Linear Integer Arithmetic

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Linear Integer Arithmetic is also called Presburger Arithmetic. There are three views on it.

Semi-linear Sets

Let $S \subseteq \mathbb{N}$. S is called:

- *periodic* if there is a $p > 0$ such that, for all x, $x \in S$ iff $x + p \in S$;
- *ultimately periodic* if there is a N and $p > 0$ such that, for all $x \geq n$, $x \in S$ iff $x + p \in S$.

p and N are called the *period* and the *offset* respectively.

Proposition.

A set $S \subseteq \mathbb{N}$ is ultimately periodic iff it is a union of finitely many arithmetic progressions.

Ultimately periodic sets are closed under complement, intersection (the LCM of periods is the new period and the larger offset is the new offset) and union.

Definition. Linear Sets

A set $S \subseteq \mathbb{Z}^d$ is called linear if there exists $\boldsymbol{b} \in \mathbb{Z}^d$ and a finite set $P = \{\boldsymbol{p}_{\mathbf{1}}, ..., \boldsymbol{p}_k\} \subseteq \mathbb{Z}^d$ such that

$$S \stackrel{\text{def}}{=} \left\{\boldsymbol{b} + \sum_{i=1}^{k} \lambda_i \boldsymbol{p}_i : \lambda_1, ..., \lambda_k \in \mathbb{N}\right\}. \quad (1)$$

Programming community uses the term “integer cone” for cases where $\boldsymbol{b} = \mathbf{0}$.

Definition. Semi-linear Sets

A set is *semi-linear* if it is a union of finitely many linear sets. Semi-linear set is a generalization of the concept of ultimately periodic set.