The Sound of Numbers

An exploration of the relationship between music and mathematics

Submitted in partial completion of a Capstone Thesis Project

St. Anne's-Belfield School

Charlottesville, VA

by

Brehanu Bugg

William Nichols

Mentor

Zach Minster

Second Reader

March 25th, 2022

Acknowledgements

To Mr. William Nichols for your insightful feedback and always pushing me to do my best. I have become a better scholar because of you. To Mr. Zach Minster for being my mentor since middle school and fostering my love of mathematics and computer science. This study would not have been possible without you. To Mom and Dad for always supporting my endeavors, both academic and artistic. I could not have done this study without you two.

This thesis is dedicated to you all.

Contents

1 Introduction

1.1 Abstract

Mathematics and music have much in common—such as frequency, rhythm, and chords—but it's hard to see how they overlap. This Capstone will demonstrate how mathematics influences musical composition using set theory notation and its principles. Set theory is a branch of mathematics that deals with non-sequential and non-repeating lists of numbers, and applying transformations to those lists. We will explore how manipulating sets of notes affects the resulting sound by creating axioms (or rules that I define as a baseline based on my research) for set compositions with my findings. The goal of this study is to write a composition using only numbers and logic, not through listening and intuition. The culminating final project will be notes on translating sets to music as well as a three-to-five page piano solo, created solely through set theory and the principles derived from its relationship to music theory. This model for translating mathematical sets to compositions will provide a deeper understanding of mathematics' relationship to music. This study hopes to interest mathematics-centric students in composition and interest music theory-centric students in set theory.

1.2 Motivations

In Honors Integrated Mathematics 2 and 3, we designed roller coasters in Demos—a popular online graphing calculator—using what we learned in our graphing unit. We used linear, quadratic, polynomial, exponential, and trigonometric equations to influence specific portions of the coaster to understand the different equation types. Visualizing graphical equations in this artistic manner helped us to understand the different types of equations and their components better than simply reapplying our knowledge on a test. We were struck by how we combined two different fields—visual art and mathematics—to create a novel piece of art that deepened our understanding of graphing. This Capstone will offer this same perspective shift, but for music instead of visual art. By approaching musical composition logically, mathematics-centric people will be able to exercise their artistic expression even with zero musical background. Likewise, music-centric people will be able to see notes not as abstract ideas, but numbers that can be manipulated with simple operations and equations. Both types of people will learn more about the other subject, and find a new appreciation for their similarities.

1.3 Structure of Thesis

This thesis is structured similar to an undergraduate thesis. The next section will define all of the key terms and definitions that readers will need to know—some of the terms will include examples. We highly recommend returning back to this page whenever you encounter an unfamiliar term.

Next, we will cover miniature sets and discuss our compositions from quarters I and II. These sections will act more like textbook chapters, explaining our thinking, teaching set and music theory concepts, and providing concrete examples with set theory notation.

Finally, we conclude by walking through our final composition, *Ordinal Garden*. Here, we will discuss theme exploration, process of composition, and the analysis of our compositional choices.

Since compositions are meant to be heard, we will provide the audio recordings to each composition covered. We will also include the sheet music. (Add instructions here later.)

2 Key Terms and Definitions

Definition (Music Theory). *The study of relations between notes, rhythm, and structure through a musical language designed to evoke feeling and express meaning.*

Definition (Set Theory). *The study of collections of objects, usually numbers but not always. This field is often paired with predicate logic, a mathematical field that describes complicated relationships and inferences through symbols.*

Definition (Set). *A non-repeating, non-sequential collection of objects, usually numbers. Mathematicians apply different operations on these sets or draw conclusions about them using predicate logic.*

Definition (Pitch-Class Set). *A music-focused variation of a mathematical set that allows repeating objects, usually notes, and is sequential. This way, mathematicians and music theorists can model sheet music where they could not with a regular set.*

For brevity, this study will use "set" as shorthand for "pitch-class set."

Definition (Function). *A unique input-output mapping. A function has a name and, given inputs, returns a unique output.*

ex. The given function f to a graph a parabola is shown below.

$$f(x) = x^2$$

The function has a name, f, and an input, x, which is then modified on the right-hand side of the equal sign. Now defined, we can substitute any number for x to get the result.

$$f(5) = 5^2 = 25$$

Definition (Pitch). *How high or low a sound is based on some frequency.*

Definition (Frequency). *The speed of the vibration, which therefore influences the pitch.*

Definition (Note). *A letter assigned to a pitch for easier identification.*

In western music, we have twelve unique notes: A, A♯/B♭, B, C, C♯/D♭, D, D♯/E♭, E, F, F♯/G♭, G, and G♯/A♭. A—specifically the middle A on a standard piano—has a frequency of 440.0 Hz. However, instead of calling it by its numerical frequency, we assign it the single letter, A.

Definition (Octave). *A note twelve semitones above itself; twice the frequency.*

As noted above, middle A has a frequency of 440.0 Hz. The A above it will have the frequency 880.0 Hz. Likewise, the A below it will have the frequency 220.0 Hz. The higher the frequency, the higher the pitch, and vice-versa. However, it is still the same note of A, just an octave apart.

Definition (Interval)**.** *The distance between two pitches.*

$$I(a,b) = \begin{cases} b - a, & \text{if } a < b \\ 12 - (a - b), & \text{if } a > b \end{cases}$$

This function takes in two notes, a and b, and evaluates one of two equations. Note: a and b are the numeric representation of the notes in question. Learn more in **4.2.1**.

The function I is an example of a *conditional function* since the result depends on some logical expression being true. In this example, either note a is greater than note b, or vice-versa. Depending on which statement is true, you compute the corresponding equation.

This function has to be conditional since sometimes notes “wrap around.” For example, the distance from A♯/B♭to D is not 8, but rather 4. Ordering matters.

Definition (Half-Step or Semitone)**.** *An interval of distance 1. In other words,* $I(a,b) = 1$.

Definition (Whole Step or Tone)**.** *An interval of distance 2. In other words,* $I(a,b) = 2$.

Definition (Structural Set)**.** *A function, given a note or set of notes, that produces a new set according to pre-defined intervals.*

Definition (Stepping Numbers)**.** *Numbers—representing notes—that go between adjacent notes if the interval surpasses four.*

Our brain generally likes listening to music that has a smooth melody. In other words, the intervals between adjacent notes are not too far apart. However, larger intervals can sometimes be unavoidable in compositions created via a mathematical sequence.

Suppose we have the following set that is a subset of a larger piece.

$$\{0,\ 7,\ 2\}$$

Here, we jump from C to G to D. Since the interval from 0 to 7 and the interval from 7 to 2 are larger than 4, we need stepping notes in between to smooth out the melody.

We first find the average between 0 and 7, which is 3.5. We will round down to 3. This works since $I(0, 3) < 4$ and $I(3, 7) < 4$. The average from 2 and 7 is 4.5, which we will round down to 4. $I(7, 4) < 4$ and $I(4, 2) < 4$, meaning that this works.

The new subset with the stepping notes is as shown below.

$$\{\, 0, 3,\ 7,\ 4,\ 2 \,\}$$

3 Literature Review

Mathematics and music have an inherently beautiful, yet not always immediately discernible, relationship with each other. Pythagoras noticed their similarities in his theory *musica universalis*, or "music of the spheres." When set theory—a branch of mathematics that deals with lists of numbers—is applied to music theory—the study of explaining and composing music—numbers and sound blend together in harmony. In this study, we seek to understand how set theory can be used to influence musical notation and as a compositional tool.

Set theory is not a standalone subject, but rather draws in concepts from many other areas of mathematics. In *Discrete Mathematics with Applications*, Susanna Epp covers set theory as well as related topics under the discrete mathematics umbrella, such as logic, relations, and counting and probability. In the preface, Epp has a "List of Symbols" section, which briefly covers potential gaps in our knowledge: a mathematical glossary. From this source, she teaches the set of all real, integer, rational, and natural numbers, and explains how the numbers in each set relate to each other and how they are referenced to limit scope.

When making advances in any field of study, it is essential to be well versed in the fundamentals before moving on to the higher-level, abstract material. In *Sets, Invariance and Partitions*, Daniel Starr approaches sets from a macro to micro level, by defining what a set is and basic operations one can perform on a set, before moving on to more abstract concepts such as the hexachord theorem and binary-notated sets. Starr reviews essential terms like transposition, inversion, and complements, while also introducing new concepts such as modular arithmetic—we will integrate these terms and concepts into our work. The author rapidly goes through the fundamentals to explore the less applicable concepts, such as using binary to find recurring set-classes. This source primarily focuses on non-applicable theory, and therefore there are no concrete examples to reference.

Understanding set theory notation allows mathematicians and composers to express complex ideas through the use of symbols. In *The Joy of Sets*, Keith Devlin explores many facets of set theory including Zermelo-Fraenkel axioms, ordinal and cardinal numbers, and pure set theory. In the opening chapter, Devlin teaches the basic symbols set theory is founded on, then moves on to relations, functions, and ordinals. This text taught us the definition of what a set is and some of its basic characteristics of notation, cardinality, and relations. Even with the example problems, the concepts quickly become very high-level, touching on Borel sets, trees, and other ideas not relevant to answering this study's essential question. That—coupled with the fact that this text was specifically designed for an undergraduate mathematics course—made this text only somewhat useful.

Transforming notes into numbers allows composers to see the three pillars of variety from a macro and micro perspective: symmetry, proportion, and balance. In *Pitch-Class Set Theory in Music and Mathematics*, Michael James McNeilis explains in detail the mathematical and logical reasoning behind his six original compositions. McNeilis explores concepts such as mutation, scales, and instrumentation through different mathematical ideas like the golden ratio, pi, and Markov chains. The author's detailed explanation of his approach to composing these pieces helps to relate the mathematical and musical components with the creative component. Additionally, his thought process clearly demonstrates mastery and provides a template that our study could mirror. This source does not have proper set theory notation or mathematical equations of any kind, and goes not explore how one can represent higher-level music theory numerically.

Demystifying the field of pitch-class set theory—which is filled with complicated terminology to describe very specific but simple behaviors—is paramount as to provide new researchers a foundation for new findings. In *A Primer for Atonal Set Theory*, Joseph Straus establishes a basic foundation by describing all of the main key terms with examples. He defines sixteen terms in depth, the most relevant to our study being pitch class, pitch intervals, pitch-class sets, set class, and complement relations. Straus' most important idea was that this field is not a rigid theory, but a collection of ideas to be applied: "It is best understood not as a rigidly prescribed practice, but as an array of flexible tools for discovering and interpreting musical relationships" (2). While comprehensive in definitions, this source lacks any set theory or mathematical notation.

Clear, concise, and intelligible communication is key to understanding complex ideas and furthering research and development in any particular field. In *Logic, Set Theory, Music Theory*, John Rahn stresses the importance of defining every musical idea in terms of set theory: "Any music theory—and theory of any sort, in fact—can be constructed as an extension of an axiomatic set theory" (115). The author begins by defining how to identify if an element "x" is a note or a rest. Instead of using set theory symbolic notation, Rahn uses longform English to clearly articulate what is happening. He then goes on to define time-adjacency, pitch-adjacency, and neighbor notes. The way Rahn approaches definitions—first broadly, then introducing sub-definitions to hone in on the specific details—is the model we will use for our written thesis. This process, coupled with longform English notation, helps to map out Rahn's thought process as he explains these complex concepts. This article does not use symbolic set theory notation.

Understanding what a set class constitutes is crucial to applying set and musical theory topics to new compositions. In *An Intervallic Definition of Set Class*, Christopher Hasty declares that the purpose of set classes is to show the relationships between its constituents (elements, or, in this case, pitches) to other sets. Hasty explains that we compare pitches by their interval: how far away two notes are. We can use intervals to equate sets in four different ways: J-equivalence, K-equivalence, L-equivalence, and M-equivalence. Each of these are defined using set theory logic, but in English notation. Shifting the focus away from equivalence, the author explores the applications of transposition, inversion, and other musical and mathematical operations one can apply to sets. These operations are one-to-one mappings—meaning that they are functions that can be defined mathematically. When translating notes on a staff to a pitch-class set, Hasty looks at the notes on the staff vertically (across staffs) rather than horizontally. His focus on operations and intervals will provide this study with the tools to analyze and manipulate pitch-sets. This article lacks symbolic notation, instead choosing to dive into higher-level matrix mathematics that does not have useful applications for this study. Additionally, it is at times confusing to follow Hasty's thought process, as every time he defines what a set class is, he disproves it later on.

In analyzing Iannis Xenakis' solo piano composition *Herma*, Ronald Squibbs applies set and probability theory to deconstruct the rhythmic structure and harmonic melody. Squibbs argues that other researchers analyzing *Herma* overlook rhythm in favor of pitch, leaving them blind to the ways the temporal structure affects the music (8). He describes the referential set $\mathbb{R}$ as all of the pitches on a standard 88-key piano keyboard, and subsets A, B, and C to describe smaller themes that culminate in set F. Squibbs describes thematic development as a combination of primary sets and their complements breaking down to smaller ideas called motives. Continuing with his analysis of pitch, the author defines stochastic composition as the application of probability theory to the process of musical composition. Independent random probability events change intervals through exponential, linear, and uniform distributions. The temporal analysis of *Herma* is what *Musical Composition as Applied Mathematics: Set Theory and Probability in Iannis Xenakis's* Herma emphasizes most. Squibbs claims that many other researchers analyze the pitches in this composition, but not the temporal structure. As we create our own compositions, we will keep in mind the way Squibb ties together probability theory, thematic development, stochastic composition, and intervals to describe the structure of any piece and how these choices affect the overall sound and intention of said pieces. He also places special emphasis on the fact that not all of *Herma* was automated: there existed remnants of human involvement. This article does not generalize its findings to apply to other works. Furthermore, it makes no use of mathematical notation whether that be set theory, group-theory, or probability-theory in nature.

In *Musical Logic. A Contribution to the Theory of Music*, Hugibert Ries breaks down his definition of musical logic into two types: Harmonic Logic, and Metric Logic. Harmonic Logic deals with different types of cadences (and their makeup of thetic, antithetic, and synthetic elements), secondary harmonies, expanded cadences, modulation, and forbidden fifths. Metric Logic deals with measure, phrase, and period; accents; deceptive cadences; and rhythm. In essence, Ries highlights that these musical rules, which at times appear random, are designed to make logical sense to our brain and ears. Breaking these rules should only happen if it makes sense for the composition at that time. We now have an understanding for when to break the rules in a way that makes sense for our compositions. This article is very comprehensive, but it either does not define some terms or overcomplicates them, making reading this source laborious and confusing.

Milton Babbitt was a well-known composer and mathematician who heavily influenced twelve-tone theory and inspired future composers like Allen Forte. In *A MUSICAL SET THEORY*, Howard Wilcox and Pozzi Escot analyzed Babbitt's piece *Wierdersehen*, specifically the song cycle *Du*, to break apart the composer's intentions from a musical standpoint. They argue that knowing what modifications one can make to a sequence of notes is imperative to drastically changing the atmosphere and tone of a composition. Wilcox and Escot focused on these different modifications: transposition, raising notes up or down by a specific number of semitones; retrograde, flipping the order of notes; inversion, flipping notes vertically on the staff; and retrograde inversion, a combination of regular retrograde and regular inversion. To provide familiar variety in our composition, we will make use of these modifications in the form of mathematical operations. They performed all of these operations on sample hexachords—chords with six notes—before analyzing Babbitt's use in his composition. Their article revealed that even with one chord or one sequence of notes, you can create at the bare minimum 4 new chords or sequences of notes easily. The diagrams are hard to read and interpret. Additionally, this article lacked definitions for important terms, which made understanding it difficult.

In *Scale Theory, Serial Theory and Voice Leading*, Dmitri Tymoczko describes efficient voice leading to mean that "notes should be distributed among individual musical voices so that no voice moves very far as harmonies change" (1). He argues that to create the most pleasurable listening experience, composers should strive to make voices as efficient as possible, which can be modeled using set theory (1). Tymoczko begins by establishing the proper set notation, distinguishing between curly brace notation—for when the order of items in a set does not matter—and parenthetical notation—for when the order of items in a set do matter. From there, we are introduced to the arrow notation to represent progression from chord to chord. For our study, we will use Tymoczoko's detailed description of transposition, inversion, and set ordering notation for modifying pitch-class sets and establishing standard symbolic notation. Nevertheless, his other dense definitions were difficult to understand. Furthermore, he quickly goes into high-level matrix mathematics, rendering over half of this article too advanced for this study.

Set theory helps to prove very abstract questions such as inclusion relation: if set A is a subset of set B if and only if every element of A is in set B. In this context, “element” is a pitch. In *A Theory of Set-Complexes for Music*, Allen Forte begins by defining commonplace terms such as ordered versus unordered pitch-sets, interval-sets, and interval vectors. He then returns to his essential question about inclusion relation, and begins to prove it using the terms he defined before and set theory. The author’s mathematical proofs are in longform English notation, making them easy to read; we will do the same in our written thesis. This article is very concise—at times, too concise—and contains very short sentences. This helps to accelerate the reading pace; however, this article is meant for an undergraduate mathematics course as it reads similarly to *The Joy of Sets*, contains complex notation with minimal explanation, and hardly makes use of concrete examples.

Allen Forte’s Contribution to Music Theory provides a brief introduction to Allen Forte, often referred to as the father of pitch-class set theory, who laid the foundation for relating musical sets and analyzing them. Joseph Straus explains that Forte had four main focuses in his musical-related studies: pitch-class set theory, Schenkerian theory, theories of motivic relations, and applying Shenkerian theory to American songs (3). In regards to pitch-class set theory, Forte made significant contributions to five subareas: similarity relations (which has resemblance to *An Intervallic Definition of Set Class*), set-class complexes, rhythm, linear organization, and octatonicism. This source helpfully enumerates Forte’s main points of focus, which will be helpful in providing historical context to those attending our presentation that are new to musical set theory.

At this stage, we now have a solid understanding of sets, different ways to represent and mutate pitch-class sets, and how to avoid sounding smarter than we are for our formal thesis. Our first round of mini-sets, which we created before this literature review, lacked focus. Now, we can begin to incorporate symbolic notation and ideas of transposition, retrograde, inversion, and retrograde inversion into our second quarter mini-sets.

4 Miniature Sets and Exploration

4.1 Miniature Sets Explained

This study ends with a final composition. To accomplish this goal required application of research in an engaging way. We developed minature sets, often abbreviated as mini-sets.

For quarters I and II, we composed short pieces that applied what we were learning at that time. The only requirements were at least three mini-sets per quarter, with each set having at least three measures.

Quarter I mini-sets lacked focus and a clear direction. We composed these prior to the literature review, where we examined the research in pitch-class set theory and learned what we can do with it. The quarter II mini-sets are targeted, use specific terminology, and, at times, custom notation.

In quarter III, we began the final composition. We used a similar process to the quarters I and II mini-sets. But, the resulting complexity is well above the previous quarter's compositions due to the numerous instruments and attention to theme.

4.2 Quarter I.

Since these were the first mini-sets, we experimented with both translating notes to numbers and basic set manipulation using the well-known children's nursery rhyme, *Twinkle, Twinkle Little Star*.

4.2.1 Set I. — *Twinkle, Twinkle Little Star*

In order to compose more complex pieces, we had to start with the most basic question: How can we translate notes to numbers? Sets in mathematics are notated with curly braces. An empty set has a pair of curly braces with nothing inside of them.

$$\{\ \}$$

Pitch-class set theorists still needed a way to represent alphabetical characters into numbers. In western music, there are twelve unique pitches: A, A♯/B♭, B, C, C♯/D♭, D, D♯/E♭, E, F, F♯/G♭, G, and G♯/A♭. Changing the frequency of a pitch changes how high or low it sounds. C2 is a lower sounding C than C5. They are the same pitch, just different frequencies. Because of this, we only need twelve numbers to represent these twelve pitches. The frequency does not matter to our study. Below is the mapping of notes to numbers that is the standard in pitch-class set theory.

0	1	2	3	4	5	6	7	8	9	T	E
C	C#/Db	D	D#/Eb	E	F	F#/Gb	G	G#/Ab	A	A#/Bb	B

For the sake of brevity, we only took the first two lines of the nursey rhyme: "Twinkle, twinkle little star, how I wonder what you are." The pitch-class set below represents these lyrics.

$$T_{\text{original}} \rightarrow \{\,0\,,0\,,7\,,7\,,9\,,9\,,7\,,5\,,5\,,4\,,4\,,2\,,2\,,0\,\}$$

Note: we are storing the set—everything in-between the curly braces—in the variable T as shown by the arrow notation. We use a subscript of original since the following mini-sets will build off of this one.

We can see that most of the numbers are repeated twice with the exception of 7, which is note G. This pitch-class set's cardinality—a mathematician's way of saying "length of"—is 14, since there are fourteen numbers in it.

We have found a way to represent music as a pitch-class set. Now, we can manipulate it with mathematics.

4.2.2 Set II. — (Dark) *Twinkle, Twinkle Little Star*

From the previous mini-set, we noticed that notes are repeated twice. To experiment, we replaced the second occurance of each repeating note pair with the average of itself to the next note.

For example, take the first three numbers from the previous mini-set like so.

$$\{\,0\,,0\,,7\,\}$$

We replace the second 0 with the average of itself and 7. This gives us 3.5, but we will round down. Repeating this process throughout the entire set yields the following pitch-class set.

$$T_{\text{dark}} \rightarrow \{\,0\,,3\,,7\,,8\,,9\,,8\,,7\,,5\,,4\,,4\,,3\,,2\,,1\,,0\,\}$$

When played, each pitch sounds wrong. We lost the double note property of the original song, but this "averaged" note illicits this limbo effect: as a listener, you know its wrong, but the notes feel almost right. This creates some bearable dissonance, and provides a dark turn on the classic nursery rhyme.

4.2.3 Set III. — (Motif) ***Twinkle, Twinkle Little Star***

The first four notes of *Twinkle, Twinkle Little Star* are instantly recognizable. Those two different notes repeated twice with that specific interval has a strong connection in the brain to this particular song. For this mini-set, we wanted to expand this four-note phrase to create a spin on this nursery rhyme.

$$T_{\text{motif}} \to \{\, 0\,,0\,,7\,,7\,,0\,,4\,,7\,,4\,,0\,,3\,,7\,,3\,,0 \,\}$$

This mini-set is a mix of the previous two. The first four notes are from T_{original}. The last four notes up to the last 0—or C—are from T_{dark}, except instead of going up to 8—or G♯–we went down to 3—or D♯. The middle notes are similar to the first three notes of T_{dark} with the following exceptions.

- Instead of rounding down to 3—or G♯—we rounded up to 4—or E.
- Instead of rounding up to 8—or E♯—we rounded down to 4—or E.

Some of these creative choices were purely arbitrary. However, this mini-set was a mix of the previous two's ideologies. The song is still recognizable as *Twinkle, Twinkle Little Star* due to the four-note motif at the beginning, yet it is still off-putting because our expectations are not met due to the changed notes.

4.3 Quarter II.

4.3.1 Collatz Conjecture

The Collatz conjecture, named after Lothar Collatz in 1937, is one of the many unsolved problems in mathematics, and one of the most famous. It has other names such as the Ulam conjecture, Kakutani's problem, Thwaites conjecture, Hasse's algorithm, the Syracuse problem and the $3n+1$ problem. The name is irrelevant; all of these conjectures/algorithms/problems describe the same phenomenon.

Say that you start with any positive integer (whole number) which we call n. There are two rules you apply to this number. If n is even, divide n by 2. Otherwise, multiply n by 3 and add 1. The mathematical function describing this is defined as such.

$$C(n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \mod 2 \\ 3n+1, & \text{if } n \equiv 1 \mod 2 \end{cases}$$

The left-hand statement before the "if" tells you what to do if the right-hand statement after the "if" is true. The word "mod"—short for *modulus*—is a mathematical operation that tells you the remainder after division. Here, we are using "mod 2" to test if a number is even or odd. If "n mod 2" is 0, then the number is even, and therefore follows the even rule of dividing itself by 2. Otherwise, n is odd and therefore follows the odd rule of $3n + 1$.

The Collatz conjecture becomes more interesting when the output is then fed back into the function as input. Here are two examples of the sequence playing out with different starting numbers.

$$n = 12 \rightarrow \{\, 12\,, 6\,, 3\,, 10\,, 5\,, 16\,, 8\,, 4\,, 2\,, 1\,\}$$
$$n = 19 \rightarrow \{\, 19\,, 58\,, 29\,, 88\,, 44\,, 22\,, 11\,, 34\,, 17\,, 52\,, 26\,, 13\,, 40\,, 20\,, 10\,, 5\,, 16\,, 8\,, 4\,, 2\,, 1\,\}$$

Starting with an even n of 12, we divide it by 2. Now, our n is 6. What if we put this new n back in to $C(n)$? Well, 6 is even, so we divide it by 2 again. Now, our n is 3. 3 is an odd number, so we apply the odd rule of $3n + 1$ on it. Now, our n is 10. 10 is even, so the output is 5. 5 is odd, so we multiply it by 3 and add 1 to get a new n of 16. 16 is even, so we divide it by 2. 8 is even, so we divide it by 2. 4 is even, so we divide it by 2. 2 is even, so we divide it by 2. 1 is odd, so we multiply it by 3 and add 1. However, when we do this, we end up back at 4. This creates a loop, so we prematurely "end" the sequence on 1.

It is important to note that this function does not return a set, just a single number. We repeated this sequence and constructed our own set using the results. We can use set builder notation to change this function into a set as so.

$$\{\, C(n) \mid n \in \mathbb{Z}^+ \,\}$$

Definition (Set Builder Notation). *A shorthand for creating a set that follows some mathematical process or algorithm. This is done to prevent writing out the entire set, which can be impossible in some cases.*

Set builder notation follows the structure as shown below.

$$\{\, \text{what to do} \mid \text{under what conditions to do so} \,\}$$

The $C(n)$ reference the Collatz function defined in equation I. The bar means only do what is on the left of it if the conditions on the right of the bar are true. The "e" symbol, $\in$, means "exist in." Additionally, the $\mathbb{Z}$ represents the set of all integers (whole numbers). The superscript + symbol means only positive numbers. So $\mathbb{Z}^+$ means only positive integers. The contents to the right of the bar means n values that exist in the positive integers. In other words, only run the Collatz function on n if and only if n is a positive integer.

We constructed a computer program to create a sequence given any starting number n. It then takes that sequence and converts it into an audio file with the same note-to-number mapping as shown in the first quarter mini-sets. For this mini-set, we chose $n = 41$ due to the length of the piece and the interesting melody and phrases.

Listening to the piece, we noticed the repeating 10 (A♯/B♭) 11 (B) phrase which was present in other pieces with different starting numbers. We also noticed that for each piece we played, the song always ended with the sequence 8 (G♯), 4 (E), 2 (D), 1 (C♯/D♭), which was expected since each set ended with that sequence of numbers. Even out of an extremely structured composition, we could still hear numerous musical ideas.

4.3.2 Sunbeam Sonata

The first movement of Ludwig van Beethoven's famous *Moonlight Sonata* is well-known for evoking feelings of unease. Although the movement only uses three notes, they way they repeat make the listener apprehensive.

For this mini-set, we created the opposite of *Moonlight Sonata*: a *Sunbeam Sonata*. Our piece still has the same rhythmic structure as the original, but with different notes. We used four transformations for this composition.

Definition (Transposition)**.** *The process of increasing or decreasing a note by n semitones where n is an integer.*

We can define transposition mathematically as so.

$$T(a, n) = a + n$$

where a is the note in question (in numerical form) and n is the number of semitones to change a by. Note that n can also be negative, which would decrease the note. Therefore, the sign of n is significant.

If we were to increase 4—note E—by 3 semitones, we would write $T(4, 3)$. This yields the result 7, which is note G.

Definition (Inversion)**.** *Inverting note(s) so that the order is changed.*

In other words, inversion is counting the same number of half steps in the opposite direction. The note C is mapped to 0. Four semitones above C is 4—note E. Four semitones below C is 8—note G♯/A♭. Therefore, 4 and 8 are inversions of each other since they are both the same number of semitones away from 0—note C.

We can define inversion mathematically as so.

$$V(n) = 12 - n$$

where n is the note we want to get the inverted counterpart of. We use V for the function name instead of I since the latter is used for finding the interval between two notes.

Definition (Retrograde)**.** *Reversing a sequence of notes.*

Given the set $\{4, 6, 8\}$, the retrograde would be $\{8, 6, 4\}$. Retrograde is the easiest operation to understand, but the most difficult to define mathematically. There is no mathematical definition for "reverse," therefore, there is no mathematical definition for retrograde.

However, we can use the following definition in the context of our study

$$R(s) = \text{reverse } s$$

where s is a set of pitches.

Definition (Retrograde Inversion)**.** *A combination of inversion and retrograde, in that order.*

For this mini-set, we made use of these four operations—transposition, inversion, retrograde, and retrograde inversion—to create a piece that was similar to *Moonlight Sonata*, but was still unique; the operations changed the sound of the piece while still keeping it cohesive.

We found the first movement online and we copied the first five-and-a-half measures into our music notation software. From there, we experimented with the operations—sometimes using multiple operations on a subset of notes—until we arrived at a combination that we were satisfied with.

On the sheet music, you will see *Moonlight Sonata* first, and *Sunbeam Sonata* second. Above the notes in our composition, you will see letters corresponding to what operation we did: **T** for transposition, **V** for inversion, and **R** for retrograde. Parentheses and a number specify how much we applied that operation. For example $T(5)$ means we transposed every note in that particular subset up by five semitones.

4.3.3 A Polynomic Jingle

The Collatz conjecture composition and the *Sunbeam Sonata* composition were complete opposites: one strictly followed a mathematical process and the other was subjected to arbitrary decisions regarding what sounded "good." For the final mini-set, we wanted to create a piece that was a combonation of these two processes.

We decided to write a jingle for a polynomial equation. A polynomial is an n-termed equation and is defined like so

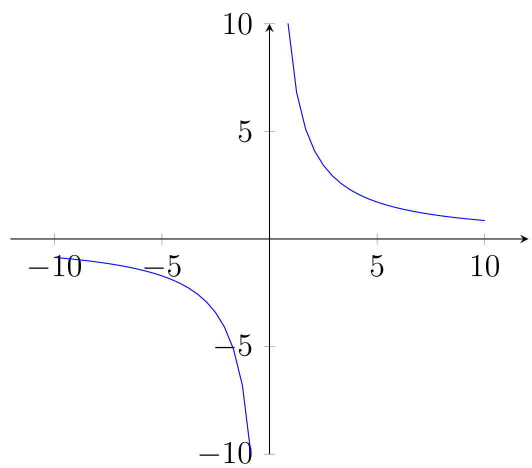
$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + c$$

where the a values are coefficients, the n values link specific coefficients to the variables with the same exponent, and c is the constant, non-variable value. Coefficients are the numbers that appear in front of a variable like the 2 in $2x$.

We wanted to create a composition that "followed" a polynomial of our choosing. We chose the following equation

$$p(x) = \frac{6}{0.7x}$$

The numbers were chosen at random, but the fractional format where the x variable was in the bottom was deliberately chosen. This produces the following graph



We wanted to be able to plug in any integer in to $p(x)$—except for 0 since that will be undefined—and play that note. We decided to plug in numbers from negative 10 to positive 10; what follows are our results. Note: in reality, $p(-10) \neq 11$, but we mapped its value to be between 0–11 so that we could assign a note to it.

Negatives	**Positives**
$p(-10) = 11$	$p(10) = 0$
$p(-9) = 11$	$p(9) = 0$
$p(-8) = 10$	$p(8) = 1$
$p(-7) = 10$	$p(7) = 1$
$p(-6) = 10$	$p(6) = 1$
$p(-5) = 10$	$p(5) = 1$
$p(-4) = 9$	$p(4) = 2$
$p(-3) = 9$	$p(3) = 2$
$p(-2) = 7$	$p(2) = 4$
$p(-1) = 3$	$p(1) = 8$

We can define this process in set builder notation.

$$\{\, p(x) \mid (x \in \mathbb{Z}) \wedge (x \neq 0) \,\} \text{ on the domain } [-10, 10] \,\}$$

We noticed that for some x, we got the same answer. For example, $p(-8)$, $p(-7)$, $p(-6)$, and $p(-5)$ all yield 10. We discarded duplicates so that there was only one unique result per x.

We noticed that this piece was similar to the Collatz conjecture one in terms of compositional journey: it strictly followed some mathematical formula. Since the piece sounded haunted, we decided to manually re-order the notes until it sounded "good." The notes were chosen mathematically, but ordered in such a way that was pleasing to the ear.

5 Ordinal Garden

5.1 Theme Exploration

This study began as an exploration between two different academic areas: art (music theory) and mathematics (set theory.) Throughout our research, we heard the natural beauty that came from mathematical order. For our final composition, we used the imagery of a garden to connect all of our observations from the year.

Nature, like music, can also be modeled mathematically. Famous patterns—such as the Fibonacci sequence—can be found all around us, from the spirals on a sunflower to movement of water in a stream. As life imitates art, so too does nature imitate mathematics.

Definition (Ordinal (adj))**.** *relating to a thing's position in a series.*

Since we are dealing with sets, in which order does matter, and sequences, it is logical to use the term ordinal for the title of the final composition.

In this section, we will explain the connections to nature we chose, the process of composition, and the mathematical and musical connections we observed.

5.2 Connections to Nature

5.2.1 Sequences

Definition (Fibonacci Sequence)**.** *A recursive list of numbers where the last number is the sum of the two before it. We can define it recursively two different ways as below.*

$$\begin{aligned} u_0 &= 1 \\ u_1 &= 1 \\ u_n &= u_{n-1} + u_{n-2} \\ n &> 1 \end{aligned} \qquad f(n) = \begin{cases} n & \text{if } (n \equiv 1) \vee (n \equiv 2) \\ f(n-1) + f(n-2) & \text{otherwise} \end{cases}$$

The left-hand side is in recursion notation, and the right-hand side is a piecewise function that is self-referencing ($f(n)$ is called within the definition itself.)

To find the first 10 numbers in the Fibonacci sequence, you would be evaluating u_{10} or $f(10)$. Both versions produce the following sequence.

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55\}$$

5.2.2 φ (Phi)

Definition (Phi)**.** *An irrational mathematical constant known as the "golden mean," "divine proportion," and "God's fingerprint" for its aesthetically pleasing ratio. It is referenced with the Greek letter "phi" or φ. It can be computed using the formula below.*

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

In nature, φ can be found in the number of petals on a flower, the distance between one's eyes to the bottom of their chin, and the shape of spirals in the galaxy. The first twelve digits of φ are shown below.

$$1.61803398874\ldots$$

Dividing adjacent numbers in the Fibonacci sequence yields a closer and closer estimate of φ as the numbers get sufficiently large. Because of this property, the Fibonacci sequence and φ are often seen within the same context.

5.3 Process of Composition

We composed *Ordinal Garden* using the Fibonacci sequence and the digits of φ to immitate nature. We created four primary movements in this composition:

- Approaching and entering the garden gates
- Playing in the stream
- Taking a nap under an oak tree
- Approaching and exiting the garden gates

Additionally, we experimented with different instrumentation to get a richer sound. We achieved this by using the low—and high—pitched instruments listed below.

- Violin (high-pitched)

- Contrabass (low-pitched)
- Piano (treble clef, high-pitched; bass clef, low-pitched)

We made three programs to create the melodies for each of the movements. While the melody line is being played, the three remaining parts—one of the piano clefs, contrabass, and/or violin—need to play something that is different than the melody. We use set theory operations to build off of and support the melody line.

5.4 Analysis of Creative Choices

5.4.1 Setup

A Day in the Garden will be in the C Major scale—no flats or sharps—and will have a time signature of $\frac{4}{4}$. The tempo will be ♩ = 60.

5.4.2 Section I. — *Approaching and entering the garden gate.*

Let A be the set containing the first fifty-four numbers of the Fibonacci sequence such that any adjacent numbers in the sequence that have an interval greater than four are separated by x amount of stepping numbers.

$$A \to \{\, f(n) \mod 7 \mid \left(n \in \mathbb{Z}^{+}\right) \land (2 \leq n \leq 10) \,\}$$

We cannot codify the interval skip clause, but it is up to the transcriber to fully realize both the statement and equation above while translating these ideas to more traditional music notation.

The piano's treble clef will play A where each number is a quarter note. The piano's bass clef will play every other note an octave apart of A, starting with the first. Note that this can be expressed as follows.

$$B \to \forall n. \in A \iff (n \equiv 1 \mod 2)$$

On the seventh measure, the violin and contrabass will accompany the piano. Let the violin play A in conjunction with the piano's treble clef with the same tempo. Let the contrabass play B in conjunction with the piano's bass clef with the same tempo.

Starting on the seventh measure, the piano's treble clef will add a note a third above the original melody line. This will not always be possible, but it is the general trend. It can be expressed as follows.

$$C \to \{\, n + 3 \mid \forall n. \in A[7:] \,\}$$

The square brackets after A are custom notation that specifies only the numbers in the sequence starting on the seventh measure and beyond.

Starting on the seventh measure, the piano's treble clef will play A and C at the same time.

5.4.3 Section II. — *Playing in the stream.*

Let P be the set containing the first 78 digits of the golden ratio, φ, such that any adjacent numbers that have an interval greater than four are separated by x number of stepping notes.

$$P \rightarrow \{\, n \mid \forall n. \in \varphi[:78] \,\}$$

Note: the bracket notation—as seen in *Approaching the garden gate*—now shows the colon first proceeded by the number, signifying up to the seventy-eigth digit in this case.

Let the violin play the melody C, switching between quarter and half dotted notes up to the transcriber's discretion.

Let the piano's treble clef play an alternative version of C such that it is the retrograde of each measure with the same rhymthm as the violin line. This cannot be expressed mathematically.

Let the contrabass and the piano's bass clef play every fourth beat of C on whole notes an octave apart. We say beat here instead of number since the transcriber can choose the rhythm of C as stated above.

5.4.4 Section III. — *A nap under an oak tree.*

$$u_0 = 1,\ u_1 = 1$$
$$u_a = u_{a-1} + u_{a-2},\ \text{such that } a > 2$$

Let O be the set containing the first thirty numbers of the Fibonacci sequence built in the non-recursive way—as defined above—such that any adjacent numbers in the sequence that have an interval greater than four are separated by x amount of stepping numbers.

$$O \rightarrow \{\, u_n \mod 7 \mid 2 \leq n \leq 30 \,\}$$

Let the piano's treble clef play O such that stepping notes for intervals larger than five are eighth notes and non-stepping notes are quarter notes.

Let the contrabass and the piano's bass clef both play half notes, completing—if applicable—the triad of the first and third beat of the measure in question in O. The violin sound does not fit this section's theme of a peaceful nap, so some of the triads will not be fully constructed as it is up to the transcriber's discretion.

5.4.5 Section IV. — *Approaching and exiting the garden gate.*

Let E be the set containing the retrograde of set A as a symbol of a full circle as defined below

$$E \to R(A)$$

where the $R(x)$ notation stands in place for retrograde, alerting the transcriber to take the retrograde of the input, A.

Note that this operation will also include order of instrumentation as well. In the opening theme, A, the violin and contrabass came in at the seventh measure. Here, in E, they will start instantly, and end by the seventh measure.

We observed that the musical number 4—note E—appears frequently in the piece. It is the first note heard from the opening theme, and is prominent in every section thereafter. To offer a sense of closure, we will play note E at the same time on multiple octaves.

Let F be the set containing the single note E two octaves apart with a length of six beats.

$$F \to \{\{4, 16\}\}$$

We use an inner set to represent that those notes be played at the same time. An octave above 4—E—is twelve plus that note, hence the 16. To conclude this composition, and to re-iterate the common note E, let both clefs on the piano play F.

6 Concluding Discoveries

As we conclude this study, it is important to explain why all of this research was important. Music theory and mathematics are complex fields in their own right, so what insight does combining them provide? The essential question for this study was, *How does examining music through set theory provide a greater insight in musical composition*? In this section, we will synthesize our year-long discoveries.

We came up with five reasons to justify studying this field: structure, perspective, artistic freedom, representation, and fun. None of these reasons are more favorable than another. There are many more reasons and sub-reasons, but these are the most applicable in the context of this year's study.

Structure. Because we can quantify musical intervals, we can quickly identify the distance between any two pitches. This can be harder to visualize on a traditional musical staff, especially as the interval gets larger. Structural sets provide structure by laying out a foundation. This can be in the form of describing what a scale looks like to any quality of triads. Given the structure, the composer can quickly compose without needing to think.

Perspective. Due to the stereotypes surrounding mathematics and music, it is not surprising that the former garners a reputation of rigidity whereas the latter is described as fluid and artistic. We found the opposite to be true. Music theory has many rules and extremely specific characteristics that can not be disregarded. You can model anything with mathematics, and you are free to create your own notation and rules so long as you can explain and justify them. We fostered a new appreciation of mathematics because of this much-needed flexibility and ability to model whatever the music needed.

Artistic Freedom. Mathematics has an abundance of numerical sequences, formulas, and algorithmic processes that we now have the ability to listen to. Instead of creating a composition from scratch—not necessarily organically—a composer can start off with the Catalan sequence, for example, and then build from there. Without this freedom to port over all of these mathematical sequences, algorithms, and concepts, we could have never listened to the Collatz sequence.

Representation. This study can be summarized with one word: representation. How can we represent complex, musical ideas in terms of numbers? How can we manipulate sets representing a musical idea, and what impact will that have on the resulting sound? How can we structure this idea so that the computer will be able to generate and play the result? We raised all of these questions during the study. We found that notation does not matter, justification does. We found that the complexity of the mathematical process does not matter, but how you arrange it does. Representing abstract ideas, especially in a blend of academic disciplines, is not easy, but allows composers to quickly compose wonderfully sounding compositions that follow some sort of mathematical law.

Entertainment. We found it exhilarating to discover a way to play a mathematical idea, especially in the beginning of this study. There are not many compositional-focused resources on this topic, so many findings were on accident or a result of "what would this do?" This kept our research fun and engaging.

We came into this study with the hopes of getting mathematics-centric students interested in music theory and music theory-centric students interested in mathematics. We believe that we achieved this goal. By composing music using set theory, one can enjoy the immediate satisfaction of hearing your composition without needed to know advanced music theory or waiting for inspiration from a muse. In the process, you learn about both fields in a fun, engaging manner.

Here ends our study, *The Sound of Numbers.*

7 References

Devlin, Kevin. *The Joy of Sets: Fundamentals of Contemporary Set Theory*. Springer New York, 1993.

Forte, Allen. "A Theory of Set-Complexes for Music." *Journal of Music Theory*, vol. 8, no. 2, 1964, pp. 136–83. https://doi.org/doi.org/10.2307/843079.

Hasty, Christopher F. "An Intervallic Definition of Set Class." *Journal of Music Theory*, vol. 31, no. 2, 1987, pp. 183–204. https://doi.org/doi.org/10.2307/843707.

McNeilis, Michael James. "Pitch-Class Set Theory in Music and Mathematics." *University of Liverpool*, 2017, pp. ii–119.

Pearsall, Edward. "Interpreting Music Durationally: A Set-Theory Approach to Rhythm." *Perspectives of New Music*, vol. 35, no. 1, 1997, pp. 205–30. https://doi.org/doi.org/10.2307/833685.

Rahn, John. "Logic, Set Theory, Music Theory." *College Music Symposium*, vol. 19, no. 1, 1979, pp. 114–27.

Ries, Hugibert. "Musical Logic. A Contribution to the Theory of Music." *Journal of Music Theory*, vol. 44, no. 1, 2000, pp. 100–26. https://doi.org/doi.org/10.2307/3090671.

Rogi-Francoli, Miguel A. "A Theory of Pitch-Class-Set Extension in Atonal Music." *College Music Symposium*, vol. 41, 2001, pp. 57–90.

Squibbs, Ronalds. "Musical Composition as Applied Mathematics: Set Theory and Probability in Iannis Xenaki's Herma." *University of Connecticut Department of Music*, 2000. t.archive.bridgesmathart.org/2000/bridges2000-141.pdf.

Starr, Daniels. "Sets, Invariance and Partitions." *Journal of Music Theory*, vol. 22, no. 1, 1978, pp. 1–42. https://doi.org/doi.org/10.2307/843626.

Straus, Joseph N. "A Primer for Atonal Set Theory." *College Music Symposium*, vol. 31, 1991, pp. 1–26.

———. "Allen Forte's Contribution to Music Theory." *Music Theory Spectrum*, vol. 37, no. 1, 2015, pp. 3–5.

Susanna S., Epp. *Discrete Mathematics with Applications*. 4th, Brooks/Cole, 2010, vulms.vu.edu.pk/Courses/MTH202/Downloads/2Discrete%20Mathematics%20with%20Applications%20by%20Susanna%20S.%20Epp%20-%204th%20Edition.pdf.

Tymoczko, Dmitri. "Scale Theory, Serial Theory and Voice Leading." *Music Analysis*, vol. 27, no. 1, 2008, pp. 1–49.

Wilcox, Howard, and Escot Pozzi. "A MUSICAL SET THEORY." *Theory and Practice*, vol. 4, no. 2, 1979, pp. 17–37.