

Separating the Wholeness axioms

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Large cardinal axioms

- Large cardinals are means to gauge the strength of extensions of ZFC.
- Since the beginning of set theory, set theorists defined stronger notion of large cardinals (Inaccessible, Mahlo, Weakly compact, Measurable, Woodin, Supercompact, etc.)
- Large cardinals stronger than measurable cardinals are usually defined in terms of elementary embedding.

Elementary embedding

Definition

Let $M \subseteq V$ be a transitive class. A map $j: V \rightarrow M$ is elementary if for every formula $\phi(\vec{x})$ over the language $\{\in\}$,

$$\phi(\vec{a}) \leftrightarrow \phi^M(j(\vec{a})).$$

κ is a critical point of j if κ is the least ordinal moved by j , i.e., $j(\kappa) > \kappa$.

Reinhardt embedding

Reinhardt introduced the following ‘eventual’ form of a large cardinal axiom:

Definition

A cardinal κ is a Reinhardt cardinal if it is the critical point of $j: V \rightarrow V$.

An elementary embedding $j: V \rightarrow V$ is called a Reinhardt embedding.

Icarian fate of Reinhardt cardinals

However, Reinhardt cardinals cannot exist over ZFC:

Theorem (Kunen 1971, ZFC)

There is no elementary embedding $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$. As a corollary, there is no elementary embedding $j: V \rightarrow V$.

(If we take $\lambda = \sup_{n < \omega} j^n(\kappa)$, then $j \upharpoonright V_{\lambda+2}: V_{\lambda+2} \rightarrow V_{\lambda+2}$.)

(Non-in)consistent weakening of Reinhardtness

Set theorists studied the non-inconsistent weakening of Reinhardt cardinals:

Definition

- 1 $I_3(\lambda)$: There is an elementary $j: V_\lambda \rightarrow V_\lambda$.
- 2 $I_2(\lambda)$: There is an Σ_1 -elementary[†] $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$.
- 3 $I_1(\lambda)$: There is an elementary $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$.
- 4 $I_0(\lambda)$: There is an elementary $j: L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$.

They are not known to be inconsistent over ZFC.

[†]A formula is Σ_1 if it takes the form $\exists x\phi(x)$, where every quantifier in ϕ is bounded.

Other weakening

The obvious weakening is Reinhardt embedding without Choice.
The consistency of ZF with $j: V \rightarrow V$ is yet to be known, but

Theorem (Schiltzenberg 2020)

If ZFC + $I_0(\lambda)$ is consistent, then so is ZF + ($j: V_{\lambda+2} \rightarrow V_{\lambda+2}$).

Other weakening

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Theorem (Schultzenberg 2020)

If ZFC + $I_0(\lambda)$ is consistent, then so is ZF + ($j: V_{\lambda+2} \rightarrow V_{\lambda+2}$).

We can also consider Reinhardtness over a weaker theory, like ZFC without Power set:

Theorem (Matthews 2023)

ZFC + $I_1(\lambda)$ proves the consistency of $\text{ZFC}^- + \exists j: V \rightarrow V$.

Here ZFC^- is a technical variant of 'ZFC without Power set.'

Formulating a Reinhardt embedding

An elementary embedding $j: V \rightarrow V$ is a proper class and not a set. That is, we cannot quantify over j .

To formulate j over ZFC, let us take the following approach:

Definition

ZFC $_j$ is the theory over the language $\{\in, j\}$ with the following axioms:

- 1 Usual axioms of ZFC,
- 2 Axiom schema of Separation and Replacement are allowed for formulas over $\{\in, j\}$.
- 3 $j: V \rightarrow V$ is elementary: For every formula $\phi(\vec{x})$ over the language $\{\in\}$, we have

$$\phi(\vec{x}) \leftrightarrow \phi(j(\vec{x})).$$



The Wholeness axiom

Corazza introduced the Wholeness axiom by restricting Replacement to formulas over $\{\in, j\}$:

Definition

WA is the combination of the following statement:

- 1 Axiom schema of Separation for formulas over $\{\in, j\}$.
- 2 $j: V \rightarrow V$ is elementary.

$I_3(\lambda)$ proves the consistency of WA; In fact, if $I_3(\lambda)$ holds, then V_λ is a model of ZFC + WA.

Weaker variants of WA

Definition

A formula $\phi(x)$ is Σ_n^j if it takes of the form

$$\exists v_0 \forall v_1 \cdots Qx_{v-1} \psi(v_0, v_1, \dots, v_{n-1}, x),$$

where ψ is a formula over the language $\{\in, j\}$ in which every quantifier is bounded.

If ψ does not mention j , then we say ϕ is Σ_n .

Definition

WA_n is obtained from WA by restricting Separation schema to Σ_n^j -formulas.

Theorem (Hamkins 1999)

WA_0 *does not prove* WA_1 .

Hamkins asked whether WA_1 proves the consistency of WA_0 .

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The main idea is to construct a truth predicate satisfying $ZFC + WA_0$.

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My proof does not take this form.

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ZFC proves the consistency of its finite fragment.

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ZFC proves the consistency of its finite fragment.

- 1 Since there are finitely many formulas, there is n such that every formula of the fragment is Σ_n .
- 2 ZFC can define the truth predicate \models_{Σ_n} for Σ_n -formulas.[†]
- 3 By the reflection principle, we can find α such that V_α respects \models_{Σ_n} . Hence V_α satisfies the fragment we fixed.

[†]In fact, KP suffices.

We want to mimic a similar argument to prove the consistency of $\text{ZFC} + \text{WA}_0$.

To do this, we must define a truth predicate that can capture every axiom of $\text{ZFC} + \text{WA}_0$.

Lemma ($\text{ZFC} + \text{WA}_0$)

Let $j: V \rightarrow V$ be the elementary embedding. If κ is the least ordinal moved by j , and if $\phi(x)$ is a formula over $\{\in\}$, then

$$\forall x \in V_\kappa [\phi(x) \leftrightarrow V_\kappa \models \phi(x)].$$

In other words, V_κ is an ‘elementary substructure’ of V .

Lemma (ZFC + WA₀)

Let j and κ be as before. If we let $\kappa_0 = \kappa$, $\kappa_{n+1} = j(\kappa_n)$, then

- 1 $\langle \kappa_n \mid n < \omega \rangle$ is Σ_1^j -definable.
- 2 $\langle \kappa_n \mid n < \omega \rangle$ is cofinal over the class of all ordinals: That is, for every ordinal α there is $n < \omega$ such that $\alpha < \kappa_n$.

These two lemma allow us to define a ‘truth predicate’ for formulas over $\{\in\}$:

Definition

$$\models_{\Sigma_\infty} \phi(x) \iff \exists n < \omega (x \in V_{\kappa_n} \wedge V_{\kappa_n} \models \phi(x)).$$

Extending the truth predicate

\models_{Σ_∞} covers every axiom of ZFC, but it is ‘too simple’ to cover WA₀ since \models_{Σ_∞} does not take any formulas with j .

Definition

A class of $\Delta_0^j(\Sigma_\infty)$ formulas is the least class of formulas containing formulas in $\{\in\}$ closed under

- 1 Boolean connectives ($\wedge, \vee, \neg, \rightarrow$), and
- 2 Bounded quantifiers, which take of the form $\forall u \in j^n(x)$ or $\exists u \in j^n(x)$.

We can define the truth predicate $\models_{\Delta_0^j(\Sigma_\infty)}$ for $\Delta_0^j(\Sigma_\infty)$ formulas in a Σ_1^j way, in which we will omit the details.

The unreachable

Recall that we are mimicking the following argument:

- 1 Since there are finitely many formulas, there is n such that every formula of the fragment is Σ_n .
- 2 ZFC can define the truth predicate \models_{Σ_n} for Σ_n -formulas.
- 3 By the reflection principle, we can find α such that V_α respects \models_{Σ_n} . Hence V_α satisfies the fragment we fixed.

The unreachable

Recall that we are mimicking the following argument:

- 1 Since \models_{Σ_∞} is Σ_1^j -definable, every axiom of ZFC + WA₀ is finitely axiomatizable.
- 2 We can define $\models_{\Delta_0^j(\Sigma_\infty)}$.
- 3 Do we have a reflection argument?

The latter step won't work because we do not have Replacement for j -formulas.

Strong soundness: What shines the darkness

To get around the issue, we need a proof-theoretic tool:

Definition

Let term_V be the class of all terms generated from constant symbols $\{c_x \mid x \in V\}$ corresponding to the class of all sets with a function symbol j .

Let Form_V be the class of all formulas over $\{\in, j\}$, with terms from term_V .

For a set X of sentences over $\{\in, j\}$, let S_V^X be the least class containing X and closed under subformulas, term substitution, and Boolean combinations.

Definition

Let X be a set of sentences over $\{\in, j\}$. A class function

$T : \text{Form}_V \cup S_V^X \rightarrow V$ is a weak class model for X if

- 1 $T(j(t)) = j(T(t))$ for $t \in \text{term}_V$.
- 2 T respects the Tarskian truth definition, i.e.,
 - For terms s, s' and t, t' , if $T(s) = T(s')$, $T(t) = T(t')$, then $T(\ulcorner s = t \urcorner) = T(\ulcorner s' = t' \urcorner)$ and $T(\ulcorner s \in t \urcorner) = T(\ulcorner s' \in t' \urcorner)$.
 - $T(\ulcorner \neg \sigma \urcorner) = 1 - T(\ulcorner \sigma \urcorner)$.
 - If \circ is a logical connective, then $T(\ulcorner \phi \circ \psi \urcorner) = 1$ if and only if $T(\ulcorner \phi \urcorner) \circ T(\ulcorner \psi \urcorner) = 1$.
 - If Q is a quantifier, then $T(\ulcorner Qx\phi(x) \urcorner) = 1$ if and only if $Qx[T(\ulcorner \phi(x) \urcorner) = 1]$ holds.*

*It applies only when $\ulcorner Qx\phi(x) \urcorner \in S_V^X$.

The main feature of a weak class model is that it evaluates the truth of a class of formulas even if the class is not closed under quantifiers.

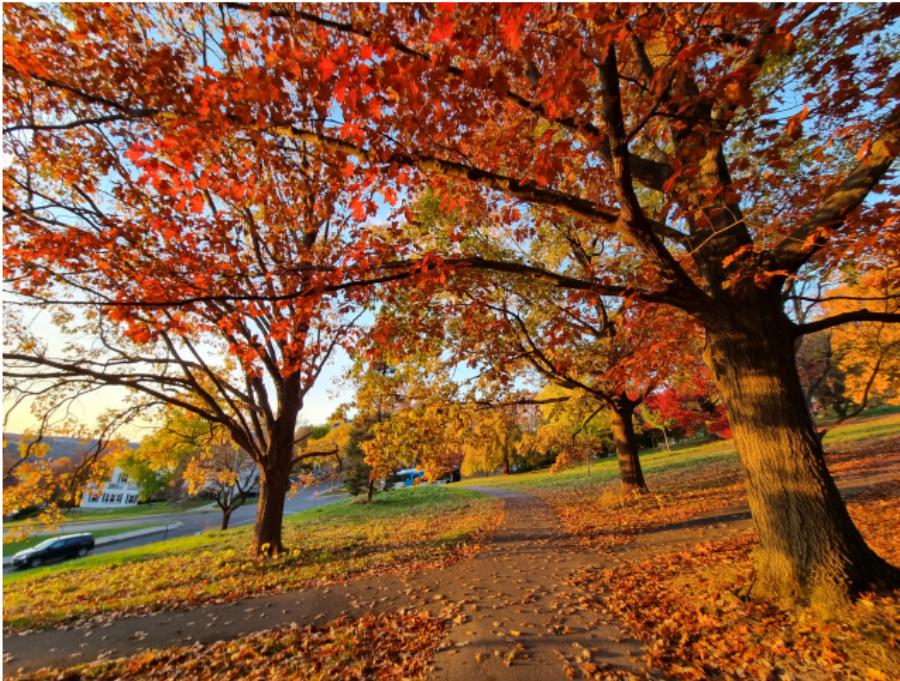
The following lemma says a weak class model is enough to establish the consistency:

Lemma (Strong Soundness, ZFC + WA₁)

If there is a Π_1^j -definable weak class model for X , then X is consistent.

We can construct a Π_1^j -definable class model of ZFC + WA₀ from $\models_{\Delta_0^j(\Sigma_\infty)}$.

Questions



Thank you!