Homework 9 Numerical Analysis

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Question 1

Now we are trying to use the way for Trapezoidal Rule error to derive the error function of Simpsons Rule. $x_{n-1} + h = x_n$

$$
\begin{aligned}
E^S(f) &= \int_{x_0}^{x_n} f(x)dx - \sum_{i=0}^{\frac{n-2}{2}} \int_{x_{2i}}^{x_{2i+2}} P_2(x)dx = \sum_{i=0}^{\frac{n-2}{2}} \int_{x_{2i}}^{x_{2i+2}} f(x) - P_2(x)dx \\
&= \sum_{i=0}^{\frac{n-2}{2}} \int_{x_{2i}}^{x_{2i+2}} \Psi_2(x) \frac{f^{(3)}(c_x)}{3!} dx \\
&= \sum_{i=0}^{\frac{n-2}{2}} \frac{f^{(3)}(c_x)}{3!} \int_{x_{2i}}^{x_{2i+2}} \Psi_2(x)dx \\
&= \sum_{i=0}^{\frac{n-2}{2}} \frac{f^{(3)}(c_x)}{3!} \int_{x_{2i}}^{x_{2i+2}} (x - x_{2i})(x - x_{2i+1})(x - x_{2i+2})dx \\
&= \sum_{i=0}^{\frac{n-2}{2}} \frac{f^{(3)}(c_x)}{3!} \int_{x_{2i}}^{x_{2i+2}} (x - x_{2i})(x - x_{2i} - h)(x - x_{2i} - 2h)d(x - x_i) \\
&= \sum_{i=0}^{\frac{n-2}{2}} \frac{f^{(3)}(c_x)}{3!} \int_{0}^{2h} x(x-h)(x-2h)dx \\
&= \sum_{i=0}^{\frac{n-2}{2}} \frac{f^{(3)}(c_x)}{3!} \int_{0}^{2h} x^3 - 3hx^2 + 2h^2x \quad dx \\
&= \sum_{i=0}^{\frac{n-2}{2}} \frac{f^{(3)}(c_x)}{3!} \left(\frac{x^4}{4} - hx^3 + h^2x^2\right)\Big|_0^{2h} \\
&= \sum_{i=0}^{\frac{n-2}{2}} \frac{f^{(3)}(c_x)}{3!} (4h^4 - 8h^4 + 4h^4) = 0
\end{aligned}
\tag{1}
$$

That is the error will vanish in this manner.

Question 2

2a. Taylor expansion at x_1

$$f(x) = f(x_1) + \frac{f'(x_1)}{1!}(x-x_1) + \frac{f''(x_1)}{2!}(x-x_1)^2 + \frac{f'''(x_1)}{3!}(x-x_1)^3 + \frac{f^{(4)}(c)}{4!}(x-x_1)^4 \tag{2}$$

2b. Integrate Taylor expansiion

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)dx = &\int_{x_0}^{x_2} (f(x_1) + \frac{f'(x_1)}{1!}(x-x_1) + \frac{f''(x_1)}{2!}(x-x_1)^2 \\
&+ \frac{f'''(x_1)}{3!}(x-x_1)^3 + \frac{f^{(4)}(c)}{4!}(x-x_1)^4) \quad dx \\
= &\int_{x_0}^{x_2} (f(x_1) + \frac{f'(x_1)}{1!}(x-x_1) + \frac{f''(x_1)}{2!}(x-x_1)^2 \\
&+ \frac{f'''(x_1)}{3!}(x-x_1)^3 + \frac{f^{(4)}(c)}{4!}(x-x_1)^4) \quad d(x-x_1) \\
= &\int_{-h}^{h} (f(x_1) + \frac{f'(x_1)}{1!}x + \frac{f''(x_1)}{2!}x^2 + \frac{f'''(x_1)}{3!}x^3 + \frac{f^{(4)}(c)}{4!}x^4)dx \\
&\text{Odd functions will be cancelled out because of identical positive and negative area} \\
= &\int_{-h}^{h} (f(x_1) + \frac{f''(x_1)}{2!}x^2 + \frac{f^{(4)}(c)}{4!}x^4)dx \\
&=(f(x_1)x + \frac{f''(x_1)}{3!}x^3 + \frac{f^{(4)}(c)}{5!}x^5)\Big|_{-h}^{h} \\
&=2f(x_1)h + \frac{f''(x_1)}{3}h^3 + 2\frac{f^{(4)}(c)}{5!}h^5
\end{aligned} \tag{3}$$

2c.Function of $f''(x_1)$

By Taylor expansion at x_1, we have

$$f(x) = f(x_1) + \frac{f'(x_1)}{1!}(x-x_1) + \frac{f''(x_1)}{2!}(x-x_1)^2 + \frac{f'''(x_1)}{3!}(x-x_1)^3 + \frac{f^{(4)}(c)}{4!}(x-x_1)^4 \tag{4}$$

substitute x_0 and x_2 into equation above,

$$\begin{aligned}
f(x_0) &= f(x_1) + \frac{f'(x_1)}{1!}(x_0-x_1) + \frac{f''(x_1)}{2!}(x_0-x_1)^2 + \frac{f'''(x_1)}{3!}(x_0-x_1)^3 + \frac{f^{(4)}(c)}{4!}(x_0-x_1)^4 \\
&= f(x_1) - f'(x_1)h + \frac{f''(x_1)}{2}h^2 - \frac{f'''(x_1)}{3!}h^3 + \frac{f^{(4)}(c)}{4!}h^4
\end{aligned} \tag{5}$$

$$
\begin{aligned}
f(x_2) &= f(x_1) + \frac{f'(x_1)}{1!}(x_2 - x_1) + \frac{f''(x_1)}{2!}(x_2 - x_1)^2 + \frac{f'''(x_1)}{3!}(x_2 - x_1)^3 + \frac{f^{(4)}(c)}{4!}(x_2 - x_1)^4 \\
&= f(x_1) + f'(x_1)h + \frac{f''(x_1)}{2}h^2 + \frac{f'''(x_1)}{3!}h^3 + \frac{f^{(4)}(c)}{4!}h^4
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
f(x_0) + f(x_2) &= 2f(x_1) + f''(x_1)h^2 + \frac{f^{(4)}(c)}{12}h^4 \\
\rightarrow f''(x_1) &= \frac{1}{h^2}(f(x_0) - 2f(x_1) + f(x_2)) - \frac{f^{(4)}(c)}{12}h^2
\end{aligned}
\tag{7}
$$

2d. Error of Simpson Rule

$$
\begin{aligned}
&\text{Result from 2b.} \\
&\int_{x_0}^{x_2} f(x)dx = 2f(x_1)h + \frac{f''(x_1)}{3}h^3 + 2\frac{f^{(4)}(c)}{5!}h^5 \\
&\text{Formula of Simpson Rule} \\
&\int_{x_0}^{x_2} P_2(x)dx = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))
\end{aligned}
\tag{8}
$$

Substitute $f''(x_1) = \frac{1}{h^2}(f(x_0) - 2f(x_1) + f(x_2)) - \frac{f^{(4)}(c)}{12}h^2$ into results from 2b.

$$
\begin{aligned}
&\int_{x_0}^{x_2} f(x)dx = 2f(x_1)h + \frac{\frac{1}{h^2}(f(x_0) - 2f(x_1) + f(x_2)) - \frac{f^{(4)}(c)}{12}h^2}{3}h^3 + 2\frac{f^{(4)}(c)}{5!}h^5 \\
&\int_{x_0}^{x_2} f(x)dx = 2f(x_1)h + \frac{1}{3}(f(x_0) - 2f(x_1) + f(x_2))h - \frac{f^{(4)}(c)}{36}h^5 + \frac{f^{(4)}(c)}{60}h^5 \\
&\int_{x_0}^{x_2} f(x)dx = \frac{1}{3}(f(x_0) + 4f(x_1) + f(x_2))h - \frac{f^{(4)}(c)}{90}h^5 \\
&\int_{x_0}^{x_2} f(x)dx = \int_{x_0}^{x_2} P_2(x)dx - \frac{f^{(4)}(c)}{90}h^5 \\
\rightarrow &\int_{x_0}^{x_2} f(x) - P_2(x)dx = -\frac{f^{(4)}(c)}{90}h^5
\end{aligned}
\tag{9}
$$

2e. Error function in Riemann Form and error estimation with large n

Let $x_i + h = x_{i+1}$ for $i = 0, 1, 2, ..., n$, and $h = \frac{x_n - x_0}{n}$. When n is sufficiently large, on $[x_{2i}, x_{2i+2}]$, $f^{(4)}(x) \approx f^{(4)}(c_i)$. Then

$$\begin{aligned} f^{(3)}(2i+2) - f^{(3)}(2i) &= \int_{x_{2i}}^{x_{2i+2}} f^{(4)}(x)dx \\ &\approx \int_{x_{2i}}^{x_{2i+2}} f^{(4)}(c_i)dx \\ &= f^{(4)}(c_i) \int_{x_{2i}}^{x_{2i+2}} dx = 2hf^{(4)}(c_i) \end{aligned} \tag{10}$$

$$\begin{aligned} \int_{x_0}^{x_n} f(x) - P_2(x)dx = \sum_{i=0}^{\frac{n-2}{2}} \int_{x_{2i}}^{x_{2i+2}} f(x) - P_2(x)dx &= \frac{h^5}{4!} \sum_{i=0}^{\frac{n-2}{2}} f^{(4)}(c_i) \\ &= -\frac{h^4}{180} \sum_{i=0}^{\frac{n-2}{2}} 2hf^{(4)}(c_i) \\ &\approx -\frac{h^4}{180} \sum_{i=0}^{\frac{n-2}{2}} (f^{(3)}(2i+2) - f^{(3)}(2i)) \\ &= -\frac{h^4}{180}(f^{(3)}(x_n) - f^{(3)}(x_0)) \end{aligned} \tag{11}$$

2f.Show Theorem 5.2.5

Using the result from 2e, by changing $a = x_0$ and $b = x_n$. For any function f on intervel $[a, b]$, we will get the asymptotic estimation of the error of Simpson Rule,

$$E^S(f) \approx -\frac{h^4}{180}(f^{(3)}(b) - f^{(3)}(a)) \tag{12}$$