Homework2

EE688. Optimal Control Theory

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Contents

1 Solution of Problem 2: KKT conditions

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 + \theta x_2 \\ \text{subject to} \quad & 1 - x_1 x_2 \le 0 \\ & -x_1, -x_2 \le 0 \end{aligned}$$

1.1 (a)

Yes. The problem is convex!

Reasons:

- $J(x) = x_1^2 + \theta x_2$ is a quadratic function in the form of $x^T Q x + c^T x$ (convex function) [1].
- For the constraints, we can show that the intersection set of the constraints is convex[2] $S = \{(x_1, x_2) \in \mathcal{R}^2 : x_1 x_2 \ge 1, x_1, x_2 \ge 0\}$

  Proof: For simplicity, let us denote $z_1 = (x_1, y_1) \in S$, $z_2 = (x_2, y_2) \in S$. So, in order to prove that S is a convex set, we need to show that $\forall\ 0 \le \lambda \le 1$, $z_3 = (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in S$.

  That means we need to show that $(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2) \ge 1, \forall\ 0 \le \lambda \le 1$

  Please note that $x_1 y_1 \ge 1$ as $z_1 \in S$, $x_2 y_2 \ge 1$ as $z_2 \in S$

  AM-GM inequality source[3]

$$\begin{aligned} (\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2) = \\ &= \lambda x_1 \lambda y_1 + \lambda x_1 (1-\lambda) y_2 + (1-\lambda) x_1 (1-\lambda) y_2 + (1-\lambda) x_2 \lambda y_1 \\ &\text{(From the note)} \\ &\ge \lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)(x_1 y_2 + x_2 y_1) \\ &\text{(Using AM-GM inequality that states} : x + y \ge 2\sqrt{xy}, \text{s.t. } x, y \ge 0) \\ &\text{(Means that } (x_1 y_2 + x_2 y_1) \ge 2\sqrt{x_1 y_1\ x_2 y_2} \ge 2) \\ &\ge \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) \\ &= (\lambda + (1-\lambda))^2 = 1 \end{aligned}$$

□

That proves $(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2) \ge 1$ and $z_3 \in S$ s.t. $z_1, z_2 \in S$, that set S is a convex set

[1] Reference: link
[2] Reference: link
[3] Reference: link

1.2 (b)

First, formulating the Lagrangian

Lagrangian:

$$\mathcal{L}(x,u) = x_1^2 + \theta x_2 - u_1 x_1 x_2 + u_1 - u_2 x_1 - u_3 x_2$$

- Stationary conditions

$$\nabla_x \mathcal{L} = 0$$

$$\begin{aligned} 2x_1 - u_1 x_2 - u_2 &= 0 \quad (a) \\ \theta - u_1 x_1 - u_3 &= 0 \quad (b) \end{aligned} \tag{1}$$

- Complementary Slackness

$$\begin{aligned} u_1 - u_1 x_1 x_2 &= 0 \quad (a) \\ -u_2 x_1 &= 0 \quad (b) \\ -u_3 x_2 &= 0 \quad (c) \end{aligned} \tag{2}$$

- Dual feasibility

$$\begin{aligned} u_1 &\geq 0 \quad (a) \\ u_2 &\geq 0 \quad (b) \\ u_3 &\geq 0 \quad (c) \end{aligned} \tag{3}$$

- Primal feasibility

$$\begin{aligned} 1 - x_1 x_2 &\leq 0 \quad (a) \\ -x_1 &\leq 0 \quad (b) \\ -x_2 &\leq 0 \quad (c) \end{aligned} \tag{4}$$

1.3 (c)

Multiplying Eq. 1(a) both sides with x_1, we get the following:

$$\begin{aligned} 2x_1^2 - u_1 x_1 x_2 - u_2 x_1 &= 0 && (u_2 x_1 = 0 \;\; (2(b))) \\ 2x_1^2 - u_1 x_1 x_2 &= 0 && (u_1 x_1 x_2 = u_1 \;\; (2(a))) \\ 2x_1^2 - u_1 &= 0 \\ 2x_1^2 &= u_1 \end{aligned}$$

Multiplying Eq. 1(b) both sides with x_2, we get the following:

$$\begin{aligned}
\theta x_2 - u_1 x_{1_x} 2 - u_3 x_2 &= 0 && (u_3 x_2 = 0 \quad (2(c))) \\
\theta x_2 - u_1 x_{1_x} 2 &= 0 && (u_1 x_1 x_2 = u_1 \quad (2(a))) \\
\theta x_2 - u_1 &= 0 \\
\theta x_2 &= u_1
\end{aligned}$$

Therefore,

$$x_2 = \frac{2x_1^2}{\theta}$$

Furthermore, from Eq. 2(a)

$$\begin{aligned}
u_1 - u_1 x_1 x_2 &= 0 \\
u_1(1 - x_1 x_2) &= 0
\end{aligned}$$

Thus, there are two cases:

- $u_1 = 0$, then $x_1 = 0, x_2 = 0$, however, this contradicts the original constraints of $x_1 x_2 \geq 1$
- $1 - x_1 x_2 = 0$ then, $x_1 x_2 = 1 = \frac{2x_1^3}{\theta}$ and it follows that $x_1 = (\frac{\theta}{2})^{\frac{1}{3}}, x_2 = (\frac{2}{\theta})^{\frac{1}{3}}$

From the sufficient condition of KKT: The problem is convex (a), KKT conditions are satisfied (b,c). Therefore, $x_1 = x_1^* = (\frac{\theta}{2})^{\frac{1}{3}}, x_2 = x_2^* = (\frac{2}{\theta})^{\frac{1}{3}}$ are the optimal solutions

2 Solution of Problem 2: LQR

2.1 (a)

The source code is adapted from the lecture codes, and attached in the solution zip file. The simulation timesteps = simulation time / dt = 10s/0.1s = 100 timestep

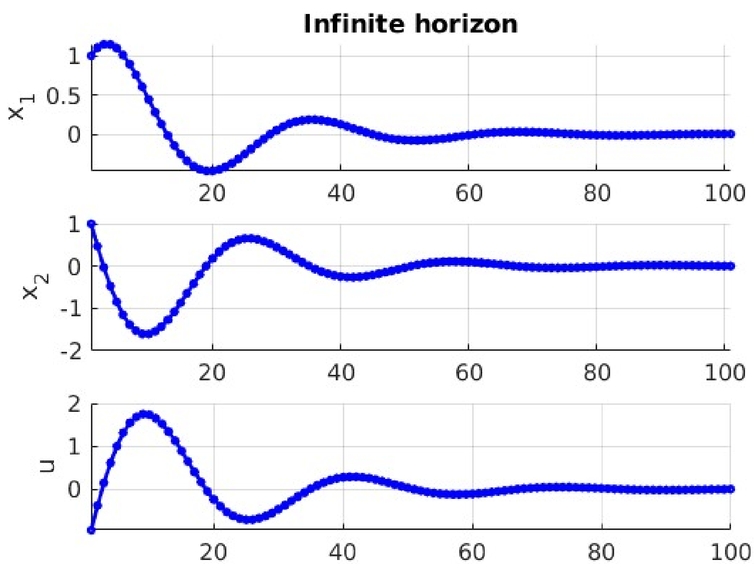
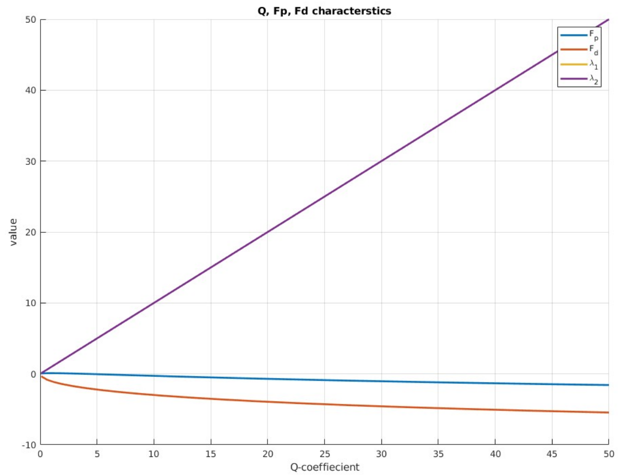


Figure 1: Infinite horizon

2.2 (b)

First, let us analysis with increasing Q. We created Q as an identity matrix multiplied by a coefficient $Q = q \cdot diag([1, 1])$ that is tested from 0.1 to 50 with discrete step = 100.



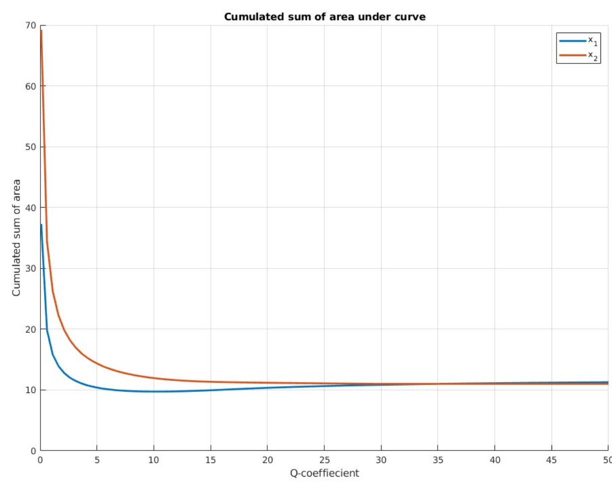
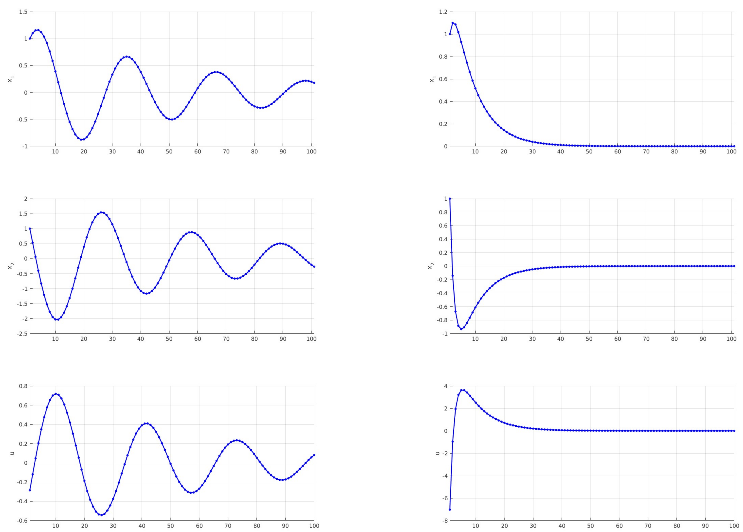


Figure 2: Changing Q matrix
Top figure shows the characteristics of Q matrix (eigenvalues) as it is diagonal, both eigenvalues are equal. Furthermore, it shows the changes of F_p and F_d as PD controller. Bottom figure shows the cumulative area under the curve of the states of the system for each Q

The following figure shows the plots of the system with the lowest and highest values for Q (q=0.1, 50)

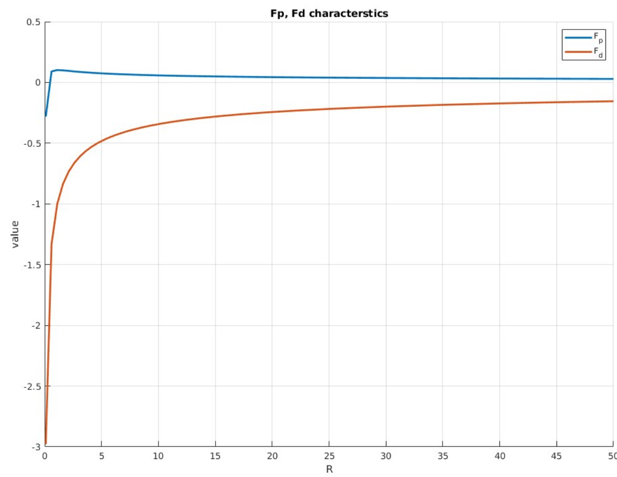


(a) q=0.1 (b) q=50

Figure 3: System plots for extreme values of q

We can see the larger q, the lower F_p, F_d and the less the area under the curve and the less the oscillations.

Second, let us analysis with increasing R that is tested from 0.1 to 50 with discrete step = 100.



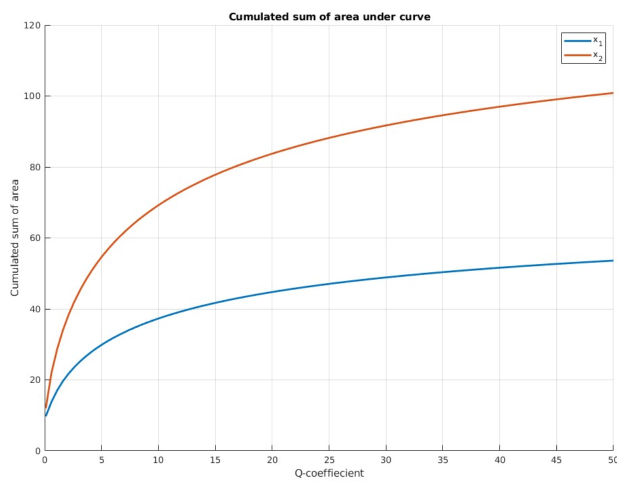
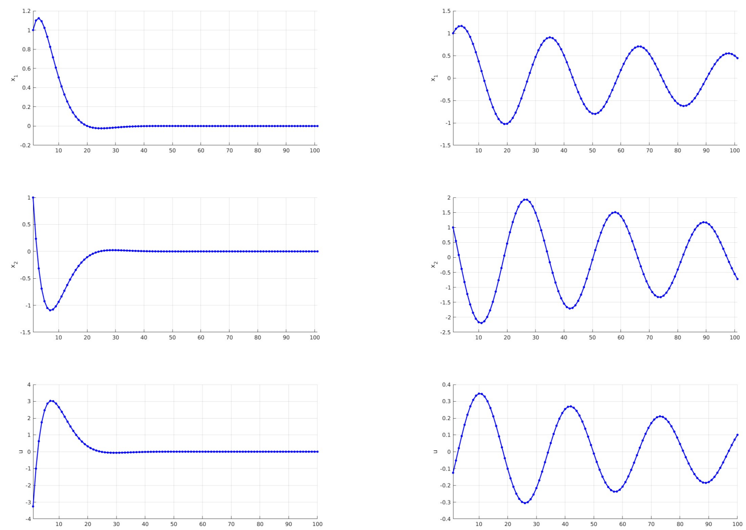


Figure 4: Changing R matrix
Top figure shows the characteristics of R. Furthermore, it shows the changes of F_p and F_d as PD controller. Bottom figure shows the cumulative area under the curve of the states of the system for each Q

The following figure shows the system plots with the lowest and highest values for R (R=0.1, 50)

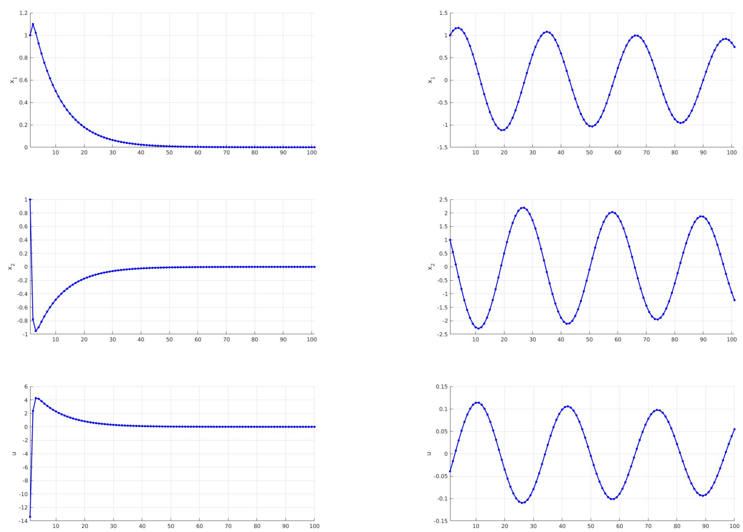


(a) R=0.1 (b) R=50

Figure 5: System plots for extreme values of R

We can see that the lower R, the lower F_p, F_d, the less the area under the curve, and the fewer the oscillations.

Finally, we can see the plots of the system with both good and bad combinations of Q and R.



(a) q=50,R=0.1 (Best)

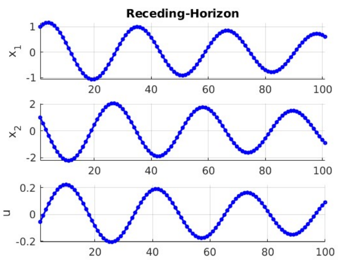
(b) q=0.1,R=50 (Worst)

Figure 6: System plots for extreme values of Q and R

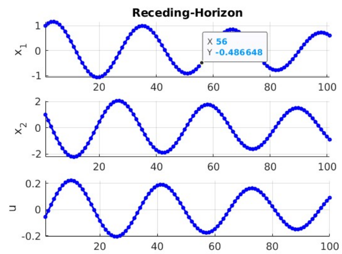
$Q = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}, R = 0.1$ are the best parameters to reduce the oscillations.

2.3 (c)

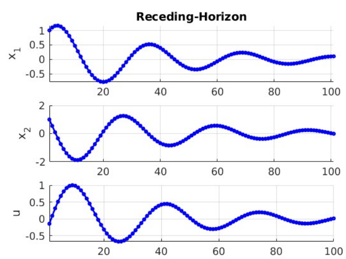
Here, we test with different N for the finite-horizon.



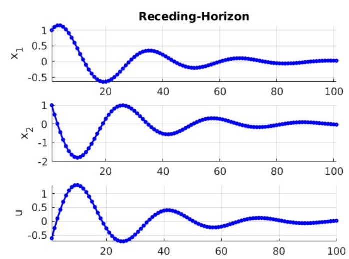
(a) N=1



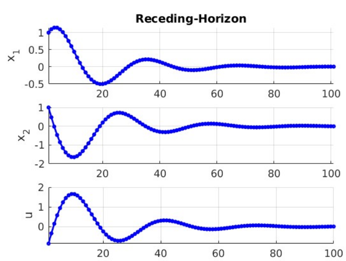
(b) N=5



(c) N=10



(d) N=15



(e) N=30

Figure 7: Receding-horizon strategy
Each figure has a different number of N steps for the horizon.

The receding-horizon is about applying the finite-horizon (Horizon length = N) at each time-step by solving the RDE (Riccati Difference Equation) and applying only the first part of the input of the computed horizon. On the other side, the infinite horizon solves the DARE (Discrete-time Algebraic Riccati Equation) requires satisfying two conditions of (A,B) is controllable and Q,R positive definite, which guarantees that the convergence to the unique positive definite matrix P. Therefore, the receding-horizon is used to approximate the infinite-horizon optimal control problem.

The larger the N, the receding-horizon strategy gets closer to the infinite-horizon solution as the matrix P converges to P_∞ when N gets large. But, with small N, the P is not converging to the P_∞ that is why the the above-mentioned plots of (a),(b), (c) are not similar to the plot in P2.(a) with infinite-horizon.

Therefore, the larger N, closer to the infinite-horizon solution, but more computation time. As larger N, needs more computation loop at each timestep to compute the P_0, thus it is necessary to choose the N, that it is not so big that takes too much computation and too small that it performs bad in terms of control.

3 Solution of Problem 3: MPC

3.1 (a)

It is completed in the Matlab code, the code is attached, such that:

$$A = \begin{bmatrix} 1.5431 & 1.1752 \\ 1.1752 & 1.5431 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.5431 \\ 1.1752 \end{bmatrix}$$

3.2 (b)

The source code is adapted from the lecture codes to fit the problem formulation, and attached to the solution zip file.

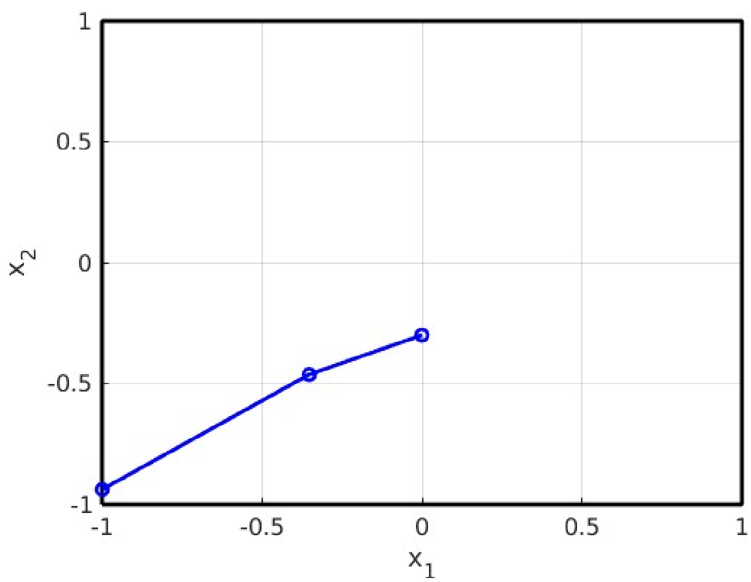


Figure 8: Infeasible after 3 time-steps

The problem becomes infeasible to solve after 3 time-steps as it is shown in Fig. 8.

3.3 (c)

The source code is adapted from the lecture codes to fit the problem formulation, and attached to the solution zip file.

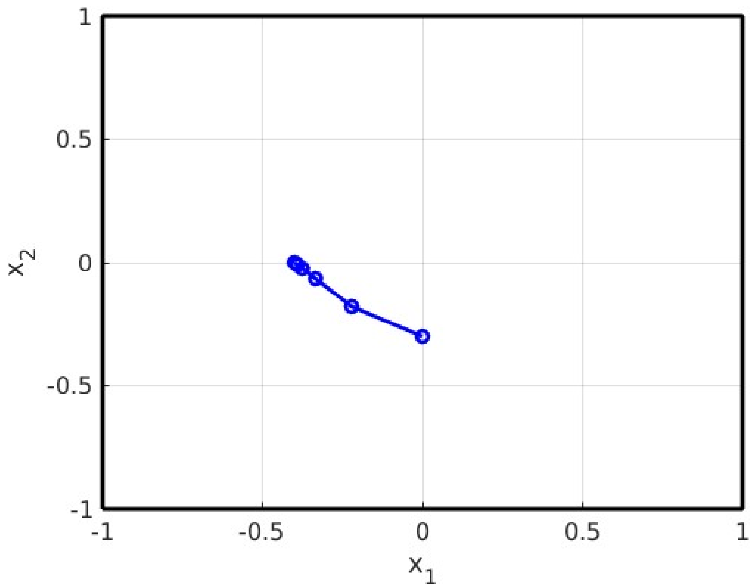


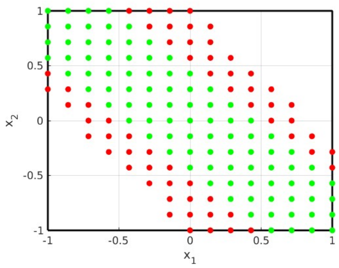
Figure 9: Using control invariant set with simulation of 20 time-steps

Using the control invariant set, it becomes feasible using the same initial state as in (b).

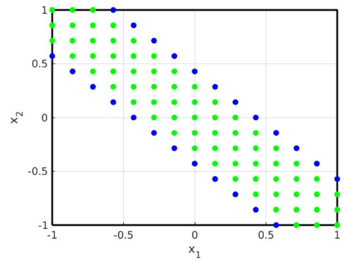
3.4 (d)

The source code is adapted from the lecture codes to fit the problem formulation, and attached to the solution zip file.

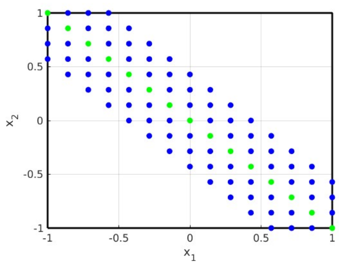
Here, we are interested in Fig. (a) as it shows the feasibility of the controller for initial state, such that the initial state is swept across the discretized region and tested for the solution of (b) and (c).



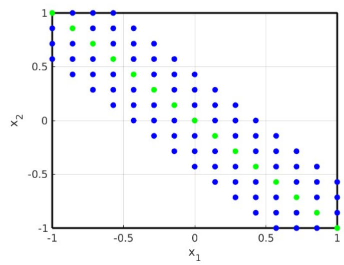
(a) 1 time-step



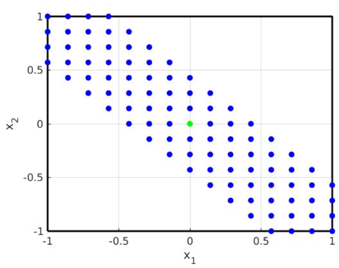
(b) 2 time-steps



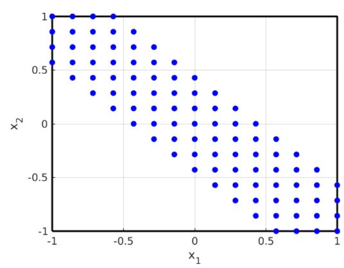
(c) 5 time-steps



(d) 10 time-steps



(e) 15 time-steps



(f) 100 time-steps

Figure 10: Regions of feasibility
Each figure has a different number of steps to solve the problem. After exceeding this number of steps, the problem is considered feasible for this number of steps and it is plotted, if it has terminated due to the feasiblity, it is not plotted. The blue means the feasible region using the control invariant set in (c), red means the feasible region from (b), and green means the intersected feasible region of red and blue

The rest of figures, shows how the feasibility region is decreased when we check whether the problem is feasible or not till certain number of steps. For example, Fig. 10 (c) shows the region where P3.b and P3.c solutions starts feasibly and continue to be feasible till the time-step=5. But Fig. 10 (a) only shows the feasibility of P3.b and P3.c only in the initial condition.

From these plots, we can see that, P3.b if it starts initially feasible, it gets infeasible while the simulation progresses (Decreasing of the red dots and the intersection (green dots)) when we increased the simulation time). However, P3.c it is constant feasible region for all the time-steps.

Therefore, the feasible region of P3.c is the positively invariant for the closed-loop system as the system starts inside the feasible region and all the future states will be inside that feasible region.