MODULE 5 : Graph theory & its Applications

MODULE 1 : Set & Number Theory

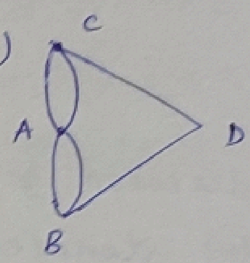
MODULE 4 : Relations

MODULE 3 : Functions

MODULE 2 : fundamentals of Logic

MODULE-5 Graph Theory & its Applications

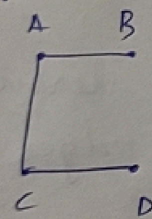
No solution for this (konigsberg Bridge problem) as all can be covered but all bridges can be cannot be covered without repetitions.



- Points are called vertex and lines joining the vertices are called edges
- A graph is a diagram where some points & edges can be shown.

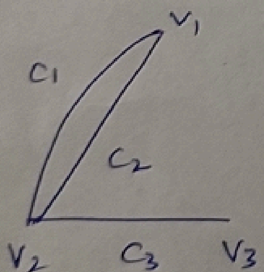
Graph: A graph is a pair (V, E). The element of 'V' are called vertices & elements of 'E' are called unidirected edges & its denoted by $g = (V, E)$

Ex:



$V = \{A, B, C, D\}$

$E = \{AB, AC, CD\}$



$\cdot V_4$, isolated vertex

$V = \{V_1, V_2, V_3, V_4\}$

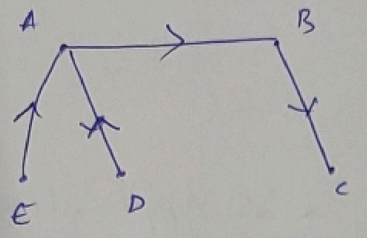
$E = \{C_1, C_2, C_3\}$

- Directed graph:

A directed graph (digraph) is a pair (V, E) where V is a non empty set & E is a set of ordered pair of elements taken from the set V

The elements of V are called vertices & elements of E are called directed edges & is denoted by $\boxed{D = (V, E)}$

Ex:



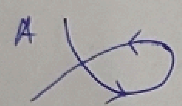
$V = \{A, B, C, D, E\}$

$E = \{AB, BC, DA, EA\}$

- In the edge BC ⇒ vertex B = initial vertex
  Vertex C = terminal vertex

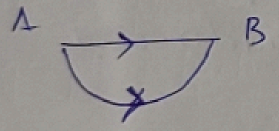
- A directed ~~gp~~ edge begining and ending at same vertex is called denoted by A A and is called directed loop

Ex



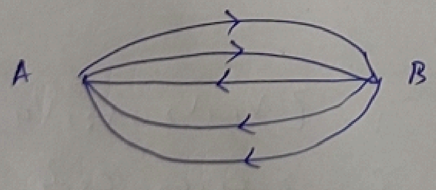
edge = 1
degree = 2

- Parallel Directed Edges:



Two directed edges having the same initial & terminal vertex are called parallel directed edges.

- If more directed edges having the same initial vertices & terminal vertices are called multiple directed edges



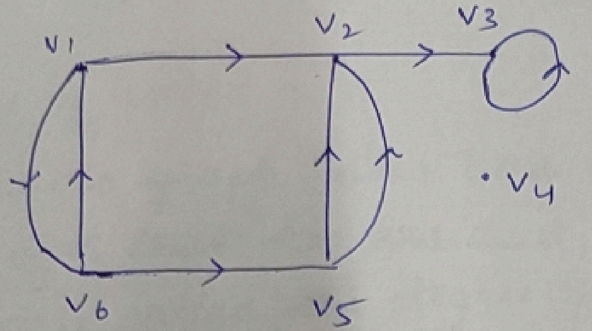
⇒ INDEGREE & OUTDEGREE:

- If v is a vertex of digraph D, then the no. of edges for which v is the initial vertex is called the 'outgoing'/ 'out' degree of v

- The no of edges for which v is the terminal vertex is called the 'incoming'/'in' degree of v

- The outdegree is denoted by: $d^+(v)$ or $od(v)$
- The indegree is denoted by: $d^-(v)$ or $id(v)$

Ex

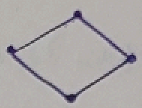


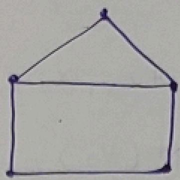
$d^+(v_1) = 2$	$d^-(v_1) = 1$
$d^+(v_2) = 1$	$d^-(v_2) = 3$
$d^+(v_3) = 1$	$d^-(v_3) = 2$
$d^+(v_4) = 0$	$d^-(v_4) = 0$
$d^+(v_5) = 2$	$d^-(v_5) = 1$
$d^+(v_6) = 2$	$d^-(v_6) = 1$

- Simple graph :

A graph which contains neither loop nor multiple edges.

Ex :

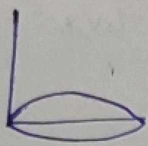


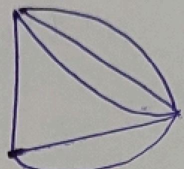


- Multi-graph :

A graph which contains multiple edges but no loops

Ex :

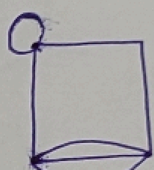


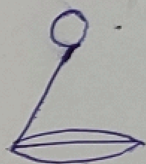


- General graph :

A graph which contains edges, or multiple edges or loop

Ex :





- Null Graph

A graph which contains no edges

Ex :

- Trivial Graph :

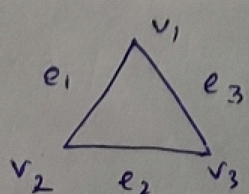
It is a null graph with only one vertex.

Ex :

- Order of a graph - The no. of vertices in a graph (Order = $|V|$)
- Size of a graph - The no. of edges in a graph (Size = $|E|$)

- Incidence - When a vertex v of a graph G is an end vertex of an edge e of the graph G then we say that the edge e is incident on vertex v.

Ex :



e_1 is incident on v_1 & v_2

e_2 is incident on v_2 & v_3

e_3 is " v_3 & v_1

- Two non parallel edges are said to be adjacent edges if they are incident on a common vertex.

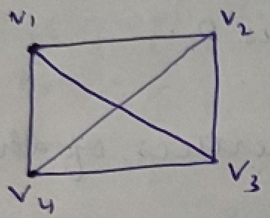
Two vertices are adjacent if there is an edge between them

- Degree of vertex - A

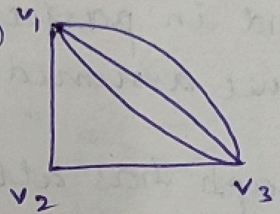
Let G be a graph and v be a vertex of G. Then the no. of edges of G that are incident on v ~~as~~ with the loops counted twice is called the degree of the vertex v..

It is denoted by deg(v) or d(v)

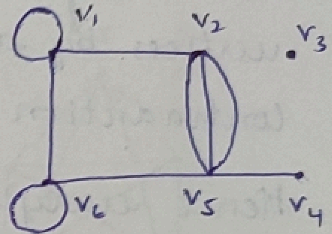
Ex. ①



②



③



$\deg(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$

$\Sigma \deg(v_i) = 12 = 2 * 6$

$d(v_1) = 4$

$d(v_2) = 2$

$d(v_3) = 4$

$\Sigma \deg(v_i) = 10 = 5 * 2$

$d(v_1) = d(v_6) = 4$

$d(v_2) = \cancel{d(v_5)} = 4$

$d(v_5) = 5$

$d(v_3) = 0$

$d(v_4) = 1$

$\Sigma \deg(v_i) = 18$

(Q) State Hand shaking property

Sum of the degrees of all the vertices in a graph is an even no and this no is equal to the twice the no. of edges in the graph

$$\Sigma \deg(v_i) = 2 \times |E|$$

(Q) Verify HandShaking property for the above graphs

1) $\Sigma \deg(v_i) = 12$

$|E| = 2 * 6 = 12$

2) $\Sigma \deg(v_i) = 10$

$|E| = 5 * 2 = 10$

3) $\Sigma \deg(v_i) = 18$

$|E| = 9 * 2$.

(Q) Show that every ~~sing~~ simple graph has atleast 2 vertices of same degree

Proof: Let G be a simple graph with n vertices. Suppose all the vertices have different degrees.

Then since every vertex must have a degree and since all such degrees ~~b/w~~ must be b/w $\underset{A}{0}, 1, \ldots, \underset{B}{(n-1)}$

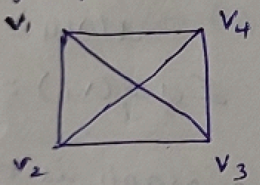
Let A be the vertex whose degree is 0 and B be the vertex whose degree is n-1. This implies B joined to all other vertices by an edge and in particular to A which is a contradiction because we assumed deg(A) is 0.

Hence every simple graph has atleast 2 vertices of the same degree.

⇒ Cubic graph:

It is a graph in which all vertices have a degree 3.

Ex:



(Q) Show that every cubic graph has an even no. of vertices.

Let G be a cubic graph with n vertices. Since it

$\therefore \sum deg(v_i) = 3 \times n$.

Let E be no. of edges $|E| = m$

$\sum deg(v_i) = 2m \Rightarrow 3n = 2m$

$\therefore$ n has to be even.

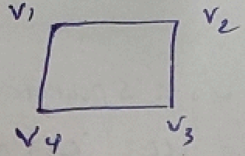
- Regular graph:

A graph in which all the vertices are of same degree

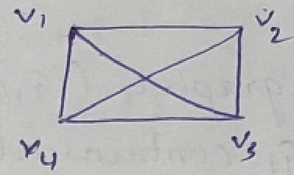
- A regular graph in which all vertices are of degree k is called k regular graph

Ex: 

1 Regular graph



2 regular graph

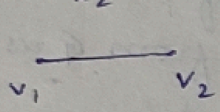


3 regular graph

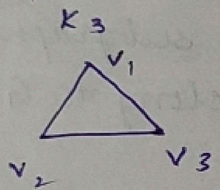
- Complete graph -

A simple graph of order greater than equal to two in which there is an edge b/w any pair of vertices is called a complete graph of order n and is denoted by K_n and K_n is a regular graph of degree $n-1$

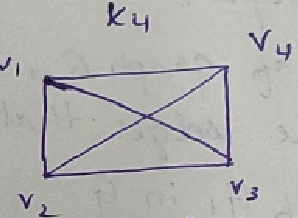
Ex:



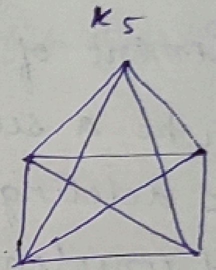
$deg(v_i) = 1$



$deg(v_i) = 2$



$deg(v_i) = 3$

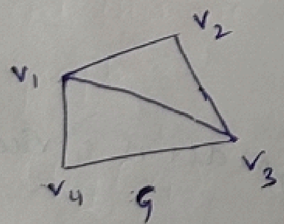


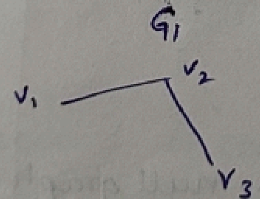
- Subgraph of a graph:

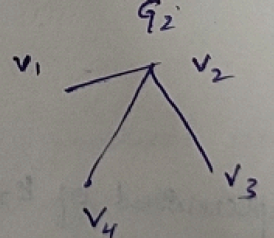
Given 2 graphs G and G_1, G_1 is a subgraph of G if the following conditions hold

1. all the vertices and all the edges of G_1 are in G.
2. Each edge of G_1 has the same end points in G as in G_1

Ex:







Here, G_1 is a subgraph of G but not G_2

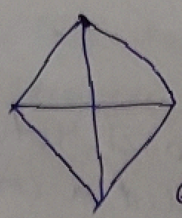
NOTE:

- Every graph is a subgraph of itself
- Every simple graph is a subgraph of complete graph K_n

- Spani subgraph:

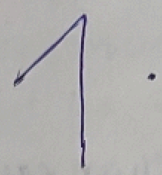
A subgraph G_1 of a graph G ($G_1 \subset G$) is a spanic subgraph of G whenever vertex set of G_1 contains all the vertices of G.

Ex:



G(V, E)

G_1



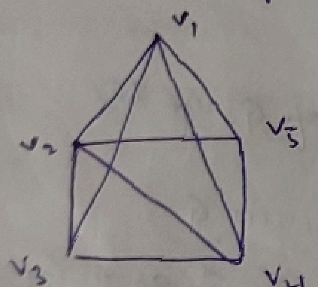
Spanic subgraph

- Complement of a subgraph:

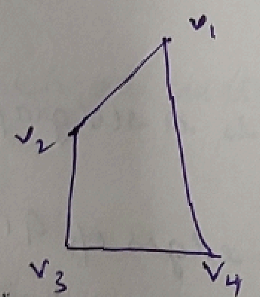
Let G_1 be a subgraph of graph G, the subgraph of G obtained by ~~disgher~~ deleting all the edges that belong to G_1 from G is called complement of G_1 in G.

It is denoted by $G - G_1$ (or) $\overline{G_1}$

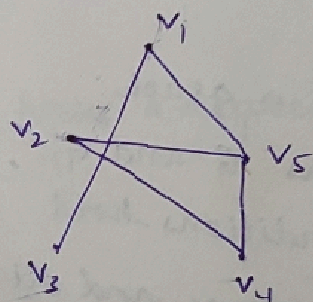
Ex:



G



G_1



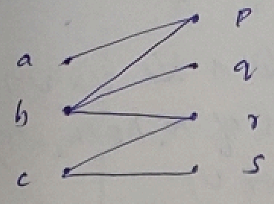
$G - G_1$

NOTE:

1. $\overline{\overline{G}} = G$
2. $\overline{K_n}$ i.e complement of K_n is null graph of order n & vice versa

- Bipartite graph:

A simple graph G is such that its vertex set V is the union of two mutally disjoint nonempty sets ~~(V= V1,~~ V_1 and V_2 ($V = V_1 \cup V_2$) which are such that edge in G joins a vertex in V_1 and V_2, then G is called Bipartite graph and denoted by $G(V_1, V_2, E)$

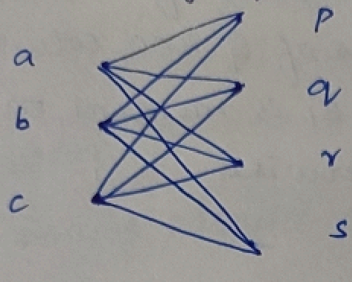
Ex: 

$V_1 = \{a, b, c\}$

$V_2 = \{p, q, r, s\}$

- Complete Bipartite graph:

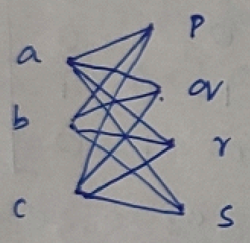
A Bipartite graph $G(V_1, V_2, E)$ is called complete Bipartite graph if there is an edge from every vertex of V_1 to every vertex in V_2

Ex: 

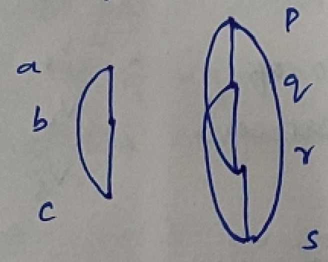
$K_{3,4}$

- It is denoted by $K_{m,n}$ where m = no of vertices in V_1, n = no of vertices in V_2

(Q) Draw the complement of complete bipartitie graph $K_{3,3}$



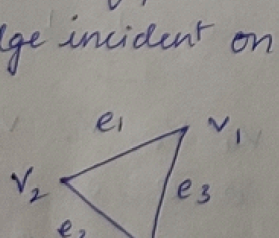
$K_{3,4}$



$\overline{K_{3,4}}$

- Vertex of ~~pendent~~ degree 1 is called pendent vertex.

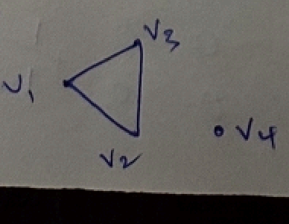
Edge incident on pendent vertex is called pendent edge



v_4 - pendent vertex

e_4 - pendent edge

- Isolated vertex is a vertex in the graph G which is not an end vertex of any edge of the graph i.e a vertex with degree 0 (zero).

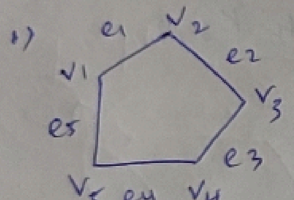


⇒ Adjacency matrix for a simple graph:
Let G be a simple graph then adjacency matrix A is $v \times v$ matrix such that v = no of vertices the rows and columns of A both represent the vertices of G with $A_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$

* For simple graph ⇒ A, M will be either 1 (or) 0

⇒ Incidence matrix:
Incidence matrix is $v \times e$ matrix M such that v = no. of vertices, e = no of edges the rows of M represents the vertices of G and columns represents the edges and $M_{ij} = \begin{cases} 1 & \text{if } e_i \text{ is incident on } v_j \\ 0 & \text{otherwise} \end{cases}$

(Q) Find the adjacency and incidence matrix for the following graphs.

1)


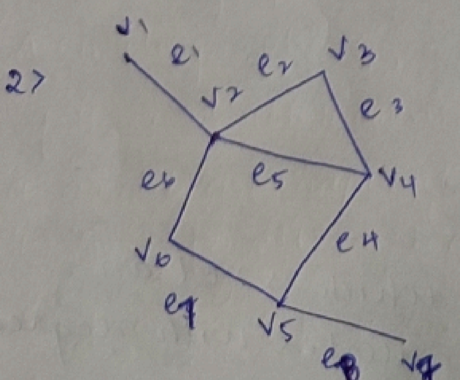
Adjacency matrix A =

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

rows = col's = no. of vertices

Incidence matrix M =

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

2)


A =

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

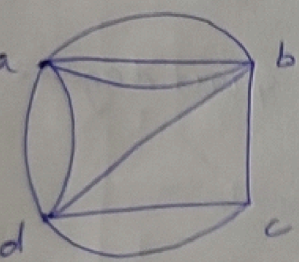
⇒ PSEUDOGRAPH:

A graph having parallel edges and self loops in it is called as a pseudograph.

⇒ Adjacency matrix for pseudograph:

Let $G(V, E)$ be a pseudo graph with n vertices then the adjacency matrix $A = [a_{ij}]_{n \times n}$ of G is an $n \times n$ symmetric matrix from v_1 to v_n and defined by the elements a_{ij} = no. of ~~vertices~~ edges adjacent to v_i to v_j

(Q) Find the adjacency matrix for the following graph



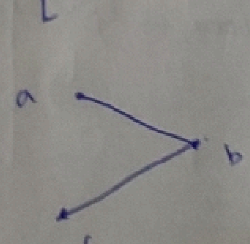
$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 2 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

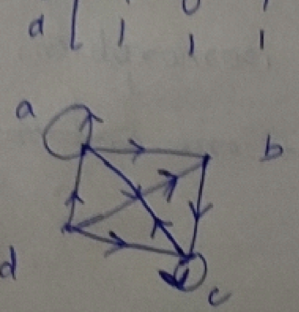
(Q) Draw a graph of the given adjacency matrix

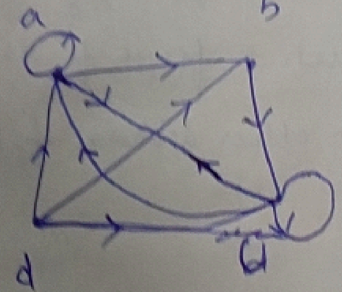
1) $\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$

2) $\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$

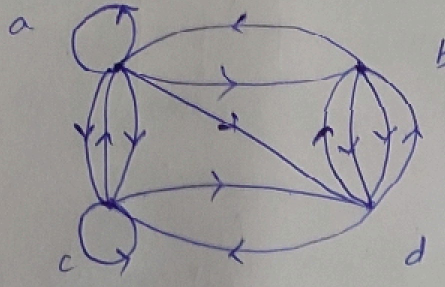
Soln
=







(Q) Find the adjacency matrix of the given directed multi graph



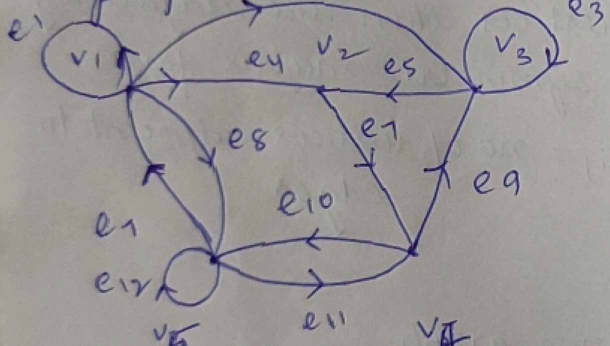
$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

⇒ FIRST THEOREM OF DI-GRAPH:

In directed graph,

$$\sum_{i=1}^{n} \deg^{\overset{\text{out deg}}{+}}(v_i) = \sum_{i=1}^{n} \deg^{\overset{\text{in degree}}{-}}(v_i) = |E|$$

(a) Verify $\sum_{i=1}^{n} \deg^{+}(v_i) = \sum_{i=1}^{n} \deg^{-}(v_i) = |E|$ in the following graph



$$\sum_{i=1}^{n} \deg^{+}(v_i) = 4 + 1 + 2 + 2 + 3$$
$$= 12$$

$$\sum_{i=1}^{5} \deg^{-}(v_i) = 2 + 2 + 3 + 2 + 3$$
$$= 12$$

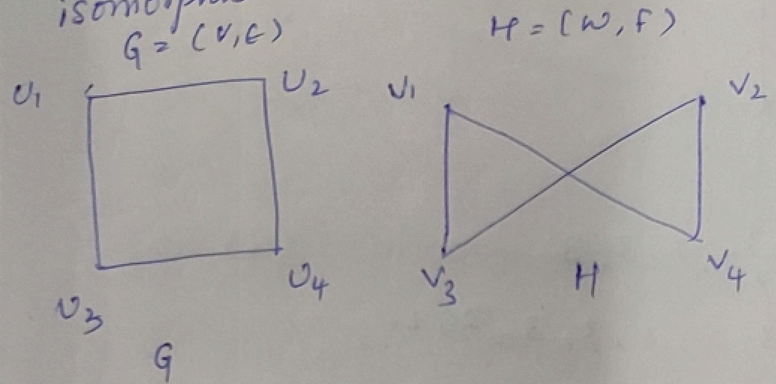
$$|E| = 12$$

* Loops should be considered for both indegree & out degree.

⇒ Isomorphic graph:

- A simple graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one to one and onto func. from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all $a, b \in V_1$
- Such a func 'f' is called isomorphism.
- In other words, f is one to one and onto and preserves adjacency

(Q) Show that the graphs $G = (V, E)$ $H = (W, F)$ shown in the fig are isomorphic.



(2) $\xrightarrow{f}$ Let f be the func or mapping as

(3) $U_1 \to V_1$
$U_2 \to V_3$
$U_3 \to V_4$
$U_4 \to V_2$

2 Graphs have same no. of vertices and edges and same degree sequence

(1)

$G \to \deg(U_1) = 2$ $\quad H \to \deg(V_1) = 2$

$\deg(U_2) = 2$ $\quad \deg(V_2) = 2$

$\deg(U_3) = 2$ $\quad \deg(V_3) = 2$

$\deg(U_4) = 2$ $\quad \deg(V_4) = 2$

(4)

Therefore; U_1 is adj U_2 ✓ $\quad U_2$ is adj U_4 ✓

V_1 is adj V_3 $\quad V_3$ is adj V_2

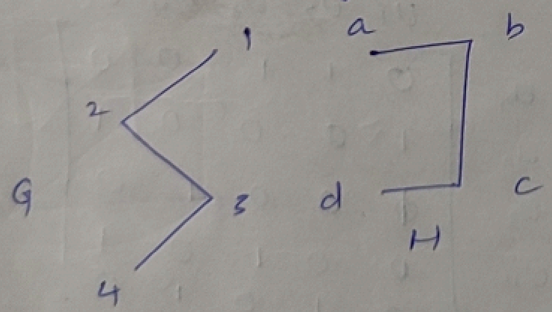
U_1 is adj U_3 ✓ $\quad U_3$ is adj U_4 ✓ → 4 edges 4 "is adj"

V_1 is adj V_4 $\quad V_4$ is adj V_2

∴ Adjacency is preserved and G, H are isomorphic

(Q) Show that the given graphs are isomorphic

(1)



Let f be the func or mapping as

$1 \to a$ $\quad$ $1 \to d$
$2 \to b$ (or) $2 \to c$
$3 \to c$ $\quad$ $3 \to b$
$4 \to d$ $\quad$ $4 \to a$

$\deg(1) = 1 = \deg(4)$ $\quad \deg(a) = \deg(d) = 1$

$\deg(2) = \deg(3) = 2$ $\quad \deg(b) = \deg(c) = 2$

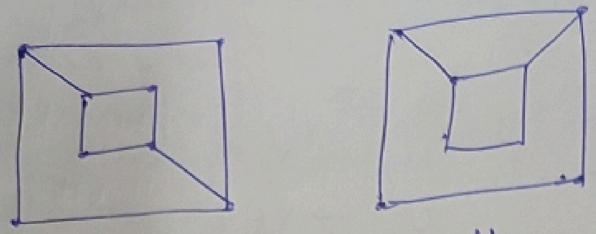
∴ same no of vertices and edges and same degree sequence.

∴ Adjacency is preserved

∴ G & H are isomorphic

$\{1, 2\} \in E(G)$
$\{a, b\} \in E(H)$
$\{2, 3\} \in E(G)$
$\{b, c\} \in E(H)$
$\{3, 4\} \in E(G)$
$\{c, d\} \in E(H)$

(2)



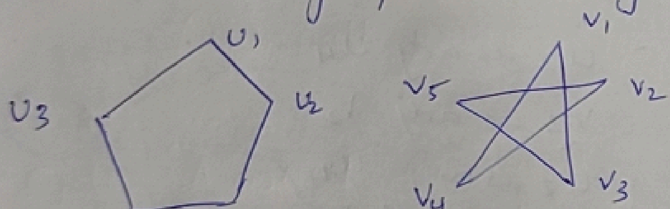
G H

In G, the deg sequence 3, 2, 3, 2, 2, 2, 3, 3

In H, the degree sequence 3, 3, 2, 2, 3, 3, 2, 2

In Graph G no vertices of deg 2 are adjacent while in the graph H vertices of degree 2 are adjacent. Because isomorphism preserves adjacency of vertices, the graphs are not isomorphic

(Q) Prove that graph G & $\bar{G}$ given below are isomorphic



U4 U5 $\bar{G}$

G

The 2 graph have same no of vertices and edges and same degree sequence.

$u_1 \to v_1$

$u_2 \to v_3$

$u_3 \to v_4$

$u_4 \to v_2$

$u_5 \to v_5$

$$A(G) = \begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 1 & 0 & 0 \\ u_2 & 1 & 0 & 0 & 0 & 1 \\ u_3 & 1 & 0 & 0 & 1 & 0 \\ u_4 & 0 & 0 & 1 & 0 & 1 \\ u_5 & 0 & 1 & 0 & 1 & 0 \end{array}$$

$$B(\bar{G}) = \begin{array}{c|ccccc} & v_1 & v_3 & v_4 & v_2 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 0 & 0 \\ v_3 & 1 & 0 & 1 & 0 & 0 \\ v_4 & 1 & 0 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 1 & 0 & 1 \\ v_5 & 0 & 1 & 0 & 1 & 0 \end{array}$$

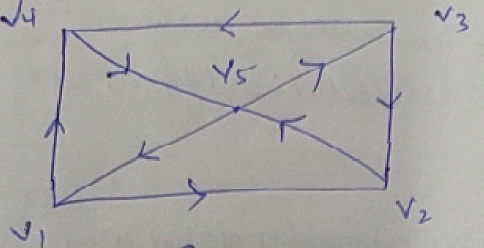
images of $u_1, u_2 \ldots u_5$

Since adjacency matrices of G & $\bar{G}$ are same i.e $A(G) = B(\bar{G})$

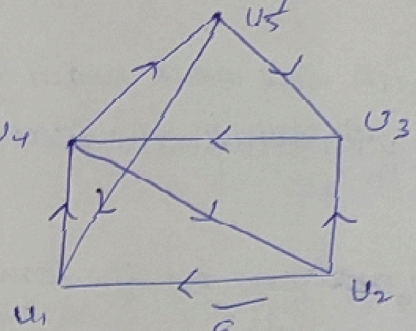
Hence adjacency is preserved.

and G, $\bar{G}$ are isomorphic to each other

(Q) Show that the give Di-Graphs are isomorphic.



G



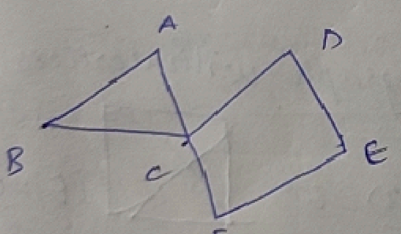
$\bar{G}$

G	in (deg^-)	out (deg^+)
v_1	1	2
v_2	2	1
v_3	1	2
v_4	2	1
v_5	2	2

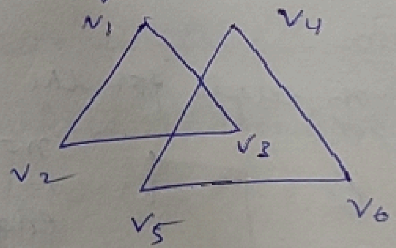
$\bar{G}$	deg^-	deg^+
u_1	2	1
u_2	1	2
u_3	2	1
u_4	2	2
u_5	1	2

$f(v_1) = u_5$, $f(v_2) = u_1$, $f(v_3) = u_2$, $f(v_4) = u_3$, $f(v_5) = u_4$

⇒ Connected and Disconnected graphs:

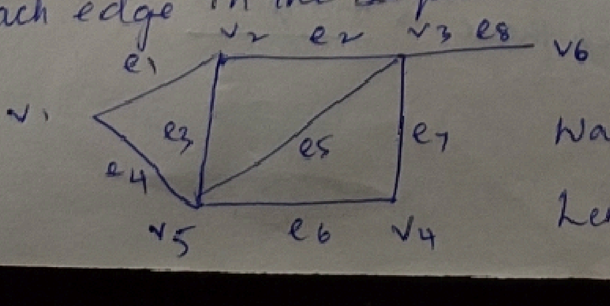


Connected graph



Disconnected graph.

WALK: Let G be a graph, consider a finite alternating sequence of vertices and edges of the form $v_i e_j v_{i+1} e_{j+1} \cdots e_k v_m$ which begins and ends with vertices and which is such that each edge in the sequence is incident on the vertices preceeding



Circuit → $v_1 e_1 v_2 e_2 v_3 e_5 v_5 e_4 v_1$

Cycle → $v_1 e_1 v_2 e_2 v_3 e_5 v_5 e_4 v_1$

Walk → $v_5 e_3 v_2 e_2 v_3 e_8 v_6$

Length → 3 Trail → $v_1 e_1 v_2 e_3 v_5 e_6 v_4$

Length of the walk is the no. of edges appearing in the walk.

- The vertex with which the walk begin is called the initial vertex and on the vertex with which the walk end is called the terminal vertex.

- A walk that begin and end at the same vertex is called a closed walk. Otherwise it is a open walk.

- <u>Trail</u>: Trail is an open walk with no edge appearing more than once.

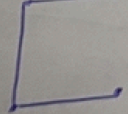
- Circuit: It is a closed walk with no edge appearing more than once.

- Path: A trail in which no vertex appears more than once (open walk with no edge, no vertex repeating)

- Cycle - A <u>circuit</u> in which the terminal vertex doesn't appear as an internal vertex and no internal vertex is repeated.

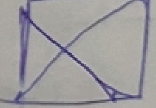
- <u>Remark</u>: The max no. of edges for the simple graph with vertices n is $\frac{n(n-1)}{2}$

Ex



$n=4$

edges $\leq \frac{4(3)}{2} = 6$

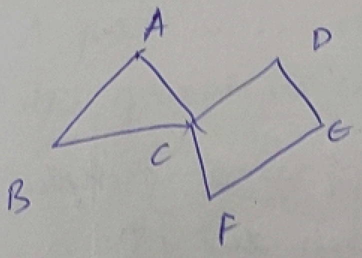


$n=4$

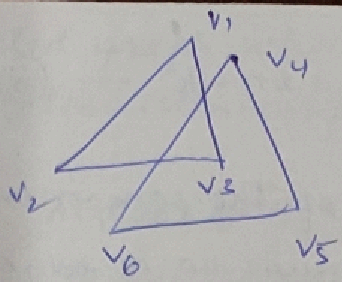
$|E| = \underline{6}$

⇒ Connected graph:
A graph is said to be connected if there exists atleast one path b/w every 2 distinct vertices in graph

⇒ Disconnected graph:
A graph is said to be disconnected if atleast one pair of distinct vertices between which there is <u>no path</u>.



connected graph



Disconnected graph

<u>NOTE</u>: It is evident that every graph G consists of one or more ~~comp~~ connected graph each such connected graph is called component of G. Connected graph contains one component. Disconnected graph contains more than one component. Number of components is denoted by $K(G)$

<u>THEOREM</u>: A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges

Let n_1 be the no of vertices in the first component.
Let n_2 be the no of vertices in the 2nd component
⋮
n_k be the no of vertices in the k^{th} component

Then $n_1+n_2+\cdots+n_k = n$ where n = total no. of vertices of G — ①

$$(n_1-1)+(n_2-1)+\cdots(n_k-1) = n-k$$

Squaring on both sides

$$(n_1-1)^2+(n_2-1)^2+\cdots+(n_k-1)^2 + S = (n-k)^2 \text{ — ②}$$

where S is the sum of the product of the form $2ab$ or $2n_{i-1}n_{j-1}$ where $i=1,2,\cdots k$; $j=1,2\cdots k$ and $i \neq j$

Since each of $n_1, n_2 \text{---} n_k$ is greater than or equal to 1 we have $\Rightarrow S \geq 0$

② $\Rightarrow (n_1-1)^2+(n_2-1)^2+\cdots+(n_k-1)^2 \leq (n-k)^2$

$$n_1^2+n_2^2+\cdots+n_k^2+k-2(n_1+n_2+\cdots+n_k) \leq (n-k)^2$$

$$n_1^2+n_2^2+\cdots+n_k^2 - 2(n)+k \leq (n-k)^2$$

$$n_1^2+n_2^2+\cdots+n_k^2 \leq (n-k)^2+2n-k$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2+k^2-2nk+2n-k \quad — (3)$$

$\because$ G is a simple graph each of the component of G is a simple graph. Therefore the maximum no. of edges which the i^{th} component can have $\frac{(n_i)(n_i-1)}{2}$

$\therefore$ The maximum no. of edges which the graph G can have is N (say)

$$N = \frac{1}{2}\sum_{i=1}^{k} (n_i)(n_i-1) \quad — (4)$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - n_i$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \qquad \left(\because \sum_{i=1}^{k} n_i = n\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - n\right)$$

$$\leq \frac{1}{2}\left(n^2+k^2-2nk+2n-k-n\right)$$

$$N \leq \frac{1}{2}\left(n^2+k^2-2nk+n-k\right)$$

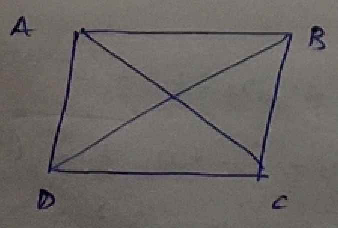
$$N \leq \frac{1}{2}\left((n-k)^2+(n-k)\right)$$

$$\boxed{N \leq \frac{1}{2}(n-k)(n-k+1)}$$

The no. of edges in G cannot exceed $\frac{(n-k)(n-k+1)}{2}$

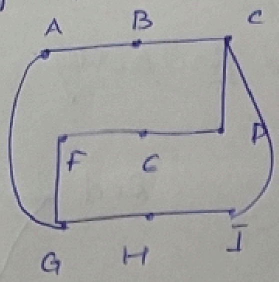
$\Rightarrow$ Hamiltonian cycle and Hamiltonian path:

Let G be connected graph if there is a cycle of G that contains all the vertices of G then that cycle is called a Hamiltonian cycle in G. A graph that contains Hamiltonian cycle is called Hamiltonian graph.



ABCDA - is a Hamilton cycle

A path in a connected graph which includes every vertex of the graph is called Hamiltonian path in the graph.



ABCDEFGHI is a Hamilton path but graph is not containing Hamilton cycle

2) EULER CIRCUIT:

G is connected graph if there is a circuit in G that contains all the edges of G then the circuit is called Euler's circuit in G.

If there is a trail in G that contains all the edges of G then the trail is called an Euler Trail in G.

⇒ Eulerian Graph:

The graph is Eulerian if there is a closed trail which includes every edge of the graph G.

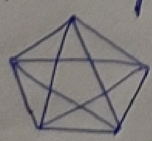
- Theorem: A graph is Eulerian if all the vertices have even degree

- DIRAC THEOREM:

If in a simple connected graph with n vertices ($n \geq 3$) the degree of every vertex is greater than equal to $n/2$ then the graph is Hamilton.

(Q) Which of the graphs are Eulerian

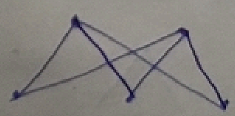
(i) The complete graph K_5



$deg(v_i)$ is even

✓

(ii) The complete bipartite graph ($K_{2,3}$)

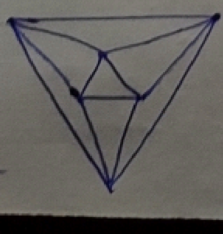


✗

(iii) The graph of octahedral

$deg(v_i)$ is even

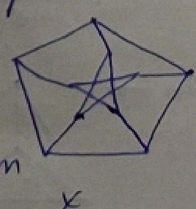
Hence, it is a Eulerian graph



(iv) Peterson graph

10 vertices

15 edges

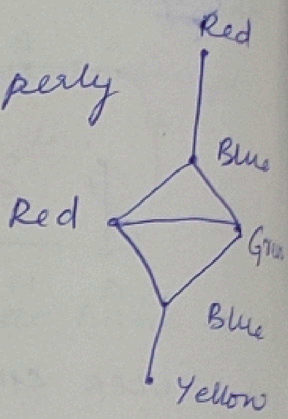


Not a Eulerian graph

✗

⇒ Graph coloring :

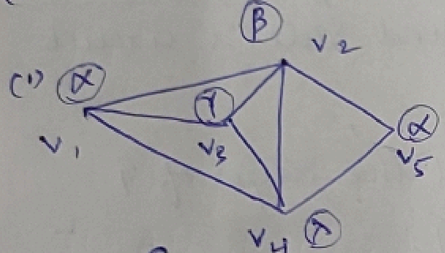
Proper coloring - Adjacent vertices have no same color.
Chromatic number - minimum no. of colours required properly color the graph.



$\chi(G)$ - symbol for chromatic number

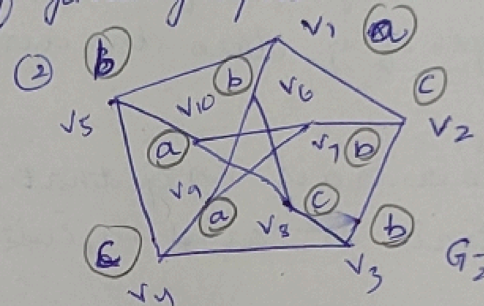
(Q) Find the chromatic number of given graphs

(1)



G_1

(2)

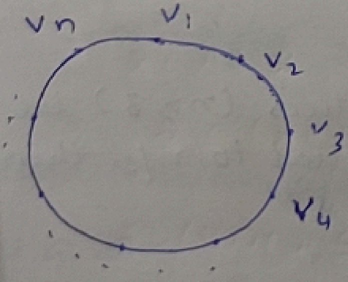


G_2 - peterson graph

$\chi(G_1) = 4$

(Q) ~~Write the theorem stat~~

Prove that a graph of order $(n \geq 2)$ consisting of a single cycle is 2-chromatic if n is even and 3-chromatic if n is odd



n is even

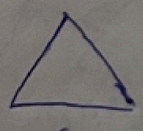
⇒ $v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6$
$\alpha \quad \beta \quad \alpha \quad \beta \quad \alpha \quad \beta$

n is odd

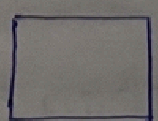
⇒ $v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7$
$\alpha \quad \beta \quad \alpha \quad \beta \quad \alpha \quad \beta \quad \gamma$

⇒ Cycle Wheel

C_n ⇒

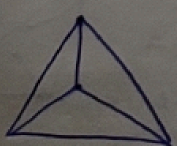


C_3

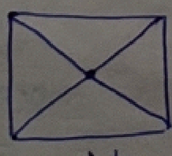


C_4

W_n →



W_4



W_5

MODULE-1: Set Theory & Number Theory

- A set ~~of~~ is collection of well defined objects. Each object in the set is called a member or element of the set

Ex: Books, cities, animals, flowers etc.

- Power set of $A = P(A)$ = Set of all subsets of $A = 2^n$

Ex: $A = \{1, 2, 3, 4\} \Rightarrow P(A) = 2^4 = 16$

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

(Q) For any 3 Sets A, B, C prove that

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Sol

(i) Let $x \in A \cup (B \cap C)$ then

$\Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ —— ①

~~(ii) Let $x \in A \cap (B \cup C)$~~

Now takey any $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A \cup (B \cap C)$

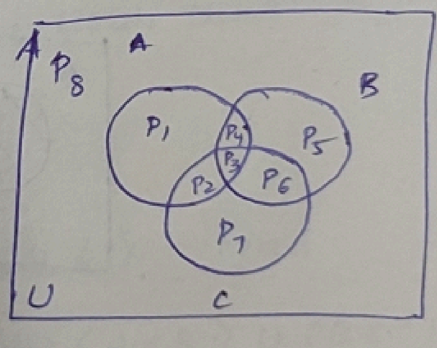
$\Rightarrow (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ —— ②

From ① & ② $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(Q) Using Venn diagram for any three sets A, B, C. Prove that

(i) $\overline{(A\cup(B)\cap C)} = (\bar{A}\cap\bar{B})\cup\bar{C}$

(ii) $\overline{(A\cap B)\cap C} = (\bar{A}\cup\bar{B})\cup\bar{C}$



$A = P_1\cup P_2\cup P_3\cup P_4$

$B = P_3\cup P_4\cup P_5\cup P_6$

$C = P_2\cup P_3\cup P_6\cup P_7$

$A\cup(B\cap C) = P_1\cup P_2\cup P_3\cup P_4\cup P_6$

Sol Consider the following venn diagram

(i) $\overline{A\cup(B\cap C)} = P_5\cup P_7\cup P_8$

RHS :

$\bar{A} = P_5\cup P_6\cup P_7\cup P_8$

$\bar{B} = P_1\cup P_2\cup P_7\cup P_8$

$\bar{C} = P_1\cup P_4\cup P_5\cup P_8$

$(\bar{A}\cap\bar{B}) = P_7\cup P_8 \Rightarrow (\bar{A}\cap\bar{B})\cup\bar{C} = P_1\cup P_4\cup P_5\cup P_7\cup P_8$

LHS

$\Rightarrow (A\cup B)\cap C = P_2\cup P_3\cup P_6 \Rightarrow \overline{(A\cup B)\cap C} = P_1\cup P_4\cup P_5\cup P_7\cup P_8$

$\therefore$ LHS = RHS

(ii) $\overline{(A \cap B) \cap C} = (\bar{A} \cup \bar{B}) \cup \bar{C}$

LHS:

$(A \cap B) \cap C = P_3 \Rightarrow \overline{(A \cap B) \cap C} = P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8$

RHS:

$(\bar{A} \cup \bar{B}) \cup \bar{C} = P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8$

⇒ Relative complement:

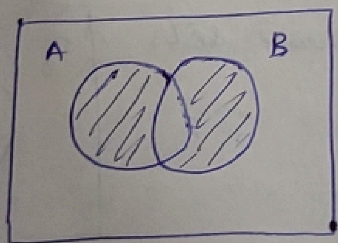
Given 2 sets A, B, the set of all elements that belongs to B but not to A is called complement of A relative to B or relative complement of A in B and is denoted by

$B - A = \{x \mid x \in B \text{ and } x \notin A\}$ where $B - A \neq A - B$.

Symmetric difference:

For two sets A & B the relative complement of $A \cap B$ in $A \cup B$ is called the symmetric difference of A and B. Denoted by

$A \Delta B = (A \cup B) - (A \cap B) = \{x \mid x \in A \cup B \text{ and } x \notin A \cap B\}$



$A \Delta B$

$\{ - A \rightarrow \cap \bar{A} \}$

(Q) Prove the following

(i) $A \Delta B = (B \cap \bar{A}) \cup (A \cap \bar{B}) = (B - A) \cup (A - B)$

(ii) If $A \Delta C = B \Delta C$ then $B = C$

Sol

(i) $A \Delta B = (A \cup B) - (A \cap B)$

$= (A \cup B) \cap \overline{(A \cap B)}$

$= (A \cup B) \cap (\bar{A} \cup \bar{B})$

$= \{(A \cup B) \cap \bar{A}\} \cup \{(A \cup B) \cap \bar{B}\}$

$= (A \cap \bar{A}) \cup (B \cap \bar{A})\} \cup \{(A \cap \bar{B}) \cup (B \cap \bar{B})\}$

$= \{\phi \cup (B \cap \bar{A})\} \cup \{(A \cap \bar{B}) \cup \phi\}$

$= (B \cap \bar{A}) \cup (A \cap \bar{B})$

$= \underline{\underline{RHS}}$

(ii) $A \Delta C = (C \cap \bar{A}) \cup (A \cap \bar{C})$ {from (i)}

$B \Delta C = (C \cap \bar{B}) \cup (B \cap \bar{C})$

(Given $A \Delta C = B \Delta C$)

- Take $x \in A$ suppose $x \in C$ then $x \notin (C \cap \bar{A})$ and $x \notin A \cap \bar{C}$ and such $x \notin (C \cap \bar{A}) \cup (A \cap \bar{C})$. This means $x \notin A \Delta C$. Since $A \Delta C = B \Delta C$ then $x \notin B \Delta C$. Therefore $x \in B$ because if $x \notin B$ so that $x \in C \cap \bar{B}$ which yields $x \in B \Delta C$ a contradiction. Thus if $x \in C$ we have $A \subseteq B$

- Suppose $x \in A$ and $x \notin C$. Then $x \in \bar{C}$ so that $x \in (A \cap \bar{C})$. This yields $x \in A \Delta C$. Since $A \Delta C = B \Delta C$ it follows that $x \in B \Delta C$. Therefore, $x \in C \cap \bar{B}$ or $x \in B \cap \bar{C}$ Since $x \notin C$ then $x \in B \cap \bar{C}$. Then $x \in B$. Thus if $x \notin C$ we have $A \subseteq B$. In a similar way it follows that $B \subseteq A$

$\therefore \underline{\underline{A = B}}$.

(Q) Let $A = \{1, 3, 5\}$ $B = \{2, 3\}$ and $C = \{4, 6\}$. Find

(i) $(A \cup B) \times C$ (ii) $(A \times B) \cap (B \times A)$ (iii) $(A \times B) \cup (B \times C)$

$\underline{\underline{Sol}}$ (i) $\{1, 2, 3, 5\} \times \{4, 6\}$

$= \{(1,4), (1,6), (2,4), (2,6), (3,4), (3,6), (5,4), (5,6)\}$

(ii) $\{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\} \cap$

$\{(2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\}$

$= \{(3,3)\}$

(iii) $\{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\} \cup \{(2,4), (2,6), (3,4), (3,6)\}$

$= \ldots\ldots$

(Q) Prove that (i) $A \times (B - C) = (A \times B) - (A \times C)$

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Sol : (i) $A \times (B - C) = \{(x, y) \mid x \in A \text{ and } y \in B - C\}$

$= \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } y \notin C\}$

$= \{(x, y) \mid (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \notin C)\}$

$= \{(x, y) \mid (x, y) \in A \times B \text{ and } (x, y) \notin A \times C\}$

$= \{(x, y) \mid (x, y) \in (A \times B) - (A \times C)\}$

$= (A \times B) - (A \times C)$

(ii) $A \times (B \cup C) = \{(x, y) \mid x \in A \text{ and } y \in (B \cup C)\}$

$= \{(x, y) \mid x \in A \text{ and } (y \in B \text{ or } y \in C)\}$

$= \{(x, y) \mid (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) \mid (x, y) \in A \times B \text{ or } (x, y) \in A \times C\}$

$= \{(x, y) \mid (x, y) \in (A \times B) \cup (A \times C)\}$

$= (A \times B) \cup (A \times C)$

2. Prove that for any three sets A, B, C

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Sol : $A \times (B \cap C) = \{(x, y) \mid x \in A \text{ and } y \in (B \cap C)\}$

$= \{(x, y) \mid x \in A \text{ and } (y \in B \text{ and } y \in C)\}$

$= \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } x \in A \text{ and } y \in C\}$

$= \{(x, y) \mid (x, y) \in A \times B \text{ and } (x, y) \in A \times C\}$

$= \{(x, y) \mid (x, y) \in (A \times B) \cap (A \times C)\}$

$= (A \times B) \cap (A \times C)$

NOTE: Cartesian product of A and B

$A \times B = \{(x, y) \mid x \in A \quad y \in B\}$

(Q) Prove that no. of elements present in $A \times B$ = product of no of elements in A & B

$|A \times B| = |A||B|$ cordinality.

Sol) $|A \times B| = |A||B|$

Let $|A| = m$, $|B| = n$

$A = \{a_1, a_2 \cdots a_m\}$ $B = \{b_1, b_2 \cdots b_n\}$

$A \times B = \{(a_1, b_1), (a_1, b_2) \cdots (a_m, b_n)\}$

There are m ways to choose $x \in A$ for the first position in the ordered pair and n ways to choose $y \in B$ in the second position So by multiplication principle there are $m \times n$ ways to form an ordered pair. Therefore cordinality of $A \times B$

$|A \times B| = mn$

$|A \times B| = |A||B|$

⇒ Computer Representation of sets:

This is a method of storing elements using an arbitary ordering of the elements of the universal set i.e if the elements of universal set U be represented by $a_1, a_2 \cdots a_n$ to represent a subset A of U with the bit string of length n where i^{th} bit in the string is 1 if $a_i \in A$ and 0 if $a_i \notin A$

⇒ Coding Theory:

Consider the set $Z_2 = \{0, 1\}$ and the additive group $(Z_2, +)$ where + denotes addition modulo 2 (Addition. Then for any positive integer n we have $Z^n = Z_2 \times Z_2 \times \cdots \times Z_2$ (n terms)

$= \{(a_1, a_2 \cdots a_n) | a_1, a_2 \cdots a_n \in Z_2\}$

Every element of Z_2^n is called as n-tuple $(a_1\, a_2\, a_3 \cdots a_n)$ In which every entry belongs to Z_2 so that it is 0 or 1. For simplicity n-tuple as $a_1 a_2 \cdots a_n$. This form is called as a word or a string in Z_2^n and each a_i (= 0 or 1) is called a bit or component or individual signal. Here n is called length of the string.

(Q) Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

What bit string represent the subset of all odd integers in U, subset of all even integers in U and the complement of the odd integers.

Sol
Set of even integers = 0101010101 (length = n)

Set of odd integers = 1010101010

Set of complement of odd integers = 0101010101

(Q) The bit string for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 1111100000 and 1010101010 resp. Use bit strings to find the union and intersection of the sets

Sol
The union of 1111100000 and 1010101010 = 1111101010

Set corresponding to union = $\{1, 2, 3, 4, 5, 7, 9\}$

The instersection of 1111100000 and 1010101010 = 1010100000

(1111100000 ∧ 1010101010)

Set corresponding to intersection = $\{1, 3, 5\}$

(Q) If $A = \{x \mid 3x^2 - 7x - 6 = 0\}$ and $B = \{x \mid 6x^2 - 5x - 6 = 0\}$

Find $A \cap B$ and $A \cup B$.

Sol

$$3x^2 - 7x - 6 = 0$$
$$3x^2 - 9x + 2x - 6 = 0$$
$$3x(\cancel{x}x - 3) + 2(x - 3) = 0$$
$$(x-3)(3x+2) = 0$$
$$x = 3, -2/3$$
$$\therefore A = \{3, -2/3\}$$

$$6x^2 - 5x - 6 = 0$$
$$6x^2 - 9x + 4x - 6 = 0$$
$$3x(2x - 3) + 2(2x - 3) = 0$$
$$(3x+2)(2x-3) = 0$$
$$x = -2/3, 3/2$$
$$B = \{-2/3, 3/2\}$$

$$\therefore A \cap B = \{-2/3\}$$
$$A \cup B = \{3, -2/3, 3/2\}$$

(Q) If $A = \{\alpha, \beta\}$, $B = \{1, 2, 3\}$. Find (1) $A \times B$ (2) $B \times A$ (3) $A \times A$

(4) $B \times B$ (5) $(A \times B) \times A$ (6) $(A \times B) \times B$ (7) $(A \times B) \cap (B \times A)$

⇒ Addition principles of set theory

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

(Q). Set A has 4 elements and set B has 7 elements. What can be the minimum no. of elements in $A \cup B$

$|A| = 4$, $|B| = 7$ ⇒ $|A \cup B| = |A| + |B| - |A \cap B|$

$= 4 + 7 - |A \cap B|$

$|A \cup B|$ is mininum when $|A \cap B|$ is maximum i.e

when $A \subset B$ and therefore $|A \cap B| = 4$

$\therefore$ $|A \cup B| = 4 + 7 - 4 = \underline{\underline{7}}$

(Q) A certain company has 100 programmers. Of these 47 can program in Fortan, 35 in pascal, 20 in cobol, 23 in Fortan and Pascal, 12 in cobol and Forton, 11 in pascal and cobol; 5 in all languages.
How many can program in none of all these languages

Sol:

$A \rightarrow$ Fortan ⇒ $|A| = 47$ $\quad$ $|A \cap B| = 23$

$B \rightarrow$ Pascal ⇒ $|B| = 35$ $\quad$ $|A \cap C| = 12$

$C \rightarrow$ Cobol ⇒ $|C| = 20$ $\quad$ $|B \cap C| = 11$ and $|A \cap B \cap C| = 5$

$|\overline{A \cup B \cup C}| = U - |A \cup B \cup C|$

$= 100 - (47 + 35 + 20 - 23 - 12 - 11 + 5)$

$= 39$

(Q) In an examination 70% of students passed in english, 65% in mathematics, 27% failed in both the subjects and 248 passed in both the subjects. Find the total no. of students

Sol

Let A and B be the set of students who passed in english and math.

$|A| = 70\%$; $|B| = 65\%$, $\bar{A} \cap \bar{B} = 27\% = \overline{A \cup B}$

$\Rightarrow A \cup B = U - (\overline{A \cup B})$

$|A \cap B| = 248$

$= 100 - 27 = 73\%$,

$\Rightarrow |A \cup B| = |A| + |B| - |A \cap B| = 248$

$\Rightarrow |A \cap B| = |A| + |B| - |A \cup B|$

$248 = (65 + 70 - 73)\%$

$248 = 62\%$ of total

$248 = \frac{62}{100} \times \text{Total}$

$\text{Total} = 400$

(Q) The professor has 2dozen textbooks on computer science and is concerned about their coverage of topics say A compilers, B data structures and C operating system. Following are the no's of books that contain material on these topics.

$|A| = 8$ $|B| = 13 = |C|$, $|A \cap B| = 5$, $|A \cap C| = 3$, $|B \cap C| = 6$

$|A \cap B \cap C| = 2$

(i) How many of the books include material on exactly one of these topics.

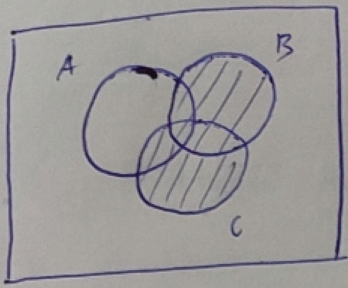
(ii) How many do not deal with any of the topics

continued....

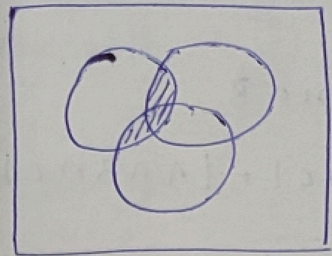
(Q) Prove using Venn Diagram A, B, C

$\overline{A \cap (B \cup C)} = \bar{A} \cup (\bar{B} \cap \bar{C})$

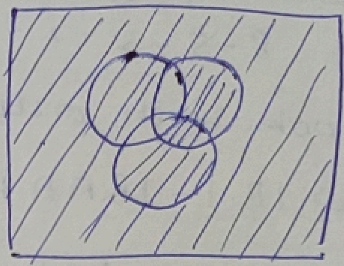
Proof: LHS:



$B \cup C$

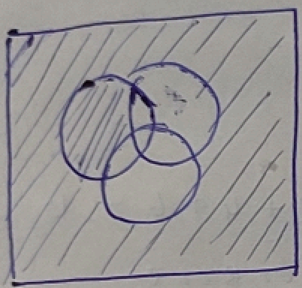


$A \cap (B \cup C)$

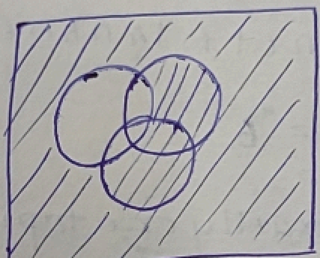


$\overline{A \cap (B \cup C)}$

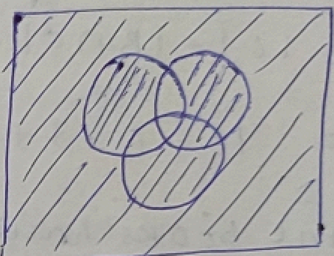
RHS:



$\bar{B} \cap \bar{C}$



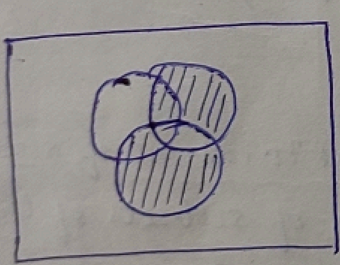
$\bar{A}$



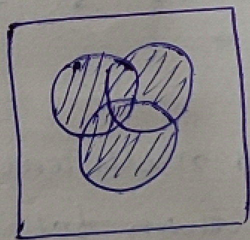
$\bar{A} \cup (\bar{B} \cap \bar{C})$

(Q) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

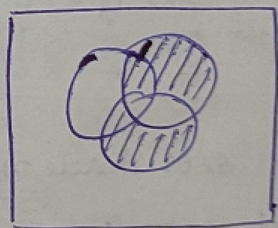
LHS:



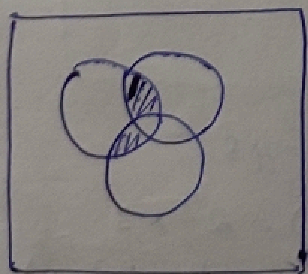
$B \Delta C = B \cup C - B \cap C$



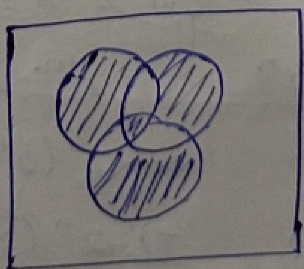
$A \cup (B \Delta C)$



$A \cap (B \Delta C)$



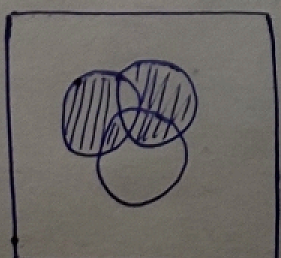
$A \cap (B \Delta C)$



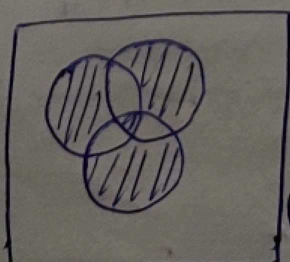
$A \Delta (B \Delta C)$

$\therefore$ LHS = RHS

RHS:



$A \Delta B$



$(A \Delta B) \Delta C$

Sol : (i) Books with only compilers (topic A)

$= $ coordinality of only $|A|$

$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 8 - 5 - 3 + 2 = 2$

Books with only topic B

$= |B| - |B \cap A| - |B \cap C| + |A \cap B \cap C|$

$= 13 - 5 - 6 + 2 = 4$

Books with only topic C

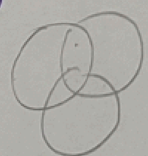
$= |C| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 13 - 6 - 3 + 2 = 6$

$\therefore$ Total books having exactly one topic $= 2 + 4 + 6 = 12$

$=$ only A or B or C

Books containing



(ii) None of topics $= 24 - |A \cup B \cup C|$

$= 24 - (|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 24 - (8 + 13 + 13 - 5 - 3 - 6 + 2)$

$= 2$

(Q) Two finite sets have m and n elements. The total no. of subsets of the first set is 48 more than total no. of subsets of the second set.

$|P(A)| = |P(B)| + 48$

$|A| = m, \quad |B| = n$

$P(A) = 2^m, \quad P(B) = 2^n$

$\Rightarrow 2^m = 2^n + 48$ $\quad m > n$

$\Rightarrow 2^m - 2^n = 48$

$\Rightarrow 2^n(2^{m-n} - 1) = 48$

$\Rightarrow 2^n(2^{m-n} - 1) = 2^4 \times 3$

$= 2^4 \times (2^2 - 1)$

$\therefore n = 4$ and $m = 6$

(Q) Show that in every graph the no. of odd degree vertices is even

Proof: Consider a graph with n vertices; k of these are of odd degree.

So that the remaining $n-k$ are of even degree. Therefore

$$\sum_{i=1}^{k} deg(v_i) + \sum_{i=k+1}^{n} deg(v_i) = \sum_{i=1}^{n} deg(v_i) \quad \text{—①}$$

By Handshaking property.

Graph Theory

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

$$① \Rightarrow \underbrace{\sum_{i=1}^{k} deg(v_i)}_{\text{(odd deg)}} + \underbrace{\sum_{i=k+1}^{n} deg(v_i)}_{\text{(even deg)}} = \underbrace{2|E|}_{\text{(even no)}} \quad \text{—②}$$

From ② $\Rightarrow$ RHS is even no and $\sum_{i=k+1}^{n} deg(v_i)$ is even. Therefore $\sum_{i=1}^{k} deg(v_i)$ should be even no. $\Rightarrow$ v_i should be even no.

$\therefore$ The no. of odd degree vertices (k) should be even.

(Q) State and prove Demorgan's Law. Also prove these laws by using Venn Diagram.

(i) $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (ii) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(i) LHS $= \overline{A \cap B} = \{x \mid x \in \overline{A \cap B}\}$

$= \{x \mid x \notin A \cap B\}$

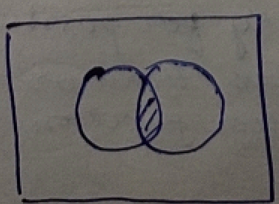
$= \{x \mid x \notin A \text{ or } x \notin B\}$

$= \{x \mid x \in \overline{A} \text{ or } x \in \overline{B}\}$

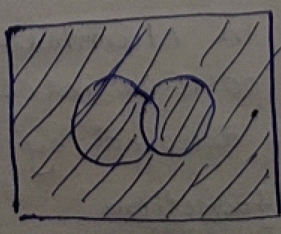
$= \{x \mid x \in \overline{A} \cup \overline{B}\}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

LHS:

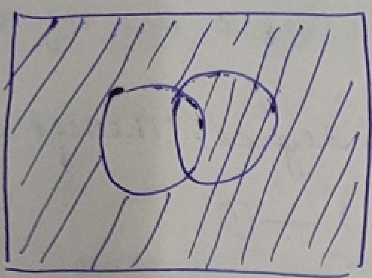


$A \cap B$

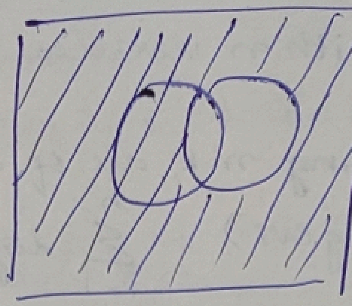


$\overline{A \cap B}$

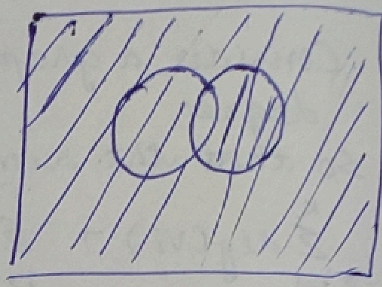
RHS:



$\bar{A}$



$\bar{B}$



$\bar{A} \cup \bar{B}$

② $\overline{A \cup B} = \{x \mid x \in \overline{A \cup B}\}$

$= \{x \mid x \notin A \cup B\}$

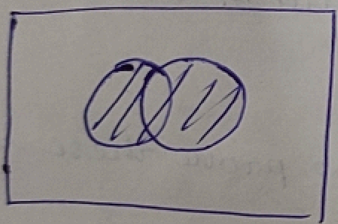
$= \{x \mid x \notin A \text{ and } x \notin B\}$

$= \{x \mid x \in \bar{A} \text{ and } x \in \bar{B}\}$

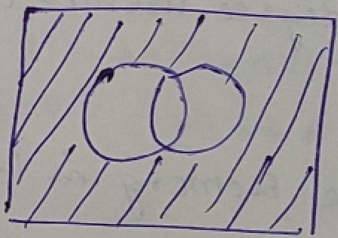
$= \{x \mid x \in \bar{A} \cup \bar{B}\}$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

LHS=

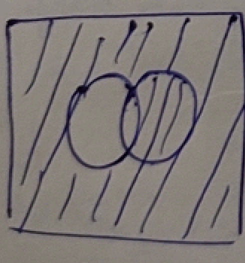


$A \cup B$

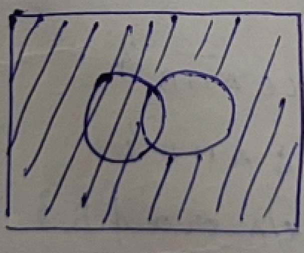


$\overline{A \cup B}$

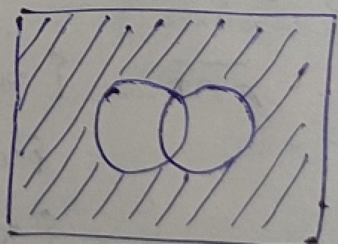
RHS:



$\bar{A}$



$\bar{B}$



$\bar{A} \cap \bar{B}$

(Q) How many integers are there between 1 and 200 which are divisible by 2, 3 & 5

NOTE: Standard result in Number theory for positive integers a and b, the number of positive integers less than or equal to a and divisible by b is denoted by the floor function

$\left\lfloor \frac{a}{b} \right\rfloor$ = floor functions

which is greatest integer $\leq a/b$

NOTE:

Sol: Let $S = \{1, 2, \cdots 200\}$

Then $|S| = 200$

Let A_1, A_2, A_3 are subsets of S whose elements are divisible by 2, 3 & 5 resp.

$|A_1| = \lfloor \frac{200}{2} \rfloor = \lfloor 100 \rfloor = 100$

$|A_2| = \lfloor \frac{200}{3} \rfloor = \lfloor 66.66 \rfloor = 66$

$|A_3| = \lfloor \frac{200}{5} \rfloor = \lfloor 40 \rfloor = 40$

$|A_1 \cap A_2| = \lfloor \frac{200}{6} \rfloor = \lfloor 33.33 \rfloor = 33$

$|A_2 \cap A_3| = \lfloor \frac{200}{15} \rfloor = \lfloor 13.33 \rfloor = 13$

$|A_1 \cap A_3| = \lfloor \frac{200}{10} \rfloor = 20$

$|A_1 \cap A_2 \cap A_3| = \lfloor \frac{200}{3} \rfloor = \lfloor 6.66 \rfloor = 6$

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$= 100 + 66 + 40 - 33 - 20 - 13 + 6$