Tufts University
Department of Mathematics
Fall 2018

MA 126: Numerical Analysis

Homework 10 (v1.0) [1]

Assigned Friday 23 November 2018
Due Friday 30 November 2018 at 3 pm

In this assignment, we will suppose that you would like to find a Gaussian quadrature scheme that will perform integrals of the form

$$I[f] := \int_0^1 \log\left(\frac{1}{x}\right) f(x)\, dx,$$

for functions f that are continuous on $[0, 1]$.

1. Using the methodology of Problem 2 on Homework 8, define the inner product

$$(f, g) := \int_0^1 \log\left(\frac{1}{x}\right) f(x)g(x)\, dx,$$

and define the orthogonal polynomials $Q_j(x)$ as those obtained by sequential orthogonalization, beginning with the set of monomials $\{1,\, x,\, x^2,\, x^3, \ldots\}$. Find the first four of these, $Q_0(x)$, $Q_1(x)$, $Q_2(x)$, and $Q_3(x)$. Choose the normalization constant so that $Q_j(1) = 1$ in each case. It is recommended that you use symbolic algebra to do the integrals, but be sure to express your answers exactly as polynomials with *rational* coefficients.

2. Find the roots of the following polynomials in $[0, 1]$:

   (a) $Q_1(x)$

   (b) $Q_2(x)$

   (c) $Q_3(x)$

   You may use symbolic algebra to aid you if you wish. Your answers for the roots of Q_1 and Q_2 should be exact. You may find the roots of Q_3 numerically, and express your answers as approximate decimals, accurate to at least six significant digits.

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3. In this problem, we seek Gaussian quadrature schemes of the form

$$I[f] \approx \sum_{j=1}^{n} w_j f(x_j).$$

where $I[f]$ is as defined in the preamble to this problem set. As usual, we do this by demanding that

$$I[x^k] = \sum_{j=1}^{n} w_j x_j^k$$

be exact for $k = 0, \ldots, 2n-1$. In this way, find the weights w_j and nodes x_j for

(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

You may use symbolic algebra to aid you if you wish. Your answers for the weights and nodes should be exact for $n = 1$ and $n = 2$. You may find the weights and nodes for $n = 3$ numerically, and express your answers as approximate decimals, accurate to at least six significant digits.

4. Apply the Gaussian quadrature scheme obtained above to $f(x) = e^{-x}$ for

(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

In all cases, you should express your answers as decimals, accurate to at least six significant digits. Compare your results with the application of the Trapezoidal Rule, also accurate to at least six significant digits. How many (equal-width) subintervals were needed to obtain that accuracy with the Trapezoidal Rule?