SYSTEM OF LINEAR EQUATIONS

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Real-world problems can be approximated as and resolved by systems of linear equations:

$$Ax = b; \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where one of $\{x, b\}$ is the input and the other is the output.

Computer science is intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations:

1. Linear programming. Many important management decisions today are made on the basis of linear programming models that use hundreds of variables. The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.
2. Electrical networks. Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.
3. Artificial intelligence. Linear algebra plays a key role in everything from scrubbing data to facial recognition.
4. Signals and signal processing. From a digital photograph to the daily price of a stock, important information is recorded as a signal and processed using linear transformations.
5. Machine learning. Machines (specifically computers) use linear algebra to learn about anything from online shopping preferences to speech recognition.

What you would learn, from Linear Algebra:

1. How to Solve Systems of Linear Equations.
2. Matrix Algebra (Matrix Inverse and Factorizations).
3. Determinants.
4. Vector Spaces.
5. Eigenvalues and Eigenvectors.

1.1. SYSTEM OF LINEAR EQUATIONS

DEFINITION

Definition. *A linear equation in the variable* $x_1, x_2, \ldots, x_n$ *is an equation that can be written in the form:*

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where b *and the coefficients* $a_1, a_2, \ldots, a_n$ *are real or complex numbers.*

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Example.

$$(a)\begin{cases} 4x_1 - x_2 & = 3, \\ 2x_1 + 3x_2 & = 5 \end{cases} \qquad (b)\begin{cases} 2x + 3y - 4z & = 2, \\ x - 2y + z & = 1 \\ 3x + y - 2z & = -1. \end{cases}$$

SOLUTION OF THE SYSTEM

- A **solution of the system** is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.
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- Two linear systems are called **equivalent** if they have the same solution set.
  For example, above (a) is equivalent to

$$\begin{cases} 2x_1 - 2x_2 & = -2, \\ 2x_1 + 3x_2 & = 5. \end{cases} \qquad R_1 \longleftarrow (R_1 - R_2)$$

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Example. Consider the case of two equations in two unknowns.

$$(a)\begin{cases} -x+y &= 1, \\ -x+y &= 3 \end{cases} \quad (b)\begin{cases} x+y &= 1, \\ x-y &= 2. \end{cases} \quad (c)\begin{cases} -2x+y &= 2, \\ -4x+2y &= 4. \end{cases}$$

EXISTENCE AND UNIQUENESS QUESTIONS

Two Fundamental Questions about a Linear System

1. (Existence): Is the system consistent; that is, does at least one solution exist?
2. (Uniqueness): If a solution exists, is it the only one; that is, is the solution unique?

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Most systems in real-world are consistent (existence) and they produce the same output for the same input (uniqueness).

MATRIX FORM

Consider a simple system of 2 linear equations:

$$\begin{cases} -2x_1 + 3x_2 &= -1, \\ x_1 + 2x_2 &= 4. \end{cases} \qquad (1)$$

Such a system of linear equations can be treated much more conveniently and efficiently with matrix form. In matrix form, (1) reads

$$\underbrace{\begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}}_{\textbf{coefficient matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

The essential information of the system can be recorded compactly in a rectangular array called a augmented matrix:

$$\begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & 4 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS

Tools. Three Elementary Row Operations

1. Replacement: Replace one row by the sum of itself and a multiple of another row
$$R_i \leftarrow R_i + k \cdot R_j, \quad i \neq j.$$
2. Interchange: Interchange two rows
$$R_i \leftrightarrow R_j, \quad i \neq j.$$
3. Scaling: Multiply all entries in a row by a nonzero constant
$$R_i \leftarrow k \cdot R_i.$$

SOLVING (1)

System of linear equations **Matrix form**

$$\begin{cases} -2x_1 + 3x_2 = -1 & ① \\ x_1 + 2x_2 = 4 & ② \end{cases} \qquad \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & 4 \end{bmatrix}$$

① ↔ ②: (interchange)

$$\begin{cases} x_1 + 2x_2 = 4 & ① \\ -2x_1 + 3x_2 = -1 & ② \end{cases} \qquad \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & -1 \end{bmatrix}$$

② ← ② + 2 · ①: (replacement)

$$\begin{cases} x_1 + 2x_2 = 4 & ① \\ 7x_2 = 7 & ② \end{cases} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 7 & 7 \end{bmatrix}$$

② ← ②/7: (scaling)

$$\begin{cases} x_1 + 2x_2 = 4 & ① \\ x_2 = 1 & ② \end{cases} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

① ← ① − 2 · ②: (replacement)

$$\begin{cases} x_1 = 2 & ① \\ x_2 = 1 & ② \end{cases} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

At the last step:

$$\text{LHS: solution}: \begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases} \qquad \text{RHS}: \left[\begin{array}{c|c} \mathbf{I} & \begin{matrix} 2 \\ 1 \end{matrix} \end{array}\right]$$

Definition. Two matrices are **row equivalent** if there is a sequence of EROs that transforms one matrix to the other.

Example. Solve the following system of linear equations, using the 3 EROs. Then, determine if the system is consistent.

$$\begin{cases} x_2 - 4x_3 = 8, \\ 2x_1 - \quad 3x_2 + 2x_3 = 1 \\ 4x_1 - \quad 8x_2 + 12x_3 = 1. \end{cases}$$

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True or false.

a. Every elementary row operation is reversible.
b. Elementary row operations on an augmented matrix never change the solution of the associated linear system.
c. Two linear systems are equivalent if they have the same solution set.
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Ans: T, T, T, F.

1.2. ROW REDUCTION AND ECHELON FORMS

ECHELON FORMS

Terminologies.

1. A **nonzero row** in a matrix is a row with at least one nonzero entry;
2. A **leading entry** of a row is the left most nonzero entry in a nonzero row.
3. A **leading 1** is a leading entry whose value is 1.

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Echelon form. A rectangular matrix is in **an echelon form** if it has following properties

1. All nonzero rows are above any rows of all zeros;
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Row reduced echelon form. If a matrix in an echelon form satisfies 4 and 5 below, then it is in the **row reduced echelon form** (RREF), or the **reduced echelon form** (REF)

4. The leading entry in each nonzero row is 1;
5. Each leading 1 is the only nonzero entry in its column.

Example. The matrix

$$\begin{bmatrix} -2 & 3 & -1 & 1 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is in echelon form, and the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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Example. The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Any nonzero matrix may be row reduced (that is, transformed by elementary row operations) into more than one matrix in echelon form, using di erent sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

Uniqueness of the Reduced Echelon Form

Theorem

Each matrix is row equivalent to one and only one reduced echelon matrix.

Definition.

1. A **pivot position** is a location in A that corresponds to a leading 1 in the reduced echelon form of A;
2. A **pivot column** is a column of A that contains a pivot position.

See Example 2, 3 p.39-42.

Remark. **PIVOT POSITIONS.** Once a matrix is in an echelon form, further row operations do not change the positions of leading entries. Thus, the leading entries become the leading 1's in the reduced echelon form.

Definition.

3. In the system $A\mathbf{x} = \mathbf{b}$ the variables that correspond to pivot columns (in $[A : b]$) are **basic variables**.
4. In the system $A\mathbf{x} = \mathbf{b}$, the variables that correspond to non-pivotal columns are **free variables**.

See Example 4, p.43-44.

Steps to reduce to reduced echelon form.

1. Start with the **leftmost non-zero column**. This is a **pivot column**. The pivot is at the top;
2. Choose a nonzero entry in the pivot column as a pivot. If necessary, **interchange rows** to move a nonzero entry into the pivot position.
3. Use **row replacement** operations to make zeros in all positions beow the pivot.
4. Ignore row and column containing the pivot and ignore all rows above it. Apply Steps 1–3 to the remaining submatrix. Repeat this until there are no more rows to modify.
5. Start with right most pivot and work upward and left to make zeros above each pivot. If pivot is not 1, make it 1 by a **scaling** operation.

Example. Row reduce the matrix into reduced echelon form.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

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Solution. $\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$

$\xrightarrow{R_2 \leftarrow R_2/5} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 3R_2} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$

$\xrightarrow{R_3 \leftarrow R_3/-5;\ \text{above the pivot} \to 0} \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 4R_2} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Combination of Steps 1–4 is call the forward phase of the row reduction, while Step 5 is called the backward phase.

THE GENERAL SOLUTION OF LINEAR SYSTEMS

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
For example, that the augmented matrix of a linear system has been changed into the equivalent R.E.F:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{cases} x_1 - 5x_3 & = 1; \\ x_2 + x_3 & = 4; \\ 0 & = 0. \end{cases} \qquad (2)$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**. The other variable, x_3, is called a **free variable**.

Whenever a system is consistent, as in (2), the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables. This operation is possible because the reduced echelon form places each basic variable in one and only one equation. In (2), solve the first equation for x_1 and the second for x_2. (Ignore the third equation; it offers no restriction on the variables.)

$$\begin{cases} x_1 = 1 + 5x_3; \\ x_2 = 4 - x_3; \\ x_3 \text{ is free.} \end{cases} \qquad (3)$$

The statement "x_3 **is free**" means that you are free to choose any value for x_3. Once that is done, the formulas in (3) determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is $(1, 4, 0)$; when $x_3 = 1$, the solution is $(6, 3, 1)$.

- Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.
- The solution in (3) represents all the solutions of the system (2), which is called the **general solution** of the system.

Properties.

- Any nonzero matrix **may be row reduced** (i.e., transformed by elementary row operations) into **more than one matrix in echelon form**, using different sequences of row operations.
- Once a matrix is in an echelon form, further row operations do not change the **pivot positions**.
- Each matrix is row equivalent to **one and only one reduced echelon matrix**.
- A linear system is consistent if and only if the right most column of the augmented matrix is not a pivot column–i.e., if and only if **an echelon form of the augmented matrix has no row of the form** $[0 \cdots 0 \ b]$ with b nonzero.
- If a linear system is **consistent**, then the solution set contains either (i) a **unique solution**, when there are **no free variables**, or (ii) **infinitely many solutions**, when there is **at least one free variable**.

Example. Choose h and k such that the system has no solution; uniqe solution; many solutions

$$\begin{cases} x_1 - 3x_2 & = 1; \\ 2x_1 + hx_2 & = k. \end{cases}$$

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Example. True-or-False

(a) The row reduction algorithm applies to only to augmented matrices for a linear system.

(b) If one row in an echelon form of an augmented matrix is $[0\ 0\ 0\ 0\ 2\ 0]$, then the associated linear system is inconsistent.

(c) The pivot positions in a matrix depend on whether or not row interchanges are used in the row reduction process.

(d) Reducing a matrix to an echelon form is called the forward phase of the row reduction process.

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Ans: F, F, F, T.

EXISTENCE AND UNIQUENESS QUESTIONS

Theorem (Existence and Uniqueness Solution)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \quad \cdots \quad 0 \quad b], \qquad b \neq 0.$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

(1) Write the augmented matrix of the system.
(2) Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
(3) Continue row reduction to obtain the reduced echelon form.
(4) Write the system of equations corresponding to the matrix obtained in step 3.
(5) Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.3. VECTOR EQUATIONS

This section connects equations involving vectors to ordinary systems of equations.
Vector will mean an ordered list of numbers.

COLUMN VECTOR

Definition. A matrix with only one column is called a **column vector**, or simply a **vector**.

Examples:

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \in \mathbb{R}^{\mathbf{2}}; \quad \mathbf{w} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \in \mathbb{R}^{\mathbf{3}}.$$

- The set of all vectors with two entries is denoted by $\mathbb{R}^2$.
- The set of all vectors with three entries is denoted by $\mathbb{R}^3$.

We can identify a geometric point (a, b) with a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ position vector. So we may regard $\mathbb{R}^2$ as the set of all points in the plane.

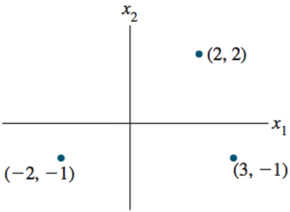

FIGURE 1 Vectors as points.

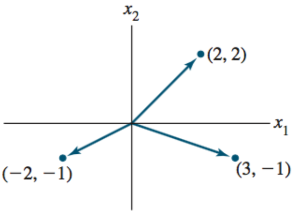

FIGURE 2 Vectors with arrows.

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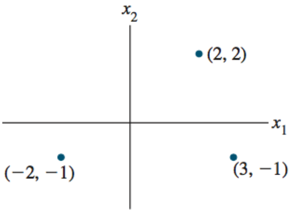

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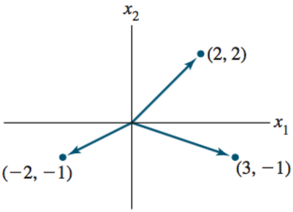

FIGURE 2 Vectors with arrows.

The geometric visualization of a vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is often aided by including an arrow (directed line segment) from the origin $(0, 0$ to the point $(3, -1)$, as in F.2.
In this case, the individual points along the arrow itself have no special significance

Vectors are mathematical objects having direction and length. We may try to (1) compare them, (2) add or subtract them, (3) multiply them by a scalar, (4) measure their length, and (5) apply other operations to get information related to angles.

1. **Equality of vectors**: Two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are equal if and only if corresponding entries are equal, i.e., $u_i = v_i$, $i = 1, 2$.
2. **Addition**: Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then
$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$
3. **Scalar multiple**: Let $c \in \mathbb{R}$, a scalar. Then
$$c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}.$$

VECTOR IN $\mathbb{R}^n$

For $n \in \mathbb{N}$, $\mathbb{R}^n$ denotes the collection of all lists (or ordered n-tuples) of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

The vector operations in $\mathbb{R}^2$, including the parallelogram rule, are also applicable for vectors in $\mathbb{R}^3$ and $\mathbb{R}^n$, in general.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

LINEAR COMBINATIONS AND SPAN

- Given vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$y = c_1\mathbf{v_1} + \ldots + c_p\mathbf{v_p}$$

  is called a **linear combination** of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ with **weights** $c_1, c_2, \ldots, c_p$.

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- If $\mathbf{v_1}, \ldots, \mathbf{v_p}$ in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v_1}, \ldots, \mathbf{v_p}$ is denoted by Span $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or generated) by $\mathbf{v_1}, \ldots, \mathbf{v_p}$. That is, Span $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v_1} + \ldots + c_p\mathbf{v_p}$$

  with $c_1, c_2, \ldots, c_p$ scalars.

VECTOR EQUATION

A **vector equation** is of the form

$$x_1\mathbf{a_1} + \ldots + x_n\mathbf{a_n} = \mathbf{b},$$

where $\mathbf{a_1}, \ldots, \mathbf{a_n}, \mathbf{b}$ are vectors and $x_1, x_2, \ldots, x_n$ are weights.

- A vector equation $x_1\mathbf{a_1} + \ldots + x_n\mathbf{a_n} = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a_1} \quad \mathbf{a_2} \quad \ldots \quad \mathbf{a_n} \quad \mathbf{b}]. \qquad (*)$$

- In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a_1}, \ldots, \mathbf{a_n}$ if and only if there exists a solution to the linear system $A\mathbf{x} = \mathbf{b}$ corresponding to the matrix $(*)$.

1.3. MATRIX EQUATIONS

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector.

PRODUCT OF A MATRIX AND A VECTOR

Let $A = [\mathbf{a_1}\ \mathbf{a_2}\ \ldots\ \mathbf{a_n}]$ be an $m \times n$ matrix and $x \in \mathbb{R}$, then the **product** of A and $\mathbf{x}$, denoted by $A\mathbf{x}$, is **the linear combination of the columns** of A using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = [\mathbf{a_1}\ \mathbf{a_2}\ \ldots\ \mathbf{a_n}].\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a_1} + \ldots + x_n\mathbf{a_n}.$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

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Example:

a. $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

b. $$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$

DEFINITION

A matrix equation is of the form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is a column vector of size $m \times 1$.

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Matrix equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Linear system

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= -1 \end{aligned}$$

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$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= -1 \end{aligned}$

Note that any system of linear equations, or any vector equation, can be written as an equivalent matrix equation in the form $A\mathbf{x} = \mathbf{b}$.

Theorem

If A *is an* $m \times n$ *matrix, with columns* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$*, and if* $b \in \mathbb{R}^m$*, the matrix equation*

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

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which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a_1}\ \mathbf{a_2}\ \ldots\ \mathbf{a_n}\ \mathbf{b}].$$

THEOREM

Theorem

If A is an $m \times n$ matrix, with columns $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$, and if $b \in \mathbb{R}^m$, the matrix equation

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which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

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Above theorem provides a powerful tool for gaining insight into problems in linear algebra, because a system of linear equations may now be viewed in three different but equivalent ways: as a matrix equation, as a vector equation, or as a system of linear equations.

Existence of Solutions

The definition of $A\mathbf{x}$ leads directly to the following useful fact.

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent (all true, or all false).

(i) *For each $\mathbf{b} \in \mathbb{R}^{\mathbf{m}}$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*

(ii) *Each $\mathbf{b} \in \mathbb{R}^{\mathbf{m}}$ is a linear combination of columns of A.*

(iii) *The columns of A span $\mathbb{R}^m$.*

(iv) *A has a pivot position in every row.*

Warning: Theorem is about a coeffcient matrix, not an augmented matrix.

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Note that this theorem is one of the most useful theorems in this chapter. Statements (i), (ii), and (iii) are equivalent because of the definition of $A\mathbf{x}$ and what it means for a set of vectors to span $\mathbb{R}^m$.

THANK YOU FOR YOUR ATTENTION!