Math 123 Homework 6

Hanyuan Zhu

October 25, 2018

Question 1

(a)

$$vol(C) = \sum_{\substack{x_i \in C \\ x_j \in V}} W_{ij} = \sum_{\substack{x_i \in C \\ x_j \in N(x_i)}} W_{ij} \tag{1}$$

$N(x_i)$ is the set of neighorhood of x_i, $(x_j | w_{ij} \neq 0, for \quad x_i)$

$$D_{ii} = \sum_{x_j \in N(x_i)} W_{ij} \tag{2}$$

$$(Df^C)_i = \begin{cases} -\sum_{x_j \in N(x_i)} W_{ij} \sqrt{vol(\bar{C})/volC)} & x_i \in C \\ \sum_{x_j \in N(x_i)} W_{ij} \sqrt{vol(C)/vol\bar{C})} & x_i \in \bar{C} \end{cases} \tag{3}$$

$$\begin{aligned}
\langle Df^C \cdot \mathbb{1} \rangle &= \sum_{x_i \in V} (Df^C)_i = \sum_{x_i \in C} (Df^C)_i + \sum_{x_i \in \bar{C}} (Df^C)_i \\
&\stackrel{(3)}{=} -\sum_{\substack{x_i \in C \\ x_j \in V}} W_{ij} \sqrt{vol(\bar{C})/vol(C)} + \sum_{\substack{x_i \in \bar{C} \\ x_j \in V}} W_{ij} \sqrt{vol(C)/vol(\bar{C})} \\
&= -\sqrt{vol(\bar{C})/vol(C)} \cdot \sum_{\substack{x_i \in C \\ x_j \in V}} W_{ij} + \sqrt{vol(C)/vol(\bar{C})} \cdot \sum_{\substack{x_i \in \bar{C} \\ x_j \in V}} W_{ij} \\
&\stackrel{(1)}{=} -\sqrt{vol(\bar{C})/vol(C)} \cdot vol(C) + \sqrt{vol(C)/vol(\bar{C})} \cdot vol(\bar{C}) \\
&= -\sqrt{vol(\bar{C}) vol(C)} + \sqrt{vol(C) vol(\bar{C})} \\
&= 0
\end{aligned} \tag{4}$$

(b)

$$
\begin{aligned}
(f^C)^T D f^C &= \sum_{x_i \in V} f^C(x_i)(\sum_{x_j \in N(x_i)} W_{ij}) f^C(x_i) \\
&= \sum_{x_i \in V} (f^C(x_i))^2 (\sum_{x_j \in N(x_i)} W_{ij}) \\
&= \sum_{x_i \in C} (f^C(x_i))^2 (\sum_{x_j \in N(x_i)} W_{ij}) + \sum_{x_i \in \bar{C}} (f^C(x_i))^2 (\sum_{x_j \in N(x_i)} W_{ij}) \\
&= vol(C)\frac{vol(\bar{C})}{vol(C)} + vol(\bar{C})\frac{vol(C)}{vol(\bar{C})} \\
&= vol(\bar{C}) + vol(C) \\
&= \sum_{\substack{x_i \in \bar{C} \\ x_j \in V}} W_{ij} + \sum_{\substack{x_i \in C \\ x_j \in V}} W_{ij} = \sum_{\substack{x_i \in V \\ x_j \in V}} W_{ij} \\
&= vol(V)
\end{aligned}
\tag{5}
$$

(c)

$$
(f^C)^T L f^C = (f^C)^T (D - W) f^C = (f^C)^T D f^C - (f^C)^T W f^C \tag{6}
$$

We know that

$$
(f^C)^T D f^C = vol(V) \tag{7}
$$

And

$$
\begin{aligned}
(f^C)^T W f^C &= \sum_{\substack{x_i \in V \\ x_j \in V}} W_{ij} f_i^C f_j^C \\
&= \sum_{\substack{x_i \in C \\ x_j \in C}} W_{ij} f_i^C f_j^C + \sum_{\substack{x_i \in \bar{C} \\ x_j \in \bar{C}}} W_{ij} f_i^C f_j^C + \sum_{\substack{x_i \in C \\ x_j \in \bar{C}}} W_{ij} f_i^C f_j^C + \sum_{\substack{x_i \in \bar{C} \\ x_j \in C}} W_{ij} f_i^C f_j^C
\end{aligned}
\tag{8}
$$

Notice that

$$
f_i^C f_j^C = \begin{cases} \frac{vol(\bar{C})}{vol(C)} & x_i \in C and x_j \in C \\ \frac{vol(C)}{vol(\bar{C})}, & x_i \in \bar{C} and x_j \in \bar{C} \\ -1 & x_i \in \bar{C} and x_j \in C, or x_i \in C and x_j \in \bar{C} \end{cases}
$$

, and we define $\sum_{\substack{x_i \in C \\ x_j \in \bar{C}}} W_{ij} = \sum_{\substack{x_i \in \bar{C} \\ x_j \in C}} W_{ij} =: \frac{1}{2} vol(B)$.

Then (8) turns out to be

$$
\begin{aligned}
(8) &= \sum_{\substack{x_i \in C \\ x_j \in C}} W_{ij} \frac{vol(\bar{C})}{vol(C)} + \sum_{\substack{x_i \in \bar{C} \\ x_j \in \bar{C}}} W_{ij} \frac{vol(C)}{vol(\bar{C})} - \sum_{\substack{x_i \in C \\ x_j \in \bar{C}}} W_{ij} - \sum_{\substack{x_i \in \bar{C} \\ x_j \in C}} W_{ij} \\
&= \frac{vol(\bar{C})}{vol(C)} \sum_{\substack{x_i \in C \\ x_j \in C}} W_{ij} + \frac{vol(C)}{vol(\bar{C})} \sum_{\substack{x_i \in \bar{C} \\ x_j \in \bar{C}}} W_{ij} - \frac{1}{2} vol(B) - \frac{1}{2} vol(B) \\
&= \frac{vol(\bar{C})}{vol(C)} (\sum_{\substack{x_i \in C \\ x_j \in V}} W_{ij} - \sum_{\substack{x_i \in C \\ x_j \in \bar{C}}} W_{ij}) + \frac{vol(C)}{vol(\bar{C})} (\sum_{\substack{x_i \in \bar{C} \\ x_j \in V}} W_{ij} - \sum_{\substack{x_i \in \bar{C} \\ x_j \in C}} W_{ij}) - vol(B) \\
&= \frac{vol(\bar{C})}{vol(C)} (vol(C) - \frac{1}{2} vol(B)) + \frac{vol(C)}{vol(\bar{C})} (vol(\bar{C}) - \frac{1}{2} vol(B)) - vol(B) \\
&= vol(\bar{C}) - \frac{1}{2} vol(B)) \frac{vol(\bar{C})}{vol(C)} + vol(C) - \frac{1}{2} vol(B) \frac{vol(C)}{vol(\bar{C})} - vol(B) \\
&= vol(V) - \frac{1}{2} vol(B) (\frac{vol(\bar{C})}{vol(C)} + \frac{vol(C)}{vol(\bar{C})} + 2) \\
&= vol(V) - \frac{1}{2} vol(B) (\sqrt{\frac{vol(\bar{C})}{vol(C)}} + \sqrt{\frac{vol(C)}{vol(\bar{C})}})^2 \\
&= vol(V) - \frac{1}{2} vol(B) \frac{(\sqrt{vol(\bar{C})}\sqrt{vol(\bar{C})} + \sqrt{vol(C)}\sqrt{vol(C)})^2}{vol(\bar{C}) vol(C)} \\
&= vol(V) - \frac{1}{2} vol(B) \frac{vol(V)^2}{vol(\bar{C}) vol(C)} \\
&= vol(V) - \frac{1}{2} vol(B) vol(V) \frac{vol(V)}{vol(\bar{C}) vol(C)} \\
&= vol(V) - \frac{1}{2} vol(B) vol(V) \frac{vol(\bar{C}) + vol(C)}{vol(\bar{C}) vol(C)} \\
&= vol(V) - \frac{1}{2} vol(B) vol(V) (\frac{1}{vol(\bar{C})} + \frac{1}{vol(C)}) \\
&= vol(V) - vol(V) Ncut(C, \bar{C})
\end{aligned}
\tag{9}
$$

, where $Ncut(C, \bar{C}) = \sum_{\substack{x_i \in C \\ x_j \in \bar{C}}} W_{ij} (\frac{1}{vol(\bar{C})} + \frac{1}{vol(C)}) = \frac{1}{2} vol(B) (\frac{1}{vol(\bar{C})} + \frac{1}{vol(C)})$

Substitute (7) and (9) into (6), we have

$$
\begin{aligned}
(f^C)^T L f^C = (f^C)^T (D - W) f^C &= (f^C)^T D f^C - (f^C)^T W f^C \\
&= vol(V) - (vol(V) - vol(V) Ncut(C, \bar{C})) \\
&= vol(V) Ncut(C, \bar{C})
\end{aligned}
\tag{10}
$$

Question 2

(a) Because

$$L_{ij} = \begin{cases} -W_{ij} & i \neq j \\ \sum_{\substack{x_j \in V \\ i \neq j}} W_{ij} & i = j \end{cases}$$

, and

$$\lim_{\sigma \to 0} W_{ij} = \lim_{\sigma \to 0} e^{-\frac{||x_i - x_j||_2^2}{\sigma^2}} = 0$$

, then

$$\lim_{\sigma \to 0} L_{ij} = 0$$

.

Therefore, all eigenvalues are 0s and any $v \in \mathbb{R}^n / \vec{0}$ are the eighenvectors.

(b) Since

$$\lim_{\sigma \to \infty} W_{ij} = \lim_{\sigma \to \infty} e^{-\frac{||x_i - x_j||_2^2}{\sigma^2}} = 1$$

, then

$$\lim_{\sigma \to \infty} L_{ij} = \begin{cases} -1, & i \neq j \\ n-1, & i = j \end{cases}$$

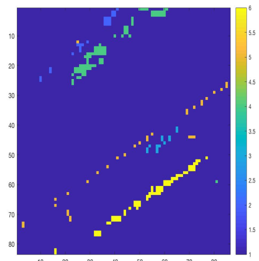
It is equivalent to a unweighted complete graph. Therefore, it has one egenvalue at 0, and other eigenvalues at n. The eigenvector of 0 is $\mathbb{1}$ and all other eigenvector are distinct.

Question 3

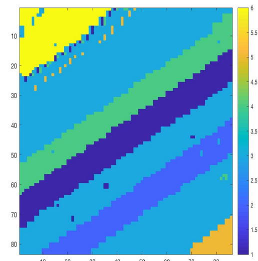
(a) We choose sigma = 50000 here and k =6. When the knn is about 15 to 25, the clusters looks similar to grountruth.

Firstly, it is clear that our result shows less infomation than grountruth, because the groundtruth has 14 categories, while we only have 6 clusters (in eigenspace) for kmean method.

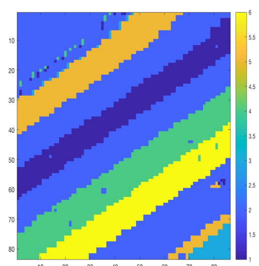
Secondly, When knn is too small, the weighted degree of different nodes various too much. Some points in " very dense" area become outliers in eighenspace, which results in kmean producing poor clusters(lots of clusters are outliers only).



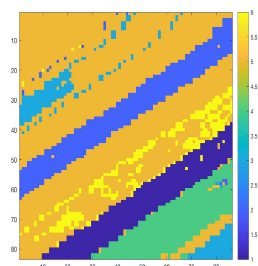
(a) knn=5



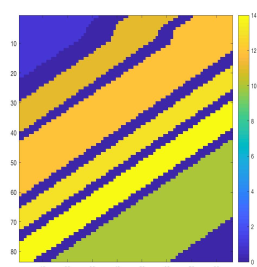
(b) knn=15



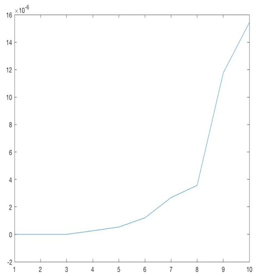
(a) knn=25



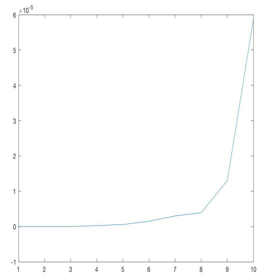
(b) knn=50



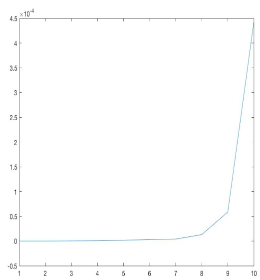
(c) groundtruth



(a) sigma=500



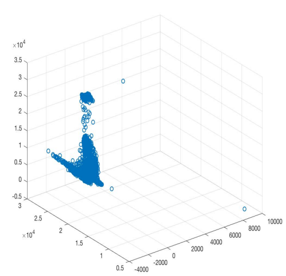
(b) sigma=5000



(c) sigma=50000

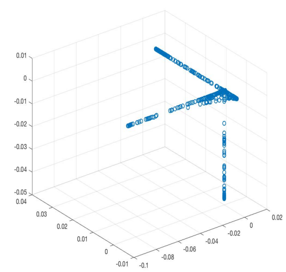
(b) As σ increase, the eigengaps increase about same order as σ does.

(c) The PCA plot represent a frequency space where those pixel points varies the most. I call the space as a frequency space is because that the subspace is a linear transformation from the original frequency space.



(a) PCA

The 3d plot of the first three Laplacian eigenvectors shows that the pixel points can be easily categorized into 3 clusters by kmeans. This 3d space is subspace of graph eigenspace. (knn=15 sigma = 50000)



(a) Eigenspace

Attachment : Codes

0.0.1 Question3a:spectral clustering

```
clear all;
load('SalinasA_corrected.mat');
close all;

data1D = reshape(salinasA_corrected,[],204);


%% Sparse construction
sigma=50;

W_sparse=sparse(size(data1D,1),size(data1D,1));
Knn=15;
NN=zeros(Knn,size(data1D,1));

for i=1:size(data1D,1)
    NN(:,i)=knnsearch(data1D,data1D(i,:),'k',Knn);
    for j=1:Knn
       W_sparse(i,NN(j,i))=exp(-norm(data1D(i,:)-data1D(NN(j,i),:)).^2/sigma^2);
%       W_sparse(i,NN(j,i))=(norm(data1D(i,:)-data1D(NN(j,i),:)));
    end
end

D_sparse=diag(sum(W_sparse,2));

L_sparse=eye(size(W_sparse))-D_sparse^(-1)*W_sparse;

[EigVecsSparse,EigValsSparse]=eigs(L_sparse,10,'sr');
EigValsSparse=diag(EigValsSparse);

close all;

%% Display Ng, Jordan, Weiss clustering
```

```
Labels=kmeans(EigVecsSparse(:,1:6),6,'Replicates',100);

im = image(reshape(Labels,83,86),'CDataMapping','scaled');
colorbar

%% sigma vs egienvalues
plot(fliplr(EigValsSparse'))
```

0.0.2 PCA

```
clear all;
load('SalinasA_corrected.mat');
close all;

data1D = reshape(salinasA_corrected,[],204);
pcaM=pca(data1D);
projData=data1D*pcaM;

pca1=reshape(projData(:,1),83,86);

pca1=reshape(projData(:,1),83,86);
image(pca1,'CDataMapping','scaled');

pca2=reshape(projData(:,2),83,86);
image(pca2,'CDataMapping','scaled');

pca3=reshape(projData(:,3),83,86);
image(pca3,'CDataMapping','scaled');
```