Question1

We have

$$\begin{aligned} m_0 &= Q_0 = 1 \\ m_1 &= x \\ m_2 &= x^2 \\ m_3 &= x^3 \end{aligned} \tag{1}$$

To compute Q_1, we use mathematica to get $(m_1, Q_0) = \int_0^1 In(\frac{1}{x})xdx = \frac{1}{4}$, and $(Q_0, Q_0) = \int_0^1 In(\frac{1}{x})dx = 1$

$$Q_1 = c(m_1 - \frac{(m_1, Q_0)}{(Q_0, Q_0)}Q_0) = c(x - \frac{1}{4}) \tag{2}$$

To nomalized it, $Q_1(1) = 1$, then $c = \frac{4}{3}$,

$$Q_1 = \frac{4}{3}x - \frac{1}{3}$$

To compute Q_2, we use mathematica to get

$$\begin{aligned} (m_2, Q_0) &= \int_0^1 In(\frac{1}{x})x^2dx = \frac{1}{9} \\ (m_2, Q_1) &= \int_0^1 In(\frac{1}{x})x^2(x - \frac{1}{4})dx = \frac{5}{144} \\ (Q_1, Q_1) &= \int_0^1 In(\frac{1}{x})(x - \frac{1}{4})^2dx = \frac{7}{144} \end{aligned} \tag{3}$$

$$Q_2 = c(m_2 - \frac{(m_2, Q_0)}{(Q_0, Q_0)}Q_0 - \frac{(m_2, Q_1)}{(Q_1, Q_1)}Q_1) = c(x^2 - \frac{5}{7}x + \frac{17}{252}) \tag{4}$$

To nomalized it, $Q_2(1) = 1$, then $c = \frac{252}{89}$,

$$Q_2 = \frac{252}{89}x^2 - \frac{180}{89}x + \frac{17}{89}$$

To compute Q_3, we use mathematica to get

$$\begin{aligned} (m_3, Q_0) &= \int_0^1 In(\frac{1}{x})x^2dx = \frac{1}{16} \\ (m_3, Q_1) &= \int_0^1 In(\frac{1}{x})x^2(x - \frac{1}{4})dx = \frac{39}{1600} \\ (m_3, Q_2) &= \int_0^1 In(\frac{1}{x})x^2(x - \frac{1}{4})dx = \frac{23}{6720} \\ (Q_2, Q_2) &= \int_0^1 In(\frac{1}{x})(x - \frac{1}{4})^2dx = \frac{647}{226800} \end{aligned} \tag{5}$$

$$
\begin{aligned}
Q_3 &= c(m_3 - \frac{(m_3, Q_0)}{(Q_0, Q_0)}Q_0 - \frac{(m_3, Q_1)}{(Q_1, Q_1)}Q_1 - \frac{(m_3, Q_2)}{(Q_2, Q_2)}Q_2) \\
&= c(x^3 - \frac{3105}{2588}x^2 + \frac{5751}{16175}x + \frac{4679}{258800})
\end{aligned} \tag{6}
$$

To nomalized it, $Q_3(1) = 1$, then $c = \frac{258800}{35637}$,

$$
Q_3 = \frac{258800}{35637}x^3 - \frac{103500}{11879}x^2 + \frac{30672}{11879}x - \frac{4679}{35637}
$$

Question2

$$
Q_1 = \frac{4}{3}x - \frac{1}{3} = 0
$$

$$
x = \frac{1}{4}
$$

$$
Q_2 = \frac{252}{89}x^2 - \frac{180}{89}x + \frac{17}{89} = 0
$$

$$
x_1 = \frac{15 - \sqrt{106}}{42} \text{ and } x_2 \frac{15 + \sqrt{106}}{42}
$$

$$
Q_3 = \frac{258800}{35637}x^3 - \frac{103500}{11879}x^2 + \frac{30672}{11879}x - \frac{4679}{35637} = 0
$$

$$
x_1 = 0.766880
$$

$$
x_2 = 0.063891
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$$
x_3 = 0.368997
$$