Graphs – Part 1

Discrete Mathematic

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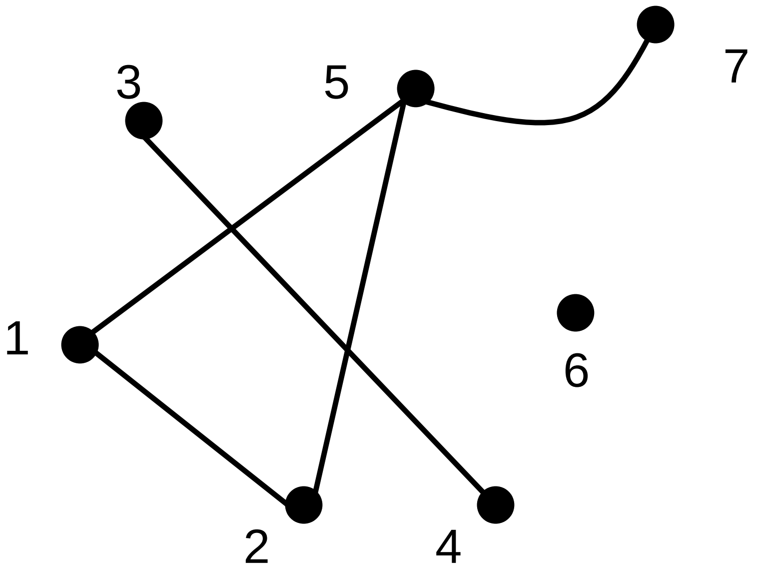
Definition 1: Graph

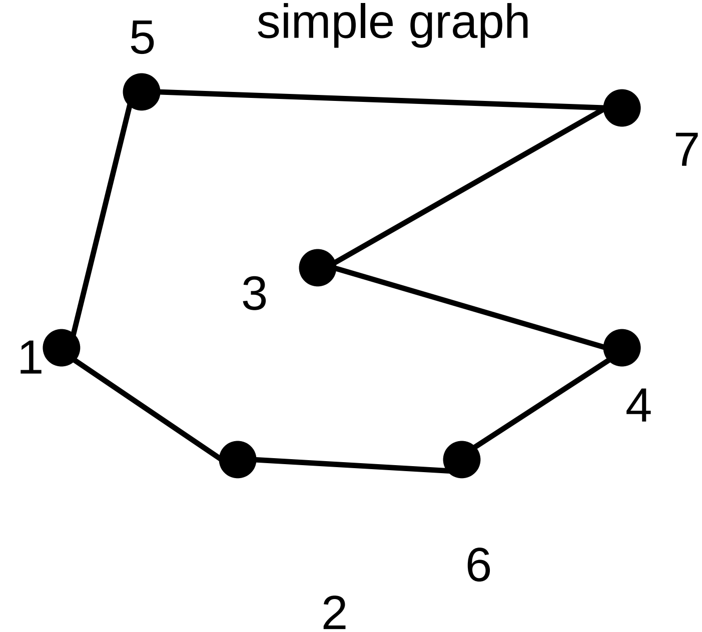
- A graph $G = (V,E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.
- There are different types of graph such as: simple graph, multi-graphs, undirected graph, directed graph,

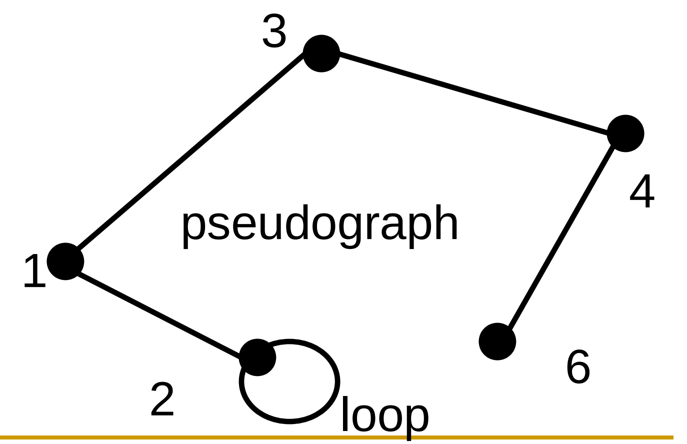
Definition 1: Graph

undirected graph



simple graph





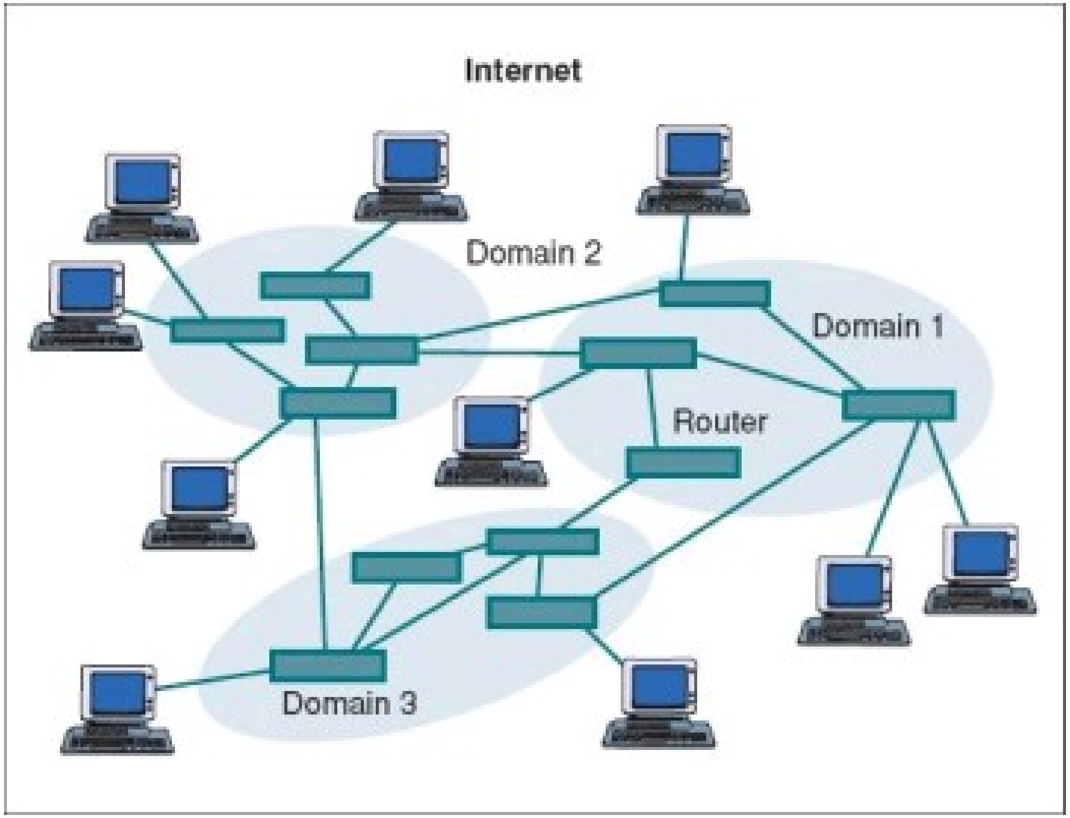
$V(G)$: Vertex set of a graph G

$E(G)$: Edge set of a graph G

The graph on $V = \{1, \ldots, 7\}$ with edges set

$E = \{\{1, 2\}, \{1, 5\}, \{2, 5\}, \{3, 4\}, \{5, 7\}\}$

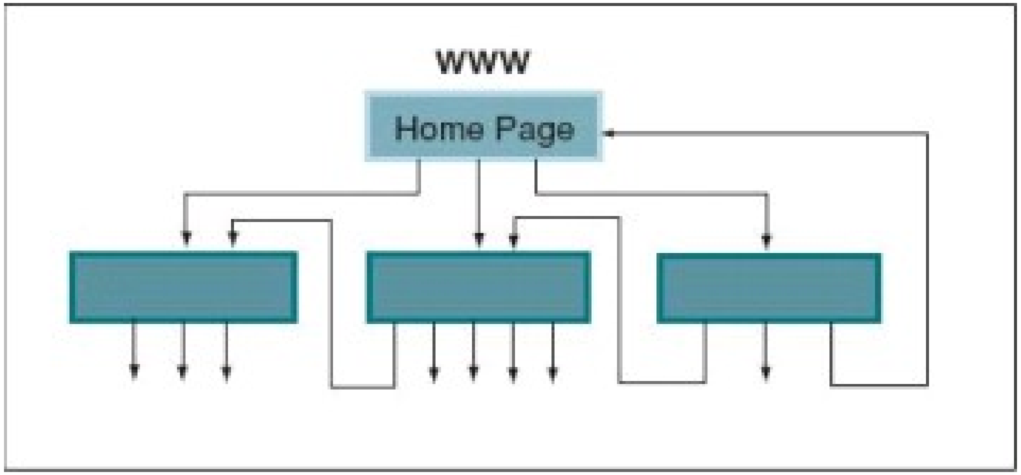
Example: Computer network



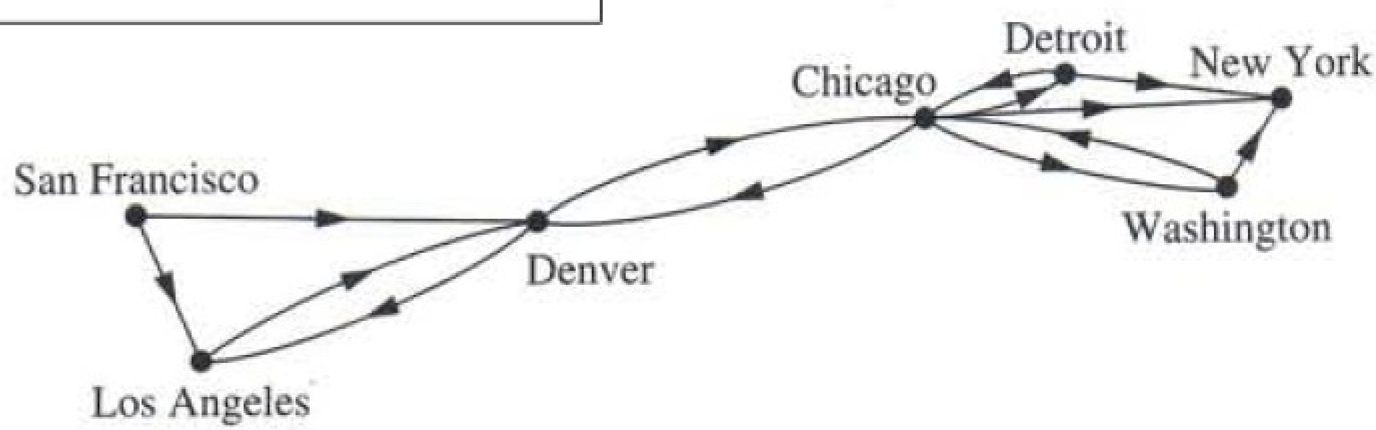
Definition 2: Directed graph

- A directed graph (V,E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices.
- The directed edge associated with the ordered pair (u,v) is said to start at u and end at v.

Example: WWW

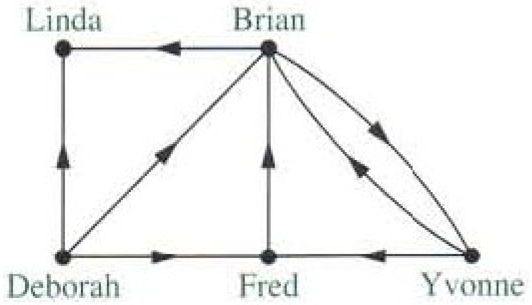


directed
multigraph

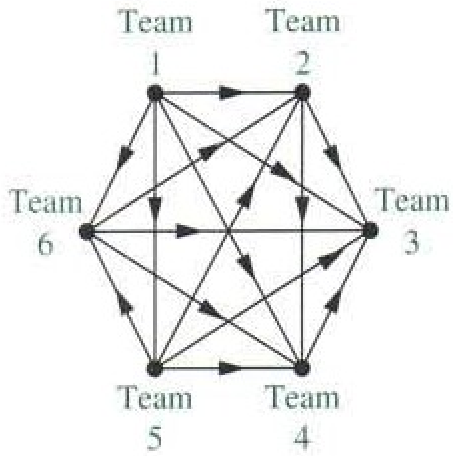


Example: A Communication Network

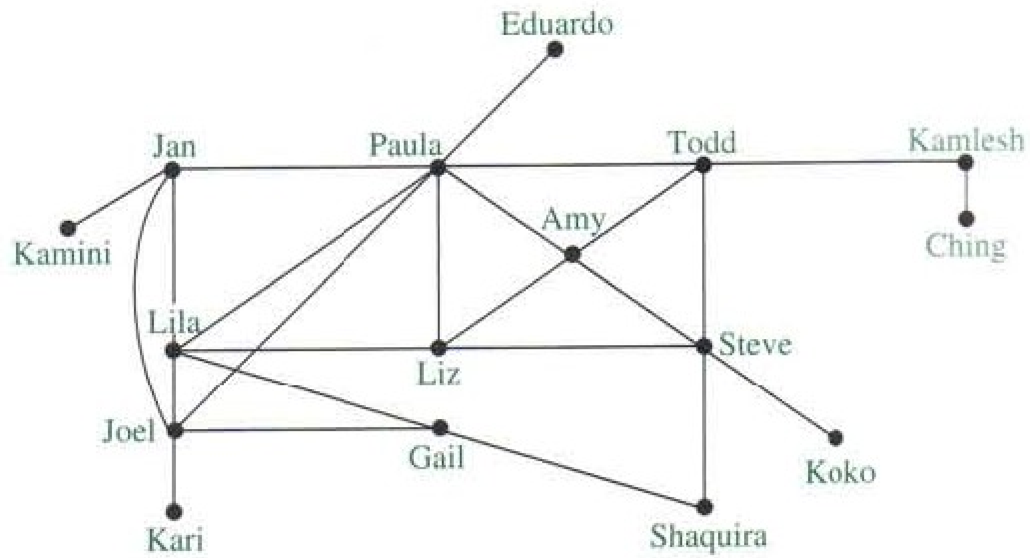
Graph Models



An Influence Graph



Round-Robin Tournament

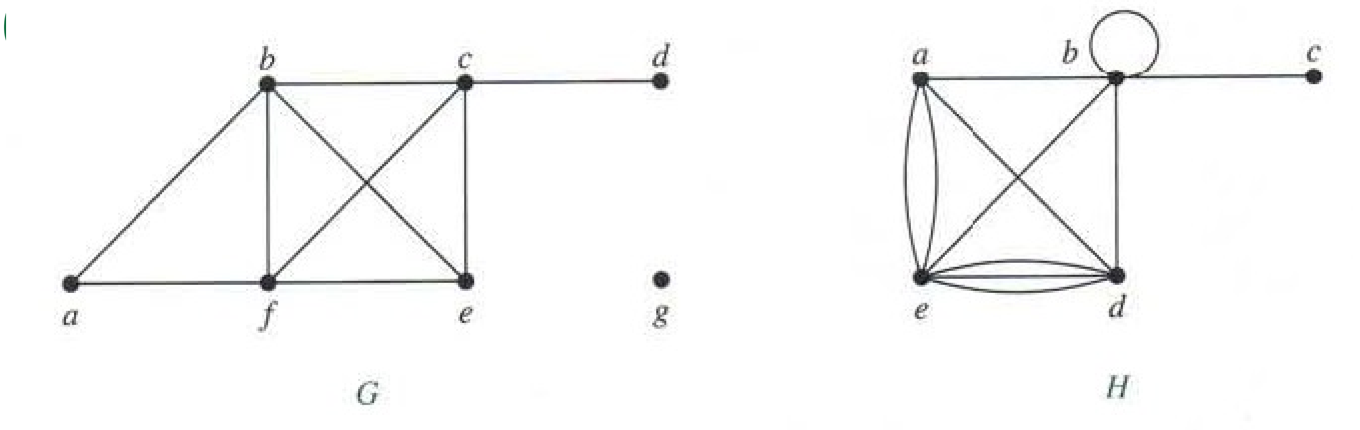


Acquaintanceship graph

Basic Terminology

- Adjacent (or neighbors)
  - Two vertices *u* and *v* in an undirected graph *G* are called *adjacent* (or neighbors) in *G* if *u* and *v* are endpoints of an edge of *G*. If edge e is associated with {u,v}, the edge e is called *incident* with the vertices u and v.
- Degree
  - The degree of a vertex *v* in a graph *G* is the number of edges incident with *v*, denoted by deg(*v*).

Example 1: Calculate the d



- Graph G:

  deg(a) = 2, deg(b)=4, deg(c)=4, deg(d)=1,deg(g)=0,deg(f)=4,deg(e)=3.
- Graph H: deg(a)=4,deg(b)=deg(e)=6,deg(c)=1,deg(d)=5

Theorem 1: The Handshaking

- Let G = (V,E) be an graph with e edges. Then

$$2e = \sum_{v \in V} \deg(v)$$

It is true for present of multipe edges and loops.

Question: How many edges are there in a graph with 10 vertices each of degree six?

Theorem 2

- A graph has an even number of vertices of odd degree.

$$2e = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

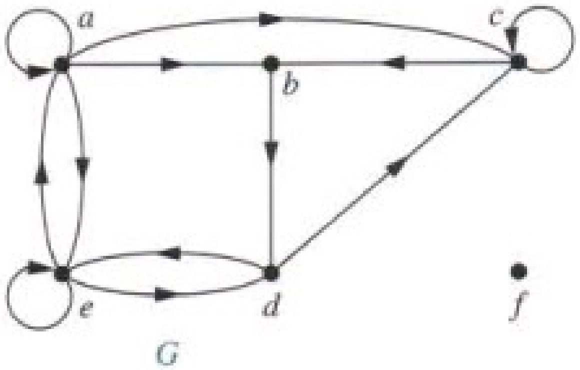
  - V1: Set of vertices of even degree.
  - V2: Set of vertices of odd degree.
  - Both terms must be even, then the theorem 2 is proofed.

Note

- When (u,v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u.
- The vertex u is called the initial vertex of (u,v), and v is called the terminal or end vertex of (u,v). The initial vertex and terminal vertex of a loop are the same.

In-degree and out-degree

- In a graph with directed edges in in-degree of a vertex *v*, denoted by $\deg^-(v)$, is the number of edges with *v* as their terminal vertex. The out-degree of *v*, denoted by $\deg^+(v)$, is the number of edges with *v* as their initial vertex.

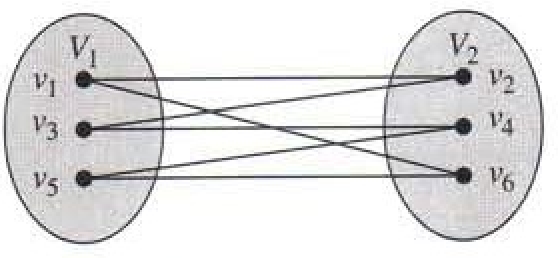


$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

Bipartite Graph

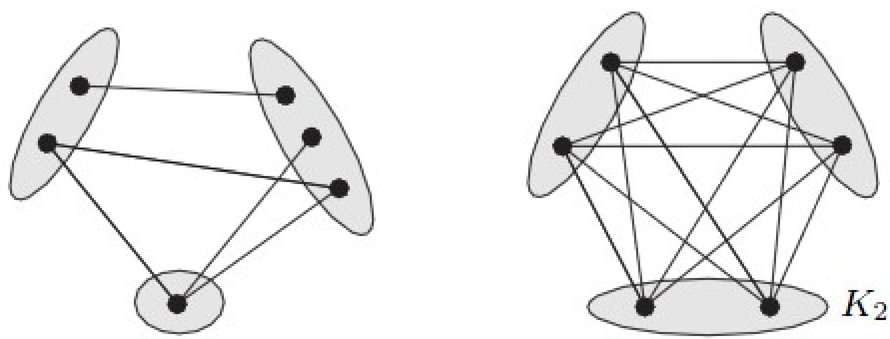
- Let $r \geq 2$ be an integer. A graph $G = (V, E)$ is called *r-partite* if V admits a partition into *r* classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. Instead of '*2-partite*' one usually says *bipartite*.

An example of bipartite graph



R-partite Graph

- An r-partite graph in wich every two vertices from different partition classes are adjacent is called complete. The complete r-partite graphs for all r together are complete multipartite graphs.



$K_{2,2,2} = K_2^3$

Examples of r-partite graph

Representing Graph

- A graph can be represented without edges by using a list of all edges of the graph or adjacency lists.

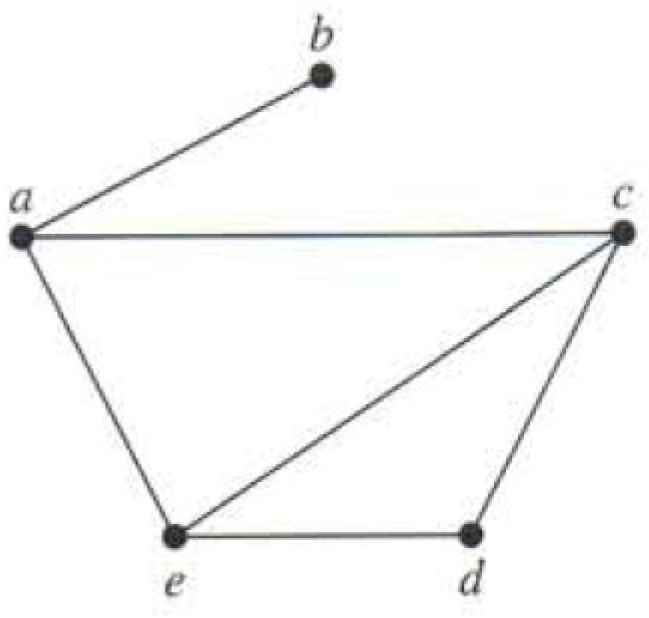


TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	*b*, *c*, *e*
b	*a*
c	*a*, *d*, *e*
d	*c*, *e*
e	*a*, *c*, *d*

Representing Graph – contd.

- **Adjacency matrices:** A or A_G of G with respect to this listing of the vertices.
- Suppose that $G = (V, E)$ is a simple graph where total number of vertices in G is $|V| = n$. Matrix A is $n \times n$ matrix, denoted as

$$A = [a_{ij}]_1^n,\ \text{where } a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of G} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{c c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

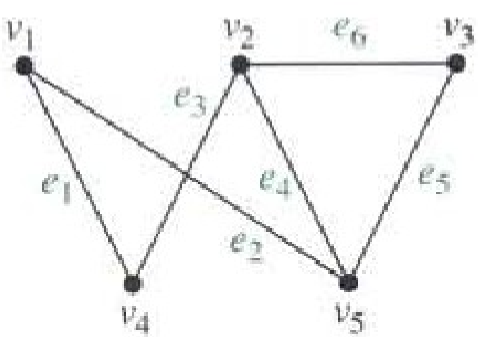
edge $\{v_1,v_2\}$, or {a,b}

no edge between c and d

Representing Graph – contd.

- **Incidence matrices:** A or A_G of G with respect to this listing of the vertices.
- Suppose that $G = (V, E)$ is a undirected graph, $\{v_1, v_2, \ldots, v_n\}$ are vetices, and $\{e_1, e_2, \ldots, e_m\}$ are edges. Matrix M is $n \times m$ matrix, denoted as

$$B = [m_{ij}]_{n\times m}, \text{ where } b_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

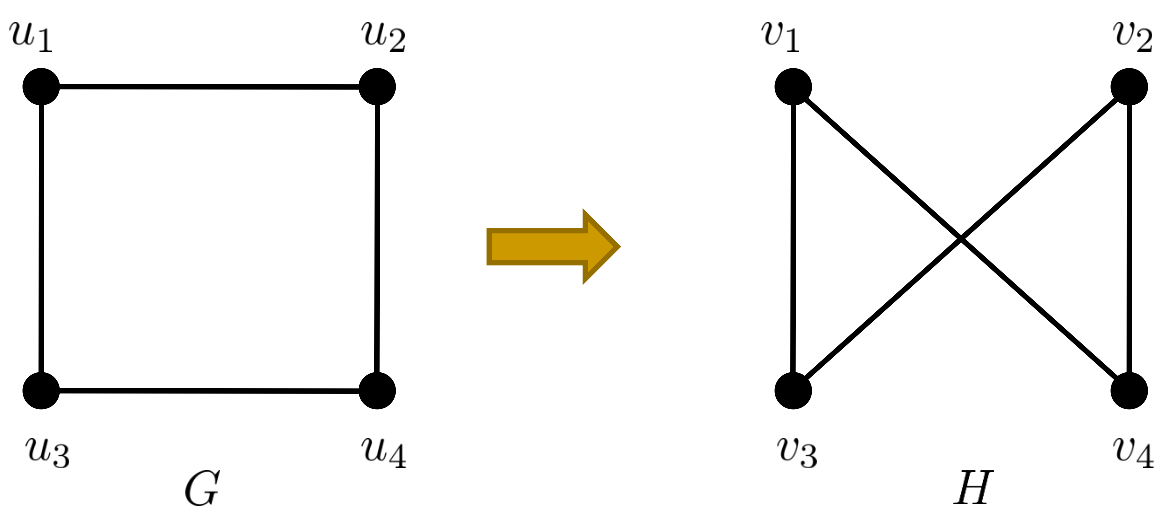


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & \boxed{1} & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

e_3, e_4, e_6 are incident with v_2

Isomorphism of Graph

- Simple graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is one-to-one and onto function f from $V_1 \Rightarrow V_2$ if a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 $\forall a, b \in V_1$
  Function f is called an *isomorphism*.

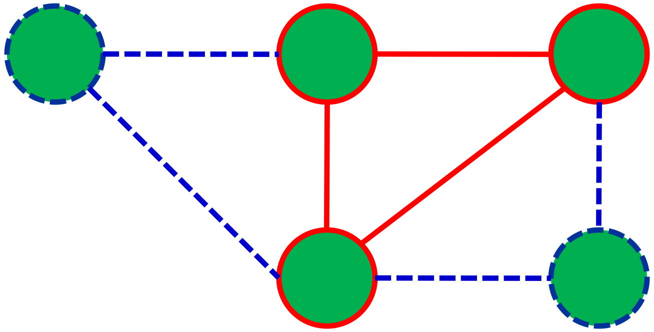


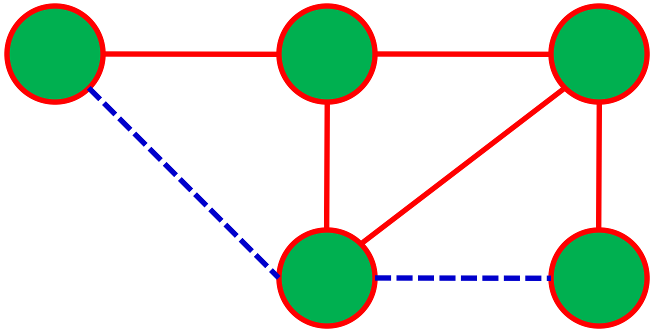
$$f(u_1) = v_1$$
$$f(u_2) = v_2$$
$$f(u_3) = v_3$$
$$f(u_4) = v_4$$

G and H are isomorphic.

Subgraph and Spanning Subgraph

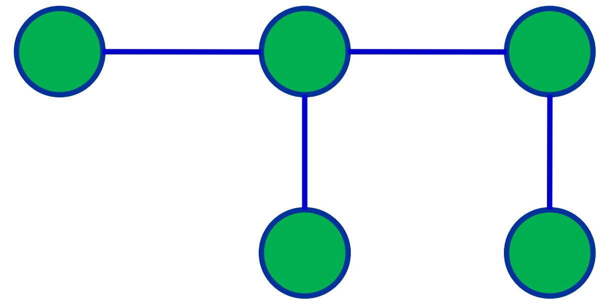
- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G.
  - The edges of S are a subset of the edges of G.
- A spanning subgraph of G is a subgraph that contains all the vertices of G.



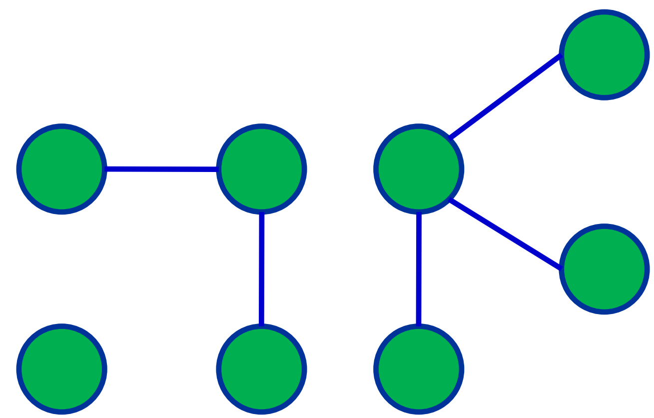


Trees and Forests

- A tree, is an undirected graph T such that
  - T is connected.
  - T has no cycles.



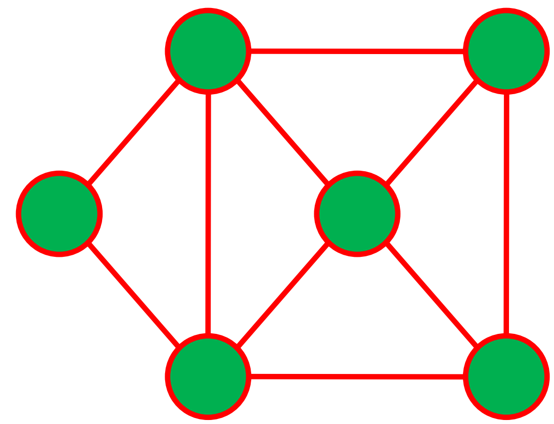
- A forest is an undirected graph without cycles



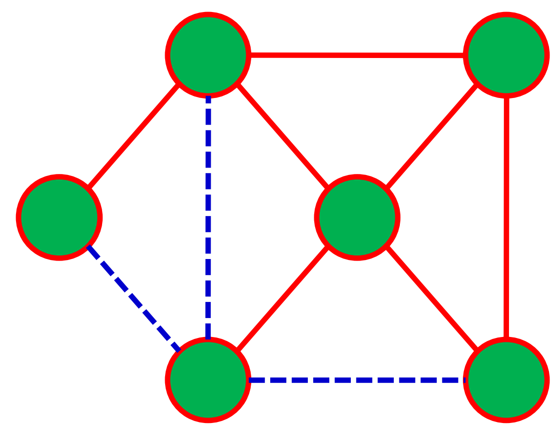
- The connected components of a forest are trees.

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree. Spanning tree is not unique unless the graph is a tree.
- A spanning forest of a graph is a spanning subgraph that is a forest.



Graph



Spanning tree

Walk, Trail, Path, Closed Walk,

- **Definition**

Let G be a graph, and let v and w be vertices in G.

A **walk from v to w** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for all $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from v to v** consists of the single vertex v.

A **trail from v to w** is a walk from v to w that does not contain a repeated edge.

A **path from v to w** is a trail that does not contain a repeated vertex.

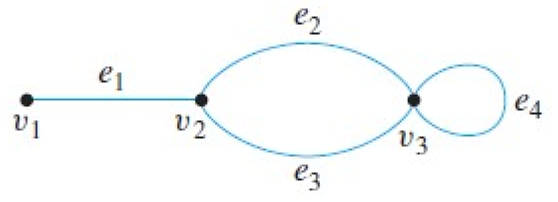
A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

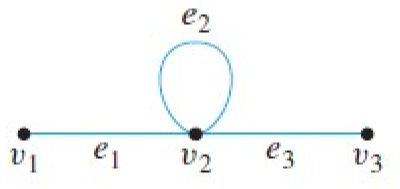
A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

Notion of Walk

a. In the graph below, the notation $e_1e_2e_4e_3$ refers unambiguously to the following walk: $v_1e_1v_2e_2v_3e_4v_3e_3v_2$. On the other hand, the notation e_1 is ambiguous if used to refer to a walk. It could mean either $v_1e_1v_2$ or $v_2e_1v_1$.



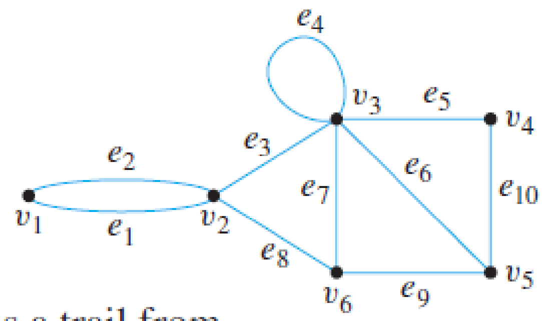
b. In the graph of part (a), the notation v_2v_3 is ambiguous if used to refer to a walk. It could mean $v_2e_2v_3$ or $v_2e_3v_3$. On the other hand, in the graph below, the notation $v_1v_2v_2v_3$ refers unambiguously to the walk $v_1e_1v_2e_2v_2e_3v_3$.



Note that if a graph G does not have any parallel edges, then any walk in G is uniquely determined by its sequence of vertices.

Walks, Trail Paths, and Circuits

In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.



a. $v_1 e_1 v_2 e_3 v_3 e_4 v_3 e_5 v_4$ b. $e_1 e_3 e_5 e_5 e_6$ c. $v_2 v_3 v_4 v_5 v_3 v_6 v_2$

d. $v_2 v_3 v_4 v_5 v_6 v_2$ e. $v_1 e_1 v_2 e_1 v_1$ f. v_1

a. This walk has a repeated vertex but does not have a repeated edge, so it is a trail from v_1 to v_4 but not a path.

b. This is just a walk from v_1 to v_5. It is not a trail because it has a repeated edge.

c. This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, so it is a circuit. Since the vertex v_3 is repeated in the middle, it is not a simple circuit.

d. This walk starts and ends at v_2, contains at least one edge, does not have a repeated edge, and does not have a repeated vertex. Thus it is a simple circuit.

e. This is just a closed walk starting and ending at v_1. It is not a circuit because edge e_1 is repeated.

f. The first vertex of this walk is the same as its last vertex, but it does not contain an edge, and so it is not a circuit. It is a closed walk from v_1 to v_1. (It is also a trail from v_1 to v_1.)

Euler Circuits

• Definition

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Theorem 10.2.2

If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Theorem 10.2.4

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has positive even degree.

Hamiltonian Circuits

- **Definition**

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

Proposition 10.2.6

If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

Exercises

- All exercises in chapter 10 of textbook [1], from page 675-678, page 703-706.

- [1] Rosen, K.H, *Discrete Mathematics and its Applications*, 7th ed., McGraw-Hill, Inc., 2011.