Homework 7

Hanyuan Zhu

Nov 2, 2018

Question1

In interval $[1, 5]$, the interpolation passes 5 points $(1, 3), (2, 1), (3, 2), (4, 3), (5, 2)$

a.

$$y = \begin{cases} \frac{1-3}{2-1}(x-2)+1 = -2x+4, & x \in [1,2] \\ \frac{2-1}{3-2}(x-3)+2 = x-1, & x \in (2,3] \\ \frac{3-2}{4-3}(x-4)+3 = x-1, & x \in (3,4] \\ \frac{2-3}{5-4}(x-5)+2 = -x+7, & x \in (4,5] \end{cases} \tag{1}$$

b. Let the second derivative, $s''(x)$, at x_1, x_2, x_3, x_4, x_5 be M_1, M_2, M_3, M_4, M_5 respectively.

Also, it is a not-a-knot boundary condition, witch means additional condition:

$$M_2 = \frac{M_1 + M_3}{2} \text{ and } M_4 = \frac{M_3 + M_5}{2} \tag{2}$$

In addition, we have 3 equations

$$\begin{aligned} \frac{1}{6}M_1 + \frac{2}{3}M_2 + \frac{1}{6}M_3 &= (2-1) - (1-3) = 3 \\ \frac{1}{6}M_2 + \frac{2}{3}M_3 + \frac{1}{6}M_4 &= (3-2) - (2-1) = 0 \\ \frac{1}{6}M_3 + \frac{2}{3}M_4 + \frac{1}{6}M_5 &= (2-3) - (3-2) = -2 \end{aligned} \tag{3}$$

Substitute equation(2) into (3)

$$\begin{aligned} &\frac{1}{6}M_1 + \frac{2}{3}\frac{M_1 + M_3}{2} + \frac{1}{6}M_3 = 3 \\ &\frac{1}{6}\frac{M_1 + M_3}{2} + \frac{2}{3}M_3 + \frac{1}{6}\frac{M_3 + M_5}{2} = 0 \\ &\frac{1}{6}M_3 + \frac{2}{3}\frac{M_3 + M_5}{2} + \frac{1}{6}M_5 = -2 \end{aligned} \tag{4}$$

To simplify above,

$$
\begin{aligned}
&M_1 + M_3 = 6 \\
&M_1 + 10M_3 + M_5 = 0 \\
&M_3 + M_5 = -4
\end{aligned}
\tag{5}
$$

To solve above, we get $M_1 = 6\frac{1}{4}, M_3 = -\frac{1}{4}, M_5 = -3\frac{3}{4}$
Substitute back to the integral form of $s''(x)$, we have

$$
y = \begin{cases} -\frac{13}{24}x^3 + \frac{19}{4}x^2 - \frac{299}{24}x + \frac{45}{4}, & x \in [1,3] \\ -\frac{7}{24}x^3 + \frac{5}{2}x^2 - \frac{137}{24}x + \frac{9}{2}, & x \in (3,5] \end{cases}
\tag{6}
$$

Question2

No.

To be a cubic spline, $s''(x)$, $s'(x)$ and $s(x)$ should agree at boundaries. Since we have

$$
s''(x) = \begin{cases} 6(x+1) & x \in [-2,-1] \\ 6ax + 2b & x \in (-1,1] \\ 2 & x \in (1,2] \end{cases}
\tag{7}
$$

To solve $s''(x)$ at $x = -1$ and $x = 1$

$$
\begin{aligned}
0 &= -6a + 2b \\
2 &= 6a + 2b
\end{aligned}
\tag{8}
$$

we get unique solution $a = \frac{1}{6}, b = \frac{1}{2}$.

Then substitute $a = \frac{1}{6}, b = \frac{1}{2}$ in to $s'(x)$

$$
s'(x) = \begin{cases} 3(x+1)^2 & x \in [-2,-1] \\ \frac{1}{2}x^2 + x + c & x \in (-1,1] \\ 2(x-1) & x \in (1,2] \end{cases}
\tag{9}
$$

We have equations at boundary $x = -1$ and $x = 1$.

$$
\begin{aligned}
0 &= \frac{1}{2} - 1 + c \\
0 &= \frac{1}{2} + 1 + c
\end{aligned}
\tag{10}
$$

Apparently, there is no solution for c.

Question3

Lets change variable, substituting $x = \cos\theta$.

$$T_n(x) = T_n(\cos\theta) = \cos n\theta$$

$$
\begin{aligned}
\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx &= \int_{-1}^{1} \frac{\cos n\theta \cos m\theta}{\sqrt{1-\cos^2\theta}} d\cos\theta \\
&= -\int_{-1}^{1} \frac{\cos n\theta \cos m\theta}{\sin\theta} \sin\theta d\theta \\
&= -\int_{-1}^{1} \cos n\theta \cos m\theta d\theta \\
&= -\int_{-1}^{1} \frac{\cos(n+m)\theta + \cos(n-m)\theta}{2} d\theta \\
&= -\frac{1}{2}\left(\int_{-1}^{1} \cos(n+m)\theta d\theta + \int_{-1}^{1} \cos(n-m)\theta d\theta\right)
\end{aligned} \tag{11}
$$

let $\alpha = n+m, \beta = n-m$, substitute into (11)

$$
\begin{aligned}
(11) &= -\frac{1}{2}\left(\int_{-1}^{1} \cos(\alpha)\theta d\theta + \int_{-1}^{1} \cos(\beta)\theta d\theta\right) \\
&= -\frac{1}{2}\left(\frac{1}{\alpha}\int_{-1}^{1} \cos(\alpha\theta) d(\alpha\theta) + \frac{1}{\beta}\int_{-1}^{1} \cos(\beta\theta) d(\beta\theta)\right) \\
&= -\frac{1}{2}\left(\frac{1}{\alpha}\int_{-\alpha}^{\alpha} \cos\theta d\theta + \frac{1}{\beta}\int_{-\beta}^{\beta} \cos\theta d\theta\right) \\
&= -\frac{1}{2}\left(\frac{1}{\alpha}\sin\theta|_{-\alpha}^{\alpha} + \frac{1}{\beta}\sin\theta|_{-\beta}^{\beta}\right) \\
&= 0
\end{aligned} \tag{12}
$$

because sin function is odd function.

Question5

a. To approximate $f(x) = x^3$ on interval $x \in [0, a]$ by Chebyshev polynomials. Because Chebyshev polynomials is defined on interval $[-1, 1]$, we need to rescale the function $f(x) = x^3$ to $g(z) = (\frac{a}{2}z + \frac{a}{2})^3$, where $z \in [-1, 1]$. Then we solve $T_2(z) = 0$, we get

$$
\begin{aligned}
z_1 &= \cos\left(\frac{\pi}{4}\right) \\
z_2 &= \cos\left(\frac{3\pi}{4}\right)
\end{aligned} \tag{13}
$$

Substitute back to $g(z)$, we have

$$
\begin{aligned}
g(z_1) &= \frac{a^3}{8}(\cos\left(\frac{\pi}{4}\right) + 1)^3 \\
g(z_2) &= \frac{a^3}{8}(\cos\left(\frac{3\pi}{4}\right) + 1)^3
\end{aligned} \tag{14}
$$

Then the linear line m(z) is

$$\begin{aligned}
m(z) &= \frac{g(z_1)-g(z_2)}{z_1-z_2}(z-z_1)+g(z_1) \\
m(z) &= \frac{\frac{a^3}{8}(\cos\frac{\pi}{4}+1)^3-\frac{a^3}{8}(\cos\frac{3\pi}{4}+1)^3}{\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{3\pi}{4}\right)}(z-\cos\left(\frac{\pi}{4}\right))+\frac{a^3}{8}(\cos\left(\frac{\pi}{4}\right)+1)^3 \\
m(z) &= \frac{a^3}{8}\frac{(\frac{\sqrt{2}}{2}+1)^3-(1-\frac{\sqrt{2}}{2})^3}{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}(z-\frac{\sqrt{2}}{2})+\frac{a^3}{8}(\frac{\sqrt{2}}{2}+1)^3 \\
m(z) &= \frac{7}{16}a^3z+\frac{5}{16}a^3
\end{aligned} \tag{15}$$

Then we substitute $z=(x-\frac{a}{2})\frac{2}{a}$ to $m(z)$,

$$m(x)=\frac{7}{16}a^3(\frac{2}{a}x-1)+\frac{5}{16}a^3$$

$$m(x)=\frac{7}{8}a^2x-\frac{2}{16}a^3$$

b. To approximate $f(x)=x^3$ on interval $x\in[-a,a]$, let $f(az)=g(z)=a^3z^3$, where $z\in[-1,1]$. Then we solve $T_2(z)=0$, we get $z_1=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ and $z_2=\cos\left(\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

Then, $g(z_1)=\frac{1}{2\sqrt{2}}$ and $g(z_2)=-\frac{1}{2\sqrt{2}}$

The line is $m(z)=\frac{\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}(z-\frac{1}{\sqrt{2}})+\frac{1}{2\sqrt{2}}$,

$$m(z)=\frac{z}{2}$$

$$m(x)=\frac{x}{2a}$$

Question6

a. To approximate $f(x)=x^3$ on interval $x\in[0,a]$ by least square. We let the linear approximation be $g(x)=a_1x+a_0$, then we define least square intergral below

$$\begin{aligned}
I(x) &= \int_0^a [f(x)-g(x)]^2dx \\
&= \int_0^a [x^3+a_1x+a_0]^2dx \\
&= \int_0^a (x^6-2a_1x^4-2a_0x^3+a_1^2x^2+2a_1a_0x+a_0^2)dx
\end{aligned} \tag{16}$$

We take PDE with respect to a_1, a_0

$$
\begin{aligned}
\frac{\partial I(x)}{\partial a_1} &= \int_0^a (x^4 - a_1 x^2 - a_0 x)dx \\
&= (\frac{1}{5}x^5 - \frac{a_1}{3}x^3 - \frac{a_0}{2}x^2)|_0^a = 0 \\
\frac{\partial I(x)}{\partial a_0} &= \int_0^a (x^3 - a_1 x - a_0)dx \\
&= (\frac{1}{4}x^4 - \frac{a_1}{2}x^2 - a_0 x)|_0^a = 0
\end{aligned} \tag{17}
$$

Solve equations (17), we get $a_1 = \frac{9}{10}a^2, a_0 = -\frac{1}{5}a^3$ Then the approximation is

$$
g(x) = \frac{9}{10}a^2 x - \frac{1}{5}a^3
$$

b. To approximate $f(x) = x^3$ on interval $x \in [-a, a]$, we using the same method as part a. We solve below to get the a_1 and a_0.

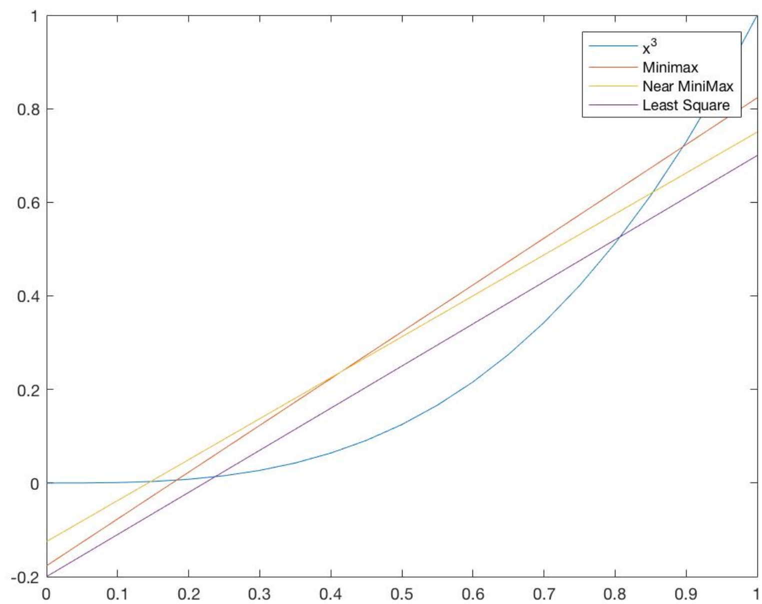
$$
\begin{aligned}
&(\frac{1}{5}x^5 - \frac{a_1}{3}x^3 - \frac{a_0}{2}x^2)|_{-a}^a = 0 \\
&\rightarrow \frac{2}{5}a^5 - \frac{2a_1}{3}a^3 = 0 \\
&\rightarrow a_1 = \frac{3}{5}a^2 \\
&(\frac{1}{4}x^4 - \frac{a_1}{2}x^2 - a_0 x)|_{-a}^a = 0 \\
&\rightarrow -2aa_0 = 0 \\
&\rightarrow a_0 = 0
\end{aligned} \tag{18}
$$

Then the approximation is

$$
g(x) = \frac{3}{5}a^2 x
$$

Question7

a. Approximations on [0,1]



b. Approximations on [-1,1]