MODULE-4: RELATIONS

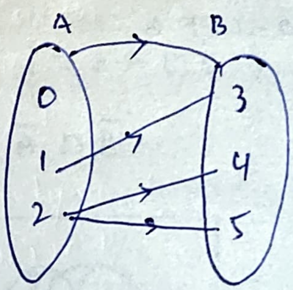
- Let A and B be any two sets then a subset of $A \times B$ is called a binary relation or just a relation from A to B. Thus R is a relation from A to B then R is a set of ordered pairs (a, b) where $a \in A$ and $b \in B$ then R is a relation from A to B. Also if $(a, b) \in R$ then we say that "a is related b" or aRb.

Ex: $A = \{0, 1, 2\}$ $B = \{3, 4, 5\}$

Let $R = \{(1, 3), (2, 4), (2, 5)\}$

$R \subset A \times B \Rightarrow R$ is a relation from A to B

This can be represented using arrow diagram



NOTE:

If A is a set with m elements and B is a set with n elements then $A \times B$ will have mn elements.

$\therefore$ The power set or set of all students of $A \times B$ will have 2^{mn} elements. Hence there are 2^{mn} relations from A and B.

Ex: If $|A| = 3$ & $|B| = 2$ then $|A \times B| = (3)(2) = 6$

then there exists 2^6 relations from A to B.

Zero matrix:

Consider the sets $A = \{a_1, a_2 \cdots a_m\}$ and $B = \{b_1, b_2 \cdots b_n\}$

Let R be a relation from A to B so that $R \subseteq A \times B$. Then let us put $m_{ij} = (a_i, b_j)$ where $a_i \in A$, $b_j \in B$ and $1 \le i \le m$, $1 \le j \le n$ and assign the values 1 or 0 according to the following rule

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

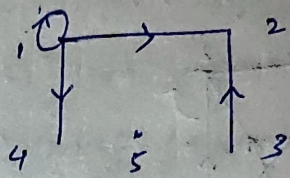
The $m \times n$ matrix formed by these m_{ij}'s is called Relation matrix for R or zero-one matrix denoted by M_R or $M(R)$

⇒ Digraph of a relation:

Let R be a relation on a finite set. Then R can be represented by pictorically using bullets and arrows as given below.

Ex: $A = \{1, 2, 3, 4, 5\}$ and $R = \{(1,1), (1,2), (1,4), (3,2)\}$

The digraph is as shown below.



(Q) Let $A = \{1, 2\}$, $B = \{p, q, r, s\}$ and let R from A to B be is defined by $R = \{(1,q), (1,r), (2,p), (2,s)\}$. Write the matrix of R.

Sol: M_R in $M(R) = \begin{array}{c|cccc} & p & q & r & s \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{array}$

(Q) Let $A = \{1, 2, 3, 4\}$ and R be the relation on A defined by xRy if and only if x divides y

(i) Write R as a set of order pairs

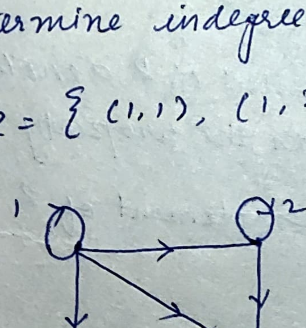
(ii) Draw the digraph of R

(iii) Write the matrix R.

(iv) Determine indegree and outdegrees of the vertices in the digraph

Sol (i) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

(ii)



(iii) $M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}$

(iv) $id(1) = 1$ $\quad$ $od(1) = 4$.

$id(2) = 2$ $\quad$ $od(2) = 2$

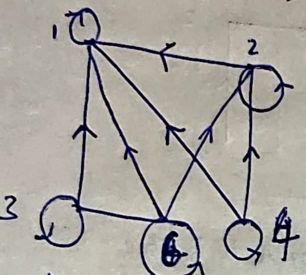
$id(3) = 2$ $\quad$ $od(3) = 1$

$id(4) = 3$ $\quad$ $od(4) = 1$

(Q) Let $A = \{1, 2, 3, 4, 6\}$ and R is a relation on A defined by ~~ARB~~ aRb if and only a is a multible of b.
Represent the relation R as a matrix and draw its digraph.

Sol

$$R = \{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)\}$$



(Q) <u>Operations on Relations:</u>

- ~~Given R_r~~ Union:
  Let R_1 and R_2 be 2 relations from set A to B. The union of R_1 and R_2 denoted by $R_1 \cup R_2$ is defined as a relation from A to B with property that $(a, b) \in R_1 \cup R_2$ if and only if $(a, b) \in R_1$ or $(a, b) \in R_2$

- Intersection of relation:
  The intersection R_1 and R_2 denoted by $R_1 \cap R_2$ is defined as a relation from A to B with property that $(a, b) \in R_1 \cap R_2$ if and only if $(a, b) \in R_1$ and $(a, b) \in R_2$

- Complement of Relation:
  Given a relation R from A to B the complement of R denoted by $\bar{R}$ is defined as a relation from A to B with the property that $(a, b) \in \bar{R}$ if $(a, b) \notin R$ or $\bar{R}$ is the complement of R in the universal set $A \times B$

- Converse of a Relation
  Given a relation R from A to B the converse of R is denoted by R^c is defined as a relation from B to A with the property that $(a, b) \in R^c$ if and only if $(b, a) \in R$

(Q) Determine the relation R from a set A to B as described by the following matrix.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} p & q & r \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Sol

Let $A = \{a, b, c, d\}$ $B = \{p, q, r\}$

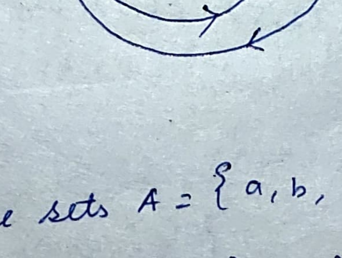
$R = \{(a, p), (a, r), (b, p), (b, q), (c, r), (d, p)\}$

(Q) Let $A = \{u, v, x, y, z\}$ and R to be a relation on R where matrix is as given below. Determine R and the digraph of the matrix.

$$\begin{array}{c} \\ u \\ v \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} u & v & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Sol $R = \{(u, v), (u, x), (v, u), (v, x), (x, u), (x, v), (x, z),$
$(y, u), (y, z), (z, x), (z, y)\}$

Digraph:



(Q) Consider the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ and $R = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$ and $S = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$ from A to B. Determine $\bar{R}$, $\bar{S}$, $R \cup S$, $R \cap S$, R^c, S^c. Write their corresponding matrix representation

Sol: $A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2),$
$(c,3)\}$

$\bar{R} = (A \times B) - R = \{(a,2), (a,3), (b,2), (b,3), (c,1)\}$

$\bar{S} = (A \times B) - S = \{(a,3), (b,3), (c,1), (c,2), (c,3)\}$

$R \cup S = \{(a,1), (b,1), (c,2), (c,3), (a,2), (b,2)\}$

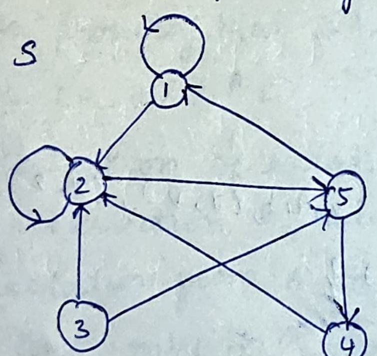
$R \cap S = \{(a,1), (b,1)\}$

$R^c = \{(1,a), (1,b), (2,c), (3,c)\}$

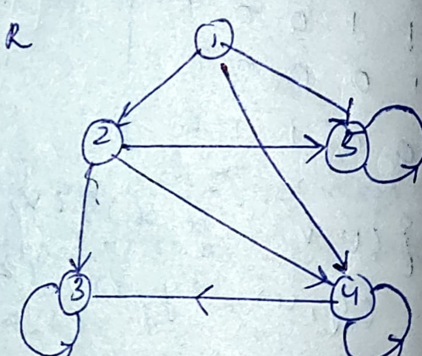
$S^c = \{(1,a), (2,a), (1,b), (2,b)\}$

(Q) Let $A = \{1,2,3,4,5\}$ and R and S be relations on A whose correspond-ing digraphs are given below. Find $\bar{R}$, R^c, $R \cup S$, $R \cap S$. Hence write the corresponding matrices.

~~R~~ S



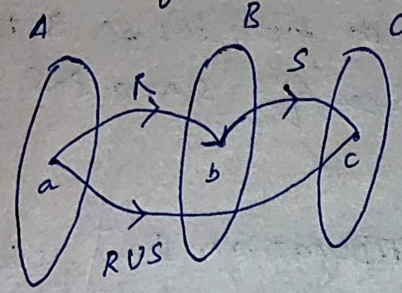
R



Sol: $R = \{(1,2), (1,5), (1,4), (2,5), (2,3), (2,4), (3,3),$
$(4,3), (4,4), (5,5)\}$

$S = \{(1,1), (1,2), (2,2), (2,5), (3,2), (3,5), (4,2),$
$(5,4), (5,1)\}$

⇒ Composition of relations :



$R \circ S = \{ (a,c) \mid a \in A,\ c \in C$ and there exists $b \in B$ with $(a,b) \in R,\ (b,c) \in S \}$

Let R be a relation from set A to B and S be the relation from B to C then the product or the composition of R and S from set A to set C is denoted by $R \circ S$ and is defined as

$$R \circ S = \{ (a,c) \mid a \in A,\ c \in C \text{ and there exists } b \in B \text{ with } (a,b) \in R,\ (b,c) \in S \}$$

(Q) Let $A = \{a, b, c\}$ and R, S be relations on A whose matrices are given below. Find the composite relations $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$ and write their digraphs

$$M(R) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M(S) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol :

$R = \{ (a,a), (a,c), (b,a), (b,b), (b,c), (c,b) \}$

$S = \{ (a,a), (b,b), (b,c), (c,a), (c,c) \}$

$R \circ S = \{ (a,a), (a,c), (b,a), (b,b), (b,c), (c,b), (c,c) \}$

$S \circ R = \{ (a,a), (a,c), (b,a), (b,b), (b,c), (c,a), (c,c), (c,b) \}$

$R \circ R = \{ (a,a), (a,c), (a,b), (b,a), (b,c), (b,b), (c,a), (c,b), (c,c) \}$

$S \circ S = \{ (b,c), (b,a), (b,c), (c,a) \}$

For ROR, SOS → can't pair with the same elements

Theorem: Let R be a relation from a set $A = \{a_1, a_2, \dots a_m\}$ to a set $B = \{b_1, b_2, \dots b_n\}$ and S be a relation from the set B to a set $C = \{c_1, c_2, \dots c_p\}$. Then the matrices of R, S and RoS satisfy the identity $\boxed{M(R) \times M(S) = M(R o S)}$.

(Here the symbol × denotes matrix multiplication. Since relation matrices contain 0 and 1 only; while multiplying two relations, the rule of matrix multiplication is employed with the stipulation that $\boxed{1+1=1}$)

Corollary: If R is a relation on a set $A = \{a_1, a_2, \dots a_m\}$ then $M(R^2) = (M(R))^2$

Theorem: $Ro(SoT) = (RoS)oT$

(Q) For the relations R_1 and R_2 defined on the sets $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, $C = \{5, 6, 7\}$

(1) R_1 and R_2 $R_1 = \{(1, x), (2, x), (3, y), (3, z)\}$

$R_2 = \{(w, 5), (x, 6)\}$. Find $M(R_1)$, $M(R_2)$ and $M(R_1 o R_2)$

Also verify that $M(R_1 o R_2) = M(R_1) \cdot M(R_2)$

Sol:
$R_1 o R_2 = \{(1, 6), (2, 6)\}$

$$M(R_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad M(R_2) = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(R_1 o R_2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

~~To ver~~

To verify $M(R_1 o R_2) = M(R_1) \cdot M(R_2)$

$$M(R_1)\cdot M(R_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M(R_1 o R$$

(Q) If $A = \{1, 2, 3, 4\}$ and R, S are relations on A defined by

$R = \{(1,2), (1,3), (2,4), (4,4)\}$

$S = \{(1,1), (1,2), (1,3), (1,4), (2,3), (2,4)\}$.

Find ROS, SOR, R^2, S^2 write their matrices.

($R^2 = ROR$ and $S^2 = SOS$)

(Q) Let $A = \{a, b, c\}$ and R, S be the relations on A whose matrices are given as $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Find ROS, SOR, SOS and their matrices.

Sol: $R = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,b)\}$

$S = \{(a,a), (b,b), (b,c), (c,a), (c,c)\}$

⇒ Properties of Relations:

1. A relation R on A is said to be reflexive R on A if $(a, a) \in R$ for all $a \in A$.

2. A relation R on A is said to be irreflexive relation if $(a,a) \notin R$ for any $a \in A$

3. A relation R on A is said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $(a,b) \in R$.

4. Asymmetric or Not symmetric : R is not symmetric if there exists $a, b \in A$ such that $(a,b) \in R$ but $(b,a) \notin R$

5. Antisymmetric relation: A relation R on set A is said to be antisymmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a = b$

Ex: The relation '$\leq$' or '$\geq$' are antisymmetric because

$a \leq b$ and $b \leq a \Rightarrow a = b$

$a \geq b$ and $b \geq a \Rightarrow a = b$

6. Transitive relation:

A relation R on set A is said to be transitive relation if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ for all $a, b, c \in A$

NOTE: 1. Equivalence Relation → If a relation satisfies
(i) reflexive (ii) symmetric (iii) transitive.

2. Assymmetric (Not symmetric) and anti symmetric relations are not one and the same.

3. A relation can be both symmetric and antisymmetric

4. A relation can be neither symmetric nor anti-symmetric

5. In the digraph of an antisymmetric relation for two different vertices A & B there cannot be a bidirectional edge between A & B

(Q) For a fixed integer $n > 1$ Prove that a relation "congruent modulo n" is an equivalence relation on the set of all integers Z

Sol For $a, b \in Z$ we say that "a is congruent to b modulo n" i.e $a \equiv b \pmod{n}$ if $a-b$ is a multiple of n or $a-b = kn$ for some $k \in Z$ ~~or~~ $n / a-b$ (n divides $a-b$)

(1) To prove congruent modulo n is reflexive

consider $a \equiv a \pmod{n}$ then $a - a = 0$ can be multiple of n

or $a - b = 0n$ or $n / a-a$ or $n / 0$ $\left[\frac{0}{n}\right]$

$\therefore$ R is reflexive

(2) To prove R symmetric

Consider $aRb \Rightarrow a \equiv b \pmod{n} \Rightarrow n / a-b \Rightarrow n / b-a \Rightarrow b \equiv a \pmod{n}$

$\therefore$ R is symmetric

(3) To prove R transitive

Let aRb and bRc then $a \equiv b \pmod n$ and $b \equiv c \pmod n$

$\Rightarrow n / a-b$ and $n / b-c$

$\Rightarrow n / (a-b)+(b-c)$

$\Rightarrow n / ac \Rightarrow a \equiv c \pmod n$

$\therefore aRc \Rightarrow R$ is transitive

Hence R is an Equivalance relation

(Q) On the set of all integers Z defined by the relation R by aRb if $ab >$
Show that R is an equivalence relation.

Sol. $\forall a \in Z$, aRa because $\forall a > 0$ then $a \cdot a > 0$ and if $a < 0$ then $a \cdot a > 0$. R is reflexive.

- $\forall a, b \in Z$, if aRb then $ab > 0 \Rightarrow ba > 0 \Rightarrow bRa$.
  $\therefore$ R is symmetric

- $\forall a, b, c \in Z$ if aRb, bRc then $ab > 0$ and $bc > 0$

  $\Rightarrow a, b, c > 0$ or $a, b, c < 0$

  $\Rightarrow ac > 0$ or $ac > 0$

  $\Rightarrow aRc \Rightarrow R$ is transitive

Therefore R is an equivalence relation.

(Q) Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$ be a relation on A. Verify that R is an equivalence relation.

Sol Clearly R is reflexive because aRa $\forall a \in A$ and R is symmetric because $(1,2), (3,4) \in R$ then $(2,1), (4,3) \in R$

(Q) Let $A = \{1, 2, 3\}$. Determine the nature of the following relations on A

(i) $R_1 = \{(1,2), (2,1), (1,3), (3,1)\}$

Sol: R is symmetric and irreflexive but neither reflexive nor transitive

(ii) $R_2 = \{(1,1), (2,2), (3,3), (2,3)\}$

R is reflexive and transitive but not symmetric

(iii) $R_3 = \{(1,1), (2,2), (3,3)\}$

R is reflexive and symmetric

(iv) $R_4 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

R is reflexive and symmetric

⇒ Partial orders

A relation R on a set A is said to be partial ordering relation or a partial order on A then

(i) R is reflexive

(ii) R is antisymmetric

(iii) R is transitive on A

⇒ Posets: A set A with the partial order R defined on it is called a partially ordered set or ordered set or Poset denoted by a pair (A, R)

Eg: "$\leq$", "$\geq$", divisibility relation on Z^+ are partial order relations

Property: The digraph of partially order has no cycle of length greater than 1

⇒ Hasse Diagram: Since a partially order is reflexive at every vertex in the digraph there would be a cycle of length 1

- While drawing digraph we need not to exhibit such cycles explicitly. They will be automatically understood.
- If in the digraph of a partial order there is an edge from vertex a to b and there is an edge from vertex b to c then there should be an edge from vertex a to c (because it is transitive)

As such we need not exhibit an edge from a to c explicitly ... be understood.

- To simplify the format of digraph of partial order we represent the vertices by dots and draw the digraph in such a way that all edges point upward. With this convention we need not put arrows in the edges.
- The digraph of a partial order drawn by adopting the conventions indicated in the above paragraph is called a poset diagram or Hasse Diagram for the partial order.

(Q) Draw the Hasse diagram of the positive divisors of 36.

Sol: $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

The relation R of divisibility i.e. aRb "if a divides b" (a/b) is a partial order on this set. The Hasse diagram for this partial order is required.

1 is related to 1, 2, 3, 4 36

2 " 2, 4, 6, 12, 18, 36

3 " 3, 6, 9, 12, 18, 36

4 " 4, 12, 18, 36

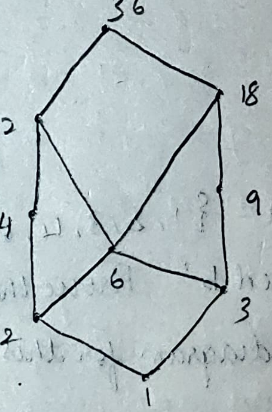
6 " 6, 12, 18, 36

9 " 9, 18, 36

12 " 12, 36

18 " 18, 36

36 " 36.



Hasse diagram of D_{36}

(Q) If R is a relation on the set $A = \{1, 2, 3, 4\}$ defined by xRy if $x|y$. Prove that (A, R) is a poset and draw Hasse diagram.

Sol: $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

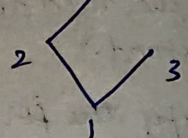
To show that R is a partial order on A

(i) It is reflexive (i.e xRx i.e $x|x$)

(ii) It is antisymmetric (xRy but y is not related to x ⟹ $x|y$ but y does not divide x

(iii) It is transitive ($x|y$ and $y|z \Rightarrow x/z$)

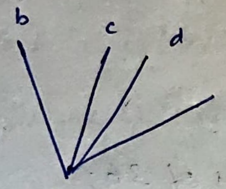
From the relation, (A, R) is a poset.



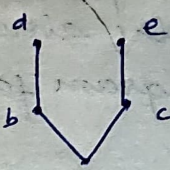
Hasse diagram

(Q) Find the matrix of the partial order relation where Hasse diagram is given

(1)



(2)



$R = \{ (a,b), (a,c), (a,d), (a,e), (a,a) \}$

i.e $= \{ (a,a), (a,b), (a,c), (a,d), (a,e),$
$(b,b), (c,c), (d,d), (e,e) \}$

$R = \{ (a,a), (a,b), (a,c),$
$(a,d), (a,e), (b,b), (b,d),$
$(c,c), (c,e), (d,d),$
$(e,e) \}$

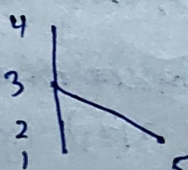
(Q) ① Let $A = \{1, 2, 3, 4, 6, 12\}$. Define relation R by aRb iff 'a divided b'. Prove that a is partial order on A. Draw the Hasse diagram for this.

(Q) Draw the Hasse diagram of the relation R on $A = \{1, 2, 3, 4, 5\}$

① where matrix is $M(R) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

③

(Q) Determine the matrix of the partial order whose Hasse diagram is given below.



3-soln $R = \{ (1,1$
$(2$

$M(R)$

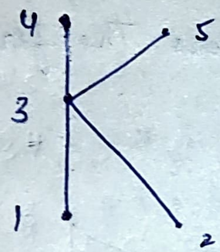
③-Soln $R = \{(1,$
$(3.$

(Q) Find x an
(i) $(2x, x$
(ii) $(4-2,$
(iii) $(2x-$
(iv) $(x+2$
(v) (x, y)
(vi) $(x, y$

3- soln $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (5,3), (5,4)\}$

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

③- Son $R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4) (5,5)\}$



(Q) Find x and y in each of the following

(i) $(2x, x+y) = (6, 1)$

(ii) $(y-2, 2x+1) = (x-1, y+2)$

(iii) $(2x-3, 3y+1) = (5, 7)$

(iv) $(x+2, 4) = (5, 2x+y)$

(v) $(x, y) = (x^2, y^2)$

(vi) $(x, y) = (y^2, x^2)$

MODULE 1: NUMBER THEORY

1. Establish the validity of the number pattern.

(i) $1 \cdot 9 + 2 = 11$

$12 \cdot 9 + 3 = 111$

$123 \cdot 9 + 4 = 1111$

$1234 \cdot 9 + 5 = 11111$

$12345 \cdot 9 + 6 = 111111$

$123456 \cdot 9 + 7 = 1111111$

Sol Now let us study the pattern.

The LHS of each eqn in a sum of two numbers. The first number is a product of the no. $123 \cdot n$ and 9.

The value of n in the first eqn is 1

" second " 2

" third " 3

Now let us look at the second addends on the LHS: 2, 3, 4, 5 ... It is an increasing sequence begining with 2, so that second addends in the ~~next eqn~~ n^{th} eqn is $n+1$ ones

Thus a pattern emerges and we are ready to state it explicitly: The first no in the n^{th} line is $123 \ldots n$ the second no is always 9. The second addend is $n+1$ and the RHS is $n+1$ ones

So the next two lines are

$1234567 \cdot 9 + 8 = 11111111$

$12345678 \cdot 9 + 9 = 111111111$

Now the validity of the no pattern we would like to prove that

$123 \ldots n \times 9 + (n+1) = 1 + 11 \ldots 1$ ($n+1$ ones)

$LHS \Rightarrow 123 \ldots n \times 9 + (n+1)$

$= 9\left[1 \times (10)^{n-1} + 2(10^{n-2}) + 3(10^{n-3}) + \ldots + n(10^0)\right] + (n+1)$

$= (10-1)\left[1 \times (10)^{n-1} + 2(10^{n-2}) + 3(10^{n-3}) + \ldots + n(10^0)\right] + (n+1)$

$= 1(10^n) + 2(10^{n-1}) + \ldots + n(10^1) - \left[1(10)^{n-1} + 2(10^{n-2}) + \ldots + n(10^0)\right]$

$+ (n+1)$

$= 1(10^n) + 10^{n-1} + 10^{n-2} + \dots + 10 - n + n + 1$

$= 10^n + 10^{n-1} + \dots + 10 + 1$

$= \underbrace{1111\dots1}_{n+1 \text{ terms}}$

$=$ RHS

(Q) Add two more rows in the following pattern conjecture a formula for the n^{th} row and prove it.

$9 \cdot 9 + 7 = 88$

$98 \cdot 9 + 6 = 888$

$987 \cdot 9 + 5 = 8888$

$9876 \cdot 9 + 4 = 88888$

$98765 \cdot 9 + 3 = 888888$

Sol The next two lines / rows

$987654 \cdot 9 + 2 = 8888888$

$9876543 \cdot 9 + 1 = 88888888$

The general pattern is like $1 \le n \le 8$

$987\dots(10 \cdot n) \cdot 9 + (8-n) = \underbrace{88\dots8}_{(n+1) \text{ eights}}$

LHS: $987\dots(10-n) \cdot 9 + (8-n)$

$= 9[987\dots(10-n)] + (8-n)$

$= (10-1)[9(10^{n-1}) + 8(10^{n-2}) + \dots (11-n)(10^1) + (10-n)(10^0)] + (8-n)$

$- \cancel{[9(10^{n-1}) + 8(10^{n-2}) + \dots + (11-n)(10^1) + (10-n)(10^0)]}$

$= [9(10^n) + 8(10^{n-1}) + \dots (11-n) 10^2 + (10-n) 10^1)] -$

$[9(10^{n-1}) + 8(10^{n-2}) + \dots + (11-n)(10^1) + (10-n)(10^0)] + (8-n)$

$= 9(10^n) - [10^{n-1} + 10^{n-2} + \dots + 10] - (10-n) + (8-n)$

$= 9(10^n) - [10^{n-1} + 10^{n-2} + \dots + 10] - 1 - 1$

$= 9(10^n) - [10^{n-1} + 10^{n-2} + \dots + 10 + 1] - 1$

$= (10-1)10^n - [10^{n-1} + 10^{n-2} + \dots + 10 + 1] - 1$

$= 10 \cdot 10^n - [10^n + 10^{n-1} + 10^{n-2} + \dots + 10 + 1] - 1$

$\left(\text{Since } \sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1} \quad (r \neq 1) \text{ Here } r = 10 \right)$

$$LHS = 10^{n+1} - \left[\frac{10^{n+1}-1}{9} \right] - 1 = \frac{9 \cdot 10^{n+1} - 10^{n+1} + 1 - 9}{9}$$

$$= \frac{8 \cdot 10^{n+1} - 8}{9} = \frac{8 \cdot (10^{n+1} - 1)}{9}$$

But $10^{n+1} - 1 = 999\ldots9$ $(n+1 \cdot 9\text{'s})$

$$LHS = \frac{8(99 - 9)}{9} = 8(111\ldots1)$$

$$= 888\ldots8 \quad (n+1 \text{ } 8\text{'s})$$

$$= RHS$$

⇒ PRIME AND COMPOSITE NO'S:

⇒ 1 is neither prime nor composite.

⇒ A positive integer > 1 is a prime if its only +ve factors are 1 and itself.

⇒ A positive integer > 1 that is not a prime is composite.

Theorem:

Every integer $n \geq 2$ has a prime factor.

Proof: By induction

The given statement is true when $n = 2$, now assume that the result is true for every +ve integer $n \leq k$, where $k \geq 2$ consider the integer $k+1$.

Case 1: If $k+1$ is a prime, then $k+1$ is a prime factor of itself.

Case 2: If $k+1$ is not a prime, $k+1$ must be composite So it must have a factor $d \leq k$. Then by induction hypothesis, d has a prime factor p.

Then $\Rightarrow d | k+1$ and $P | d$ or

$P | d$ and $d | k+1$

By transivity property for divisibility $[a/b, b/c \Rightarrow a/c]$

$P | k+1$ $\Rightarrow k+1$ has a prime factor p.

$\therefore$ Every integer ≥ 2 has a prime factor.

(Q) Determine whether 1601 is a prime no.

Sol: First list all prime $\leq \sqrt{1601}$

$\sqrt{1601} = 40$

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37

Since none of them is a ~~pr~~ factor of 1601 ~~means~~

1601 is the only prime factor

$\therefore$ 1601 is prime no.

NOTE :

Let $P_1, P_2, \ldots P_t$ be the primes $\leq \sqrt{n}$ then

$$\pi(n) = n-1+\pi(\sqrt{n}) - \sum_i \left|\frac{n}{P_i}\right| + \sum_{i<j} \left|\frac{n}{P_{ij}}\right| - \sum_{i<j<k} \left[\frac{n}{P_i P_j P_k}\right] + \cdots + (-1)^t \left[\frac{n}{P_1 P_2 \cdots P_t}\right]$$

(Q) Find the no. of primes ≤ 100 by using the above result.

Sol Here $n = 100$ $\pi(\sqrt{n}) = \pi(\sqrt{100}) = \pi(10) =$ no. of primes $\le 10 = 4$

which are 2, 3, 5, 7 we call $p_1 = 2$, $P_2 = 3$, $P_3 = 5$, $P_4 = 7$

$$\therefore \pi(100) = 100 - 1 + 4 - \left[\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{7} \right\rfloor\right] + \left[\left\lfloor \frac{100}{2\cdot3} \right\rfloor + \left\lfloor \frac{100}{2\cdot5} \right\rfloor + \left\lfloor \frac{100}{2\cdot7} \right\rfloor + \left\lfloor \frac{100}{3\cdot5} \right\rfloor + \left\lfloor \frac{100}{3\cdot7} \right\rfloor + \left\lfloor \frac{100}{5\cdot7} \right\rfloor\right] - \left[\left\lfloor \frac{100}{2\cdot3\cdot5} \right\rfloor + \left\lfloor \frac{100}{2\cdot3\cdot7} \right\rfloor + \left\lfloor \frac{100}{3\cdot5\cdot7} \right\rfloor\right] + \left\lfloor \frac{100}{2\cdot3\cdot5\cdot7} \right\rfloor$$

$$= 103 - [50 + 33 + 20 + 14] + [16 + 10 + 7 + 6 + 4 + 2] - [3 + 2 + 1 + 0] + 0$$

$$= 25.$$

NOTE: For every +ve integer n there are n consecutive integers that are composite no.

$(n+1)! + 2$, $(n+1)! + 3$ .... $(n+1)! + (n+1)$

(Q) Find 6 consecutive integers that are composite.

Sol $n = 6$ (For every +ve integer n, there are n consecutive integ

$(6+1)! = 5040$

$\therefore$ 5042, 5043, 5044, 5045, 5046, 5047 are 6 consecutive integers.

(Q) Find the no. of +ve integers less than or equal to 2076 which are divisible by neither 4 nor 5.

Sol $A = \{x \in N \mid x \le 2076$ and divisible by 4 $\}$

$B = \{x \in N \mid x \le 2076$ and divisible by 5 $\}$

No. of +ve integers ≤ 2076 divisible by 4 or 5

$= |A \cup B| = |A| + |B| - |A \cap B|$

$$= \left\lfloor \frac{2076}{4} \right\rfloor + \left\lfloor \frac{2076}{5} \right\rfloor - \left\lfloor \frac{2076}{4\cdot5} \right\rfloor$$

$= 519 + 415 - 103$

$= 831$

Among first 2076 +ve integers, there are $2076 - 831 = 1245$ integers are not divisible by 4 or 5.

(Q) Find the no. of +ve integers ≤ 3000 which are divisible by 3, 5 or 7

Sol $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

$$= \left\lfloor \frac{3000}{3} \right\rfloor + \left\lfloor \frac{3000}{5} \right\rfloor + \left\lfloor \frac{3000}{7} \right\rfloor - \left\lfloor \frac{3000}{3 \cdot 5} \right\rfloor - \left\lfloor \frac{3000}{3 \cdot 7} \right\rfloor - \left\lfloor \frac{3000}{5 \cdot 7} \right\rfloor$$

$$+ \left\lfloor \frac{3000}{3 \cdot 5 \cdot 7} \right\rfloor$$

$= 1629$

(Q) Express 10110_{two} in base ~~Two~~ Ten.

Sol $10110_2 = (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (1 \times 2) + (0 \times 2^0)$

$= 16 + 4 + 2 \quad = 22$

NOTE: A, B, C ..... to represent the digits ten, eleven, twelve .... resp.

Ex. Base - 16

0 1 2 3 4 5 6 7 8 9 A B C D E F

(Q) Express $(3ABC)_{16}$ in base ten

Sol $(3ABC)_{16} = 3 \times 16^3 + 10 \times 16^2 + 11 \times 16^1 + 12 \times 16^0$

$= 15036$

(Q) Express 3014 in base eight

Sol The largest power of 8 that is contained in 3014 is $8^3 = 512$

Apply division algorithm with 3014 as dividend and 512 as divisor.

$3014 = 5 \cdot 512 + 454$

Now look at 454. it lies b/w 64 and 512

The largest power of 8 within 454 is 8^2.

$\therefore 454 = 7 \times 64 + 6$

$6 = 6 \times 8^0 + 0 \times 8^1$

$3014 = 5(512) + 7(64) + 6$

$= 5(8^3) + 7(8^2) + 0 \times 8^1 + 6 \times 8^0$

$= (5706)_8$

Note: Base 16 is called Hexa Decimal system

(Q) Represent 15036 in the hexa decimal system that is in base 16.

Sol By division algorithm

$15036 = 939 \cdot 16 + \underline{12}$

$939 = 58 \cdot 16 + \underline{4}$

$58 = 3 \cdot 16 + \underline{10}$

$3 = 0 \cdot 16 + 3$

$\therefore 3(10)(11)(12) = 3ABC = 3 \times 16^3 + 10 \times 16^2 + 11 \times 16^1 + 12 \times 16^0$

$= 12288 + 2560 + 176 + 12$

$= (15036)_{10}$

(Q) Find the primes such that the digits in their decimal values alternate b/w 0s and is begining and ending with 1

Sol Ex: 1010101

Suppose N is a prime of the disired form and it contains n ones.

Then $N = 10^{n-2} + 10^{2n-4} + \cdots + 10^2 + 1$

Since,

$$\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r-1}, \quad r \neq 1$$

Here $r = 10^2$

then $N = \dfrac{10^{2n} - 1}{10^2 - 1} = \dfrac{(10^n - 1)(10^n + 1)}{99}$

Ex $n = 4$

$N = 1010101$

$= 1 \times 10^6 + 0 \times 10^5 + 1 \times 10^4 + 0 \times 10^3 + 1 \times 10^2 + 0 \times 10^1 + 1 \times 10^0$

$= 10^6 + 0 + 10^4 + 0 + 10^2 + 1$

If $n = 2$ then $N = \dfrac{(10^2 - 1)(10^2 + 1)}{99}$

$= \dfrac{99(101)}{99} = 101$ is a prime no.

$n>2$, then $10^n - 1799$ and $10^n + 1799$, then N has non-trivial factors, So N is composite. Thus 101 is the only prime with the desired properties.