Real functions of one real variable

Definition

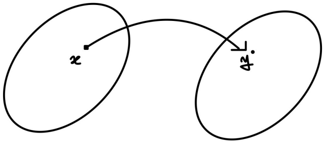
Given $D \subset \mathbb{R}$, a *real function of one real variable* is a rule associating to every element x in D a real number, denoted with $f(x)$. In symbols:

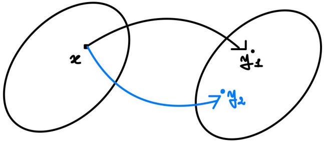
$$f : D \to \mathbb{R}, \quad x \mapsto f(x)$$

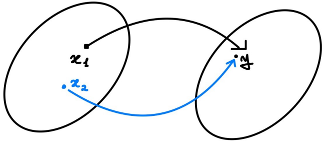
(Also $f : D \subset \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$).

Then a function is given when we know 3 things: 1) the domain D, 2) the codomain (arrival set), and 3) the law assigning every element of the domain to a **unique** element of the codomain.

Is it a function?







Examples

- Linear functions: $f : \mathbb{R} \to \mathbb{R}$, $(y =)\ \ f(x) = 2x - 1$
- Quadratic functions: $f : \mathbb{R} \to \mathbb{R}$, $\ f(x) = x^2 + 1$
- Rational functions: $f : \mathbb{R} - \{2\} \to \mathbb{R}$, $\ f(x) = \frac{1-x}{x-2}$
- Irrational functions: $f : [0, 1] \to \mathbb{R}$, $\ f(x) = \sqrt{x(1-x)}$
- Exponential functions: $f : \mathbb{R} \to \mathbb{R}$, $\ f(x) = e^{x-2}$, or $f(x) = (\frac{1}{2})^x$
- Logarithmic functions: $f : (0, +\infty) \to \mathbb{R}$, $\ f(x) = \ln(x)$;
  $f : (2, +\infty) \to \mathbb{R}$, $\ f(x) = \log_4(x - 2)$;
- Other functions: $f : \mathbb{R} \to \mathbb{R}$, $\ f(x) = e^{x^2+x+1}$; $f : \mathbb{R} \to \mathbb{R}$,
  $f(x) = \ln\left(\frac{2}{1+x^2}\right)$

Homework from textbook.

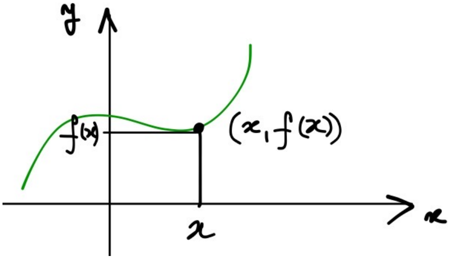
Solve exercises in section 4.2, from 1 to 12.

Graph

Definition

The *graph* of a function $f : D \to \mathbb{R}$, is the followiing subset of the plane $\mathbb{R}^2$

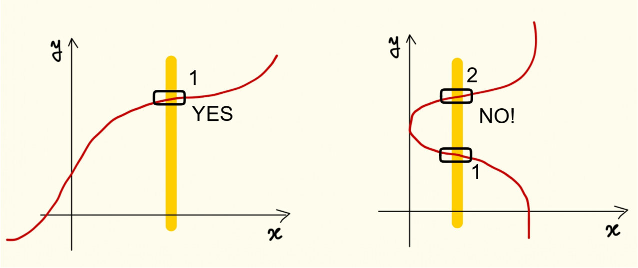
$$G_f = \{(x, y) : x \in D,\ y = f(x)\} = \{(x, f(x)) : x \in D\}$$



The Graph is a subset of $\mathbb{R}^2$.

Vertical line test

To check whether a certain curve in the plane is the graph of a function, use "vertical line test".



Exercise 1

Consider the curves of equations:

$y = x^2$

$x^2 + y^2 = 1$

$x = y^2$

$2 - x^3 + y = 0$

Plot them and establish if the curve is the graph of a function

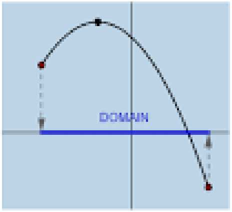
Range

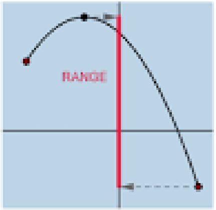
Definition

The *range* of a function $f : D \to \mathbb{R}$, (denoted with R_f or $f(D)$), in symbols

$$R_f = \{y : x \in D, y = f(x)\} = \{f(x) : x \in D\}$$

is the subset of $\mathbb{R}$ of values attained by f.





The Domain and the Range are subsets of $\mathbb{R}$.

Exercise 2

Find the range of the following functions:

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ $\quad [0, +\infty)$

2. $f : [0, 1) \to \mathbb{R}$, $f(x) = 1 - x$ $\quad (0, 1]$

Surjective functions or "onto"

Surjective functions or "onto"

Consider a real function of real variable with domain $D \subset \mathbb{R}$ and codomain $C \subset \mathbb{R}$, $f : D \to C$. Then f is called *surjective or "onto"* when C coincide with the range of f, $C = R_f$. In formulas:

$$\forall y \in C, \exists x \in D : y = f(x).$$

In the previous examples:

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is NOT onto, but
   $f : \mathbb{R} \to [0, +\infty)$, $f(x) = x^2$ is onto;
2. $f : [0, 1) \to \mathbb{R}$, $f(x) = 1 - x$ is NOT onto, but
   $f : [0, 1) \to (0, 1]$, $f(x) = 1 - x$ is onto.

How do we obtain a surjective function from any given function? We restrict the arrival set to the range.

Natural Domain

Definition

The *natural domain* of a function is the largest subset of $\mathbb{R}$ where a given rule $f(x)$ is defined.

Exercise 3

Compute the (natural) domain of the following functions:

(a) $f(x) = \sqrt{1 - x^2}$

(b) $f(x) = \frac{1-2x}{x^2-4}$

(c) $f(x) = \sqrt{\frac{1-x}{x-4}\frac{1}{x-2}}$

(d) $f(x) = \ln(3 - x)$

(e) $f(x) = \frac{1}{\ln(3-x)}$

(f) $f(x) = \ln(\frac{1-2x}{x^2-4})$

(g) $f(x) = \sqrt{\frac{\ln(1-x)}{x-4}} \ln\left(\frac{1}{x-2}\right)$

Solution to Exercise 3(f)

$f(x) = \ln(\frac{1-2x}{x^2-4})$

Most common troublesome functions

- Ratios: $\frac{1}{x}$ is defined when $x \neq 0$ implies that ratios are defined when the denominator is nonzero.
- Roots: since $\sqrt{x}$, $\sqrt[2n]{x}$ are defined when $x \geq 0$, then roots with even exponents are defined when their argument is positive or null;
- Logs: since $\ln x, \log_a x$ are defined when $x > 0$, then logs are defined when their argument is striclty positive;

Exercise 4

Describe and compute the range, and plot the graph of the following functions:

1. a) $f : \mathbb{R} \to \mathbb{R}, \ f(x) = 2x - 1$;
   b) $f : [0,1] \to \mathbb{R}, \ f(x) = 2x - 1$
   c) $f : [0,1) \to \mathbb{R}, \ f(x) = 2x - 1$
2. a) $f : \mathbb{R} \to \mathbb{R}, \ f(x) = 1 - 2x$;
   b) $f : (0,1] \to \mathbb{R}, \ f(x) = 1 - 2x$
3. a) $f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^2 + 1$; b) $f : [-1,0] \to \mathbb{R}, \ f(x) = x^2 + 1$
4. $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 1 - 2x & x \geq 1 \\ 1 + x^2 & x < 1 \end{cases}$

Homework from textbook.
Solve exercises for section 4.2, from 12 to 15, exercises for section 4.3.

Composite Functions [SHS 5.2]

Definition

If $A, B, C \subset \mathbb{R}$, and $f : A \to B$ and $g : f(B) \to C$, the *composite* function $g \circ f$ ("g after f") is given by

$$g \circ f : A \to C, \quad g \circ f(x) = g(f(x))$$

Remark Composition of functions is non commutative.

Example:
Compute $g \circ f$ where $f(x) = \frac{1}{x}$; $g(x) = e^x$.
Show that $f \circ g \neq g \circ f$.

Exercise 5

Compute the composite functions $g \circ f$, $f \circ g$, and their natural domain:

1. $f(x) = \frac{1-x}{x^2-3}$; $g(x) = e^x$
2. $f(x) = \ln(7-x)$; $g(x) = \frac{1}{x-2}$
3. $f(x) = \frac{1-x}{x^2-3}$; $g(x) = \sqrt{x+2}$
4. $f(x) = x^2 - 1$; $g(x) = \sqrt{x+1}$

One-to-One functions

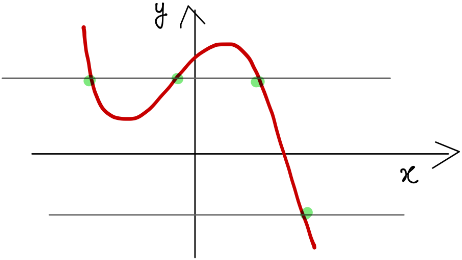
Definition

A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is one-to-one when it assigns different values of the domain to different values of the range:

$$(x_1, x_2 \in \mathbb{R}, \quad x_1 \neq x_2) \Rightarrow (f(x_1) \neq f(x_2))$$

Horizontal line test

To check from the graph whether a certain function is one-to-one, use the horizontal line test.



Exercise 6

Establish whether the following functions are one-to-one.

1. $f : \mathbb{R} \to \mathbb{R},\ \ f(x) = 2x - 1$
2. a) $f : \mathbb{R} \to \mathbb{R},\ \ f(x) = x^2$; b) $f : [0, +\infty) \to \mathbb{R},\ \ f(x) = x^2$
3. $f : \mathbb{R} \to \mathbb{R},\ \ f(x) = 2x^2 - 8$;
4. $f : \mathbb{R} \to \mathbb{R},\ \ f(x) = e^{x+1}$
5. $f : \mathbb{R} - \{4\} \to \mathbb{R},\ \ f(x) = \frac{1-x}{x-4}$
6. $f : \mathbb{R} \to \mathbb{R},\ \ f(x) = e^{x^2+1}$
7. $f : \mathbb{R} \to \mathbb{R},\ \ f(x) = x^2 e^x$ (use a calculator to plot the graph and then decide whether it is one to one or not)

Inverse function [SHS 5.3]

Definition

If $A, B \subset \mathbb{R}$ and $f : A \to B$, the function $g : B \to A$ is its *inverse* if

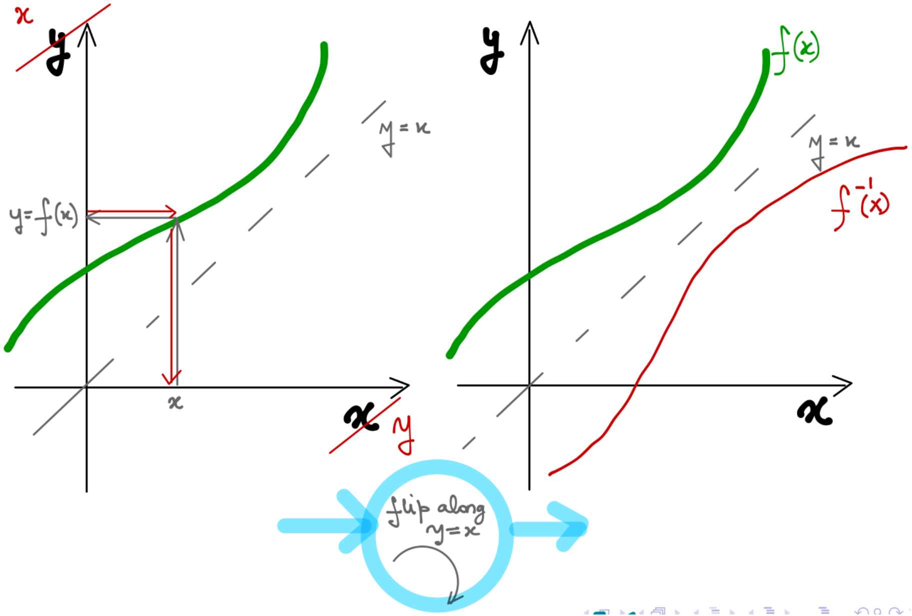
$$y = f(x) \Leftrightarrow x = g(y), \ \forall x \in A, \forall b \in B$$

Remarks:
1) The inverse is usually denoted with $f^{-1}(x)$.
2) From the definition, $f(g(x)) = g(f(x)) = x$ (inverse functions commute, and their compound function is the identity function)

Theorem

- A function is invertible if and only if it is one-to-one and onto.
- As a consequence a one-to-one function is invertible on its range.

Graph of Inverse Functions



Exercise 7

Compute the inverse function to

$$f : \mathbb{R} - \{4\} \to \mathbb{R} \quad f(x) = \frac{1-x}{x-4}$$

possibly restricting the arrival set to the range of the function.
Solution

Homework

Exercise 8
Establish whether the functions of exercise 6 are invertible, and if they are compute their inverse (if possible).

Homework from textbook.
Solve exercises for section 5.3.

Further Homework
Read all of Chapters 4 and 5 from the book, be sure to be confident with the contents, and solve all exercises there contained. In particular be confident with the following classes of functions: linear, quadratic, power, exponential, logarithm. Study also how to graph circles, ellipses, hyperbolas.