EE688. Optimal Control Theory

Homework 1

Assigned: Sep 14, 2022
Due: Sep 26, 2022

Problem 1: Linearization and Discretization

Consider a mixing tank with constant supply temperatures T_C and T_H where $T_C < T_H$. Let the inputs be the two flow rates $q_C(t)$ and $q_H(t)$ into the tank, so $q_C(t) \geq 0$ and $q_H(t) \geq 0$. With the height $h(t)$ and temperature $T_T(t)$, the equations for the tank are

$$\begin{aligned} \dot{h}(t) &= \frac{1}{A_T}\left(q_C(t) + q_H(t) - c_D A_o \sqrt{2gh(t)}\right) \\ \dot{T}_T(t) &= \frac{1}{h(t)A_T}\left(q_C(t)[T_C - T_T(t)] + q_H(t)[T_H - T_T(t)]\right), \end{aligned} \tag{1}$$

where A_T, A_o, c_D, g are the system parameters. Let the state vector x and the input vector u be defined as

$$x(t) := \begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix}, \quad u(t) := \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix}.$$

(a) Show that any height $\bar{x}_1 = \bar{h} > 0$ and any tank temperature $\bar{x}_2 = \bar{T}_T$ satisfying $T_C \leq \bar{T}_T \leq T_H$ is a possible equilibrium point.

(b) Find the linearized model of (1) at an equilibrium point $(\bar{x}_1, \bar{x}_2)$ with the corresponding equilibrium input $(\bar{u}_1, \bar{u}_2)$.

(c) Suppose that $T_C = 10, T_H = 90, A_T = 3, A_o = 0.05, c_D = 0.7$ and $g = 10$. At $(\bar{h}, \bar{T}_T) = (1, 25)$, find the linearized model.

(d) Find the discrete-time linear model of the solution of (c) by applying the zero-order hold method. Let the sampling time be $0.05\,\mathrm{s}$.

Problem 2: Stability

Consider the discrete-time linear system

$$x_{k+1} = Ax_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$. In the following, we will prove the following statement:

$$\begin{gathered}\text{The discrete-time linear system is aymptotically stable.}\\ \Updownarrow \\ \text{There is a quadratic Lyapunov function that proves it.}\end{gathered}$$

(a) ($\Leftarrow$) Show that if the matrix P is symmetric and positive definite and

$$A^\intercal PA - P = -Q \tag{3}$$

for some symmetric and positive definite matrix Q, then the linear system (2) is asymptotically stable.

(b) Suppose the system (2) is asymptotically stable, and a matrix Q is given. Show that the discrete-time Lyapunov equation (3) uniquely determines P.
(Hint: Given Q, $P = \sum_{t=0}^{\infty}(A^\intercal)^t Q A^t$ is the solution of (3). This exists if and only if (2) is asymptotically stable.)

(c) ($\Rightarrow$) Show that if the system (2) is asymptotically stable, for any symmetric and positive definite matrix Q there exists a symmetric and positive definite matrix P that satisfies (3). (Therefore, $V(x) = x^\intercal Px$ is a quadratic Lyapunov function.)

(d) Suppose

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix}.$$

Use the Lyapunov stability theorem to prove that the system is asymptotically stable.

Problem 3: Convexity Preserving Operations

Suppose $J_1 : \mathbb{R}^n \to \mathbb{R}$ and $J_2 : \mathbb{R}^n \to \mathbb{R}$ are convex functions.

(a) Show that the sub-level set $\{z \in \mathbb{R}^n : J_1(z) \leq \alpha\}$ for some $\alpha \in \mathbb{R}$ is a convex set.

(b) Show that $J_1 + J_2$ is a convex function.

(c) Show that the composition of J_1 with an affine map is convex, that is, $J_1(Az + b)$ is a convex function.

(d) Show that the pointwise maximum of J_1 and J_2 is a convex function, that is, $J(z) := \max\{J_1(z), J_2(z)\}$ is a convex function.

Problem 4: Lagrange Duality

Consider the optimization problems.

P1

$$\begin{aligned} \min_x \quad & 0.5(x_1^2 + x_2^2) \\ \text{subject to} \quad & x_1 - 1 \geq 0. \end{aligned}$$

P2

$$\begin{aligned} \min_x \quad & x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 - 1 \leq 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

P3

$$\begin{aligned} \min_x \quad & -x \\ \text{subject to} \quad & x - 0.5 \leq 0 \\ & x \in \{0, 1\}. \end{aligned}$$

For each of the optimization problems above, answer the following questions.

(a) Is the problem a convex optimization problem? Provide reasons for your answer.

(b) Formulate the dual problem.
(Hint: Define the dual function as $\min_{x_1 \geq 0, x_2 \geq 0} L(x, u)$ for P2 and as $\min_{x \in \{0,1\}} L(x, u)$ for P3.)

(c) Is the dual problem a convex optimization problem? Provide reasons for your answer.

(d) Solve the dual problem.

(e) Compute the duality gap.
(Hint: the optimal costs of P1, P2, P3 are 0.5, -1, 0, respectively.)

(f) Check that when the problem is convex and Slater's condition is satisfied, the duality gap is zero (strong duality).