03/10/2023 Recitation #2 Handout

1. Public Goods

1.0. Setting:

- The set of agents $N = \{1, 2, \dots, n\}$.
- Public good consumption y
- Private good consumption $\mathbf{x} = (x_1, \dots, x_n)$
- Agent i's utility $u_i(x_i, y) = v_i(y) + x$ (assume $v_i'' \leq 0$)
- The cost of producing public good $c(y)$ (assume $c' > 0$, $c'' \geq 0$)

1.1. Pareto optimal level of public good

Pareto optimality (+ interior solution) $\implies$ Samuelson condition $\sum_i v_i'(y) = c'(y)$.

1.2. Private provision of public good

Consider case where public good provided by means of private purchases by consumers.
A Walrasian equilibrium is a price vector $(p^*, 1)$[1] and an allocation $(y^*, \mathbf{x}^*)$:

1. Consumers optimize:
$$y_i^* \in \underset{y_i \geq 0}{\operatorname{argmax}}\ v_i\left(y_i + \textstyle\sum_{j \neq i} y_j\right) - p^* y_i + \theta_i \Pi_0(y_0^*, p^*) \implies v_i'\left(y_i^* + \textstyle\sum_{j \neq i} y_j^*\right) \leq p^*$$
2. Firm optimizes:
$$y_0^* \in \underset{y_0 \geq 0}{\operatorname{argmax}}\ \Pi_0(y_0, p^*) := p^* y_0 - c(y_0) \implies p^* \leq c'(y_0^*)$$
3. Market clears: $y_0^* = \sum_i y_i^*$ (and $x_i^* = -p^* y_i^* + \theta_i \Pi_0(y_0^*, p^*)$, $\forall i$)

To sump up, $v_i'(y^*) \leq p^*$, $\forall i$. Draw a graph.

1.3. Internalizing the externality

Consider case where every consumer reports the total amount of the public good she will consume.
A Lindahl equilibrium is a price vector $(\mathbf{p}^*, 1)$ such that $\mathbf{p}^* = (p_1^*, p_2^*, \dots, p_n^*)$ and an allocation $(y^*, \mathbf{x}^*)$:

1. Consumers optimize:
$$y_i^* \in \underset{y_i \geq 0}{\operatorname{argmax}}\ v_i(y_i) - p_i^* y_i + \theta_i \Pi_0(y_0^*, \mathbf{p}^*) \implies v_i'(y_i^*) \leq p_i^*$$
2. Firm optimizes:
$$y_0^* \in \underset{y_0 \geq 0}{\operatorname{argmax}}\ \Pi_0(y_0, \mathbf{p}^*) := \textstyle\sum_i p_i^* y_0 - c(y_0) \implies \textstyle\sum_i p_i^* \leq c'(y_0^*)$$
3. Market clears: $y_0^* = y_1^* = \dots = y_n^*$ (and $x_i^* = -p_i^* y_i^* + \theta_i \Pi_0(y_0^*, \mathbf{p}^*)$, $\forall i$)

To sum up, with interior solution, $\sum_i v_i'(y^*) = c'(y^*)$. [Nice outcome: P.O. & IR] Draw a graph.

[1] WLOG, we can normalize the price for the private good to be 1.

2. Social Choice

2.0. Definitions

Example 1. *Two teaching assistants decide the color of printing papers for the midterm exam.*

- SWF vs SCF/SCC
- Axioms on SWF: Paretian, IIA
- Axioms on SCF: Weak Paretian, monotonicity
- dictatorial

2.1. Arrow's impossibility theorem

We've proved Muller-Satterthwaite theorem (in terms of SCF) during the lecture. Refer to Reny (2000) *Arrow's Theorem and the Gibbard-Satterthwaite Theorem: A Unified Approach.*
Key assumptions:
URIP3: U(niversal domain) - R(ational) - I(IA) - P(aretian) - 3 (alternatives)

- U: think about majority rule under single-peaked preferences
- R: think about Condorcet cycles
- I: think about Borda count
- P: think about constant SCF
- 3: think about majority rule with $|X| = 2$

3. Exercises

1. (HW) Consider the problem of public goods. Prove that at the Walrasian equilibrium, the only contributor is the agent with highest $v_j'(y^*)$.
2. (HW) Check that in the Kolm triangle, if $p_1 + p_2 = 1$, then price hyperplanes coincide.
3. (HW) Consider the domain of single-peaked preferences and let F be the SWF under pairwise majority voting. Is F transitive? (What happens if n is odd/even?) Could F have cycles "below the top"?
4. (HW) How to construct from F (SWF) to f (SCF) and to F' (SWF)? Is F the same as F'?