HW

1. Public Goods & Externalities

HW 1. *Prove that at the Walrasian equilibrium, the only contributor is the agent with highest $v'_j(y^*)$.*

HW 2. *Prove that Lindahl equilibrium is IR.*

HW 3. *Check that in the Kolm triangle, if $p_1 + p_2 = 1$, then price hyperplanes coincide.*

2. Social Choice

HW 4. *Consider the domain of single-peaked preferences and let F be pairwise majority voting SWF. Is F transitive? What happens if n is odd/even? Could F have cycles "below the top"?*

HW 5. *Consider the domain of single-peaked preferences and let F be the SWF under pairwise majority voting. Is F transitive? (What happens if n is odd/even?) Could F have cycles "below the top"?*

HW 6. *We've shown the proof of Muller-Satterthwaite theorem using social choice function. Go through the similar proof but with social welfare function, Paretian and IIA. (Read Reny's paper)*

HW 7. *Prove or disprove*

    1. F is IIA + PAR $\implies$ f MONO +WP

    2. F is dictatorial $\implies$ f dictatorial

    3. f MONO +WP $\implies$ F is IIA + PAR

    4. f dictatorial $\implies$ F is dictatorial

HW 8. *We know how to construct from F (SWF) to f (SCF) and to F' (SWF). Is F the same as F'?*

3. Mechanism Design

HW 9. *We've shown that the revelation principle holds for single agent utility maximization.*

$$F(\theta) \supseteq g\left(\mu_\Gamma(\theta)\right) \supseteq g\left(\hat{\mu}_\Gamma(\theta)\right) = h(\theta) \in h\left(\mu_{\hat{\Gamma}}(\theta)\right)$$

    1. Is $h(\theta) = h(\mu_{\hat{\gamma}}(\theta))$ for every θ?

    2. If $\Gamma = (S, g)$ weakly implements F, then $\hat{\Gamma} = (\Theta, h \equiv g \circ \hat{\mu}_\Gamma)$ weakly implements F.

HW 10. *If f is single-valued, then if $\exists \Gamma$ that weakly implements f, then f is IC. The contrapositive to this result is: If f is not IC, then there is no Γ that weakly implements f. Could there exist a Γ that partially implements f?*

HW 11.

Theorem 1. *Suppose SC_+ and uniformly bounded MRS.*
$f(\theta) = (x(\theta), t(\theta))$ is IC, if and only if

1. *$\frac{dx}{d\theta} \geq 0$, $\forall \theta$*

2. *$\frac{dt}{d\theta} = -\left(\frac{du/dx}{du/dt}\right) \cdot \frac{dx}{d\theta}$, $\forall \theta$*

Prove the "if" part of this theorem by contradiction.

HW 12. *Consider a quasi-linear setting:*

- *Principal: $u_0(x,t) = v_0(x) - t$*

- *Agent: $u_1(x,t|\theta) = v_1(x|\theta) + t$*

- *MRS: $\frac{\partial v_1}{\partial x} > 0$; SC_+: $\frac{\partial^2 v_1}{\partial x \partial \theta} > 0$*

(First-best) $\qquad\qquad \max\limits_{x(\cdot),t(\cdot)} \quad v_0(x(\theta)) - t(\theta) \quad s.t. \ [IR] \ v_1(x(\theta)|\theta) + t(\theta) \geq 0, \ \forall \theta$

- *IR will be binding at every θ.*

- *$x \in \arg\max_{x(\cdot)} \{v_0(x(\theta)) + v_1(x(\theta)|\theta)\}$, so x is P.O.*

- *x is not IC (agent has incentive to lie downwards):*

$$u_1(x(\hat\theta), t(\hat\theta)|\theta) = v_1(x(\hat\theta)|\theta) \underbrace{-v_1(x(\hat\theta)|\hat\theta)}_{=t(\hat\theta) \ (binding \ IR)}$$

$$[\hat\theta] \quad \frac{\partial v_1(x(\hat\theta)|\theta)}{\partial x}\frac{dx(\hat\theta)}{d\hat\theta} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial x}\frac{dx(\hat\theta)}{d\hat\theta} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial \theta} = 0$$

$$\Longleftrightarrow \quad \underbrace{\left[\frac{\partial v_1(x(\hat\theta)|\theta)}{\partial x} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial x}\right]}_{\geq 0}\frac{dx(\hat\theta)}{d\hat\theta} = \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial \theta} \implies \quad By \ SC_+, \ \theta \geq \hat\theta$$

Here, we need to show that the first best solution satisfies $dx(\hat\theta)/d\hat\theta \geq 0$.

(Second-best)

$$\max\limits_{x(\cdot),t(\cdot)} \quad \int_{\underline\theta}^{\hat\theta} [v_0(x(\theta)) - t(\theta)]\, g(\theta)d\theta \quad s.t. \ [IR] \ v_1(x(\theta)|\theta) + t(\theta) \geq 0, \ \forall \theta$$

$$[IC] \ v_1(x(\theta)|\theta) + t(\theta) \geq v_1(x(\hat\theta)|\theta) + t(\hat\theta), \ \forall \theta, \hat\theta$$

1. *Verify that the first best solution satisfies $dx(\hat\theta)/d\hat\theta \geq 0$.*

2. *Verify that the second best solution satisfies $dx(\hat\theta)/d\hat\theta \geq 0$.*

3. *In class, we solve the second best solution with SC_+. What happens if we assume SC_- instead?*

HW 13. *In the example of regulating a firm, using integration by parts to rewrite the double integral*

$$\int \int_{\theta}^{\bar{\theta}} \phi'(\tau - c(\tau)) d\tau g(\theta) d\theta.$$

HW 14.

Theorem 2. *With linear preferences, f is BIC, if and only if*

1. *V_i is non-decreasing in θ_i*

2. *$T_i(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} V_i(\tau) d\tau - \theta_i V_i(\theta_i) + u_i(\underline{\theta}_i)$*

Prove that the "if" part of this theorem by contradiction.

HW 15. *We've shown that the optimal mechanism is equivalent to first price auction with reserve price. With $\sum_i x_i \leq 1$, the planner can keep the good. What would be the optimal mechanism with $\sum_i x_i = 1$?*

HW 16. 1. *Prove that the revelation principle holds for BNE.*

2. *Could there be any bad equilibrium? YES (Read Example 4.1.2 in Dasgupta, Hammond & Maskin (1979))*

HW 17. 1. *Prove that the revelation principle holds for dominant strategy equilibrium.*

HW 18. *We can write transfers in different ways:*
$DSIC \implies [FOC] \frac{\partial t_i}{\partial \hat{\theta}_i} = -\frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \hat{\theta}_i} \implies t_i(\theta_i, \hat{\theta}_{-i}) = t_i(\underline{\theta}_i, \hat{\theta}_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \hat{\theta}_i} d\tau$

$DSIC \implies [using\ Envelope\ Theorem]\ t_i(\theta_i, \hat{\theta}_{-i}) = u_i(\underline{\theta}_i, \hat{\theta}_{-i}) + v_i(y(\theta_i, \hat{\theta}_{-i})|\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} \frac{dv_i}{d\theta_i} d\tau$

Are they the same? Why?

HW 19. *If we add marginal cost K and consider budget balance $\sum_i t_i = -Ky$, how should we modify VCG/Groves mechanisms?*

HW 20. *Is Arrow-d'Aspremont-Gerard-Varet mechanism DSIC?*

HW 21. *Show that in the Groves & Ledyard mechanism, every NE m^* gives a P.O. outcome (+BB).*