Realy Analysis Homework 5

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Oct 26 2018

Question 12

proof To show $K = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{Z}^+\}$ is compact, that is to show any open cover $\cup_\alpha G_\alpha$ of K has a finite subcover $\cup_{i=0}^s G_{\alpha_i}$, for some $s \in \mathbb{N}$.

Let G_{α_0} contains 0, then there exist some $\epsilon > 0$, such that $N_\epsilon(0) \in G_{\alpha_0}$. we let $s \leq \frac{1}{\epsilon}$, such that, for any $n > s$, $n > \frac{1}{\epsilon}$. Therefore, because $\epsilon > \frac{1}{n}$ if $n > s$, $\frac{1}{n} \in N_\epsilon(0)$, that is G_{α_0} covers $\{\frac{1}{n} | n > s\}$. And let G_{α_i} contains $1/i$, and $i = 1, 2, ..., s$. Then $\cup_{i=0}^s G_{\alpha_i}$ is an open finite subcover of K. Therefore K is compact.

Question 13

To construct such set, we can utilize the compact set $K = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{Z}^+\}$ in Question 12.

K is a countable set. We can construct a similar set whose limit point is the element of K, that is 1/n. That is, $\{\frac{1}{n}\} \cup \{\frac{1}{n} + \frac{1}{m} | m \in \mathbb{Z}^+\}$ has only limit point $\frac{1}{n}$, so it is closed and bounded within $\frac{1}{n}$ to $1 + \frac{1}{n}$.

Therefore $\{0\} \cup \{\frac{1}{n} | n \in \mathbb{Z}^+\} \cup \{\frac{1}{n} + \frac{1}{m} | n \in \mathbb{Z}^+, m \in \mathbb{Z}^+\}$ is closed and bounded, that is compact. And the set of limit point is K, which is countable.

Question 14

$G_{\alpha_i} = (\frac{1}{i}, 1 - \frac{1}{i})$, where $i = 3, 4, ...$. Such $\cup_\alpha G_\alpha$ is a open cover of $(0, 1)$ without a finite subcover. In order to cover $(0, 1)$, i has to go up to ∞.

Question 15

These falses can be easily found by constructing a "closed but unbounded" set or a "bounded but open" set.

First of all, we make "closed but unbounded" subsets $CU_i = [i, \infty) \subset \mathbb{R}^+ \cup \{0\}, i \in \mathbb{N}$. For any choices of set $\{i\}$, $j = max\{i\}$, $\cap_i CU_i = CU_j \neq \emptyset$. However, as i goes up to infinity, the lower bound becomes infinity and there is no element larger than infinity. $\cap_i^\infty CU_i = \emptyset$.

Secondly, we make bounded open set $BO_i = (0, \frac{1}{i}) \subset (0, 1)$, where $i \in \mathbb{Z}^+$. Intersection of any finite subsets is the smallest subset among those. However the as i go to infinity, $\frac{1}{n}$ approach the limit point 0. Therefore $\cap_i^\infty BO_i = \emptyset$.

By the same manner above, we can see that **corollary** fails if we replacing the "compact" by "closed" or "bounded", since $CU_i \supset CU_{i+1}$ and $BOi \supset BO_{i+1}$.

Question 16

To show E is closed and bounded in Q, but it is not compact. Frist of all, we want to show E is closed, that is to show the complement set Q/E is open.

Any $x \in Q/E$, that is $x \in Q$ and such x satisfies $x^2 \leq 2$ or $x^2 \geq 3$. Because $x \in Q$, the complement set is eqivalent to $\{x \in Q | x^2 < 2 \quad or \quad x^2 > 3\}$.

Let show the set $Q_2 = \{x \in Q | x^2 < 2\}$ is open first. for any $x \in Q_2$, we can always have such $\epsilon = min(\frac{\sqrt{2}-x}{2}, \frac{\sqrt{2}+x}{2})$. Let $y \in N_\epsilon(x)$, then $y = x + d$, where $|d| = d|x - y| < \epsilon$.

$$\begin{aligned} y^2 &= (x+d)^2 \leq (|x| + d|x-y|^2) \\ &< (|x| + \epsilon)^2 = (|x| + \frac{\sqrt{2} - |x|}{2})^2 = (\frac{\sqrt{2} + |x|}{2})^2 \\ &< \frac{\sqrt{2} + \sqrt{2}}{2} = \sqrt{2}^2 = 2 \end{aligned} \tag{1}$$

, that is any such $y \in N_\epsilon x$, $y \in Q_2$. Q_2 is open.

Then, by the same method, we can show $Q_3 = \{x \in Q | x^2 > 3\}$ is open. For $\epsilon = max(\frac{x-\sqrt{3}}{2}, \frac{-x-\sqrt{3}}{2})$. For any $y \in N_\epsilon(x)$, we have

$$\begin{aligned} y^2 &= (x+d)^2 = (x - (-d))^2 \\ &\geq ||x| - |-d||^2 = ||x| - |d||^2 = ||x| - d(x-y)|^2 \\ &> ||x| - \epsilon|^2 = ||x| - \frac{|x| - \sqrt{3}}{2}|^2 = (\frac{|x| + \sqrt{3}}{2})^2 \\ &> (\frac{\sqrt{3} + \sqrt{3}}{2})^2 = 3 \end{aligned} \tag{2}$$

, that is any such $y \in N_\epsilon x$, $y \in Q_3$. Q_3 is open.

$Q/E = Q_2 \cup Q_3$ is open. Then E is **closed** in Q.

And E is **bounded** by $[-\sqrt{3}, \sqrt{3}]$.

Now we show E is not compact by constrcuting an open cover of E, $G_{\alpha_n} = \{x | 2 + \frac{1}{n} < x^2 < 3, n \in \mathbb{N} \& n \geq 2\}$. To cover entire E, that is any $p \in \{x \in Q | 2 < x^2 < 3\}, p \in G_{\alpha_n}$ for some n. Such n has go to infinity, that is there is no finite subcover of $\cup_i^{infty} G_{\alpha_i}$. E is **not compact**.

To show E is open in Q We can utilize the same method for showing the openness of complement of E. By choosing $\epsilon = min\frac{\sqrt{3}-x}{2}, \frac{-x-\sqrt{3}}{2}, \frac{x-\sqrt{2}}{2}$. For any $y \in N_\epsilon(x)$, we can have $2 < y^2 < 3$. Therefore, E is open in Q.

Question 22

First of all, we show Q^k is dense in R^k. Let $r = R^k = (r_i | i = 1, 2, ..., k)$. By Theorem 1.20, "Q is dense in R", we know , for abitary ϵ, $\exists q_i \in (r_i, r_i + \epsilon)$ such that $|q - r| = (\frac{\sum_{i=1}^{k}(q_i - r_i)^p}{k})^{\frac{1}{p}} < (\frac{\sum_{i=1}^{k}(\epsilon)^p}{k})^{\frac{1}{p}} = \epsilon$ There always exists a rational point $q \in Q^k$, $q \in N_\epsilon(r)$ for all r and arbitary ϵ.

Therefore, the complement set of Q^k in R^k has empty interior, that is the closure of Q^k in R^k is R^k, or Q^k is **dense** in R^k.

Then we want to show Q^k is countable. It is easy to see $Q^2 \sim \cup_\alpha Q_\alpha$, where $\alpha \in Q$. We know Q is countable so Q^2 is countable. We can construct Q^{i+1} in same way, $Q^{i+1} \sim \cup_\alpha Q_\alpha^i$, where $\alpha \in Q$. By deduction method, we see any Q^i, $i = 1, 2, ...$, is countable. Therefore Q^k is a **countable** set. R^k is separable.

Question 23

We construct a base of X by X's countable dense subset,$Q = \{q_i | i \in \mathbb{N}\}$ and some rational distance r.

$$V_{\alpha_i} = N_r(q_i) \tag{3}$$

V_{α_i} is the neighborhood of q_i with some rational radius r. Now we are going to show $\{V_\alpha\}$ is a base of X. Let G be any open subset of X. For every $x \in G \subset X$, because G is open, there exists $\epsilon > 0$ such that $N_\epsilon(x) \subset G$. Since Q is dense in X, we can find some $q_k \in N_{\frac{\epsilon}{10}}(x)$. In order to let $x \in N_r(q_i)$, $r > d|q_i, x|$, and in order to let $N_r(q_i) \subset N_\epsilon(x)$, $r < \frac{\epsilon}{10}$. Thus $x \in N_r(q_i) \subset N_\epsilon(x) \subset G$.

Therefore $\{N_r(q_i) | i \in \mathbb{N}\}$ is a countable base of X.

Question 1

Since we know $A_i \subset A_{i-1}$ and all $A_i \neq \emptyset$, then for any finite choices set of $\{i\}$, $j = max\{i\}$, $\cap_i A_i = A_j \neq \emptyset$.

By Theorem 2.36, because A_is are compact sets and have the property showed above , $\cap_{i=1}^{\infty} A_i \neq \emptyset$.

Question 2

To prove it by contradiction, assume A, a uncountable subet set of $\mathbb{R}$, has no condensation point, that is for every $x \in A$, there exists such r that the neighborhood $N_r(x) \cap A$ is at most countable. Then $\{N_r(x) | x \in A\}$ is an open cover of A. Here we use the hint that "every open cover of a subset of R has an at most countable subcover." $A \subset \cup_i N_r(x_i)$ is at most countable union. Then $A = A \cap (\cup_i N_r(x_i)) = \cup_i (A \cap N_r(x_i))$ is at most countable, because it is countably union of $N_r(x) \cap A$, at most countable sets. This contradicts to A is uncountable. Therefore A has at least one condensation point.

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We have showed in Question 23, that $\mathbb{R}$ has a countable base $\{G_{q_i} | i \in \mathbb{N}\}$.

Because $A \subset \mathbb{R}$ and A is uncountable, $A = A \cap R = A \cap (\cup_i^{\infty} G_{q_i}) = \cup_i^{\infty}(A \cap G_{q_i})$ is uncountable. Therefore there exist some i such that $A \cap G_{q_i}$ is uncountable, because coutably union of at most countable set is countable. Then for any $x \in A$ with any $r > 0$, we can choose such a G_{q_i} that $A \cap G_{q_i}$ is uncountable and $x \in G_{q_i} \subset N_r(x)$, $N_r(x) \cap A$ is uncountable.