

Simplified Square Span Program

1 Background

1.1 Span Programs

A Span Program is a linear-algebraic model of computation introduced by Karchmer and Wigderson[2][4].

Definition 1. A SP over a field \mathbb{F} consists of a nonzero target vector \mathbf{t} over \mathbb{F} , a set of vectors $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, a partition of the indices $\mathcal{I} = \{1, \dots, m\}$ into two sets $\mathcal{I}_{labeled}$ and \mathcal{I}_{free} , and a further partition of $\mathcal{I}_{labeled}$ as $\cup_{i \in [n], j \in \{0,1\}} \mathcal{I}_{ij}$.

The SP is said to “compute” a function f if the following is true for all input assignment $u \in \{0,1\}^n$: the target vector is in the span of the vectors that “belong” to the input assignments u – namely, the vectors with indices in $\mathcal{I}_u = \mathcal{I}_{free} \cup_i \mathcal{I}_{i,u_i}$ – iff $f(u) = 1$. The size of the span program is m .

1.2 Circuit Checker Function

Suppose C is a Boolean circuit that computes a function f .

Definition 2. Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a function whose Boolean circuit C has s gates. Let $N = n + s$. Suppose $\phi : \{0,1\}^N \rightarrow \{0,1\}$ is a function that outputs ‘1’ iff the input is a valid assignment of C ’s wires (wires that fan out are considered one wire) with output wire set to ‘1’. We say that ϕ is the circuit checker function for f .

2 Verifiable Computation

A public verifiable computation (VC) scheme allows a computationally limited client to outsource the computation of a function F on input u to an untrusted worker, and then verify the correctness of the returned result $F(u)$. Critically, the outsourcing and verification procedures must be significantly more efficient for the client than performing the computation by itself.[5]

A public verifiable computation scheme provides *public delegation*, which allows arbitrary parties to submit inputs for delegation; and *public verifiability*, which allows arbitrary parties (not just the delegator) to verify the correctness of the results returned by the worker. The following definition captures these two properties.

Definition 3. A public verifiable computation scheme \mathcal{VC} consists of a set of three polynomial-time algorithms (KeyGen, Compute, Verify) defined as follows:

- $(EK_F, VK_F) \leftarrow \text{KeyGen}(F, 1^\lambda)$: The randomized key generation algorithm takes the function F to be outsourced and security parameter λ . It outputs a public evaluation key EK_F , and a public verification key VK_F
- $(y, \pi_y) \leftarrow \text{Compute}(EK_F, u)$: The deterministic worker algorithm uses the public evaluation key EK_F and input u . It outputs $y \leftarrow F(u)$ and a proof π_y of y 's correctness.
- $\{0, 1\} \leftarrow \text{Verify}(VK_F, u, y, \pi_y)$: Given the verification key VK_F , the deterministic verification algorithm outputs 1 if $F(u) = y$, and 0 otherwise.

3 Simplified Square Span Program

We define SSSP somewhat similarly to SSP.

Definition 4. A simplified square span program (SSSP) Q over the field \mathbb{F} consists of $m + 1$ polynomials $v_0(x), v_1(x), \dots, v_m(x)$ and a target polynomial $t(x)$ such that $\deg(v_i(x)) \leq \deg(t(x))$ for all $i = 0, \dots, m$.

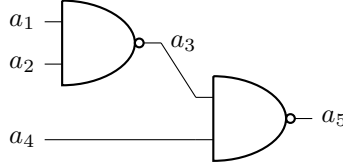
$$t(x) \text{ divides } \left(\sum_{i=0}^m a_i v_i(x) \right)^2 - 1$$

Specifically, $a_0 = 1$.

We say that Q verifies a boolean function $f : \{0, 1\}^l \rightarrow \{0, 1\}$ if it accepts exactly those inputs $\mathbf{a} \in \mathbb{F}^l$ that satisfy $\mathbf{a} \in \{0, 1\}^l$ and $f(\mathbf{a}) = 1$. We may see f as a binary circuit.

3.1 Transformation From Circuit Satisfiability to SSSPs

Consider a circuit consisting of two NAND gates, as is shown below:



It satisfies that

$$a_3 = \neg(a_1 \wedge a_2)$$

$$a_5 = \neg(a_3 \wedge a_4)$$

To guarantee $a_1, a_2, \dots, a_5 \in \{0, 1\}$, we use the constraints

$$(2a_i - 1)^2 = 1, \quad i = 1, 2, \dots, 5$$

To linearize[3] the NAND gate with input a_1, a_2 and output a_3 , writing \bar{c} for $1 - c$, we have

$$\begin{aligned}
a_3 = \neg(a_1 \wedge a_2) &\iff a_1 + a_2 - 2a_3 \in \{0, 1\} \\
&\iff (2(a_1 + a_2 - 2(1 - a_3)) - 1)^2 = 1 \\
&\iff (2a_1 + 2a_2 + 4a_3 - 5)^2 = 1
\end{aligned}$$

Similarly, we have

$$(2a_3 + 2a_4 + 4a_5 - 5)^2 = 1$$

The satisfiability of the circuit can therefore be represented by 7 quadratic equations:

$$\begin{aligned}
(2a_1 - 1)^2 &= 1 & (2a_2 - 1)^2 &= 1 & \dots & (2a_5 - 1)^2 &= 1 \\
(2a_1 + 2a_2 + 4a_3 - 5)^2 &= 1 \\
(2a_3 + 2a_4 + 4a_5 - 5)^2 &= 1
\end{aligned}$$

Corresponding to $(\mathbf{a}V)^2 = 1$.

Then we can represent the constraints as

$$\mathbf{a}V = (1, a_1, a_2, a_3, a_4, a_5) \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -5 & -5 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 4 \end{pmatrix}$$

To get a SSSP, let p be a prime and r_1, r_2, \dots, r_7 be 7 distinct elements in \mathbb{Z}_p . Pick degree 5 polynomials $v_0(x), v_1(x), \dots, v_5(x)$ such that

$$\begin{pmatrix} v_0(r_1) & v_0(r_2) & v_0(r_3) & v_0(r_4) & v_0(r_5) & v_0(r_6) & v_0(r_7) \\ v_1(r_1) & v_1(r_2) & v_1(r_3) & v_1(r_4) & v_1(r_5) & v_1(r_6) & v_1(r_7) \\ v_2(r_1) & v_2(r_2) & v_2(r_3) & v_2(r_4) & v_2(r_5) & v_2(r_6) & v_2(r_7) \\ v_3(r_1) & v_3(r_2) & v_3(r_3) & v_3(r_4) & v_3(r_5) & v_3(r_6) & v_3(r_7) \\ v_4(r_1) & v_4(r_2) & v_4(r_3) & v_4(r_4) & v_4(r_5) & v_4(r_6) & v_4(r_7) \\ v_5(r_1) & v_5(r_2) & v_5(r_3) & v_5(r_4) & v_5(r_5) & v_5(r_6) & v_5(r_7) \end{pmatrix} = V = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -5 & -5 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 4 \end{pmatrix}$$

Let $t(x) = (x-r_1)(x-r_2)\dots(x-r_7)$ to get a SSSP $(v_0(x), v_1(x), \dots, v_5(x), t(x))$ for the circuit such that

$$t(x) \text{ divides } \left(\sum_{i=0}^5 a_i v_i(x) \right)^2 - 1, \text{ where } a_0 = 1$$

If and only if a_1, a_2, \dots, a_5 satisfy the circuit.

4 Building Verifiable Computation from SSSPs

4.1 Bilinear groups

We use the following notation[1]:

1. \mathbb{G} and \mathbb{G}_T are two (multiplicative) cyclic groups of prime order q .
2. G is a generator of \mathbb{G} .
3. e is a bilinear map $e : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_T$. That is, for all $U, V \in \mathbb{G}$ and $a, b \in \mathbb{Z}$, we have $e(U^a, V^b) = e(U, V)^{ab}$. We also require that $e(G, G)$ is a generator of \mathbb{G}_T .

We say that \mathbb{G} is a **bilinear group** if there exists a group \mathbb{G}_T and a bilinear map as above.

4.2 Verifiable Computation from SSSPs

We will now construct succinct and perfect NIZK argument of knowledge for any functions l_u, l_w and families $\{\mathcal{R}\}_\lambda$ of relations R of pairs $(u, w) \in \{0, 1\}^{l_u(\lambda)} \times \{0, 1\}^{l_w(\lambda)}$ that can be computed by polynomial size circuits with $m(\lambda)$ wires and $n(\lambda)$ gates for a total size of $d(\lambda) = m(\lambda) + n(\lambda)$

- $(\sigma, \tau) \leftarrow \text{Setup}(1^\lambda, R)$: Run $gk := (p, \mathbb{G}, \mathbb{G}_T, e) \leftarrow \mathcal{G}(1_\lambda)$. Parse R as a boolean circuit $C_R : \{0, 1\}^{l_u} \times \{0, 1\}^{l_w} \rightarrow \{0, 1\}$. Generate a SSSP $Q = (v_0(x), \dots, v_m(x), t(x))$ that verifies C_R over \mathbb{Z}_p . Pick $G \leftarrow \mathbb{G}^*$ and $\beta, s \leftarrow \mathbb{Z}_p^*$ such that $t(s) \neq 0$. Return

$$\begin{aligned}\sigma &= (gk, G, \dots G^{s^d}, \{G^{\beta v_i(s)}\}_{i > l_u}, G^{\beta t(s)}, Q) \\ \tau &= (\sigma, \beta, s)\end{aligned}$$

- $\pi \leftarrow \text{Prove}(\sigma, u, w)$: Parse u as $(a_1, \dots, a_{l_u}) \in \{0, 1\}^{l_u}$ and use w to compute a_{l_u+1}, \dots, a_m such that $t(x)$ divides $\left(\sum_{i=0}^m a_i v_i(x)\right)^2 - 1$.

Let

$$h(x) = \frac{\left(\sum_{i=0}^m a_i v_i(x)\right)^2 - 1}{t(x)}$$

Use linear combinations of the elements in σ to compute

$$\begin{aligned}H &= G^{h(s)} & V_w &= G^{\sum_{i > l_u}^m a_i v_i(s)} \\ B_w &= G^{\beta \left(\sum_{i > l_u}^m a_i v_i(s)\right)} & V &= G^{\sum_{i=0}^m a_i v_i(s)}\end{aligned}$$

and return $\pi = (H, V_w, B_w, V)$.

- $\{0, 1\} \leftarrow \text{Verify}(\sigma, u, \pi)$: Parse u as $(a_1, \dots, a_{l_u}) \in \{0, 1\}^{l_u}$ and π as $(H, V_w, B_w, V) \in \mathbb{G}^4$. Return 1 if and only if

$$e(H, G^{t(s)}) = e(V^2/G) \quad e(V_w, G^\beta) = e(B_w, G).$$

5 Extractor

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References

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