EE688. Optimal Control Theory

Homework 2

Assigned: October 4, 2022
Due: October 12, 2022

Problem 1: Karush-Kuhn-Tucker Conditions

Consider the following nonlinear program:

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 + \theta x_2 \\ \text{subject to} \quad & x_1 x_2 \geq 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

where $\theta > 0$ is a parameter. The goal of this problem is to find the optimal solution as a function of θ.

(a) For a fixed value of θ, is this problem convex?

(b) Write down the (θ-dependent) KKT conditions for this problem.

(c) Using the KKT conditions, find an analytic expression for the optimal $x_1^\star(\theta)$ and $x_2^\star(\theta)$.

Problem 2: Linear Quadratic Regulator

Consider the 2-nd order system

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = u$$

where $\omega_n = 2$ and $\zeta = 0.1$. This system can be approximated by the following discrete-time dynamics

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -\omega_n^2\Delta t & 1 - 2\zeta\omega_n\Delta t \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} u(k)$$

where $x_1(k) \approx y(k\Delta t)$, $x_2(k) \approx \dot{y}(k\Delta t)$, and $\Delta t = 0.1$.

(a) Using MATLAB, design an infinite-horizon linear quadratic regulator for this system where $Q = I$ and $R = 1$. Simulate the closed-loop system over $10\,s$ with the starting point $x(0) = (1, 1)^\intercal$.
(Hint: Use the MATLAB command `dare`)

(b) For this plant the LQR controller is a proportional-derivative (PD) controller

$$u(t) = \begin{bmatrix} F_p & F_d \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} F_p & F_d \end{bmatrix} \begin{bmatrix} y(k\Delta t) \\ \dot{y}(k\Delta t) \end{bmatrix}.$$

How do the choices of Q and R affect the PD gains F_p and F_d? Find a Q such that the closed-loop system does not oscillate.

(c) Now use a receding-horizon strategy to approximate the infinite-horizon controller. Using MATLAB, solve the unconstrained finite-control problem

$$\min_{u_0,\ldots,u_{N-1}} \quad x_N' Q x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \tag{1}$$

$$\text{s.t.} \quad x_{k+1} = A x_k + B u_k \tag{2}$$

$$x_0 = x(t) \tag{3}$$

at each time t to obtain a input sequence $u_0^*, \ldots, u_{N-1}^*$. Apply u_0^* and repeat at time $t+1$. How does this control strategy compare with the infinite-horizon linear quadratic regulator? How does the horizon N affect the relative performance? Provide reasons for your answers.

Problem 3: Model predictive control

Consider a linearized inverted pendulum system which can be stabilized by the torque u,

$$\ddot{\theta} - \theta = u.$$

It can be written in the state-space representation as

$$\dot{x} = \frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} x + \begin{bmatrix}0 \\ 1\end{bmatrix} u.$$

The system is subject to the constraints $x \in \mathcal{X}$ and $u \in \mathcal{U}$ where

$$\mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix}-1 \\ -1\end{bmatrix} \leq x \leq \begin{bmatrix}1 \\ 1\end{bmatrix} \right\}$$
$$\mathcal{U} = \{u \in \mathbb{R}^1 : -0.5 \leq u \leq 0.5\}.$$

(a) Discretize the system with sampling time $\Delta t = 1$ using the zero-order hold method and derive the discrete-time linear system $x_{k+1} = Ax_k + Bu_k$.
(Hint: Use the MATLAB command `c2d`.)

(b) Simulate the closed-loop system with the controller obtained by solving the one-step ($N = 1$) model predictive control problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & (Ax + Bu)'P(Ax + Bu) + u'Ru \\ \text{subject to} \quad & Ax + Bu \in \mathcal{X} \\ & u \in \mathcal{U}. \end{aligned}$$

Use $P = \text{diag}([1, 0.1])$, $R = 10^3$, and $N = 1$. Using the initial condition $x_0 = [0, -0.3]'$, simulate and describe the closed-loop behavior.

(c) Suppose we are given a set

$$\mathcal{C} = \left\{ x \in \mathcal{X} : \begin{bmatrix}1 & 1 \\ -1 & -1\end{bmatrix} x \leq \begin{bmatrix}0.4 \\ 0.4\end{bmatrix} \right\},$$

and it is control invariant (Proof in a later lecture). Simulate the closed-loop system with the controller obtained by solving the one-step ($N = 1$) model predictive control problem with control invariant terminal constraint

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & (Ax + Bu)'P(Ax + Bu) + u'Ru \\ \text{subject to} \quad & Ax + Bu \in \mathcal{C} \\ & u \in \mathcal{U}. \end{aligned}$$

Use the same cost function and initial state as in (b). Simulate for at least 20 time-steps and describe the closed-loop behavior.

(d) Plot the feasible regions of the model predictive control problems in (b) and (c) by, for example, discretizing the state space. Which set is larger? Describe the behavior of each closed-loop system if it starts inside the corresponding feasible region. Which feasible region is positively invariant for the closed-loop system?