# Homework3

EE688. Optimal Control Theory

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## Contents

# 1 Solution of Problem 1: Receding Horizon Control

## 1.1 (a)

We can show that $\mathcal{C}$ is control invariant set if we can prove that $\mathcal{C} \subseteq \text{Pre}(\mathcal{C})$ then $\mathcal{C}$ is control invariant set according to lecture 12

As the definition of $\mathcal{C}$ as follows:

$$\mathcal{C} = \left\{ x \in \mathcal{X} : \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} x \leq \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \right\}$$

And according to the definition $x \in \mathcal{X}$ which needs to satisfy $\mathcal{X} = \left\{ x \in \mathrm{R}^2 : \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq x \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ ($\mathcal{C}$ is a subset of $\mathcal{X}$; $\mathcal{C} \subset \mathcal{X}$).This has been taken into consideration in the code.

Therefore, the following plot shows the $\mathcal{C}$ and $\text{Pre}(\mathcal{C})$

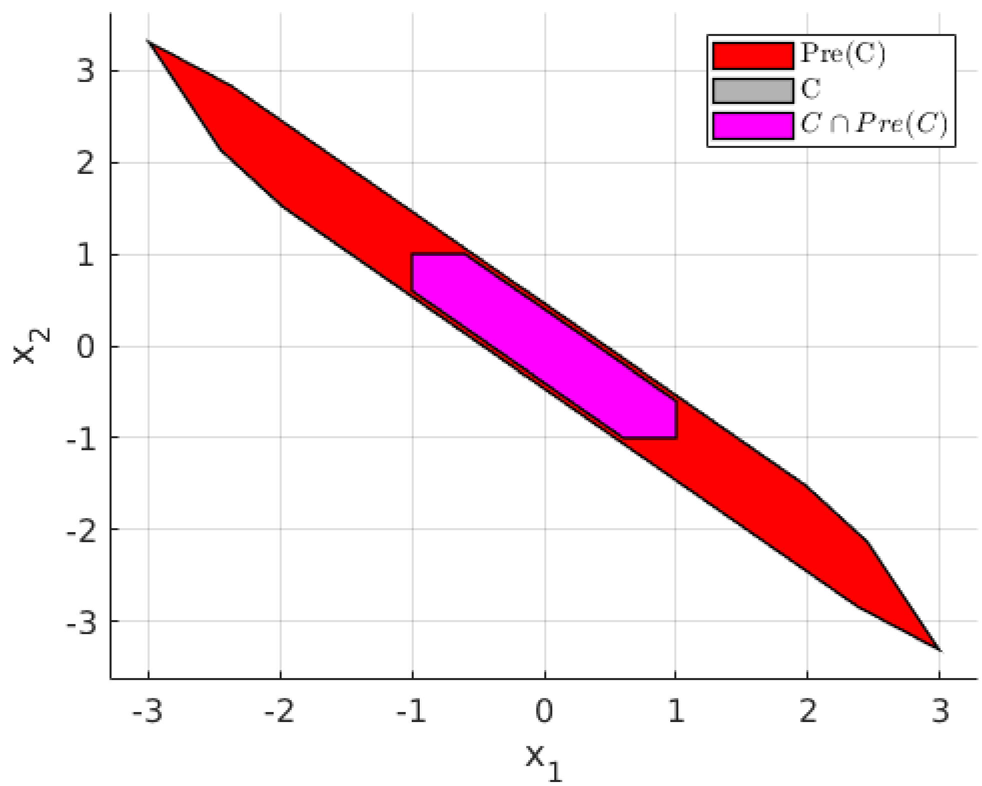



Figure 1: Plots of $\mathcal{C}$ with $\text{Pre}(\mathcal{C})$

We can see that $\mathcal{C} \subseteq \text{Pre}(\mathcal{C})$ and it means that $C$ is a control invariant set.

## 1.2 (b)

As it is one-step problem, we compute the initial feasible states as one-step backward reachable set $\mathcal{K}_1(\mathcal{C}) = \mathcal{X}_0$ such that the terminal constraint is $\mathcal{C}$; the control invariant set.

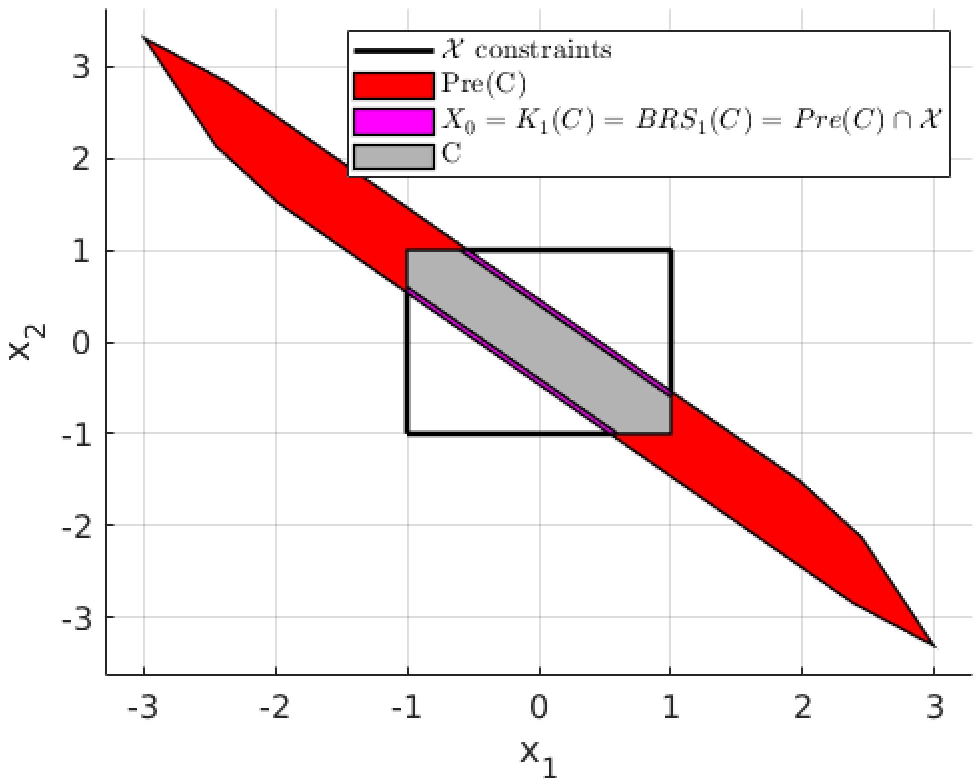



Figure 2: Plots of $\mathcal{C}$ and $\mathcal{K}_1(\mathcal{C}) = \mathcal{X}_0$

We can see that the grey area which is $\mathcal{C}$ resides inside the magenta shape which is $\mathcal{X}_0$, therefore, $\mathcal{X}_0$ is a superset of $\mathcal{C}$ which is equivalent to $\mathcal{C} \subseteq \mathcal{X}_0$

The reason is that if a set is control invariant then $\mathcal{C} \subseteq \text{Pre}(\mathcal{C})$ according to lecture 12. Then by computing $\mathcal{K}_1(\mathcal{C}) = \mathcal{X}_0 = \text{Pre}(\mathcal{C}) \cap \mathcal{X}$ and $\mathcal{C} \subseteq \text{Pre}(\mathcal{C})$ and according to the definition of $\mathcal{C}$ is a subset of $\mathcal{X}$; $\mathcal{C} \subset \mathcal{X}$. Then, that justify the result that we have obtained which is $\mathcal{C} \subseteq \mathcal{X}_0$.

## 1.3 (c)

In the code, we find $P$ by solving the discrete algebraic Ricatti equation using dare function. As we are using a specific controller. Then we find $F_\infty =$

$-(B^T P_+ R)^{-1} B^T P_\infty A$. Then, we use it to compute the precursor set until the convergence and it will be the maximal positively invariant set $\mathcal{O}_{\max}$.

$$P_\infty = 1000 \cdot \begin{bmatrix} 2.1659 & 2.1645 \\ 2.1645 & 2.1644 \end{bmatrix}$$

And the following figure shows the plot of $O_{\max}$

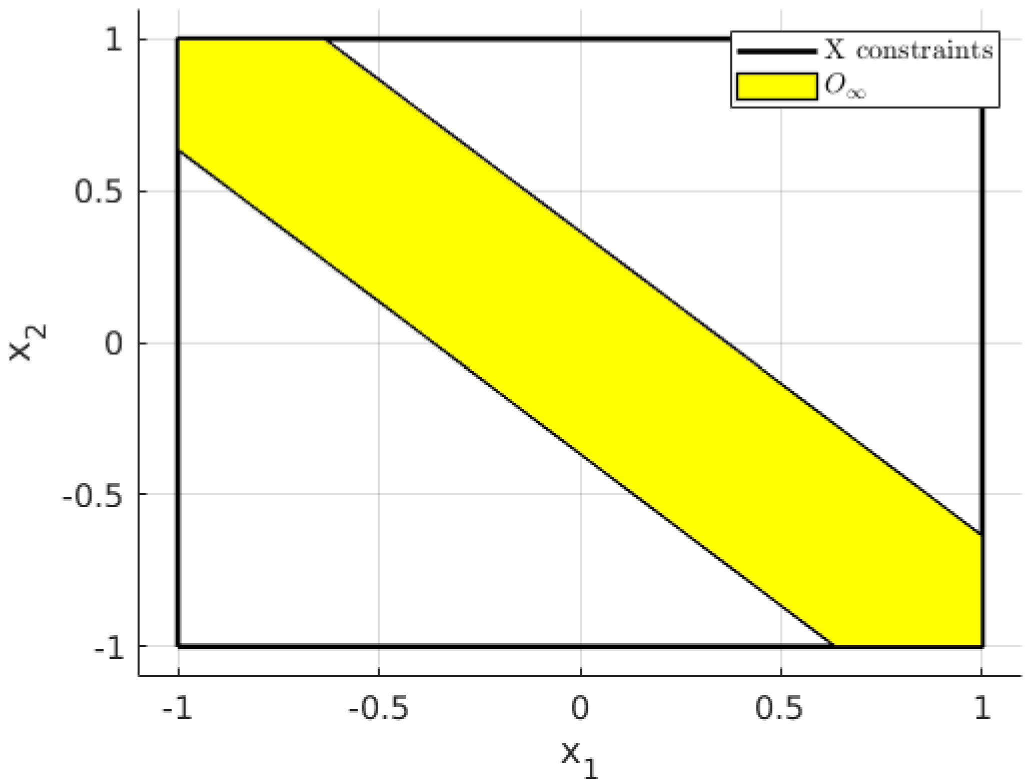



Figure 3: Plot of $O_{\max}$

The following plot shows the simulation of solving the system with $O_{\max}$. We can see that the system converges to the origin

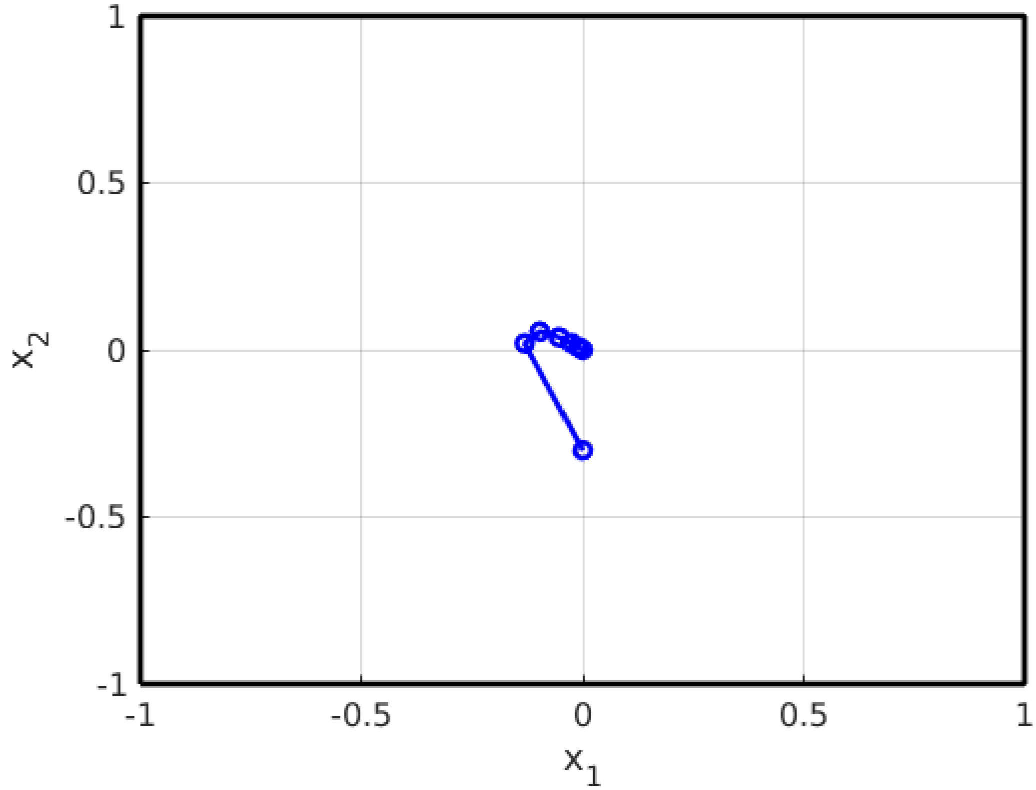



Figure 4: System simulation

As $P = P_\infty, Q, R$ are positive definite which means that the stage cost and terminal costs are positive-definite $q(x, u) = u^T Ru, p(x_N) = x_N^T P_\infty x_N$.

As, $F = F_\infty, P = P_\infty$, then according to the proof in Lecture 10, they satisfy $p(Ax_N + Bv) - p(x_N) \leq -q(x_N, v)$ assuming $v = Fx$. That means that the terminal cost function is a control-Lyapunov function

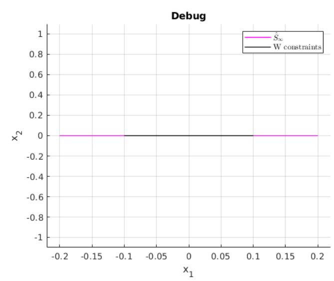
Furthermore, we have $X_f = \mathcal{O}_{\max}$ such that it is positive invariant set.

The previous reasons and the fulfilled conditions enabled the system to be recursive feasible and the origin of the closed loop system under the receding horizon control law is asymptotically stable with the domain of attraction. Therefore, we observed the previous behavior of reaching the origin.

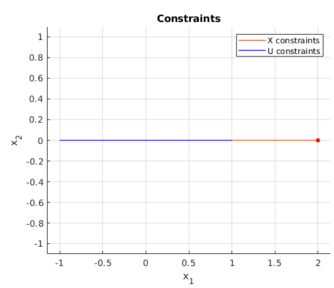
# 2 Solution of Problem 2: Tube-based MPC

## 2.1 (a)

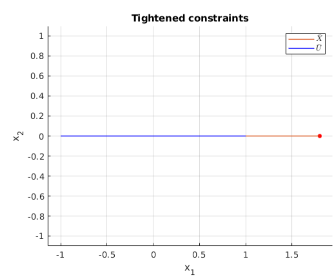
In the code, I have constructed the sets according to the description of the sets in the assignment.




(a) Plotting $S_\infty$ and W constraints




(b) The original constraints




(c) Tightened constraints

Figure 5: Tightened state and input constraints

The original constraints:

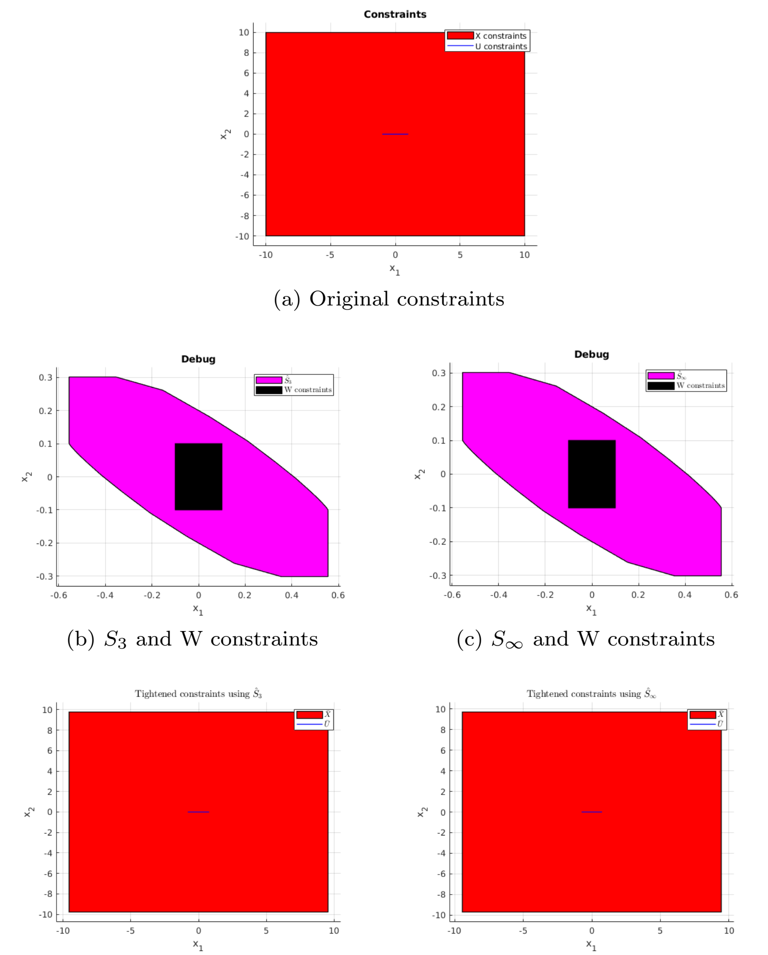
$$\mathcal{X} := \{x : x \leq 2\}$$
$$\mathcal{U} := \{u : -1 \leq u \leq 1\}$$

The tightened constraints are showed in the previous plot and they are described with the following set construction as well

$$\bar{\mathcal{X}} := \{x : x \leq 1.8\}$$
$$\bar{\mathcal{U}} := \{u : -1 \leq u \leq 1\}$$

## 2.2 (b)

We have showed the affect of $S_\infty$ (with many number of iterations) and $S_3$ on tightning the constraints.




(a) Original constraints

(b) $S_3$ and W constraints

(c) $S_\infty$ and W constraints

(d) Tightened state and input constraints using $S_3$

(e) Tightened state and input constraints using $S_\infty$

Figure 6: Tightened state and input constraints

The tightened constraints are showed in the previous plot and they are described with the following set construction as well

$$\bar{\mathcal{X}} := \left\{ x \in \mathrm{R}^2 : \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 9.56 \\ 9.772 \\ 9.56 \\ 9.772 \end{bmatrix} \right\}$$

$$\bar{\mathcal{U}} := \{u : -0.7904 \leq u \leq 0.7904\}$$

## 2.3 (c)

Using the tightened bounds using $S_3$ from the previous point with MPC controller. We minimize the cost function with horizon =3 using the nominal system (The system without the disturbances) and we can plot the nominal trajectory

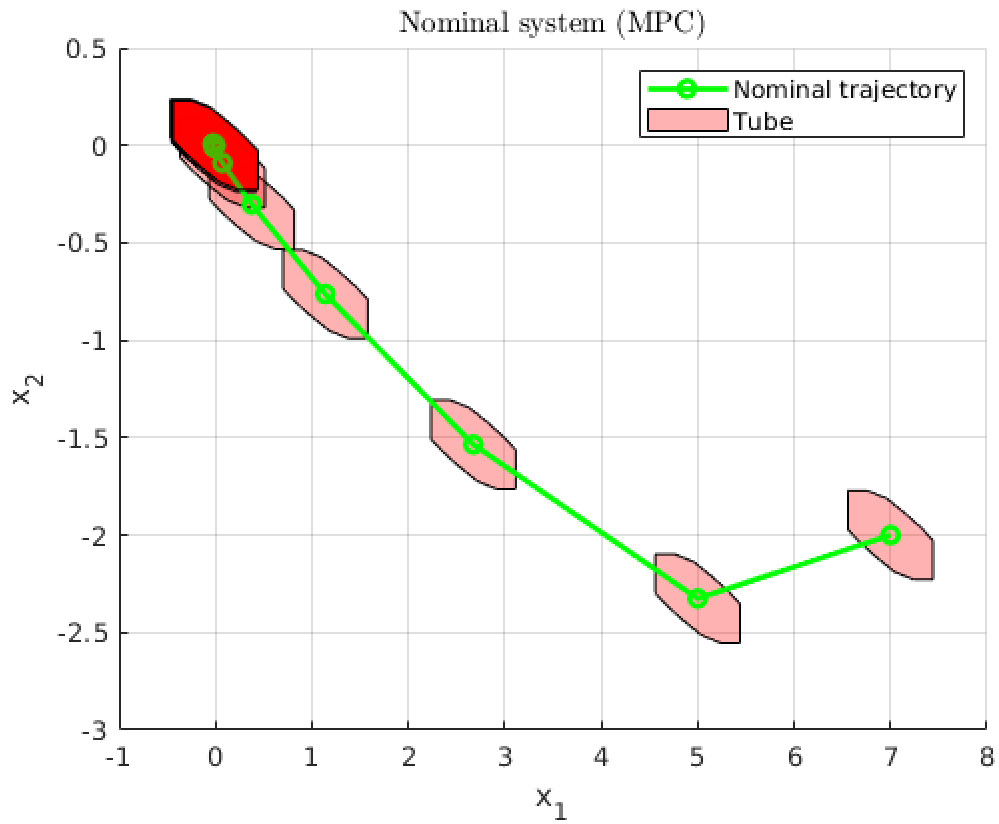



Figure 7: Nominal trajectory

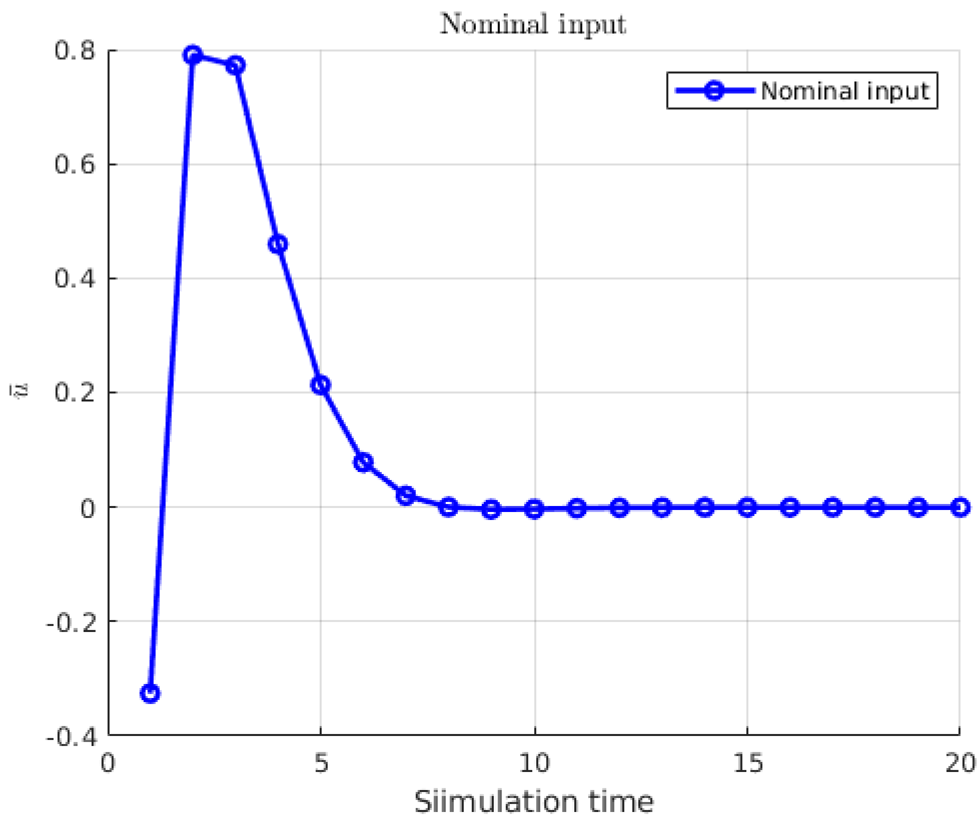



Figure 8: Nominal input

## 2.4 (d)

Using the nominal generated trajectory and input with linear feedback controller for trajectory tracking. we can show the system simulation as follows

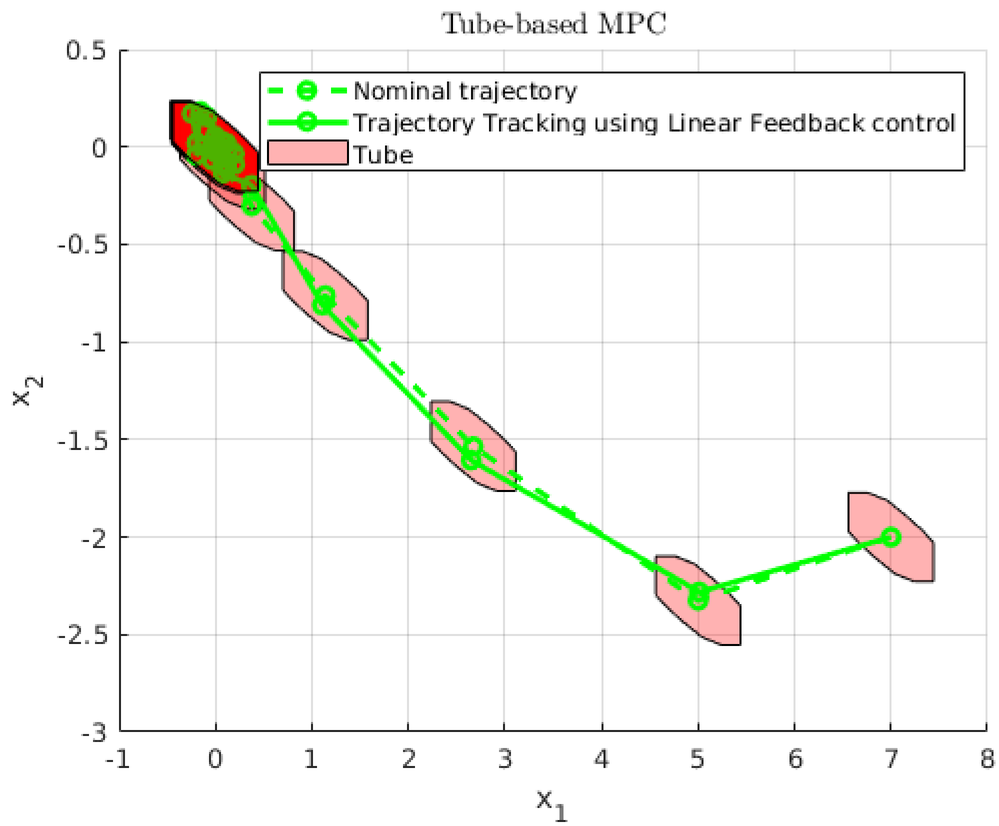



Figure 9: Tube-based MPC

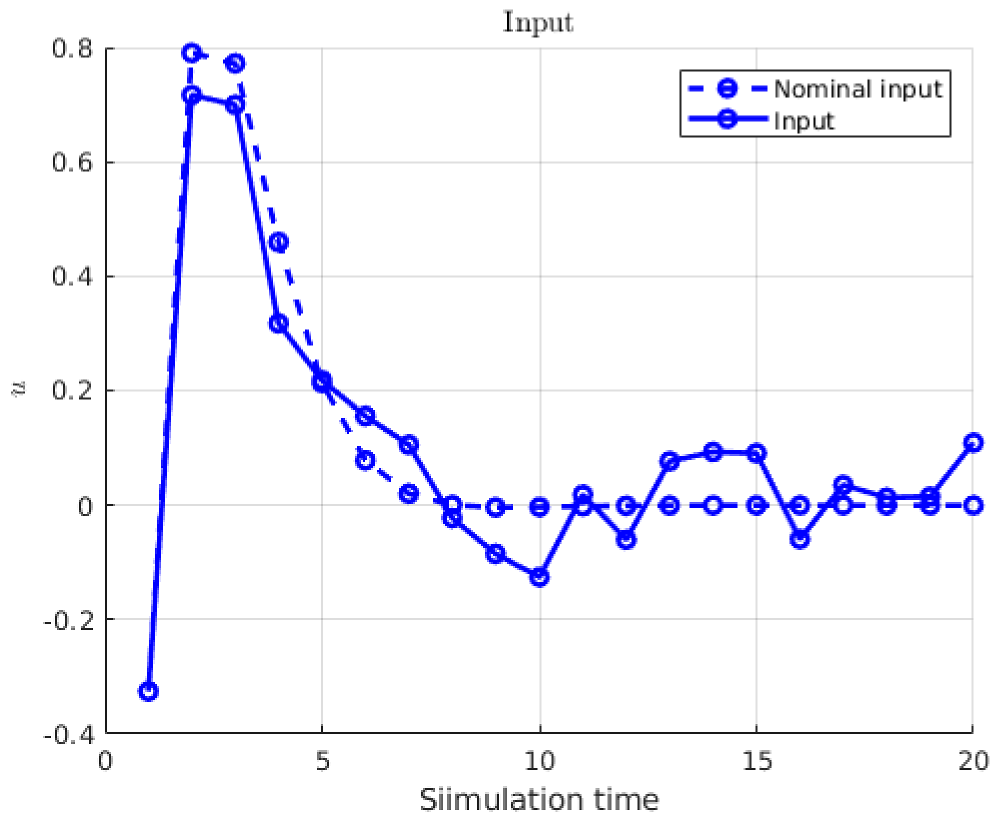



Figure 10: Input

## 2.5 (e)

Showing the error bounds while plotting $S_3$, we show that all the errors resides within the tube that constructed using $S_3$ and used to tightened the bounds of input and states constraints.

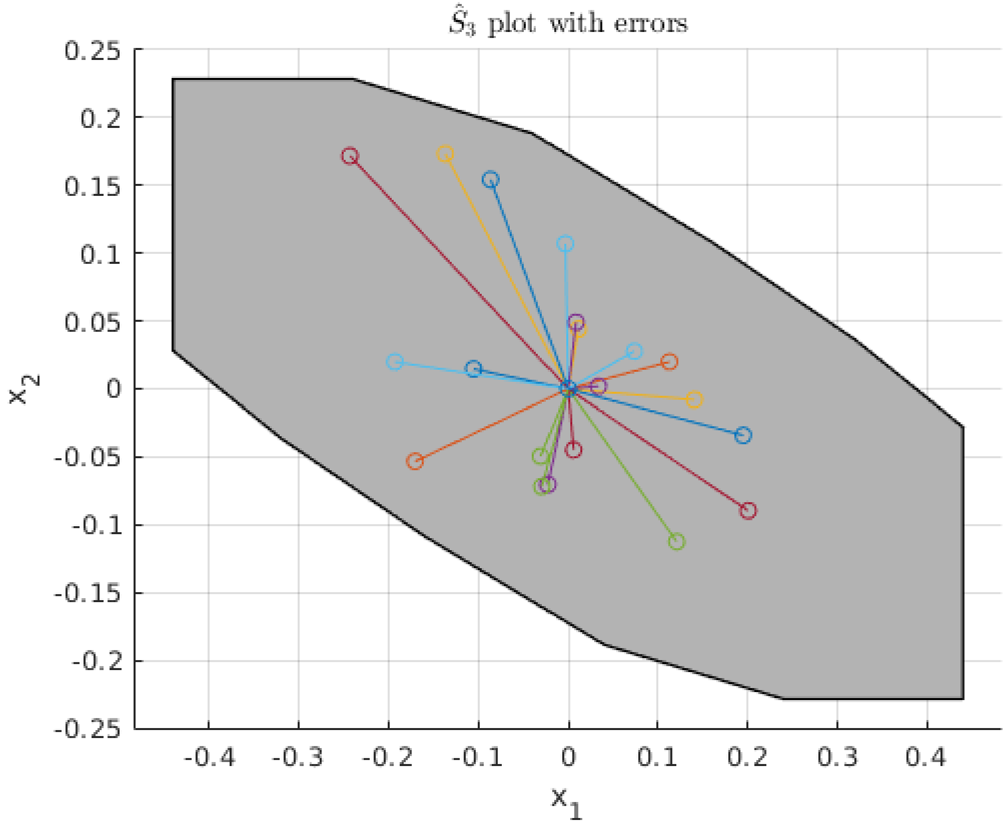



Figure 11: Errors with the tube bounds $S_3$

We can observe that all the errors does not exceed the tube bounds.