Matrix Algebra:

Inverse matrix

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The inverse matrix of a $n \times n$ matrix A denoted by $n \times n$ matrix A^{-1}. The inverse matrix satisfies:

$$AA^{-1} = A^{-1}A = I,$$

where $I = I_n$, the $n \times n$ identity matrix.

Note:

- Square matrices may have inverses. When a matrix A has an *inverse*, we say it is *invertible*.
- In fact, matrix A^{-1} is unique determine by A.
- A matrix that is not invertible is called a *singular matrix*, and an invertible matrix is called a *nonsingular matrix*.

Theorem

- *If A is an invertible matrix, then A^{-1} is invertible and:* $(A^{-1})^{-1} = A$;
- *If A and B are invertible matrices, then so is AB:* $(AB)^{-1} = B^{-1}A^{-1}$;
- *If A is invertible then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

For example:

Solve the problem

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where x_1, x_2,y_1, y_2 are unknown.

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The corresponding linear system of equations:

$$ax_1 + by_1 = 1, \qquad cx_1 + dy_1 = 0,$$

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The solution for the inverse matrix is found to be:

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

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The term $ad - bc$ is just the definition of the determinant of the two-by-two matrix:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Theorem

Let matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $\det A = ad - bc \neq 0$, *then* A *is invertible and:*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If $\det A = ad - bc = 0$, *then* A *is not invertible.*

Theorem says that 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Notice that the inverse of a two-by-two matrix, in words, is found by switching the diagonal elements of the matrix, negating the off-diagonal elements, and dividing by the determinant.

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$$A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

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$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 5/2 & -3/2 \end{pmatrix}.$$

Theorem

If A is an invertible $n \times n$ *matrix, then for each* $\mathbf{b}$ *in* R^n, *the equation* $A\mathbf{x} = \mathbf{b}$ *has the unique solution*

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For example:

Use the inverse of the matrix to solve the system:

$$3x_1 + 4x_2 = 3; \qquad 5x_1 + 6x_2 = 7.$$

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This system is equivalent to $A\mathbf{x} = \mathbf{b}$, so:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -3 & 2 \\ 5/2 & -3/2 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix. The next example illustrates the three kinds of elementary matrices.

- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.
- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

SOLUTION Verify that

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, \quad E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

Addition of -4 times row 1 of A to row 3 produces E_1A. (This is a row replacement operation.) An interchange of rows 1 and 2 of A produces E_2A, and multiplication of row 3 of A by 5 produces E_3A. ■

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Solution. To transform E_1 into I, add $+4$ times row 1 to row 3. The elementary matrix that does this is

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{pmatrix}$$

Algorithm for finding the inverse of a matrix

Theorem

An $n \times n$ matrix A is an invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

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If we place A and I side by side to form an augmented matrix $[A \;\; I]$, then row operations on this matrix produce identical operations on A and on I. By above theorem, either there are row operations that transform A to I_n and I_n to A^{-1} or else A is not invertible.

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If we place A and I side by side to form an augmented matrix $[A \ \ I]$, then row operations on this matrix produce identical operations on A and on I. By above theorem, either there are row operations that transform A to I_n and I_n to A^{-1} or else A is not invertible.

Algorithm. Row reduce the augmented matrix $[A \ \ I]$. If A is row equivalent to I, then $[A \ \ I]$ is row equivalent to $\left[I \ \ A^{-1}\right]$. Otherwise, A does not have an inverse.

For example: Find the inverse of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$$

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SOLUTION

$$[A \quad I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

It is a good idea to check the final answer:

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is not necessary to check that $A^{-1}A = I$ since A is invertible. ■

The Invertible Matrix Theorem

Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1) *A is an invertible matrix.*
2) *A is row equivalent to the $n \times n$ identity matrix.*
3) *A has n pivot positions.*
4) *The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5) *The columns of A form a linearly independent set.*
6) *The linear transformation $\mathbf{x} \longleftrightarrow A\mathbf{x}$ is one-to-one.*
7) *The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each b in $\mathbb{R}^n$.*
8) *The columns of A span in $\mathbb{R}^n$.*
9) *The linear transformation $\mathbf{x} \longleftrightarrow A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.*
10) *There is a $n \times n$ matrix C such that $CA = I$.*
11) *There is a $n \times n$ matrix D such that $AD = I$.*
12) *A^T is an invertible matrix.*

1. Use determinants to determine which of the following matrices are invertible.

a. $\begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix}$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

3. If A is an invertible matrix, prove that $5A$ is an invertible matrix.

THANK YOU FOR YOUR ATTENTION!