ĐỀ CƯƠNG TOÁN CAO CẤP

Ôn trọng tâm chương 2 và 3

I. LINEAR EQUATIONS IN LINEAR ALGEBRA

1. …
2. *Row reduction and echelon form*
3. *Vector equations*

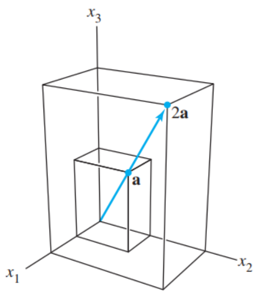

FIGURE 6
Scalar multiples.

Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries. They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity. The vectors $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $2\mathbf{a}$ are displayed in Figure 6.

Vectors in $\mathbb{R}^n$

If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$. (The number of entries in $\mathbf{0}$ will be clear from the context.)

Equality of vectors in $\mathbb{R}^n$ and the operations of scalar multiplication and vector addition in $\mathbb{R}^n$ are defined entry by entry just as in $\mathbb{R}^2$. These operations on vectors have the following properties, which can be verified directly from the corresponding properties for real numbers. See Practice Problem 1 and Exercises 33 and 34 at the end of this section.

Vector in R2 => 2 entries (41/579)

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) **by** $\mathbf{v}_1, \dots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \dots, c_p$ scalars.

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \qquad (4)$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \qquad (5)$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \qquad (6)$$

4. *The matrix equation Ax=b*

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

5. ***Solution sets of linear system***

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Nghiệm tầm thường (trivial solution) => x1 =x2=...=xn = 0
Nghiệm không tầm thường (nontrivial solution) => có ít nhất 1 free variable

Parametric vector form

Parametric Vector Form

The original equation (1) for the plane in Example 2 is an *implicit* description of the plane. Solving this equation amounts to finding an *explicit* description of the plane as the set spanned by $\mathbf{u}$ and $\mathbf{v}$. Equation (2) is called a **parametric vector equation** of the plane. Sometimes such an equation is written as

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \text{ in } \mathbb{R})$$

to emphasize that the parameters vary over all real numbers. In Example 1, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line. Whenever a solution set is described explicitly with vectors as in Examples 1 and 2, we say that the solution is in **parametric vector form**.

6. **Ứng dụng của … Trong cuộc sống (:>, chắc là k thi đâu)**
7. **Linear independence**

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (2)$$

- ***Det A = 0 => Ma trận A không nghịch đảo và là phụ thuộc tuyến tính***
- ***Det A # 0 => Ma trận A nghịch đảo và là độc lập tuyến tính***

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Có số cột > số hàng => phụ thuộc tuyến tính (linear dependence)

If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Có cột 0 => phụ thuộc tuyến tính (linear dependence)

Có free variables => nghiệm không tầm thường => phụ thuộc tuyến tính. Ngược lại, không có free variables => trivial solution =>độc lập tuyến tính

8. ***Linear transformation***
9. ***The matrix of a linear transformation***

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

T-onto ⇔ Mọi b thuộc Rm: T(x) = b có ít nhất 1 nghiệm

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

T-one-to-one ⇔ Mọi b thuộc Rm: T(x) = b có nhiều nhất 1 nghiệm (tức là có thể không có nghiệm)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

10. ***Ứng dụng của …***

II. MATRIX ALGEBRA

1. ***Matrix operations***

Nhân hai ma trận:

- A cỡ m*n (m hàng, n cột), B cỡ n*p (n hàng, p cột) => AB là ma trận cỡ m*p
- Hàng * cột

WARNINGS:

1. In general, $AB \neq BA$.

2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)

3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

Ma trận chuyển vị (Transpose of a matrix)

- Given an m n matrix A, the transpose of A is the n m matrix, denoted by AT , whose columns are formed from the corresponding rows of A. **(đổi hàng thành cột, cột thành hàng)**

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Ma trận I (ma trận đơn vị (identity) cỡ m*m với 1 một đường chéo hướng đông nam các pivots = 1)

Identity Matrices

$1 \times 1 \quad [1]$

$2 \times 2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$3 \times 3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

etc.

2. *Inverse of a matrix*

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

A matrix that is *not* invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

EXAMPLE 1 If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $C = A^{-1}$. ■

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The simple proof of Theorem 4 is outlined in Exercises 25 and 26. The quantity $ad - bc$ is called the **determinant** of A, and we write

$$\det A = ad - bc$$

Theorem 4 says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

PROOF Take any $\mathbf{b}$ in $\mathbb{R}^n$. A solution exists because if $A^{-1}\mathbf{b}$ is substituted for $\mathbf{x}$, then $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$. So $A^{-1}\mathbf{b}$ is a solution. To prove that the solution is unique, show that if $\mathbf{u}$ is any solution, then $\mathbf{u}$, in fact, must be $A^{-1}\mathbf{b}$. Indeed, if $A\mathbf{u} = \mathbf{b}$, we can multiply both sides by A^{-1} and obtain

$$A^{-1}A\mathbf{u} = A^{-1}\mathbf{b}, \quad I\mathbf{u} = A^{-1}\mathbf{b}, \quad \text{and} \quad \mathbf{u} = A^{-1}\mathbf{b}$$ ■

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

SOLUTION

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

3. *Characteristics of… (Tính chất của ma trận nghịch đảo)*

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

4. <u>Partitioned matrix (loại)</u>
5. <u>Matrix factorizations (loại) (LU)</u>

6. The leontief input-output…. (lạ quá, chịu)
7. Ứng dụng…
8. ***Subspaces of Rn***

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.
b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

EXAMPLE 1 If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^n$ and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace of $\mathbb{R}^n$. To verify this statement, note that the zero vector is in H (because $0\mathbf{v}_1 + 0\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$). Now take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

Then

$$\mathbf{u}+\mathbf{v} = (s_1+t_1)\mathbf{v}_1 + (s_2+t_2)\mathbf{v}_2$$

which shows that $\mathbf{u}+\mathbf{v}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and hence is in H. Also, for any scalar c, the vector $c\mathbf{u}$ is in H, because $c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$. ■

If $\mathbf{v}_1$ is not zero and if $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ simply span a *line* through the origin. So a line through the origin is another example of a subspace.

[1] Sections 2.8 and 2.9 are included here to permit readers to postpone the study of most or all of the next two chapters and to skip directly to Chapter 5, if so desired. *Omit* these two sections if you plan to work through Chapter 4 before beginning Chapter 5.

Matrix Ax = w is inconsistent/ has a solution => w is not in subspace of R3

Subspace spanned by a set => linear combination ⇔ consistent

- ***Col space***

The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

If $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, with the columns in $\mathbb{R}^m$, then Col A is the same as Span$\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Example 4 shows that the **column space of an $m \times n$ matrix is a subspace of** $\mathbb{R}^m$. Note that Col A equals $\mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$. Otherwise, Col A is only part of $\mathbb{R}^m$.

EXAMPLE 4 Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Determine whether $\mathbf{b}$ is in the column space of A.

SOLUTION The vector $\mathbf{b}$ is a linear combination of the columns of A if and only if $\mathbf{b}$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$, that is, if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution. Row reducing the augmented matrix $[\,A \quad \mathbf{b}\,]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we conclude that $A\mathbf{x} = \mathbf{b}$ is consistent and $\mathbf{b}$ is in Col A. ■

We have : A = [v1 v2 v3]

Span {v1, v1, v2} = Col A

- ***Null space***

DEFINITION

The **null space** of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

When A has n columns, the solutions of $A\mathbf{x} = \mathbf{0}$ belong to $\mathbb{R}^n$, and the null space of A is a subset of $\mathbb{R}^n$. In fact, Nul A has the properties of a *subspace* of $\mathbb{R}^n$.

THEOREM 12

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Determine if p is in Nul A?

=> Tính: A * p (A: m hàng n cột, p: n cột p hàng)

- A* p = 0 => p is in Null A
- Otherwise, p is not in Null A

- ***A basic of subspace***

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

EXAMPLE 5 The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem. One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $\mathbf{e}_1, \dots, \mathbf{e}_n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See Figure 3. ■

- ***Basic of null*** (Biểu diễn x theo dạng general x = xn*vn + xm*vm +..., thì (vn vm v… là basic for null space of the given matrix)

EXAMPLE 6 Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

SOLUTION First, write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}}$$

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} \tag{1}$$

Equation (1) shows that Nul A coincides with the set of all linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. That is, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ generates Nul A. In fact, this construction of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ automatically makes them linearly independent, because equation (1) shows that $\mathbf{0} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$ only if the weights x_2, x_4, and x_5 are all zero. (Examine entries 2, 4, and 5 in the vector $x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$.) So $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a *basis* for Nul A. ■

- ***Basic for col space of the matrix*** (sau khi biến đổi ma trận được các cột có chứa pivot, thì tập hợp các cột đó (trước khi biến đổi) chính là basic of Col given matrix)

EXAMPLE 7 Find a basis for the column space of the matrix

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SOLUTION Denote the columns of B by $\mathbf{b}_1, \dots, \mathbf{b}_5$ and note that $\mathbf{b}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_2$ and $\mathbf{b}_4 = 5\mathbf{b}_1 - \mathbf{b}_2$. The fact that $\mathbf{b}_3$ and $\mathbf{b}_4$ are combinations of the pivot columns means that any combination of $\mathbf{b}_1, \dots, \mathbf{b}_5$ is actually just a combination of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_5$. Indeed, if $\mathbf{v}$ is any vector in Col B, say,

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4 + c_5\mathbf{b}_5$$

then, substituting for $\mathbf{b}_3$ and $\mathbf{b}_4$, we can write $\mathbf{v}$ in the form

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3(-3\mathbf{b}_1 + 2\mathbf{b}_2) + c_4(5\mathbf{b}_1 - \mathbf{b}_2) + c_5\mathbf{b}_5$$

which is a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_5$. So $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5\}$ spans Col B. Also, $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_5$ are linearly independent, because they are columns from an identity matrix. So the pivot columns of B form a basis for Col B. ■

EXAMPLE 8 It can be verified that the matrix

$$A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_5\,] = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

is row equivalent to the matrix B in Example 7. Find a basis for Col A.

SOLUTION From Example 7, the pivot columns of A are columns 1, 2, and 5. Also, $\mathbf{b}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_2$ and $\mathbf{b}_4 = 5\mathbf{b}_1 - \mathbf{b}_2$. Since row operations do not affect linear dependence relations among the columns of the matrix, we should have

$$\mathbf{a}_3 = -3\mathbf{a}_1 + 2\mathbf{a}_2 \quad \text{and} \quad \mathbf{a}_4 = 5\mathbf{a}_1 - \mathbf{a}_2$$

Check that this is true! By the argument in Example 7, $\mathbf{a}_3$ and $\mathbf{a}_4$ are not needed to generate the column space of A. Also, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ must be linearly independent, because any dependence relation among $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_5$ would imply the same dependence relation among $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_5$. Since $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5\}$ is linearly independent, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is also linearly independent and hence is a basis for Col A. ■

The argument in Example 8 can be adapted to prove the following theorem.

> The pivot columns of a matrix A form a basis for the column space of A.

- ***Ma trận A khả nghịch (det A # 0) => các cột sinh ra R n***

  Ma trận A cỡ n*n (Ma trận vuông A cỡ n)

  + có n pivot => độc lập tuyến tính (linear independence)
  + Mỗi hàng đều có pivot => Sinh ra không gian R n

● Ngược lại, … Không là cơ sở, phụ thuộc tuyến tính (không có pivot ở mọi cột), không sinh R n (không có pivot ở mọi hàng)

- ***Ma trận n hàng có pivot ở mọi hàng => Hệ vectơ sinh ra R n nhưng nếu nó phụ thuộc tuyến tính (không phải cột nào cũng có pivot thì là phụ thuộc tuyến tính) thì không là cơ sở***
- Ví dụ:

	1	4

A=	0	-13
	0	0
Ma trận A độc lập tuyến tính, nhưng không sinh ra R3.		

9. Dimensions and Rank

The Dimension of a Subspace

It can be shown that if a subspace H has a basis of p vectors, then every basis of H must consist of exactly p vectors. (See Exercises 27 and 28.) Thus the following definition makes sense.

> The **dimension** of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.[2]

The space $\mathbb{R}^n$ has dimension n. Every basis for $\mathbb{R}^n$ consists of n vectors. A plane through $\mathbf{0}$ in $\mathbb{R}^3$ is two-dimensional, and a line through $\mathbf{0}$ is one-dimensional.

EXAMPLE 2 Recall that the null space of the matrix A in Example 6 in Section 2.8 had a basis of 3 vectors. So the dimension of Nul A in this case is 3. Observe how each basis vector corresponds to a free variable in the equation $A\mathbf{x} = \mathbf{0}$. Our construction always produces a basis in this way. So, to find the dimension of Nul A, simply identify and count the number of free variables in $A\mathbf{x} = \mathbf{0}$. ■

> The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

Since the pivot columns of A form a basis for Col A, the rank of A is just the number of pivot columns in A.

Hạng của given matrix = số dòng khác 0 của given matrix (sau khi biến đổi thành matrix bậc thang)

EXAMPLE 3 Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

SOLUTION Reduce A to echelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix A has 3 pivot columns, so rank $A = 3$.

The Rank Theorem

If a matrix A has n columns, then rank A + dim Nul $A = n$.

The following theorem is important for applications and will be needed in Chapters 5 and 6. The theorem (proved in Section 4.5) is certainly plausible, if you think of a p-dimensional subspace as isomorphic to $\mathbb{R}^p$. The Invertible Matrix Theorem shows that p vectors in $\mathbb{R}^p$ are linearly independent if and only if they also span $\mathbb{R}^p$.

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Rank and the Invertible Matrix Theorem

The various vector space concepts associated with a matrix provide several more statements for the Invertible Matrix Theorem. They are presented below to follow the statements in the original theorem in Section 2.3.

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$

o. dim Col $A = n$

p. rank $A = n$

q. Nul $A = \{\mathbf{0}\}$

r. dim Nul $A = 0$

- ***Dim (nul A): số lượng vector trong basis of Nul A (free variable)***
- ***Dim (col A) = Rank A***
- ***Nếu A là ma trận vuông cấp n thì A khả nghịch khi và chỉ khi rank(A) = n***

Định lý 2.22 (Định lý Kronecker-Capelli). *Hệ 2.3 có nghiệm khi và chỉ khi*

$$r\left(\overline{A}\right) = r(A),$$

trong đó $\overline{A}$ là ma trận bổ sung tức là ma trận A thêm cột b, $\overline{A} = [Ab]$.

Hệ quả 2.23. 1. *Nếu $r\left(\overline{A}\right) \neq r(A)$ thì hệ 2.3 vô nghiệm*

2. *Nếu $r\left(\overline{A}\right) = r(A) = n$ thì hệ 2.3 có nghiệm duy nhất*

3. *Nếu $r\left(\overline{A}\right) = r(A) < n$ thì hệ 2.3 có vô số nghiệm*

III. DET

1. Giới thiệu

- **Cách tính …**
- Cách 1: truyền thống DET A = ...
- Cách 2: biến đổi thành ma trận chéo => nhân các leading entries của mỗi hàng

2. Tính chất

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

A square matrix A is invertible if and only if $\det A \neq 0$.

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

3. *Cramer's rule, diện tích, thể tích*

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

EXAMPLE 3 Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

SOLUTION The nine cofactors are

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

The adjugate matrix is the *transpose* of the matrix of cofactors. [For instance, C_{12} goes in the (2, 1) position.] Thus

$$\operatorname{adj} A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

Một định thức có 1 hàng (hoặc 1 cột) toàn số 0 thì bằng 0.
Một định thức có 2 hàng (hoặc 2 cột) tỉ lệ thì bằng 0.

- ***Determinants as Area or Volume….***

IV. VECTOR SPACES

Tương tự II. Matrix algebra

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]
b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

We call Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Given any subspace H of V, a **spanning** (or **generating**) **set** for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

The next example shows how to use Theorem 1.

Subspace spanned by a set => linear combination => consistent

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

This calculation shows that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vectors indicated above. Thus H is a subspace of $\mathbb{R}^4$ by Theorem 1.

The Null Space of a Matrix

Consider the following system of homogeneous equations:

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0 \end{aligned} \tag{1}$$

In matrix form, this system is written as $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \tag{2}$$

Recall that the set of all $\mathbf{x}$ that satisfy (1) is called the **solution set** of the system (1). Often it is convenient to relate this set directly to the matrix A and the equation $A\mathbf{x} = \mathbf{0}$. We call the set of $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{0}$ the **null space** of the matrix A.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

EXAMPLE 3 Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

SOLUTION The first step is to find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of free variables. Row reduce the augmented matrix $[\, A \quad \mathbf{0} \,]$ to *reduced* echelon form in order to write the basic variables in terms of the free variables:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free. Next, decompose the vector giving the general solution into a linear combination of vectors where *the weights are the free variables*. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{\mathbf{u}}{\underset{\uparrow}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}} + x_4 \underset{\mathbf{v}}{\underset{\uparrow}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}} + x_5 \underset{\mathbf{w}}{\underset{\uparrow}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}}$$

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} \tag{3}$$

Every linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is an element of Nul A and vice versa. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for Nul A. ■

Two points should be made about the solution of Example 3 that apply to all problems of this type where Nul A contains nonzero vectors. We will use these facts later.

The Column Space of a Matrix

Another important subspace associated with a matrix is its column space. Unlike the null space, the column space is defined explicitly via linear combinations.

> The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\, \mathbf{a}_1 \ \cdots \ \mathbf{a}_n \,]$, then
>
> $$\text{Col } A = \text{Span}\,\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

Since Span $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a subspace, by Theorem 1, the next theorem follows from the definition of Col A and the fact that the columns of A are in $\mathbb{R}^m$.

> The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Note that a typical vector in Col A can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

The notation $A\mathbf{x}$ for vectors in Col A also shows that Col A is the *range* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. We will return to this point of view at the end of the section.

EXAMPLE 4 Find a matrix A such that $W = \text{Col}\, A$.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$$

SOLUTION First, write W as a set of linear combinations.

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Second, use the vectors in the spanning set as the columns of A. Let $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

Then $W = \text{Col}\, A$, as desired. ∎

Recall from Theorem 4 in Section 1.4 that the columns of A span $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$. We can restate this fact as follows:

> The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

EXAMPLE 5 Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

a. If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

b. If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

SOLUTION

a. The columns of A each have three entries, so Col A is a subspace of $\mathbb{R}^k$, where $k = 3$.

b. A vector $\mathbf{x}$ such that $A\mathbf{x}$ is defined must have four entries, so Nul A is a subspace of $\mathbb{R}^k$, where $k = 4$. ∎

When a matrix is not square, as in Example 5, the vectors in Nul A and Col A live in entirely different "universes." For example, no linear combination of vectors in $\mathbb{R}^3$ can produce a vector in $\mathbb{R}^4$. When A is square, Nul A and Col A do have the zero vector in common, and in special cases it is possible that some nonzero vectors belong to both Nul A and Col A.

EXAMPLE 6 With A as in Example 5, find a nonzero vector in Col A and a nonzero vector in Nul A.

SOLUTION It is easy to find a vector in Col A. Any column of A will do, say, $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

To find a nonzero vector in Nul A, row reduce the augmented matrix $[\, A \quad \mathbf{0}\,]$ and obtain

$$[\, A \quad \mathbf{0}\,] \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, if $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $x_1 = -9x_3$, $x_2 = 5x_3$, $x_4 = 0$, and x_3 is free. Assigning a nonzero value to x_3—say, $x_3 = 1$—we obtain a vector in Nul A, namely, $\mathbf{x} = (-9, 5, 1, 0)$. ∎

EXAMPLE 7 With A as in Example 5, let $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

a. Determine if $\mathbf{u}$ is in Nul A. Could $\mathbf{u}$ be in Col A?

b. Determine if $\mathbf{v}$ is in Col A. Could $\mathbf{v}$ be in Nul A?

SOLUTION

a. An explicit description of Nul A is not needed here. Simply compute the product $A\mathbf{u}$.

$$A\mathbf{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously, $\mathbf{u}$ is *not* a solution of $A\mathbf{x} = \mathbf{0}$, so $\mathbf{u}$ is not in Nul A. Also, with four entries, $\mathbf{u}$ could not possibly be in Col A, since Col A is a subspace of $\mathbb{R}^3$.

b. Reduce $[\, A \quad \mathbf{v}\,]$ to an echelon form.

$$[\, A \quad \mathbf{v}\,] = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

At this point, it is clear that the equation $A\mathbf{x} = \mathbf{v}$ is consistent, so $\mathbf{v}$ is in Col A. With only three entries, $\mathbf{v}$ could not possibly be in Nul A, since Nul A is a subspace of $\mathbb{R}^4$. ■

DEFINITION Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

EXAMPLE 3 Let A be an invertible $n \times n$ matrix—say, $A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_n\,]$. Then the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem. ■

EXAMPLE 4 Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix, I_n. That is,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$ (Figure 1). ■

EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

SOLUTION Since there are exactly three vectors here in $\mathbb{R}^3$, we can use any of several methods to determine if the matrix $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,]$ is invertible. For instance, two row replacements reveal that A has three pivot positions. Thus A is invertible. As in Example 3, the columns of A form a basis for $\mathbb{R}^3$. ■

EXAMPLE 6 Let $S = \{1, t, t^2, \dots, t^n\}$. Verify that S is a basis for $\mathbb{P}_n$. This basis is called the **standard basis** for $\mathbb{P}_n$.