

HW

1. Public Goods & Externalities

Question 1. Consider the problem of public goods. Prove that at the Walrasian equilibrium, the only contributor is the agent with highest $v'_j(y^*)$.

Question 2. Show that Lindahl equilibrium is IR.

Question 3. Check that in the Kolm triangle, if $p_1 + p_2 = 1$, then price hyperplanes coincide.

2. Social Choice

Question 4. Consider the domain of single-peaked preferences and let F be pairwise majority voting SWF. Is F transitive? What happens if n is odd/even? Could F have cycles “below the top”?

Question 5. Consider the domain of single-peaked preferences and let F be the SWF under pairwise majority voting. Is F transitive? (What happens if n is odd/even?) Could F have cycles “below the top”?

Question 6. We’ve shown the proof of Muller-Satterthwaite theorem using social choice function. Go through the similar proof but with social welfare function, Paretian and IIA. (Read Reny’s paper)

Question 7. Prove or disprove

1. F is IIA + PAR $\implies f$ MONO + WP
2. F is dictatorial $\implies f$ dictatorial
3. f MONO + WP $\implies F$ is IIA + PAR
4. f dictatorial $\implies F$ is dictatorial

Question 8. How to construct from F (SWF) to f (SCF) and to F' (SWF)? Is F the same as F' ?

3. Mechanism Design

Question 9. We’ve shown that the revelation principle holds for single agent utility maximization.

$$F(\theta) \supseteq g(\mu_\Gamma(\theta)) \supseteq g(\hat{\mu}_\Gamma(\theta)) = h(\theta) \in h(\mu_{\hat{\Gamma}}(\theta))$$

1. Is $h(\theta) = h(\mu_{\hat{\gamma}}(\theta))$ for every θ ?
2. If $\Gamma = (S, g)$ weakly implements F , then $\hat{\Gamma} = (\Theta, h \equiv g \circ \hat{\mu}_\Gamma)$ weakly implements F .

Question 10. Suppose f is single-valued. If there exists Γ that weakly implements f , then f is IC. The contra-positive to this result is: If f is not IC, then there is no Γ that weakly implements f . Could there exist a Γ that partially implements f ?

Question 11.

Theorem 1. Suppose SC_+ and uniformly bounded MRS.
 $f(\theta) = (x(\theta), t(\theta))$ is IC, if and only if

1. $\frac{dx}{d\theta} \geq 0, \forall \theta$
2. $\frac{dt}{d\theta} = - \left(\frac{du/dx}{du/dt} \right) \cdot \frac{dx}{d\theta}, \forall \theta$

Prove the “if” part of this theorem by contradiction.

Question 12. Consider a quasi-linear setting:

- Principal: $u_0(x, t) = v_0(x) - t$
- Agent: $u_1(x, t|\theta) = v_1(x|\theta) + t$
- MRS: $\frac{\partial v_1}{\partial x} > 0$; SC_+ : $\frac{\partial^2 v_1}{\partial x \partial \theta} > 0$

$$(\text{First-best}) \quad \max_{x(\cdot), t(\cdot)} \quad v_0(x(\theta)) - t(\theta) \quad \text{s.t.} \quad [IR] \quad v_1(x(\theta)|\theta) + t(\theta) \geq 0, \quad \forall \theta$$

- IR will be binding at every θ .
- $x \in \arg \max_{x(\cdot)} \{v_0(x(\theta)) + v_1(x(\theta)|\theta)\}$, so x is P.O.
- x is not IC (agent has incentive to lie downwards):

$$\begin{aligned} u_1(x(\hat{\theta}), t(\hat{\theta})|\theta) &= v_1(x(\hat{\theta})|\theta) \underbrace{-v_1(x(\hat{\theta})|\hat{\theta})}_{=t(\hat{\theta}) \text{ (binding IR)}} \\ [\hat{\theta}] \quad &\frac{\partial v_1(x(\hat{\theta})|\theta)}{\partial x} \frac{dx(\hat{\theta})}{d\hat{\theta}} - \frac{\partial v_1(x(\hat{\theta})|\hat{\theta})}{\partial x} \frac{dx(\hat{\theta})}{d\hat{\theta}} - \frac{\partial v_1(x(\hat{\theta})|\hat{\theta})}{\partial \theta} = 0 \\ \iff \quad &\underbrace{\left[\frac{\partial v_1(x(\hat{\theta})|\theta)}{\partial x} - \frac{\partial v_1(x(\hat{\theta})|\hat{\theta})}{\partial x} \right]}_{\geq 0} \frac{dx(\hat{\theta})}{d\hat{\theta}} = \frac{\partial v_1(x(\hat{\theta})|\hat{\theta})}{\partial \theta} \implies \quad \text{By } SC_+, \quad \theta \geq \hat{\theta} \end{aligned}$$

Here, we need to show that the first best solution satisfies $dx(\hat{\theta})/d\hat{\theta} \geq 0$.

(Second-best)

$$\max_{x(\cdot), t(\cdot)} \quad \int_{\theta}^{\hat{\theta}} [v_0(x(\theta)) - t(\theta)] g(\theta) d\theta \quad \text{s.t.} \quad [IR] \quad v_1(x(\theta)|\theta) + t(\theta) \geq 0, \quad \forall \theta$$

$$[IC] \quad v_1(x(\theta)|\theta) + t(\theta) \geq v_1(x(\hat{\theta})|\theta) + t(\hat{\theta}), \quad \forall \theta, \hat{\theta}$$

1. Verify that the first best solution satisfies $dx(\hat{\theta})/d\hat{\theta} \geq 0$.
2. Verify that the second best solution satisfies $dx(\hat{\theta})/d\hat{\theta} \geq 0$.

3. In class, we solve the second best solution with SC_+ . What happens if we assume SC_- instead?

Question 13. In the example of regulating a firm, using integration by parts to rewrite the double integral

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} \phi'(\tau - c(\tau)) d\tau g(\theta) d\theta.$$

Question 14.

Theorem 2. With linear preferences, f is BIC, if and only if

1. V_i is non-decreasing in θ_i
2. $T_i(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} V_i(\tau) d\tau - \theta_i V_i(\theta_i) + u_i(\theta_i)$

Prove that the “if” part of this theorem by contradiction.

Question 15. We’ve shown that the optimal mechanism is equivalent to first price auction with reserve price. With $\sum_i x_i \leq 1$, the seller can keep the good. What would be the optimal mechanism with $\sum_i x_i = 1$?

Question 16. 1. Prove that the revelation principle holds for BNE.

2. Could there be any bad equilibrium? YES (Read Example 4.1.2 in Dasgupta, Hammond & Maskin (1979))

Question 17. 1. Prove that the revelation principle holds for dominant strategy equilibrium.

Question 18. We can write transfers in different ways:

$$DSIC \implies [FOC] \frac{\partial t_i}{\partial \theta_i} = -\frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \theta_i} \implies t_i(\theta_i, \hat{\theta}_{-i}) = t_i(\theta_i, \hat{\theta}_{-i}) - \int_{\theta_i}^{\theta_i} \frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \theta_i} d\tau$$

$$DSIC \implies [\text{using Envelope Theorem}] t_i(\theta_i, \hat{\theta}_{-i}) = u_i(\theta_i, \hat{\theta}_{-i}) - v_i(y(\theta_i, \hat{\theta}_{-i}) | \theta_i) + \int_{\theta_i}^{\theta_i} \frac{dv_i}{d\theta_i} d\tau$$

Are they the same? Why?

Question 19. If we add marginal cost K and consider budget balance $\sum_i t_i = -Ky$, how should we modify VCG/Groves mechanisms?

Question 20. Is Arrow-d’Aspremont-Gerard-Varet mechanism DSIC?

Question 21. Show that in the Groves & Ledyard mechanism, every NE m^* gives a P.O. outcome (+BB).