Set Theory and Logic

Han Wang*

July 24, 2020

1 Roadmaps

Primitive Concepts (A notion of belonging); Designations and Sentences; Operations on Sets (union, intersection); Subsets (if $\cdots$, then $\cdots$); Providing Equivalences (sufficiency, necessity); Negation ("complement" with the notion of universal set); Expressions with Variables; Collections of Sets; Sets Defined by Sentences (Russell's Paradox).

2 Counter-intuitive facts

Axiom 1. *(Extensionality) Consider two sets A and B. Then $A = B$ iff $x \in A \Leftrightarrow x \in B$.*

Axiom 2. *(Abstraction) Given a set X, the set $\{x \in X | p(x)\}$ exists.*

Remark: The names of these axioms come from "Zermelo's fix to Russell's Paradox" by Nicolas G. Belmonte.

Definition 1. *Let $\mathcal{C}$ be a collection of sets.*

1. $x \in \cup_{X \in \mathcal{C}} X \Leftrightarrow \exists_{X \in \mathcal{C}} x \in X$;
2. $x \in \cap_{X \in \mathcal{C}} X \Leftrightarrow \forall_{X \in \mathcal{C}} x \in X$.

Proposition 1.

1. $\cup_{X \in \emptyset} X = \emptyset$;
2. $\cap_{X \in \emptyset} X$ *is not a set.*

Sketch for Proposition 1.1: It follows naturally from the definition of an empty set. The empty set, $\emptyset$, contains no elements, hence the sentence $x \in \emptyset$ is always false.

Proof. for Proposition 1.2: (By contradiction) Suppose $A = \cap_{X \in \emptyset} X$ is a set. Construct $B = \{Y \in A | Y \notin Y\}$. (By axiom 2, such B exists.) Hence, $B \in B \Leftrightarrow (B \in A$ and $B \notin B)$. We know that $B \notin B$ and $B \notin A$ must be true. However, $B \notin A \Leftrightarrow \text{not}(B \in \cap_{X \in \emptyset} X) \Leftrightarrow \text{not}(\forall_{X \in \emptyset} B \in X) \Leftrightarrow \exists_{X \in \emptyset} B \notin X$. The last sentence is false, implying that $B \notin A$ is false, so we find a contradiction. We conclude A is not a set.

Solution to the non-existence problem: Restrict our attention to subsets of a fixed universal set E. We can define, for an arbitrary collection of sets $\mathcal{C} \subset \mathcal{P}(E)$, $\cap_{X \in \mathcal{C}} X = \{x \in E | \forall_{X \in \mathcal{C}} x \in X\}$. Then, $\cap_{X \in \emptyset} X = E$. The orignal proof doesn't work under the new definition. $B \notin A \Leftrightarrow \text{not}(B \in \cap_{X \in \emptyset} X) \Leftrightarrow \text{not}(B \in E \text{ and } \forall_{X \in \emptyset} B \in X) \Leftrightarrow B \notin E$ or $\exists_{X \in \emptyset} B \notin X$, hence we see that $B \notin A$ can be true, if $B \notin E$ is true. There's no contradiction. To illustrate, think of an example: Suppose $E = \{\{0\}, \{1\}\}$. Then $A = B = E$. Clearly, $\{\{0\}, \{1\}\} \notin \{\{0\}, \{1\}\}$.

Remark for Proposition 1.2: This is known as Russell's Paradox (1901). The source of trouble is Axiom of Abstraction. Zermelo (1908) fixs Russell's Paradox by modifying the axiom, $(\exists B)(\forall x)(x \in B \Leftrightarrow p(x))$, to be $(\exists B)(\forall x)(x \in B \Leftrightarrow x \in E \ \&\ p(x))$. A simple derivation can have a great impact!

*This note borrows heavily from "Set theory and logic: fundamental concepts" by Dr. Joao P. Santos.