

# Set Theory and Logic

Han Wang\*

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## 1 Roadmaps

Primitive Concepts; Designations and Sentences; Operations on Sets; Subsets; Providing Equivalences; Negation; Expressions with Variables; Collections of Sets; Sets Defined by Sentences.

## 2 Counter-intuitive facts

**Axiom 1.** (*Extensionality*) Consider two sets  $A$  and  $B$ . Then  $A = B$  iff  $x \in A \Leftrightarrow x \in B$ .

**Axiom 2.** (*Abstraction*) The set  $\{x \in X | p(x)\}$  exists.

Remark: The names of these axioms come from “Zermelo’s fix to Russell’s Paradox” by Nicolas G. Belmonte.

**Definition 1.** Let  $\mathcal{C}$  be a collection of sets.

1.  $x \in \cup_{X \in \mathcal{C}} X \Leftrightarrow \exists X \in \mathcal{C} x \in X$ ;
2.  $x \in \cap_{X \in \mathcal{C}} X \Leftrightarrow \forall X \in \mathcal{C} x \in X$ .

**Proposition 1.** Under Axiom 2,

1.  $\cup_{X \in \emptyset} X = \emptyset$ ;
2. (*Russell’s Paradox, 1901*)  $\cap_{X \in \emptyset} X$  is not a set.

Sketch for Proposition 1.1: It follows naturally from the definition of an empty set. The empty set,  $\emptyset$ , contains no elements, hence the sentence  $x \in \emptyset$  is always false.

Proof. for Proposition 1.2: (By contradiction) Suppose  $A = \cap_{X \in \emptyset} X$  is a set. Construct  $B = \{Y \in A | Y \notin Y\}$ . (By axiom 2, such  $B$  exists.) Hence,  $B \in B \Leftrightarrow (B \in A \text{ and } B \notin B)$ . We know that  $B \notin B$  and  $B \notin A$  must be true. However,  $B \notin A \Leftrightarrow \text{not}(B \in \cap_{X \in \emptyset} X) \Leftrightarrow \text{not}(\forall X \in \emptyset B \in X) \Leftrightarrow \exists X \in \emptyset B \notin X$ . The last sentence is false, implying that  $B \notin A$  is false, so we find a contradiction. We conclude  $A$  is not a set.

Solution to the non-existence problem: Restrict attention to subsets of a fixed universal set  $E$ . We can define, for an arbitrary collection of sets  $\mathcal{C} \subset \mathcal{P}(E)$ ,  $\cap_{X \in \mathcal{C}} X = \{x \in E | \forall X \in \mathcal{C} x \in X\}$ . Then,  $\cap_{X \in \emptyset} X = E$ . The original proof doesn’t work under the new definition.  $B \notin A \Leftrightarrow \text{not}(B \in \cap_{X \in \emptyset} X) \Leftrightarrow \text{not}(B \in E \text{ and } \forall X \in \emptyset B \in X) \Leftrightarrow B \notin E \text{ or } \exists X \in \emptyset B \notin X$ , hence we see that  $B \notin A$  can be true, if  $B \notin E$  is true. There’s no contradiction. To illustrate, think of an example: Suppose  $E = \{\{0\}, \{1\}\}$ . Then  $A = B = E$ . Clearly,  $\{\{0\}, \{1\}\} \notin \{\{0\}, \{1\}\}$ .

Remark for the second claim: The source of trouble is Axiom 2. Zermelo (1908) fixes Russell’s Paradox by modifying Axiom 2,  $(\exists B)(\forall x)(x \in B \Leftrightarrow p(x))$ , to be  $(\exists B)(\forall x)(x \in B \Leftrightarrow x \in E \ \& \ p(x))$ . A simple derivation with great impacts!

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\*This note borrows heavily from “Set theory and logic: fundamental concepts” by Dr. Joao P. Santos.