Project Euler.net

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problem-902

Permutation powers

Problem Statement

A permutation π of $\{1, \ldots, n\}$ can be represented in **one-line notation** as

$$\pi(1), \pi(2), \ldots, \pi(n).$$

If all $n!$ permutations are written in lexicographic order then $\operatorname{rank}(\pi)$ is the position of π in this 1-based list.

For example,

$$\operatorname{rank}(2, 1, 3) = 3$$

because the six permutations of $\{1, 2, 3\}$ in lexicographic order are

$$1, 2, 3 \quad 1, 3, 2 \quad 2, 1, 3 \quad 2, 3, 1 \quad 3, 1, 2 \quad 3, 2, 1.$$

For a positive integer m, define a permutation of $\{1, \ldots, n\}$ with $n = \frac{m(m+1)}{2}$ as follows:

$$\sigma(i) = \begin{cases} \frac{k(k-1)}{2} + 1, & \text{if } i = \frac{k(k+1)}{2},\ k \in \{1, \ldots, m\}, \\ i + 1, & \text{otherwise}, \end{cases}$$

$$\tau(i) = \big((10^9 + 7)i \bmod n\big) + 1,$$
$$\pi(i) = \tau^{-1}\big(\sigma(\tau(i))\big),$$

where τ^{-1} denotes the inverse permutation of τ.

Define

$$P(m) = \sum_{k=1}^{m!} \operatorname{rank}(\pi^k),$$

where π^k denotes the permutation arising from applying π k times.

For example,

$$P(2) = 4, \quad P(3) = 780, \quad P(4) = 38810300.$$

Task: Find $P(100)$. Give your answer modulo $10^9 + 7$.

Solution

from 1 to n find π(i) and store it into list. for perticular i-th term π^l(i) will be $\pi(i)$ where l is cycle length. store it in a list for all i from 1 to n.

$$P(m) = \sum_{k=1}^{m!} \mathrm{rank}(\pi^k)$$

$$\mathrm{rank}(\pi^t) = 1 + \sum_{i=1}^{n} c_i (n-i)!$$

$$c_i = \#\{\pi^t(j) < \pi^t(i) \mid \pi^t(j) \text{ not among } \pi^t(1), \pi^t(2), \ldots, \pi^t(i-1)\}$$

So computing a single rank costs $O(n^2)$. Doing this for π^t with $t = 1, \ldots, m!$ is impossible for $m = 100$.

$$P = \sum_{t=1}^{m!} \mathrm{rank}(\pi^t) = \sum_{i=1}^{m!} 1 + \sum_{i=1}^{n} (n-i)! \sum_{t=1}^{m!} c_i^t$$

$$A = \sum_{t=1}^{m!} 1$$

where the first term can be computed separately (denoted as Term A). Now we have to reduce the second term. Let

$$\phi_i = \sum_{t=1}^{m!} c_i^t$$

$$c_i^t = \#\{\pi^t(j) < \pi^t(i) \mid \pi^t(j) \in \{\pi^t(i+1), \ldots, \pi^t(n)\}\}$$

$$c_i^t = \pi^t(i) - 1 - \{\pi^t(j) < \pi^t(i) \text{ and } \pi^t(j) \in \{\pi^t(1), \ldots, \pi^t(i-1)\}\}$$

$$\phi_i = \sum_{t=1}^{m!} \big(\pi^t(i) - 1\big) - \sum_{t=1}^{m!} \# \big\{\pi^t(j) < \pi^t(i) \text{ and } \ \pi^t(j) \in \pi^t(1),\ \pi^t(2), \ldots, \pi^t(i-1)\big\}$$

After certain cycle, each ith term repeats, of cycle length $l_i$then $\pi^{l_i}(i) = \pi(i)$ For ith term, cycle length is l_i

$$\begin{aligned} P &= \sum_{t=1}^{m!} 1 + \sum_{i=1}^{n} (n-i)! \left(\frac{m!}{l_i} \sum_{t=1}^{l_i} \big(\pi^t(i) - 1\big) \right) \\ &\quad - \sum_{i=1}^{n} (n-i)! \sum_{t=1}^{m!} \sum_{j=1}^{i-1} 1 \left\{\pi^t(j) < \pi^t(i)\right\} \\ &= A + B - C \end{aligned}$$

A & B term: Now we can calculate with exponentially decaying time complexity.
Only C term we have to reduce:

$$C = \sum_{i=1}^{n} (n-i)! \sum_{t=1}^{m!} \sum_{j=1}^{i-1} 1\left\{\pi^t(j) < \pi^t(i)\right\}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{i-1} (\cdots) \equiv \sum_{j=1}^{n} \sum_{i=j+1}^{n} (\cdots)$$

$$C = \sum_{j=1}^{n} \sum_{i=j+1}^{n} (n-i)! \sum_{t=1}^{m!} 1\left\{\pi^t(j) < \pi^t(i)\right\}$$

$$A_{ij} = \sum_{t=1}^{m!} 1\left\{\pi^t(j) < \pi^t(i)\right\}$$

$$C = \sum_{1 \le j < i \le n} (n-i)!\, A_{ij}$$

If we took i & j pair and cycle length of i and jth is l_i & l_j respectively:

$$\pi^{l_i}(i) = \pi(i)$$

Let $\mathrm{lcm}(l_i, l_j) = l_{ij}$

$$\pi^{l_{ij}}(i) = \pi(i) \quad \text{and} \quad \pi^{l_{ij}}(j) = \pi(j)$$

$$C = \frac{m!}{l_{ij}} \sum_{1 \le j < i \le n} (n-i)! \sum_{t=1}^{l_{ij}} 1\left\{\pi^t(j) < \pi^t(i)\right\}$$

Now all terms A, B, C can computationally be calculated

$$A = \sum_{t=1}^{m!} 1 = m!$$

$$B = \sum_{i=1}^{n} (n-i)! \frac{m!}{l_i} \sum_{t=1}^{l_i} (\pi^t(i) - 1)$$

$$C = \sum_{1 \le j < i \le n} (n-i)! \frac{m!}{l_{ij}} \sum_{t=1}^{l_{ij}} 1\left\{\pi^t(j) < \pi^t(i)\right\}$$

$$\boxed{P = A + B - C}$$

$$\boxed{P(100) = 343557869}$$