HW

1. Public Goods & Externalities

HW 1. *Prove that at the Walrasian equilibrium, the only contributor is the agent with highest* $v_j'(y^*)$.

HW 2. *Prove that Lindahl equilibrium is IR.*

HW 3. *Check that in the Kolm triangle, if* $p_1 + p_2 = 1$, *then price hyperplanes coincide.*

2. Social Choice

HW 4. *Consider the domain of single-peaked preferences and let* F *be pairwise majority voting SWF. Is* F *transitive? What happens if* n *is odd/even? Could* F *have cycles "below the top"?*

HW 5. *Consider the domain of single-peaked preferences and let* F *be the SWF under pairwise majority voting. Is* F *transitive? (What happens if* n *is odd/even?) Could* F *have cycles "below the top"?*

HW 6. *We've shown the proof of Muller-Satterthwaite theorem using social choice function. Go through the similar proof but with social welfare function, Paretian and IIA. (Read Reny's paper)*

HW 7. *Prove or disprove*

1. F *is IIA + PAR* $\implies$ f *MONO +WP*
2. F *is dictatorial* $\implies$ f *dictatorial*
3. f *MONO +WP* $\implies$ F *is IIA + PAR*
4. f *dictatorial* $\implies$ F *is dictatorial*

HW 8. *We know how to construct from* F *(SWF) to* f *(SCF) and to* F' *(SWF). Is* F *the same as* F'*?*

3. Mechanism Design

HW 9. *We've shown that the revelation principle holds for single agent utility maximization.*

$$F(\theta) \supseteq g\left(\mu_\Gamma(\theta)\right) \supseteq g\left(\hat{\mu}_\Gamma(\theta)\right) = h(\theta) \in h\left(\mu_{\hat{\Gamma}}(\theta)\right)$$

1. *Is* $h(\theta) = h(\mu_{\hat{\gamma}}(\theta))$ *for every* θ*?*
2. *If* $\Gamma = (S, g)$ *weakly implements* F, *then* $\hat{\Gamma} = (\Theta, h \equiv g \circ \hat{\mu}_\Gamma)$ *weakly implements* F.

HW 10. *Suppose* f *is single-valued. If there exists* Γ *that weakly implements* f, *then* f *is IC. The contra-positive to this result is: If* f *is not IC, then there is no* Γ *that weakly implements* f. *Could there exist a* Γ *that partially implements* f*?*

HW 11.

Theorem 1. *Suppose SC_+ and uniformly bounded MRS.*
$f(\theta) = (x(\theta), t(\theta))$ is IC, if and only if

1. $\frac{dx}{d\theta} \geq 0$, $\forall \theta$

2. $\frac{dt}{d\theta} = -\left(\frac{du/dx}{du/dt}\right) \cdot \frac{dx}{d\theta}$, $\forall \theta$

Prove the "if" part of this theorem by contradiction.

HW 12. *Consider a quasi-linear setting:*

- *Principal:* $u_0(x, t) = v_0(x) - t$
- *Agent:* $u_1(x, t|\theta) = v_1(x|\theta) + t$
- *MRS:* $\frac{\partial v_1}{\partial x} > 0$; SC_+: $\frac{\partial^2 v_1}{\partial x \partial \theta} > 0$

$$\text{(First-best)} \qquad \max_{x(\cdot), t(\cdot)} \quad v_0(x(\theta)) - t(\theta) \quad s.t. \text{ [IR] } v_1(x(\theta)|\theta) + t(\theta) \geq 0, \ \forall \theta$$

- *IR will be binding at every θ.*
- $x \in \arg\max_{x(\cdot)} \{v_0(x(\theta)) + v_1(x(\theta)|\theta)\}$, *so x is P.O.*
- *x is not IC (agent has incentive to lie downwards):*

$$u_1(x(\hat{\theta}), t(\hat{\theta})|\theta) = v_1(x(\hat{\theta})|\theta) \quad \underbrace{-v_1(x(\hat{\theta})|\hat{\theta})}_{=t(\hat{\theta}) \text{ (binding IR)}}$$

$$[\hat{\theta}] \quad \frac{\partial v_1(x(\hat{\theta})|\theta)}{\partial x} \frac{dx(\hat{\theta})}{d\hat{\theta}} - \frac{\partial v_1(x(\hat{\theta})|\hat{\theta})}{\partial x} \frac{dx(\hat{\theta})}{d\hat{\theta}} - \frac{\partial v_1(x(\hat{\theta})|\hat{\theta})}{\partial \theta} = 0$$

$$\iff \underbrace{\left[\frac{\partial v_1(x(\hat{\theta})|\theta)}{\partial x} - \frac{\partial v_1(x(\hat{\theta})|\hat{\theta})}{\partial x}\right]}_{\geq 0} \frac{dx(\hat{\theta})}{d\hat{\theta}} = \frac{\partial v_1(x(\hat{\theta})|\hat{\theta})}{\partial \theta} \implies \quad \text{By } SC_+, \ \theta \geq \hat{\theta}$$

Here, we need to show that the first best solution satisfies $dx(\hat{\theta})/d\hat{\theta} \geq 0$.

(Second-best)

$$\max_{x(\cdot), t(\cdot)} \quad \int_{\underline{\theta}}^{\hat{\theta}} [v_0(x(\theta)) - t(\theta)]\, g(\theta) d\theta \quad s.t. \text{ [IR] } v_1(x(\theta)|\theta) + t(\theta) \geq 0, \ \forall \theta$$

$$\text{[IC] } v_1(x(\theta)|\theta) + t(\theta) \geq v_1(x(\hat{\theta})|\theta) + t(\hat{\theta}), \ \forall \theta, \hat{\theta}$$

1. *Verify that the first best solution satisfies $dx(\hat{\theta})/d\hat{\theta} \geq 0$.*
2. *Verify that the second best solution satisfies $dx(\hat{\theta})/d\hat{\theta} \geq 0$.*

3. *In class, we solve the second best solution with SC_+. What happens if we assume SC_- instead?*

HW 13. *In the example of regulating a firm, using integration by parts to rewrite the double integral*

$$\int \int_{\theta}^{\bar{\theta}} \phi'(\tau - c(\tau)) d\tau g(\theta) d\theta.$$

HW 14.

Theorem 2. *With linear preferences, f is BIC, if and only if*

1. *V_i is non-decreasing in θ_i*
2. *$T_i(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} V_i(\tau) d\tau - \theta_i V_i(\theta_i) + u_i(\underline{\theta}_i)$*

Prove that the "if" part of this theorem by contradiction.

HW 15. *We've shown that the optimal mechanism is equivalent to first price auction with reserve price. With $\sum_i x_i \leq 1$, the planner can keep the good. What would be the optimal mechanism with $\sum_i x_i = 1$?*

HW 16. 1. *Prove that the revelation principle holds for BNE.*

2. *Could there be any bad equilibrium? YES (Read Example 4.1.2 in Dasgupta, Hammond & Maskin (1979))*

HW 17. 1. *Prove that the revelation principle holds for dominant strategy equilibrium.*

HW 18. *We can write transfers in different ways:*
DSIC $\implies$ [FOC] $\frac{\partial t_i}{\partial \hat{\theta}_i} = -\frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \hat{\theta}_i} \implies t_i(\theta_i, \hat{\theta}_{-i}) = t_i(\underline{\theta}_i, \hat{\theta}_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \hat{\theta}_i} d\tau$

DSIC $\implies$ [using Envelope Theorem] $t_i(\theta_i, \hat{\theta}_{-i}) = u_i(\underline{\theta}_i, \hat{\theta}_{-i}) + v_i(y(\theta_i, \hat{\theta}_{-i})|\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} \frac{dv_i}{d\theta_i} d\tau$

Are they the same? Why?

HW 19. *If we add marginal cost K and consider budget balance $\sum_i t_i = -Ky$, how should we modify VCG/Groves mechanisms?*

HW 20. *Is Arrow-d'Aspremont-Gerard-Varet mechanism DSIC?*

HW 21. *Show that in the Groves & Ledyard mechanism, every NE m^* gives a P.O. outcome (+BB).*