

On a cofinal Reinhardt embedding without powerset

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Large cardinals

- Large cardinals are a means to gauge the strength of extensions of ZFC.
- Since the beginning of set theory, set theorists have defined stronger notions of large cardinals (Inaccessible, Mahlo, Weakly compact, Measurable, Woodin, Supercompact, etc.)
- Large cardinals stronger than measurable cardinals are usually defined in terms of elementary embedding.

Reinhardt embedding

Reinhardt defined an ‘ultimate’ form of a large cardinal axiom:

Definition

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 $j: V \rightarrow V$.

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This poor axiom destined an Icarian fate:

Theorem (Kunen 1971)

ZFC proves there is no Reinhardt embedding.

In fact, there is no elementary embedding $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$.

(Not in)consistent weakenings

Set theorists studied the non-inconsistent weakening of Reinhardt cardinals:

Definition

- $I_3(\lambda)$: There is an elementary $j: V_\lambda \rightarrow V_\lambda$.
- $I_2(\lambda)$: There is a Σ_1 -elementary $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$.
- $I_1(\lambda)$: There is an elementary $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$.
- $I_0(\lambda)$: There is an elementary $j: L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$.

They are not known to be inconsistent over ZFC.

What about other options?

We may have a consistent version of Reinhardt embedding over a weakening of ZFC.

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We do not know the consistency of ZF with a Reinhardt embedding, but

Theorem (Schlutzenberg 2024)

If $\text{ZFC} + \text{I}_0$ is consistent, then so is

$$\text{ZF} + \text{DC}_\lambda + \exists j: V_{\lambda+2} \rightarrow V_{\lambda+2}.$$

ZFC without powerset

The option we will examine is when we drop the axiom of powerset.

Remark

In ZFC without Replacement, the following are equivalent:

- 1 Replacement
- 2 Collection: For every family of proper classes $\{C_x \mid x \in I\}$ indexed by a set I , we have a family of sets $\{\hat{C}_x \mid x \in I\}$ such that $\hat{C}_x \subseteq C_x$.
- 3 Reflection principle.

It is no longer valid if we drop the Axiom of Powerset.

ZFC without Powerset can be weird

Theorem (Gitman-Hamkins-Johnstone 2011)

Let ZFC^- be ZFC without Powerset. Then each of the following is consistent with ZFC^- :

- 1 ω_1 is singular.
- 2 Every set of reals is countable, but ω_1 exists.
- 3 There are sets of reals of size ω_n for $n < \omega$, but none of size ω_ω .
- 4 The failure of Łoś's theorem.

However, ZFC^- , ZFC without Powerset but Collection, is free from these ill-behaviors.

Formulating a Reinhardt embedding

Let us formulate a set theory with a Reinhardt embedding j .
 j is a ‘proper class,’ but it cannot be definable:

Theorem (Suzuki 1999)

ZF proves there is no definable elementary embedding $j: V \rightarrow V$.

Hence, we must introduce a new symbol for a Reinhardt embedding.

Definition

ZFC_j is a first-order theory over the language $\{\in, j\}$ with the following axioms:

- 1 Axioms of ZFC.
- 2 Axiom schemes over the new language $\{\in, j\}$.

ZFC_j^- is defined similarly. Also, $j: V \rightarrow V$ is the combination of the following assertions:

- 1 $\exists x(j(x) \neq x)$.
- 2 An axiom scheme for the elementarity of j for $\{\in\}$ -formulas:
If $\psi(\vec{x})$ is a formula without j , then

$$\forall x[\phi(\vec{x}) \leftrightarrow \phi(j(\vec{x}))].$$

Matthews' result

Richard Matthews proved that $\text{ZFC}_j^- + j: V \rightarrow V$ is consistent:

Theorem (Matthews 2022)

$\text{ZFC} + I_1$ proves there is a transitive model of $\text{ZFC}_j^- + j: V \rightarrow V$.

However, Matthews' model does not satisfy

Definition

An embedding $j: V \rightarrow V$ is cofinal if for every set a , there is b such that $a \in j(b)$.

In fact, Hayut proved that ZFC_j^- is inconsistent with a cofinal Reinhardt embedding.

A Cofinal Reinhardt embedding

Question

Is $\text{ZF}_j^- + j: V \rightarrow V$ consistent with the cofinality of j ?

Theorem (J.)

$\text{ZFC} + I_0$ proves there is a transitive model of ZF_j^- with a cofinal $j: V \rightarrow V$.

Matthews' proof

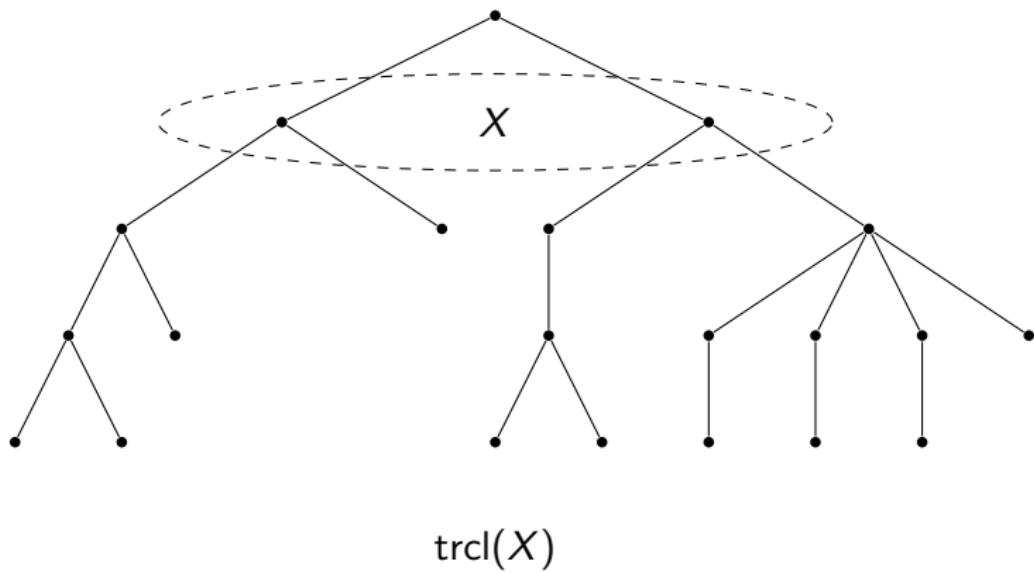
Let us sketch the main idea of (a variant of) a proof of Matthews' result.

Observation

Let λ be a strong limit cardinal, and let H_{λ^+} be the set of all hereditarily size $< \lambda^+$ sets:

$$H_{\lambda^+} = \{x : |\text{TC}(x)| < \lambda^+\}.$$

Then we can code every member of H_{λ^+} into a tree of size λ .



The tree for X is: $\{\langle x_0, x_1, \dots, x_n \rangle \mid X \ni x_0 \ni x_1 \ni \dots \ni x_n\}$.

Tree coding

For every well-founded tree T over V_λ , we can associate a set $t(T)$.

Lemma

For a well-founded tree T , let us define

$$t(T) = \{t(T \downarrow \langle x \rangle) \mid \langle x \rangle \in T\}.$$

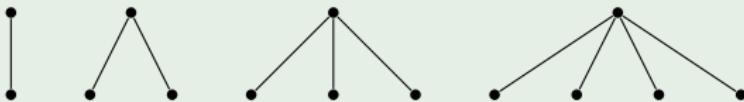
Then we have the following:

- 1 *If T is a well-founded tree over V_λ , then $t(T) \in H_{\lambda^+}$.*
- 2 *Every member of H_{λ^+} has a form $t(T)$.*

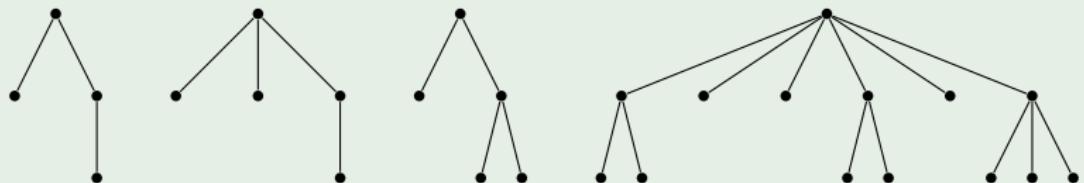
Note that even $1 = \{0\}$ has different ways for tree coding, even up to isomorphism.

Example

All of these code are the same set $1 = \{0\}$:

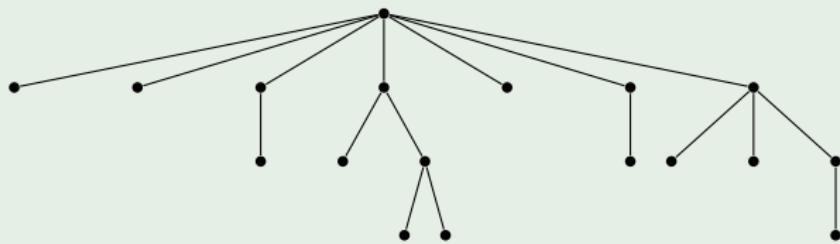


Also, all of these code the same set $2 = \{0, 1\}$:

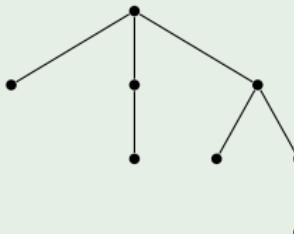


Example

The following tree codes $3 = \{0, 1, 2\}$:



But we also have a simpler tree coding 3:



Which trees are ‘equal’

Definition

Let S and T be well-founded trees. Define $S =^* T$ if and only if there is a binary relation $R \subseteq S \times T$ such that $\langle \langle \rangle, \langle \rangle \rangle \in R$, and $\langle \sigma, \tau \rangle \in R$ iff

- 1 $\forall \langle u \rangle \in (S \downarrow \sigma) \exists \langle v \rangle \in (T \downarrow \tau) [(\sigma \frown \langle u \rangle, \tau \frown \langle v \rangle) \in R]$, and
- 2 and vice versa.

We say $S \in^* T$ iff there is $\langle u \rangle \in T$ such that $S = T \downarrow \langle u \rangle$.

Theorem

If S, T are well-founded, then $S =^* T$ iff $t(S) = (T)$. Also, $S \in^* T$ iff $t(S) \in (T)$.

Tree interpretation

We can pull a formula over H_{λ^+} into $V_{\lambda+1}$:

Definition

Let ϕ be a formula. Define ϕ^t as follows:

- 1 $(x \in y)^t \equiv (x \in^* y)$. $(x = y)^t \equiv (x =^* y)$.
- 2 $(\phi \circ \psi)^t \equiv \phi^t \circ \psi^t$. $(\neg \phi)^t \equiv \neg \phi^t$. ($\circ = \wedge, \vee, \rightarrow$.)
- 3 For a quantifier Q,

$$(Qx\phi(x))^t \equiv QT[T \text{ is a well-founded tree over } V_\lambda \rightarrow \phi^t(T)].$$

Lemma

For every formula ϕ and well-founded trees T_0, \dots, T_{m-1} over V_λ , we have

$$H_{\lambda^+} \models \phi(t(T_0), \dots t(T_{m-1})) \iff V_{\lambda+1} \models \phi^t(T_0, \dots T_{m-1}).$$

Pushing $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$ into H_{λ^+}

Theorem

Let $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$ be an I_1 -embedding. For a well-founded tree T over V_λ , define

$$k(t(T)) = t(j(T)).$$

Then k is well-defined and an elementary embedding $H_{\lambda^+} \rightarrow H_{\lambda^+}$.

Corollary

(H_{λ^+}, k) is a model of $ZFC_j^- + j: V \rightarrow V$.

Finding a cofinal embedding

The resulting embedding is not cofinal by a Kunen inconsistency-type argument.

To get a cofinal elementary embedding, we start from a base model with a stronger property.

Definition (Goldberg-Schlutzenberg 2021)

Let $j: V_{\lambda+n} \rightarrow V_{\lambda+n}$ be an elementary embedding. We say j is cofinal if every $a \in V_{\lambda+n}$ is contained in $j(b)$ for some $b \in V_{\lambda+n}$...

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... Is it a correct definition?

Cofinal embedding over $V_{\lambda+n}$

Such b may not exist when a has the largest rank. However, we can still state $a \in j(b)$ for a ‘small’ subset b of $V_{\lambda+n}$:

Definition

Let $a \in V_{\lambda+n}$ be a binary relation. For $i \in \text{dom}(a)$, define

$$(a)_i = \{x \mid \langle i, x \rangle \in a\}.$$

Also, for $a, b \in V_{\lambda+n}$, define

$$(a : b) = \{(a)_i \mid i \in b\}.$$

The correct definition of a cofinality over $V_{\lambda+n}$

Definition (Goldberg-Schlutzenberg 2021)

$j: V_{\lambda+n} \rightarrow V_{\lambda+n}$ is cofinal if for every $a \in V_{\lambda+n}$ there is $b, c \in V_{\lambda+n}$ such that $a \in (j(b) : j(c))$.

Theorem (Goldberg-Schlutzenberg 2021)

$j: V_{\lambda+n} \rightarrow V_{\lambda+n}$ is cofinal iff n is even.

In particular, $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$ is cofinal.

Flat pairing

The previous definitions of $(a : b)$ and $(a)_i$ also have a ‘flaw’ since the usual Kuratowski ordered pair $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$ raises the rank by +2.

Hence, we have to use Quine-Rosser flat pairing instead of the usual pairing function.

Definition

Let

$$s(x) = \begin{cases} 2x + 1 & x \in \omega, \\ x & \text{otherwise.} \end{cases}$$

Define $f_0(a) = s[a]$ and $f_1(a) = s[a] \cup \{0\}$, then

- 1 f_0, f_1 are one-to-one.
- 2 $\text{ran } f_0 \cap \text{ran } f_1 = \emptyset$.

Define $\langle a, b \rangle = f_0[a] \cup f_1[b]$.

We also need a flat tuple to define trees, whose definition is similar.

Where to find $V_{\lambda+2}$?

We turn $V_{\lambda+2}$ with an elementary embedding $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$ into a transitive model of ZF_j^-
 $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$ is inconsistent with ZFC. But...

Theorem (Schlutzenberg 2024)

Let $i: L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ be an I_0 -embedding. If
 $j = i \upharpoonright V_{\lambda+2}^{L(V_{\lambda+1})}$, then $L(V_{\lambda+1}, j)$ satisfies

- 1 $\text{ZF} + \text{DC}_\lambda + I_0(\lambda)$.
- 2 $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$ is elementary.
- 3 $V_{\lambda+2} \subseteq L(V_{\lambda+1})$.

The model

Now let us work over Schlutzenberg's model $L(V_{\lambda+1}, j)$, which is a choiceless model. H_{λ^+} or similar notions do not work well without Choice.

Definition

Let X be a set. $H(X)$ is the union of all transitive sets M such that M is a surjective image of a member of X .

$H(X)$ is a transitive set, and every non-empty set in $H(X)$ is a surjective image of a member of X .

The Collection Principle

We can prove that $H(V_{\lambda+2})$ satisfies all axioms of ZF^- except for Collection. For Collection, we need the Collection principle:

Definition (Goldberg)

We say $V_{\lambda+1}$ satisfies the Collection principle if every binary relation $R \subseteq V_\lambda \times V_{\lambda+1}$ has a subrelation $S \subseteq R$ of the same domain such that $\text{ran } S$ is a surjective image of $V_{\lambda+1}$.

Theorem (Essentially by Goldberg)

$L(V_{\lambda+1}, j)$ thinks $V_{\lambda+2}$ satisfies the Collection principle.

Theorem

The Collection principle for $V_{\lambda+2}$ implies $H(V_{\lambda+2})$ satisfies Collection.

$L(V_{\lambda+1}, j)$ satisfies the Collection principle for $V_{\lambda+2}$, so $H(V_{\lambda+2}) \models \text{ZF}^-$ in this model.

The main result

Again, we can define the tree interpretation t satisfying

$$H(V_{\lambda+2}) \vDash \phi(t(T_0), \dots, t(T_{m-1})) \iff V_{\lambda+2} \vDash \phi^t(T_0, \dots, T_{m-1}).$$

Then we can push $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$ to $k: H(V_{\lambda+2}) \rightarrow H(V_{\lambda+2})$ by letting $k(t(T)) = t(j(T))$.

Theorem

In $L(V_{\lambda+1}, j)$, $k: H(V_{\lambda+2}) \rightarrow H(V_{\lambda+2})$ is a cofinal elementary embedding.

Proof.

Every set in $H(V_{\lambda+2})$ is of the form $t(T)$ for some well-founded tree over $V_{\lambda+1}$.

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$T \in V_{\lambda+2}$, so by the cofinality of j , we can find sets $a, b \in V_{\lambda+2}$ such that $T \in (j(a) : j(b))$.

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Then define

$$T' = \{\langle x \rangle^\frown \sigma \mid x \in b \wedge \sigma \in (a)_x \wedge (a)_x \text{ is a well-founded tree}\}$$

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$T \in (j(a) : j(b))$ implies there is $z \in j(b)$ such that $T = (j(a))_z$.
Hence $T \in^* j(T')$. □

Comparing the two proofs

Matthews' proof	My proof
Working over ZFC + I ₁	Schlutzenberg's model
I ₁ embedding $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$.	An embedding $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$
Turn $V_{\lambda+1}$ to H_{λ^+}	Turn $V_{\lambda+2}$ to $H(V_{\lambda+2})$
Collection holds by Choice	by Collection Principle
A model of ZFC _j ⁻	A model of ZF _j ⁻ with a cofinal j

Questions

Question

How strong is the theory ZF_j^- with a cofinal $j: V \rightarrow V$? For example, does it imply the consistency of $ZFC + I_1$?

Question

Does ZF_j^- with a cofinal $j: V \rightarrow V$ prove λ^+ or $V_{\lambda+1}$ exists, for $\lambda = \sup_{n < \omega} j^n(\text{crit } j)$?

(Note: $V_{\lambda+1} \in H(V_{\lambda+2})$ in the Schlutzenberg's model.)

Any other Questions?



Thank you!