Matrix Algebra:

Vector Spaces

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In this lectures, we learn about vector spaces. A vector space consists of a set of vectors and a set of scalars that is closed under vector addition and scalar multiplication and that satisfies the usual rules of arithmetic.

We will learn some of the vocabulary and phrases of linear algebra, such as linear independence, span, basis and dimension. We will also learn about the four fundamental subspaces of a matrix.

A vector space consists of a set of vectors and a set of scalars. Although vectors can be quite general, for the purpose of this course we will only consider vectors that are real column matrices, and scalars that are real numbers.

For the set of vectors and scalars to form a vector space, the set of vectors must be closed under vector addition and scalar multiplication. That is, when you multiply any two vectors in the set by real numbers and add them, the resulting vector must still be in the set.

Definition

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Our main interest in vector spaces is to determine the vector spaces associated with matrices. There are four fundamental vector spaces of an $n \times n$ matrix A. They are called the **null space**, **the column space**, **the row space**, and **the left null space**.

Using only these axioms, we can show that the zero vector in Axiom 4 is unique, and the vector $-\mathbf{u}$, called the negative of $\mathbf{u}$, in Axiom 5 is unique for each $\mathbf{u}$ in V.

For each $\mathbf{u}$ in V and scalar c,

$$0\mathbf{u} = \mathbf{0} \tag{1}$$

$$c\mathbf{0} = \mathbf{0} \tag{2}$$

$$-\mathbf{u} = (-1)\mathbf{u} \tag{3}$$

Examples of Vector spaces:

a) $\mathbb{R}^n$, $n \geq 1$, are the premier examples of vector spaces.

b) Let $\mathbb{P}_n = \{\mathbf{p}(t) = a_0 + a_1t^1 + \cdots + a_nt^n\}$, $n \geq 1$. For $\mathbf{p}(t) = a_0 + a_1t^1 + \cdots + a_nt^n$ and $\mathbf{q}(t) = b_0 + b_1t^1 + \cdots + b_nt^n$, define

$$\begin{aligned}(\mathbf{p}+\mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) = (a_0+b_0) + (a_1+b_1)t^1 + \cdots + (a_n+b_n)t^n \\ (c\mathbf{p})(t) &= c\mathbf{p}(t) = ca_0 + ca_1t^1 + \cdots + ca_nt^n\end{aligned}$$

Then $\mathbb{P}_n$ is a vector space, with the usual polynomial addition and scalar multiplication.

c) Let $V = \{$all real-valued functions defined on a set $D\}$. Then, V is a vector space, with the usual function addition and scalar multiplication.

Linear independence

An indexed set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is said to be linearly independent if for any scalars $x_1, x_2, \dots, x_n$, the equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots + x_n\mathbf{u}_n = 0$$

has only the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

What this means is that one is unable to write any of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ as a linear combination of any of the other vectors.

The vectors $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ are linearly dependent if there exist weights $c_1, c_2, \dots, c_n$, not all zero, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n = 0. \qquad (LD)$$

Equation (LD) is called a linear dependence relation among $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ when the weights are not all zero.

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

SOLUTION

a. We must determine if there is a nontrivial solution of equation (1) above. Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, x_1 and x_2 are basic variables, and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution of (1). Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent (and not linearly independent).

b. To find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \qquad - 2x_3 &= 0 \\ x_2 + \ x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free. Choose any nonzero value for x_3—say, $x_3 = 5$. Then $x_1 = 10$ and $x_2 = -5$. Substitute these values into equation (1) and obtain

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$$

This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. ■

Example:

As an example consider whether the following three three-by-one column vectors are linearly independent:

$$\mathrm{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathrm{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathrm{w} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

Indeed, they are not linearly independent, that is, they are *linearly dependent*, because w can be written in terms of u and v. In fact, $\mathrm{w} = 2\mathrm{u} + 3\mathrm{v}$.

Now consider the three three-by-one column vectors given by

$$\mathrm{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathrm{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathrm{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These three vectors are linearly independent because you cannot write any one of these vectors as a linear combination of the other two. If we go back to our definition of linear independence, we can see that the equation

$$a\mathrm{u} + b\mathrm{v} + c\mathrm{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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For simple examples, visual inspection can often decide if a set of vectors are linearly independent. For a more algorithmic procedure, place the vectors as the rows of a matrix and compute the reduced row echelon form. If the last row becomes all zeros, then the vectors are linearly dependent, and if not all zeros, then they are linearly independent.

Linear Independence of Matrix Columns

Let a matrix $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$ instead of a set of vectors.
The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = 0.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.
Thus we have the following important fact.

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Subspaces

In many problems, a vector space consists of an appropriate subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Theorem.

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V.

We call Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. Given any subspace H of V, a **spanning** (or **generating**) **set** for H is a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Examples of Subspaces:

a) $H = \{0\}$: the **zero subspace**

b) Let $\mathbb{P} = \{\text{all polynomials with real coefficients defined on } \mathbb{R}\}$. Then, $\mathbb{P}$ is a subspace of the space $\{\text{all real-valued functions defined on } \mathbb{R}\}$.

c) The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, because $\mathbb{R}^2 \not\subset \mathbb{R}^3$.

d) Let $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$. Then H is a subspace of $\mathbb{R}^3$.

Note:
In applications of linear algebra, subspaces of $\mathbb{R}^n$ usually arise in one of two ways:

a) as the set of all solutions to a homogeneous linear system;
b) as the set of all linear combinations of certain vectors.

Null Space

The null space of an $m \times n$ matrix A, written as $\operatorname{Nul} A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\operatorname{Nul} A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

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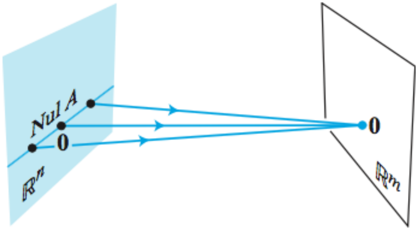

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Example: Let H be the set of all vector in $\mathbb{R}^4$, whose coordinates a, b, c, d satify the equations

$$\begin{cases} a - 2b + 5c & = d, \\ c - a & = b. \end{cases}$$

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Solution. Rewrite the above equations as

$$\begin{cases} a - 2b + 5c - d & = & 0 \\ -a - b + c & = & 0 \end{cases}$$

Then $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is the solution of $\begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \mathbf{x} = 0$. Thus **the collection of these solutions** is a subspace.

The Column Space

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The column space of an $m \times n$ matrix A, written as $\operatorname{Col} A$, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$, then

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A typical vector in $\operatorname{Col} A$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

$$\operatorname{Col} A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^n$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$.

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Note: $\operatorname{Col} A$ is the range of the linear transformation $\mathbf{x} \to A\mathbf{x}$.

The Contrast Between Null A and Col A

Let $A \in \mathbb{R}^{m \times n}$

1. $\mathit{Nul}\,A$ is a subspace of $\mathbb{R}^n$.	1. $\mathit{Col}\,A$ is a subspace of $\mathbb{R}^m$.
2. $\mathit{Nul}\,A$ is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$).	2. $\mathit{Col}\,A$ is explicitly defined; that is, you are told how to build vectors in $\mathit{Col}\,A$.
3. It takes time to find vectors in $\mathit{Nul}\,A$. Row operations on $[A \ \ \mathbf{0}]$ are required.	3. It is easy to find vectors in $\mathit{Col}\,A$. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between $\mathit{Nul}\,A$ and the entries in A.	4. There is an obvious relation between $\mathit{Col}\,A$ and the entries in A, since each column of A is in $\mathit{Col}\,A$.
5. A typical vector $\mathbf{v}$ in $\mathit{Nul}\,A$ has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in $\mathit{Col}\,A$ has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in $\mathit{Nul}\,A$. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in $\mathit{Col}\,A$. Row operations on $[A \ \ \mathbf{v}]$ are required.
7. $\mathit{Nul}\,A = \{\mathbf{0}\}$ $\Leftrightarrow$ the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. $\mathit{Col}\,A = \mathbb{R}^m$ $\Leftrightarrow$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.
8. $\mathit{Nul}\,A = \{\mathbf{0}\}$ $\Leftrightarrow$ the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is *one-to-one*.	8. $\mathit{Col}\,A = \mathbb{R}^m$ $\Leftrightarrow$ the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Basis Vectors

The span of the set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ *is the vector space consisting of all linear combinations of* $\mathbf{b}_1, \ldots, \mathbf{b}_p$.

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Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if

(i) *$\mathcal{B}$ is a linearly independent set, and*

(ii) *the subspace spanned by $\mathcal{B}$ coincides with H; that is,*

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The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself. Thus a basis of V is a linearly independent set that spans V. Observe that when $H = V$, condition (ii) includes the requirement that each of the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ must belong to H, because $\mathrm{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ contains $\mathbf{b}_1, \ldots, \mathbf{b}_p$.

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Theorem

The pivot columns of a matrix A form a basis for $\operatorname{Col} A$.

EXAMPLE 8 Find a basis for Col B, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SOLUTION Each nonpivot column of B is a linear combination of the pivot columns. In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$. By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span Col B. Let

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since $\mathbf{b}_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent (Theorem 4). Thus S is a basis for Col B. ■

What about a matrix A that is *not* in reduced echelon form? Recall that any linear dependence relationship among the columns of A can be expressed in the form $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x}$ is a column of weights. (If some columns are not involved in a particular dependence relation, then their weights are zero.) When A is row reduced to a matrix B, the columns of B are often totally different from the columns of A. However, the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have exactly the same set of solutions. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$, then the vector equations

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \quad \text{and} \quad x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n = \mathbf{0}$$

also have the same set of solutions. That is, the columns of A have *exactly the same linear dependence relationships* as the columns of B.

THANK YOU FOR YOUR ATTENTION!