04/14/2023 Recitation #4 Handout

1. Definitions

- Agent's utility: $u(x,t|\theta)$
- MRS: $\frac{\partial u/\partial x}{\partial u/\partial t}$ ($du = \partial u/\partial x dx + \partial u/\partial t dt = 0$ implies $dt/dx = -MRS$.)
- SC_+: $\frac{\partial}{\partial \theta} MRS > 0$, $\forall x, t, \theta$ (clockwise rotation); SC_-: $\frac{\partial}{\partial \theta} MRS < 0$, $\forall x, t, \theta$
- SCF f is implementable if there's a $\Gamma = (S, g)$ that implements f, i.e., $f(\theta) \in g(\mu_\Gamma(\theta))$, $\forall \theta$.
- SCF f is IC if truth-telling is an equilibrium of $\Gamma_f = (\Theta, f)$, i.e., $\theta \in \mu_{\Gamma_f}(\theta)$, $\forall \theta$.

2. Characterization of IC
If revelation principle holds, then f is implementable $\iff$ f is IC. Thus, we can restrict attention to direct mechanisms.

Theorem 1. *If $f(\theta) = (x(\theta), t(\theta))$ is IC, then*

1. $\left[\frac{\partial}{\partial \theta}\left(\frac{\partial u/\partial x}{\partial u/\partial t}\right)\right] \cdot \frac{dx}{d\theta} \geq 0$, $\forall \theta$

2. $\frac{dt}{d\theta} = -\left(\frac{\partial u/\partial x}{\partial u/\partial t}\right) \cdot \frac{dx}{d\theta}$, $\forall \theta$

When does the converse hold? (1) does t always exist? – "add a uniform boundedness assumption on MRS" (2) local SOC doesn't imply global SOC – "add SC_+"

Theorem 2. *Suppose SC_+ and uniformly bounded MRS.*
$f(\theta) = (x(\theta), t(\theta))$ is IC, if and only if

1. $\frac{dx}{d\theta} \geq 0$, $\forall \theta$

2. $\frac{dt}{d\theta} = -\left(\frac{du/dx}{du/dt}\right) \cdot \frac{dx}{d\theta}$, $\forall \theta$

HW: prove Theorem 2 by contradiction.
Application: quadratic scoring rule.
3. Optimal mechanism
Consider a quasi-linear setting:

- Principal: $u_0(x,t) = v_0(x) - t$
- Agent: $u_1(x,t|\theta) = v_1(x,t|\theta) + t$
- $\frac{\partial v_0}{\partial x} \geq 0$, $\frac{\partial v_1}{\partial x} \geq 0$; $\frac{\partial v_1}{\partial \theta} > 0$; $\frac{\partial^2 v_0}{\partial x^2} \leq 0$, $\frac{\partial^2 v_1}{\partial x^2} \leq 0$
- MRS: $\frac{\partial v_1}{\partial x}$; SC_+: $\frac{\partial^2 v_1}{\partial x \partial \theta} > 0$

(First-best) $$\max_{x(\cdot),t(\cdot)} \quad v_0(x(\theta)) - t(\theta) \quad \text{s.t. [IR] } v_1(x(\theta)|\theta) + t(\theta) \geq 0,\ \forall\theta$$

- IR will be binding at every θ.
- $x \in \arg\max_{x(\cdot)} \{v_0(x(\theta)) + v_1(x(\theta)|\theta)\}$, so x is P.O.
- x is not IC (agent has incentive to lie downwards):

$$u_1(x(\hat\theta), t(\hat\theta)|\theta) = v_1(x(\hat\theta)|\theta) \quad \underbrace{-v_1(x(\hat\theta)|\hat\theta)}_{=t(\hat\theta)\ \text{(binding IR)}}$$

$$[\hat\theta] \quad \frac{\partial v_1(x(\hat\theta)|\theta)}{\partial x}\frac{dx(\hat\theta)}{d\hat\theta} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial x}\frac{dx(\hat\theta)}{d\hat\theta} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial \theta} = 0$$

$$\iff \underbrace{\left[\frac{\partial v_1(x(\hat\theta)|\theta)}{\partial x} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial x}\right]}_{\geq 0}\frac{dx(\hat\theta)}{d\hat\theta} = \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial \theta} \implies \quad \text{By } SC_+,\ \theta \geq \hat\theta$$

HW: verify that $dx(\hat\theta)/d\hat\theta \geq 0$.

(Second-best)

$$\max_{x(\cdot),t(\cdot)} \quad \int_{\underline{\theta}}^{\hat\theta} [v_0(x(\theta)) - t(\theta)]\, g(\theta) d\theta \quad \text{s.t. [IR] } v_1(x(\theta)|\theta) + t(\theta) \geq 0,\ \forall\theta$$

$$\text{[IC] } v_1(x(\theta)|\theta) + t(\theta) \geq v_1(x(\hat\theta)|\theta) + t(\hat\theta),\ \forall\theta,\hat\theta$$

- By Theorem 2, IC means that for every θ, $x'(\theta) > 0$ and $t(\theta) = \int_{\underline{\theta}}^{\theta} \frac{\partial v_1(\tau|\tau)}{\partial x} d\tau - v_1(x(\theta)|\theta) + c$.

$$\left(\frac{du(\theta)}{d\theta} = \frac{\partial v_1(x(\theta)|\theta)}{\partial\theta} \implies u(\theta) = \int_{\underline{\theta}}^{\theta} \frac{\partial v_1(x(\tau)|\tau)}{\partial\theta} d\tau + c\right)$$

- If $\frac{\partial v_1}{\partial\theta} > 0$, then IR for $\underline{\theta}$ implies IR for every θ. (because $u(\theta|\theta)$ is increasing in θ)

$$\max_{x(\cdot),t(\cdot)} \quad \int_{\underline{\theta}}^{\hat\theta} \left[v_0(x(\theta)) + v_1(x(\theta)) - \int_{\underline{\theta}}^{\theta} \frac{\partial v_1(x(\tau)|\tau)}{\partial x} d\tau - 0\right] g(\theta) d\theta \quad \text{s.t. } x'(\theta) > 0$$

$$\max_{x(\cdot),t(\cdot)} \quad \int_{\underline{\theta}}^{\hat\theta} \left[v_0(x(\theta)) + v_1(x(\theta)) - \frac{\partial v_1}{\partial\theta}\frac{1-G(\theta)}{g(\theta)} - 0\right] g(\theta) d\theta \quad \text{s.t. } x'(\theta) > 0$$

We will ignore the constraint $x'(\theta) > 0$ for now and check later.

$$\forall \theta : [x] \quad \frac{\partial v_0}{\partial x} + \frac{\partial v_1}{\partial x} = \underbrace{\frac{1 - G(\theta)}{g(\theta)} \frac{\partial^2 v_0}{\partial \theta^2}}_{\geq 0}$$

- $\bar{\theta}$: No distortion at the top.
- $\underline{\theta}$: IR binds.
- Other θ: IC binds $\implies u_1(\theta) > 0 \implies$ IR has slack.