GRAPH THEORY

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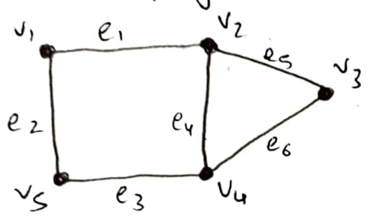
SUBJECT - MATHS

1 (a) Define terms with examples →

(i) Graph (ii) Multi Graph (iii) Simple Graph

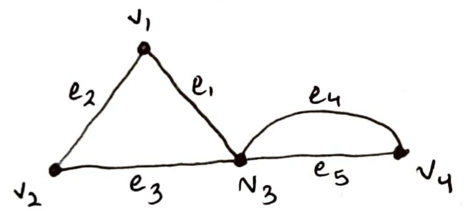
Ans] (i) A graph G is mathematical structure consisting of two sets V and E where V is a non empty set of vertices and E is a non empty set of edges.

Example:

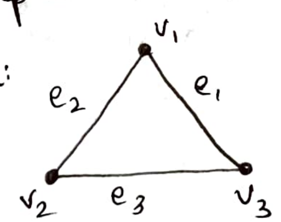


(ii) Multigraph → A graph does not contain any self loop but contain multiedge is called multigraph.

Example:



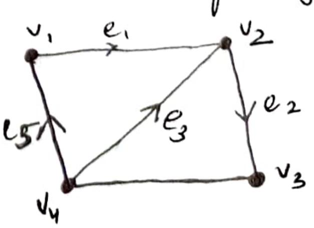
(iii) Simple Graph - A graph does not contain any self loop and multiedge. Example:



(b) Define the following terms with example:

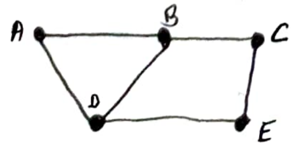
(i) Directed graph (ii) Undirected Graph

Ans] (i) Directed Graph → A graph consist the direction of edges then this is called directed graph. Ex:



(ii) Undirected Graph → A graph which is not directed then its called undirected graph.

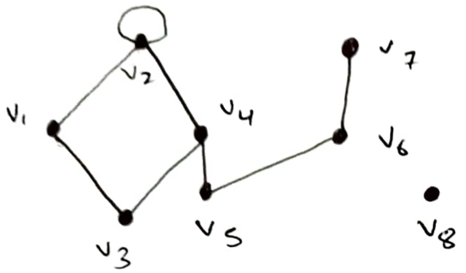
Example:



2. a) Define with example
(i) Degree of vertex (ii) Isolated vertex

Ans] (i) Degree of Vertex: The degree of vertex v in a graph G written as $d(v)$ is equal to number of edges which are incident on v with self loop counted twice.

Example:



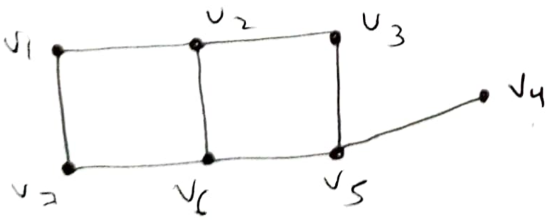
$d(v_1) = 2$, $d(v_2) = 4$, $d(v_3) = 2$
$d(v_4) = 3$, $d(v_5) = 2$, $d(v_6) = 2$
$d(v_7) = 1$, $d(v_8) = 0$

(ii) Isolated vertex → A vertex having degree 0 is called isolated vertex. Ex: v_8 is a isolated vertex.

2 (b) Define with example
(i) Bipartite Graph (ii) Complete Bipartite graph.

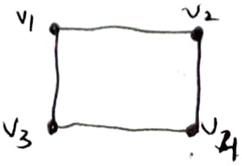
Ans] (i) Bipartite Graph → If the vertex set V of a graph G can be partitioned into two non empty disjoint subsets X and Y in such a way that edge of G has one end in X and one end in Y. Then G is called bipartite.

Ex:



$X = \{v_1, v_6, v_3, v_4\}$
$Y = \{v_7, v_2, v_5\}$

(ii) Complete Bipartite Graph → If every vertex in X is disjoint with every vertex in Y, then it is called a complete bipartite graph. If X and Y condition m & n vertices then this graph is denoted as $K_{m,n}$. Example.

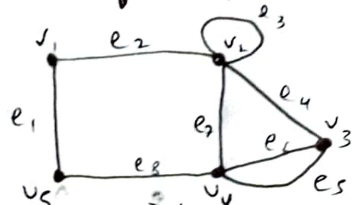


$X = \{v_1, v_4\}$
$Y = \{v_2, v_3\}$
Denoted by $K_{2,2}$

3. (a) Define with example:

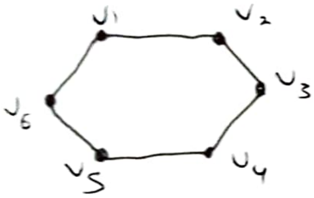
(i) Pseudograph (ii) Cycle Graph (iii) Wheel Graph

(i) Pseudograph: A graph contain both self loop and multiedge is called pseudograph. Example:



(ii) Cycle graph → It is a graph that consists of a single cycle or some number of vertices in a closed chain.

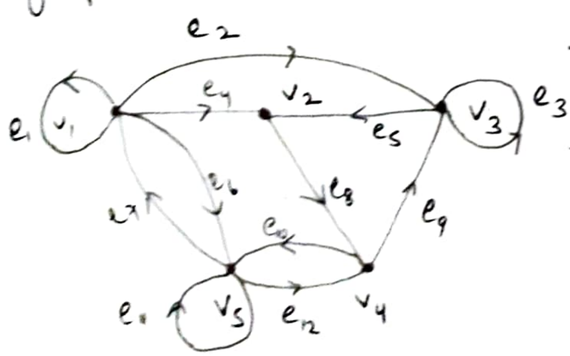
Ex:



(iii) Wheel Graph: A graph formed by connecting a single universal vertex to all vertices of a cycle.

Ex:

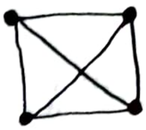
3(b) Verify $\sum_{i=1}^{n} deg^{+}(v_i) = \sum_{i=1}^{n} deg^{-}(v_i) = |E| = e$ in the following graph.



	$deg^{+}(v_i)$	$deg^{-}(v_i)$
(6) v_1	2	4
(3) v_2	2	1
(5) v_3	3	2
(4) v_4	~~3~~ 2	~~3~~ 2
(6) v_5	3	3
(24)	12	12

4(a) Draw the graphs

(i) K_n for $1 \le n \le 4$

Ans) $n = 1$ , $n = 2$ , $n = 3$ , $n = 4$ 

(ii) (a) C_3 (b) C_4 (c) C_5, (d) C_6. (e) C_8

Ans) (a) 

C_3

(b) 

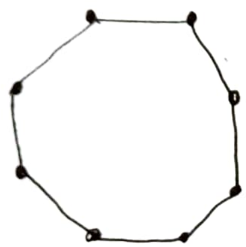
C_4

(c) 

C_5

(d) 

C_6

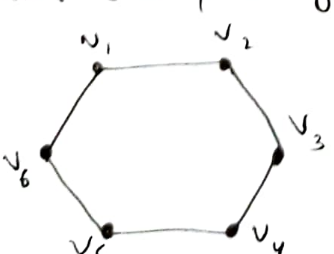
(e) 

C_8

4(b) Is C_3 and C_6 bipartite - Explain?

Ans] C_3 is not a bipartite graph as every vertex in C_3 is connected to every other vertex.

C_6 is a bipartite graph as we can partition the vertex set into $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ so that every edge of C_6 connects a vertex in V_1 & V_2.



5(a) Draw the complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$, $K_{2,6}$

Ans] (i) $K_{2,3}$ →

(ii) $K_{3,3}$ →

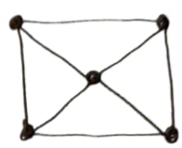
(iii) $K_{3,5}$ →

(iv) $K_{2,6}$

5(b) Draw the graphs of (a) W_3, (b) W_4, c) W_5 d) W_6 e) W_7

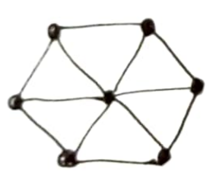
Ans] (i) W_3 →

(ii) W_4 →

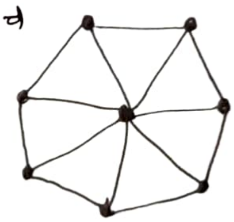


(iii) W_5 →

(iv) W_6 →



(v) W_7 →



6(a) Define complement graph and if the simple graph G has v vertices and edges how many edges does $\bar{G}$ have?

Ans) Complement of a graph G is defined as a simple graph with the same vertex set as G and where two vertices u and v are adjacent only when they are not adjacent in G

• $V(G) = V(G')$, • $E(G') = \frac{n(n-1)}{2} - E(G)$

n: total vertices in a graph

6(b) State Handshaking property. Show that every simple graph has two vertices of the same degree

[Ans] Handshaking property states in any graph that sum of degree of all the vertices is twice the number of edges contained in it.

$$\boxed{\sum_{i=1}^{n} d(v_i) = 2|E|}$$

Contradictory proof → Assume graph has n vertices. Each of those is connected to either 0, 1, 2 . . . . n − 1 other vertices. If any of the vertices is connected to n − 1 vertices, then its connected to all the others so there cannot be a vertex connected to 0 others. Thus its impossible to have a graph with n vertices where one is vertex has 0 and another has degree n − 1. Thus the vertices can have at most n − 1 different degrees but since there are n vertices, at least two must have the same degree.

7(a) Define adjacency matrix and incidence matrix.

[Ans] (a) Adjacency Matrix : Let a_{ij} denote the no. of edges (v_i, v_j) then $A = [a_{ij}]_{m \times m}$ is called adjacency matrix of G if

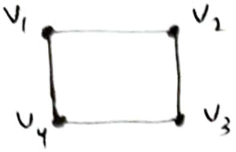
$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

(b) ~~Inc~~ Incidence Matrix : Let G be a graph with m vertices $v_1, v_2, \ldots, v_m$ and n edges $e_1, e_2 \ldots e_n$. Let a matrix $M = [m_{ij}]_{m \times n}$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident on the } e_j \\ 0 & \text{if } v_i \text{ is not incident on } e_j \\ 2 & \text{if } v_i \text{ is an end of the loop } e_j \end{cases}$$

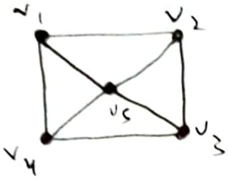
7 (b) Write the adjacency matrix of C_4 and W_4.

(i) $C_4 \Rightarrow$



$$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_2 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

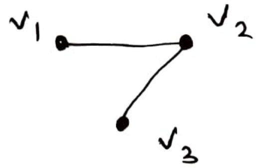
(ii) $W_4 \Rightarrow$



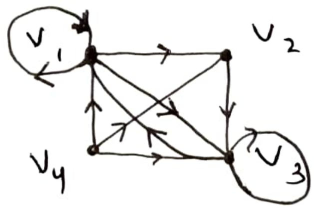
$$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

8 (a) Draw the graph of the given adjacency matrix.

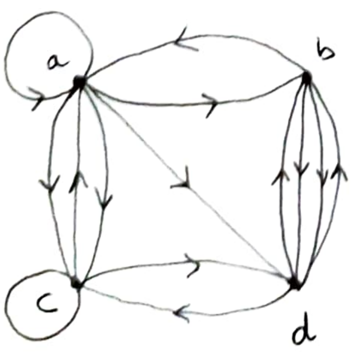
(i) $$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow$$



(ii) $$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \Rightarrow$$



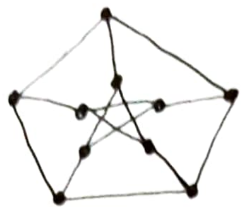
8 (b) Find the adjacency matrix of the given directed multigraph.



$$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

9(a) Draw Petersen graph.

Ans] The petersen graph is an undirected graph with 10 vertices and 15 edges



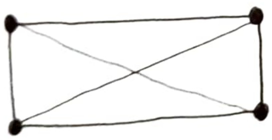
9(b) Is there a simple graph with 1, 1, 3, 3, 3, 4, 6, 7 as the degree of its vertices.

Ans] There is no graph with degree sequence (1, 1, 3, 3, 3, 4, 6, 7)

10(a) Explain a regular graph with example

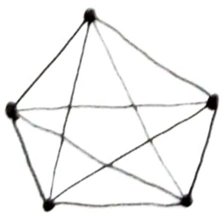
Ans] 10(a) A regular graph where each vertex has the same number of neighbours i.e. every vertex has the same degree or valency.

Ex:

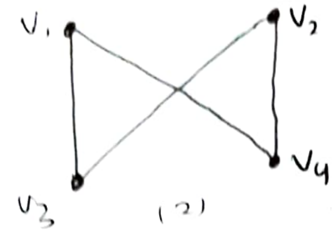


10(b) How many vertices does a regular graph of degree 4 with 10 edges have?

Ans] 5 vertices



11(a) Define isomorphism of graphs. Show that the graphs G = (V, E) and H = (W, F) shown in the figures are isomorphic.



Ans] Two graphs G_1 & G_2 are isomorphic if →

- The no. of vertices are same
- The no. of edges are same
- An equal number of vertices with given degree
- vertex correspondence & edge correspondance valid.

(ii)

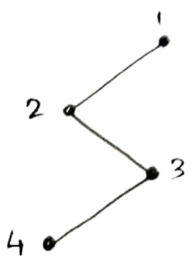
- In figure (1) and (2), equal no. of vertices are there = 4
- Equal no. of edges = 4
- Degree of vertices →

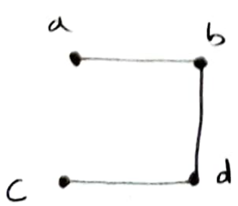
$v_1 = 2$, $v_1 = 2$
$v_2 = 2$, $v_2 = 2$
$v_3 = 2$, $v_3 = 2$
$v_4 = 2$, $v_4 = 2$

- Correspondance

v_1 —— v_1
v_2 —— v_4
v_3 —— v_2
v_4 —— v_3

(b) Show that the two graphs shown in the figure are isomorphic.





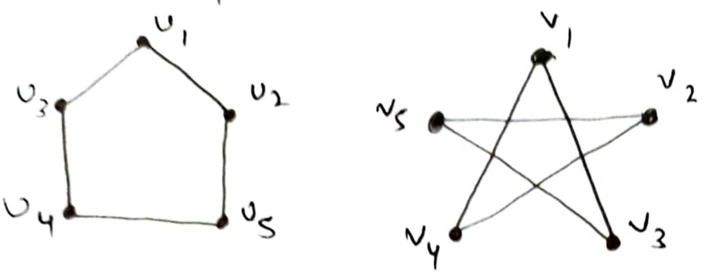
(Ans)
- Equal no. of vertices = 4
- Equal no. of edges = 3
- Degree

$1 \to 1$, $a = 1$
$2 \to 2$, $b = 2$
$3 \to 2$, $d = 2$
$4 \to 1$, $c = 1$

- correspondence

$1 \to a$
$2 \to b$
$3 \to d$
$4 \to c$

12 (a) Show that the graphs G and G' shown in the figure are isomorphic



Ans) 12 (a)

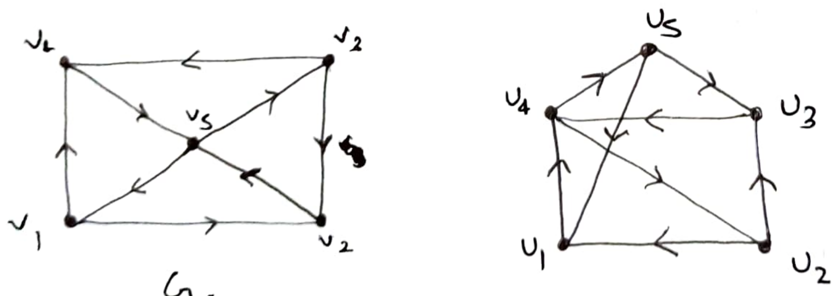
- Equal no. of vertices = 5
- Equal no. of edges = 5
- Degree of vertex

$u_1 = 2$	, $v_1 = 2$
$u_2 = 2$	, $v_2 = 2$
$u_5 = 2$	, $v_3 = 2$
$u_4 = 2$	, $v_4 = 2$
$u_3 = 2$	, $v_5 = 2$

- correspondence

$u_1 \rightarrow v_1$
$u_2 \rightarrow v_3$
$u_3 \rightarrow v_4$
$u_4 \rightarrow v_2$
$u_5 \rightarrow v_5$

12 (b) Show that the digraphs are isomorphic.



Ans)

- Equal No. of vertices = 5
- equal No. of edges = 8
- Degree of vertices

$v_1 = 3$	, $u_1 = 3$
$v_2 = 3$	, $u_2 = 3$
$v_3 = 3$	, $u_3 = 3$
$v_4 = 3$	, $u_4 = 4$
$v_5 = 4$	, $u_5 = 3$

- correspondence

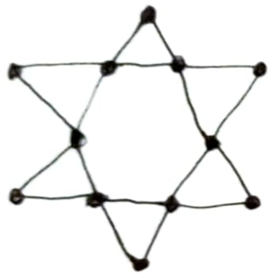
$v_1 \rightarrow u_2$
$v_2 \rightarrow u_3$
$v_3 \rightarrow u_5$
$v_4 \rightarrow u_1$
$v_5 \rightarrow u_4$

13 (a) Define with an example each

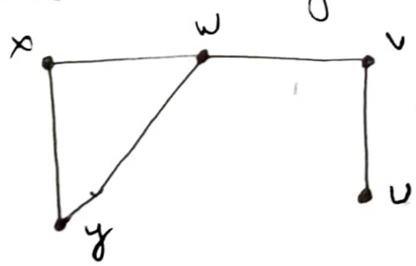
(i) Euler circuit (ii) Euler Path (iii) Euler Trail

13a] Ans) (i) Euler Circuit → A circuit in a graph is said to be an eulerian circuit if it traverses each edge in the graph once & only once

Eg



(ii) Eulerian Path → A path in a graph is said to be eulerian path if it traverses each edge in the graph once and only once. Ex :



(iii) Euler Trail → A trail than can traverse each edge only once. Ex.

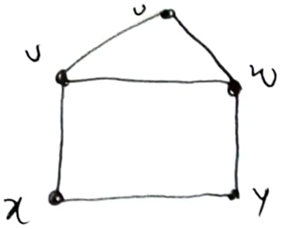
13 (b) Define with an example each

(i) A Hamiltonian path → A path which contain every vertex of a graph G exactly once is called Hamiltonian graph

Ex

(ii) Hamiltonian Circuit → A circuit that passes through each of the vertices in a group G exactly once except the starting vertex and end vertex is called Hamiltonian circuit.

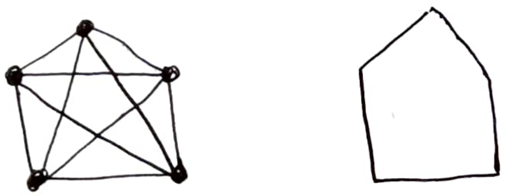
Ex :



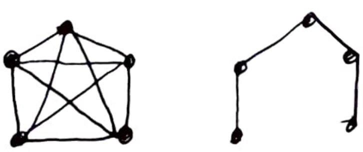
14 (a) Define with an example : (a) subgraph of a graph
(b) spanning sub graph.

Ans] (a) Subgraph of a graph → Let $G(V, E)$ be a graph. Let V' be a subset of V and let E' be a subset of E whose end point belong to V'. Then $G(V', E')$ is a graph and called a subgraph of $G(V, E)$.

Ex ·



(b) Spanning sub Graph → A spanning subgraph is a subgraph which contains all the vertices in G. A spanning subgraph need not contain all the edges in G.



14 (b) Show that the complement of a bipartite graph need not be a bipartite graph.

Ans] $K_{3,3}$

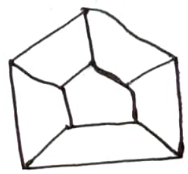
Therefore the complement of a bipartite graph is not bipartite.

(15) Which of the following graphs are Eulerian?

(i) The complete graph K_5.  , Yes its Eulerian

(ii) The complete bipartite graph $K_{2,3}$.  , No, its not eulerian

(iii) The graph of the Octahedron.  , yes its eulerian

(iv) The peterson graph.  Its not eulerian

(16) Prove that a simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

[Ans] Let $n_1, n_2, \dots n_k$ be the no. of vertices with k components of G.

$\sum n_i = n_1 + n_2 + \dots n_k = n$

Squaring on both sides

$[(n_1-1) + (n_2-1) + (n_3-1) + \dots (n_k-1)]^2 = (n-k)^2$

$(n_1-1)^2 + (n_2-1)^2 + \dots (n_k-1)^2 + s = (n-k)^2$

where s is the sum of ~~these~~ terms which is in form of $2(n_i-1)(n_j-1)$ for $i = 1, 2, 3 \dots k$, $j = 1, 2, 3 \dots k$, $i \neq j$

$(n_1-1)^2 + (n_2-1)^2 + (n_3-1)^2 + \dots + (n_k-1)^2 \leq (n-k)^2$

since $s > 0$

$(n_1^2 + n_2^2 + n_3^2 \dots n_k^2) + k - 2n_1 - 2n_2 - 2n_3 \dots 2n_k \leq (n-k)^2$

$(n_1^2 + n_2^2 + \dots + n_k^2) + k - 2n \leq (n-k)^2$

$$\sum_{i=1}^{k} n_i^2 \le p^2 + k^2 - 2kp + 2p - k$$

$$\sum_{i=1}^{k} n_i^2 \le p^2 - (k-1)(2p-k)$$

Since the graph G is simple each of its components are also simple. Hence the maximum no. of edges in each component is $\frac{n_i(n_i - 1)}{2}$

$${}^{n}C_2 = \frac{n!}{(n-2)!\,2!} = \frac{(n)(n-1)}{2}$$

$${}^{n_i}C_2 = \frac{n_i(n_i - 1)}{2}$$

$\sum_{i=1}^{k} \frac{(n_i)(n_i - 1)}{2}$ edge in each component

$$\to \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \quad \frac{1}{2}\sum_{i=1}^{k} n p_i^2 - \frac{1}{2} n$$

$$\frac{1}{2}\sum^{n} p_i^2 - \frac{1}{2} n \le \frac{1}{2}\left[n^2 - (k-1)(2n-k)\right] - \frac{1}{2} n$$

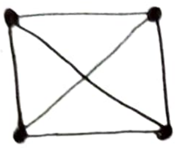
$$\le \frac{1}{2} n^2 - \frac{(k-1)(2n-k)}{2} - \frac{1}{2}$$

$$\le \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right]$$

$$\boxed{\le \frac{1}{2}(n-k)(n-k+1)}$$

17 ⓐ (i) Give an example of a graph which has a Hamiltonian circuit but not an euler circuit.

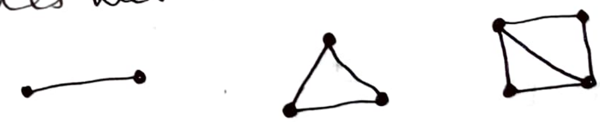
(ii) Given an example of a graph which has an Euler circuit but not a hamiltonian circuit.

Ans] (i)  (ii) 

17 ⓑ Show that K_n has a hamilton circuit whenever $n \geq 3$.

[Ans] Let $v_1 - v_2 \cdots v_n$ be any way of listing the vertices in order. Then $v_1 - v_2 - v_3 \cdots v_n - v_1$ is a hamiltonian circuit covering up all the ~~int~~ edges in the periphery and the vertex.

18 ⓐ Define graph coloring of a graph with an example.

Ans] Painting all the vertices of a graph with colors such ~~that~~ no two adjacent vertices have the same color is called coloring of graph. Ex: 

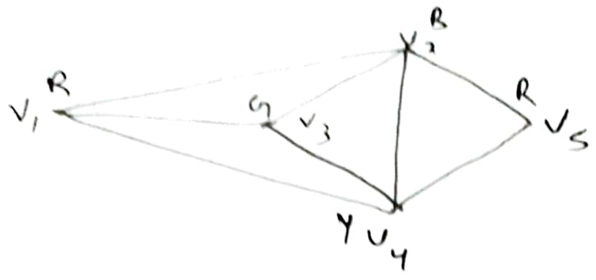
18 ⓑ Define chromatic number of a graph with an example.

Ans] The least number of colors required for coloring of a graph G is called its chromatic number.

$\rightarrow 2$,

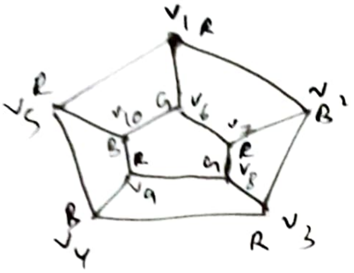

$\rightarrow 3$

19(a) Find the chromatic number of the graph shown below



Ans) 4

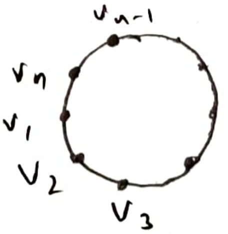
19(b) Find the chromatic number of the graph below.



Ans) 3

20(a) Prove that a graph of order $(n \geq 2)$ consisting of a single cycle is 2-chromatic if n is even and 3-chromatic if n is odd.

Ans) Consider the graph



Obviously the graph cannot be properly coloured with a single colour. Assign two colors alternatively to the vertices, starting with v_1. That is the odd vertices, v_1, v_3, v_5 etc will have a color ~~of~~ α and even vertices v_2, v_4, v_6 etc will have a different color β.

Suppose n is even. Then the vertex v_n is an even vertex and therefore will have the color β and the graph gets properly coloured. Therefore the graph is 2-chromatic.

Suppose n is odd. Then the vertex v_n is an odd vertex and therefore will have the colour α and the graph is not properly coloured. To make it properly coloured, it is enough if v_n is assigned a third color γ. Thus in this case the graph is 3-chromatic.

20 (b) Prove that every connected simple planar graph G is 6-colorable.

Ans] We prove the result by induction on the number of vertices. For $v \le 6$, we can suppose we have a graph such that $v \le 6$. For $v \le 6$, we can give each vertex a different color and use ≤ 6 colors

Now assume that any simple planar graph on $v = n$ vertices can be properly ~~to~~ colored with six colors.

Let G be any simple planar graph on $v = n+1$ vertices. From our lemma above, we know that G must have some vertex w of degree ≤ 5. Remove w from G to form G'. G' has $v =$ n vertices and we may apply our induction hypothesis to know it can be properly colored in 6 colors. Properly color G' with ≤ 6 colors. Now, we can think of this as coloring all of G except w. But since w has degree at most ~~to~~ 5, one of the 6 colors will not be used for any of the neighbors of w and we can finish coloring G.