Very large set axioms over Constructive set theories

Hanul Jeon

Cornell University

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This work is joint work with Richard Matthews (Leeds)

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Measurable cardinals

- Defined in terms of ultrafilters
- Scott (1961) associated measurable cardinals with elementary embeddings:

Definition

Let $M \subseteq V$ be a transitive class.

- $j\colon V \to M$ is an elementary embedding if j respects all first-order formulas, that is,

$$V \models \phi(\vec{a}) \iff M \models \phi(j(\vec{a})).$$

- The critical point $\operatorname{crit} j$ of an elementary embedding $j\colon V \to M$ is the least ordinal moved by j.

Theorem (Scott 1961)

A cardinal κ is measurable if and only if it is a critical point of some elementary embedding $j\colon V \to M$.

Theorem (Scott 1961)

Measurable cardinals do not exist in L.

Climbing to the large cardinal hierarchy

Generalizations of measurable cardinals were extensively studied by various people. (For example, Solovay-Reinhardt-Kanamori 1978).

One of these examples include:

Definition

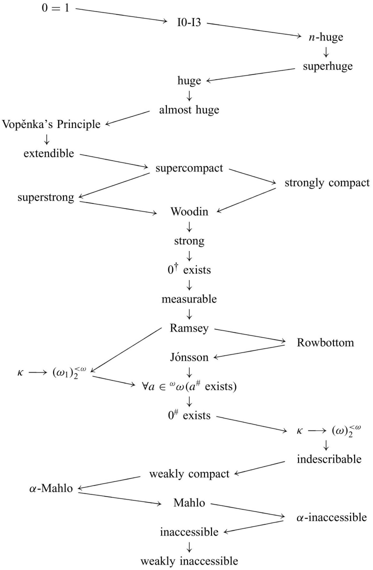
A cardinal κ is <u>extendible</u> if for each α we can find an elementary embedding $j\colon V_{\kappa+\alpha} \to V_\zeta$ for some ζ with $\operatorname{crit} j = \kappa$.

Definition (Rank-into-rank embeddings)

A cardinal κ is

1. I_3 if κ is a critical point of $j\colon V_\lambda \to V_\lambda$,
2. I_2 if κ is a critical point of $j\colon V \to M$ such that $V_\lambda \subseteq M$,
3. I_1 if κ is a critical point of $j\colon V_{\lambda+1} \to V_{\lambda+1}$,
4. I_0 if κ is a critical point of $j\colon L(V_{\lambda+1}) \to L(V_{\lambda+1})$,

where $\lambda = \sup_{n<\omega} j^n(\kappa)$ is the first ordinal above κ fixed by j.


0 = 1
I0-I3
n-huge
superhuge
huge
almost huge
Vopěnka's Principle
extendible
supercompact
strongly compact
superstrong
Woodin
strong
$0^\dagger$ exists
measurable
Ramsey
Rowbottom
Jónsson
$\kappa \longrightarrow (\omega_1)_2^{<\omega}$
$\forall a \in {}^\omega\omega(a^\# \text{ exists})$
$0^\#$ exists
$\kappa \longrightarrow (\omega)_2^{<\omega}$
indescribable
weakly compact
α-Mahlo
Mahlo
α-inaccessible
inaccessible
weakly inaccessible

Reinhardt's dream

The attempt to find a stronger notion of large cardinal bore the notion now known as a Reinhardt cardinal.

Definition

A Reinhardt cardinal is a critical point of an elementary embedding $j\colon V \to V$.

Flying too close to the sun – Kunen Inconsistency

Kunen (1971) proved that Reinhardt's notion cannot be realized over ZFC:

Theorem (Kunen Inconsistency theorem, 1971)

Work over ZFC, there is no Reinhardt cardinals. In fact, there is no elementary embedding $j\colon V_{\lambda+2} \to V_{\lambda+2}$

It is still open whether a Reinhardt cardinal is compatible with ZF, that is, in the choiceless context.

Night Flying: Choiceless large cardinals

Some difficulties in working without the axiom of choice:

- Not all cardinal correspond to an ordinal.
- Not all successor cardinal are regular.
- Constructions and proofs become harder.
- Needs to consider more before applying the known results.

Super Reinhardtness

Definition

A cardinal κ is

- Super Reinhardt if for each ordinal α we can find an elementary embedding $j\colon V \to V$ such that $\operatorname{crit} j = \kappa$ and $j(\kappa) > \alpha$.
- A-super Reinhardt for a class A if for each ordinal α we can find an elementary embedding for A-formulas $j\colon V \to V$ such that $\operatorname{crit} j = \kappa$ and $j(\kappa) > \alpha$.

Super Reinhardtness was defined by Woodin in 1983, and extensively studied in the mid-2010s in the context of Woodin's HOD-dichotomy.

Total Reinhardtness

Definition

- We call <u>Ord is total Reinhardt</u> if for every class A we can find a cardinal κ which is A-super Reinhardt.
- κ is <u>total Reinhardt</u> if $(V_\kappa, V_{\kappa+1}) \models \mathrm{Ord}$ is total Reinhardt.

Theorem (Goldberg)

The following two theories are equiconsistent over $\mathsf{ZF}+\mathsf{DC}$*:*

1. *For some ordinal* λ*, there is an elementary embedding* $V_{\lambda+2} \to V_{\lambda+2}$
2. $\mathsf{ZFC}+\mathsf{I}_0$.

Theorem (Goldberg)

Work over $\mathsf{ZF}+\mathsf{DC}$*, if there is an elementary embedding* $j\colon V_{\lambda+3} \to V_{\lambda+3}$*, then we have the consistency of* $\mathsf{ZFC}+\mathsf{I}_0$.

The Wholeness axiom

Before defining the Wholeness axiom, let us analyze the definition of a Reinhardt cardinal:

Definition

ZFC with a Reinhardt cardinal is a theory comprising:

- Language: $\in$ and a unary function symbol j,
- Axioms: Usual axioms of ZFC, with the elementarity of j, and Separation and Replacement for <u>j-formulas</u>.

Definition (Corazza 2000)

The Wholeness Axiom WA is obtained by restricting Replacement to formulas with <u>no</u> j.

We can further weaken WA as follows:

Definition

The theory Basic Theory of Elementary Embedding (BTEE) is claim that $j\colon V \to V$ is a elementary embedding. BTEE does not subsume Separation and Replacement for j-formulas.

That is, we obtain BTEE by dropping Separation for j-formulas from WA.

We can strengthen BTEE by adding TI_j, the transfinite induction for j-formulas.

Theorem (Corazza)

Work over ZFC,

1. $\mathrm{I}_3 \implies \mathrm{Con}(\mathrm{ZFC}+\mathrm{WA})$ *and* $\mathrm{WA} \implies$ *a proper class of extendibles.*
2. $0^\sharp \implies L \models \mathrm{BTEE}$ *and* $\mathrm{BTEE} \implies$ *n-ineffable cardinal for each (meta-)natural n.*

(A cardinal κ is n-effable if for every $f : [\kappa]^n \to 2$ there is a stationary S subset of κ such that $f \restriction [S]^n$ is constant.)

Weakening set theory: the theory ZFC^-

Definition

ZFC^- is obtained by dropping Powerset and replacing Replacement to Collection from ZFC.

Collection is the following statement:

$$\forall a[\forall x \in a \exists y \phi(x, y) \to \exists b \forall x \in a \exists y \in b \phi(x, y)].$$

- Collection is stronger than Replacement, and they are equivalent if we assume Powerset.
- ZFC$-$, a mere ZFC without Powerset, is ill-behaved. (Gitman-Hamkins-Johnstone 2016)

Reinhardt embeddings over ZFC^-

Work over ZFC^-_j, the theory obtained by adding j and allowing j to the axiom schemes of ZFC^-. The following result shows a Reinhardt embedding is compatible with ZFC^-:

Theorem (Matthews)

ZFC *proves the followings are equivalent:*

- *There is an elementary embedding* $j\colon H_{\lambda^+} \to H_{\lambda^+}$, *and*
- *There is an elementary embedding* $k\colon V_{\lambda+1} \to V_{\lambda+1}$.

Especially, if λ *is* I_1, *then* (H_{λ^+}, j) *is a model of* ZFC^-_j *with a non-trivial elementary embedding* $j\colon V \to V$ *and* $V_{\mathrm{crit}\, j}$ *exists.*

However, a Reinhardt embedding cannot be cofinal:

Definition

An elementary embedding $j\colon V \to V$ is cofinal if for each x we can find y such that $x \in j(y)$.

Theorem (Matthews)

Work in ZFC_j^-, *if* $j\colon V \to V$ *is a non-trivial* Σ_0*-elementary embedding and* $V_{\operatorname{crit} j}$ *exists, then* j *cannot be cofinal.*

IZF and CZF: A brief history

- (H. Friedman, 1973) Intuitionistic ZF (IZF) with the double-negation translation between IZF and ZF.
- Various attempts to formalize the foundation for Bishop-styled constructive mathematics.
- Myhill's constructive set theory CST.
- (Aczel, 1978) Constructive ZF and its type-theoretic interpretation.

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$.
2. Pairing: $\{a, b\}$ exists.
3. Union: $\bigcup a$ exists.
4. Separation: $\{x \in a \mid \phi(x)\}$ exists,
5. Replacement: $\{F(x) \mid x \in a\}$ exists if F is a class function.
6. Power set: $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ exists.
7. Regularity: Every set has a $\in$-minimal element.
8. Infinity: $\mathbb{N}$ exists.

Axioms of IZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$.
2. Pairing: $\{a, b\}$ exists.
3. Union: $\bigcup a$ exists.
4. Separation: $\{x \in a \mid \phi(x)\}$ exists.
5. Collection: if $\forall x \in a \exists y \phi(x, y)$, then there is b such that $\forall x \in a \exists y \in b \phi(x, y)$
6. Power set: $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ exists.
7. Set Induction: $\forall a[[\forall x \in a \phi(x)] \to \phi(a)] \to \forall a \phi(a)$
8. Infinity: $\mathbb{N}$ exists.

Axioms of CZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$.
2. Pairing: $\{a, b\}$ exists.
3. Union: $\bigcup a$ exists.
4. Bounded Separation: $\{x \in a \mid \phi(x)\}$ exists if ϕ is bounded.
5. Strong Collection: if $\forall x \in a \exists y \phi(x, y)$, then there is b such that $\forall x \in a \exists y \in b \phi(x, y)$ and $\forall y \in b \exists x \in a \phi(x, y)$.
6. Subset Collection: There is a full subset of $\mathrm{mv}(a, b)$.
7. Set Induction: $\forall a[[\forall x \in a \phi(x)] \to \phi(a)] \to \forall a \phi(a)$
8. Infinity: $\mathbb{N}$ exists.

Differences between IZF and CZF

IZF

1. Full separation
2. Powerset
3. Impredicative
4. Equiconsistent with ZF

and more...

CZF

1. Bounded separation
2. Subset collection
3. Allows type-theoretic interpretation
4. Far more weaker than ZF

Large set axioms

- Ordinals over constructive set theories are not well-behaved. (e.g., every ordinal is well-ordered $\implies$ Excluded Middle for Δ_0-formulas, $\alpha \subseteq \beta$ does not imply $\alpha \in \beta$ or $\alpha = \beta$.)
- We define large cardinal properties over constructive set theories by mimicking the structural properties of H_κ and V_κ.

Multi-valued functions

The notion of multi-valued function provides a syntactic sugar for Collection.

Definition

Let A and B be classes. We call a relation R of domain A and codomain B a multi-valued function from A to B. (Notation: $R\colon A \rightrightarrows B$)
If $R\colon A \rightrightarrows B$ and $R^{-1}\colon A \rightrightarrows B$, we write $R\colon A \leftrightarrows B$, and we call B a subimage of R.

Multi-valued functions replace functions in CZF-context.

Definition (Collection, restatement)

If $R\colon a \rightrightarrows V$ is a first-order definable class multi-valued function with parameters, then we can find a set codomain b of R. (That is, $R\colon a \rightrightarrows b$.)

Definition (Strong Collection, restatement)

If $R\colon a \rightrightarrows V$ is a first-order definable class multi-valued function with parameters, then we can find a set subimage b of R. (That is, $R\colon a \leftrightarrows b$.)

Definition (Subset Collection, equivalent formulation)

For given sets a, b, we can find a set c such that if $r\colon a \rightrightarrows b$, then c contains a subimage of r. (i.e., there is $d \in c$ such that $r\colon a \leftrightarrows d$.)

Defining large sets

With some pain, we can prove

Lemma (ZFC)

A cardinal κ is regular if and only if for $a \in H_\kappa$ and $f : a \to H_\kappa$, we have $\operatorname{Im} f \in H_\kappa$.

By mimicking the above result, we have

Definition

A transitive set K is regular if $a \in K$ and $r : a \rightrightarrows K$, then we can find some subimage $b \in K$ of r (that is, $r : a \leftrightarrows b$.)

Similarly, we can see

Lemma (ZFC)

A cardinal κ is inaccessible if and only if κ is regular and H_κ satisfies:

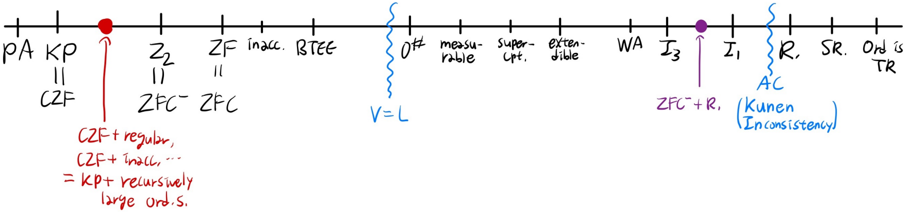
1. $\omega \in H_\kappa$, H_κ *is closed under union and intersection, and*
2. *if* $a, b \in H_\kappa$, *then* $c := \{\operatorname{Im} f \mid f\colon a \to b\} \in H_\kappa$.

Definition

A set K is <u>inaccessible</u> if K is regular and

1. $\omega \in K$, K is closed under union and intersection, and
2. if $a, b \in K$, then we can find $c \in K$ such that we can always find a subimage of $r\colon a \rightrightarrows b$, $r \in K$ from c.

Consistency hierarchy



Large sets and elementary embeddings

Let us consider elementary embeddings over CZF:

Definition

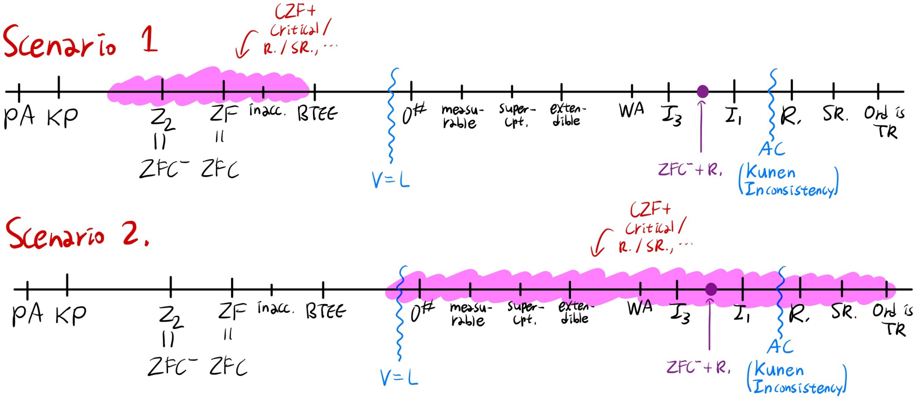
Let $j \colon V \to M$ be an elementary embedding. A set K is a critical point of j if K is the 'least' set lifted by j in the sense that $j(x) = x$ for all $x \in K$ and $K \in j(K)$.

Definition

A set K is critical if K is inaccessible and a critical point of an elementary embedding $j \colon V \to M$.

Question: the consistency strength of CZF with a critical cardinal.

Two scenarios



A Lower bound

Theorem (J., Matthews, CZF)

Let K be a critical point of a Σ_0-elementary embedding $j\colon V \to M$ such that K satisfies Δ_0-separation. Then $K \models \mathsf{IZF}$.

(Note: the above theorem does not require Separation, Strong Collection or Set Induction for j-formulas.)

Theorem

CZF *with a critical set proves the consistency of* ZFC + BTEE

Reinhardt embeddings

Definition

An inaccessible set K is a Reinhardt set if K is a critical point of $j\colon V \to V$.

Theorem

CZF *with a Reinhardt cardinal proves* $\mathrm{Con}(\mathrm{ZF} + \mathrm{WA})$.

Go beyond the Reinhardtness

Definition

- An inaccessible set K is <u>super Reinhardt</u> if for every set a we can find an elementary embedding $j\colon V \to V$ such that K is a critical point of j and $a \in j(K)$.
- An inaccessible set K is <u>K-super Reinhardt</u> if for every set a we can find an A-elementary embedding $j\colon V \to V$ such that K is a critical point of j and $a \in j(K)$.

Theorem

CZF *with a super Reinhardt set proves the consistency of* ZF *with a Reinhardt cardinal.*

Definition

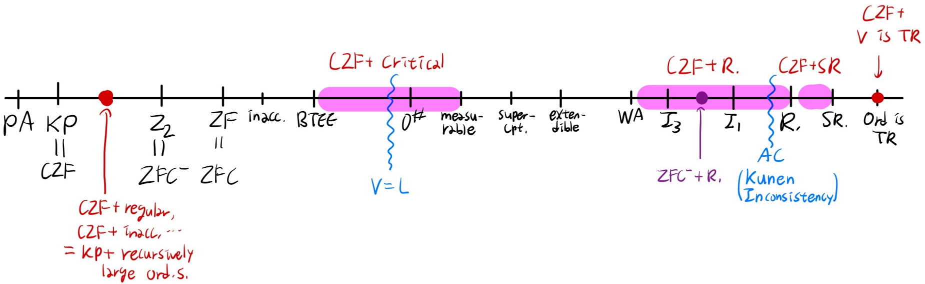
V is total Reinhardt (abbr. V is TR) is the following claim: for every class A, there is an A-super Reinhardt set K.

Theorem

CZF *with 'V is TR' proves all axioms of* IZF. *Furthermore, the following two theories are equiconsistent:*

- CZF + *'V is TR,' and*
- ZF + *'V is TR.'*

(The exact definition for super/total Reinhardtness require the formulation of constructive second-order set theory Appendix)


CZF+ V is TR
CZF+ Critical
CZF+R.
CZF+SR
PA
KP = CZF
Z2 = ZFC⁻
ZF = ZFC
inacc.
BTEE
0#
measurable
super-cpt.
extendible
WA
I3
I1
R.
SR.
Ord is TR
V=L
ZFC⁻+R.
AC (Kunen Inconsistency)
CZF+regular,
CZF+inacc, ...
= KP+ recursively large ord.s.

A rough sketch for the proofs

The proof divides into 2-3 main steps:

1. Internal analysis of the given large set axiom. Usually produces a model of $\mathsf{IZF} + \mathsf{X}$.
2. Double-negation translation: Friedman-styled translation, Gambino's Heyting-valued model, or their combinations. The resulting lower bound is of the form $\mathsf{Con}(\mathsf{ZF} + \mathsf{X})$
3. If possible, derive the consistency strength in terms of ZFC with large cardinal axioms.

Open problems

1. Non-trivial upper bounds for the consistency strength.
2. Better lower bounds. (For example, can we derive $\mathrm{Con}(\mathrm{ZFC}+\mathrm{WA})$ from $\mathrm{Con}(\mathrm{ZF}+\mathrm{WA})$?)
3. Defining other large set notions (e.g., supercompactness and extendibles) and analyzing their consistency strength.
4. Questions regarding machinery in the paper, e.g., second-order constructive set theory.

Questions

Thank you!

Constructive second-order set theory

Definition (Constructive Gödel-Bernays set theory, CGB)

CGB is defined over the two-sorted languages (sets and classes) with the following axioms:

- Axioms of CZF for sets.
- Every set is a class, and every element of a class is a set.
- Class Extensionality: two classes are equal if they have the same set members.
- Elementary Comprehension: if $\phi(x, p, C)$ is a first-order formula with a class parameter C, then there is a class A such that $A = \{x \mid \phi(x, p, C)\}$.

Definition (CGB, Continued)

- Class Set Induction:
  $\forall^1 A\big[[\forall^0 x(\forall^0 y \in x(y \in A) \to x \in A)] \to \forall^0 x(x \in A)\big]$.
- Class Strong Collection:
  $\forall^1 R \forall^0 a[R\colon a \rightrightarrows V \to \exists^0 b(R\colon a \leftrightarrows b)]$.

Definition (Intuitionistic Gödel-Bernays set theory, IGB)

IGB is obtained by adding the following axioms to CGB:

- Axioms of IZF for sets.
- Class Separation: if A is a class and a is a set, then $A \cap a$ is a set.

Note that CGB and IGB are conservative extensions of CZF and IZF respectively.

The definition of an elementary embedding $j\colon V \to M$ requires quantifying over formulas ϕ:

$$\phi(\vec{a}) \iff \phi^M(j(\vec{a})).$$

We resolve this problem by introducing the infinite conjunction $\bigwedge$.

Definition (CGB with the infinite connectives, CGB_∞)

CGB_∞ has the same axiom with CGB, but defined over the first-order intuitionistic logic with the infinite connectives $\bigwedge$ and $\bigvee$.

Super Reinhardtness and 'V is TR' is defined over CGB_∞. Also, CGB_∞ is a conservative extension of CGB.
(Back to main.)