for which area formulas may not be known? We investigate the questions using geometry, in a procedure called **Riemann summation**.

4.2 Antiderivatives as Areas

Riemann Sums (continued):

In the following figure, $[a, b]$ is divided into four subintervals, each having width $\Delta x = (b - a)/4$.



The heights of the rectangles are $f(x_1)$, $f(x_2)$, $f(x_3)$ and $f(x_4)$.

4.2 Antiderivatives as Areas

Riemann Sums (concluded):

The area of the region under the curve is approximately the sum of the areas of the four rectangles:

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

We can denote this sum with summation, or sigma, notation, which uses the Greek capital letter sigma, or Σ:

$$\sum_{i=1}^{4} f(x_i)\Delta x, \quad \text{or} \quad \textstyle\sum_{i=1}^{4} f(x_i)\Delta x.$$

This is read "the sum of the product $f(x_i)\Delta x$ from $i = 1$ to $i = 4$." To recover the original expression, we substitute the numbers 1 through 4 successively for i in $f(x_i)\Delta x$ and write plus signs between the results.

4.2 Antiderivatives as Areas

Example 3: Write summation notation for

$$2 + 4 + 6 + 8 + 10.$$

Note that we are adding consecutive values of 2.

$$2+4+6+8+10=\sum_{i=1}^{5} 2i$$

4.2 Antiderivatives as Areas

Quick Check 2

Write the summation notation for each expression

a.) $5+10+15+20+25$ Note that we are adding consecutive multiples of 5. Thus,

$$5+10+15+20+25 = \sum_{i=1}^{5} 5i$$

b.) $33+44+55+66$ Note that we are adding consecutive multiples of 11. Thus,

$$33+44+55+66 = \sum_{i=3}^{6} 11i$$

4.2 Antiderivatives as Areas

Example 4: Write summation notation for:

$$g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{19})\Delta x$$

$$g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{19})\Delta x = \sum_{i=1}^{19} g(x_i)\Delta x$$

4.2 Antiderivatives as Areas

Example 5: Express $\sum_{i=1}^{4} 3^i$ without using summation notation.

$$\sum_{i=1}^{4} 3^i = 3^1 + 3^2 + 3^3 + 3^4 = 120$$

4.2 Antiderivatives as Areas

Quick Check 3

Express $\sum_{i=1}^{6}\left(i^2+i\right)$ without using summation notation.

$$\sum_{i=1}^{6}\left(i^2+i\right)=\left(1^2+1\right)+\left(2^2+2\right)+\left(3^2+3\right)+\left(4^2+4\right)+\left(5^2+5\right)+\left(6^2+6\right)$$

$$=2+6+12+20+30+42$$

$$=112$$

4.2 Antiderivatives as Areas

Example 6: Express $\sum_{i=1}^{30} h(x_i)\Delta x$ without using summation notation.

$$\sum_{i=1}^{30} h(x_i)\Delta x = h(x_1)\Delta x + h(x_2)\Delta x + \cdots + h(x_{30})\Delta x$$

4.2 Antiderivatives as Areas

Example 7: Consider the graph of

$$f(x) = 600x - x^2$$

over the interval [0, 600].

a) Approximate the area by dividing the interval into 6 subintervals.

b) Approximate the area by dividing the interval into 12 subintervals.

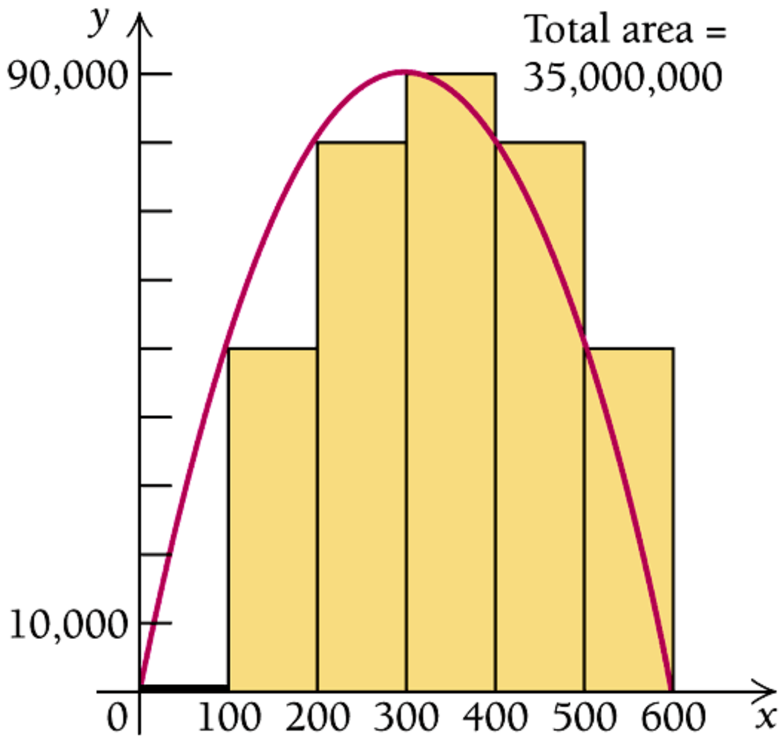
4.2 Antiderivatives as Areas

Example 7 (continued):

a) We divide [0, 600] into 6 intervals of size

$$\Delta x = \frac{600-0}{6} = 100,$$

with x_i ranging from $x_1 = 0$ to $x_6 = 500$.



4.2 Antiderivatives as Areas

Example 7 (continued):

Thus, the area under the curve is approximately

$$\sum_{i=1}^{6} f(x_i)\Delta x = f(0)\cdot 100 + f(100)\cdot 100 + f(200)\cdot 100$$

$$+ f(300)\cdot 100 + f(400)\cdot 100 + f(500)\cdot 100$$

$$= 0\cdot 100 + 50{,}000\cdot 100 + 80{,}000\cdot 100$$

$$+ 90{,}000\cdot 100 + 80{,}000\cdot 100 + 50{,}000\cdot 100$$

$$= 35{,}000{,}000$$

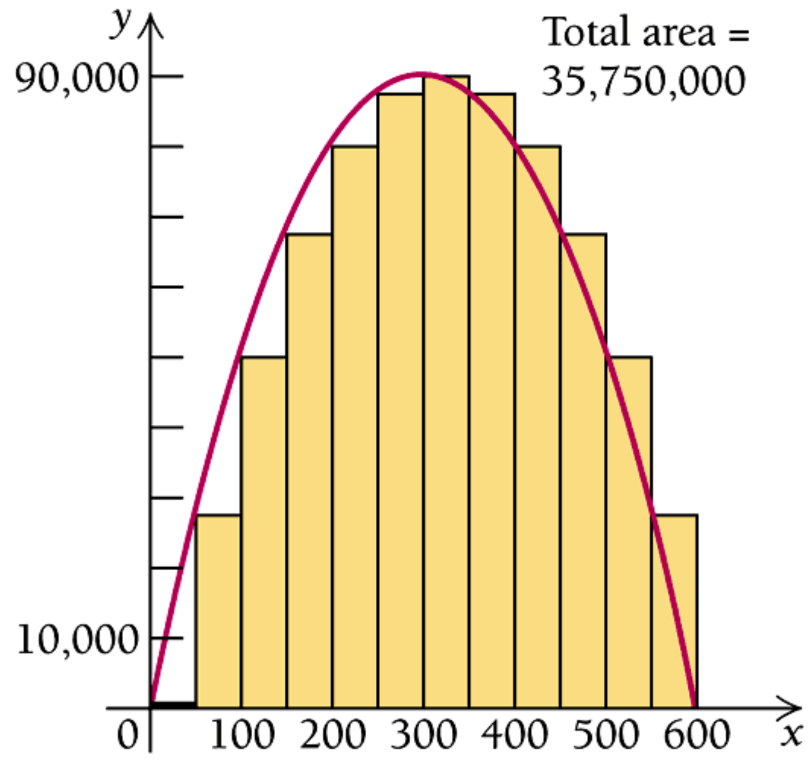
4.2 Antiderivatives as Areas

Example 7 (continued):

b) We divide [0, 600] into 12 intervals of size

$$\Delta x = \frac{600 - 0}{12} = 50,$$

with x_i ranging from $x_1 = 0$ to $x_{12} = 550$.



4.2 Antiderivatives as Areas

Example 7 (concluded):

Thus, the area under the curve is approximately

$$\sum_{i=1}^{12} f(x_i)\Delta x = f(0)\cdot 50 + f(50)\cdot 50 + f(100)\cdot 50 + f(150)\cdot 50$$

$$+ f(200)\cdot 50 + f(250)\cdot 50 + f(300)\cdot 50 + f(350)\cdot 50$$

$$+ f(400)\cdot 50 + f(450)\cdot 50 + f(500)\cdot 50 + f(550)\cdot 50$$

$$= 0\cdot 50 + 27{,}500\cdot 50 + 50{,}000\cdot 50 + 67{,}500\cdot 50$$

$$+80{,}000\cdot 50 + 87{,}500\cdot 50 + 90{,}000\cdot 50 + 87{,}500\cdot 50$$

$$+80{,}000\cdot 50 + 67{,}500\cdot 50 + 50{,}000\cdot 50 + 27{,}500\cdot 50$$

$$= 35{,}750{,}000$$

4.2 Antiderivatives as Areas

Section Summary

- The area under a curve can often be interpreted in a meaningful way.

- The units of the area are found by multiplying the units of the input variable by the units of the output variable. It is crucial that the units are consistent.

- Geometry can be used to find areas of regions formed by graphs of linear functions.

- A *Riemann sum* uses rectangles to approximate the area under a curve. The more rectangles, the better approximation.

- The *definite integral* $\int_a^b f(x)\,dx$, is a representation of the exact area under the graph of a continuous function $y = f(x)$, where $f(x) \geq 0$, over an interval $[a, b]$.

4.3 Area and Definite Integrals

OBJECTIVE

- Find the area under the graph of a nonnegative function over a given closed interval.
- Evaluate a definite integral.
- Solve applied problems involving definite integrals.

4.3 Area and Definite Integrals

To find the area under the graph of a nonnegative, continuous function f over the interval $[a, b]$:

1. Find any antiderivative $F(x)$ of $f(x)$. (The simplest is the one for which the constant of integration is 0.)
2. Evaluate $F(x)$ using b and a, and compute $F(b) - F(a)$. The result is the area under the graph over the interval $[a, b]$.

4.3 Area and Definite Integrals

Example 1: Find the area under the graph of $y = x^2 + 1$ over the interval [– 1, 2].

1. Find any antiderivative $F(x)$ of $f(x)$. We choose the simplest one.

$$F(x) = \frac{x^3}{3} + x$$

Example 1 (concluded):

2. Substitute 2 and –1, and find the difference $F(2) - F(-1)$.

$$
\begin{aligned}
F(2) - F(-1) &= \left[\frac{2^3}{3} + 2\right] - \left[\frac{(-1)^3}{3} + (-1)\right] \\
&= \frac{8}{3} + 2 - \left[-\frac{1}{3} - 1\right] \\
&= \frac{8}{3} + 2 + \frac{1}{3} + 1 \\
&= 6
\end{aligned}
$$

4.3 Area and Definite Integrals

Quick Check 1

Refer to the function in Example 1.

a.) Calculate the area over the interval $[0,5]$.

b.) Calculate the area over the interval $[-2,2]$.

c.) Can you suggest a shortcut for part (b)?

a.) $f(x)=x^2+1$, so $F(x)=\dfrac{x^3}{3}+x$.

Substitute 0 and 5, and find the difference $F(5)-F(0)$.

$$F(5)-F(0)=\left(\frac{5^3}{3}+5\right)-\left(\frac{0^3}{3}+0\right)=\frac{125}{3}+5=46\frac{2}{3}$$

4.3 Area and Definite Integrals

Quick Check 1 Concluded

b.) Calculate the area over the interval $[-2, 2]$.

$$F(2) - F(-2) = \left(\frac{2^3}{3} + 2\right) - \left(\frac{(-2)^3}{3} + (-2)\right) = 4\frac{2}{3} - \left(-4\frac{2}{3}\right)$$

$$= 9\frac{1}{3}$$

c.) Can you suggest a shortcut for part (b)?

Note that $F(-2) = -F(2)$. Then we have

$$F(2) - F(-2) = F(2) - (-F(2)) = 2F(2).$$

Thus we would integrate from 0 to 2, then double the results. This is because the graph of *f* is symmetric with the y-axis.

4.3 Area and Definite Integrals

DEFINITION:

Let f be any continuous function over the interval $[a, b]$ and F be any antiderivative of f. Then, the **definite integral** of f from a to b is

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

4.3 Area and Definite Integrals

Example 2: Evaluate $\int_a^b x^2 dx.$

Using the antiderivative $F(x) = x^3/3$, we have

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

It is convenient to use an intermediate notation:

$$\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

4.3 Area and Definite Integrals

Example 3: Evaluate each of the following:

a) $\int_{-1}^{4} (x^2 - x)\ dx$;

b) $\int_{0}^{3} e^x\ dx$;

c) $\int_{1}^{e} \left(1 + 2x - \frac{1}{x}\right) dx$ (assume $x > 0$).

Example 3 (continued):

a) $$\int_{-1}^{4} (x^2 - x)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{4}$$

$$= \left(\frac{4^3}{3} - \frac{4^2}{2}\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)$$

$$= \frac{64}{3} - \frac{16}{2} - \left(-\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{64}{3} - 8 + \frac{1}{3} + \frac{1}{2} = 14\frac{1}{6}$$

4.3 Area and Definite Integrals

Example 3 (continued):

b) $\int_0^3 e^x \, dx = \left[e^x\right]_0^3 = e^3 - e^0$

$= e^3 - 1$

≈ 19.086

4.3 Area and Definite Integrals

Example 3 (concluded):

c) $$\int_1^e \left(1+2x-\frac{1}{x}\right)dx = \left[x+x^2-\ln x\right]_1^e$$

$$= (e+e^2-\ln e)-(1+1^2-\ln 1)$$

$$= (e+e^2-1)-(1+1-0)$$

$$= e+e^2-1-1-1$$

$$= e+e^2-3$$

$$\approx 7.107$$

4.3 Area and Definite Integrals

Quick Check 2

Evaluate each of the following:

a.) $\int_{2}^{4}\left(2x^{3}-3x\right)dx$

b.) $\int_{0}^{\ln 4} 2e^{x}\,dx$

c.) $\int_{1}^{5}\frac{x-1}{x}\,dx$

4.3 Area and Definite Integrals

Quick Check 2 Continued

a.) $\int_2^4 \left(2x^3 - 3x\right)dx = \left[\frac{1}{2}x^4 - \frac{3}{2}x^2\right]_2^4$

$$= \left(\frac{1}{2}(4)^4 - \frac{3}{2}(4)^2\right) - \left(\frac{1}{2}(2)^4 - \frac{3}{2}(2)^2\right)$$

$$= (128 - 24) - (8 - 6)$$

$$= 102$$

4.3 Area and Definite Integrals

Quick Check 2 Continued

b.) $\int_0^{\ln 4} 2e^x dx = \left[2e^x\right]_0^{\ln 4} = \left(2e^{\ln 4}\right) - \left(2e^0\right) = 2(4) - 2(1) = 6$

c.) $\int_1^5 \frac{x-1}{x} dx = \int_1^5 \left(1 - \frac{1}{x}\right) dx = \left[x - \ln x\right]_1^5 = (5 - \ln 5) - (1 - \ln 1)$

$= (5 - \ln 5) - (1 - 0) = 4 - \ln 5 \approx 2.39$

THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If a continuous function f has an antiderivative F over $[a, b]$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\ dx = F(b) - F(a).$$

4.3 Area and Definite Integrals

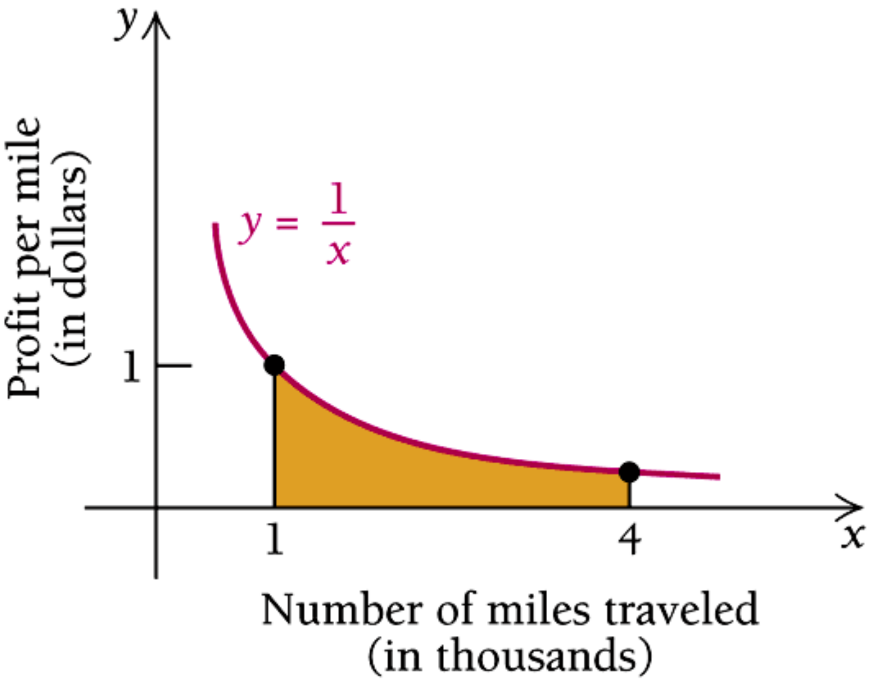
Example 4: Suppose that y is the profit per mile traveled and x is number of miles traveled, in thousands. Find the area under $y = 1/x$ over the interval [1, 4] and interpret the significance of this area.

$$\int_1^4 \frac{dx}{x} = \left[\ln x\right]_1^4$$

$$= \ln 4 - \ln 1$$

$$= \ln 4 - 0$$

$$\approx 1.3863$$



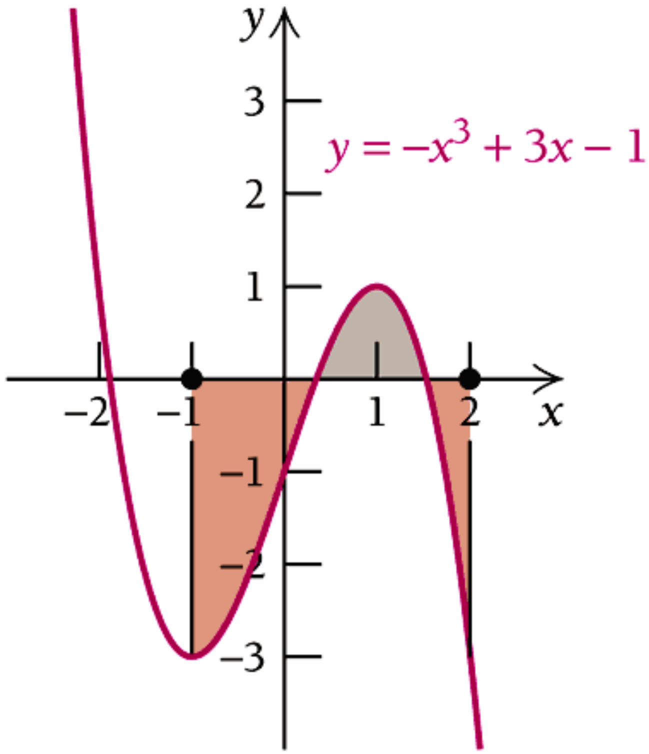
4.3 Area and Definite Integrals

Example 4 (concluded):

The area represents a total profit of $1386.30 when the miles traveled increase from 1000 to 4000 miles.

4.3 Area and Definite Integrals

Example 5: Consider $\int_{-1}^{2}(-x^3+3x-1)\,dx$. Predict the sign of the integral by examining the graph, and then evaluate the integral.



From the graph, it appears that there is considerably more area below the x-axis than above. Thus, we expect that the sign of the integral will be negative.

4.3 Area and Definite Integrals

Example 5 (concluded):

Evaluating the integral, we have

$$\int_{-1}^{2}\left(-x^3+3x-1\right)dx=\left[-\frac{x^4}{4}+\frac{3}{2}x^2-x\right]_{-1}^{2}$$

$$=\left(-\frac{2^4}{4}+\frac{3}{2}\cdot 2^2-2\right)-\left(-\frac{(-1)^4}{4}+\frac{3}{2}\cdot(-1)^2-(-1)\right)$$

$$=\left(-4+6-2\right)-\left(-\frac{1}{4}+\frac{3}{2}+1\right)$$

$$=0-2\frac{1}{4}=-2\frac{1}{4}$$

4.3 Area and Definite Integrals

Example 6: Northeast Airlines determines that the marginal profit resulting from the sale of x seats on a jet traveling from Atlanta to Kansas City, in hundreds of dollars, is given by

$$P'(x) = \sqrt{x} - 6.$$

Find the total profit when 60 seats are sold.

4.3 Area and Definite Integrals

Example 6 (continued):

We integrate to find $P(60)$.

$$P(60) = \int_0^{60} P'(x)\,dx$$

$$= \int_0^{60} \left(\sqrt{x} - 6\right) dx$$

$$= \left[\frac{2}{3}x^{3/2} - 6x\right]_0^{60}$$

$$= \left(\frac{2}{3} \cdot 60^{3/2} - 6 \cdot 60\right) - \left(\frac{2}{3} \cdot 0^{3/2} - 6 \cdot 0\right)$$

$$\approx -50.1613$$

4.3 Area and Definite Integrals

Example 6 (concluded):

When 60 seats are sold, Northeast's profit is –$5016.13. That is, the airline will lose $5016.13 on the flight.

4.3 Area and Definite Integrals

Quick Check 3

Referring to Example 6, find the total profit of Northeast Airlines when 140 seats are sold.

From Example 6, we have $P'(x) = \sqrt{x} - 6.$

We integrate to find $P(140)$.

$$P(140) = \int_0^{140} \left(\sqrt{x} - 6\right) dx = \left[\frac{2}{3}x^{3/2} - 6x\right]_0^{140}$$

$$= \left(\frac{2}{3}(140)^{3/2} - 6(140)\right) - \left(\frac{2}{3}(0)^{3/2} - 6(0)\right) \approx 264.3349$$

When 140 seats are sold, Northeast Airlines makes

$$\$100 \cdot 264.3349 = \$26.433.49.$$

4.3 Area and Definite Integrals

Example 7: A particle starts out from some origin. Its velocity, in miles per hour, is given by

$$v(t) = \sqrt{t} + t,$$

where t is the number of hours since the particle left the origin. How far does the particle travel during the second, third, and fourth hours (from $t = 1$ to $t = 4$)?

Example 7 (continued):

Recall that velocity, or speed, is the rate of change of distance with respect to time. In other words, velocity is the derivative of the distance function, and the distance function is an antiderivative of the velocity function. To find the total distance traveled from $t = 1$ to $t = 4$, we evaluate the integral

$$\int_1^4 \left(\sqrt{t} + t\right) dt.$$

4.3 Area and Definite Integrals

Example 7 (concluded):

$$\int_1^4 \left(\sqrt{t} + t\right) dt = \int_1^4 \left(t^{1/2} + t\right) dt$$

$$= \left[\frac{2}{3}t^{3/2} + \frac{1}{2}t^2\right]_1^4$$

$$= \frac{2}{3}\cdot 4^{3/2} + \frac{1}{2}\cdot 4^2 - \left(\frac{2}{3}\cdot 1^{3/2} + \frac{1}{2}\cdot 1^2\right)$$

$$= \frac{16}{3} + \frac{16}{2} - \frac{2}{3} - \frac{1}{2} = \frac{14}{3} + \frac{15}{2} = \frac{73}{6}$$

$$= 12\frac{1}{6}\text{ mi.}$$

4.3 Area and Definite Integrals

Section Summary

- The exact area between the x-axis and the graph of the nonnegative continuous function $y = f(x)$ over the interval $[a,b]$ is found by evaluating the *definite integral*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is an antiderivative of f.

- If a function has areas both below and above the x-axis, the definite integral gives the net total area, or the difference between the sum of the areas above the x-axis and the sum of the areas below the x-axis.

4.3 Area and Definite Integrals

Section Summary Concluded

- If there is more area above the x-axis than below, the definite integral will be positive.
- If there is more area below the x-axis than above, the definite integral will be negative.
- If the areas above and below the x-axis are the same, the definite integral will be 0.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

OBJECTIVE

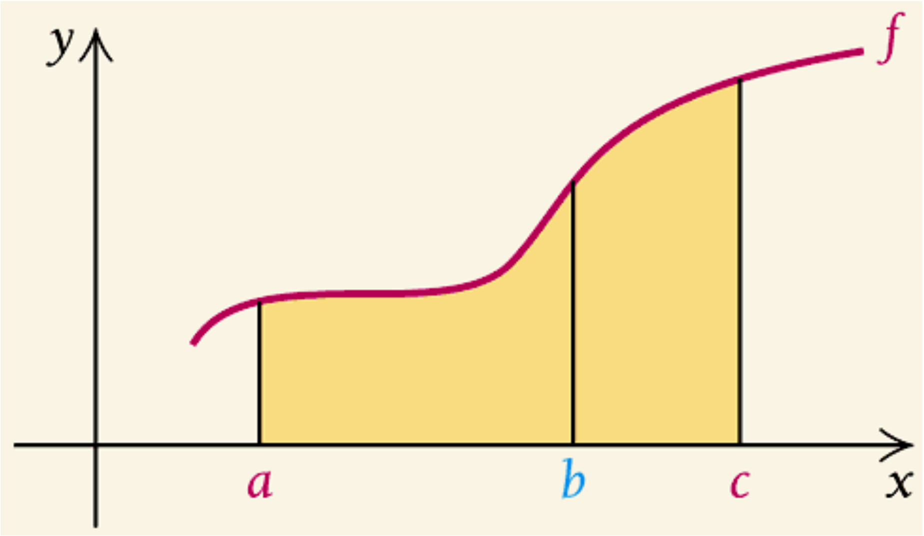
- Use the properties of definite integrals to find the area between curves.
- Solve applied problems involving definite integrals.
- Determine the average value of a function.
- Find the moving average of a function

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

THEOREM 5

For $a < b < c$, $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$

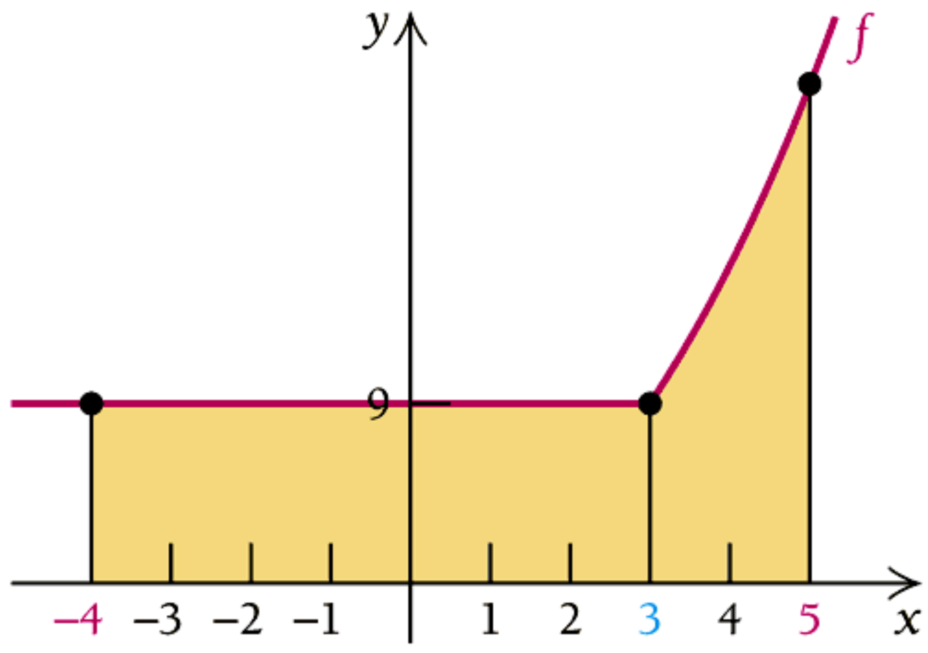
For any number b between a and c, the integral from a to c is the integral from a to b plus the integral from b to c.



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 1: Find the area under the graph of $y = f(x)$ from –4 to 5, where

$$f(x) = \begin{cases} 9, & \text{for } x < 3, \\ x^2, & \text{for } x \geq 3. \end{cases}$$



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 1 (concluded):

$$\begin{aligned}
\int_{-4}^{5} f(x)\,dx &= \int_{-4}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx \\
&= \int_{-4}^{3} 9\ dx + \int_{3}^{5} x^2\,dx \\
&= 9\left[x\right]_{-4}^{3} + \left[\frac{x^3}{3}\right]_{3}^{5} \\
&= 9\left(3-(-4)\right) + \left(\frac{5^3}{3} - \frac{3^3}{3}\right) \\
&= 95\frac{2}{3}
\end{aligned}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 1

Find the area under the graph of $y = g(x)$ from -3 to 6, where

$$g(x) = \begin{cases} x^2, & \text{for } x \le 2 \\ 8 - x, & \text{for } x > 2 \end{cases}$$

$$\int_{-3}^{6} g(x)\,dx = \int_{-3}^{2} g(x)\,dx + \int_{2}^{6} g(x)\,dx = \int_{-3}^{2} x^2\,dx + \int_{2}^{6} (8 - x)\,dx$$

$$= \left[\frac{1}{3}x^3\right]_{-3}^{2} + \left[8x - \frac{1}{2}x^2\right]_{2}^{6}$$

$$= \left(\frac{1}{3}(2)^3\right) - \left(\frac{1}{3}(-3)^3\right) + \left(\left(8(6) - \frac{1}{2}(6)^2\right) - \left(8(2) - \frac{1}{2}(2)^2\right)\right)$$

$$= \left(\frac{8}{3} + \frac{27}{3}\right) + (30 - 14) = 11\frac{2}{3} + 16 = 27\frac{2}{3}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 2

Evaluate $\int_0^7 |2x-1|\,dx.$

The function $f(x) = |2x-1|$ is defined piecewise as follows

$$f(x) = |2x-1| = \begin{cases} 2x-1, & \text{for } x \geq \dfrac{1}{2} \\ 1-2x, & \text{for } x < \dfrac{1}{2} \end{cases}.$$

From here we can use our piecewise integration technique to solve.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 2 Concluded

$$\int_0^7 |2x-1|\,dx = \int_0^{1/2} (1-2x)\,dx + \int_{1/2}^7 (2x-1)\,dx$$

$$= \left[x - x^2\right]_0^{1/2} + \left[x^2 + x\right]_{1/2}^7$$

$$= \left(\left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right) - \left(0 - 0^2\right)\right) + \left(\left(7^2 - 7\right) - \left(\left(\frac{1}{2}\right)^2 - \frac{1}{2}\right)\right)$$

$$= \left(\frac{1}{4} - 0\right) + \left(42 + \frac{1}{4}\right)$$

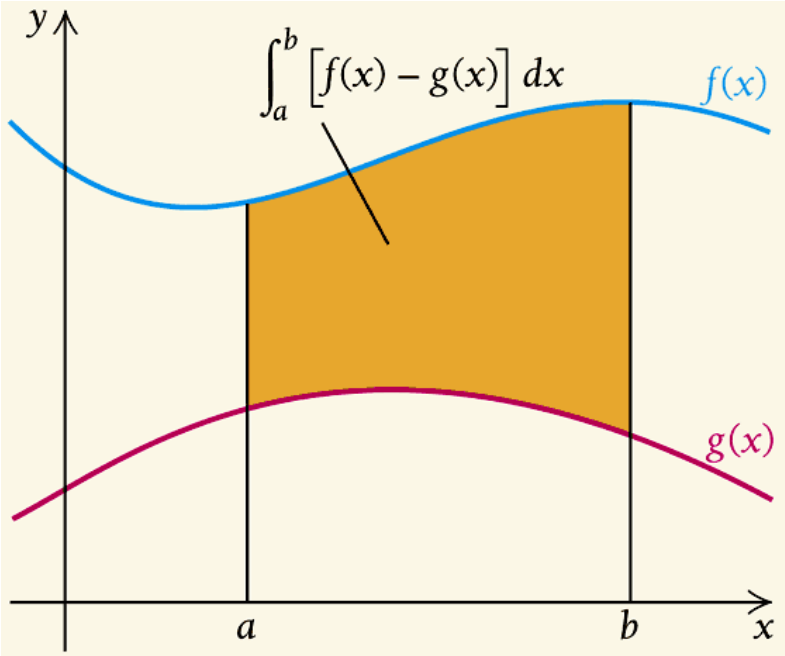
$$= 42\frac{1}{2}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

THEOREM 6

Let f and g be continuous functions and suppose that $f(x) \geq g(x)$ over the interval $[a, b]$. Then the area of the region between the two curves, from $x = a$ to $x = b$, is

$$\int_a^b \left[f(x) - g(x) \right] dx.$$



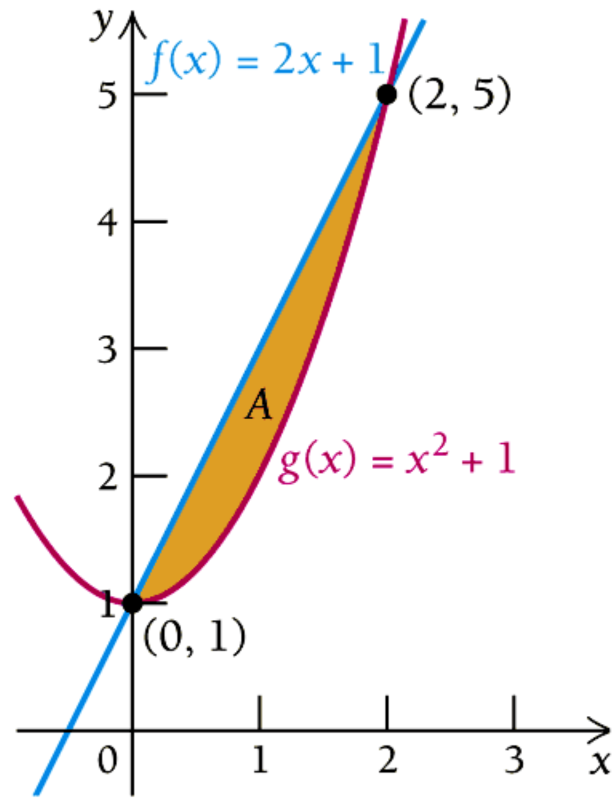
4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 2: Find the area of the region that is bounded by the graphs of

$f(x) = 2x + 1$ and

$g(x) = x^2 + 1.$

First, look at the graph of these two functions.



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 2 (continued):

Second, find the points of intersection by setting $f(x) = g(x)$ and solving.

$$\begin{aligned} f(x) &= g(x) \\ 2x+1 &= x^2+1 \\ 0 &= x^2-2x \\ 0 &= x(x-2) \end{aligned}$$

$$x = 0 \quad \text{or} \quad x = 2$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 2 (concluded):

Lastly, compute the integral. Note that on [0, 2], $f(x)$ is the upper graph.

$$\int_0^2 \left[(2x+1)-(x^2+1)\right]\,dx = \int_0^2 (2x - x^2)\,dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= \left(2^2 - \frac{2^3}{3}\right) - \left(0^2 + \frac{0^3}{3}\right)$$

$$= 4 - \frac{8}{3} - 0 + 0 = \frac{4}{3}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 3: A clever college student develops an engine that is believed to meet all state standards for emission control. The new engine's rate of emission is given by

$$E(t) = 2t^2,$$

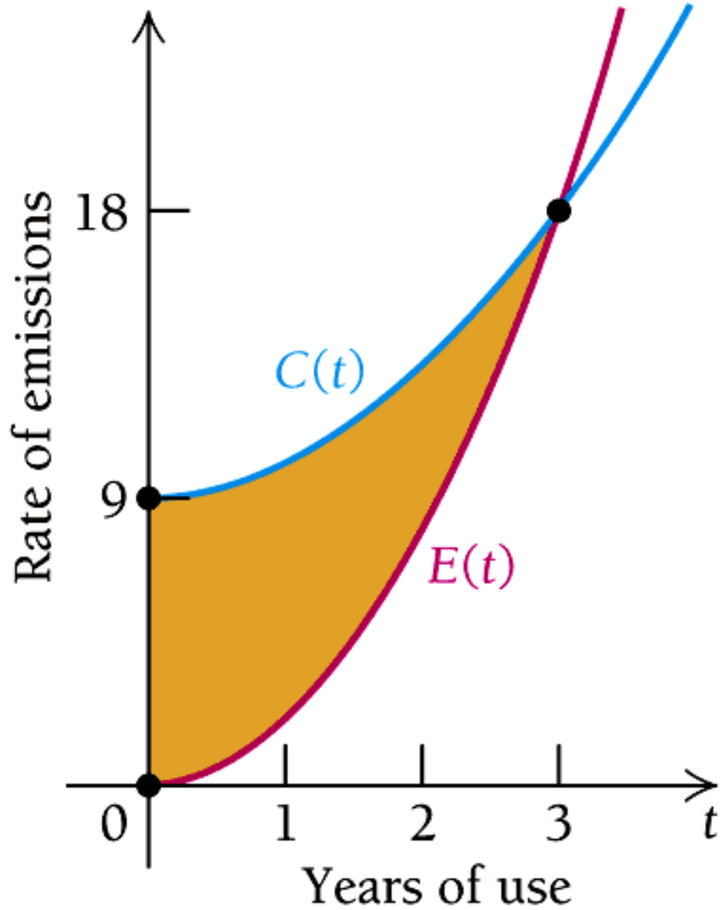
where $E(t)$ is the emissions, in billions of pollution particulates per year, at time t, in years. The emission rate of a conventional engine is given by

$$C(t) = 9 + t^2.$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 3 (continued):

The graphs of both curves are shown at the right.



a) At what point in time will the emission rates be the same?

b) What is the reduction in emissions resulting from using the student's engine?

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 3 (continued):

a) The emission rates will be the same when $E(t) = C(t)$.

$$\begin{aligned} 2t^2 &= 9 + t^2 \\ t^2 - 9 &= 0 \\ (t-3)(t+3) &= 0 \\ t = 3 \quad \text{or} \quad t &= -3 \end{aligned}$$

Since negative time has no meaning in this problem, the emission rates will be the same when $t = 3$ yr.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 3 (concluded):

b) The reduction in emissions is represented by the area between the graphs of $E(t)$ and $C(t)$ from $t = 0$ to $t = 3$.

$$\int_0^3 \left[(9 + t^2 - 2t^2\right]\,dt = \int_0^3 (9 - t^2)\,dt = \left[9t - \frac{t^3}{3}\right]_0^3$$

$$= \left(9 \cdot 3 - \frac{9^3}{3}\right) - \left(9 \cdot 0 - \frac{0^3}{3}\right)$$

$$= 27 - 9 = 18$$

$$= 18 \text{ billion pollution particulates per year.}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

DEFINITION:

Let f be a continuous function over a closed interval $[a, b]$. Its **average value**, $\boldsymbol{y_{\text{av}}}$, over $[a, b]$ is given by

$$y_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

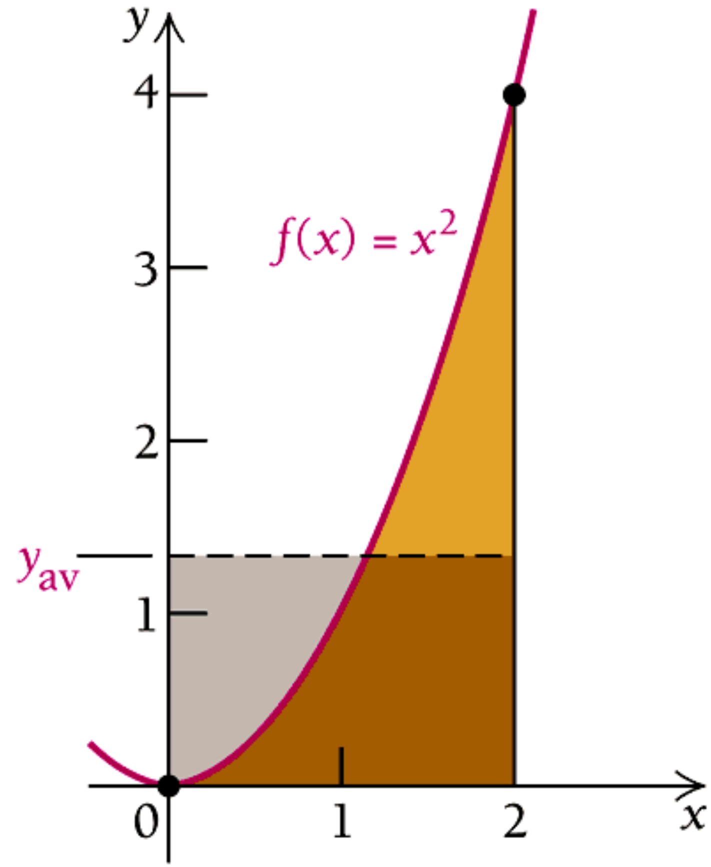
Example 4: Find the average value of $f(x) = x^2$ over the interval [0, 2].

$$\frac{1}{2-0}\int_0^2 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\left(\frac{2^3}{3} - \frac{0^3}{3}\right)$$

$$= \frac{1}{2}\cdot\frac{8}{3}$$

$$= \frac{4}{3} = 1\frac{1}{3}$$



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 5: Rico's speed, in miles per hour, t minutes after entering the freeway is given by

$$v(t) = -\frac{1}{200}t^3 + \frac{3}{20}t^2 - \frac{3}{8}t + 60, \quad t \leq 30.$$

From 5 min after entering the freeway to 25 min, what was Rico's average speed? How far did he travel over that time interval?

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 5 (continued):

The average speed is

$$\frac{1}{25-5}\int_{5}^{25}\left(-\frac{1}{200}t^{3}+\frac{3}{20}t^{2}-\frac{3}{8}t+60\right)dt$$

$$=\frac{1}{20}\left[-\frac{1}{800}t^{4}+\frac{1}{20}t^{3}-\frac{3}{16}t^{2}+60t\right]_{5}^{25}$$

$$=\frac{1}{20}\left(\frac{53,625}{32}-\frac{9625}{32}\right)$$

$$=68\frac{3}{4}\text{ mph.}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 5 (concluded):

To find how far Rico traveled over the time interval [5, 25], we first note that *t* is given in minutes, not hours. Since 25 min – 5 min = 20 min is 1/3 hr, the distance traveled over [5, 25] is

$$\frac{1}{3} \cdot 68\frac{3}{4} = 22\frac{11}{12}\text{ mi.}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 3

The temperature, in degrees Fahrenheit, in Minneapolis on a winter's day is modeled by the function

$$f(x) = -0.012x^3 + 0.38x^2 - 1.99x - 10.1,$$

where x is the number of hours from midnight $(0 \le x \le 24)$. Find the average temperature in Minneapolis during this 24-hour period.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 3 Concluded

The average temperature is

$$= \frac{1}{24-0}\int_0^{24}\left(-0.012x^3 + 0.38x^2 - 1.99x - 10.1\right)dx$$

$$= \frac{1}{24}\left[-\frac{0.012}{4}x^4 + \frac{0.38}{3}x^3 - \frac{1.99}{2}x^2 - 10.1x\right]_0^{24}$$

$$= \frac{1}{24}\left(-\frac{0.012}{4}(24)^4 + \frac{0.38}{3}(24)^3 - \frac{1.99}{2}(24)^2 - 10.1(24)\right)$$

$$= \frac{1}{24}\left(-995.328 + 1751.04 - 573.12 - 242.4\right)$$

$$\approx -2.5$$

So the average temperature is approximately -2.5°F.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Definition

Let f be a continuous function over $[a, b]$. The **moving average function** of f is given by

$$f_{av}(x) = \frac{1}{L}\int_{x}^{x+L} f(t)\,dt,$$

where L is a constant representing a fixed interval of time.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6: Business: Moving Average. The weekly revenue, $R(x)$, in thousands of dollars, of Trux Rentals x weeks since the start of the year is given by

$$R(x) = 0.05x^4 - 1.2x^3 + 9.6x^2 - 28.6x + 38.3,$$

Where $0 \le x \le 10.$ Find the moving average over 3-week intervals.

Solution: We have $L = 3$. To find the moving average for the first 3 weeks, we set $x = 0$:

$$R_{av}(x) = \frac{1}{L}\int_x^{x+L} R(t)\,dt = R_{av}(0) = \frac{1}{3}\int_0^{0+3} R(t)\,dt = \frac{1}{3}\int_0^{3} R(t)\,dt$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 continued:

$$R_{av}(0) = \frac{1}{3}\int_0^3 (0.05t^4 - 1.2t^3 + 9.6t^2 - 28.6t + 38.3)\,dt$$

$$= \frac{1}{3}\left[0.01t^5 - 0.3t^4 + 3.2t^3 - 14.3t^2 + 38.3t\right]_0^3$$

$$\approx \frac{1}{3}(50.73) = 16.91.$$

Thus, over the first 3 weeks of the year, Trux Rentals had an average revenue of \$16,910 per week.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 continued:

The moving average over the second 3 weeks—that is, between $x = 1$ and $x = 4$ is:

$$R_{av}(1) = R_{av}(1) = \frac{1}{3}\int_{1}^{1+3} R(t)\,dt = \frac{1}{3}\int_{1}^{4} R(t)\,dt$$

$$= \frac{1}{3}\int_{1}^{4}(0.05t^4 - 1.2t^3 + 9.6t^2 - 28.6t + 38.3)dt$$

$$= \frac{1}{3}\Big[0.01t^5 - 0.3t^4 + 3.2t^3 - 14.3t^2 + 38.3t\Big]_1^4$$

$$\approx \frac{1}{3}\big(37.73\big) = 11.91.$$

Thus, over these weeks, Trux Rentals had an average revenue of \$16,910 per week.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 continued:

The moving average over successive 3 week intervals can be found in a similar fashion.

$$\text{Weeks } 2-5\text{:}\quad R_{av}(2) = \frac{1}{3}\int_{2}^{5} R(t)\,dt \approx 12.41 \quad \text{or } \$12{,}410/\text{week}$$

$$\text{Weeks } 3-6\text{:}\quad R_{av}(3) = \frac{1}{3}\int_{3}^{6} R(t)\,dt \approx 14.81 \quad \text{or } \$14{,}810/\text{week}$$

$$\text{Weeks } 4-7\text{:}\quad R_{av}(4) = \frac{1}{3}\int_{4}^{7} R(t)\,dt \approx 16.71 \quad \text{or } \$16{,}710/\text{week}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 continued:

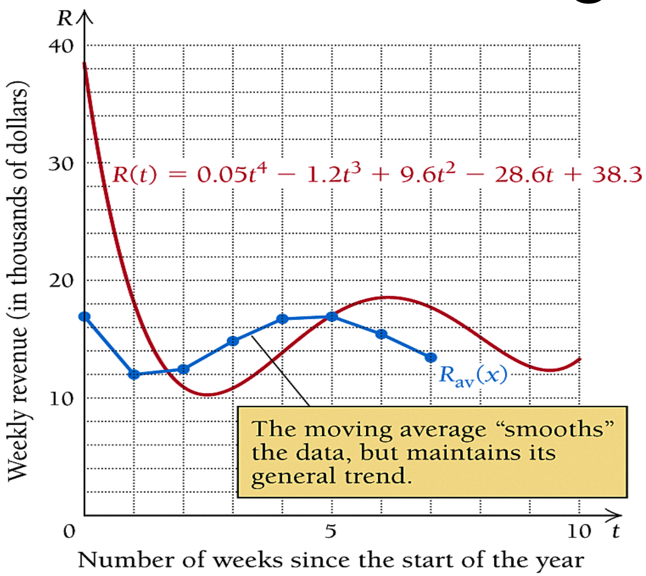
Weeks 5 – 8: $R_{av}(5) = \frac{1}{3}\int_5^8 R(t)\, dt \approx 16.91$ or \$16,910/week

Weeks 6 – 9: $R_{av}(6) = \frac{1}{3}\int_6^9 R(t)\, dt \approx 15.41$ or \$15,410/week

Weeks 7 – 10: $R_{av}(7) = \frac{1}{3}\int_7^{10} R(t)\, dt \approx 13.41$ or \$13,410/week

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 concluded: The graph below displays R, along with a graph of the moving average, R_{av}, over these seven 3-week periods of time. The graph of the moving average retains the general shape of the graph of R but reduces the fluctuations, making the general trend easier to see.



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Section Summary

- The *additive property of definite integrals* states that a definite integral can be expressed as the sum of two (or more) other definite integrals. If f is continuous on $[a, c]$ and we chose b such that $a < b < c$, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

- The area of a region bounded by the graphs of two functions, $f(x)$ and $g(x)$, where $f(x) \geq g(x)$ over an interval $[a, b]$, is

$$A = \int_a^b \left[f(x) - g(x) \right] dx.$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Section Summary Concluded

- The *average value* of a continuous function f over an interval $[a,b]$ is

$$y_{AV} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

- Let f be a continuous function over $[a, b]$. The **moving average function** of f is given by

$$f_{av}(x) = \frac{1}{L}\int_x^{x+L} f(t)\,dt,$$

where L is a constant representing a fixed interval of time.

4.5 Integration Techniques: Substitution

OBJECTIVE

- Evaluate integrals using substitution.
- Solve applied problems involving integration by substitution.

4.5 Integration Techniques: Substitution

The following formulas provide a basis for an integration technique called *substitution*.

A. $\int u^r\,du = \dfrac{u^{r+1}}{r+1} + C,\quad$ assuming $r \neq -1$

B. $\int e^u\,du = e^u + C$

C. $\int \dfrac{1}{u}\,du = \ln|u| + C;\quad$ or

$$\int \frac{1}{u}\,du = \ln u + C,\quad u > 0$$

(Unless noted otherwise, we will assume $u > 0$.)

4.5 Integration Techniques: Substitution

Example 1: For $y = f(x) = x^3$, find dy.

$$\frac{dy}{dx} = f'(x) = 3x^2$$

$$dy = 3x^2 dx$$

4.5 Integration Techniques: Substitution

Example 2: For $u = F(x) = x^{2/3}$, find du.

$$\frac{du}{dx} = F'(x) = \frac{2}{3}x^{-1/3}$$

$$du = \frac{2}{3}x^{-1/3}dx$$

4.5 Integration Techniques: Substitution

Example 3: For $y = f(x) = e^{x^2}$, find dy.

$$\frac{dy}{dx} = f'(x) = e^{x^2} \cdot 2x$$

$$dy = 2xe^{2x}dx$$

4.5 Integration Techniques: Substitution

Quick Check 1

Find each differential.

a.) For $y = \sqrt{x}$, find dy.

b.) For $y = \dfrac{1}{x^3}$, find dy.

c.) For $u = x^2 - 3x$, find du.

d.) For $u = 4x - 3$, find du.

4.5 Integration Techniques: Substitution

Quick Check 1 Continued

a.) For $y = \sqrt{x}$, find dy.

$$\frac{dy}{dx} = y'$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$dy = \frac{1}{2\sqrt{x}}dx$$

b.) For $u = x^2 - 3x$, find du.

$$\frac{du}{dx} = u'$$

$$\frac{du}{dx} = 2x - 3$$

$$du = (2x - 3)dx$$

4.5 Integration Techniques: Substitution

Quick Check 1 Concluded

c.) For $y = \dfrac{1}{x^3}$, find dy.

$$\frac{dy}{dx} = y'$$

$$\frac{dy}{dx} = -\frac{3}{x^4}$$

$$dy = -\frac{3}{x^4}dx$$

d.) For $u = 4x - 3$, find du.

$$\frac{du}{dx} = u'$$

$$\frac{du}{dx} = 4$$

$$du = 4dx$$

4.5 Integration Techniques: Substitution

Example 4: Evaluate: $\int 3x^2(x^3+1)^{10}\,dx.$

Note that $3x^2$ is the derivative of x^3. Thus,

$$\int 3x^2(x^3+1)^{10}\,dx = \int (x^3+1)^{10}\,3x^2\,dx$$

$$\xrightarrow{\text{Substitution}} \quad \begin{aligned} u &= x^3+1 \\ du &= 3x^2\,dx \end{aligned} \quad \longrightarrow \quad = \int u^{10}\,du$$

$$= \frac{u^{11}}{11} + C$$

$$\xleftarrow{\text{Reversing the Substitution}}$$

$$= \frac{(x^3+1)^{11}}{11} + C$$

4.5 Integration Techniques: Substitution

Quick Check 2

Evaluate: $\int 4x\left(2x^2+3\right)^3\,dx.$

Note that $4x$ is the derivative of $2x^2$. Thus,

$$\int 4x\left(2x^2+3\right)^3 dx = \int \left(2x^2+3\right)^3 4x\,dx \quad \xrightarrow{\text{Substitution}} \quad \begin{aligned} u &= 2x^2+3 \\ du &= 4x\,dx \end{aligned}$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{1}{4}\left(2x^2+3\right)^4 + C \quad \xleftarrow{\text{Reverse the Substitution}}$$

4.5 Integration Techniques: Substitution

Example 5: Evaluate: $\int \frac{2x\ dx}{1+x^2}$.

$$\int \frac{2x\ dx}{1+x^2} \xrightarrow{\text{Substitution}} \begin{aligned} u &= 1+x^2 \\ du &= 2x\ dx \end{aligned}$$

$$= \int \frac{du}{u}$$

$$= \ln u + C \quad \text{(Reverse the substitution)}$$

$$= \ln(1+x^2) + C$$

4.5 Integration Techniques: Substitution

Example 6: Evaluate: $\int \frac{2x\ dx}{\left(1+x^2\right)^2}$.

$$\int \frac{2x\ dx}{\left(1+x^2\right)^2} \xrightarrow{\text{Substitution}} \begin{aligned} u &= 1+x^2 \\ du &= 2x\ dx \end{aligned}$$

$$= \int \frac{du}{u^2}$$

$$= \int u^{-2} du$$

$$= -u^{-1} + C$$

$$= -\frac{1}{1+x^2} + C$$

4.5 Integration Techniques: Substitution

Quick Check 3

Evaluate: $\int \frac{6x^2}{\sqrt{3+2x^3}}\,dx.$

$$\int \frac{6x^2}{\sqrt{3+2x^3}}\,dx = \int \frac{6x^2}{\left(3+2x^3\right)^{\frac{1}{2}}}dx \xrightarrow{\text{Substitution}} \begin{aligned} u &= 3+2x^3 \\ du &= 6x^2\,dx \end{aligned}$$

$$= \int \frac{1}{u^{\frac{1}{2}}}$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{3+2x^3} + C \xleftarrow{\text{Reverse the Substitution}}$$

4.5 Integration Techniques: Substitution

Example 7: Evaluate: $\int \frac{\ln(3x)\ dx}{x}$.

$$\int \frac{\ln(3x)\ dx}{x} \xrightarrow{\text{Substitution}} \begin{aligned} u &= \ln(3x) \\ du &= \frac{1}{x}\ dx \end{aligned}$$

$$= \int u\ du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{\left(\ln(3x)\right)^2}{2} + C$$

4.5 Integration Techniques: Substitution

Example 8: Evaluate: $\int xe^{x^2}\,dx.$

$$\int xe^{x^2}\,dx = \frac{1}{2}\int 2xe^{x^2}\,dx$$

$$= \frac{1}{2}\int e^{x^2}(2x\ dx) \qquad = \frac{1}{2}\int e^u\,du$$

$$\xrightarrow{\text{Substitution}} \quad u = x^2 \qquad = \frac{1}{2}e^u + C$$

$$du = 2x\ dx$$

$$= \frac{1}{2}e^{x^2} + C$$

4.5 Integration Techniques: Substitution

Quick Check 4

Evaluate: $\int x^2 e^{4x^3}\,dx$.

$$\int x^2 e^{4x^3} = \frac{1}{12}\int 12x^2 e^{4x^3}\,dx \xrightarrow{\text{Substitution}} \begin{aligned} u &= 4x^3 \\ du &= 12x^2 \end{aligned}$$

$$= \frac{1}{12}\int e^u\,du$$

$$= \frac{1}{12}e^u + C$$

$$= \frac{1}{12}e^{4x^3} + C \xleftarrow{\text{Reverse the Substitution}}$$

4.5 Integration Techniques: Substitution

Example 9: Evaluate: $\int \frac{dx}{x+3}$.

$$\int \frac{dx}{x+3} \xrightarrow{\text{Substitution}} \begin{aligned} u &= x+3 \\ du &= dx \end{aligned}$$

$$= \int \frac{du}{u}$$

$$= \ln u + C$$

$$= \ln(x+3) + C$$

4.5 Integration Techniques: Substitution

Example 10: Evaluate: $\int_0^1 5x\sqrt{x^2+3}\,dx.$

We first find the indefinite integral and then evaluate the integral over [0, 1].

$$\int 5x\sqrt{x^2+3}\,dx = \frac{5}{2}\int 2x\sqrt{x^2+3}\,dx$$

$$= \frac{5}{2}\int\left(x^2+3\right)^{1/2}(2x\,dx)$$

$$\xrightarrow{\text{Substitution}} \quad u = x^2+3, \quad du = 2x\,dx$$

$$= \frac{5}{2}\int u^{1/2}du$$

$$= \frac{5}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{5}{3}(x^2+3)^{3/2} + C$$

4.5 Integration Techniques: Substitution

Example 10 (concluded):

Then, we have

$$\int 5x\sqrt{x^2+3}\;dx = \left[\frac{5}{3}(x^2+3)^{3/2}+C\right]_0^1$$

$$= \left(\frac{5}{3}(1^2+3)^{3/2}+C\right)-\left(\frac{5}{3}(0^2+3)^{3/2}+C\right)$$

$$= \frac{5}{3}(4)^{3/2}+C-\frac{5}{3}(3)^{3/2}-C$$

$$\approx 4.673$$

4.5 Integration Techniques: Substitution

Section Summary

- Integration by *substitution* is the reverse of applying the Chain Rule of Differentiation.
- The substitution is reversed after the integration has been performed.
- Results should be checked using differentiation.

4.6 Integration Techniques: Integration by Parts

OBJECTIVE

- Evaluate integrals using the formula for integration by parts.
- Solve applied problems involving integration by parts.

4.6 Integration Techniques: Integration by Parts

THEOREM 7

The Integration-by-Parts Formula

$$\int u\ dv = uv - \int v\ du$$

4.6 Integration Techniques: Integration by Parts

Tips on Using Integration by Parts:

1. If you have had no success using substitution, try integration by parts.
2. Use integration by parts when an integral is of the form

$$\int f(x)g(x)\ dx.$$

Match it with an integral of the form

$$\int u\ dv$$

by choosing a function to be $u = f(x)$, where $f(x)$ can be differentiated, and the remaining factor to be $dv = g(x)\ dx$, where $g(x)$ can be integrated.

4.6 Integration Techniques: Integration by Parts

3. Find du by differentiating and v by integrating.
4. If the resulting integral is more complicated than the original, make some other choice for $u = f(x)$ and $dv = g(x)\,dx$.
5. To check your solution, differentiate.

4.6 Integration Techniques: Integration by Parts

Example 1: Evaluate: $\int \ln x \, dx$.

Let $u = \ln x$ and $dv = dx$.

Then,

$$du = \frac{1}{x}\, dx \quad \text{and} \quad v = x.$$

4.6 Integration Techniques: Integration by Parts

Example 1 (concluded):

Then, the Integration-by-Parts Formula gives

$$\begin{aligned}\int \ln x \, dx &= (\ln x)x - \int x\left(\frac{1}{x}\right)dx \\ &= x\ln x - \int dx \\ &= x\ln x - x + C\end{aligned}$$

4.6 Integration Techniques: Integration by Parts

Quick Check 1

Evaluate: $\int xe^{3x}dx.$

Let $u = x$ and $dv = e^{3x}dx.$

Then, $du = dx$ and $v = \frac{1}{3}e^{3x}.$

Then, the integration by parts formula gives

$$\int xe^{3x}dx = \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}dx$$

$$= \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$$

4.6 Integration Techniques: Integration by Parts

Example 2: Evaluate: $\int x \ln x \; dx$.

Let's examine several choices for u and dv.

Attempt 1: Let $u = 1$ and $dv = x \ln x \; dx$.

This will not work because we do not know how to integrate $dv = x \ln x \; dx$.

4.6 Integration Techniques: Integration by Parts

Example 2 (continued):

Attempt 2: Let $u = x\ln x$ and $dv = dx.$

Then $du = \left[x \cdot \dfrac{1}{x} + 1\left(\ln x\right)\right] dx$ and $v = x.$

Using the Integration-by-Parts Formula, we have

$$\begin{aligned}\int (x\ln x)\,dx &= (x\ln x)x - \int x\left((1+\ln x)\,dx\right)\\ &= x^2\ln x - \int (x + x\ln x)\,dx\end{aligned}$$

This integral seems more complicated than the original.

Example 2 (continued):

Attempt 3: Let $u = \ln x$ and $dv = x\ dx.$

Then $du = \dfrac{1}{x}\ dx$ and $v = \dfrac{x^2}{2}.$

Using the Integration-by-Parts Formula, we have

$$\begin{aligned}\int (x \ln x)\,dx &= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\left(\frac{1}{x}\right)dx \\ &= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx \\ &= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C\end{aligned}$$

4.6 Integration Techniques: Integration by Parts

Example 3: Evaluate: $\int x\sqrt{5x+1}\,dx.$

Let $\quad u = x \quad$ and $\quad dv = (5x+1)^{1/2}\,dx.$

Then $\quad du = 1\cdot dx \quad$ and $\quad v = \dfrac{2}{15}(5x+1)^{3/2}.$

Using the Integration by Parts Formula gives us

$$\begin{aligned}\int x(\sqrt{5x+1})\,dx &= x\cdot\frac{2}{15}(5x+1)^{3/2} - \int \frac{2}{15}(5x+1)^{3/2}\,dx \\ &= x\cdot\frac{2}{15}(5x+1)^{3/2} - \frac{2}{25}\cdot\frac{2}{15}(5x+1)^{5/2} + C \\ &= \frac{2}{15}x(5x+1)^{3/2} - \frac{4}{375}(5x+1)^{5/2} + C\end{aligned}$$

4.6 Integration Techniques: Integration by Parts

Quick Check 2

Evaluate: $\int 2x\sqrt{3x-2}\,dx$.

Let $u = 2x$ and $dv = \sqrt{3x-2}\,dx$.

Then, $du = 2\,dx$ and $v = \frac{2}{9}(3x-2)^{3/2}$.

Using the integration by parts formula, we get:

$$\int 2x\sqrt{3x-2}\,dx = \frac{4}{9}x(3x-2)^{3/2} - \int \frac{2}{9}(3x-2)^{3/2} \cdot 2\,dx$$

$$= \frac{4}{9}x(3x-2)^{3/2} - \frac{4}{135}(3x-2)^{5/2} + C$$

4.6 Integration Techniques: Integration by Parts

Example 4: Evaluate: $\int_1^2 \ln x \, dx$

Note that we already found the indefinite integral in Example 1. Now we evaluate it from 1 to 2.

$$\begin{aligned}
\int_1^2 \ln x \, dx &= \left[x \ln x - x\right]_1^2 \\
&= (2\ln 2 - 2) - (1 \cdot \ln 1 - 1) \\
&= 2\ln 2 - 2 - 0 + 1 \\
&= 2\ln 2 - 1 \\
&\approx 0.386
\end{aligned}$$

4.6 Integration Techniques: Integration by Parts

Example 5: Life Science: Population Decay. The rate of change in a population of bacteria t hours after an antibiotic treatment begins is estimated by $P'(t) = te^{-0.04t}$, where $P'(t)$ is the rate of change of the population, in thousands of bacteria per hour. Assume that when the treatment begins, the population consists of 1,500,000 bacteria.

a) Find the population of bacteria after 24 hours of treatment.

b) What is the total change in the population of bacteria from 24 to 36 hours after the treatment begins?

4.6 Integration Techniques: Integration by Parts

Example 5 continued:

Solution:

Let $u = t$ so that $dv = e^{-0.04t}dt$. So, $du = dt$ and $v = -\frac{1}{0.04}e^{-0.04t}$

$$P(t) = \int te^{-0.04t}dt = t\left(-\frac{1}{0.04}e^{-0.04t}\right) - \int\left(-\frac{1}{0.04}e^{-0.04t}\right)dt$$

$$= -25te^{-.04t} + 25\int\left(e^{-0.04t}\right)dt$$

$$= -25te^{-.04t} + 25\left(-\frac{1}{0.04}e^{-0.04t}\right) + C$$

$$= -25te^{-.04t} - 625e^{-0.04t} + C$$

4.6 Integration Techniques: Integration by Parts

Example 5 continued:
Solution:
To find C, we evaluate P(0):

$$P(0) = 1500 = -25(0)e^{-0.04(0)} - 625e^{-0.04(0)} + C$$

$$1500 = -625 + C$$

$$C = 2125$$

Thus the population of bacteria after t hours is given by:

$$P(t) = -25te^{-.04t} - 625e^{-0.04t} + 2125$$

4.6 Integration Techniques: Integration by Parts

Example 5 continued:

Solution:

a) The population of bacteria after 24 hours is therefore

$$P(24) = -25(24)e^{-.04(24)} - 625e^{-0.04(24)} + 2125$$

$$\approx 1655.956$$

Thus the population of bacteria after 24 hours is approximately 1,655,956 bacteria.

4.6 Integration Techniques: Integration by Parts

Example 5 concluded:

Solution:

b) The total change in the population of bacteria between 24 and 36 hours is given by

$$\int_{24}^{36} P'(t)\,dt = P(36) - P(24)$$

$$P(36) = -25(36)e^{-.04(36)} - 625e^{-0.04(36)} + 2125 \approx 1763.685$$

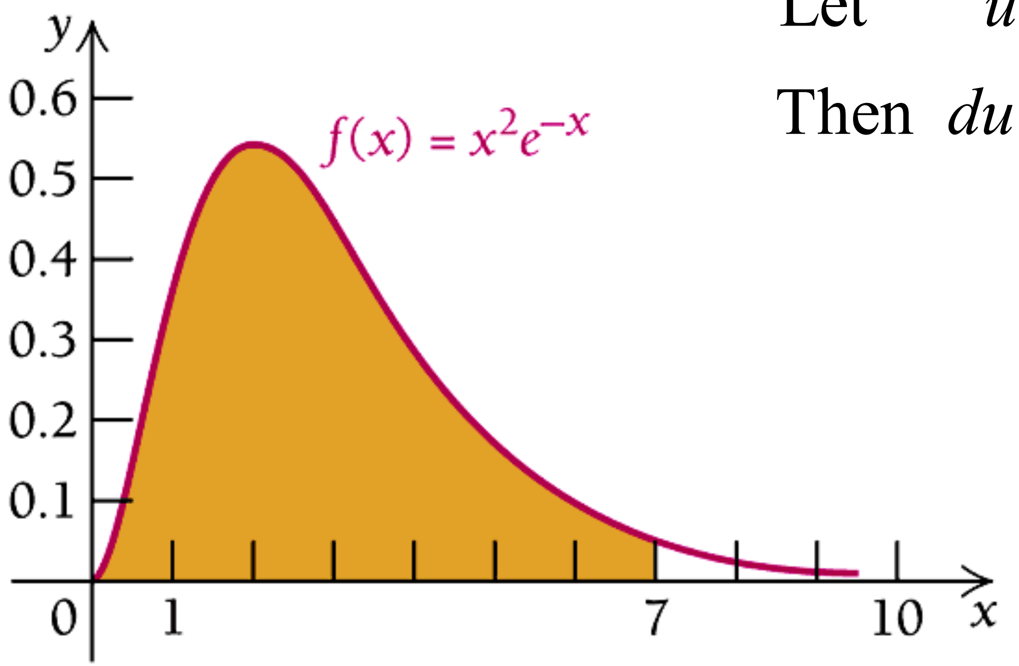
$$P(24) = -25(24)e^{-.04(24)} - 625e^{-0.04(24)} + 2125 \approx 1655.956$$

$$P(36) - P(24) = 1763.685 - 1655.956 = 107.729$$

The total change in the population of bacteria between 24 and 36 hours is approximately 107,729 bacteria.

4.6 Integration Techniques: Integration by Parts

Example 6: Evaluate $\int_0^7 x^2 e^{-x}\,dx$ to find the area of the shaded region shown below.



Let $u = x^2$ and $dv = e^{-x}dx.$

Then $du = 2x\ dx$ and $v = -e^{-x}.$

4.6 Integration Techniques: Integration by Parts

Example 6 (continued):

Using the Integration-by-Parts Formula gives us

$$\begin{aligned}\int x^2(e^{-x}dx) &= x^2(-e^{-x}) - \int -e^{-x}(2x\ dx) \\ &= -x^2e^{-x} + \int 2xe^{-x}dx\end{aligned}$$

To evaluate the integral on the right, we can apply integration by parts again, as follows.

Let $u = 2x$ and $dv = e^{-x}dx.$

Then $du = 2\ dx$ and $v = -e^{-x}.$

4.6 Integration Techniques: Integration by Parts

Example 6 (continued):

Using the Integration-by-Parts Formula again gives us

$$\begin{aligned}\int 2x(e^{-x}dx) &= 2x(-e^{-x}) - \int -e^{-x}(2\ dx) \\ &= -2xe^{-x} - 2e^{-x} + C\end{aligned}$$

Then, we can substitute this solution into the formula on the last slide.

$$\begin{aligned}\int x^2 e^{-x} dx &= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C \\ &= -e^{-x}(x^2 + 2x + 2) + C\end{aligned}$$

4.6 Integration Techniques: Integration by Parts

Example 6 (concluded):

Then, we can evaluate the definite integral.

$$\int_0^7 x^2 e^{-x}\,dx = \left[-e^{-x}(x^2+2x+2)+C\right]_0^7$$

$$= \left[-e^{-7}(7^2+2\left(7\right)+2)\right]-\left[-e^{-0}(0^2+2\left(0\right)+2)\right]$$

$$= -65e^{-7}+2$$

$$\approx 1.94$$

4.6 Integration Techniques: Integration by Parts

Quick Check 3

Evaluate: $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx.$

$$\text{Let } u = x \text{ and } dv = \frac{1}{\sqrt{x+1}}\,dx.$$

$$\text{Then, } du = dx \text{ and } v = 2\left(x+1\right)^{1/2}.$$

Using the integration by parts formula, we get:

$$\int \frac{x}{\sqrt{x+1}}\,dx = 2x\left(x+1\right)^{1/2} - \int 2\left(x+1\right)^{1/2}\,dx$$

$$= 2x\left(x+1\right)^{1/2} - \frac{4}{3}\left(x+1\right)^{3/2} + C$$

4.6 Integration Techniques: Integration by Parts

Quick Check 3 Concluded

Then we can evaluate the definite integral.

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \left[2x(x+1)^{1/2} - \frac{4}{3}(x+1)^{3/2}\right]_0^3$$

$$= \left(2(3)(3+1)^{1/2} - \frac{4}{3}(3+1)^{3/2}\right) - \left(2(0)(0+1)^{1/2} - \frac{4}{3}(0+1)^{3/2}\right)$$

$$= \left(6(4)^{1/2} - \frac{4}{3}(4)^{3/2}\right) - \left(-\frac{4}{3}(1)^{3/2}\right)$$

$$= \left(12 - \frac{4\cdot 8}{3}\right) + \frac{4}{3} = \left(\frac{36}{3} - \frac{32}{3}\right) + \frac{4}{3} = \frac{8}{3} \text{ or } 2\frac{2}{3}$$

4.6 Integration Techniques: Integration by Parts

Example 7: Evaluate $\int x^3 e^x dx.$

When you try integration by parts on this problem, you will notice a pattern. Using tabular integration can greatly simplify your work.

4.6 Integration Techniques: Integration by Parts

Example 7 (continued):

$f(x)$ and Repeated Derivatives	Sign of Product	$g(x)$ and Repeated Integrals
x^3	$(+)$	e^x
$3x^2$	$(-)$	e^x
$6x$	$(+)$	e^x
6	$(-)$	e^x
0		e^x

4.6 Integration Techniques: Integration by Parts

Example 7 (concluded):

Add the products along the arrows, making the alternating sign changes, to obtain

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$$

4.6 Integration Techniques: Integration by Parts

Section Summary

- The *Integration-by-Parts Formula* is the reverse of the Product Rule for differentiation:

$$\int u \cdot dv = uv - \int v \cdot du.$$

- The choices for u and dv should be such that the integral $\int v \cdot du$ is simpler than the original integral. If this does not turn out to be the case, other choices should be made.

- *Tabular integration* is useful in cases where repeated integration by parts is necessary.

4.7 Numerical Integration

OBJECTIVE

- Solve problems using numerical integration methods such as Riemann sums, the Midpoint Rule, the Trapezoidal Rule, and Simpson’s Rule.

4.7 Numerical Integration

Numerical Integration is usually used in the following situations:

- The antiderivative of the integrand may be difficult or impossible to determine directly, or
- The data may not conveniently "fit" into a common function that can then be integrated.

4.7 Numerical Integration

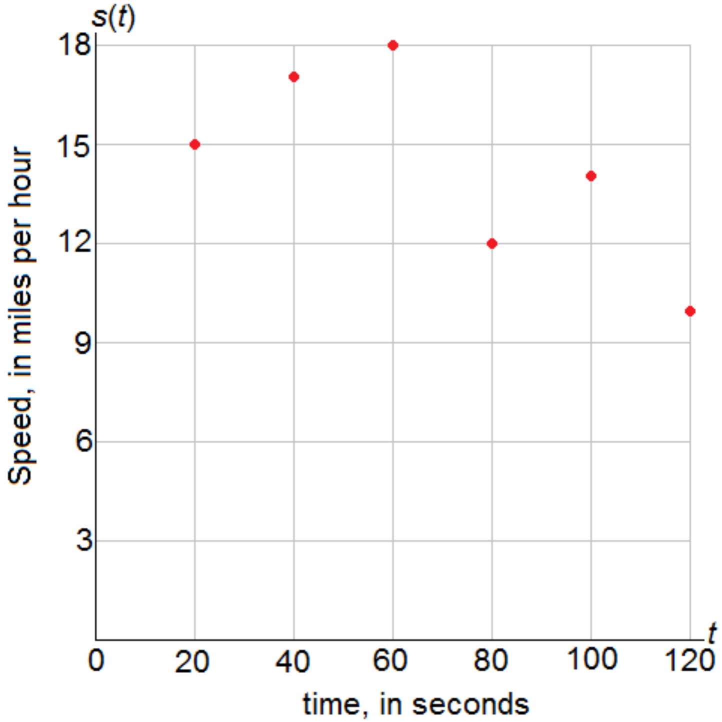
Example 1: Jeff is running on a jogging track. His speed (in miles per hour) is shown on his belt-clip pedometer. Jeff takes six readings at 20-second intervals. From this data, determine approximately how far (in miles) Jeff has run in those two minutes (the six 20-second intervals).

Time:	20 sec	40 sec	60 sec	80 sec	100 sec	120 sec
Speed:	15 mph	17 mph	18 mph	12 mph	14 mph	10 mph

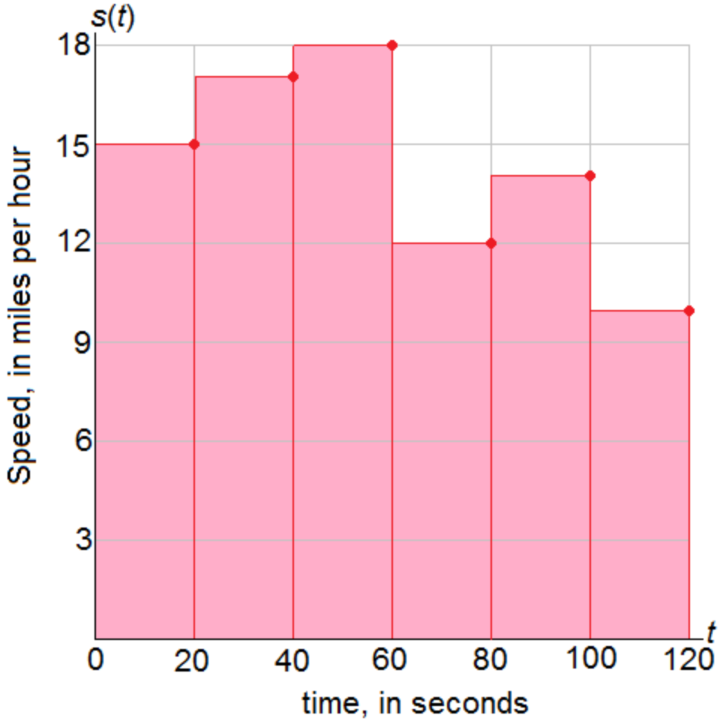
4.7 Numerical Integration

Example 1 continued:

Solution: We start by plotting the data:



Rectangular bars are then drawn, each the height of one data point:



4.7 Numerical Integration

Example 1 continued:

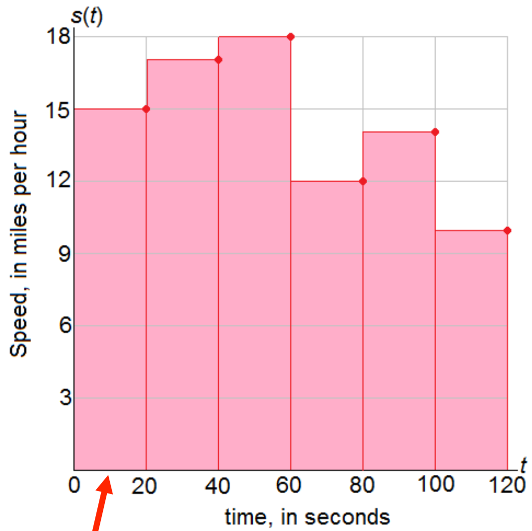
The area of each bar represents the approximate distance Jeff has run during those 20 seconds. Thus, the sum of the areas in each bar is the total approximate distance that Jeff ran during the two minutes. First, we need to be consistent with units. We have "seconds" along the horizontal axis and "miles per hour" along the vertical axis. We convert 20 seconds into hours:

$$20 \text{ seconds} \times (1 \text{ hour}/3600 \text{ seconds}) = 1/180 \text{ hour.}$$

Thus, each bar is "1/180 hour" wide.

4.7 Numerical Integration

Example 1 concluded:



Each bar is 1/180 hr wide.

Interval	Width	Height	Area (width x height)
[0,20]	1/180 hr	15 mi/hr	(1/180)(15) = 0.08333 mi
[20,40]	1/180 hr	17 mi/hr	(1/180)(17) = 0.09444 mi
[40,60]	1/180 hr	18 mi/hr	(1/180)(18) = 0.1 mi
[60,80]	1/180 hr	12 mi/hr	(1/180)(12) = 0.06667 mi
[80,100]	1/180 hr	14 mi/hr	(1/180)(14) = 0.07778 mi
[100,120]	1/180 hr	10 mi/hr	(1/180)(10) = 0.05556 mi

Note that when the units are multiplied, the "hr" units will simplify away, leaving "miles" (mi).

From the table, we then add the values in the right column:

$$0.08333 + 0.09444 + 0.1 + 0.0667 + 0.07778 + 0.05556 \approx 0.4777$$

Jeff ran approximately 0.478 mile in those two minutes.

4.7 Numerical Integration

Definition: Riemann Sums

Suppose we want to approximate a value of $\int_a^b f(x)\,dx$.

We subdivide the interval $[a,b]$ into n equally-sized subintervals. Let these n subintervals be given by

$$[a,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},b]$$

each subinterval has a width $\Delta x = \dfrac{b-a}{n}$.

4.7 Numerical Integration

Definition: Riemann Sums continued

Rectangles are erected over each subinterval. We have a choice in that we can define the height of each subinterval as the functional value at the left endpoint of each subinterval. The heights would be

$$f(a), f(x_1), f(x_2), \ldots, f(x_{n-1}).$$

The area of each rectangle is the width multiplied by the heights. Thus, the total area under these rectangles in which we used the left endpoint to define the heights is

$$L_n = \Delta x \cdot f(a) + \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \ldots + \Delta x \cdot f(x_{n-1}).$$

4.7 Numerical Integration

Definition: Riemann Sums concluded

If we use the <u>right</u> endpoint of each subinterval, then the function values are

$$f(x_1), f(x_2), f(x_3), \ldots, f(b),$$

and their sum is

$$R_n = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \ldots + \Delta x \cdot f(b).$$

Often, the average of these figures is calculated to give a better approximation.

4.7 Numerical Integration

Example 2: Find the approximate value of $\int_0^3 \sqrt{e^x + 1}\, dx$, using $n = 6$ equal-sized subintervals.

Solution: We will find L_6 first. The subintervals are [0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5] and [2.5, 3].

Each subinterval has a width $\Delta x = \dfrac{3-0}{6} = 0.5.$

4.7 Numerical Integration

Example 2 continued:

In each subinterval, we use the **left** endpoint to find the heights, given by $f(x)=\sqrt{e^x+1}$

Subinterval	Width	Height (x = left endpoint)	Area
$[0, 0.5]$	0.5	$f(0)=\sqrt{e^0+1}\approx 1.4142$	$0.5\cdot 1.4142=0.7071$
$[0.5, 1]$	0.5	$f(0.5)=\sqrt{e^{0.5}+1}\approx 1.6275$	$0.5\cdot 1.6275=0.81375$
$[1, 1.5]$	0.5	$f(1)=\sqrt{e^1+1}\approx 1.9283$	$0.5\cdot 1.9283=0.96415$
$[1.5, 2]$	0.5	$f(1.5)=\sqrt{e^{1.5}+1}\approx 2.3413$	$0.5\cdot 2.3413=1.17065$
$[2, 2.5]$	0.5	$f(2)=\sqrt{e^2+1}\approx 2.8964$	$0.5\cdot 2.8964=1.4482$
$[2.5, 3]$	0.5	$f(2.5)=\sqrt{e^{2.5}+1}\approx 3.6308$	$0.5\cdot 3.6308=1.8154$

The sum of the areas in the last column is $L_6=6.91925$.

4.7 Numerical Integration

Example 2 concluded:

Now we find R_6. In each subinterval, we use the **right** endpoint to find the heights, given by $f(x)=\sqrt{e^x+1}$.

Subinterval	Width	Height (x = right endpoint)	Area
[0, 0.5]	0.5	$f(0.5)=\sqrt{e^{0.5}+1}\approx 1.6275$	$0.5\cdot 1.6275=0.81375$
[0.5, 1]	0.5	$f(1)=\sqrt{e^{1}+1}\approx 1.9283$	$0.5\cdot 1.9283=0.96415$
[1, 1.5]	0.5	$f(1.5)=\sqrt{e^{1.5}+1}\approx 2.3413$	$0.5\cdot 2.3413=1.17065$
[1.5, 2]	0.5	$f(2)=\sqrt{e^{2}+1}\approx 2.8964$	$0.5\cdot 2.8964=1.4482$
[2, 2.5]	0.5	$f(2.5)=\sqrt{e^{2.5}+1}\approx 3.6308$	$0.5\cdot 3.6308=1.8154$
[2.5, 3]	0.5	$f(3)=\sqrt{e^{3}+1}\approx 4.5919$	$0.5\cdot 4.5919=2.29595$

The sum of the areas in the last column is $R_6=8.5081$.

4.7 Numerical Integration

Definition: The Midpoint Rule

Let $[a,b]$ be subdivided into n equally-sized subintervals with widths $\Delta x = b - a/n$.

The approximate value of $\int_a^b f(x)\,dx$ is given by M_n, where

$$M_n = \Delta x\left(f\left(\frac{a+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+\ldots+f\left(\frac{x_{n-1}+b}{2}\right)\right)$$

4.7 Numerical Integration

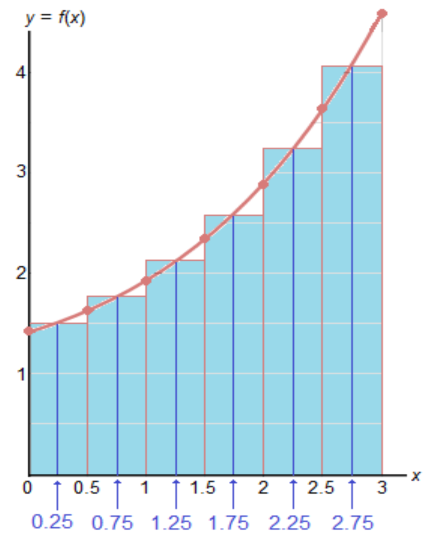
Example 3: Use the Midpoint Rule to find the approximate value of $\int_0^3 \sqrt{e^x + 1}\, dx$, using $n = 6$ equal-sized subintervals.

Solution: The midpoint of the interval [0, 0.5] is $x = 0.25$, and for the interval [0.5, 1], it is $x = 0.75$, and so on. These midpoints are used to find $f(x_n)$ in the nth subinterval which are listed in the table on the next slide.

4.7 Numerical Integration

Example 3 concluded:

Subinterval	Width	Height (x = midpoint)	Area
[0, 0.5]	0.5	$f(0.25) = \sqrt{e^{0.25} + 1} \approx 1.5113$	$0.5 \cdot 1.5113 = 0.75565$
[0.5, 1]	0.5	$f(0.75) = \sqrt{e^{0.75} + 1} \approx 1.7655$	$0.5 \cdot 1.7655 = 0.88275$
[1, 1.5]	0.5	$f(1.25) = \sqrt{e^{1.25} + 1} \approx 2.1190$	$0.5 \cdot 2.1190 = 1.0595$
[1.5, 2]	0.5	$f(1.75) = \sqrt{e^{1.75} + 1} \approx 2.5990$	$0.5 \cdot 2.5990 = 1.2995$
[2, 2.5]	0.5	$f(2.25) = \sqrt{e^{2.25} + 1} \approx 3.2385$	$0.5 \cdot 3.2385 = 1.61925$
[2.5, 3]	0.5	$f(2.75) = \sqrt{e^{2.75} + 1} \approx 4.0795$	$0.5 \cdot 4.0795 = 2.03975$



The sum of the areas in the last column is $M_6 = 7.6564$. This is a slightly better approximation of the definite integral, $\int_0^3 \sqrt{e^x + 1}\, dx = 7.67548$.

4.7 Numerical Integration

Definition: The Trapezoidal Rule

Let $[a,b]$ be subdivided into n equally-sized subintervals with widths $\Delta x = b - a/n$.

The approximate value of $\int_a^b f(x)dx$ is given by T_n, where

$$T_n = \Delta x\left(\frac{f(a)}{2} + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \frac{f(b)}{2}\right)$$

4.7 Numerical Integration

Example 4: Use the Trapezoidal Rule to find the approximate value of $\int_0^3 \sqrt{e^x + 1}\, dx$, using $n = 6$ equal-sized subintervals.

Solution: From previous examples we know:

$f(0) \approx 1.4142$ $\quad f(0.5) \approx 1.6275$ $\quad f(1) \approx 1.9283$

$f(1.5) \approx 2.3413$ $\quad f(2) \approx 2.8964$ $\quad f(2.5) \approx 3.6308$

$f(3) \approx 4.5919$ $\qquad$ Thus,

$$T_6 = \frac{1}{2}\left(\frac{1.4142}{2} + 1.6275 + 1.9283 + 2.3413 + 2.8964 + 3.6308 + \frac{4.5919}{2}\right)$$

$$T_6 = 7.714.$$

4.7 Numerical Integration

Definition: Simpson's Rule

Let $[a, b]$ be subdivided into n equally-sized subintervals with widths $\Delta x = b - a/n$. Here, n is an even number.

The approximate value of $\int_a^b f(x)\,dx$ is given by S_n, where

$$S_n = \frac{b-a}{3n}\left(f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(b)\right)$$

4.7 Numerical Integration

Example 5: Use Simpson's Rule to find the approximate value of $\int_0^3 \sqrt{e^x + 1}\, dx$, using $n = 6$ equal-sized subintervals.

Solution: From previous examples we know:

$f(0) \approx 1.4142 \quad f(0.5) \approx 1.6275 \quad f(1) \approx 1.9283$

$f(1.5) \approx 2.3413 \quad f(2) \approx 2.8964 \quad f(2.5) \approx 3.6308$

$f(3) \approx 4.5919$ and $\dfrac{b-a}{3n} = \dfrac{3-0}{3(6)} = \dfrac{3}{18} = \dfrac{1}{6}$ Thus,

$$S_6 = \frac{1}{6}\left(1.4142 + 4(1.6275) + 2(1.9283) + 4(2.3413) + 2(2.8964) + 4(3.6308) + 4.5919\right)$$

$S_6 = 7.67565.$

4.7 Numerical Integration

Section Summary

- The process of numerically approximating the value of $\int_a^b f(x)\,dx$ using geometry is called *numerical integration*.

  We subdivide the interval $[a,b]$ into n equally-sized subintervals. Let these n subintervals be given by

  $[a,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},b]$

  each subinterval has a width $\Delta x = \dfrac{b-a}{n}$.

4.7 Numerical Integration

Section Summary continued

We then use any of the following techniques:

- *Riemann sums*: Rectangles are erected over each subinterval. Using left endpoints we can approximate the integral using:

$$L_n = \Delta x \cdot f(a) + \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \ldots + \Delta x \cdot f(x_{n-1}).$$

Using right endpoints we can approximate the integral using:

$$R_n = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \ldots + \Delta x \cdot f(b).$$

4.7 Numerical Integration

Section Summary continued

- *Midpoint Rule:* Using rectangles, the height of each rectangle is evaluated at the midpoint of the subinterval. The sum of the areas of all rectangles is denoted as *Mn*.

$$M_n = \Delta x\left(f\left(\frac{a+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+\ldots+f\left(\frac{x_{n-1}+b}{2}\right)\right)$$

- *Trapezoidal Rule:* Trapezoids are drawn over each subinterval. The sum of the areas of all the trapezoids is denoted *Tn*, where

$$T_n = \Delta x\left(\frac{f(a)}{2}+f(x_1)+f(x_2)+\ldots+f(x_{n-1})+\frac{f(b)}{2}\right)$$

4.7 Numerical Integration

Section Summary concluded

- *Simpson's Rule:* The interval $[a, b]$ is divided into n subintervals, where n is even. The graph of f is then approximated by $n/2$ parabolas, and the area under f is approximated by S_n, where

$$S_n = \frac{b-a}{3n}\left(f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(b)\right)$$