HANU
HANOI UNIVERSITY

FACULTY OF INFORMATION TECHNOLOGY
DEPARTMENT OF COMPUTER SCIENCE

Fall, 2023

DISCRETE MATHEMATIC

LEC-03:
Induction

Problem

$$\forall k \in N, P(k)$$

Example

$$\forall n \in N, A = P(1+r)^n$$

A **compounded amount (including interest)**
P **principle (the original amount)**
r **annual rate**
n **number of year**

More examples

$$\forall k \in N, \sum_{i=0}^{k} i = \frac{k(k+1)}{2}$$

$$\forall n \in N, n > 1, n! < n^n$$

Three steps

<u>Base Case</u>: Prove that $P(0)$ is true.

<u>Inductive Hypothesis</u>: Assume that $P(k)$ is true.

<u>Inductive Step</u>: Prove that $P(k+1)$ is true.

Example 1

$$\forall k \in N, \sum_{i=0}^{k} i = \frac{k(k+1)}{2}$$

Base Case: *P*(0)

$$\sum_{0}^{0} i = \frac{0(0+1)}{2}$$

Inductive Hypothesis: *P*(*k*)

$$\sum_{0}^{k} i = \frac{k(k+1)}{2}$$

Inductive Step: *P*(*k*+1)

$$\sum_{0}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

$$\sum_{0}^{k+1} i = \left(\sum_{0}^{k} i\right) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)(\frac{k}{2} + 1)$$

$$= \frac{(k+1)(k+2)}{2}$$

Example 2

$$\forall n \in N, n > 1, n! < n^n$$

Base Case: *P*(2)

$$2! < 2^2$$

Inductive Hypothesis: *P*(*k*)

$$k! < k^k$$

Inductive Step: *P*(*k*+1)

$$(k+1)! < (k+1)^{k+1}$$

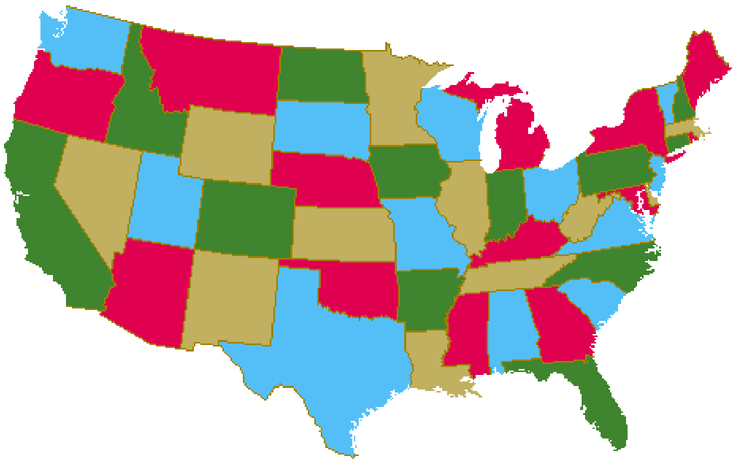
$$(k+1)! = (k+1)k!$$

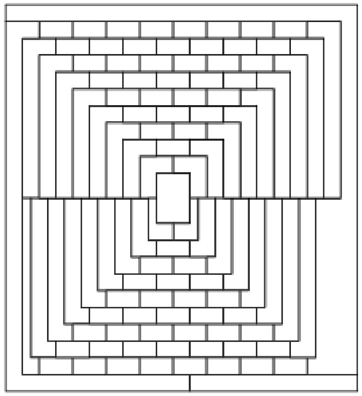
$$< (k+1)k^k$$

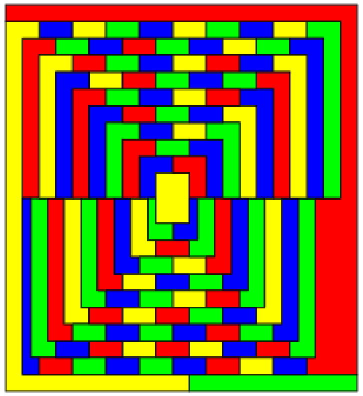
$$< (k+1)(k+1)^k$$

$$= (k+1)^{k+1}$$

Four color theorem



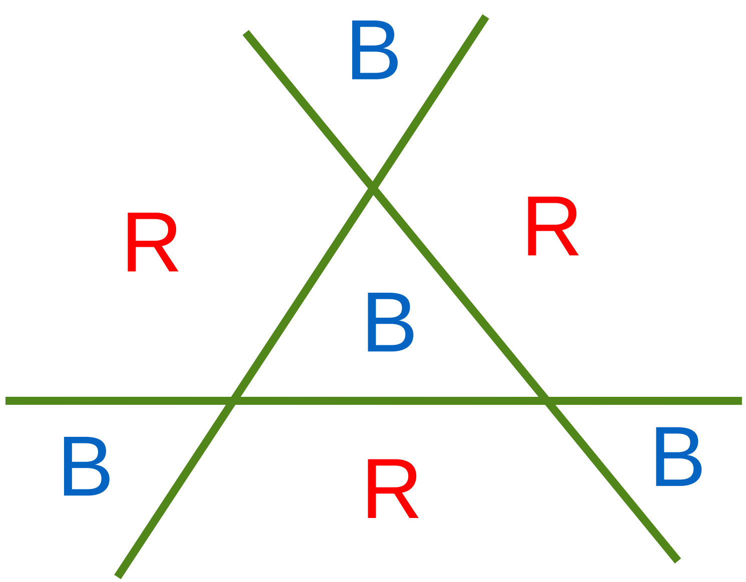




Any map can be colored with four colors such that any two adjacent countries must have different colors.
The problem was first posed in 1852. It is very difficult to prove.
It was not until 1976 that the theorem was finally proved (with the aid of a computer) by Appel and Haken.

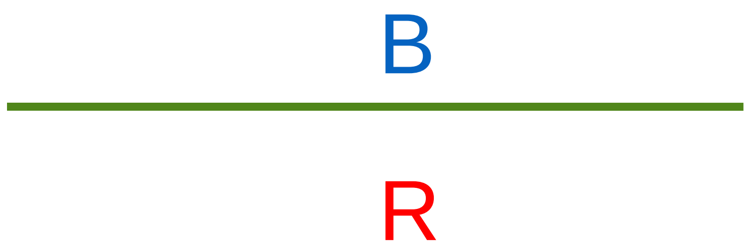
Two color theorem

Suppose a map is drawn using only lines that extend to infinity in both directions; two colors are sufficient to color the countries so that no pair of countries with a common border have the same color.



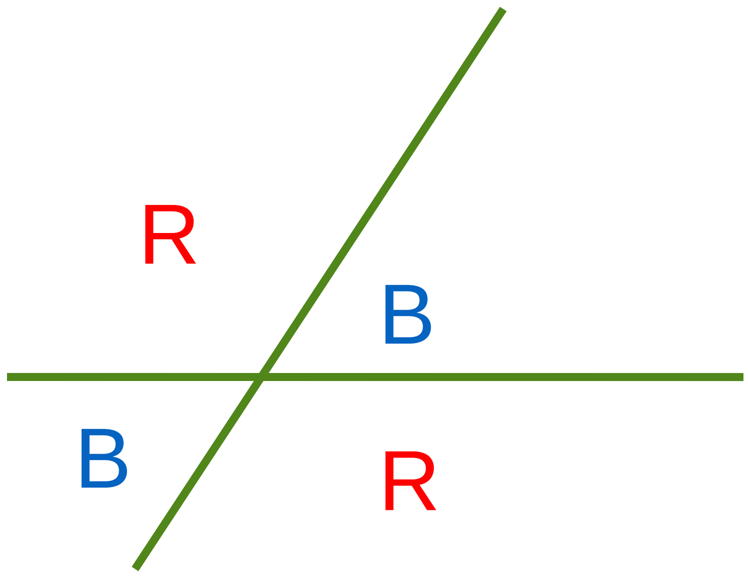
We divide the plane into regions by drawing straight lines.

Step 1: Base case



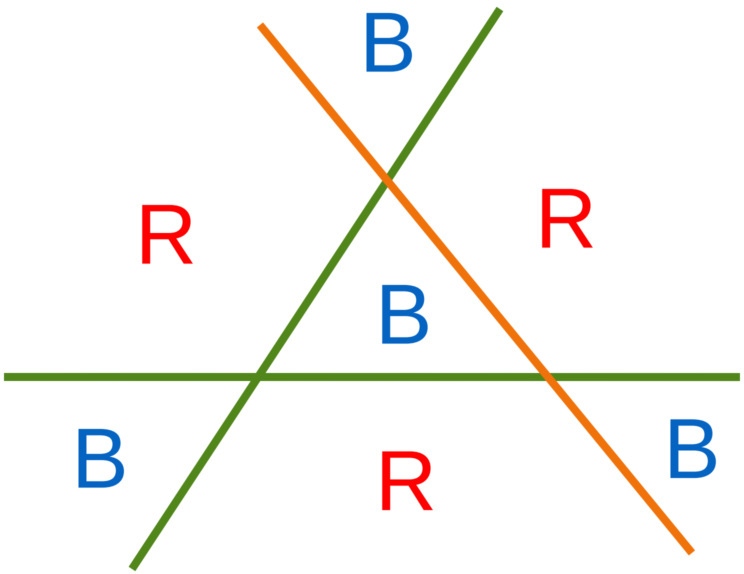
The plane is divided into 2 parts by 1 line

Step 2: Inductive hypothesis



Assume a map with n lines can be two-colored

Step 3: Inductive step



The (n+1)th line

Leave colors on one side of the (n+1)th line unchanged.
On the other side of the line, swap red <-> blue

Compound interest

Example

$$\forall n \in N, A = P(1+r)^n$$

A **compounded amount (including interest)**
P **principle (the original amount)**
r **annual rate**
n **number of year**

Base case: $P(0)$

$$A = P(1+r)^0 = P$$

In year 0 you receive no interest, the compounded amount is equal to the principle.

A compounded amount (including interest)
P principle (the original amount)
r annual rate

Inductive hypothesis: *P*(*k*)

$$A = P(1+r)^k$$

Assume this is the compounded amount after *k* year(s).

A compounded amount (including interest)
P principle (the original amount)
r annual rate

Inductive step: *P*(*k*+1)

$$A = P(1+r)^{k+1}$$

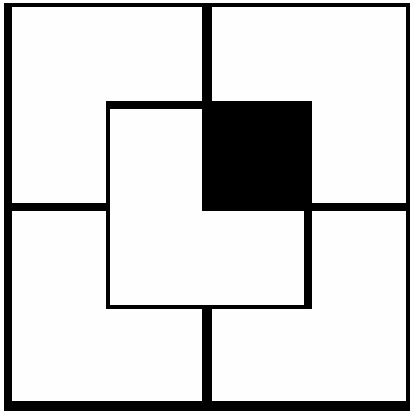
We must show this is the compounded amount after *k*+1 year(s).

$$A = P(1+r)^k + P(1+r)^k r = P(1+r)^{k+1}$$

A compounded amount (including interest)
P principle (the original amount)
r annual rate

L-shaped tiles

Example

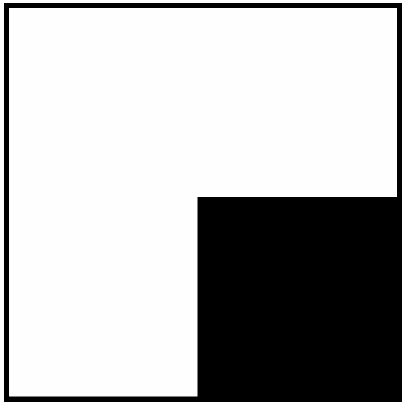


L-shaped tiles: 2 x 2 square tile with a missing 1 x 1 square.

We want to tile a 2^n x 2^n square with a missing 1 x 1 square in the middle.

Base case: $P(1)$

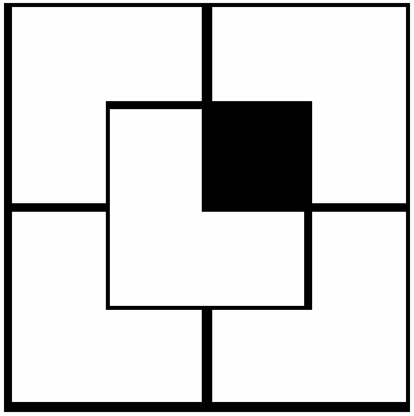
Example



A 2 x 2 square can be tiled with L-shaped tiles with a missing 1 x 1 square in the middle.

Inductive hypothesis: $P(n)$

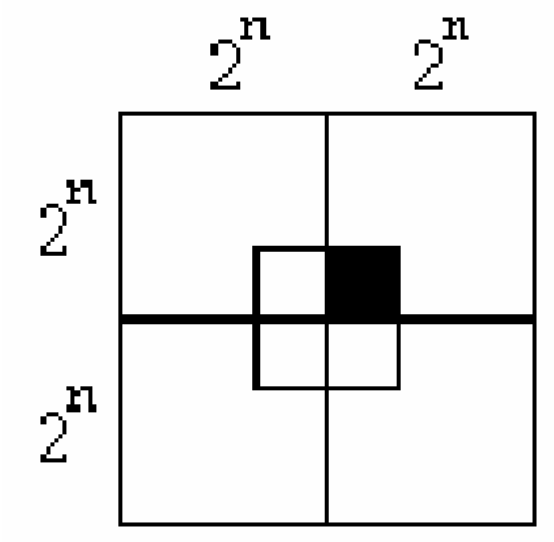
Example

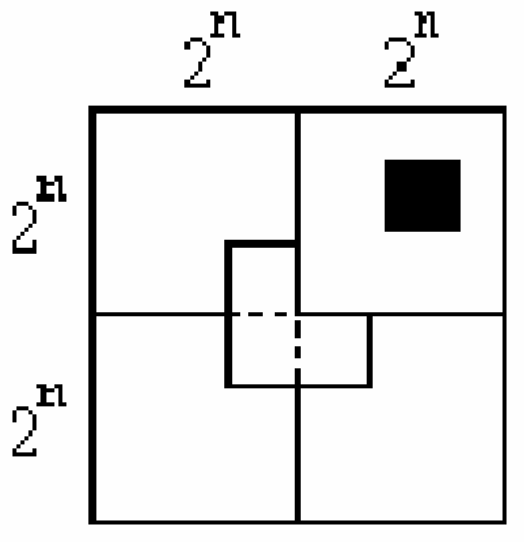


Assume we can tile a 2^n x 2^n square with a missing 1 x 1 square in the middle.

Inductive step: $P(n+1)$

Example





A $2^{n+1} \times 2^{n+1}$ square can be broken up into 4 smaller squares of size $2^n \times 2^n$.

We can tile the square with the hole being anywhere.

Strong induction

Inductive hypothesis: Instead of just assuming $P(k)$ is true, we assume a stronger statement that $P(0)$, $P(1)$, …, and $P(k)$ are all true.

Example

Every natural number $n > 1$ can be written as a product of primes.

Base Case: $P(2)$

2 is a prime number.

Inductive Hypothesis: $P(2), P(3), \ldots,$ and $P(k)$

Every natural number $2 <= n <= k$ can be written as a product of primes.

Inductive Step: $P(k+1)$

$k + 1$ can be written as a product of primes.
Case 1: $k + 1$ is a prime
Case 2: $k + 1 = xy$, $1 < x,y < k$

Example

Any integers from 8 upwards can be composed from 3 and 5.

Base Case: $P(8)$

$8 = 3 + 5$

Inductive Hypothesis: $P(8),\ldots,$and $P(k)$

Every natural number $8 \leq n \leq k$ can be composed from 3 and 5.

Inductive Step: $P(k+1)$

$k+1 = (k-2) + 3$
$P(9)$, $P(10)$

Strong induction vs. simple induction

$\forall k \in N, P(k)$

$\forall k \in N, Q(k) = P(0) \wedge P(1) \wedge \cdots P(k)$

Base Case:

$P(0)$

$Q(0) = P(0)$

Inductive Hypothesis:

$P(0) \wedge P(1) \wedge \cdots \wedge P(k)$

$Q(k) = P(0) \wedge P(1) \wedge \cdots \wedge P(k)$

Inductive Step:

$$P(0) \wedge P(1) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)$$

$$P(0) \wedge P(1) \wedge \cdots \wedge P(k) \Rightarrow P(0) \wedge P(1) \cdots \wedge P(k) \wedge P(k+1)$$

Induction and Recursion

$F(0) = 0$, and $F(1) = 1$

For $n \geq 2$, $F(n) = F(n-1) + F(n-2)$

```
int F(int n)
{
    if (n == 0) return 1;
    if (n == 1) return 1;
        return F(n-1) + F(n-2);
}
```

Exercises

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

Exercises

$$1.2 + 2.5 + 3.8 + \ldots + n(3n-1) = n^2(n+1)$$

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1).(2n+1)} = \frac{n}{2n+1}$$

Homework

Problem Set 3