04/21/2023 Recitation #5 Handout

1. Revelation Principle

Proposition 1. *If $\exists\Gamma$ that implements f in BNE, then $\hat{\Gamma}=(\Theta, f)$ truthfully implements f in BNE.*

Proof of Proposition 1. (Using notation for finite Θ_i.) By assumption, there is a Bayesian equilibrium s^* of Γ such that $g\left(s^*(\theta)\right)=f(\theta)$ for all $\theta \in \Theta$.
Suppose agent i is of type θ_i. Under Γ, if her opponents play s_{-i}^*, it is optimal for agent i to play $s_i^*\left(\theta_i\right)$. That is,

$$\sum_{\theta_{-i}} \mu\left(\theta_{-i} \mid \theta_i\right) u_i(\underbrace{g\left(s_i^*\left(\theta_i\right), s_{-i}^*\left(\theta_{-i}\right)\right)}_{f(\theta)}, \theta) \geq \sum_{\theta_{-i}} \mu\left(\theta_{-i} \mid \theta_i\right) u_i(\underbrace{g\left(\hat{s}_i, s_{-i}^*\left(\theta_{-i}\right)\right)}_{\text{some } x \in X}, \theta) \text{ for all } \hat{s}_i \in S_i .$$

In particular this is true when $\hat{s}_i=s_i^*\left(\hat{\theta}_i\right)$ for some $\hat{\theta}_i \in \Theta_i$, so that $g\left(s_i^*\left(\hat{\theta}_i\right), s_{-i}^*\left(\theta_{-i}\right)\right)=f\left(\hat{\theta}_i, \theta_{-i}\right)$.
Substituting into the previous expression thus yields

$$\sum_{\theta_{-i}} \mu\left(\theta_{-i} \mid \theta_i\right) u_i\left(f\left(\theta_i, \theta_{-i}\right), \theta\right) \geq \sum_{\theta_{-i}} \mu\left(\theta_{-i} \mid \theta_i\right) u_i\left(f\left(\hat{\theta}_i, \theta_{-i}\right), \theta\right) \text{ for all } \hat{\theta}_i \in \Theta_i .$$

This says that it is optimal for i to be truthful in $\hat{\Gamma}$ when others are truthful, or in other words, that f is Bayesian incentive compatible. □

Proposition 2. *If $\exists\Gamma$ that implements f in DSE, then $\hat{\Gamma}=(\Theta, f)$ truthfully implements f in DSE.*

Sketch of the Proof of Proposition 2. By assumption, there is a dominant strategy equilibrium s^* of Γ such that $g\left(s^*(\theta)\right)=f(\theta)$ for all $\theta \in \Theta$.
Suppose agent i is of type θ_i. Under Γ, no matter how her opponents play, it is optimal for agent i to play $s_i^*\left(\theta_i\right)$. That is,

$$u_i(g\left(s_i^*\left(\theta_i\right), s_{-i}\right), \theta) \geq u_i(g\left(\hat{s}_i, s_{-i}\right), \theta) \text{ for all } \hat{s}_i \in S_i \text{ and } s_{-i} \in S_{-i} .$$

In particular this is true when $\hat{s}_i=s_i^*\left(\hat{\theta}_i\right)$ for some $\hat{\theta}_i \in \Theta_i$ and $s_{-i}=s_{-i}^*\left(\hat{\theta}_{-i}\right)$ for some $\hat{\theta}_{-i} \in \Theta_{-i}$. □

Remark 1.

1. *In a direct mechanism, elements of Θ have two interpretations: as profiles of types and profiles of type reports. For example, when asking whether allocation $x(\theta)$ is efficient, we think of θ as a profile of types. But when asking whether agent i has an incentive to report truthfully, we think of $\theta_i \in \Theta_i$ as i's actual type and $\hat{\theta}_i \in \Theta_i$ as i's type announcement.*

2. *The revelation principle shows that for the purpose of determining which social choice functions are implementable, direct mechanisms are enough.*

3. *There are many reasons for considering more general mechanisms: (1) to rule out bad equilibria; (2) to be easier applied in practice (e.g., biding in auctions).*

2. Dominant Strategy Incentive Compatible (DSIC)

Theorem 1 (Gibbard-Satterthwaite Theorem)**.** *Suppose that X is finite with $|X| \geq 3$ and the type space include all possible weak (or strict) orderings on X. A social choice function f with $f(\Theta) = X$ is DSIC if and only if it is dictatorial.*

Proof of the theorem in the case of $\mathcal{P}$, $N = 2$ and $X = \{a, b, c\}$. The type space will be: $\Theta = \Theta_1 \times \Theta_2 = \{abc, cab, bac, bca, cab, cba\} \times \{abc, cab, bac, bca, cab, cba\}$. By assumption $f(\Theta) = \{a, b, c\}$.
($\Longrightarrow$) Here, we will prove by construction that if f is DSIC, then it is dictatorial. In the following table we assign a social alternative for every type profile:

	abc	acb	bac	bca	cab	cba
abc						
acb						
bac						
bca						
cab						
cba						

The rows denote the type of agent 1 and the columns denote the type of agent 2. First, notice that in order for f to be DSIC, it must be that whenever both players agree on their top alternative, it assigns that alternative as the outcome. If not, then both players can agree to misreport in such a way that they get their top alternative. Then, $f(abc, abc) = f(abc, acb) = f(acb, abc) = f(acb, acb) = a$, $f(bac, bac) = f(bac, bca) = f(bca, bac) = f(bca, bca) = b$, $f(cab, cab) = f(cab, cba) = f(cba, cab) = f(cba, cba) = c$
Second, let us consider $f(abc, bca)$. Notice that $f(abc, bca) \neq c$; otherwise, agent 1 has an incentive to misreport as bac and get b instead of c. So, $f(abc, bca) \in \{a, b\}$. For the rest of this analysis, let $f(abc, bca) = a$ and we will show that agent 1 is the dictator in this case. The case $f(abc, bca) = b$ ensures that agent 2 is the dictator, by a very similar argument.
If $f(abc, bca) = a$, then $f(abc, bac) = a$ (by the IC for agent 2) and $f(acb, bac) = f(acb, bca) = a$ (by the IC for agent 1). The IC for agent 2 ensures also that $f(abc, cab) = f(abc, cba) = f(acb, cab) = f(acb, cba) = a$. So far, the table should look like:

	abc	acb	bac	bca	cab	cba
abc	a	a	a	a	a	a
acb	a	a	a	a	a	a
bac			b	b		
bca			b	b		
cab					c	c
cba					c	c

Repeating the process by starting from $f(bac, cba)$ and $f(cab, bca)$, we can show that f is dictatorial (agent 1 is the dictator). $\square$

Gibbard-Satterthewaite Theorem tells us DSIC is strong. One way to break this impossibility result is by putting some domain restrictions. In the next section, we talk about different mechanisms under quasi-linear setting to implement the Pareto optimal (P.O.) level of public good.

3. Application: Solutions to Public Goods Problem

1. Groves mechanism: DSIC + P.O. y, not budget balanced
   Note: By Green & Laffont, DSIC + P.O. $y \implies$ Groves mechanism.
2. VCG mechanism: DSIC + P.O. y, budget feasible
3. AGV mechanism: BIC + P.O. y + budget balanced
4. Groves & Ledyard (indirect) mechanism: NE + P.O. y + budget balanced
5. Walker (indirect) mechanism: NE + P.O. y + budget balanced + individually rational
   Note: By Hurwitz, Nash implementable + P.O. + individually rational $\implies$ Lindahl.