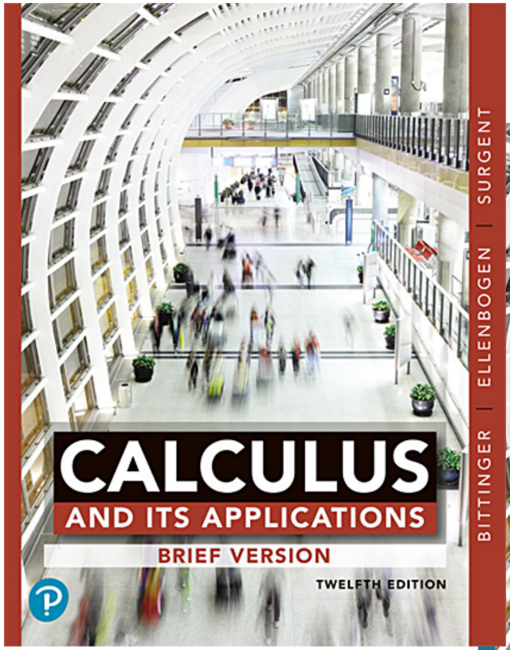
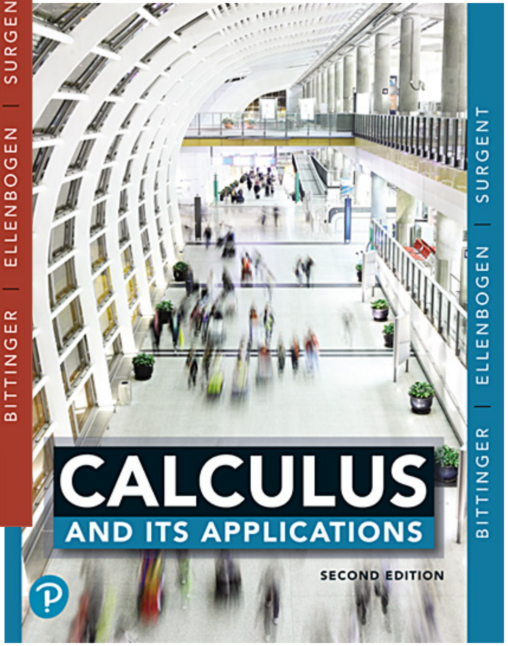
# Chapter 1

## Differentiation



# 1.1 Limits: A Numerical and Graphical Approach

## Objective

- Find limits of functions, if they exist, using numerical or graphical methods.

# 1.1 Limits: A Numerical and Graphical Approach

**Example 1:** For each sequence, determine its limit, and rewrite the sequence in the form $x \to a^-$ or $x \to a^+$.

a) 2.24, 2.249, 2.2499, 2.24999, …

c) 5.51, 5.501, 5.5001, 5.50001,…

c) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots$

# 1.1 Limits: A Numerical and Graphical Approach

**Example 1 (concluded):**

a) These numbers are approaching the limit 2.25. Since each number in the sequence is less than 2.25, we write $x \to 2.25^-$, read "$x$ approaches 2.25 from the left."

b) These numbers are approaching the limit 5.5. Since each number in the sequence is greater than 5.5, we write $x \to 5.5^+$, read "$x$ approaches 5.5 from the right."

c) These numbers are approaching the limit 1. Since each number in the sequence is less than 1, we write $x \to 1^-$, read "$x$ approaches 1 from the left."

# 1.1 Limits: A Numerical and Graphical Approach

**DEFINITION:**

As $x$ approaches $a$, the **limit** of $f(x)$ is $L$, written

$$\lim_{x \to a} f(x) = L,$$

if all values of $f(x)$ are close to $L$ for values of $x$ that are sufficiently close, but not equal to, $a$.

# 1.1 Limits: A Numerical and Graphical Approach

**THEOREM 1**

As $x$ approaches $a$, the **limit** of $f(x)$ is $L$, if the limit from the left exists and the limit from the right exists and both limits are $L$. That is, if

1) $\lim\limits_{x \to a^-} f(x) = L,$

and

2) $\lim\limits_{x \to a^+} f(x) = L,$

then

$$\lim_{x \to a} f(x) = L.$$

# 1.1 Limits: A Numerical and Graphical Approach

Quick Check 1

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$.

a) What is $f(3)$?

b) What is the limit of $f(x)$ as $x$ approaches 3?

# 1.1 Limits: A Numerical and Graphical Approach

Quick Check 1 Solution a)

1.) Since $f(x) = \dfrac{x^2 - 9}{x - 3}$, we will substitute 3 in for $x$, giving us the new equation $f(3) = \dfrac{3^2 - 9}{3 - 3}$.

2.) Solving for $f(3)$, we get $f(3) = \dfrac{3^2 - 9}{3 - 3} = \dfrac{9 - 9}{3 - 3} = \dfrac{0}{0}$.

Thus $f(3)$ does not exist.

# 1.1 Limits: A Numerical and Graphical Approach

Quick Check 1 Solution b)

First let $x$ approach $3$ from the left:

| $x \to 3^-$ | 2 | 2.5 | 2.9 | 2.99 | 2.999 |
|---|---|---|---|---|---|
| $f(x)$ | 5 | 5.5 | 5.9 | 5.99 | 5.999 |

Thus it appears that $\lim_{x \to 3^-} f(x)$ is $6$.

Next let $x$ approach $3$ from the right:

| $x \to 3^+$ | 4 | 3.5 | 3.1 | 3.01 | 3.001 |
|---|---|---|---|---|---|
| $f(x)$ | 7 | 6.5 | 6.1 | 6.01 | 6.001 |

Thus it appears that $\lim_{x \to 3^+} f(x)$ is $6$.

Since both the left-hand and right-hand limits agree, $\lim_{x \to 3} f(x) = 6$.

# 1.1 Limits: A Numerical and Graphical Approach

**Example 2:** Consider the function $H$ given by

$$H(x) = \begin{cases} 2x + 2 & for \quad x < 1 \\ 2x - 4 & for \quad x \geq 1 \end{cases}.$$

Graph the function and find each of the following limits, if they exist. When necessary, state that the limit does not exist.

a) $\lim\limits_{x \to 1} H(x)$

b) $\lim\limits_{x \to -3} H(x)$

# 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically**

First, let $x$ approach 1 from the left:

| $x \to 1^-$ | 0 | 0.5 | 0.8 | 0.9 | 0.99 | 0.999 |
|---|---|---|---|---|---|---|
| $H(x)$ | 2 | 3 | 3.6 | 3.8 | 3.98 | 3.998 |

Thus, it appears that $\lim_{x \to 1^-} H(x) = 4.$

# 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically (continued)**

Then, let $x$ approach 1 from the right:

| $x \to 1^+$ | 2 | 1.8 | 1.1 | 1.01 | 1.001 | 1.0001 |
|---|---|---|---|---|---|---|
| $H(x)$ | 0 | –0.4 | –1.8 | –1.98 | –1.998 | –1.9998 |

Thus, it appears that $\lim_{x \to 1^+} H(x) = -2$.

# 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically (concluded)**

Since 1) $\lim\limits_{x \to 1^-} H(x) = 4$

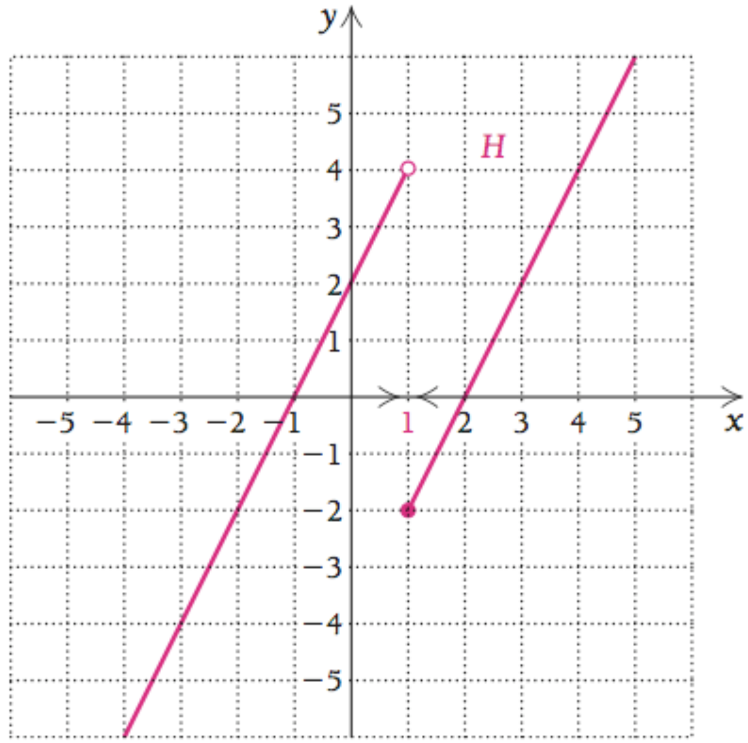
and

2) $\lim\limits_{x \to 1^+} H(x) = -2,$

then, $\lim\limits_{x \to 1} H(x)$ does not exist.

# 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Graphically**




Observe on the graph that

1) $\lim_{x \to 1^-} H(x) = 4$

and

2) $\lim_{x \to 1^+} H(x) = -2.$

Therefore,

$\lim_{x \to 1} H(x)$ does not exist.

# 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically**

First, let $x$ approach –3 from the left:

| $x \to -3^-$ | –4 | –3.5 | –3.1 | –3.01 | –3.001 |
|---|---|---|---|---|---|
| $H(x)$ | –6 | –5 | –4.2 | –4.02 | –4.002 |

Thus, it appears that $\lim_{x \to -3^-} H(x) = -4$.

# 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically (continued)**

Then, let $x$ approach $-3$ from the right:

| $x \to -3^+$ | $-2$ | $-2.5$ | $-2.9$ | $-2.99$ | $-2.999$ |
|---|---|---|---|---|---|
| $H(x)$ | $-2$ | $-3$ | $-3.8$ | $-3.98$ | $-3.998$ |

Thus, it appears that $\lim\limits_{x \to -3^+} H(x) = -4.$

# 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically (concluded)**

Since 1) $\lim_{x \to -3^-} H(x) = -4$

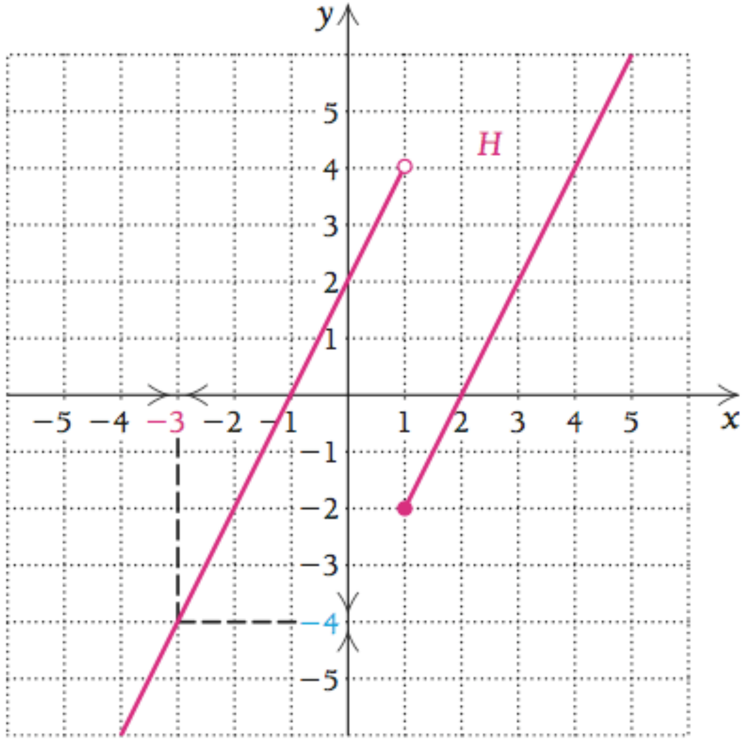
and

2) $\lim_{x \to -3^+} H(x) = -4,$

then, $\lim_{x \to -3} H(x) = -4.$

# 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Graphically**




Observe on the graph that

1) $\lim\limits_{x \to -3^-} H(x) = -4$

and

2) $\lim\limits_{x \to -3^+} H(x) = -4.$

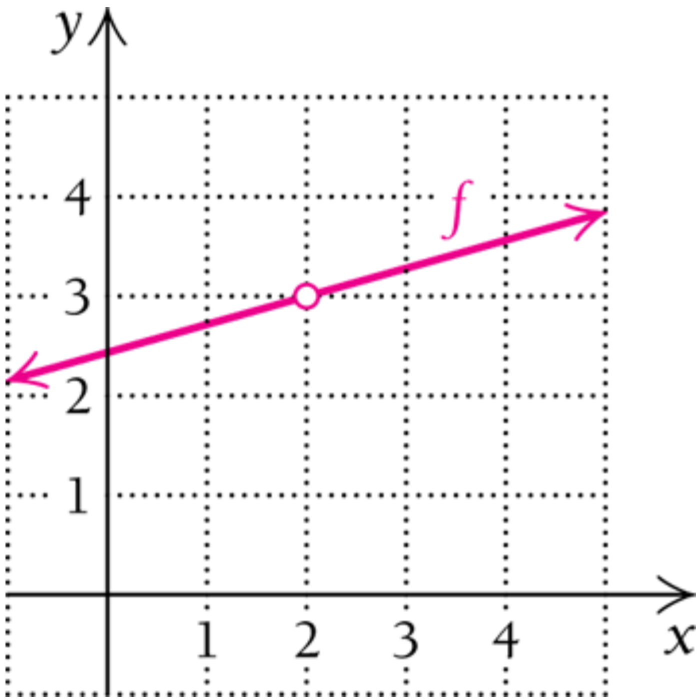
Therefore,

$$\lim_{x \to -3} H(x) = -4.$$

# 1.1 Limits: A Numerical and Graphical Approach

Quick Check 2

Calculate the following limits based on the graph of $f$.




a.) $\lim_{x \to 2^-} f(x)$

b.) $\lim_{x \to 2^+} f(x)$

c.) $\lim_{x \to 2} f(x)$

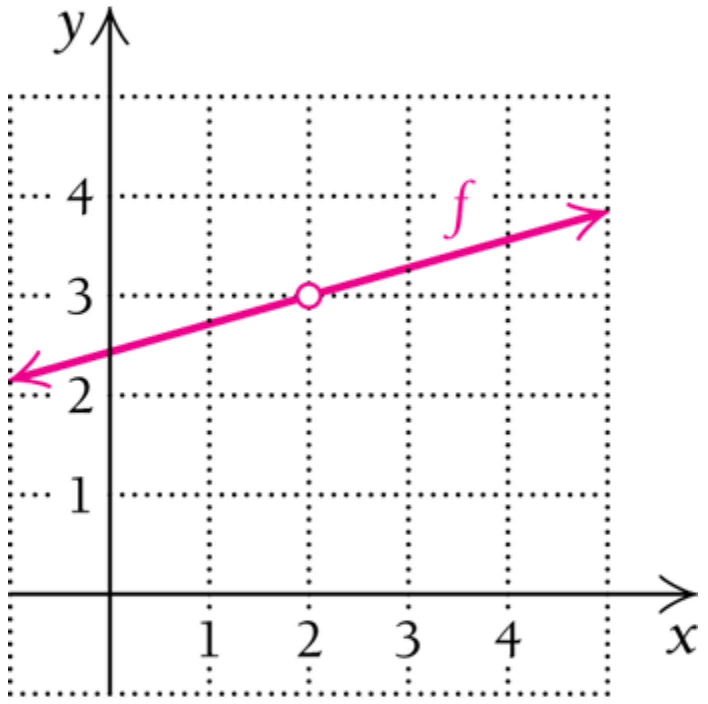
# 1.1 Limits: A Numerical and Graphical Approach

## Quick Check 2 Solution

a.) $\lim\limits_{x \to 2^-} f(x)$: By looking at the graph, as $x$ approaches $2$ from the left, we can see that the $\lim\limits_{x \to 2^-} f(x) = 3$.

b.) $\lim\limits_{x \to 2^+} f(x)$: By looking at the graph, as $x$ approaches $2$ from the right, we can see that the $\lim\limits_{x \to 2^+} f(x) = 3$.

c.) Based on the solutions to parts a.) and b.), we know that the $\lim\limits_{x \to 2} f(x) = 3$.



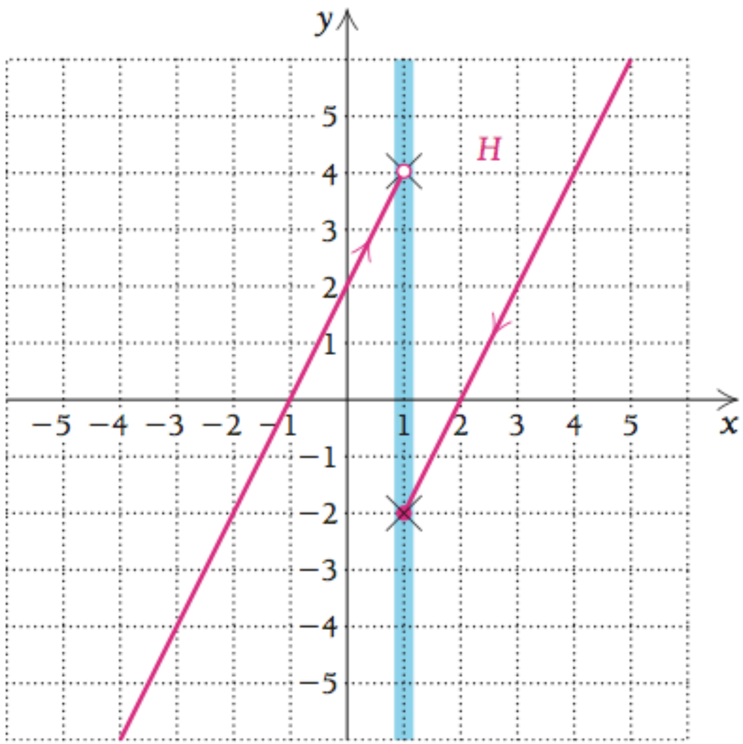
# 1.1 Limits: A Numerical and Graphical Approach

**The "Wall" Method:**

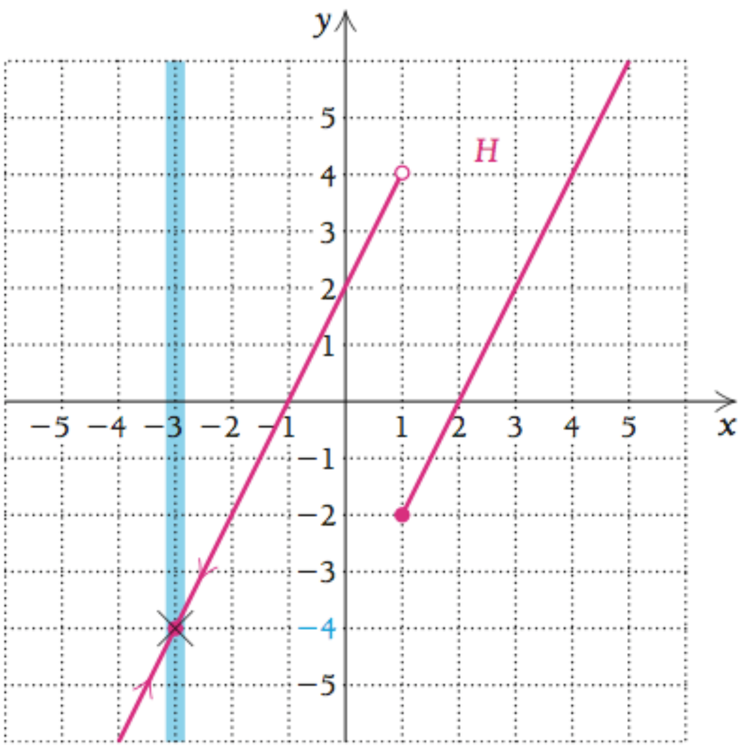
As an alternative approach to Example 1, we can draw a "wall" at $x = 1$, as shown in blue on the following graphs. We then follow the curve from left to right with pencil until we hit the wall and mark the location with an ×, assuming it can be determined. Then we follow the curve from right to left until we hit the wall and mark that location with an ×. If the locations are the same, we have a limit. Otherwise, the limit does not exist.

# 1.1 Limits: A Numerical and Graphical Approach

**Thus, for Example 2:**




$$\lim_{x \to 1} H(x) \text{ does not exist}$$




$$\lim_{x \to -3} H(x) = -4$$

# 1.1 Limits: A Numerical and Graphical Approach

**Example 3:** Consider the function $f$ given by

$$f(x) = \frac{1}{x-2} + 3.$$

Graph the function, and find each of the following limits, if they exist. If necessary, state that the limit does not exist.

a) $\lim_{x \to 3} f(x)$

b) $\lim_{x \to 2} f(x)$

# 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically**

Let $x$ approach 3 from the left and right:

| $x \to 3^-$ | 2.1 | 2.5 | 2.9 | 2.99 |
|---|---|---|---|---|
| $f(x)$ | 13 | 5 | $4.\overline{11}$ | $4.\overline{01}$ |

$\Rightarrow \lim\limits_{x \to 3^-} f(x) = 4$

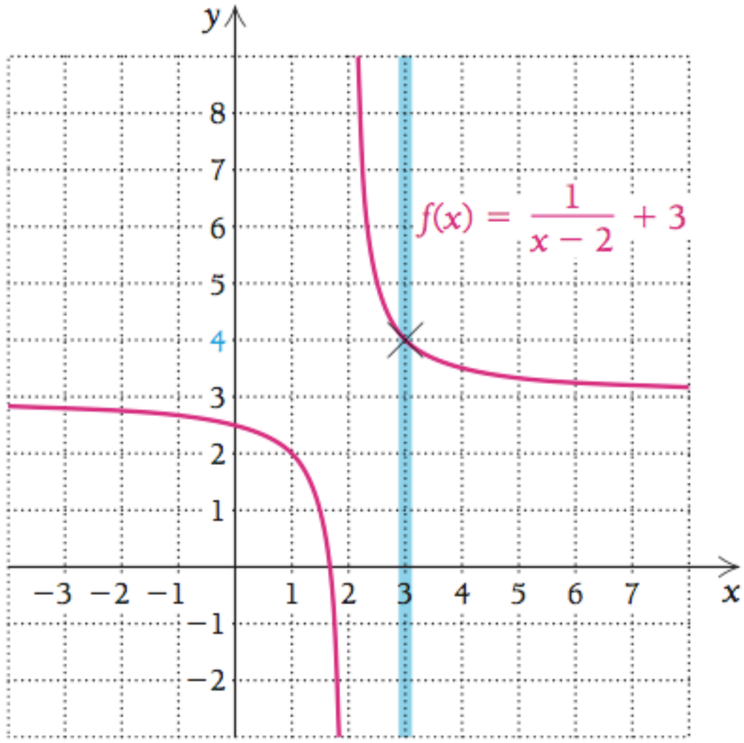
| $x \to 3^+$ | 3.5 | 3.2 | 3.1 | 3.01 |
|---|---|---|---|---|
| $f(x)$ | $3.6\overline{6}$ | $3.8\overline{3}$ | $3.90\overline{90}$ | $3.\overline{9900}$ |

$\Rightarrow \lim\limits_{x \to 3^+} f(x) = 4$

Thus, $\lim\limits_{x \to 3} f(x) = 4.$

# 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Graphically**




Observe on the graph that:

1) $$\lim_{x \to 3^-} f(x) = 4$$

and

2) $$\lim_{x \to 3^+} f(x) = 4$$

Therefore,

$$\lim_{x \to 3} f(x) = 4.$$

# 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically**

Let $x$ approach 2 from the left and right:

| $x \to 2^-$ | 1.5 | 1.9 | 1.99 | 1.999 |
|---|---|---|---|---|
| $f(x)$ | 1 | –7 | –97 | –997 |

$\Rightarrow \lim_{x \to 2^-} f(x) = -\infty$

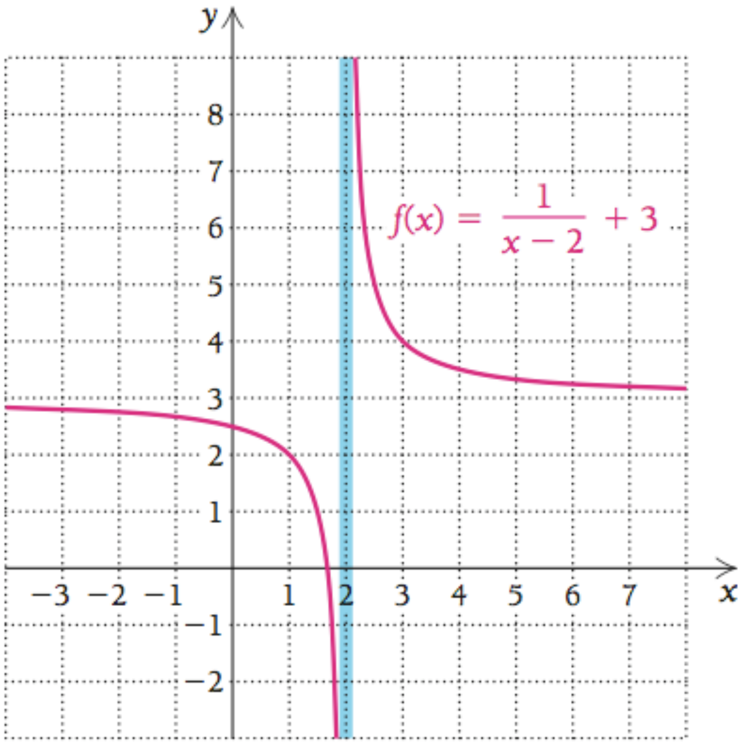
| $x \to 2^+$ | 2.5 | 2.1 | 2.01 | 2.001 |
|---|---|---|---|---|
| $f(x)$ | 5 | 13 | 103 | 1003 |

$\Rightarrow \lim_{x \to 2^+} f(x) = \infty$

Thus, $\lim_{x \to 2} f(x)$ does not exist.

# 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Graphically**




Observe on the graph that

1) $\lim_{x \to 2^-} f(x) = -\infty$

and

2) $\lim_{x \to 2^+} f(x) = \infty$.

Therefore,

$\lim_{x \to 2} f(x)$ does not exist.

# 1.1 Limits: A Numerical and Graphical Approach

**Example 4:** Consider again the function $f$ given by

$$f(x) = \frac{1}{x-2} + 3.$$

Find $\lim_{x \to \infty} f(x)$.

# 1.1 Limits: A Numerical and Graphical Approach

## Limit Numerically

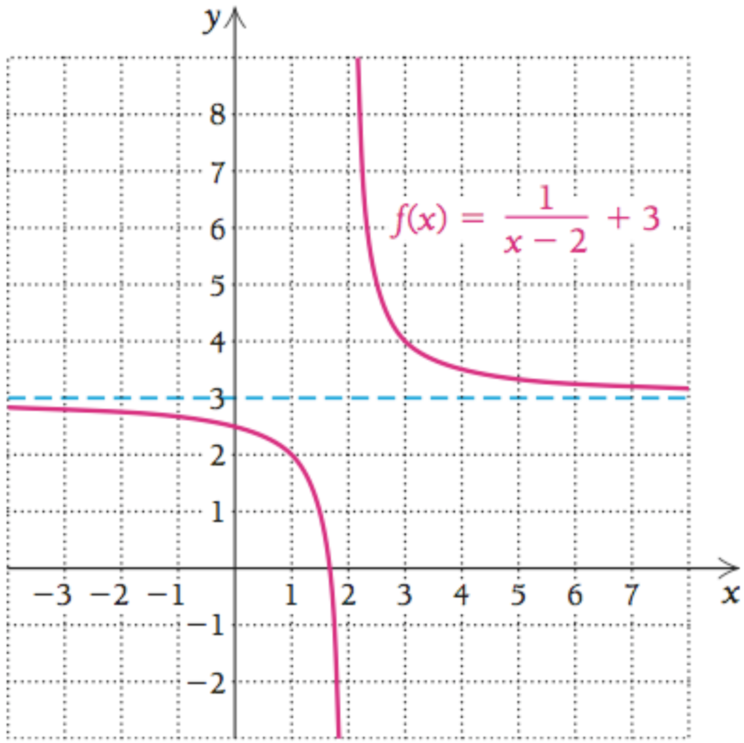
Note that you can only approach $\infty$ from the left:

| $x \to \infty$ | 5 | 10 | 100 | 1000 |
|---|---|---|---|---|
| $f(x)$ | $3.\overline{3}$ | 3.125 | 3.0102 | 3.001 |

Thus, $\lim\limits_{x \to \infty} f(x) = 3.$

# 1.1 Limits: A Numerical and Graphical Approach

## Limit Graphically




Observe on the graph that, again, you can only approach ∞ from the left.

Therefore,

$$\lim_{x \to \infty} f(x) = 3.$$

# 1.1 Limits: A Numerical and Graphical Approach

Quick Check 3

Let $h(x) = \dfrac{1}{1-x} + 6.$ Find the following limits:

a.) $\lim\limits_{x \to 1} h(x)$

b.) $\lim\limits_{x \to 2} h(x)$

c.) $\lim\limits_{x \to \infty} h(x)$

# 1.1 Limits: A Numerical and Graphical Approach

Quick Check 3 Solution

a.) $\lim_{x \to 1} h(x)$: Find the left-hand and right-hand limits as $x$ approaches $1$:

Left-hand Limit

| $x \to 1^-$ | $h(x)$ |
|---|---|
| 0 | 7 |
| 0.5 | 8 |
| 0.9 | 16 |
| 0.99 | 106 |
| 0.999 | 1006 |

Right-hand Limit

| $x \to 1^+$ | $h(x)$ |
|---|---|
| 2 | 5 |
| 1.5 | 4 |
| 1.1 | –4 |
| 1.01 | –94 |
| 1.001 | –994 |

Since the Left-Hand Limit goes to $\infty$ and the Right-Side Limit goes to $-\infty$,

the $\lim_{x \to 1} h(x) =$ does not exist.

# 1.1 Limits: A Numerical and Graphical Approach

Quick Check Solution Continued

b.) $\lim_{x \to 2} h(x)$: Find both the left-hand and right-hand limits as $x$ approaches 2.

Left-Hand Limit

| $x \to 2^-$ | $h(x)$ |
|---|---|
| 1.1 | $-4$ |
| 1.5 | 4 |
| 1.9 | $4.\overline{8}$ |
| 1.99 | $4.\overline{98}$ |
| 1.999 | $4.\overline{998}$ |

Right-Hand Limit

| $x \to 2^+$ | $h(x)$ |
|---|---|
| 3 | 5.5 |
| 2.5 | $5.\overline{3}$ |
| 2.1 | $5.\overline{09}$ |
| 2.01 | $5.\overline{0099}$ |
| 2.001 | $5.\overline{000999}$ |

Since both the Left-Side Limit and Right-Side Limit agree, the $\lim_{x \to 2} h(x) = 5$.

# 1.1 Limits: A Numerical and Graphical Approach

Quick Check Solution Concluded

c.) $\lim_{x \to \infty} h(x)$: Find the limit as $x$ approaches $\infty$:

| $x \to \infty$ | $h(x)$ |
|---|---|
| 5 | 5.75 |
| 10 | $5.\overline{8}$ |
| 100 | $5.\overline{98}$ |
| 1000 | $5.\overline{998}$ |

Since both the Left-Side Limit and Right-Side Limit agree, the $\lim_{x \to \infty} h(x) = 6$.

# 1.1 Limits: A Numerical and Graphical Approach

## Section Summary

- The *limit* of a function $f$, as $x$ approaches $a$, is written $\lim_{x \to a} f(x) = L.$

  This means that as the values of $x$ approach $a$ the corresponding values of $f(x)$ approach $L$. The value of $L$ must be a unique, finite number.

- A *left-hand limit* is written $\lim_{x \to a^-} f(x).$

  The values of $x$ are approaching $a$ from the left, that is, $x < a$.

- A *right-hand limit* is written $\lim_{x \to a^+} f(x).$

  The values of $x$ are approaching $a$ from the right, that is, $x > a$.

- If the left-hand and right-hand limits (as $x$ approaches $a$) are *not* equal, the limit does *not* exist. On the other hand, if the left-hand and right-hand limits are equal, the limit does exist.

- A limit $\lim_{x \to a} f(x)$ may exist even though the function value $f(a)$ does not. (See Example 1.)
- A limit $\lim_{x \to a} f(x)$ may exist and be different from the function value $f(a)$. (See Example 3b.)
- Graphs and tables are useful tools in determining limits.

# 1.2 Algebraic Limits and Continuity

**OBJECTIVE**

- Develop and use the Limit Principles to calculate limits.
- Determine whether a function is continuous at a point.

# 1.2 Algebraic Limits and Continuity

**LIMIT PROPERTIES:**

If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, and c is any constant, then we have the following:

**L.1**

The limit of a constant is the constant: $\lim\limits_{x \to a} c = c$

# 1.2 Algebraic Limits and Continuity

**LIMIT PROPERTIES (continued):**

**L.2** The limit of a power function is the limit of the base, raised to that power.

That is, for any positive integer *n*,

$$\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n,$$

and

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L},$$

assuming that $L \geq 0$ when $n$ is even.

# 1.2 Algebraic Limits and Continuity

**LIMIT PROPERTIES (continued):**

**L.3** The limit of a sum or difference is the sum or difference of the limits.

$$\lim_{x \to a}\left[f(x) \pm g(x)\right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M.$$

**L.4** The limit of a product is the product of the limits.

$$\lim_{x \to a}\left[f(x) \cdot g(x)\right] = \left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right] = L \cdot M.$$

# 1.2 Algebraic Limits and Continuity

**LIMIT PROPERTIES (concluded):**

**L.5** The limit of a quotient is the quotient of the limits.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}, \quad M \neq 0.$$

**L.6** The limit of a constant times a function is the constant times the limit.

$$\lim_{x \to a} \left[ cf(x) \right] = c \cdot \lim_{x \to a} f(x) = cL.$$

# 1.2 Algebraic Limits and Continuity

**Example 1:** Use the limit properties to find

$$\lim_{x \to 4}\left(x^2 - 3x + 7\right)$$

We know that $\lim_{x \to 4} x = 4.$

By Limit Property L4,

$$\lim_{x \to 4} x^2 = \lim_{x \to 4} x \cdot \lim_{x \to 4} x = 4 \cdot 4 = 16.$$

## Example 1 (concluded):

By Limit Property L6,

$$\lim_{x\to 4}\left(-3x\right) = -3\cdot\lim_{x\to 4} x = -3\cdot 4 = -12.$$

By Limit Property L1,

$$\lim_{x\to 4} 7 = 7.$$

Thus, using Limit Property L3, we have

$$\lim_{x\to 4}\left(x^2 - 3x + 7\right) = 16 - 12 + 7 = 11.$$

## THEOREM 2: LIMITS OF RATIONAL FUNCTIONS

For any rational function $F$, with $a$ in the domain of $F$,

$$\lim_{x \to a} F(x) = F(a).$$

# 1.2 Algebraic Limits and Continuity

**Example 2:** Find $\lim_{x \to 0} \sqrt{\left(x^2 - 3x + 2\right)}$

The Theorem on Limits of Rational Functions and Limit Property L2 tell us that we can substitute to find the limit:

$$\lim_{x \to 0} \sqrt{\left(x^2 - 3x + 2\right)} = \sqrt{0^2 - 3 \cdot 0 + 2} = \sqrt{2}$$

# 1.2 Algebraic Limits and Continuity

Quick Check 1

Find the following limits and note the Limit Property you use at each step:

a.) $\lim_{x \to 1} 2x^3 + 3x^2 - 6$

b.) $\lim_{x \to 4} \dfrac{2x^2 + 5x - 1}{3x - 2}$

c.) $\lim_{x \to 2} \sqrt{1 + 3x^2}$

# 1.2 Algebraic Limits and Continuity

Quick Check 1 Solution a.) $\lim_{x \to 1} 2x^3 + 3x^2 - 6$

We know that the $\lim_{x \to 1} x = 1.$

1.) $\lim_{x \to 1} x^3 = \lim_{x \to 1} x \cdot \lim_{x \to 1} x \cdot \lim_{x \to 1} x = 1 \cdot 1 \cdot 1 = 1$ Limit Property L4

2.) $\lim_{x \to 1} 2x^3 = 2(\lim_{x \to 1} x \cdot \lim_{x \to 1} x \cdot \lim_{x \to 1} x) = 2$ Limit Property L6

3.) $\lim_{x \to 1} x^2 = \lim_{x \to 1} x \cdot \lim_{x \to 1} x = 1 \cdot 1 = 1$ Limit Property L4

4.) $\lim_{x \to 1} 3x^2 = 3(\lim_{x \to 1} x \cdot \lim_{x \to 1} x) = 3$ Limit Property L6

5.) $\lim_{x \to 1} 6 = 6$ Limit Property L1

6.) Combining steps 2.), 4.), and 5.) we get

$\lim_{x \to 1} 2x^3 + 3x^2 - 6 = 2 + 3 - 6 = -1$ Limit Property L3

# 1.2 Algebraic Limits and Continuity

Quick Check 1 solution b.) $\lim_{x \to 4} \dfrac{2x^2 + 5x - 1}{3x - 2}$

We know that $\lim_{x \to 4} x = 4.$

1.) $\lim_{x \to 4} 2x^2 = 2(\lim_{x \to 4} x \cdot \lim_{x \to 4} x) = 2(4 \cdot 4) = 2 \cdot 16 = 32$ Limit Properties L4 and L6

2.) $\lim_{x \to 4} 5x = 5 \cdot \lim_{x \to 4} x = 5 \cdot 4 = 20$ Limit Property L6

3.) $\lim_{x \to 4} 1 = 1$ Limit Property L1

4.) Combine above steps: $\lim_{x \to 4} 2x^2 + 5x - 1 = 32 + 20 - 1 = 51$ Limit Property L3

5.) $\lim_{x \to 4} 3x = 3 \cdot \lim_{x \to 4} x = 3 \cdot 4 = 12$ Limit Property L6

6.) $\lim_{x \to 4} 2 = 2$ Limit Property L1

7.) Combine above steps: $\lim_{x \to 4} 3x - 2 = 12 - 2 = 10$ Limit Property L3

8.) Combine steps 4.) and 7.)

$$\lim_{x \to 4} \frac{2x^2 + 5x - 1}{3x - 2} = \frac{51}{10} = 5.1$$ Limit Property L5

# 1.2 Algebraic Limits and Continuity

Quick Check 1 solution c.) $\lim_{x \to 2} \sqrt{1 + 3x^2}$

We know that $\lim_{x \to 2} x = 2$.

| | |
|---|---|
| 1.) $\lim_{x \to 2} 1 = 1$ | Limit Property L1 |
| 2.) $\lim_{x \to 2} 3x^2 = 3(\lim_{x \to 2} x \cdot \lim_{x \to 2} x) = 3(2 \cdot 2) = 3 \cdot 4 = 12$ | Limit Properties L4 and L6 |
| 3.) Combine above steps: $\lim_{x \to 2} 1 + 3x^2 = 1 + 12 = 13$ | Limit Property L3 |
| 4.) Using step 3.) | |
| $\lim_{x \to 2} \sqrt{1 + 3x^2} = \sqrt{1 + 12} = \sqrt{13}$ | Limit Property L2 |

# 1.2 Algebraic Limits and Continuity

**Example 3:** Find $\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$

Note that the Theorem on Limits of Rational Functions does not immediately apply because –3 is not in the domain of $\frac{x^2 - 9}{x + 3}$.

However, if we simplify $\frac{x^2 - 9}{x + 3}$ first, the result can be evaluated at $x = -3$.

# 1.2 Algebraic Limits and Continuity

**Example 3 (concluded):**

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = \lim_{x \to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{x+3}}$$

$$= \lim_{x \to -3} x - 3$$

$$= -3 - 3$$

$$= -6$$

This means that the limit exists as x approaches -3, but the actual point (from previous slide) does not.

# 1.2 Algebraic Limits and Continuity

**Example 4:** Find $\lim_{x\to\infty}\left(\dfrac{x^2+4x-5}{2x^2+x+1}\right)$.

First, divide the numerator and the denominator by the highest power of the denominator, $x^2$.

$$\lim_{x\to\infty}\left(\frac{x^2+4x-5}{2x^2+x+1}\right)=\lim_{x\to\infty}\left(\frac{1+\frac{4}{x}-\frac{5}{x^2}}{2+\frac{1}{x}+\frac{1}{x^2}}\right)$$

Using our Limit Properties, we get:

$$=\frac{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{4}{x}-\lim_{x\to\infty}\frac{5}{x^2}}{\lim_{x\to\infty}2+\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{1}{x^2}}=\frac{1+0-0}{2+0+0}=\frac{1}{2}$$

# 1.2 Algebraic Limits and Continuity

**Quick Check 2:**

Find $\lim\limits_{x \to \infty} \dfrac{2x^3 + 5x^2 + 4x - 1}{3x^3 + 6x^2 - 7}$ .

First, divide the numerator and the denominator by the highest power of the denominator, $x^3$.

$$\lim_{x \to \infty} \frac{2x^3 + 5x^2 + 4x - 1}{3x^3 + 6x^2 - 7} = \lim_{x \to \infty} \left( \frac{2 + \frac{5}{x} + \frac{4}{x^2} - \frac{1}{x^3}}{3 + \frac{6}{x} - \frac{7}{x^3}} \right)$$

Using our Limit Properties, we get:

$$= \frac{\lim\limits_{x \to \infty} 2 + \lim\limits_{x \to \infty} \frac{5}{x} + \lim\limits_{x \to \infty} \frac{4}{x^2} - \lim\limits_{x \to \infty} \frac{1}{x^3}}{\lim\limits_{x \to \infty} 3 + \lim\limits_{x \to \infty} \frac{6}{x} - \lim\limits_{x \to \infty} \frac{7}{x^3}} = \frac{2 + 0 + 0 - 0}{3 + 0 - 0} = \frac{2}{3}$$

# 1.2 Algebraic Limits and Continuity

**DEFINITION:**

A function $f$ is **continuous** at $x = a$ if:

1) $f(a)$ exists, (The output at $a$ exists.)

2) $\lim_{x \to a} f(x)$ exists, (The limit as $x \to a$ exists.)

3) $\lim_{x \to a} f(x) = f(a)$. (The limit is the same as the output.)

A function is **continuous over an interval** c < x < d if it is continuous at each point in that interval.

# 1.2 Algebraic Limits and Continuity

**Example 5:** Is the function $f$ given by

$$f(x) = x^2 - 5$$

continuous at $x = 3$? Why or why not?

1) $f(3) = 3^2 - 5 = 9 - 5 = 4$

2) By the Theorem on Limits of Rational Functions,

$$\lim_{x \to 3} x^2 - 5 = 3^2 - 5 = 9 - 5 = 4$$

3) Since $\lim_{x \to 3} f(x) = f(3)$ $f$ is continuous at $x = 3$.

# 1.2 Algebraic Limits and Continuity

**Example 6:** Is the function $g$ given by

$$g(x) = \begin{cases} \frac{1}{2}x + 3, & \text{for} \quad x < -2 \\ x - 1, & \text{for} \quad x \geq -2 \end{cases}$$

continuous at $x = -2$? Why or why not?

1) $g(-2) = -2 - 1 = -3$

2) To find the limit, we look at left and right-side limits.

$$\lim_{x \to -2^-} g(x) = \frac{1}{2} \cdot -2 + 3 = -1 + 3 = 2$$

# 1.2 Algebraic Limits and Continuity

**Example 6 (concluded):**

3) $\lim_{x \to -2^+} g(x) = -2 - 1 = -3$

Since $\lim_{x \to -2^-} g(x) \neq \lim_{x \to -2^+} g(x)$ we see that the

$\lim_{x \to -2} g(x)$ does not exist.

Therefore, $g$ is not continuous at $x = -2$.

# 1.2 Algebraic Limits and Continuity

Quick Check 3

Let $g(x) = \begin{cases} 3x - 5, & \text{for } x < 2 \\ 2x + 1, & \text{for } x \geq 2 \end{cases}$ Is $g$ continuous at $x = 2$ ? Why or why not?

1.) $g(2) = 2(2) + 1 = 4 + 1 = 5$

2.) To find the limit, we look at both the left-hand and right-hand limits:

Left-hand: $\lim\limits_{x \to 2^-} g(x) = 3(2) - 5 = 6 - 5 = 1$

Right-hand: $\lim\limits_{x \to 2^+} g(x) = 2(2) + 1 = 4 + 1 = 5$

Since $\lim\limits_{x \to 2^-} g(x) \neq \lim\limits_{x \to 2^+} g(x)$ we see that $\lim\limits_{x \to 2} g(x)$ does not exist.

Therefore $g$ is not continuous at $x = 2$.

# 1.2 Algebraic Limits and Continuity

## Quick Check 4a

Let $h(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & \text{for } x\neq 3\\ 7, & \text{for } x=3\end{cases}$ Is $h$ continuous at $x=3$? Why or why not?

In order for $h(x)$ to continuous, $\lim\limits_{x\to 3} h(x)=h(3)$. So lets start by finding $\lim\limits_{x\to 3} h(x)$.

$$\lim_{x\to 3} h(x)=\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3} x+3=6$$

So the $\lim\limits_{x\to 3} h(x)=6$. However, $h(3)=7$, and thus $\lim\limits_{x\to 3} h(x)\neq h(3)$. Therefore $h(x)$ is not continuous at $x=3$.

# 1.2 Algebraic Limits and Continuity

## Quick Check 4b

Let $p(x)=\begin{cases}\dfrac{x^2-25}{x-5}, & \text{for } x\neq 5\\ c, & \text{for } x=5\end{cases}$ Determine $c$ such that $p$ is continuous at $x = 5$.

In order for $p$ to be continuous at $x=5$, $\lim_{x\to 5} p(x) = p(5) = c$. So if we find $\lim_{x\to 5} p(x)$, we can determine what $c$ is. Let's find $\lim_{x\to 5} p(x)$:

$$\lim_{x\to 5} p(x) = \lim_{x\to 5}\frac{x^2-25}{x-5} = \lim_{x\to 5}\frac{(x-5)(x+5)}{x-5} = \lim_{x\to 5} x+5 = 10$$

So $\lim_{x\to 5} p(x) = 10$. Therefore, in order for $p$ to be continuous at $x=5$, $c=10$.

# 1.2 Algebraic Limits and Continuity

Section Summary

- For a rational function for which $a$ is in the domain, the limit as $x$ approaches $a$ can be found by direct evaluation of the function at $a$.
- If direct evaluation leads to the *indeterminate form* $0/0$, the limit may still exist: algebraic simplification and/or a table and graph are used to find the limit.
- Informally, a function is *continuous* if its graph can be sketched without lifting the pencil off the paper.

# 1.2 Algebraic Limits and Continuity

Section Summary Continued

- Formally, a function is continuous at $x = a$ if:
    1. The function value $f(a)$ exists
    2. The limit as $x$ approaches $a$ exists
    3. The function value and the limit are equal
    4. This can be summarized as $\lim_{x \to a} f(x) = f(a)$.
- If any part of the continuity definition fails, then the function is *discontinuous* at $x = a$.

# 1.3 Average Rates of Change

**OBJECTIVE**

- Compute an average rate of change.
- Find a simplified difference quotient.

# 1.3 Average Rates of Change

**DEFINITION:**

As $x$ approaches $a$, the **limit** of $f(x)$ is $L$, written

$$\lim_{x \to a} f(x) = L,$$

if all values of $f(x)$ are close to $L$ for values of $x$ that are sufficiently close, but not equal to, $a$.

# 1.3 Average Rates of Change

**DEFINITION:**

The **average rate of change of $y$ with respect to $x$**, as $x$ changes from $x_1$ to $x_2$, is the ratio of the change in output to the change in input:

$$\frac{y_2 - y_1}{x_2 - x_1}, \qquad \text{where } x_2 \neq x_1.$$

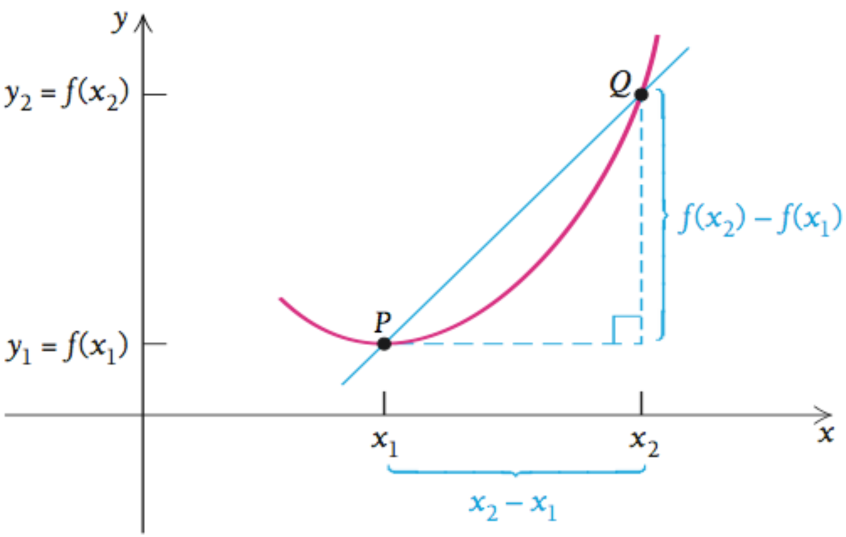
# 1.3 Average Rates of Change

**DEFINITION (concluded):**

If we look at a graph of the function, we see that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

which is both the average rate of change *and* the slope of the line from $P(x_1, y_1)$ to $Q(x_2, y_2)$.




The line through P and Q, $\overleftrightarrow{PQ}$, is called a **secant line**.

# 1.3 Average Rates of Change

**Example 1:** For $y = f(x) = x^2$ find the average rate of change as:

a) $x$ changes from 1 to 3.

b) $x$ changes from 1 to 2.

c) $x$ changes from 2 to 3.

a) When $x_1 = 1$, $y = f(x_1) = f(1) = 1^2 = 1.$

When $x_2 = 3$, $y = f(x_2) = f(3) = 3^2 = 9.$

Thus, the average rate of change is

$$\frac{9-1}{3-1} = \frac{8}{2} = 4.$$

# 1.3 Average Rates of Change

**Example 1 (concluded):**

b) When $x_1 = 1$, $y = f(x_1) = f(1) = 1^2 = 1.$

When $x_2 = 2$, $y = f(x_1) = f(2) = 2^2 = 4.$

Thus, the average rate of change is

$$\frac{4-1}{2-1} = \frac{3}{1} = 3.$$

c) When $x_1 = 2$, $y = f(x_1) = f(2) = 2^2 = 4.$

When $x_2 = 3$, $y = f(x_1) = f(3) = 3^2 = 9.$

Thus, the average rate of change is

$$\frac{9-4}{3-2} = \frac{5}{1} = 5.$$

# 1.3 Average Rates of Change

Quick Check 1

State the average rate of change for each situation in a short sentence. Be sure to include units.

a.) It rained 4 inches over a period of 8 hours.

The average rate of change is $\dfrac{4\text{ in} - 0\text{ in}}{8\text{ hr} - 0\,\text{hr}} = \dfrac{4\text{ in}}{8\text{ in}} = \dfrac{1\text{ in}}{2\,\text{hr}}$.

The average rate of rain fall was 0.5 inches of rain every hour.

b.) Your car travels 250 miles on 20 gallons of gas.

The average rate of change is $\dfrac{250\,\text{mi} - 0\text{ mi}}{20\text{ gal} - 0\text{ gal}} = \dfrac{250\text{ mi}}{20\text{ gal}} = \dfrac{25\text{ mi}}{2\text{ gal}}$.

The average miles traveled on a gallon of gas was 12.5 miles every gallon.

c.) At 2 p.m., the temperature was 82 degrees. At 5 p.m., the temperature was 76 degrees.

The average rate of change is $\dfrac{82 - 76\text{ degrees}}{5\text{ p.m.} - 2\text{ p.m.}} = \dfrac{-6\text{ degrees}}{3\text{ hours}} = -\dfrac{2\text{ degrees}}{1\text{ hour}}$.

The average change in temperature was -2 degrees every hour.

# 1.3 Average Rates of Change

Quick Check 2

For $f(x) = x^3$, find the average rate of change between:

a.) $x = 1$ and $x = 4$;

b.) $x = 1$ and $x = 2$;

c.) $x = 1$ and $x = 1.2$.

a.) When $x_1 = 1$, $y_1 = f(x_1) = f(1) = 1^3 = 1$.

When $x_2 = 4$, $y_2 = f(x_2) = f(4) = 4^3 = 64$.

Thus the rate of change is $\dfrac{64-1}{4-1} = \dfrac{63}{3} = 21$.

# 1.3 Average Rates of Change

Quick Check 2 Continued

b.) When $x_1 = 1,\ y_1 = f(x_1) = f(1) = 1^3 = 1.$

When $x_2 = 2,\ y_2 = f(x_2) = f(2) = 2^3 = 8.$

Thus the average rate of change is $\dfrac{8-1}{2-1} = 7.$

c.) When $x_1 = 1,\ y_1 = f(x_1) = f(1) = 1^3 = 1.$

When $x_2 = 1.2,\ y_2 = f(x_2) = f(1.2) = 1.2^3 = 1.728.$

Thus the average rate of change is

$$\frac{1.728-1}{1.2-1} = 3.64.$$

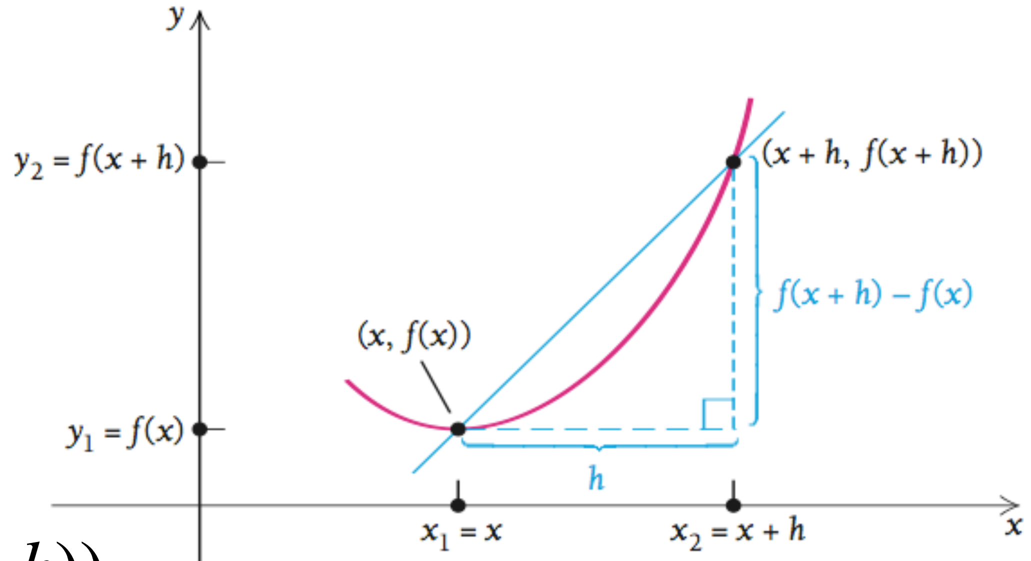
# 1.3 Average Rates of Change

**DEFINITION:**

The average rate of change of $f$ with respect to $x$ is also called the **difference quotient**. It is given by

$$\frac{f(x+h)-f(x)}{h}, \text{ where } h \neq 0.$$

The difference quotient is equal to the slope of the secant line from $(x, f(x))$ to $(x+h, f(x+h))$.



# 1.3 Average Rates of Change

**Example 2:** For $f(x) = x^2$ find the difference quotient when:

a) $x = 5$ and $h = 3$.

b) $x = 5$ and $h = 0.1$.

a) We substitute $x = 5$ and $h = 3$ into the formula:

$$\frac{f(x+h)-f(x)}{h} = \frac{f(5+3)-f(5)}{3} = \frac{f(8)-f(5)}{3}$$

$$= \frac{8^2-5^2}{3} = \frac{64-25}{3} = \frac{39}{3} = 13$$

# 1.3 Average Rates of Change

**Example 2 (concluded):**

b) We substitute $x = 5$ and $h = 0.1$ into the formula:

$$\frac{f(x+h)-f(x)}{h} = \frac{f(5+0.1)-f(5)}{0.1} = \frac{f(5.1)-f(5)}{0.1}$$

$$= \frac{5.1^2-5^2}{0.1} = \frac{26.01-25}{0.1} = \frac{1.01}{0.1} = 10.1.$$

# 1.3 Average Rates of Change

**Example 3:** For $f(x) = x^3$ find a simplified form of the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}$$

$$= \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \frac{\cancel{h}\left(3x^2 + 3xh + h^2\right)}{\cancel{h}}$$

$$= 3x^2 + 3xh + h^2, \quad h \neq 0.$$

# 1.3 Average Rates of Change

Quick Check 3

Use the result of Example 3 to calculate the slope of the secant line (average rate of change) at $x = 2$, for $h = 0.1$, $h = 0.01$, and $h = 0.001$.

Use the formula found in Example 6 $(3x^2 + 3xh + h^2, h \neq 0)$.

For $h = 0.1$: $3(2)^2 + 3(2)(0.1) + 0.1^2 = 12 + 0.6 + 0.01 = 12.61$

For $h = 0.01$: $3(2)^2 + 3(2)(0.01) + 0.01^2 = 12 + 0.06 + 0.0001 = 12.0601$

For $h = 0.001$: $3(2)^2 + 3(2)(0.001) + 0.001^2 = 12 + 0.006 + 0.000001$

$$= 12.006001$$

# 1.3 Average Rates of Change

**Example 4:** For $f(x) = \dfrac{3}{x}$ find a simplified form of the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{3}{x+h} - \dfrac{3}{x}}{h} = \frac{\dfrac{3x - 3(x+h)}{x(x+h)}}{h}$$

$$= \frac{\dfrac{\cancel{3x} - \cancel{3x} - 3h}{x(x+h)}}{h} = \frac{\dfrac{-3\cancel{h}}{x(x+h)}}{\cancel{h}}$$

$$= \frac{-3}{x(x+h)}, \quad h \neq 0.$$

# 1.3 Average Rates of Change

Section Summary

- An *average rate of change* is the slope of a line between two points.

If the points are $(x_1, y_1)$ and $(x_2, y_2)$, then the average rate of change is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

- If the two points are solutions to a single function, an equivalent form of the slope formula is $\dfrac{f(x+h)-f(x)}{h}$, where $h$ is the horizontal difference between the two $x$-values. This is called the *difference quotient*. The line connecting these two points is called a *secant line.*

# 1.3 Average Rates of Change

Section Summary Continued

- The difference quotient is the same as the slope formula. Both give the slope of the line between two points.
- The difference quotient gives the *average rate of change* between two points on a graph, represented by the secant line.
- It is preferable to simplify a difference quotient algebraically before evaluating it for particular values of $x$ and $h$.

# 1.4 Differentiation Using Limits of Difference Quotients

**OBJECTIVE**

- Find derivatives and values of derivatives
- Find equations of tangent lines

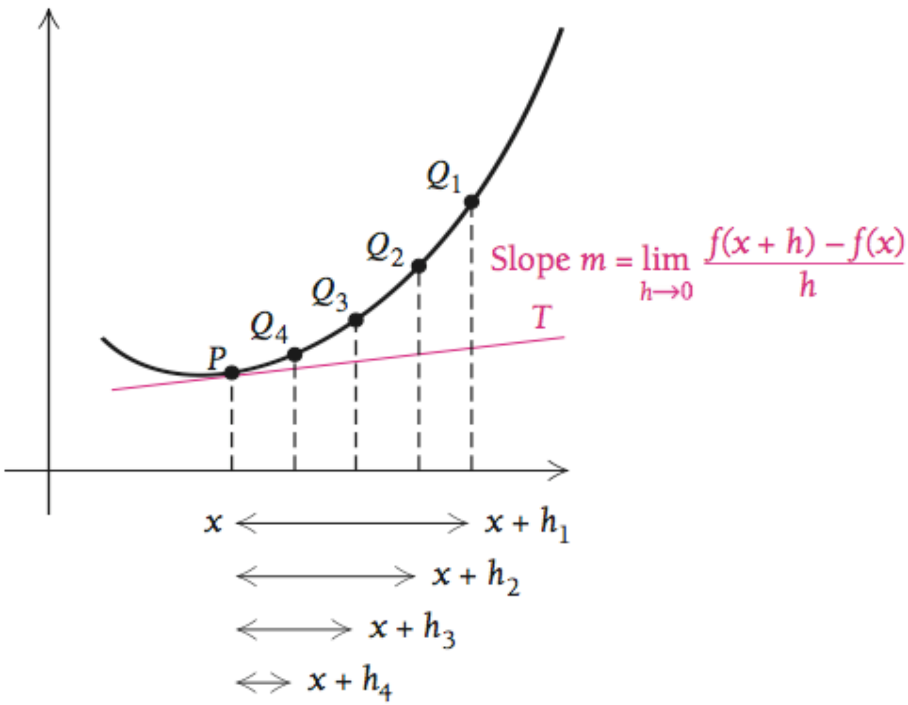
# 1.4 Differentiation Using Limits of Difference Quotients

**DEFINITION:**

The **slope of the tangent line** at $(x, f(x))$ is

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This limit is also the **instantaneous rate of change** of $f(x)$ at $x$.



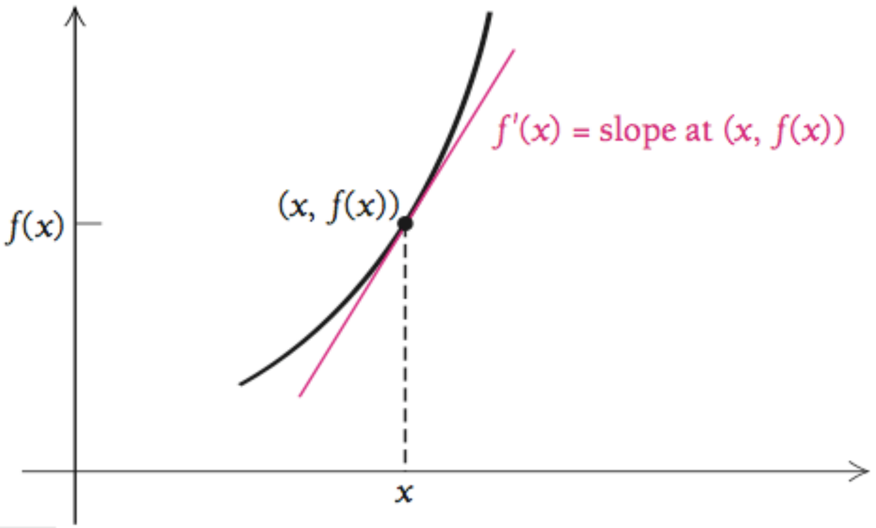
# 1.4 Differentiation Using Limits of Difference Quotients

**DEFINITION:**

For a function $y = f(x)$, its **derivative** at $x$ is the function $f'$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. If $f'(x)$ exists, then we say that $f$ is **differentiable** at $x$.



# 1.4 Differentiation Using Limits of Difference Quotients

**Example 1:** For $f(x) = x^2$, find $f'(x)$. Then find $f'(-3)$ and $f'(4)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$f'(x) = \lim_{h \to 0} 2x + h$$

$$f'(x) = 2x$$

# 1.4 Differentiation Using Limits of Difference Quotients

## Example 1 (concluded):

$$f'(x) = 2x$$

$$f'(-3) = 2(-3) = -6$$

$$f'(4) = 2(4) = 8$$

# 1.4 Differentiation Using Limits of Difference Quotients

**Example 2:** For $f(x) = x^3$, find $f'(x)$.
Then find $f'(-1)$ and $f'(1.5)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$f'(x) = 3x^2$$

# 1.4 Differentiation Using Limits of Difference Quotients

**Example 2 (concluded):**

$$f'(x) = 3x^2$$

$$f'(-1) = 3(-1)^2 = 3(1) = 3$$

$$f'(x) = 3(1.5)^2 = 3(2.25) = 6.75$$

# 1.4 Differentiation Using Limits of Difference Quotients

Quick Check 1

Use the results from Examples 1 and 2 to find the derivative $f(x) = x^3 + x^2$ and then calculate $f'(-2)$ and $f'(4)$. Interpret these results.

From Example 1, we know that the derivative of $x^2$ is 2x , and from Example 2, we know that the derivative of $x^3$ is $3x^2$ Using the Limit Property L3, we then know that $f'(x) = 3x^2 + 2x$ .

# 1.4 Differentiation Using Limits of Difference Quotients

Quick Check 1 Concluded

Now, we plug in $x = -2$ into our new derivative formula:

$$f'(-2) = 3(-2)^2 + 2(-2) \;= 12 - 4 \;= 8$$

Next, we plug in $x = 4$ into our new derivative formula:

$$f'(4) = 3(4)^2 + 2(4) = 48 + 8 = 56$$

These results mean that when $x = -2,$ the slope of the tangent line is 8, and when $x = 4,$ the slope of the tangent line is 56.

# 1.4 Differentiation Using Limits of Difference Quotients

**Example 3:** For $f(x) = \dfrac{3}{x}$:

a) Find $f'(x)$.

b) Find $f'(2)$.

c) Find an equation of the tangent line to the curve at $x = 2$.

# 1.4 Differentiation Using Limits of Difference Quotients

**Example 3 (continued):**

a) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{3}{x+h}-\frac{3}{x}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{3x-3(x+h)}{x(x+h)}}{h}=\lim_{h\to 0}\frac{\frac{\cancel{3x}-\cancel{3x}-3h}{x(x+h)}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{-3\cancel{h}}{x(x+h)}}{\cancel{h}}=\lim_{h\to 0}\frac{-3}{x(x+h)}=-\frac{3}{x^2}.$$

# 1.4 Differentiation Using Limits of Difference Quotients

**Example 3 (continued):**

b) $$f'(x) = -\frac{3}{x^2}$$

$$f'(2) = -\frac{3}{2^2} = -\frac{3}{4}$$

# 1.4 Differentiation Using Limits of Difference Quotients

**Example 3 (concluded):**

c) $x = 2,\ m = f'(2) = -\dfrac{3}{4},\ y = f(2) = \dfrac{3}{2}$

$$\begin{aligned} y &= mx + b \\ \frac{3}{2} &= -\frac{3}{4}\cdot 2 + b \\ \frac{3}{2} &= -\frac{3}{2} + b \\ 3 &= b \end{aligned}$$

Thus, $y = -\dfrac{3}{4}x + 3$

is the equation of the tangent line.

# 1.4 Differentiation Using Limits of Difference Quotients

Quick Check 2

Repeat Example 3a for $f(x) = -\frac{2}{x}$. What are the similarities in your method?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad = \lim_{h \to 0} \frac{\frac{-2}{x+h} - \frac{-2}{x}}{h} \quad = \lim_{h \to 0} \frac{\frac{-2x + 2(x+h)}{x(x+h)}}{h}$$

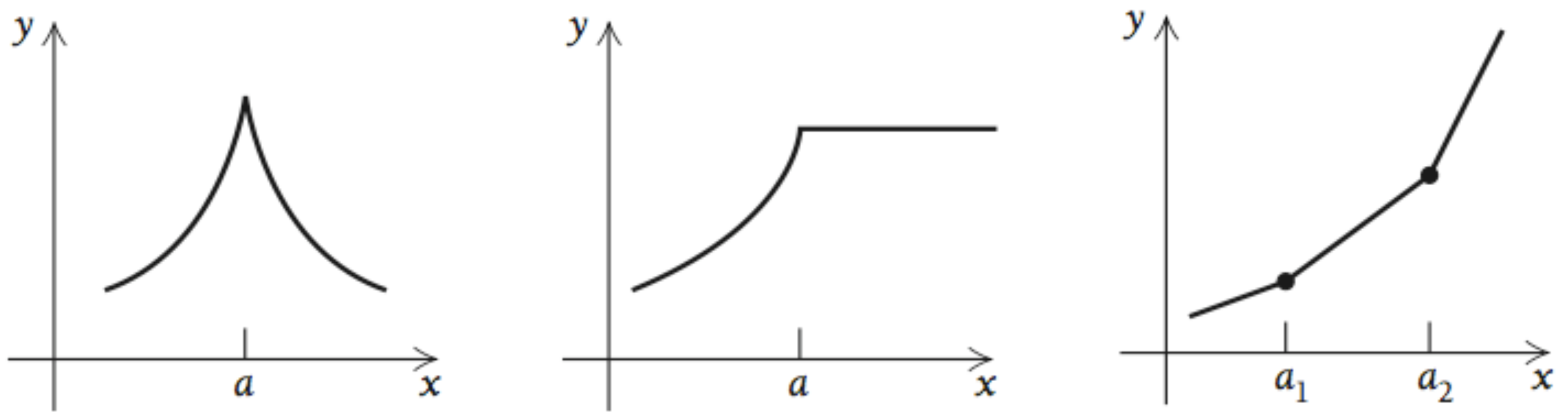
$$= \lim_{h \to 0} \frac{\frac{-2\cancel{x} + 2\cancel{x} + 2h}{x(x+h)}}{h} \quad = \lim_{h \to 0} \frac{\frac{2\cancel{h}}{x(x+h)}}{\cancel{h}} \quad = \lim_{h \to 0} \frac{2}{x(x+h)} \quad = \frac{2}{x^2}$$

Both methods had the same basic principle. You start by using the derivative formula, then you break it down until you do not have an $h$ anywhere in the equation.

# 1.4 Differentiation Using Limits of Difference Quotients

## Where a Function is Not Differentiable:

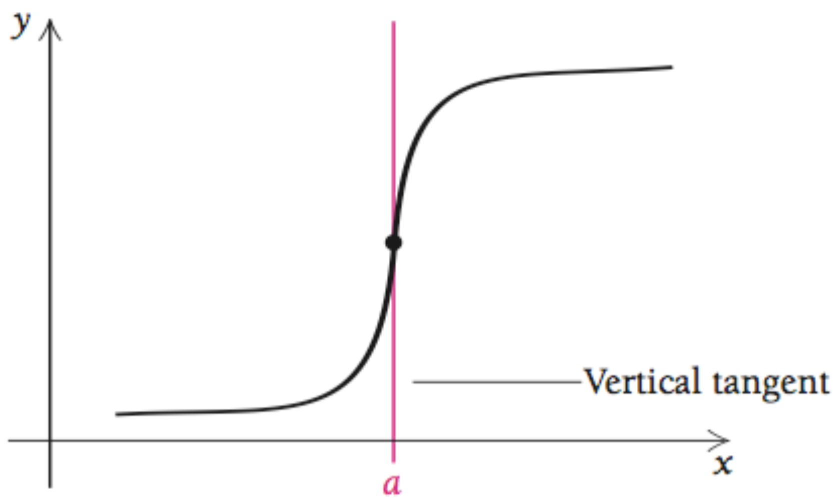
1) A function $f(x)$ is not differentiable at a point $x = a$, if there is a "corner" at $a$.



# 1.4 Differentiation Using Limits of Difference Quotients

## Where a Function is Not Differentiable:

2) A function $f(x)$ is not differentiable at a point $x = a$, if there is a vertical tangent at $a$.

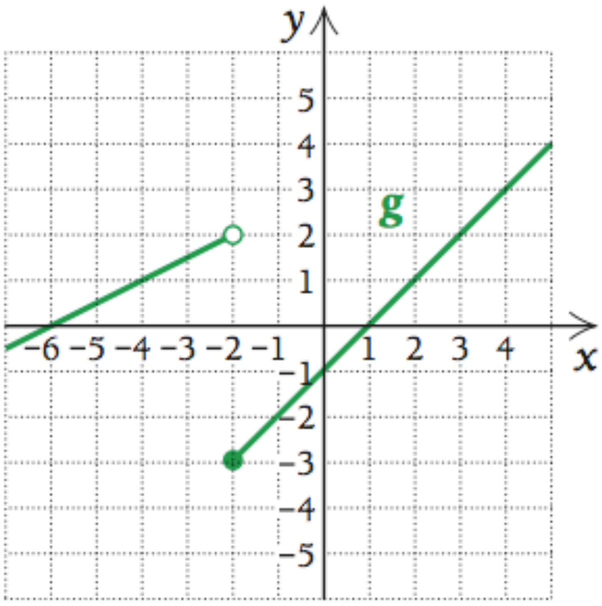


# 1.4 Differentiation Using Limits of Difference Quotients

## Where a Function is Not Differentiable:

3) A function $f(x)$ is not differentiable at a point $x = a$, if it is not continuous at $a$.

Example: $g(x)$ is not continuous at $-2$, so $g(x)$ is not differentiable at $x = -2$.



# 1.4 Differentiation Using Limits of Difference Quotients

Quick Check 3

Where is $f(x) = |x - 6|$ not differentiable? Why?

$f(x) = |x - 6|$ is not differentiable at $x = 6$. This is the vertex of the function, and is considered a corner of the function. Therefore $f(x) = |x - 6|$ is not differentiable at $x = 6$.

# 1.4 Differentiation Using Limits of Difference Quotients

Section Summary

- A *tangent line* is a line that touches a (smooth) curve at a single point, the *point of tangency*. See Fig. 3 (on p. 133) for examples of tangent lines (and lines that are not considered tangent lines).

- The *derivative* of a function $f(x)$ is defined by

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}.$$

# 1.4 Differentiation Using Limits of Difference Quotients

Section Summary Continued

- The *slope* of the tangent line to the graph of $y = f(x)$ at $x = a$ is the value of the derivative at $x = a$; that is, the slope of the tangent line at $x = a$ is $f'(a)$.
- Slopes of tangent lines are interpreted as *instantaneous rates of change.*
- The equation of a tangent line at $x = a$ is found by simplifying $y - f(a) = f'(a)(x - a)$
- If a function is differentiable at a point $x = a$, then it is *continuous* at $x = a$. That is, differentiability implies continuity.

# 1.4 Differentiation Using Limits of Difference Quotients

Section Summary Concluded

- However, continuity at a point $x = a$ does *not* imply differentiability at $x = a$. A good example is the absolute-value function, $f(x) = |x|$, or any function whose graph has a corner. Continuity alone is not sufficient to guarantee differentiability.

- A function is not differentiable at a point $x = a$ if:

  1) There is a discontinuity at $x = a$

  2) There is a corner at $x = a$, or

  3) There is a vertical tangent at $x = a$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**OBJECTIVE**

- Differentiate using the Power Rule or the Sum-Difference Rule.
- Differentiate a constant or a constant times a function.
- Determine points at which a tangent line has a specified slope.

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

## Leibniz's Notation:

When $y$ is a function of $x$, we will also designate the derivative, $f'(x)$, as

$$\frac{dy}{dx},$$

which is read "the derivative of $y$ with respect to $x$."

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

## THEOREM 3: The Power Rule

For any real number $k$,

$$\frac{dy}{dx}x^{k} = k \cdot x^{k-1}$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 1:** Differentiate each of the following:

a) $y = x^5$ b) $y = x$ c) $y = x^{-4}$

a) $\dfrac{d}{dx}x^5 = 5 \cdot x^{5-1}$ b) $\dfrac{d}{dx}x = 1 \cdot x^{1-1}$ c) $\dfrac{d}{dx}x^{-4} = -4x^{-4-1}$

$\dfrac{d}{dx}x^5 = 5x^4$ $\dfrac{d}{dx}x = 1$ $\dfrac{d}{dx}x^{-4} = -4x^{-5}$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

Quick Check 1

a.) Differentiate:

(i) $y = x^{15}$; (ii) $y = x^{-7}$

b.) Explain why $\frac{d}{dx}(\pi^2) = 0$, not $2\pi$.

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

Quick Check 1 solution

a.) Use the Power Rule for both parts: $\dfrac{d}{dx}\left[x^k\right] = kx^{k-1}$

(i) $y = x^{15}, \quad \dfrac{dy}{dx} = 15x^{14}$

(ii) $y = x^{-7}, \quad \dfrac{dy}{dx} = -7x^{-8}$

b.) The reason $\dfrac{d}{dx}(\pi^2) = 0$ and not $2\pi$, is because $\pi$ is a constant, not a variable and the derivative of any constant is 0:

$$\frac{d}{dx}c = 0$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 2:** Differentiate:

a) $y = \sqrt{x}$  b) $y = x^{0.7}$

a) $$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1}$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-\frac{1}{2}},\ or$$

$$= \frac{1}{2x^{\frac{1}{2}}},\ or$$

$$= \frac{1}{2\sqrt{x}}$$

b) $$\frac{d}{dx}x^{0.7} = 0.7 \cdot x^{0.7-1}$$

$$\frac{d}{dx}x^{0.7} = 0.7x^{-0.3}$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**THEOREM 4:**

The derivative of a constant function is 0. That is,

$$\frac{d}{dx}c = 0.$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**THEOREM 5:**

The derivative of a constant times a function is the constant times the derivative of the function. That is,

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx} f(x)$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 3:** Find each of the following derivatives:

a) $\dfrac{d}{dx}7x^4$  b) $\dfrac{d}{dx}(-9x)$  c) $\dfrac{d}{dx}\left(\dfrac{1}{5x^2}\right)$

a)
$$\begin{aligned}\frac{d}{dx}7x^4 &= 7\cdot\frac{d}{dx}x^4\\ &= 7\cdot 4x^{4-1}\\ \frac{d}{dx}7x^4 &= 28x^3\end{aligned}$$

b)
$$\begin{aligned}\frac{d}{dx}(-9x) &= -9\cdot\frac{d}{dx}x\\ &= -9\cdot 1x^{1-1}\\ \frac{d}{dx}(-9x) &= -9\end{aligned}$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 3 (concluded):**

c) $$\frac{d}{dx}\left(\frac{1}{5x^2}\right) = \frac{1}{5}\cdot\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{5}\cdot\frac{d}{dx}x^{-2}$$

$$= \frac{1}{5}\cdot -2x^{-2-1}$$

$$\frac{d}{dx}\left(\frac{1}{5x^2}\right) = -\frac{2}{5}x^{-3}, \text{ or } = -\frac{2}{5x^3}$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

Quick Check 2

Differentiate each of the following:

a.) $y = 10x^9; \ \dfrac{dy}{dx} = 9 \cdot 10x^{9-1} = 90x^8$

b.) $y = \pi x^3; \ \dfrac{dy}{dx} = 3 \cdot \pi x^{3-1} = 3\pi x^2$

c.) $y = \dfrac{2}{3x^4} = \dfrac{2}{3}x^{-4}; \ \dfrac{dy}{dx} = -4\dfrac{2}{3}x^{-4-1} = -\dfrac{8}{3}x^{-5} = -\dfrac{8}{3x^5}$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

## THEOREM 6: The Sum-Difference Rule

***Sum***: The derivative of a sum is the sum of the derivatives.

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

***Difference***: The derivative of a difference is the difference of the derivatives.

$$\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 4:** Find each of the following derivatives:

a) $\dfrac{d}{dx}(5x^3 - 7)$  b) $\dfrac{d}{dx}\left(24x - \sqrt{x} + \dfrac{5}{x}\right)$

a) $$\frac{d}{dx}(5x^3 - 7) = \frac{d}{dx}(5x^3) - \frac{d}{dx}(7)$$

$$= 5 \cdot \frac{d}{dx}x^3 - 0 = 5 \cdot 3x^{3-1}$$

$$\frac{d}{dx}(5x^3 - 7) = 15x^2$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 4 (concluded):**

b)

$$\frac{d}{dx}\left(24x - \sqrt{x} + \frac{5}{x}\right) = \frac{d}{dx}(24x) - \frac{d}{dx}\sqrt{x} + \frac{d}{dx}\left(\frac{5}{x}\right)$$

$$= 24 \cdot \frac{d}{dx}x - \frac{d}{dx}x^{\frac{1}{2}} + 5 \cdot \frac{d}{dx}x^{-1}$$

$$= 24 \cdot 1x^{1-1} - \frac{1}{2}x^{\frac{1}{2}-1} + 5 \cdot -1x^{-1-1}$$

$$= 24 - \frac{1}{2}x^{-\frac{1}{2}} - 5x^{-2}, \quad \text{or} \quad = 24 - \frac{1}{2\sqrt{x}} - \frac{5}{x^2}$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

Quick Check 3

Differentiate: $y = 3x^5 + 2\sqrt[3]{x} + \frac{1}{3x^2} + \sqrt{5}$

$$\frac{dy}{dx}\left(3x^5 + 2\sqrt[3]{x} + \frac{1}{3x^2} + \sqrt{5}\right) = \frac{dy}{dx}3x^5 + \frac{dy}{dx}2\sqrt[3]{x} + \frac{dy}{dx}\frac{1}{3x^2} + \frac{dy}{dx}\sqrt{5}$$

$$= \frac{dy}{dx}3x^5 + \frac{dy}{dx}2x^{\frac{1}{3}} + \frac{dy}{dx}\frac{1}{3}x^{-2} + \frac{dy}{dx}\sqrt{5} = 5 \cdot 3x^{5-1} + \frac{2}{3}x^{\frac{1}{3}-1} - \frac{2}{3}x^{-2-1} + 0$$

$$= 15x^4 + \frac{2}{3}x^{-2/3} - \frac{2}{3}x^{-3} = 15x^4 + \frac{2}{3\sqrt[3]{x^2}} - \frac{2}{3x^3}$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 5:** Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which the tangent line is horizontal.

Recall that the derivative is the slope of the tangent line to a curve, and the slope of a horizontal line is 0. Therefore, we wish to find all the points on the graph of $f$ where the derivative of $f$ equals 0.

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 5 (continued):**

So, for $f(x) = -x^3 + 6x^2$

$$f'(x) = -3 \cdot x^{3-1} + 6 \cdot 2x^{2-1}$$

$$f'(x) = -3x^2 + 12x$$

Setting $f'(x)$ equal to 0:

$$-3x^2 + 12x = 0$$

$$-3x(x - 4) = 0$$

$$-3x = 0 \qquad x - 4 = 0$$

$$x = 0 \qquad x = 4$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 5 (continued):**

To find the corresponding *y*-values for these *x*-values, substitute back into $f(x) = -x^3 + 6x^2$.

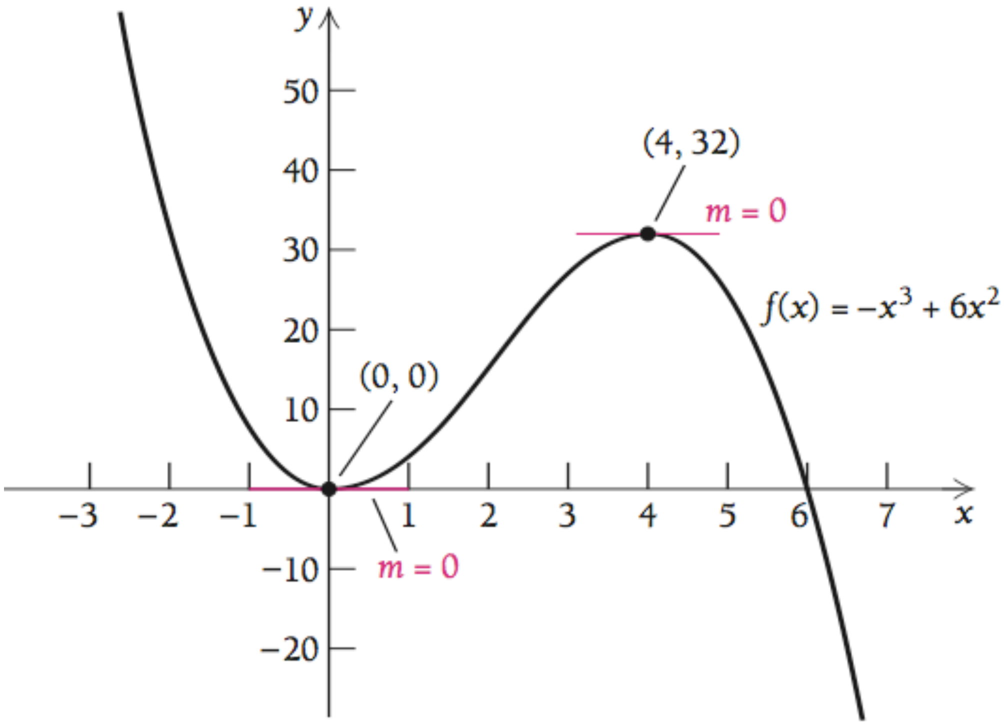
$$f(0) = -0^3 + 6 \cdot 0^2 \qquad f(4) = -4^3 + 6 \cdot 4^2$$

$$f(0) = 0 \qquad f(4) = 32$$

Thus, the tangent line to the graph of $f(x) = -x^3 + 6x^2$ is horizontal at the points (0, 0) and (4, 32).

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 5 (concluded):**



# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 6:** Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which the tangent line has slope 6.

Here we will employ the same strategy as in Example 6, except that we are now concerned with where the derivative equals 6.

Recall that we already found that $f'(x) = -3x^2 + 12x$.

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 6 (continued):**

Thus,

$$-3x^2 + 12x = 6$$

$$-3x^2 + 12x - 6 = 0$$

$$\frac{-3x^2 + 12x - 6}{-3} = \frac{0}{-3}$$

$$x^2 - 4x + 2 = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 - 8}}{2}$$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 6 (continued):**

$$x = \frac{4 \pm \sqrt{8}}{2}$$

$$x = \frac{4 \pm 2\sqrt{2}}{2}$$

$$x = 2 + \sqrt{2} \text{ and } 2 - \sqrt{2}$$

Again, to find the corresponding *y*-values, we will substitute these *x*-values into $f(x) = -x^3 + 6x^2$.

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

**Example 6 (continued):**

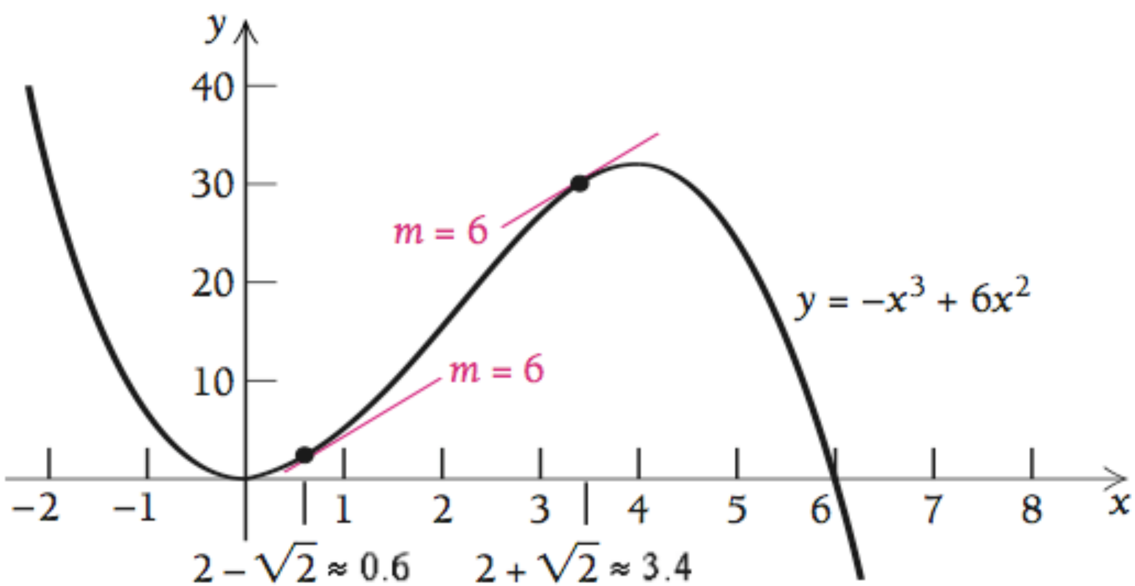
$$\begin{aligned} f\left(2+\sqrt{2}\right) &= -\left(2+\sqrt{2}\right)^3 + 6\left(2+\sqrt{2}\right)^2 \\ &= -\left(8+12\sqrt{2}+12+2\sqrt{2}\right)+6\left(4+4\sqrt{2}+2\right) \\ &= -20-14\sqrt{2}+36+24\sqrt{2} \\ &= 16+10\sqrt{2} \end{aligned}$$

Similarly, $f\left(2-\sqrt{2}\right)=16-10\sqrt{2}.$

Thus, the tangent line to $f(x)=-x^3+6x^2$ has a slope of 6 at $\left(2+\sqrt{2},\ 16+10\sqrt{2}\right)$ and $\left(2-\sqrt{2},\ 16-10\sqrt{2}\right).$

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

## Example 6 (concluded):



# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

Section Summary

- Common forms of notation for the derivative of a function are

$$y' \qquad f'(x) \qquad \frac{dy}{dx} \qquad \frac{d}{dx}f(x)$$

- The *Power Rule* for differentiation is $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$, for all real numbers $k$.
- The derivative of a constant is zero: $\frac{d}{dx}c = 0$.

# 1.5 Leibniz Notation and the Power and Sum-Difference Rules

Section Summary Concluded

• The derivative of a constant times a function is the constant time the derivative of the function:

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx} f(x)$$

• The derivative of a sum (or difference) is the sum (or difference) of the derivatives of the terms:

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

# 1.6 The Product and Quotient Rules

## OBJECTIVE

- Differentiate using the Product and the Quotient Rules.
- Use the Quotient Rule to differentiate the average cost, revenue, and profit functions.

## THEOREM 7: The Product Rule

Let $F(x) = f(x) \cdot g(x)$. Then,

$$F'(x) = \frac{d}{dx}\left[f(x) \cdot g(x)\right]$$

$$F'(x) = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x)$$

# 1.6 The Product and Quotient Rules

**Example 1:** Find $\frac{d}{dx}\left[\left(x^4-2x^3-7\right)\left(3x^2-5x\right)\right]$.

$$\frac{d}{dx}\left[\left(x^4-2x^3-7\right)\left(3x^2-5x\right)\right] =$$

$$\left(x^4-2x^3-7\right)\cdot(6x-5)+\left(3x^2-5x\right)\cdot\left(4x^3-6x^2\right)$$

# 1.6 The Product and Quotient Rules

Quick Check 1

Use the Product Rule to differentiate each of the following functions. Do not simplify.

a.) $y = (2x^5 + x - 1)(3x - 2)$

b.) $y = (\sqrt{x} + 1)(\sqrt[5]{x} - x)$

# 1.6 The Product and Quotient Rules

## Quick Check 1 Solution

a.) $y=(2x^5+x-1)(3x-2)$

Using the Product Rule: $\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\cdot\frac{d}{dx}\left[g(x)\right]+g(x)\cdot\frac{d}{dx}\left[f(x)\right]$

We get: $y'=(2x^5+x-1)(3x^{1-1}-0)+(3x-2)(5\cdot 2x^{5-1}+x^{1-1}-0)$

$$y'=3(2x^5+x-1)+(3x-2)(10x^4+1)$$

b.) $y=\left(\sqrt{x}+1\right)\left(\sqrt[5]{x}-x\right)$

Again, using the Product Rule, we get:

$$y'=(\sqrt{x}+1)(x^{\frac{1}{5}-1}-x^{1-1})+(\sqrt[5]{x}-x)(x^{\frac{1}{2}-1}+0)$$

$$y'=\left(\sqrt{x}+1\right)\left(\frac{1}{5\sqrt[5]{x^4}}-1\right)+\left(\sqrt[5]{x}-x\right)\left(\frac{1}{2\sqrt{x}}\right)$$

# 1.6 The Product and Quotient Rules

## THEOREM 8: The Quotient Rule

If $Q(x) = \dfrac{N(x)}{D(x)}$, then,

$$Q'(x) = \frac{D(x) \cdot N'(x) - N(x) \cdot D'(x)}{\left[D(x)\right]^2}$$

# 1.6 The Product and Quotient Rules

**Example 2:** Differentiate $f(x) = \dfrac{x^2 - 3x}{x - 1}.$

$$f'(x) = \frac{(x-1)(2x-3) - (x^2 - 3x)(1)}{(x-1)^2}$$

$$f'(x) = \frac{2x^2 - 5x + 3 - x^2 + 3x}{(x-1)^2}$$

$$f'(x) = \frac{x^2 - 2x + 3}{(x-1)^2}$$

# 1.6 The Product and Quotient Rules

Quick Check 2

a.) Differentiate: $f(x) = \dfrac{1-3x}{x^2+2}$. Simplify your result.

b.) Show that

$$\frac{d}{dx}\left[\frac{ax+1}{bx+1}\right] = \frac{a-b}{(bx+1)^2}$$

# 1.6 The Product and Quotient Rules

## Quick Check 2 Solution

a.) Using the Quotient Rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot\frac{d}{dx}[f(x)]-f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

We get: $$f'(x)=\frac{(x^2+2)(0-3)-(1-3x)(2x+0)}{(x^2+2)^2}$$

$$f'(x)=\frac{-3x^2-6-2x+6x^2}{x^4+4x^2+4}$$

$$f'(x)=\frac{3x^2-2x-6}{x^4+4x^2+4}$$

## Quick Check 2 Solution Concluded

b.) Using the Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot\frac{d}{dx}[f(x)]-f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$

We know that: $\frac{d}{dx}\left[\frac{ax+1}{bx+1}\right]=\frac{(bx+1)(a)-(ax+1)(b)}{\left(bx+1\right)^2}$

$$=\frac{(abx+a)-(abx+b)}{(bx+1)^2}$$

$$=\frac{abx+a-abx-b}{(bx+1)^2}$$

$$=\frac{a-b}{(bx+1)^2}$$

# 1.6 The Product and Quotient Rules

**DEFINITION:**

If $C(x)$ is the cost of producing $x$ items, then the **average cost** of producing $x$ items is $\dfrac{C(x)}{x}$.

If $R(x)$ is the revenue from the sale of $x$ items, then the **average revenue** from selling $x$ items is $\dfrac{R(x)}{x}$.

If $P(x)$ is the profit from the sale of $x$ items, then the **average profit** from selling $x$ items is $\dfrac{P(x)}{x}$.

# 1.6 The Product and Quotient Rules

**Example 3:** Paulsen's Greenhouse finds that the cost, in dollars, of growing $x$ hundred geraniums is given by $C(x) = 200 + 100 \cdot \sqrt[4]{x}$. If the revenue from the sale of $x$ hundred geraniums is given by $R(x) = 120 + 90 \cdot \sqrt{x}$, find each of the following.

a) The average cost, the average revenue, and the average profit when $x$ hundred geraniums are grown and sold.

b) The rate at which average profit is changing when 300 geraniums are being grown.

# 1.6 The Product and Quotient Rules

**Example 3 (continued):**

a) We let $A_C$, $A_R$, and $A_P$ represent average cost, average revenue, and average profit.

$$A_C(x) = \frac{C(x)}{x} = \frac{200 + 100 \cdot \sqrt[4]{x}}{x}$$

$$A_R(x) = \frac{R(x)}{x} = \frac{120 + 90 \cdot \sqrt{x}}{x}$$

$$A_P(x) = \frac{P(x)}{x} = \frac{R(x) - C(x)}{x} = \frac{120 + 90 \cdot \sqrt{x} - 200 - 100 \cdot \sqrt[4]{x}}{x}$$

$$A_P(x) = \frac{-80 + 90 \cdot \sqrt{x} - 100 \cdot \sqrt[4]{x}}{x}$$

# 1.6 The Product and Quotient Rules

**Example 3 (continued):**

b) First we must find $A_P{}'(x)$. Then we can substitute 3 (hundred) into $A_P{}'(x)$.

$$A_P{}'(x) =$$

$$= \frac{x\left(90 \cdot \frac{1}{2}x^{-\frac{1}{2}} - 100 \cdot \frac{1}{4}x^{-\frac{3}{4}}\right) - \left(-80 + 90 \cdot x^{\frac{1}{2}} - 100 \cdot x^{\frac{1}{4}}\right) \cdot 1}{x^2}$$

$$= \frac{45x^{\frac{1}{2}} - 25x^{\frac{1}{4}} + 80 - 90x^{\frac{1}{2}} + 100x^{\frac{1}{4}}}{x^2}$$

# 1.6 The Product and Quotient Rules

**Example 3 (concluded):**

$$A_P{}'(x) = \frac{80 - 45x^{\frac{1}{2}} + 75x^{\frac{1}{4}}}{x^2}$$

$$A_P{}'(3) = \frac{80 - 45(3)^{\frac{1}{2}} + 75(3)^{\frac{1}{4}}}{3^2}$$

$$A_P{}'(3) \approx 11.196$$

Thus, at 300 geraniums, Paulsen's average profit is increasing by about $11.20 per plant.

# 1.6 The Product and Quotient Rules

Section Summary

- The *Product Rule* is:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\cdot\frac{d}{dx}[g(x)]+g(x)\cdot\frac{d}{dx}[f(x)]$$

- The *Quotient Rule* is:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot\frac{d}{dx}[f(x)]-f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

- Be careful to note the order in which you write out the factors when using the Quotient Rule. Because the Quotient Rule involves subtraction and division, the order in which you perform the operations is important.

# 1.7 The Chain Rule

**OBJECTIVE**

- Find the composition of two functions.
- Differentiate using the Chain Rule.

# 1.7 The Chain Rule

**DEFINITION:**

The **composed** function $f \circ g$, the **composition** of $f$ and $g$, is defined as

$$f \circ g = f(g(x)).$$

# 1.7 The Chain Rule

**Example 1:** For $f(x) = x^3$ and $g(x) = 1 + x^2$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(1 + x^2) \\
&= (1 + x^2)^3 \\
&= 1 + 3x^2 + 3x^4 + x^6
\end{aligned}
$$

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(x^3) \\
&= 1 + (x^3)^2 \\
&= 1 + x^6
\end{aligned}
$$

# 1.7 The Chain Rule

**Example 2:** For $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = f(g(x))$$
$$= f(x - 1)$$
$$(f \circ g)(x) = \sqrt{x - 1}$$

$$(g \circ f)(x) = g(f(x))$$
$$= g(\sqrt{x})$$
$$(g \circ f)(x) = \sqrt{x} - 1$$

# 1.7 The Chain Rule

Quick Check 1

For the functions in Example 2, find:

a.) $\left(f \circ f\right)\left(x\right)$

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

b.) $\left(g \circ g\right)\left(x\right)$

$$\left(g \circ g\right)\left(x\right) = g\left(g\left(x\right)\right) = g(x-1) = \left(x-1\right)-1 = x-2$$

# 1.7 The Chain Rule

**THEOREM 9: The Chain Rule**

The derivative of the composition $f \circ g$ is given by

$$\frac{d}{dx}\bigg|[(f \circ g)(x)] = \frac{d}{dx}\big[f(g(x))\big] = f'(g(x)) \cdot g'(x).$$

# 1.7 The Chain Rule

**Example 3:** For $y = 2 + \sqrt{u}$ and $u = x^3 + 1$,

find $\dfrac{dy}{du}$, $\dfrac{du}{dx}$, and $\dfrac{dy}{dx}$.

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} \quad \text{and} \quad \frac{du}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3 + 1}}$$

# 1.7 The Chain Rule

Quick Check 2

If $y = u^2 + u$ and $u = x^2 + x$, find $\dfrac{dy}{dx}$.

We will start by finding $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$:

$$\frac{dy}{du} = 2u + 1 \qquad \frac{du}{dx} = 2x + 1$$

Next we find $\dfrac{dy}{dx}$, remembering to substitute $x^2 + x$ for $u$ when appropriate.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u+1)(2x+1) = (2(x^2+x)+1)(2x+1)$$

$$= (2x^2 + 2x + 1)(2x + 1)$$

# 1.7 The Chain Rule

**Example 4:** For $y = u^2 - 3u$ and $u = 5t - 1$, find $\dfrac{dy}{dt}$.

$$\frac{dy}{du} = 2u - 3 \quad \text{and} \quad \frac{du}{dt} = 5$$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = (2u - 3)(5)$$

$$= 10u - 15 = 10(5t - 1) - 15$$

$$= 50t - 10 - 15 = 50t - 25$$

# 1.7 The Chain Rule

**THEOREM 10: The Extended Power Rule**

Suppose that $g(x)$ is a differentiable function of $x$. Then, for any real number $k$,

$$\frac{d}{dx}\left[g(x)\right]^k = k\left[g(x)\right]^{k-1}\cdot\frac{d}{dx}g(x)$$

# 1.7 The Chain Rule

**Example 5:** Differentiate $f(x) = \left(1 + x^3\right)^{\frac{1}{2}}$.

$$\frac{d}{dx}\left(1 + x^3\right)^{\frac{1}{2}} = \frac{1}{2}\left(1 + x^3\right)^{\frac{1}{2}-1} \cdot 3x^2$$

$$= \frac{3x^2}{2}\left(1 + x^3\right)^{-\frac{1}{2}}$$

$$= \frac{3x^2}{2\sqrt{1 + x^3}}$$

# 1.7 The Chain Rule

**Example 6:**

Differentiate $f(x)=(3x-5)^4(7-x)^{10}$.

Combine Product Rule and Extended Power Rule

$$f'(x)=(3x-5)^4 10(7-x)^9(-1)+$$
$$4(3x-5)^3(7-x)^{10}(3)$$

Simplified:

$$f'(x)=2(3x-5)^3(7-x)^9(67-21x)$$

# 1.7 The Chain Rule

Quick Check 3

Differentiate: $f(x)=\dfrac{(2x^2-1)}{\left(3x^4+2\right)^2}$

We will combine both the quotient rule and the chain rule:

$$f'(x)=\frac{(3x^4+2)^2\cdot\dfrac{d}{dx}(2x^2-1)-(2x^2-1)\cdot\dfrac{d}{dx}((3x^4+2)^2)}{[(3x^4+2)^2]^2}$$

$$f'(x)=\frac{(3x^4+2)^2\cdot(4x)-(2x^2-1)\cdot(2(3x^4+2)(12x^3))}{(3x^4+2)^4}$$

$$f'(x)=\frac{4x(3x^4+2)^2-(2x^2-1)(72x^7+48x^3)}{(3x^4+2)^4}$$

$$f'\left(x\right)=\frac{-36x^5+24x^3+8x}{\left(3x^4+2\right)^3}$$

# 1.7 The Chain Rule

Section Summary

- The *composition* of $f(x)$ with $g(x)$ is written $(f \circ g)(x)$ and is defined as $(f \circ g)(x) = f(g(x))$.

- In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

- The *Chain Rule* is used to differentiate a composition of functions.

If $\quad F(x) = (f \circ g)(x) = f(g(x))$

Then $\quad F'(x) = \dfrac{d}{dx}[(f \circ g)(x)] = f'\left(g\left(x\right)\right) \cdot g'\left(x\right).$

# 1.7 The Chain Rule

Section Summary Concluded

- The *Extended Power Rule* tells us that if $y = \left[f(x)\right]^k$, then

$$y' = \frac{d}{dx}[f(x)]^k = k[f(x)]^{k-1} \cdot f'(x).$$

# 1.8 Higher Order Derivatives

**OBJECTIVE**

- Find derivatives of higher order.
- Given a formula for distance, find velocity and acceleration.

## Higher-Order Derivatives:

Consider the function given by

$$y = f(x) = x^5 - 3x^4 + x.$$

Its derivative $f'$ is given by

$$y' = f'(x) = 5x^4 - 12x^3 + 1.$$

The derivative function $f'$ can also be differentiated. We can think of the derivative $f'$ as the rate of change of the slope of the tangent lines of $f$. It can also be regarded as the rate at which $f'(x)$ is changing.

# 1.8 Higher Order Derivatives

## Higher-Order Derivatives (continued):

We use the notation $f''$ for the derivative $(f')'$.
That is,

$$f''(x) = \frac{d}{dx} f'(x)$$

We call $f''$ the *second derivative* of $f$. For

$$y = f(x) = x^5 - 3x^4 + x,$$

the second derivative is given by

$$y'' = f''(x) = 20x^3 - 36x^2.$$

## Higher-Order Derivatives (continued):

For higher-order derivatives, we use the notation $f^{(n)}(x)$ to express the $n^{th}$ derivative of $f$.

Continuing in this manner, we have

$$f^{(3)}(x) = 60x^2 - 72x, \text{the third derivative of } f,$$

$$f^{(4)}(x) = 120x - 72, \text{the fourth derivative of } f,$$

$$f^{(5)}(x) = 120, \text{the fifth derivative of } f.$$

## Higher-Order Derivatives (continued):

For $y = f(x) = x^5 - 3x^4 + x$, we have

$$\begin{aligned} f^{(3)}(x) &= 60x^2 - 72x, \\ f^{(4)}(x) &= 120x - 72, \\ f^{(5)}(x) &= 120, \\ f^{(6)}(x) &= 0, \text{ and} \\ f^{(n)}(x) &= 0, \text{ for any integer } n \geq 6. \end{aligned}$$

## Higher-Order Derivatives (continued):

Leibniz's notation for the second derivative of a function given by $y = f(x)$ is

$$\frac{d^2y}{dx^2}, \quad or \quad \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

read "the second derivative of $y$ with respect to $x$." The 2's in this notation are NOT exponents.

## Higher-Order Derivatives (concluded):

If $y = x^5 - 3x^4 + x$, then

$$\frac{dy}{dx} = 5x^4 - 12x^3 + 1,$$

$$\frac{d^2y}{dx^2} = 20x^3 - 36x^2,$$

$$\frac{d^3y}{dx^3} = 60x^2 - 72x,$$

$$\frac{d^4y}{dx^4} = 120x - 72,$$

$$\frac{d^5y}{dx^5} = 120.$$

# 1.8 Higher Order Derivatives

**Example 1:** For $y = \dfrac{1}{x}$, find $\dfrac{d^2y}{dx^2}$.

$$y = x^{-1}$$

$$\frac{dy}{dx} = -x^{-2}$$

$$\frac{d^2y}{dx^2} = 2x^{-3}, \text{ or } \frac{2}{x^3}$$

# 1.8 Higher Order Derivatives

**Example 2:** For $y = (x^2 + 10x)^{20}$, find $y'$ and $y''$.

By the Extended Chain Rule, $y' = 20(x^2 + 10x)^{19}(2x + 10)$.

Using the Product Rule and Extended Chain Rule,

$$y'' = 20(x^2 + 10x)^{19} \cdot 2 + 20(2x + 10) \cdot 19(x^2 + 10x)^{18}(2x + 10)$$

$$= 40(x^2 + 10x)^{18}\left((x^2 + 10x) + 19(x + 5)(2x + 10)\right)$$

$$= 40(x^2 + 10x)^{18}\left(x^2 + 10x + 19(2x^2 + 20x + 50)\right)$$

$$= 40(x^2 + 10x)^{18}\left(x^2 + 10x + 38x^2 + 380x + 950)\right)$$

$$y'' = 40(x^2 + 10x)^{18}\left(39x^2 + 390x + 950\right).$$

# 1.8 Higher Order Derivatives

Quick Check 1

a.) Find $y''$:

(i) $y = -6x^4 + 3x^2$

(ii) $y = \dfrac{2}{x^3}$

(iii) $y = (3x^2 + 1)^2$

b.) Find

$$\frac{d^4}{dx^4}\left[\frac{1}{x}\right]$$

# 1.8 Higher Order Derivatives

Quick Check 1 Solution

a.) For the following problems, remember that $y'' = (y')'$

(i) $y = -6x^4 + 3x^2$

$$y' = -24x^3 + 6x, \quad y'' = -72x^2 + 6$$

(ii) $y = \dfrac{2}{x^3}$

$$y' = -\frac{6}{x^4}, \qquad y'' = \frac{24}{x^5}$$

(iii) $y = (3x^2 + 1)^2$

$$y' = 2\left(3x^2 + 1\right)\left(6x\right) \quad = 36x^3 + 12x, \quad y'' = 108x^2 + 12$$

# 1.8 Higher Order Derivatives

Quick Check 1 Solution Concluded

b.) Find $\frac{d^4}{dx^4}\left[\frac{1}{x}\right]$

$$\frac{d^4}{dx^4}\left[\frac{1}{x}\right] = \frac{d^3}{dx^3}\left[-\frac{1}{x^2}\right] = \frac{d^2}{dx^2}\left[\frac{2}{x^3}\right] = \frac{d}{dx}\left[-\frac{6}{x^4}\right]$$

$$= \frac{24}{x^5}$$

# 1.8 Higher Order Derivatives

**DEFINITION:**

The **velocity** of an object that is $s(t)$ units from a starting point at time $t$ is given by

$$\text{Velocity} = v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

# 1.8 Higher Order Derivatives

**DEFINITION:**

$$\text{Acceleration} = a(t) = v'(t) = s''(t).$$

# 1.8 Higher Order Derivatives

**Example 3:** For $s(t) = 10t^2$ find $v(t)$ and $a(t)$, where $s$ is the distance from the starting point, in miles, and $t$ is in hours. Then, find the distance, velocity, and acceleration when $t = 4$ hr.

$$v(t) = s'(t) = 20t$$

$$a(t) = v'(t) = s''(t) = 20$$

$$s(4) = 10(4)^2 = 160 \text{ mi}$$

$$v(4) = 20(4) = 80 \text{ mi/hr}$$

$$a(4) = 20 \text{ mi/hr}^2$$

# 1.8 Higher Order Derivatives

Quick Check 2

A pebble is dropped from a hot-air balloon. Find how far it has fallen, how fast it is falling, and its acceleration after 3.5 seconds. Let $s(t)=16t^2$, where $t$ is in seconds, and $s$ is in feet.

Distance: $s(3.5) = 16(3.5)^2 = 16(12.25) = 196$ feet

Velocity: $v(t) = s'(t) = 32t$

$v(3.5) = 32(3.5) = 112$ feet/second

Acceleration: $a(t) = v'(t) = s''(t) = 32$

$a(3.5) = 32$ feet/second$^2$

# 1.8 Higher Order Derivatives

Section Summary

- The *second derivative* is the derivative of the first derivative of a function. In symbols, $f''(x) = \frac{d}{dx}\left[f'(x)\right]$.
- The second derivative describes the rate of change of the rate of change. In other words, it describes the rate of change of the first derivative.

# 1.8 Higher Order Derivatives

Section Summary Concluded

- A real-life example of a second derivative is *acceleration*. If $s(t)$ represents distance as a function of time of a moving object, then $v(t) = s'(t)$ describes the speed (velocity) of the object. Any change in the speed of the object is acceleration: $a(t) = v'(t) = s''(t)$
- The common notation for the *n*th derivative of a function is

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n}{dx^n} f(x).$$