Homework 8 Numerical Analysis

Hanyuan Zhu

Question 1

We are given 0 to 4th degrees of Legendre polynomials from referrence.

$$\begin{cases} P_0(x) = 1 & (P_0(x), P_0(x)) = 2 \\ P_1(x) = x & (P_1(x), P_1(x)) = \frac{3}{2} \\ P_2(x) = \frac{1}{2}(3x^2 - 1) & (P_2(x), P_2(x)) = 0.4 \\ P_3(x) = \frac{1}{2}(5x^3 - 3x) & (P_3(x), P_3(x)) = 0.285714 \\ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) & (P_4(x), P_4(x)) = \frac{2}{9} \end{cases} \tag{1}$$

3.7 (b) $f(x) = log(1 + x^2)$

$$\begin{aligned} &\int_{-1}^{1} f(x)P_0(x)dx = \int_{-1}^{1} log(1 + x^2)dx = -4 + \pi + log(4) \\ &\int_{-1}^{1} f(x)P_1(x)dx = \int_{-1}^{1} log(1 + x^2)xdx = 0 \\ &\int_{-1}^{1} f(x)P_2(x)dx = \int_{-1}^{1} log(1 + x^2)\frac{1}{2}(3x^2 - 1)dx = \frac{10}{3} - \pi \\ &\int_{-1}^{1} f(x)P_3(x)dx = \int_{-1}^{1} log(1 + x^2)\frac{1}{2}(5x^3 - 3x)dx = 0 \\ &\int_{-1}^{1} f(x)P_4(x)dx = \int_{-1}^{1} log(1 + x^2)\frac{1}{8}(35x^4 - 30x^2 + 3)dx = \frac{5\pi}{2} - \frac{118}{15} \end{aligned} \tag{2}$$

Non-Zero terms of least square approximation,

$$\begin{aligned} l(x) &= \frac{(f, P_0)}{(P_0, P_0)} + \frac{(f, P_2)}{(P_2, P_2)}P_2(x) + \frac{(f, P_4)}{(P_4, P_4)}P_4(x) \\ &= \frac{\pi + log(4) - 4}{2} + (\frac{25}{6} - \frac{5\pi}{4})(3x^2 - 1) + (\frac{45\pi}{32} - \frac{177}{40})(35x^4 - 30x^2 + 3) \\ &= \frac{log(4)}{2} - \frac{2333}{120} + \frac{191\pi}{32} + \frac{1589}{16}x^2 - \frac{3381}{32}x^4 \end{aligned} \tag{3}$$

3.7 (c) $f(x) = tan^{-1}x$

$$\begin{aligned}
\int_{-1}^{1} f(x)P_0(x)dx &= \int_{-1}^{1} tan^{-1}xdx = 0 \\
\int_{-1}^{1} f(x)P_1(x)dx &= \int_{-1}^{1} xtan^{-1}xdx \approx 0.5708 \\
\int_{-1}^{1} f(x)P_2(x)dx &= \int_{-1}^{1} tan^{-1}x\frac{1}{2}(3x^2-1)dx = 0 \\
\int_{-1}^{1} f(x)P_3(x)dx &= \int_{-1}^{1} tan^{-1}x\frac{1}{2}(5x^3-3x)dx = -0.0228612 \\
\int_{-1}^{1} f(x)P_4(x)dx &= \int_{-1}^{1} tan^{-1}x\frac{1}{8}(35x^4-30x^2+3)dx = 0
\end{aligned} \tag{4}$$

Non-Zero terms of least square approximation,

$$\begin{aligned}
l(x) &= \frac{(f,P_1)}{(P_1,P_1)}x + \frac{(f,P_3)}{(P_3,P_3)}P_3(x) \\
&\approx 0.5708\frac{2}{3}x - \frac{0.0228612}{0.285714}\frac{1}{2}(5x^3-3x) \\
&= 0.500555x - 0.2000036x^3
\end{aligned} \tag{5}$$

Question 2

a. We will show the orthogonality, $(Q_i, Q_j) = 0, i \neq j$ by induction. Given $Q_0 = M_0$,

$$\begin{aligned}
(Q_0, Q_1) &= (Q_0, M_1 - \frac{(M_1,Q_0)}{(Q_0,Q_0)}Q_0) \\
&= (Q_0, M_1) - \frac{(M_1,Q_0)}{(Q_0,Q_0)}(M_0,Q_0) \\
&= (Q_0, M_1) - \frac{(M_1,Q_0)}{(Q_0,Q_0)}(Q_0,Q_0) \\
&= (Q_0, M_1) - (Q_0, M_1) = 0
\end{aligned} \tag{6}$$

We assume $(Q_n, Q_j) = 0$, for all $j < n$,

$$\begin{aligned}
(Q_n, Q_j) &= (M_n - \sum_{i=0}^{n-1}\frac{(M_n,Q_i)}{(Q_i,Q_i)}Q_i, Q_j) \\
&= (M_n, Q_j) - \sum_{\substack{i=0 \\ i\neq j}}^{n-1}\frac{(M_n,Q_i)}{(Q_i,Q_i)}(Q_i,Q_j) - \frac{(M_n,Q_j)}{(Q_j,Q_j)}(Q_j,Q_j) \\
&= -\sum_{\substack{i=0 \\ i\neq j}}^{n-1}\frac{(M_n,Q_i)}{(Q_i,Q_i)}(Q_i,Q_j) = 0
\end{aligned} \tag{7}$$

That is $j < n, (Q_n, Q_j) = 0 \Leftrightarrow \sum_{\substack{i=0 \\ i\neq j}}^{n-1} \frac{(M_n,Q_i)}{(Q_i,Q_i)}(Q_i, Q_j) = 0$

Now we want to show $(Q_{n+1}, Q_j) = 0$, for all $j < n+1$,

$$(Q_{n+1}, Q_j) =(M_{n+1} - \sum_{i=0}^{n} \frac{(M_{n+1}, Q_i)}{(Q_i, Q_i)} Q_i, Q_j)$$

$$\begin{aligned}
&\text{Case 1}: \text{If } n \neq j,\\
&=(M_{n+1}, Q_j) - \sum_{\substack{i=0 \\ i\neq j}}^{n-1} \frac{(M_{n+1}, Q_i)}{(Q_i, Q_i)}(Q_i, Q_j)\\
&\quad - \frac{(M_{n+1}, Q_n)}{(Q_n, Q_n)}(Q_n, Q_j) - \frac{(M_{n+1}, Q_j)}{(Q_j, Q_j)}(Q_j, Q_j)\\
&=(M_{n+1}, Q_j) - 0 - \frac{(M_{n+1}, Q_n)}{(Q_n, Q_n)}0 - (M_{n+1}, Q_j)\\
&=(M_{n+1}, Q_j) - (M_{n+1}, Q_j) = 0\\
&\text{Case 2}: \text{If } n = j,\\
&=(M_{n+1}, Q_j) - \sum_{i=0}^{n-1} \frac{(M_{n+1}, Q_i)}{(Q_i, Q_i)}(Q_i, Q_n)\\
&\quad - \frac{(M_{n+1}, Q_n)}{(Q_n, Q_n)}(Q_n, Q_j)\\
&=(M_{n+1}, Q_j) - \sum_{i=0}^{n-1} \frac{(M_{n+1}, Q_i)}{(Q_i, Q_i)}0 - \frac{(M_{n+1}, Q_n)}{(Q_n, Q_n)}(Q_n, Q_n)\\
&=(M_{n+1}, Q_j) - (M_{n+1}, Q_j) = 0
\end{aligned} \tag{8}$$

Therefore, such $\{Q_n\}_{n=0}^{\infty}$ satisfies $(Q_i, Q_j) = 0, i \neq j$.

b. To get Q_1, Q_2, Q_3, Q_4 , we are given $Q_0 = M_0 = 1, M_1 = x, M_2 = x^2, M_3 = x^3, M_4 = x^4$.

If $f(x)g(x)$ is odd function,

$$\begin{aligned}
(f(x), g(x)) &= \int_{-1}^{1} f(x)g(x)dx = \int_{0}^{1} f(x)g(x)dx + \int_{-1}^{0} f(x)g(x)dx\\
&= \int_{0}^{1} f(x)g(x)dx + \int_{-1}^{0} -f(-x)g(-x)dx\\
&= \int_{0}^{1} f(x)g(x)dx + \int_{-1}^{0} f(-x)g(-x)d(-x)\\
&= \int_{0}^{1} f(x)g(x)dx + \int_{1}^{0} f(x)g(x)dx\\
&= \int_{0}^{1} f(x)g(x)dx - \int_{0}^{1} f(x)g(x)dx = 0
\end{aligned} \tag{9}$$

$$Q_1 = x - \cancelto{0}{(x,1)}\frac{1}{(1,1)} = x \tag{10}$$

$$\begin{aligned}
Q_2 &= x^2 - \cancelto{0}{(x^2,x)}\frac{x}{(x,x)} - (x^2,1)\frac{1}{(1,1)} \\
&= x^2 - \int_{-1}^{1} x^2 dx \frac{1}{\int_{-1}^{1} dx} \\
&= x^2 - \frac{2}{3}\frac{1}{2} = x^2 - \frac{1}{3}
\end{aligned} \tag{11}$$

$$\begin{aligned}
Q_3 &= x^3 - \cancelto{0}{(x^3, x^2 - \frac{1}{3})}\frac{Q_2}{\|Q_2\|^2} - (x^3, x)\frac{x}{\|x\|^2} - \cancelto{0}{(x^3,1)}\frac{1}{2} \\
&= x^3 - (x^3, x)\frac{x}{\|x\|^2} \\
&= x^3 - \frac{2}{5}\frac{3}{2}x = x^3 - \frac{3}{5}x
\end{aligned} \tag{12}$$

$$\begin{aligned}
Q_4 &= x^4 - \cancelto{0}{(x^4, x^3 - \frac{3}{5}x)}\frac{Q_3}{\|Q_3\|^2} - (x^4, x^2 - \frac{1}{3})\frac{Q_2}{\|Q_2\|^2} - \cancelto{0}{(x^4,x)}\frac{x}{\|x\|^2} - (x^3,1)\frac{1}{2} \\
&= x^4 - (x^4, x^2 - \frac{1}{3})\frac{Q_2}{\|Q_2\|^2} - (x^4,1)\frac{1}{2} \\
&= x^4 - \int_{-1}^{1} x^6 - \frac{x^4}{3} dx \frac{x^2 - \frac{1}{3}}{\int_{-1}^{1}(x^2 - \frac{1}{3})^2 dx} - \frac{1}{5} \\
&= x^4 - (\frac{2}{7} - \frac{2}{15})\frac{x^2 - \frac{1}{3}}{\frac{2}{5} - \frac{2}{9}} - \frac{1}{5} = x^4 - \frac{45}{8}\frac{16}{7 \times 15}x^2 - \frac{45}{8}\frac{16}{7 \times 15}\frac{1}{3} - \frac{1}{5} \\
&= x^4 - \frac{6}{7}x^2 + \frac{2}{7} - \frac{1}{5} = x^4 - \frac{6}{7}x^2 + \frac{3}{35}
\end{aligned} \tag{13}$$

c.

$$Q_1(1) = 1$$
$$Q_2(1) = c(1 - \frac{1}{3}) = 1 \to c = \frac{3}{2}$$
$$Q_3(1) = c(1 - \frac{3}{5}) \to c = \frac{5}{2}$$
$$Q_4(1) = c(1 - \frac{6}{7} + \frac{3}{35}) \to c = \frac{35}{8}$$

$$
\begin{aligned}
Q_1 &= x \\
Q_2 &= \frac{3}{2}(x^2 - \frac{1}{3}) \\
Q_3 &= \frac{5}{2}(x^3 - \frac{3}{5}x) \\
Q_4 &= \frac{35}{8}(x^4 - \frac{6}{7}x^2 + \frac{3}{35})
\end{aligned} \tag{14}
$$

Question 3

a. Let $f(t) = (1 - 2xt + t^2)^{-\frac{1}{2}}$, then $f'(t) = -\frac{1}{2}(1 - 2xt + t^2)^{-\frac{3}{2}}(-2x + 2t)$. The first two terms of maclaurin series of $f(t)$ is $f(t) = f(0) + f'(0)t = 1 + xt$.

Thus

$$
\begin{aligned}
P_0(x) &= f(0) = 1 \\
P_1(x) &= f'(0) = x
\end{aligned} \tag{15}
$$

This satisfies the formula in (4.115).

b. From part a., we have LHS $f'(t) = (1 - 2xt + t^2)^{-\frac{3}{2}}(x - t) = \frac{x-t}{1-2xt+t^2} f(t)$, and RHS $f'(t) = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$.

Combine two equations above,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \frac{x - t}{1 - 2xt + t^2} \sum_{n=0}^{\infty} P_n(x)t^n \\
&(1 - 2xt + t^2) \sum_{n=1}^{\infty} nP_n(x)t^{n-1} = (x - t) \sum_{n=0}^{\infty} P_n(x)t^n \\
&\sum_{n=1}^{\infty} nP_n(x)t^{n-1} - 2xt \sum_{n=1}^{\infty} nP_n(x)t^{n-1}t^2 \sum_{n=1}^{\infty} nP_n(x)t^{n-1} \\
&\qquad = x \sum_{n=0}^{\infty} P_n(x)t^n - t \sum_{n=0}^{\infty} P_n(x)t^n \\
&\sum_{n=1}^{\infty} nP_n(x)t^{n-1} - 2x \sum_{n=1}^{\infty} nP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1} \\
&\qquad = x \sum_{n=0}^{\infty} P_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1}
\end{aligned} \tag{16}
$$

Relabel the sum above,

$$\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - 2x\sum_{n=1}^{\infty} nP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n = x\sum_{n=0}^{\infty} P_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n \tag{17}$$

Note the t^0 and t^1 terms on both side can be cancelled out by given $P_2(x)$ formula of Legendre polynomial.

$$\begin{aligned}
&P_1(x) + 2tP_2(x) - 2xtP_1(x) = xP_0(x) + xtP_1(x) - tP_0(x) \\
&2tP_2(x) = (3xt-1)P_1(x) + (x-t)P_0(x) \\
&\text{Substitute } P_1(x) = x, P_0(x) = 1 \\
&2P_2(x) = 3x^2 - 1
\end{aligned} \tag{18}$$

Then the rest terms , $n \geq 2$, can be written as

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n+1)P_{n+1}(x)t^n - 2x\sum_{n=2}^{\infty} nP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n \\
&\qquad\qquad = x\sum_{n=2}^{\infty} P_n(x)t^n - \sum_{n=2}^{\infty} P_{n-1}(x)t^n \\
&(n+1)P_{n+1}(x) = 2xnP_n(x) + (1-n)P_{n-1}(x) + xP_n(x) - P_{n-1}(x) \\
&(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \\
&P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x)
\end{aligned} \tag{19}$$

Let $n = 1$, we have

$$\begin{aligned}
P_2(x) &= \frac{3}{2}xP_1(x) - \frac{1}{2}P_0(x) \\
P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2}
\end{aligned} \tag{20}$$

Thus $P_2(x), P_1(x)$, and $P_0(x)$ also satisfies the recursion. Therfore For all $n \geq 1$, we have

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x)$$

.

c. First of all, in part a., we have show $P_0(x)$ and $P_1(x)$ defined here is the same as defined in Legendre polynomials.

Secondly, by using the generating function, we derive the triple recursion relation in part b.

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x)$$

.

This is the same triple recursion relation of Legendre polynomials.

Thus any P_n derive from the same triple recursion relation by the same $P_0(x)$ and $P_1(x)$ are identical.

d. When $x = 1$, we have $f(t) = (1-t)^{-1}$. The maclaurin series of $f(t)$ is

$$f(t) = \sum_{n=0}^{\infty} t^n$$

From the RHS of generating function, we have,

$$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n$$
$$\to P_n(1) = 1$$

e. The triple recursion formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

We want to show

$$P_n(x) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} x^k$$

To show it by mathematical induction, we have $P_0 = 1$ and $P_1 = x$, Then

$$\text{By triple recursive method } P_2 = \frac{(2+1)x^2 - 1}{1+1} = \frac{3}{2}x^2 - \frac{1}{2}$$

By $P_n(x)$ formula

$$\begin{aligned} P_2 &= 2^2 \sum_{k=0}^{2} \binom{2}{k} \binom{\frac{2+k-1}{2}}{2} x^k \\ &= 2^2 \left(\binom{2}{2}\binom{\frac{3}{2}}{2} x^2 + \binom{2}{1}\binom{\frac{2}{2}}{2} x^1 + \binom{2}{0}\binom{\frac{1}{2}}{2} x^0 \right) \\ &= 2^2 \left(1 * \frac{3}{8}x^2 + 2 * 0x^1 + 1 * (-\frac{1}{8})x^0 \right) \\ &= \frac{3}{2}x^2 - \frac{1}{2} \end{aligned} \tag{21}$$

Now we assume

$$P_n(x) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} x^k$$

and

$$P_{n-1}(x) = 2^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{\frac{n+k-2}{2}}{n-1} x^k$$

By triple recursion, we have

$$\begin{aligned}
&(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \\
=&(2n+1)x2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} x^k - n2^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{\frac{n+k-2}{2}}{n-1} x^k \\
=&(2n+1)2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} x^{k+1} - 2^{n-1} \sum_{k=0}^{n-1} \left(n\binom{n-1}{k}\right) \binom{\frac{n+k-2}{2}}{n-1} x^k \\
=&(2n+1)2^n \sum_{k=1}^{n+1} \binom{n}{k-1} \binom{\frac{n+k-2}{2}}{n} x^k - 2^{n-1} \sum_{k=0}^{n-1} \left(\binom{n}{k}(n-k)\right) \binom{\frac{n+k-2}{2}}{n-1} x^k \\
=&(2n+1)2^n \sum_{k=1}^{n+1} \left(\frac{k}{n+1}\right) \binom{n+1}{k} \binom{\frac{n+k}{2}-1}{n} x^k \\
&- 2^n \sum_{k=0}^{n-1} \left(\frac{(n-k+1)(n-k)}{2(n+1)}\right) \binom{n+1}{k} \binom{\frac{n+k}{2}-1}{n-1} x^k \\
=&(2n+1)2^n \sum_{k=1}^{n+1} \left(\frac{2k}{n+k}\right) \binom{n+1}{k} \binom{\frac{n+k}{2}}{n+1} x^k \\
&- 2^n \sum_{k=0}^{n-1} \left(\frac{(n-k+1)(n-k)n}{(n+k)(n+1)}\right) \binom{n+1}{k} \binom{\frac{n+k}{2}}{n} x^k \\
=&(2n+1)2^n \sum_{k=1}^{n+1} \left(\frac{2k}{n+k}\right) \binom{n+1}{k} \binom{\frac{n+k}{2}}{n+1} x^k \\
&- 2^n \sum_{k=0}^{n-1} \left(\frac{(n-k+1)(n-k)n}{(n+k)(\frac{k-n}{2})}\right) \binom{n+1}{k} \binom{\frac{n+k}{2}}{n+1} x^k \\
=&(2n+1)2^{n+1} \sum_{k=1}^{n+1} \left(\frac{k}{n+k}\right) \binom{n+1}{k} \binom{\frac{n+k}{2}}{n+1} x^k \\
&+ 2^{n+1} \sum_{k=0}^{n-1} \left(\frac{(n-k+1)n}{n+k}\right) \binom{n+1}{k} \binom{\frac{n+k}{2}}{n+1} x^k
\end{aligned} \tag{22}$$

For terms from x^1 to x^{n-1} in RHS of (2)

$$\begin{aligned}
&= 2^{n+1}\sum_{k=1}^{n-1}\left(\frac{(2n+1)k}{n+k}+\frac{(n-k+1)n}{n+k}\right)\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k \\
&= 2^{n+1}\sum_{k=1}^{n-1}\left(\frac{(2n+1)k+(n-k+1)n}{n+k}\right)\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k \\
&= 2^{n+1}\sum_{k=1}^{n-1}\left(\frac{(n+k)(n+1)}{n+k}\right)\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k \\
&= 2^{n+1}\sum_{k=1}^{n-1}(n+1)\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k \\
&= (n+1)2^{n+1}\sum_{k=1}^{n-1}\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k
\end{aligned} \tag{23}$$

For terms x^n and x^{n+1} in RHS of (2)

$$= (2n+1)2^{n+1}\sum_{k=n}^{n+1}\left(\frac{k}{n+k}\right)\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k$$

$$= (2n+1)2^{n+1}\left(\frac{n}{2n}\right)\binom{n+1}{n}\binom{\frac{2n}{2}}{n+1}x^n + (2n+1)2^{n+1}\left(\frac{n+1}{2n+1}\right)\binom{n+1}{n+1}\binom{\frac{2n+1}{2}}{n+1}x^{n+1}$$

$$= (2n+1)2^{n+1}\left(\frac{n}{2n}\right)\binom{n+1}{n}\binom{n}{n+1}x^n + (n+1)2^{n+1}\binom{n+1}{n+1}\binom{\frac{2n+1}{2}}{n+1}x^{n+1}$$

Because $\binom{n}{n+1} = 0$ in x^n term, we can multiply any constant to the term

$$= (n+1)2^{n+1}\binom{n+1}{n}\binom{n+n}{n+1}x^n + (n+1)2^{n+1}\binom{n+1}{n+1}\binom{\frac{2n+1}{2}}{n+1}x^{n+1}$$

$$= (n+1)2^{n+1}\sum_{k=n}^{n+1}\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k \tag{24}$$

For the term x^0 in RHS of (2)

$$\begin{aligned}
&= 2^{n+1}\left(\frac{(n+1)n}{n}\right)\binom{n+1}{0}\binom{\frac{n}{2}}{n+1}x^0 \\
&= (n+1)2^{n+1}\binom{n+1}{0}\binom{\frac{n}{2}}{n+1}x^0
\end{aligned} \tag{25}$$

To sum up (3) (4) and (5), the LHS of (2)

$$(n+1)P_{n+1}(x) = (n+1)2^{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k$$

$$P_{n+1}(x) = 2^{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\binom{\frac{n+k}{2}}{n+1}x^k$$

Therefore $P_n(x) = 2^n\sum_{k=0}^{n}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k$

f. We have

$$P_n(x) = 2^n\sum_{k=0}^{n}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k$$

$$\begin{aligned}
\frac{\partial}{\partial x}P_n(x) &= 2^n\sum_{k=1}^{n}k\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^{k-1}\\
(1-x^2)\frac{\partial}{\partial x}P_n(x) &= 2^n\sum_{k=1}^{n}k\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^{k-1}\\
&\quad - 2^n\sum_{k=1}^{n}k\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^{k+1}\\
\frac{\partial}{\partial x}[(1-x^2)\frac{\partial}{\partial x}P_n(x)] &= 2^n\sum_{k=2}^{n}k(k-1)\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^{k-2} \qquad (26)\\
&\quad - 2^n\sum_{k=1}^{n}k(k+1)\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^{k}\\
\frac{\partial}{\partial x}[(1-x^2)\frac{\partial}{\partial x}P_n(x)] &= 2^n\sum_{k=0}^{n-2}(k+2)(k+1)\binom{n}{k+2}\binom{\frac{n+k+1}{2}}{n}x^{k}\\
&\quad - 2^n\sum_{k=1}^{n}k(k+1)\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^{k}
\end{aligned}$$

Note the terms as k = 0, n-1 and n,

$$\begin{aligned}
\text{k = 0 term: } & 2^n(0+2)(0+1)\binom{n}{0+2}\binom{\frac{n+0+1}{2}}{n}x^0\\
&= 2^n n(n-1)\binom{\frac{n+1}{2}}{n} = 2^n n(n-1)\frac{n+1}{1-n}\binom{\frac{n-1}{2}}{n} \qquad (27)\\
&= -n(n+1)2^n\binom{\frac{n-1}{2}}{n}
\end{aligned}$$

$$\begin{aligned}
\text{k = n-1 term: } & -2^n(n-1)n\binom{n}{n-1}\binom{n-1}{n}x^{n-1} = 0\\
\text{k = n term: } & -2^n n(n+1)\binom{n}{n}\binom{\frac{2n-1}{2}}{n}x^n \qquad (28)\\
&= -n(n+1)2^n\binom{n}{n}\binom{\frac{2n-1}{2}}{n}x^n
\end{aligned}$$

Note the terms from k = 1 to n-2 of equation (11),

$$\begin{aligned}
&\frac{\partial}{\partial x}[(1-x^2)\frac{\partial}{\partial x}P_n(x)] \\
&= 2^n \sum_{k=1}^{n-2}(k+2)(k+1)\binom{n}{k+2}\binom{\frac{n+k+1}{2}}{n}x^k \\
&- 2^n \sum_{k=1}^{n-2} k(k+1)\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k \\
&= 2^n \sum_{k=1}^{n-2}(n-k)(n-k-1)\frac{n+k+1}{2}\frac{2}{k+1-n}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k \\
&- 2^n \sum_{k=1}^{n-2} k(k+1)\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k \\
&= -2^n \sum_{k=1}^{n-2}(n-k)(n+k+1)\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k \\
&- 2^n \sum_{k=1}^{n-2} k(k+1)\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k \\
&= 2^n \sum_{k=1}^{n-2}(-(n-k)(n+k+1)-k(k+1))\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k \\
&= 2^n \sum_{k=1}^{n-2}(k^2-n^2+k-n-k-k^2)\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k \\
&= -n(n+1)2^n \sum_{k=1}^{n-2}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k \\
&= -n(n+1)P_n(x)
\end{aligned} \tag{29}$$

Question 4

4a.

$$\begin{aligned}
(1-x^2)^n &= \sum_{k=0}^{n}\binom{n}{k}(-x^2)^k = \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{2k} \\
\frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x^2)^n &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^{2k} = n!\sum_{k=\frac{n}{2}}^{n}\binom{n}{k}(-1)^k\binom{2k}{n}x^{2k-n}
\end{aligned} \tag{30}$$

Thus, the highet order term of $P_n(x)$ is $\frac{(-1)^n}{2^n}\binom{2n}{n}x^n \neq 0$, so $P_n(x)$ is a polynomials with degree n.

4b.

Given $P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n(1-x^2)^n}{\mathrm{d}x^n}$ and $P_m(x) = \frac{1}{2^m m!}\frac{\mathrm{d}^m(1-x^2)^m}{\mathrm{d}x^m}$.

First of all, we will show $\frac{\mathrm{d}^{n-k}(1-x^2)^n}{\mathrm{d}x^{n-k}}|_{-1}^{1} = 0, k \leq n$.

$$\frac{\mathrm{d}^{n-k}(1-x^2)^n}{\mathrm{d}x^{n-k}}\Big|_{-1}^{1} = \sum_{s=1}^{n-k}\sum_{k=1}^{s}(x-1)^{n-p}(x+1)^{n-s+p}DL((x-1)^n(x+1)^n)\Big|_{-1}^{1} = 0$$

where, $DL((x^ny^n)$ is linear combination of some derivates of x^ny^n

$$\begin{aligned}
\int_{-1}^{1}P_n(x)P_m(x)dx &= \frac{1}{2^n n!}\frac{1}{2^m m!}\int_{-1}^{1}\frac{\mathrm{d}^n(1-x^2)^n}{\mathrm{d}x^n}\frac{\mathrm{d}^m(1-x^2)^m}{\mathrm{d}x^m}dx \\
&= \frac{1}{2^n n!}\frac{1}{2^m m!}\Big(\cancelto{0}{\frac{\mathrm{d}^{n-1}(1-x^2)^n}{\mathrm{d}x^{n-1}}\frac{\mathrm{d}^m(1-x^2)^m}{\mathrm{d}x^m}\Big|_{-1}^{1}} \\
&\quad - \int_{-1}^{1}\frac{\mathrm{d}^{n-1}(1-x^2)^n}{\mathrm{d}x^{n-1}}\frac{\mathrm{d}^{m+1}(1-x^2)^m}{\mathrm{d}x^{m+1}}dx\Big)
\end{aligned} \tag{31}$$

Repeat the integral by parts until the following steps

$$\begin{aligned}
&= \frac{1}{2^n n!}\frac{1}{2^m m!}\Big(\cancelto{0}{\frac{\mathrm{d}^{n-m-1}(1-x^2)^n}{\mathrm{d}x^{n-m-1}}\frac{\mathrm{d}^{2m}(1-x^2)^m}{\mathrm{d}x^{2m}}\Big|_{-1}^{1}} \\
&\quad - \int_{-1}^{1}\frac{\mathrm{d}^{n-m-1}(1-x^2)^n}{\mathrm{d}x^{n-m-1}}\frac{\mathrm{d}^{2m+1}(1-x^2)^m}{\mathrm{d}x^{2m+1}}dx\Big)
\end{aligned}$$

We know $(1-x^2)^m$ is 2m degree polynomials of x, therefore $\frac{\mathrm{d}^{2m+1}(1-x^2)^m}{\mathrm{d}x^{2m+1}} = 0$. That is

$$\int_{-1}^{1}P_n(x)P_m(x)dx = -\int_{-1}^{1}\frac{\mathrm{d}^{n-m-1}(1-x^2)^n}{\mathrm{d}x^{n-m-1}}\cancelto{0}{\frac{\mathrm{d}^{2m+1}(1-x^2)^m}{\mathrm{d}x^{2m+1}}}dx = 0 \tag{32}$$