Math 135
A. Hening

Midterm Exam

Full name: ______________________________

- Outside scratch paper, formula sheets, and calculators are not allowed.
- Write your answers in the spaces following each question. If you need more room, use the reverse side of the page and check the box indicating I should look at the back.
- Circle or box important parts of your work and cross out portions you do not want evaluated.
- You may ask me questions, but unless there is an error on the exam I will probably refuse to answer.
- Unless told otherwise, **show all your work**.

Question:	1	2	3	4	Total
Points:	30	15	25	10	80
Score:					

Work on reverse ☐

1. (30 points) True/false questions. No justification required. *Read the statements carefully.* Each is worth 3 points.

   (a) True or false: The irrational numbers form a countable set.

   (b) True or false: A compact set in $\mathbb{R}$ with the regular metric is closed and bounded.

   (c) True or false: Let (X, d) be a metric space with the distance $d(p, q) = 1$ if $p \neq q$ and $d(p, p) = 0$. Then any subset $E \subset X$ is both open and closed.

   (d) True or false: The rational numbers $\mathbb{Q}$ do not have the least upper bound property: There exists $A \subset \mathbb{Q}$ such that A is bounded above but $\sup A$ does not exist.

   supA doesn't exist in Q

   (e) True or false: The only connected subsets of $\mathbb{R}$ (with the regular metric) are open, closed or half-open half-closed intervals.

   (f) True or false: The set $[0, 1] \cup [2, 3]$ is bounded, closed, and connected.

   (g) True or false: Given a subset E of a metric space (X, d), the closure $\overline{E}$ is the intersection of all closed sets that contain E.

   (h) True or false: Let $K \subset Y \subset X$ where (X, d) is a metric space. Then K is compact with respect to (Y, d) if and only if K is compact with respect to (X, d).

   (i) True or false: Let $(\mathbb{R}, d)$ be the metric space on $\mathbb{R}$ with the distance $d(p, q) = 1$ if $p \neq q$ and $d(p, p) = 0$. Then the sequence $(p_n)_{n \in \mathbb{N}}$ with $p_n = \frac{1}{n}$ converges to 0.

   (j) True or false: The rationals $\mathbb{Q}$ form a dense subset of the reals $\mathbb{R}$, that is $\overline{\mathbb{Q}} = \mathbb{R}$.

Work on reverse □

2. (15 points) (a) (5 points) Define what it means for two sets A and B to have the same cardinality.

   (b) (10 points) Prove that $\mathbb{Z} \sim \mathbb{N}$, that is $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality.

Work on reverse ☐

3. (25 points) Suppose that (X, d) is a metric space and let $E = \{x\}$ be the set containing the single point $x \in X$.

   (a) Define what it means for a set $O \subset X$ to be open.

   (b) Show, using only the definition of open, that the set $E^c = X \setminus E$ is open in X.

   (c) Define what it means for a set $K \subset X$ to be compact.

   (d) Show, using only the definition of compact, that the set $E = \{x\}$ is compact.

   (e) Generalize parts b) and d) to the setting where $E = \{x_1, \dots, x_n\}$.

Work on reverse ☐

4. (10 points) (Extra credit) Let (X, d) be a metric space. Let $p \in X$ and $r > 0$. Define $N_r(p) = \{q \in X \mid d(p, q) < r\}$ to be the neighborhood of radius $r > 0$ around p.

   (a) If $X = \mathbb{R}$ with the normal metric prove that

   $$\overline{N_r(0)} = \{q \in X \mid d(0, q) \leq r\}$$

   (b) Let (X, d) be any metric space. Is it still true that

   $$\overline{N_r(p)} = \{q \in X \mid d(p, q) \leq r\} \ ?$$

   Prove it or give a counterexample.

Extra sheet