HW

1. Public Goods & Externalities

Question 1. *Consider the problem of public goods. Prove that at the Walrasian equilibrium, the only contributor is the agent with highest $v_j'(y^*)$.*

Question 2. *Show that Lindahl equilibrium is IR.*

Question 3. *Check that in the Kolm triangle, if $p_1 + p_2 = 1$, then price hyperplanes coincide.*

2. Social Choice

Question 4. *Consider the domain of single-peaked preferences and let F be the SWF under pairwise majority voting. Is F transitive? (What happens if n is odd/even?) Could F have cycles "below the top"?*

Question 5. *We've shown the proof of Muller-Satterthwaite theorem using social choice function. Go through the similar proof but with social welfare function, Paretian and IIA. (Read Reny's paper)*

Question 6. *Prove or disprove*

1. *F is IIA + PAR $\implies$ f MONO +WP*
2. *F is dictatorial $\implies$ f dictatorial*
3. *f MONO +WP $\implies$ F is IIA + PAR*
4. *f dictatorial $\implies$ F is dictatorial*

Question 7. *How to construct from F (SWF) to f (SCF) and to F' (SWF)? Is F the same as F'?*

3. Mechanism Design

Question 8. *We've shown that the revelation principle holds for single agent utility maximization.*

$$F(\theta) \supseteq g\left(\mu_\Gamma(\theta)\right) \supseteq g\left(\hat{\mu}_\Gamma(\theta)\right) = h(\theta) \in h\left(\mu_{\hat{\Gamma}}(\theta)\right)$$

1. *Is $h(\theta) = h(\mu_{\hat{\gamma}}(\theta))$ for every θ?*
2. *If $\Gamma = (S, g)$ weakly implements F, then $\hat{\Gamma} = (\Theta, h \equiv g \circ \hat{\mu}_\Gamma)$ weakly implements F.*

Question 9. *Suppose f is single-valued. If there exists Γ that weakly implements f, then f is IC. The contra-positive to this result is: If f is not IC, then there is no Γ that weakly implements f. Could there exist a Γ that partially implements f?*

Question 10.

Theorem 1. *Suppose SC_+ and uniformly bounded MRS.*
$f(\theta) = (x(\theta), t(\theta))$ is IC, if and only if

1. $\frac{dx}{d\theta} \geq 0$, $\forall\theta$

2. $\frac{dt}{d\theta} = -\left(\frac{du/dx}{du/dt}\right) \cdot \frac{dx}{d\theta}$, $\forall\theta$

Prove the "if" part of this theorem by contradiction.

Question 11. *Consider a quasi-linear setting:*

- *Principal:* $u_0(x,t) = v_0(x) - t$
- *Agent:* $u_1(x,t|\theta) = v_1(x|\theta) + t$
- *MRS:* $\frac{\partial v_1}{\partial x} > 0$; SC_+: $\frac{\partial^2 v_1}{\partial x \partial \theta} > 0$

(First-best) $$\max_{x(\cdot),t(\cdot)} \quad v_0(x(\theta)) - t(\theta) \quad \textit{s.t. [IR]}\ v_1(x(\theta)|\theta) + t(\theta) \geq 0,\ \forall\theta$$

- *IR will be binding at every* θ.
- $x \in \arg\max_{x(\cdot)} \{v_0(x(\theta)) + v_1(x(\theta)|\theta)\}$, *so* x *is P.O.*
- x *is not IC (agent has incentive to lie downwards):*

$$u_1(x(\hat\theta), t(\hat\theta)|\theta) = v_1(x(\hat\theta)|\theta) \quad \underbrace{-v_1(x(\hat\theta)|\hat\theta)}_{=t(\hat\theta)\ \textit{(binding IR)}}$$

$$[\hat\theta] \quad \frac{\partial v_1(x(\hat\theta)|\theta)}{\partial x}\frac{dx(\hat\theta)}{d\hat\theta} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial x}\frac{dx(\hat\theta)}{d\hat\theta} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial \theta} = 0$$

$$\iff \quad \underbrace{\left[\frac{\partial v_1(x(\hat\theta)|\theta)}{\partial x} - \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial x}\right]}_{\geq 0}\frac{dx(\hat\theta)}{d\hat\theta} = \frac{\partial v_1(x(\hat\theta)|\hat\theta)}{\partial \theta} \implies \quad \textit{By } SC_+,\ \theta \geq \hat\theta$$

Here, we need to show that the first best solution satisfies $dx(\hat\theta)/d\hat\theta \geq 0$.

(Second-best)

$$\max_{x(\cdot),t(\cdot)} \quad \int_{\underline{\theta}}^{\hat\theta} [v_0(x(\theta)) - t(\theta)]\, g(\theta) d\theta \quad \textit{s.t. [IR]}\ v_1(x(\theta)|\theta) + t(\theta) \geq 0,\ \forall\theta$$

$$\textit{[IC]}\ v_1(x(\theta)|\theta) + t(\theta) \geq v_1(x(\hat\theta)|\theta) + t(\hat\theta),\ \forall\theta,\hat\theta$$

1. *Verify that the first best solution satisfies* $dx(\hat\theta)/d\hat\theta \geq 0$.

2. *Verify that the second best solution satisfies* $dx(\hat\theta)/d\hat\theta \geq 0$.

3. *In class, we solve the second best solution with* SC_+. *What happens if we assume* SC_- *instead?*

Question 12. *In the example of regulating a firm, using integration by parts to rewrite the double integral*

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} \phi'(\tau - c(\tau)) d\tau g(\theta) d\theta.$$

Question 13.

Theorem 2. *With linear preferences, f is BIC, if and only if*

1. *V_i is non-decreasing in θ_i*
2. *$T_i(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} V_i(\tau) d\tau - \theta_i V_i(\theta_i) + u_i(\underline{\theta}_i)$*

Prove that the "if" part of this theorem by contradiction.

Question 14. *We've shown that the optimal mechanism is equivalent to first price auction with reserve price. With $\sum_i x_i \leq 1$, the seller can keep the good. What would be the optimal mechanism with $\sum_i x_i = 1$?*

Question 15.

1. *Prove that the revelation principle holds for BNE.*
2. *Could there be any bad equilibrium? YES (Read Example 4.1.2 in Dasgupta, Hammond & Maskin (1979))*

Question 16.

1. *Prove that the revelation principle holds for dominant strategy equilibrium.*

Question 17. *We can write transfers in different ways:*
DSIC $\implies$ [FOC] $\frac{\partial t_i}{\partial \hat{\theta}_i} = -\frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \hat{\theta}_i} \implies t_i(\theta_i, \hat{\theta}_{-i}) = t_i(\underline{\theta}_i, \hat{\theta}_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i}{\partial y} \frac{\partial y}{\partial \hat{\theta}_i} d\tau$

DSIC $\implies$ [using Envelope Theorem] $t_i(\theta_i, \hat{\theta}_{-i}) = u_i(\underline{\theta}_i, \hat{\theta}_{-i}) - v_i(y(\theta_i, \hat{\theta}_{-i})|\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} \frac{dv_i}{d\theta_i} d\tau$

Are they the same? Why?

Question 18. *If we add marginal cost K and consider budget balance $\sum_i t_i = -Ky$, how should we modify VCG/Groves mechanisms?*

Question 19. *Is Arrow-d'Aspremont-Gerard-Varet mechanism DSIC?*

Question 20. *Show that in the Groves & Ledyard mechanism, every NE m^* gives a P.O. outcome (+BB).*