Homework 6 Real Analysis

Hanyuan Zhu

November 7, 2018

Problem 2.19

(a) A and B are closed, that is the set of their limit points, A' and B' , are in A and B, respectively, $A' \subset A, B' \subset B$.

Then $\bar{A} = A \cup A' = A$ and $\bar{B} = B \cup B' = B$.

Since A and B are disjoint, $A \cap B = \emptyset$, then we have

$$\begin{aligned} \bar{A} \cap B = A \cap B = \emptyset \\ A \cap \bar{B} = A \cap B = \emptyset \end{aligned} \tag{1}$$

Therefore A and B are separated.

(b) A and B are open and disjoint, $A \cap B = \emptyset$. To prove A and B are separated, that is to prove $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset$.

We will simplify such statement below,

$$\begin{aligned}
&(\bar{A} \cap B) \cup (A \cup \bar{B}) = \emptyset \\
\Leftrightarrow &(\bar{A} \cap B)^c \cap (A \cap \bar{B})^c = \emptyset^c = X \\
\Leftrightarrow &(\bar{A}^c \cup B^c) \cap (A^c \cup \bar{B}^c) = X \\
\Leftrightarrow &(\bar{A}^c \cap A^c) \cup (\bar{A}^c \cap \bar{B}^c) \cup (B^c \cap \bar{B}^c) \cup (B^c \cap A^c) = X \\
&\quad \text{because } \bar{A}^c \subset A^c \text{ and } \bar{B}^c \subset B^c \\
\Leftrightarrow &A^c \cup (\bar{A}^c \cap \bar{B}^c) \cup B^c \cup (B^c \cap A^c) = X \\
\Leftrightarrow &(A^c \cup B^c) \cup (\bar{A}^c \cap \bar{B}^c) \cup (B^c \cap A^c) = X \\
&\quad \text{because } \bar{A}^c \cap \bar{B}^c = (A \cup A')^c \cap (B \cup B')^c = (A^c \cap B^c) \cap (A'^c \cap B'^c) \\
&\quad \text{then } \bar{A}^c \cap \bar{B}^c \subset A^c \cap B^c \\
\Leftrightarrow &(A^c \cup B^c) \cup (A^c \cap B^c) = X \\
\Leftrightarrow &A^c \cup B^c = X \\
\Leftrightarrow &A \cap B = \emptyset
\end{aligned} \tag{2}$$

According to above, if A and B are open sets and we want to show A and B are separated, it is equivalent to show $A \cap B = \emptyset$. Since A and B are disjoint, which is $A \cap B = \emptyset$, thus A and B are separated.

(c) For some $p \in X$, $\delta > 0$, $A(p) = \{q \in X | d(p,q) < \delta\}$, $B(p) = \{q \in X | d(p,q) > \delta\}$.

Firstly, we show $A(p)$ and $B(p)$ are disjoint.

$$A(p) \cap B(p) = \{q \in X | d(p,q) < \delta \quad \& \quad d(p,q) > \delta\} = \emptyset$$

Thus $A(p)$ and $B(p)$ are disjoint.

Secondly, we shall show $A(p)$ and $B(p)$ are open. For every point $q \in A(p)$, there is $\epsilon = \frac{\delta - d(p,q)}{2}$, any $x \in N_\epsilon(q)$, $x \in A(p)$. Therefore $A(p)$ is open. In the same manner, let $\epsilon = \frac{d(p,q)-\delta}{2}$, we can see $B(p)$ is open.

Since $A(p)$ and $B(p)$ are disjoint and open, by part (b), we know $A(p)$ and $B(p)$ are separated.

(d.) First of all, we will show if metric space X is connected, it is perfect.

Lets assume if X is not perfect, there is a point $x \in X$ not a limit point of X, Therefore, we can construct two disjoint subset A and B, such that $A \cup B = X$, $\bar{A} = A = \{x\}$ and $\bar{B} = B = A^c$.

$$\bar{A} \cap B = \emptyset$$

$$A \cap \bar{B} = \emptyset$$

Above means if X is nonperfect, X is not connected. It means **if metric space X is connected, it is perfect.**

By Thm 2.43, we know a nonempty perfect set is uncountable. Therefore the connnected metric space X is uncountable.

Problem 2.20

Set E is connected

The closure $\bar{E}$ is connected To show it , We assume there exist two disjointed subset of the closure of E, $A \cup B = \bar{E}$, s.t. A and B are separated.

1. If $A \cap E \neq \emptyset$ and $B \cap E \neq \emptyset$, Then A and B are not separated by connectedness of E.
2. Therefore, $A \subset E' \backslash E$ or $B \subset E' \backslash E$. WLOG, we assume $A \subset E' \backslash E$, Then $\bar{B} = \bar{E}$, and $A \cap \bar{B} = A \neq \emptyset$. So any A and B are connected

The interior E^i could be separable Counterexample: A and B are two closed sets share some limit points in set E, and points in E are neither an interior point of A nor of B. Such as two rectangles share a borderline.

Problem 2.21

a. Firstly, we will show A_0 and B_0 are disjoint. Be definition, if $x \in A_0$ then $p(x) \in A$; since A and B are seperated, $p(x) \notin B$, that is $x \notin B_0$. And it is the same for any $x \in B_0$.

Secondly, we shall show no limit point $x \in A_0'$ of A_0 is in B_0. We assume if there exist $x \in A_0'$, $x \in B_0$. For any $\epsilon > 0$, you can find $d(x, s) = |x - s| < \epsilon$ $s \in A_0$. By definition, for some $a \in A$, and $b \in B$ $p(s) = (1 - s)a + sb = a + (b - a)s \in A$ and $p(x) = (1 - x)a + xb \in B$. $d(p(x), p(s)) = \|p(x) - p(s)\| = \|(b - a)(x - s)\| = |x - s|\|b - a\| < \|b - a\|\epsilon$. That is for any $\|b - a\|\epsilon > 0$ $p(x) \in B$, there exist $p(s) \in A$ s.t. $d(p(x), p(s)) < \|b - a\|\epsilon$, Therefore A has limit point in B, which contradict to that A and B are separated. By the same means we can should no limit point of B_0 can be found in A_0. Therefore A_0 and B_0 are separated.

b. We have showed in part a. that A_0 and B_0 are separated, $t = 0 \in A_0$ and $t = 1 \in B_0$, therefore there exist $t_0 \in (0, 1)$ s.t. $t_0 \notin A_0 \cup B_0$. By definition, $p(t_0) \notin A \cup B$.

c. Because for any convex subset E of R_k, any $a \in E$ and $b \in E$, we have $p(t) \in E$, where $p(t) = (1 - t)a + tb$ for all $t \in (0, 1)$. We have proved in part b. , by such condition, any nonempty subsets of E are connnected. Therefore E is connected.

Problem 2.24

To show metric space X is separable, that is to show there is a countable dense subset E of X.

Before to construct such subset E, we will should X has a finite base is bounded. Firstly, we assume $\{x_1, x_2 ... x_i, ...\}$ is an infinity sequence in X, and $d(x_i, x_{i+1} \leq \delta)$. Then contradiction arises here, because such a infinity subset has no limit point, $N_{\frac{\delta}{2}}(x_i) = \{x_i\}$ Therefore the subset $\{x_1, x_2 ... x_i, ... x_n\}$ must be finite for some n, and $X = \cup_i^n N_\delta(x_i)$.

Thus, for some $\delta_0 > 0$, we can have a finite subset S_{δ_0}, and $|S_{\delta_0}| = N_{\delta_0}$. By the same manner, We can construct a countable subset $E = \cup_{n=1}^{\infty} S_{\frac{\delta_0}{n}}$, because $S_{\frac{\delta_0}{n}}$ is finite and $\{1, 2, 3...\}$ is countable.

To show such E is dense in X. We know every subset, $S_{\frac{\delta_0}{n}}$, can make a finite cover $\{N_{\frac{\delta_0}{n}}(x_i) | x_i \in S_{\frac{\delta_0}{n}}\}$ of X by radius $\frac{\delta_0}{n}$. Let $x \in X$, for arbitary $\sigma > 0$, you can have $x \in N_{\frac{\delta_0}{n}}(x_i)$ with $n > \frac{\delta_0}{\sigma}$, and $x_i \in S_{\frac{\delta_0}{n}} \subset E$. So E is dense in X.

Therefore, because there is a dense countable subset E in metric space X. X is separable.

Problem 2.25

K is compact metric with a finite base. $x \in K$, $B_{\frac{1}{i}}(x)$ is the neighborhood of x with radii $\frac{1}{i}$ for some $i \in \mathbb{N}^+$. Thus for every n, K has a open cover $K = \cup_{x \in K} B_{\frac{1}{i}}(x)$, and a corresponding finite subcover is $K = \cup_{x \in E_i} B_{\frac{1}{i}}(x)$, where $E_i \subset K$ is a finite subset with cardinality $|E_i| = N_i$.

Then we can construct the countable base of metric space K by

$$B = \cup_{i \in \mathbb{N}^+} \cup_{x \in E_i} B_{\frac{1}{i}}(x)$$

We denote the center point of the jth ball corresponding to the finite subset E_i as x_j^i. The set

$$X = \{x_j^i | i \in \mathbb{N} \text{ and } j \in 1, 2, ...N_i\}$$

is countable, because for each i, N_i is finite and $\mathbb{N}$ is countable.

To show X is dense in K, for any $k \in K$ with arbitary $\delta > 0$, you have $i > \frac{1}{\delta}$ s.t. $k \in B_{\frac{1}{i}}(x)_j^i$ with some $j < N_i$.

Therefore X is a countable dense subset of K, which means K is separable.

Problem 2.29

First, we want to prove for every open set $E \subset R^1$, there exists a union of disjoint segments s.t. $\cup_i S_i = E$.

For any $x \in E$, we denote $\{S_\alpha^{(x)}\}$ as a collection of intervals , $S_\alpha^{(x)} \subset E$, that contains x, and $S^{(x)} = \cup_\alpha S_\alpha^{(x)}$. Such S^x is the maximum segments contains x, which means if $y \in S^x$, $S^x = S^y$ and if $y \notin S^x$, $S^x \cap S^x = \emptyset$ that is disjointed. For all $x \in E$, we will have a collection of distinct $T = \{S^{(x)}\}$. $E = \cup_T S^{(x)}$

Secondly, we will show such T is a countable collection. Since rational number is dense in R^1, any $S^{(x)} \subset E$ contains rational numbers. Since every disjoint(or distinct) $S^{(x)}$ contains distinct rational numbers. If T is a uncountable collection, $\{S_\alpha^{(x)}\}$, it means there are uncountably many distinct rational numbers in E, which is a contradiction to the countability of rational number. Therefore open set E is a union of at most countable collection of disjoint segments.

Problem 3.1

a. Since $\{s_n\}$ converges, it implies, for every $\epsilon > 0$, there exists N s.t. $|s_n - s_m| < \epsilon$ for any $m > N$ and $n > N$. $||s_n| - |s_m|| < |s_n - s_m| < \epsilon$ for any $m > N$ and $n > N$. Thus $\{|s_n|\}$ converges.

b. When $s_n = (-1)^n$, $\{|s_n|\}$ converge 1, but $\{s_n\}$ doesn't converge.

Problem 3.2

$$\begin{aligned}
\lim_{n\to\infty}\sqrt{n^2+n}-n &= \lim_{n\to\infty}\frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} \\
&= \frac{1}{2}
\end{aligned} \tag{3}$$

Problem 3.3

Firstly, we show $\{s_n\}$ is bounded. Since $0 < s_1 = \sqrt{2} < 2$, we assume $0 < s_n < 2$.

$$0 < s_{n+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{2+\sqrt{2}} < \sqrt{2+2} = 2$$

Therefore $\{s_n\}$ is bounded by 0 and 2.

Secondly, we show $\{s_n\}$ is monotonic.

$$s_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = s_1 \tag{4}$$

Assmue $s_n > s_{n-1}$.

$$s_n + 1 = \sqrt{2+\sqrt{s_n}} > \sqrt{2+\sqrt{s_{n-1}}} = s_n \tag{5}$$

Therefore $\{s_n\}$ is monotonically increase.
By theorem 3.14, $\{s_n\}$ converges.