Tufts University
Department of Mathematics
Fall 2018

MA 126: Numerical Analysis

Homework 9 (v1.0) [1]

Assigned Friday 9 November 2018
Due Friday 16 November 2018 at 3 pm

In class, we worked out the error estimate for the Trapezoidal Rule, from which we derived the Corrected Trapezoidal Rule. In this problem, we will conduct a similar analysis for Simpson's Rule. This is treated in Section 5.2.2 of the book, but the proof of Theorem 5.2.5 is omitted. This problem will fill in the details.

1. Recall that, for the Trapezoidal Rule, we used Eq. (4.53) for the error in one subinterval, approximating the Newton divided difference by Eq. (4.54). Moreover, we neglected the dependence of c_t on t, so that we could derive the expression in Eq. (5.26) for $\int_a^b f(x)\ dx$. We then used this to ultimately derive Eq. (5.32) for the error in the Trapezoidal Rule, from which we derived the Corrected Trapezoidal Rule.

   Show that this same approach will not work for Simpson's Rule. In particular, show that the error obtained by neglecting the dependence of c_t on t vanishes, thus indicating that we can not neglect this dependence.

2. To do a better job on deriving the error for Simpson's Rule, let's take $x_0 = a$, $x_1 = \frac{a+b}{2}$, and $x_2 = b$, and let's define $h := \frac{b-a}{2}$, so that $x_j = a + jh$. Then follow these steps:

   (a) Expand $f(x)$ in a Taylor series about $x = x_1$, retaining terms in the series up to order $(x - x_1)^3$ and an error term of order $(x - x_1)^4$.

   (b) Integrate the Taylor series and error term from $x = x_0 = a$ to $x = x_2 = b$. Use the fact that $x_2 - x_1 = x_1 - x_0 = h$ to simplify your result. When the smoke clears, you should have three terms remaining, at orders 1, 3 and 5 in h. These terms will contain derivatives of f at x_1 of orders 0, 2 and 4, respectively.

   (c) Use your result from 2(b) to write $f''(x_1)$ in terms of $f(x_0)$, $f(x_1)$, and $f(x_2)$ plus an error term of order h^2.

   (d) Substitute your result from 2(c) into your result from 2(b) to eliminate $f''(x_1)$ in that expression. Collect terms to find the error in $\int_a^b f(x)\ dx - S(f)$, where $S(f)$ denotes Simpson's Rule applied to this interval.

   (e) Now, for the *composite* version of Simpson's Rule (many subintervals), argue that this error accumulates to give you a Riemann sum. In this way, derive the error term for Simpson's Rule in the limit of large n.

   (f) From your above results, see if you can derive the essential result of Theorem 5.2.5, namely Eq. (5.36).