Tufts University
Department of Mathematics
Fall 2018

MA 126: Numerical Analysis

Homework 8 (v1.0) [1]

Assigned Friday 26 October 2018
Due Friday 2 November 2018 at 3 pm

You are likely to find it helpful to use Mathematica, or another symbolic algebra package, to help you with this assignment.

1. Atkinson & Han, Section 4.7, Problem 3b&c. The answer to 3a is in the back of the book, which will be helpful.

2. Suppose that we have a set of linearly independent entities, $\{M_n\}_{n=0}^{\infty}$, with an inner product (f, g), and hence a natural norm $\|f\| = \sqrt{(f, f)}$. The M_n could be functions, vectors, etc. Our goal is to take cumulative linear combinations of them in order to construct a new set $\{Q_n\}_{n=0}^{\infty}$ that are mutually orthogonal. More specifically we demand that each Q_n be a linear combination of only those M_m with $m \leq n$, and that $(Q_j, Q_k) = 0$ if $j \neq k$.

   (a) Begin with $Q_0 = M_0$. Show that if we define the subsequent Q_n as follows:

$$
\begin{aligned}
Q_1 &= M_1 - \frac{(M_1, Q_0)}{Q_0, Q_0} Q_0 \\
Q_2 &= M_2 - \frac{(M_2, Q_0)}{Q_0, Q_0} Q_0 - \frac{(M_2, Q_1)}{Q_1, Q_1} Q_1 \\
Q_3 &= M_3 - \frac{(M_3, Q_0)}{Q_0, Q_0} Q_0 - \frac{(M_3, Q_1)}{Q_1, Q_1} Q_1 - \frac{(M_3, Q_2)}{Q_2, Q_2} Q_2 \\
Q_4 &= M_4 - \frac{(M_4, Q_0)}{Q_0, Q_0} Q_0 - \frac{(M_4, Q_1)}{Q_1, Q_1} Q_1 - \frac{(M_4, Q_2)}{Q_2, Q_2} Q_2 - \frac{(M_4, Q_3)}{Q_3, Q_3} Q_3 \\
&\vdots \\
Q_n &= M_n - \sum_{j=0}^{n-1} \frac{(M_n, Q_j)}{Q_j, Q_j} Q_j,
\end{aligned}
$$

   then these linear combinations have the desired properties.

   (b) If the M_n are the set of monomials $M_n(x) := x^n$ on the domain $x \in [-1, +1]$, and the inner product is taken to be

$$(f, g) := \int_{-1}^{+1} dx\ f(x) g(x),$$

   then, to within multiplicative constants, the Q_n are the Legendre polynomials. Compute the first four of them in this way.

(c) Determine the multiplicative constants in front of your results by demanding that $Q_n(1) = 1$. Show that the resulting polynomials are $P_1(x)$ through $P_4(x)$, as defined in Eq. (4.115) of the book.

3. An alternative way of defining the Legendre polynomials is as the coefficients of the Maclaurin series in t of the *generating function*,

$$\left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{1}$$

In this problem, we will reconcile the above definition with that given in Section 4.7.1 of the book.

(a) Using the binomial theorem, expand the left-hand side of Eq. (1) to first order in t, and thereby identify $P_0(x)$ and $P_1(x)$. Confirm that they match the expressions in Eq. (4.115) of the book.

(b) Differentiate both sides of Eq. (1) with respect to t, and simplify the result using Eq. (1) itself, in order to obtain the triple recursion relation given in Eq. (4.117) of the book.

(c) Argue that parts (a) and (b) of this problem alone are sufficient to show that the quantities $P_n(x)$ defined in Eq. (1) are identical to those defined in the book.

(d) Set $x = 1$ in Eq. (1) and show that the left-hand side is the sum of a geometric series in order to show that $P_n(1) = 1$.

(e) Use the triple recursion formula to prove the explicit formula

$$P_n(x) = 2^n \sum_{j=0}^{n} \binom{n}{j} \binom{\frac{n+j-1}{2}}{n} x^j.$$

(f) Use the generating function in Eq. (1) to show that $P_n(x)$ obeys Legendre's differential equation,

$$\frac{d}{dx}\left[\left(1 - x^2\right) \frac{dP_n(x)}{dx}\right] = -n(n+1) P_n(x).$$

4. Prove Rodrigues' formula. This is not easy, so I have broken down the proof into steps, to guide you:

(a) First, show that $P_n(x)$ defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\left(1 - x^2\right)^n\right] \tag{2}$$

is a polynomial of degree n.

(b) Second, taking $m \leq n$ without loss of generality, show that

$$(P_m, P_n) = \int_{-1}^{+1} dx\; P_m(x) P_n(x)$$

is equal to zero if $m \neq n$ (i.e., if $m < n$). (Hint: Integrate by parts.)

(c) We have thus far shown that $\{P_n\}_{n=0}^{\infty}$ is a set of polynomials, one for each degree, that are mutually orthogonal with respect to the inner product defined in part ii above. Argue that this means that the P_n must be Legendre polynomials to within a multiplicative constant.

(d) Finally, to nail down the multiplicative constants, show that P_n defined by Eq. (2) obeys $P_n(1) = 1$.