MATH 135 HOMEWORK 4

A. HENING

Do problems 5, 6, 7, 8, 9, 10, 11 from Rudin Chapter 2 and the following problems.

1. i) If (X, d) is a metric space show that $D(x,y) = \frac{d(x,y)}{1+d(x,y)}$ also defines a metric on X and that a subset U of X is open for d if and only if it is open for D.
   ii) Let $d : X \times X \to \mathbb{R}$ be a function satisfying all the properties of a metric *except* $(d(x,y) = 0) \Rightarrow x = y$. (i.e. it is possible for two distinct points to be at distance zero from each other). Define $\sim$ on X by
   $$x \sim y \Leftrightarrow d(x,y) = 0$$
   Show that $\sim$ is an equivalence relation and that setting $D([x],[y]) = d(x,y)$, where $[x] = \{z \in X | z \sim x\}$, is well-defined on equivalence classes and makes the set of equivalence classes into a metric space.
2. (**Bonus Problem**) Define the set
   $$X := \{K \subset \mathbb{C} : \ K \text{ is bounded and closed}\}$$
   Define a function $d : X \to X$ via
   $$d(K_1, K_2) = \inf\{\delta > 0 : K_1 \subset N_\delta(K_2) \text{ and } K_2 \subset N_\delta(K_1)\}$$
   where
   $$N_\delta(K) := \bigcup_{y \in K} N_\delta(y) = \{x \in \mathbb{C} : \exists y \in K \text{ with } |x - y| < \delta\}.$$
   i) Show that d defines a metric on X.
   ii) Is d still a metric if X contains all bounded sets in $\mathbb{C}$? All closed sets?