Very large set axioms over Constructive set theories

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This work is joint work with Richard Matthews (Leeds)

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Measurable cardinals

- Defined in terms of ultrafilters
- Scott (1961) associated measurable cardinals with elementary embeddings:

Definition

Let $M \subseteq V$ be a transitive class.

- $j\colon V \to M$ is an elementary embedding if j respects all first-order formulas, that is,

$$V \models \phi(\vec{a}) \iff M \models \phi(j(\vec{a})).$$

- The critical point $\operatorname{crit} j$ of an elementary embedding $j\colon V \to M$ is the least ordinal moved by j.

Theorem (Scott 1961)

A cardinal κ is measurable if and only if it is a critical point of some elementary embedding $j\colon V \to M$.

Theorem (Scott 1961)

Measurable cardinals do not exist in L.

Generalizations of measurable cardinals were extensively studied by various people. (For example, Solovay-Reinhardt-Kanamori 1978).

$$
\begin{tikzcd}[row sep=tiny, column sep=small]
0 = 1 \arrow[dr] & & \\
& \text{I0-I3} \arrow[dr] & \\
& & n\text{-huge} \arrow[d] \\
& & \text{superhuge} \arrow[dl] \\
& \text{huge} \arrow[d] & \\
& \text{almost huge} \arrow[dl] & \\
\text{Vopěnka's Principle} \arrow[d] & & \\
\text{extendible} \arrow[dr] & & \\
& \text{supercompact} \arrow[dr] \arrow[ddl] & \\
& & \text{strongly compact} \arrow[ddl] \\
\text{superstrong} \arrow[dr] & & \\
& \text{Woodin} \arrow[d] & \\
& \text{strong} \arrow[d] & \\
& 0^\dagger \text{ exists} \arrow[d] & \\
& \text{measurable} \arrow[d] & \\
& \text{Ramsey} \arrow[dr] \arrow[dddl] & \\
& & \text{Rowbottom} \arrow[dl] \\
& \text{Jónsson} \arrow[dd] & \\
\kappa \longrightarrow (\omega_1)^{<\omega}_2 \arrow[dr] & & \\
& \forall a \in {}^{\omega}\omega (a^\# \text{ exists}) \arrow[d] & \\
& 0^\# \text{ exists} \arrow[dr] & \\
& & \kappa \longrightarrow (\omega)^{<\omega}_2 \arrow[d] \\
& & \text{indescribable} \arrow[dl] \\
& \text{weakly compact} \arrow[dl] & \\
\alpha\text{-Mahlo} \arrow[dr] & & \\
& \text{Mahlo} \arrow[dr] & \\
& & \alpha\text{-inaccessible} \arrow[dl] \\
& \text{inaccessible} \arrow[d] & \\
& \text{weakly inaccessible} &
\end{tikzcd}
$$

Reinhardt's dream

The attempt to find a stronger notion of large cardinal bore the notion now known as a Reinhardt cardinal.

Definition

A Reinhardt cardinal is a critical point of an elementary embedding $j\colon V \to V$.

Flying too close to the sun – Kunen Inconsistency

Kunen (1971) proved that Reinhardt's notion cannot be realized over ZFC:

Theorem (Kunen Inconsistency theorem, 1971)

Work over ZFC, there is no Reinhardt cardinals.
In fact, there is no elementary embedding $j\colon V_{\lambda+2} \to V_{\lambda+2}$

It is still open whether a Reinhardt cardinal is compatible with ZF, that is, in the choiceless context.

Night Flying: Choiceless large cardinals

Some difficulties in working without the axiom of choice:

- Not all cardinal correspond to an ordinal.
- Not all successor cardinal are regular.
- Constructions and proofs become harder.
- Needs to consider more before applying the known results.

Super Reinhardt cardinals

Definition

A cardinal κ is

- Super Reinhardt if for each ordinal α we can find an elementary embedding $j\colon V \to V$ such that $\operatorname{crit} j = \kappa$ and $j(\kappa) > \alpha$.
- A-super Reinhardt for a class A if for each ordinal α we can find an elementary embedding for A-formulas $j\colon V \to V$ such that $\operatorname{crit} j = \kappa$ and $j(\kappa) > \alpha$.

Super Reinhardt sets were defined by Woodin in 1983, and extensively studied in the mid-2010s in the context of Woodin's HOD-dichotomy.

Totally Reinhardt cardinals

Definition

- We call Ord is totally Reinhardt if for every class A we can find a cardinal κ which is A-super Reinhardt.
- κ is totally Reinhardt if $(V_\kappa, V_{\kappa+1}) \models \mathrm{Ord}$ is totally Reinhardt.

The Wholeness axiom

Before defining the Wholeness axiom, let us analyze the definition of a Reinhardt cardinal:

Definition

ZFC with a Reinhardt cardinal is a theory comprising:

- Language: $\in$ and a unary function symbol j,
- Axioms: Usual axioms of ZFC, with the elementarity of j, and Separation and Replacement for <u>j-formulas</u>.

Definition (Corazza 2000)

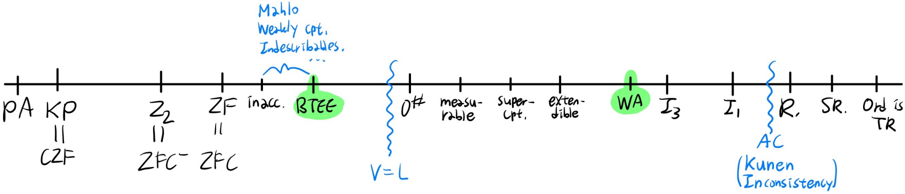
The Wholeness Axiom WA is obtained by restricting Replacement to formulas with <u>no</u> j.

We can further weaken WA as follows:

Definition

The theory Basic Theory of Elementary Embedding (BTEE) is claim that $j\colon V \to V$ is a elementary embedding. BTEE does not subsume Separation and Replacement for j-formulas.

That is, we obtain BTEE by dropping Separation for j-formulas from WA.


Mahlo
Weakly cpt.
Indescribables.
...
PA
KP
||
CZF
Z_2
||
ZFC$^-$
ZF
||
ZFC
inacc.
BTEE
V=L
$0^\#$
measurable
super-cpt.
extendible
WA
I_3
I_1
AC
(Kunen Inconsistency)
R.
SR.
Ord is TR

IZF and CZF: A brief history

- (H. Friedman, 1973) Intuitionistic ZF (IZF) with the double-negation translation between IZF and ZF.
- Various attempts to formalize the foundation for Bishop-styled constructive mathematics.
- Myhill's constructive set theory CST.
- (Aczel, 1978) Constructive ZF and its type-theoretic interpretation.

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$.
2. Pairing: $\{a, b\}$ exists.
3. Union: $\bigcup a$ exists.
4. Separation: $\{x \in a \mid \phi(x)\}$ exists,
5. Replacement: $\{F(x) \mid x \in a\}$ exists if F is a class function.
6. Power set: $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ exists.
7. Regularity: Every set has a $\in$-minimal element.
8. Infinity: $\mathbb{N}$ exists.

Axioms of IZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$.
2. Pairing: $\{a, b\}$ exists.
3. Union: $\bigcup a$ exists.
4. Separation: $\{x \in a \mid \phi(x)\}$ exists.
5. Collection: if $\forall x \in a \exists y \phi(x, y)$, then there is b such that $\forall x \in a \exists y \in b \phi(x, y)$
6. Power set: $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ exists.
7. Set Induction: $\forall a[[\forall x \in a \phi(x)] \to \phi(a)] \to \forall a \phi(a)$
8. Infinity: $\mathbb{N}$ exists.

Axioms of CZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$.
2. Pairing: $\{a, b\}$ exists.
3. Union: $\bigcup a$ exists.
4. Bounded Separation: $\{x \in a \mid \phi(x)\}$ exists if ϕ is bounded.
5. Strong Collection: if $\forall x \in a \exists y \phi(x, y)$, then there is b such that $\forall x \in a \exists y \in b \phi(x, y)$ and $\forall y \in b \exists x \in a \phi(x, y)$.
6. Subset Collection: There is a full subset of $\mathrm{mv}(a, b)$.
7. Set Induction: $\forall a[[\forall x \in a \phi(x)] \to \phi(a)] \to \forall a \phi(a)$
8. Infinity: $\mathbb{N}$ exists.

Differences between IZF and CZF

IZF

1. Full separation
2. Powerset
3. Impredicative
4. Equiconsistent with ZF

and more...

CZF

1. Bounded separation
2. Subset collection
3. Allows type-theoretic interpretation
4. Far more weaker than ZF

Large set axioms

- Ordinals over constructive set theories are not well-behaved.

Examples (CZF)

- Every ordinal is well-ordered $\implies$ Excluded Middle for Δ_0-formulas,
- $\alpha \subseteq \beta$ does not imply $\alpha \in \beta$ or $\alpha = \beta$.

- We define large cardinal properties over constructive set theories by mimicking the structural properties of H_κ and V_κ.

Multi-valued functions

The notion of multi-valued function provides a syntactic sugar for Collection.

Definition (Multi-valued function)

Let A and B be classes. We call a relation R of domain A and codomain B a multi-valued function from A to B. (Notation: $R\colon A \rightrightarrows B$)

Definition (Subimage)

If $R\colon A \rightrightarrows B$ and $R^{-1}\colon A \rightrightarrows B$, we write $R\colon A \leftrightarrows B$, and we call B a subimage of R.

Multi-valued functions replace functions in CZF-context.

From Replacement to Strong Collection

Definition (Replacement)

If $F\colon a \to V$ is a first-order definable class function, then we can find a set image b of F.

Definition (Strong Collection)

If $\boldsymbol{R\colon a \rightrightarrows V}$ is a first-order definable class **multi-valued** function, then we can find a set **sub**image b of R. (That is, $R\colon a \leftrightarrows b$.)

From Powerset to Subset Collection

Definition (Powerset, equivalent formulation)

For given sets a, b, we can find a set c such that...

If $f : a \to b$, then c contains an image of f. (i.e., $\operatorname{Im} f \in c$.)

Definition (Subset Collection, equivalent formulation)

For given sets a, b, we can find a set c such that...

If $\boldsymbol{r : a \rightrightarrows b}$, then c contains a **sub**image of r. (i.e., there is $d \in c$ such that $r : a \leftrightarrows d$.)

Defining large sets

With some pain, we can prove

Lemma (ZFC, regular cardinal)

A cardinal κ is regular if and only if for $a \in H_\kappa$ and $f : a \to H_\kappa$, we have $\operatorname{Im} f \in H_\kappa$.

By mimicking the above result, we have

Definition (CZF, regular set)

A transitive set K is regular if $a \in K$, $r \in K$ and $r\colon a \rightrightarrows K$, then we can find some subimage $b \in K$ of r (that is, $r\colon a \leftrightarrows b$.)

Similarly, we can see

Lemma (ZFC, inaccessible cardinal)

A cardinal κ is inaccessible if and only if κ is regular and H_κ satisfies:

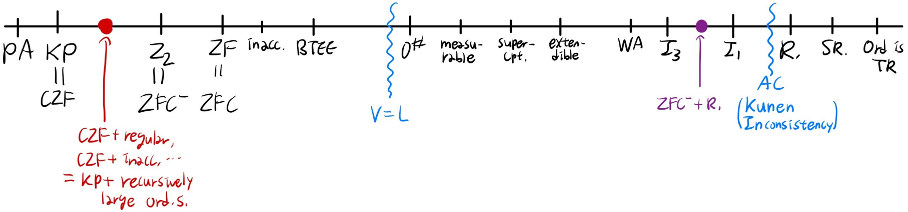
1. $\omega \in H_\kappa$*,* H_κ *is closed under union and intersection, and*
2. *if* $a, b \in H_\kappa$*, then* $c := \{\operatorname{Im} f \mid f : a \to b\} \in H_\kappa$*.*

Definition (CZF, inaccessible set)

A set K is <u>inaccessible</u> if K is regular and

1. $\omega \in K$, K is closed under union and intersection, and
2. if $a, b \in K$, then we can find $c \in K$ such that we can always find a subimage of $r : a \rightrightarrows b$, $r \in K$ from c.

Consistency hierarchy



Large sets and elementary embeddings

Let us consider elementary embeddings over CZF:

Definition

Let $j\colon V \to M$ be an elementary embedding. A set K is a critical point of j if K is the 'least' set lifted by j in the sense that

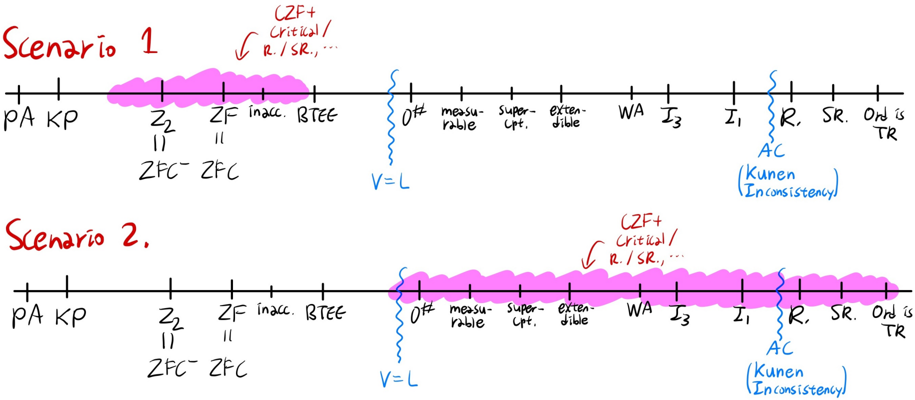
- $j(x) = x$ for all $x \in K$, and
- $K \in j(K)$

Definition

A set K is critical if K is inaccessible and a critical point of an elementary embedding $j\colon V \to M$.

Question: the consistency strength of CZF with a critical cardinal.

Two scenarios



A Lower bound

Theorem (J., Matthews, CZF)

Let K be a critical point of a Σ_0-elementary embedding $j\colon V \to M$ such that K satisfies Δ_0-separation. Then $K \models \mathrm{IZF}$.

(Note: the above theorem does not require Separation, Strong Collection or Set Induction for j-formulas.)

Theorem

CZF *with a critical set proves the consistency of* ZFC + BTEE

Reinhardt embeddings

Definition

An inaccessible set K is a Reinhardt set if K is a critical point of $j\colon V \to V$.

Theorem

CZF *with a Reinhardt cardinal proves* $\mathrm{Con}(\mathrm{ZF} + \mathrm{WA})$.

Go beyond the Reinhardtness

Definition

- An inaccessible set K is super Reinhardt if for every set a we can find an elementary embedding $j\colon V \to V$ such that K is a critical point of j and $a \in j(K)$.
- An inaccessible set K is A-super Reinhardt if for every set a we can find an A-elementary embedding $j\colon V \to V$ such that K is a critical point of j and $a \in j(K)$.

Theorem

CZF *with a super Reinhardt set proves the consistency of* ZF *with a Reinhardt cardinal.*

Definition

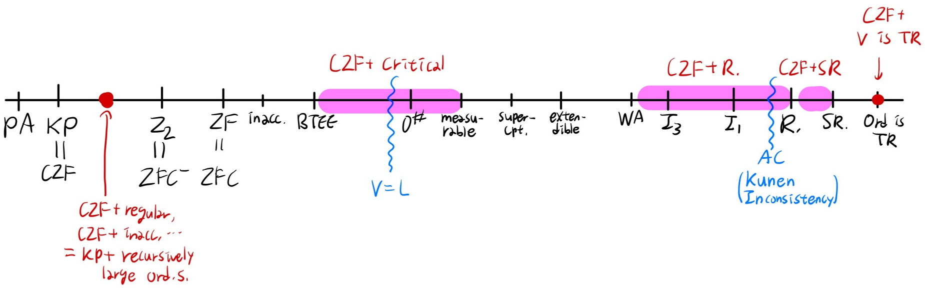
<u>V is totally Reinhardt</u> (abbr. V is TR) is the following claim: for every class A, there is an A-super Reinhardt set K.

Theorem

CZF *with 'V is TR' proves all axioms of* IZF. *Furthermore, the following two theories are equiconsistent:*

- CZF + *'V is TR,' and*
- ZF + *'V is TR.'*

(The exact definition for super/totally Reinhardt require the formulation of constructive second-order set theory Appendix)


CZF+ V is TR
CZF+ Critical
CZF+R.
CZF+SR
PA
KP
= CZF
Z2
= ZFC⁻
ZF
= ZFC
inacc.
BTEE
0#
measurable
super-cpt.
extendible
WA
I3
I1
R.
SR.
Ord is TR
V=L
AC
(Kunen Inconsistency)
CZF+ regular,
CZF+ inacc, ...
= KP+ recursively large ord.s.

A rough sketch for the proofs

The proof divides into 2-3 main steps:

1. Internal analysis of the given large set axioms. Usually produces a model of $\mathsf{IZF} + \mathsf{X}$.
2. Double-negation translation: Friedman-styled translation, Gambino's Heyting-valued model, or their combinations. The resulting lower bound is of the form $\mathrm{Con}(\mathsf{ZF} + \mathsf{X})$
3. If possible, derive the consistency strength in terms of ZFC with large cardinal axioms.

Open problems

1. Non-trivial upper bounds for the consistency strength.
2. Better lower bounds. (For example, can we derive $\mathrm{Con}(\mathsf{ZFC} + \mathsf{WA})$ from $\mathrm{Con}(\mathsf{ZF} + \mathsf{WA})$?)
3. Defining other large set notions (e.g., supercompactness and extendibles) and analyzing their consistency strength.
4. Questions regarding machinery in the paper, e.g., second-order constructive set theory.

Questions

Thank you!

Constructive second-order set theory

Definition (Constructive Gödel-Bernays set theory, CGB)

CGB is defined over the two-sorted languages (sets and classes) with the following axioms:

- Axioms of CZF for sets.
- Every set is a class, and every element of a class is a set.
- Class Extensionality: two classes are equal if they have the same set members.
- Elementary Comprehension: if $\phi(x, p, C)$ is a first-order formula with a class parameter C, then there is a class A such that $A = \{x \mid \phi(x, p, C)\}$.

Definition (CGB, Continued)

- Class Set Induction:
  $\forall^1 A\big[[\forall^0 x(\forall^0 y \in x(y \in A) \to x \in A)] \to \forall^0 x(x \in A)\big]$.
- Class Strong Collection:
  $\forall^1 R \forall^0 a[R\colon a \rightrightarrows V \to \exists^0 b(R\colon a \leftrightarrows b)]$.

Definition (Intuitionistic Gödel-Bernays set theory, IGB)

IGB is obtained by adding the following axioms to CGB:

- Axioms of IZF for sets.
- Class Separation: if A is a class and a is a set, then $A \cap a$ is a set.

Note that CGB and IGB are conservative extensions of CZF and IZF respectively.

The definition of an elementary embedding $j\colon V \to M$ requires quantifying over formulas ϕ:

$$\phi(\vec{a}) \iff \phi^M(j(\vec{a})).$$

We resolve this problem by introducing the infinite conjunction $\bigwedge$.

Definition (CGB with the infinite connectives, CGB_∞)

CGB_∞ has the same axiom with CGB, but defined over the first-order intuitionistic logic with the infinite connectives $\bigwedge$ and $\bigvee$.

Super Reinhardt sets and 'V is TR' are defined over CGB_∞. Also, CGB_∞ is a conservative extension of CGB.
(Back to main.)