Tufts University
Department of Mathematics
Fall 2018

MA 126: Numerical Analysis

Homework 1 (v1.1) [1]

Assigned Friday 7 September 2018
Due Friday 14 September 2018 at 3 pm

1. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function that is n times differentiable at a point $a \in \mathbb{R}$. The nth-order Taylor polynomial for f with center at a is defined to be

$$p_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Show that this is the unique polynomial whose first n derivatives at $x = a$ are equal to those of $f(x)$.

2. Taylor's Theorem with Remainder says that if f has continuous derivatives through order $k+1$ on an open interval I containing a, then for all $x \in I$,

$$f(x) = p_n(x) + R_n(x),$$

where the remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x. For $n = 0$, show that this reduces to the Mean Value Theorem.

3. This problem has to do with the nth order Taylor expansion of the function

$$f(x) = \frac{1}{1-x},$$

about the center $a = 0$.

   (a) What is the largest open interval I we can possibly use for this example?

   (b) Calculate the nth-order Taylor polynomial $p_n(x)$ for this example. Express your result as $\sum_{k=0}^{n} c_k x^k$, where it is your job to find the c_k.

   (c) Find a closed expression (i.e., not in terms of a sum) for the remainder $R_n(x)$.

   (d) By comparing your expression for $R_n(x)$ to Taylor's Theorem with Remainder, find an explicit expression for c for this example, in terms of n and x. Show that this value always lies between 0 and x. Be sure to treat the cases of positive and negative x separately as you show this.

   (e) Is your result valid for $x = -2$? For $x = +2$? Why or why not?

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