# HW

## 1. Public Goods & Externalities

**HW 1.** *Prove that at the Walrasian equilibrium, the only contributor is the agent with highest* $v_j'(y^*)$.

**HW 2.** *Prove that Lindahl equilibrium is IR.*

**HW 3.** *Check that in the Kolm triangle, if* $p_1 + p_2 = 1$, *then price hyperplanes coincide.*

## 2. Social Choice

**HW 4.** *Consider the domain of single-peaked preferences and let* $F$ *be pairwise majority voting SWF. Is* $F$ *transitive? What happens if* $n$ *is odd/even? Could* $F$ *have cycles "below the top"?*

**HW 5.** *Consider the domain of single-peaked preferences and let* $F$ *be the SWF under pairwise majority voting. Is* $F$ *transitive? (What happens if* $n$ *is odd/even?) Could* $F$ *have cycles "below the top"?*

**HW 6.** *We've shown the proof of Muller-Satterthwaite theorem using social choice function. Go through the similar proof but with social welfare function, Paretian and IIA. (Read Reny's paper)*

**HW 7.** *Prove or disprove*

1. $F$ *is IIA + PAR* $\implies$ $f$ *MONO +WP*
2. $F$ *is dictatorial* $\implies$ $f$ *dictatorial*
3. $f$ *MONO +WP* $\implies$ $F$ *is IIA + PAR*
4. $f$ *dictatorial* $\implies$ $F$ *is dictatorial*

**HW 8.** *We know how to construct from* $F$ *(SWF) to* $f$ *(SCF) and to* $F'$ *(SWF). Is* $F$ *the same as* $F'$*?*

## 3. Mechanism Design

**HW 9.** *We've shown that the revelation principle holds for single agent utility maximization.*

$$F(\theta) \supseteq g\left(\mu_\Gamma(\theta)\right) \supseteq g\left(\hat{\mu}_\Gamma(\theta)\right) = h(\theta) \in h\left(\mu_{\hat{\Gamma}}(\theta)\right)$$

1. *Is* $h(\theta) = h(\mu_{\hat{\gamma}}(\theta))$ *for every* $\theta$*?*
2. *If* $\Gamma = (S, g)$ *weakly implements* $F$, *then* $\hat{\Gamma} = (\Theta, h \equiv g \circ \hat{\mu}_\Gamma)$ *weakly implements* $F$.

**HW 10.** *If* $f$ *is single-valued, then if* $\exists\Gamma$ *that weakly implements* $f$, *then* $f$ *is IC. The contrapositive to this result is: If* $f$ *is not IC, then there is no* $\Gamma$ *that weakly implements* $f$. *Could there exist a* $\Gamma$ *that partially implements* $f$*?*

**HW 11.**

**Theorem 1.** *Suppose $SC_+$ and uniformly bounded MRS.*
*$f(\theta) = (x(\theta), t(\theta))$ is IC, if and only if*

1. $\frac{dx}{d\theta} \geq 0$, $\forall\theta$

2. $\frac{dt}{d\theta} = -\left(\frac{du/dx}{du/dt}\right) \cdot \frac{dx}{d\theta}$, $\forall\theta$

*Prove the "if" part of this theorem by contradiction.*

**HW 12.** *Consider a quasi-linear setting:*

- *Principal:* $u_0(x, t) = v_0(x) - t$
- *Agent:* $u_1(x, t|\theta) = v_1(x|\theta) + t$
- *MRS:* $\frac{\partial v_1}{\partial x} > 0$; *$SC_+$:* $\frac{\partial^2 v_1}{\partial x \partial \theta} > 0$

(First-best) $$\max_{x(\cdot),t(\cdot)} \quad v_0(x(\theta)) - t(\theta) \quad \textit{s.t. [IR]}\ v_1(x(\theta)|\theta) + t(\theta) \geq 0,\ \forall\theta$$

- *IR will be binding at every $\theta$.*
- $x \in \arg\max_{x(\cdot)} \{v_0(x(\theta)) + v_1(x(\theta)|\theta)\}$, *so $x$ is P.O.*
- *$x$ is not IC (agent has incentive to lie downwards):*

$$u_1(x(\hat\theta), t(\hat\theta)|\theta) = v_1(x(\hat\theta)|\theta) \quad \underbrace{-v_1(x(\hat\theta)|\hat\theta)}_{=t(\hat\theta)\ \textit{(binding IR)}}$$

$$[\hat\theta] \quad \frac{\partial v_1(x(\hat\theta)|\theta)}{\partial x}\frac{dx(\hat\theta)}{d\hat\theta} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial x}\frac{dx(\hat\theta)}{d\hat\theta} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial\theta} = 0$$

$$\iff \underbrace{\left[\frac{\partial v_1(x(\hat\theta)|\theta)}{\partial x} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial x}\right]}_{\geq 0}\frac{dx(\hat\theta)}{d\hat\theta} = \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial\theta} \implies \quad \textit{By } SC_+,\ \theta \geq \hat\theta$$

*Here, we need to show that the first best solution satisfies $dx(\hat\theta)/d\hat\theta \geq 0$.*

(Second-best)

$$\max_{x(\cdot),t(\cdot)} \quad \int_{\underline{\theta}}^{\hat\theta} [v_0(x(\theta)) - t(\theta)]\, g(\theta) d\theta \quad \textit{s.t. [IR]}\ v_1(x(\theta)|\theta) + t(\theta) \geq 0,\ \forall\theta$$

$$\textit{[IC]}\ v_1(x(\theta)|\theta) + t(\theta) \geq v_1(x(\hat\theta)|\theta) + t(\hat\theta),\ \forall\theta, \hat\theta$$

1. *Verify that the first best solution satisfies $dx(\hat\theta)/d\hat\theta \geq 0$.*
2. *Verify that the second best solution satisfies $dx(\hat\theta)/d\hat\theta \geq 0$.*

3. *In class, we solve the second best solution with $SC_+$. What happens if we assume $SC_-$ instead?*

**HW 13.** *In the example of regulating a firm, using integration by parts to rewrite the double integral*

$$\int \int_{\theta}^{\bar{\theta}} \phi'(\tau - c(\tau)) d\tau g(\theta) d\theta.$$

**HW 14.**

**Theorem 2.** *With linear preferences, $f$ is BIC, if and only if*

1. *$V_i$ is non-decreasing in $\theta_i$*
2. *$T_i(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} V_i(\tau) d\tau - \theta_i V_i(\theta_i) + u_i(\underline{\theta}_i)$*

*Prove that the "if" part of this theorem by contradiction.*

**HW 15.** *We've shown that the optimal mechanism is equivalent to first price auction with reserve price. With $\sum_i x_i \leq 1$, the planner can keep the good. What would be the optimal mechanism with $\sum_i x_i = 1$?*

**HW 16.** 1. *Prove that the revelation principle holds for BNE.*

2. *Could there be any bad equilibrium? YES (Read Example 4.1.2 in Dasgupta, Hammond & Maskin (1979))*

**HW 17.** 1. *Prove that the revelation principle holds for dominant strategy equilibrium.*

**HW 18.** *We can write transfers in different ways:*
*DSIC* $\implies$ *[FOC]* $\frac{\partial t_i}{\partial \hat{\theta}_i} = -\frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \hat{\theta}_i} \implies t_i(\theta_i, \hat{\theta}_{-i}) = t_i(\underline{\theta}_i, \hat{\theta}_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \hat{\theta}_i} d\tau$

*DSIC* $\implies$ *[using Envelope Theorem]* $t_i(\theta_i, \hat{\theta}_{-i}) = u_i(\underline{\theta}_i, \hat{\theta}_{-i}) + v_i(y(\theta_i, \hat{\theta}_{-i})|\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} \frac{dv_i}{d\theta_i} d\tau$

*Are they the same? Why?*

**HW 19.** *If we add marginal cost $K$ and consider budget balance $\sum_i t_i = -Ky$, how should we modify VCG/Groves mechanisms?*

**HW 20.** *Is Arrow-d'AGV mechanism DSIC?*

**HW 21.** *Show that in the Groves & Ledyard mechanism, every NE $m^*$ gives a P.O. outcome (+BB).*