Question 3.20

We want to show the Cauchy sequence $\{p_n\}$ converges to p, by given its subsequence $\{p_{n_k}\}$ converges to p.

Since $\{p_n\}$ is Cauchy sequence, for any $\delta > 0$, there exist such N that $n > N \& m > N$, then $d(p_m, p_n) < \delta$.

Also because subsequence $\{p_{n_k}\}$ converges to p, for any given δ above, there exist such N' that $k > N' \leq N$, then $d(d_{n_k}, p) < \delta$.

We let $\epsilon = 2\delta$. Thus, For any $\epsilon > 0$, you can construct such N' as above that $d(d_n, p) < d(p_n, p_{N'}) + d(p_{N'}, p) < \delta + \delta = \epsilon$. $\{p_n\}$ converges to p.

Question 3.21

First of all, we will show there is no distinct points in $\cap_1^\infty E_n$ if $\cap_1^\infty E_n \neq \emptyset$.

If $p \in \cap_1^\infty E_n$ and $q \in \cap_1^\infty E_n$. Then $p \in \lim_{n\to\infty} E_n$ and $q \in \lim_{n\to\infty} E_n$. Then $0 \leq d(p,q) \leq \lim_{n\to\infty} diam E_n = 0$, that is $p = q$.

Secondly, we will show $\cap_1^\infty E_n$ has at least one element.

$E_n \supset E_{n+1}$ and $\lim_{n\to\infty} diam E_n = 0$ imply E_n is Cauchy sequence for all n. Because X is a complete space, that means $\{p_m\} = E_n$ converges to p. Since E_n is closed, the limit point $p \in E_n$ for all n. Therefore there at least exists $p \in \cap_1^\infty E_n$.

To sum up, $\cap_1^\infty E_n$ consist exactly one point, which is the limit point of sequence E_n.

Question 3.22

In order to utilize the result from Problem 3.21, We want to construct a closed bounded shrinking sequence of sets, $\overline{E_n} \in G_n$

Since $\{G_n\}$ is dense and open in X, the complement set $F_n = G_n^c$ is closed and has no interier points (otherwise such a point won't be a limit point of G_n). Thus for any open set $U \subset X$, $U \not\subset F_n$ and $U \backslash F_n$ is a open set.

If $x \in U\backslash F_1$, there is a $N_{r_1}(x) \subset U\backslash F_1$. We Let $E_1 = N_{\frac{r_1}{2}}(x) \subset N_{r_1}(x) \subset U\backslash F_n$. We choose r_n by letting $N_{r_n}(x) \subset E_{n-1} \cap U\backslash F_{n-1}$, and $E_n = N_{\frac{r_n}{2}}(x)$. Then $\overline{E_{n+1}} = \overline{N_{\frac{r_{n+1}}{2}}(x)} \subset N_{r_{n+1}}(x) \subset E_n \subset \overline{E_n}$

Because of $N_{r_{n+1}}(x) \subset E_n \subset N_{\frac{r_n}{2}}(x)$, then $r_{n+1} < \frac{r_n}{2} < \frac{1}{2^{n-1}} r_1$. Therefore, as $n \to \infty$, $r_n \to 0$ and diam $\overline{E_n} \to 0$.

We know

$$\begin{aligned}
\overline{E_n} &\subset N_{r_n}(x) \subset E_{n-1} \cap (U\backslash F_n) \\
&\subset E_{n-2} \cap (U\backslash F_{n-1}) \cap (U\backslash F_n) \\
&\subset \cap_{i=1}^n (U\backslash F_i) = U\backslash(\cup_{i=1}^n F_i) \\
&= U\backslash(\cup_{i=1}^n G_i^c) = U\backslash(\cap_{i=1}^n G_i)^c \\
&= U \cap (\cap_{i=1}^n G_i)
\end{aligned} \tag{1}$$

Thus

$$\begin{aligned}\cap_{i=1}^{\infty}\overline{E_n} &\subset \cap_{i=1}^{\infty}(U \cap (\cap_{j=1}^{i} G_j)) \\ &= U \cap (\cap_{i=1}^{\infty} G_i)\end{aligned} \tag{2}$$

According to result from Problem 3.21, we know $\cap_{i=1}^{\infty}\overline{E_n} \neq \emptyset$, that is $U \cap (\cap_{i=1}^{\infty} G_i) \neq \emptyset$, then $\cap_{i=1}^{\infty} G_i \neq \emptyset$.

Question 3.23

By the hints,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_n, q_m)$$

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_n, q_m)$$

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_n, q_m)$$

Since $\{p_n\}$ $\{q_n\}$ are cauchy sequence, then for any $\frac{\epsilon}{2} > 0$, there exists N_1 and N_2 such that if $n > N_1$ and $n > N_2$ $d(p_n, p_m) < \frac{\epsilon}{2}$, also if $n > N_2$ and $n > N_2$ $d(q_n, q_m) < \frac{\epsilon}{2}$.

Then we let $N = max(N_1, N_2)$, we have For any $\epsilon > 0$, if $n > N$ and $n > N$, then $|d(p_n, q_n) - d(p_m, q_m)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So it is a Cauchy Sequenc. Because $d(p_n, q_n) \in R$, and R is a complete metric space, therefore the sequence $\{d(p_n, q_n)\}$ converges on R.

Question 3.24

3.24 a

1. $d(p_n, p_n) = |p_n - p_n| = 0$;
2. $d(q_n, p_n) = |q_n - p_n| = |p_n - q_n| = d(q_n, p_n)$;
3. By triangle inequality, let $\{r_n\}$ be sequence in X, $d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n)$, then $\lim_{n\to\infty} d(p_n, q_n) = 0$ and $\lim_{n\to\infty} d(q_n, r_n) = 0$ implies $\lim_{n\to\infty} d(p_n, r_n) = 0$.

3.24 b

In question 3.24a, we have showed that for equivalent sequences, $\lim_{n\to\infty} d(p_n, p'_n) = 0$

$\lim_{n\to\infty} d(p'_n, q'_n) \leq \lim_{n\to\infty} d(p_n, p'_n) + \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, q'_n) = \lim_{n\to\infty} d(p_n, q_n)$

Again, using the triangle inequality from the otherside, we will show $\lim_{n\to\infty} d(p'_n, q'_n) = \lim_{n\to\infty} d(p_n, q_n)$.

$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$. Therefore it is unchanged by replacing equivalent sequences.

Question 1

1 a

$f(x) = \frac{1}{2}(x + \frac{\alpha}{x})$, then let $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{1}{2}(1 - \frac{\alpha}{x^2}) = 0$ we have minimum x at $\sqrt{\alpha}$. Therefore, any $x_n > 0$, $x_n > \sqrt{\alpha}$.

1 b

In part a, we shall show $x_{n+1} > x_n$, by given $x_n > \sqrt{\alpha}$. Given $x_n > \sqrt{\alpha}$, we have

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{2}(x_n + \frac{\alpha}{x_n}) - x_n \\ &= \frac{x_n^2 + \alpha - 2x_n^2}{2x_n} \\ &= \frac{\alpha - 2x_n^2}{2x_n} < 0 \end{aligned} \tag{3}$$

Therefore the sequence is monotonically decreasing.