choices for x in (a_m, x). Therefore, the total number of possible choices for x (in all of the pairs in f) is

$$n \times (n-1) \times (n-2) \times \cdots \times \{n-(m-1)\} = \frac{n!}{(n-m)!}.$$

Thus, if $m \leq n$, there are $\{n!/(n-m)!\}$ number of one-to-one functions from A to B. This number is denoted by $P(n, m)$.

(b) Here, $m = 3$ and $\dfrac{n!}{(n-m)!} = 60$. Thus,

$$\frac{n!}{(n-3)!} = 60, \text{ or } n(n-1)(n-2) = 60.$$

Evidently, $n = 5$. Thus, $|B| = 5$. ■

Stirling Numbers of the Second kind

Let A and B be finite sets with $|A| = m$ and $|B| = n$, where $m \geq n$.* Then *the number of onto functions from A to B* is given by the *formula*:

$$p(m, n) = \sum_{k=0}^{n} (-1)^k \, ({}^nC_{n-k}) \, (n-k)^m \qquad (*)$$

The proof of this formula is omitted.

With $p(m, n)$ given by the above formula, the number $\{p(m, n)/n!\}$ is called the *Stirling number of the second kind* and is denoted by $S(m, n)$. Thus, by definition,

$$S(m, n) = \frac{p(m, n)}{n!} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, ({}^nC_{n-k}) \, (n-k)^m \qquad \text{for } m \geq n.$$

This number represents the number of ways in which it is possible to assign m distinct objects into n identical places (containers) with no place (container) left empty.

It is easy to check that $S(m, 1) = 1$ and $S(m, m) = 1$ for all $m \geq 1$.

It can be shown that the number of possible ways to assign m distinct objects to n identical places with empty places allowed is given by the formula

$$p(m) = \sum_{i=1}^{n} S(m, i), \quad \text{for } m \geq n.$$

*If $m < n$, then there is no onto function from A to B. (See Theorem 2 (Part 2).)

Example 2 *Let $A = \{1,2,3,4,5,6,7\}$ and $B = \{w,x,y,z\}$. Find the number of onto functions from A to B.*

▶ Here $m = |A| = 7$ and $n = |B| = 4$. Therefore, the number of onto functions from A to B is

$$p(7,4) = \sum_{k=0}^{4}(-1)^k\left({}^4C_{4-k}\right)(4-k)^7$$

$$= {}^4C_4 \times 4^7 - {}^4C_3(4-1)^7 + {}^4C_2(4-2)^7 - {}^4C_1(4-3)^7 + 0$$

$$= 4^7 - 4 \times 3^7 + 6 \times 2^7 - 4$$

$$= 8400.$$

Example 3 *Let $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5,6\}$.*

(a) *Find how many functions are there from A to B. How many of these are one-to-one? How many are onto?*

(b) *Find how many functions are there from B to A. How many of these are one-to-one? How many are onto?*

▶ Here, $|A| = m = 4$ and $|B| = n = 6$. Therefore:

(a) The number of functions possible from A to B is $n^m = 6^4 = 1296$.

The number of one-to-one functions possible from A to B is

$$\frac{n!}{(n-m)!} = \frac{6!}{2!} = 360.$$

There is no onto function from A to B.

(b) The number of functions possible from B to A is $m^n = 4^6 = 4096$.

There is no one-to-one function from B to A.

The number of onto functions from B to A is

$$p(6,4) = \sum_{k=0}^{4}(-1)^k({}^4C_{4-k})(4-k)^6$$

$$= 4^6 - 4 \times 3^6 + 6 \times 2^6 - 4 = 1560.$$

Example 4 *Given that $p(6,4) = 1560$ and $p(7,4) = 8400$, evaluate $S(6,4)$ and $S(7,4)$.*

▸ By definition, we have

$$S(m,n) = \frac{p(m,n)}{n!} \quad \text{for} \quad m \geq n.$$

Therefore,

$$S(6,4) = \frac{p(6,4)}{4!} = \frac{1560}{24} = 65$$

and $$S(7,4) = \frac{p(7,4)}{4!} = \frac{8400}{24} = 350$$

■

Example 5 *Evaluate $S(5,4)$ and $S(8,6)$.*

▸ By definition, we have

$$S(m,n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k \left({}^nC_{n-k}\right)(n-k)^m$$

Therefore,

$$S(5,4) = \frac{1}{4!}\sum_{k=0}^{4}(-1)^k \left({}^4C_{4-k}\right)(4-k)^5$$

$$= \frac{1}{4!}\left\{4^5 - \left({}^4C_3\right)\times 3^5 + \left({}^4C_2\right)\times 2^5 - \left({}^4C_1\right)\times 1^5\right\}$$

$$= \frac{1}{4!}\left\{4^5 - 4\times 3^5 + 6\times 2^5 - 4\right\} = \frac{240}{4!} = 10$$

and

$$S(8,6) = \frac{1}{6!}\sum_{k=0}^{6}(-1)^k \left({}^6C_{6-k}\right)(6-k)^8$$

$$= \frac{1}{6!}\left\{6^8 - \left({}^6C_5\right)\times 5^8 + \left({}^6C_4\right)\times 4^8 - \left({}^6C_3\right)\times 3^8 + \left({}^6C_2\right)\times 2^8 - \left({}^6C_1\right)\times 1^8\right\}$$

$$= \frac{1}{6!}\left\{6^8 - \left(6\times 5^8\right) + \left(15\times 4^8\right) - \left(20\times 3^8\right) + \left(15\times 2^8\right) - 6\right\}$$

$$= 266.$$

■

Example 6 *There are six programmers who can assist eight executives. In how many ways can the executives be assisted so that each programmer assists atleast one executive?*

▸ Let A denote the set of executives and B denote the set of programmers. Then the required number is equal to the number of onto functions from A to B. This number is

$$p(8,6) = (6!)\times S(8,6).$$

We note that

$$S(8,6) = \frac{1}{6!}\sum_{k-0}^{6}({}^6C_{6-k})(6-k)^8 = 266 \text{ (see Example 5)}$$

Therefore,

$$p(8,6) = (6!) \times 266 = 720 \times 266 = 191520.$$

This is the required number.

Example 7 *Find the number of ways of distributing four distinct objects among three identical containers, with some container(s) possibly empty.*

► Here, the number of objects is $m = 4$ and the number of containers is $n = 3$. Therefore, the required number is

$$p(4) = \sum_{i=1}^{3} S(4,i) = S(4,1) + S(4,2) + S(4,3).$$

Now,

$$S(4,1) = \frac{1}{1!}\sum_{k=0}^{1}(-1)^k\left({}^1C_{1-k}\right)(1-k)^4 = 1,$$

$$S(4,2) = \frac{1}{2!}\sum_{k=0}^{2}(-1)^k\left({}^2C_{2-k}\right)(2-k)^4$$

$$= \frac{1}{2}\left\{2^4 - 2\times 1^4\right\} = 7,$$

$$S(4,3) = \frac{1}{3!}\sum_{k=0}^{3}(-1)^k\left({}^3C_{3-k}\right)(3-k)^4$$

$$= \frac{1}{6}\left\{3^4 - 3\times 2^4 + 3\times 1^4\right\} = 6.$$

Thus, the required number is

$$p(4) = 1 + 7 + 6 = 14.$$

Example 8 *If m and n are positive integers with $1 \le n \le m$, prove that*

$$S(m+1,n) = S(m,n-1) + nS(m,n).$$

► Let $A = \{a_1, a_2, \ldots, a_m, a_{m+1}\}$. We consider the distribution of $m+1$ elements of A among n identical containers with no container left empty.

We note that, the elements $a_1, a_2, \ldots, a_m$ of A can be distributed among $n-1$ identical containers (with none left empty) in $S(m, n-1)$ ways. If a_{m+1} is placed in the remaining container, $S(m, n-1)$ represents the number of ways of distributing all the $m+1$ elements of A among n containers, with m elements distributed among $n-1$ containers and the remaining one element placed in the remaining container.

Next, we note that the elements $a_1, a_2, \ldots, a_m$ can be distributed among n identical containers (with none left empty) in $S(m,n)$ ways. If a_{m+1} is placed in any one of these n containers, then for each choice of the container into which a_{m+1} is placed, there are $S(m,n)$ ways of distributing all the $m+1$ elements of A among n containers. Since there are n such possible choices, there are in all $nS(m,n)$ ways of distributing all the $m+1$ elements of A among n containers with m elements distributed among n containers and the $(m+1)^{st}$ element also put into any one of these n containers.

Thus:

Total number of ways of distributing the $m+1$ elements of A among n identical containers

$$= S(m, n-1) + n\,S(m,n)$$

This proves that

$$S(m+1, n) = S(m, n-1) + n\,S(m,n). \qquad \blacksquare$$

Example 9 *Evaluate $S(8,7)$, given that $S(7,6) = 21$.*

▸ In view of the formula proved in the preceding example, we find that

$$\begin{aligned} S(8,7) &= S(7,6) + 7\,S(7,7) \\ &= 21 + 7 \times 1, \quad \text{because} \quad S(7,7) = 1 \\ &= 28. \end{aligned}$$

(7, 6) + 7 S(7-7)

$\blacksquare$

Example 10 *Given that $S(8,4) = 1701$, $S(8,5) = 1050$ and $S(8,6) = 266$, evaluate $S(10,6)$.*

▸ We have

$$S(m+1, n) = S(m, n-1) + nS(m,n).$$

Using this formula, we find that

$$\begin{aligned} S(10,6) &= S(9,5) + 6S(9,6) \\ &= \{S(8,4) + 5\,S(8,5)\} + 6 \times \{S(8,5) + 6S(8,6)\} \\ &= S(8,4) + 11\,S(8,5) + 36\,S(8,6). \end{aligned}$$

Using the given values, this yields

$$\begin{aligned} S(10,6) &= 1701 + (11 \times 1050) + (36 \times 266) \\ &= 22{,}827. \end{aligned}$$

$\blacksquare$

Exercises

1. If $A = \{2, 6, 9\}$ and $B = \{p, q, r, s, t\}$, find the number of one-to-one functions from A to B.
2. If $A = \{1, 2, 3, 4, 5\}$ and there are 6720 one-to-one functions $f : A \to B$, what is $|B|$?
3. If $A = \{w, x, y, z\}$ and $B = \{1, 2, 3\}$, find how many onto functions are there from A to B.
4. If A and B are finite sets with $|A| = 5$ and $|B| = 3$, find the number of onto functions from A to B.
5. Let $A = \{1, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, \}$.

   (i) Determine the number of functions from A to B. How many of these are one-to-one? onto?

   (ii) Determine the number of functions from B to A. How many of these are one-to-one? onto?

6. If a set A has n elements, how many one-to-one correspondences are there from A to A?
7. If $A = \{1, 2, 3, \ldots, n\}$, for some fixed $n \in \mathbb{Z}^+$, find how many bijective functions are there with $f(1) \neq 1$.
8. Verify that $S(m, 1) = 1$ and $S(m, m) = 1$ for all $m \geq 1$.
9. Prove the following: $S(5, 3) = 25$, $S(7, 2) = 63$, $S(8, 5) = 1050$.
10. Verify that $\sum_{k=0}^{6} ({}^{6}C_k)(k!)S(8, k) = 6^8$.

Answers

1. 60 2. 8 3. 36 4. 150 5. (i) 8^5, 160, 0 (ii) 5^8, 0, 126000. 6. $n!$ 7. $n! - (n-1)!$

5.3.3 Some particular functions

In this Section we define and illustrate some miscellaneous functions used in Computer Science.

(1) *Floor and Ceiling functions*

Let x be any real number. Then x is an integer or x lies between two integers. Let $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x, and $\lceil x \rceil$ denote the least integer that is greater than or equal to x. Then $\lfloor x \rfloor$ is called the ***floor*** of x and $\lceil x \rceil$ is called the *ceiling* of x.

10. Let $f : Z^+ \times Z^+ \to Z^+$ be a closed binary operation defined by $f(a.b) = gcd$ of a and b. Sho that f is commutative and associative, but does not have an identity.

11. Let A and B be sets contained in a universal set U. Show that

$$f : \mathscr{P}(\mathrm{U}) \times \mathscr{P}(\mathrm{U}) \to \mathscr{P}(\mathrm{U})$$

defined by $f(A, B) = A \cup B$ is a closed binary operation which is commutative and associative. Is there an identity for f?

12. Let $|A| = 5$. Find how many closed binary operations are there on A. How many of these are commutative?

Answers

2. $\lfloor 3.65 \rfloor = 3$, $\lfloor -3.65 \rfloor = -4$, $\lfloor -16 \rfloor = -16$,

$\lfloor 18.5 \rfloor = 18$, $\lfloor -18.5 \rfloor = -19$, $\lfloor \sqrt{5} \rfloor = 2$.

$\lceil 3.65 \rceil = 4$, $\lceil -3.65 \rceil = -3$, $\lceil -16 \rceil = -16$,

$\lceil 18.5 \rceil = 19$, $\lceil -18.5 \rceil = -18$, $\lceil \sqrt{5} \rceil = 3$.

3. (i) 0 (ii) 1 (iii) 24 (iv) 21 (v) 6 (vi) 8

8. (i), (ii), (iii): both commutative and associative
(iv) neither commutative nor associative.

11. Φ is the identity.

12. 5^{25}, 5^{15}

5.4 The Pigeonhole Principle

As an interpretation of Part (4) of Theorem 2 of Section 5.3.2, we have got the following statement:

If m pigeons occupy n pigeonholes and if m > n, then two or more pigeons occupy the same pigeonhole.

This is often restated as follows:

If m pigeons occupy n pigeonholes, where m > n, then at least one pigeonhole must contain two or more pigeons in it.

As already mentioned, this statement is known as the ***Pigeonhole Principle***.

A simple illustration of the above principle is that if 6 pigeons occupy 4 pigeonholes, then at least one pigeonhole must contain two or more pigeons in it. See Figure 5.18.

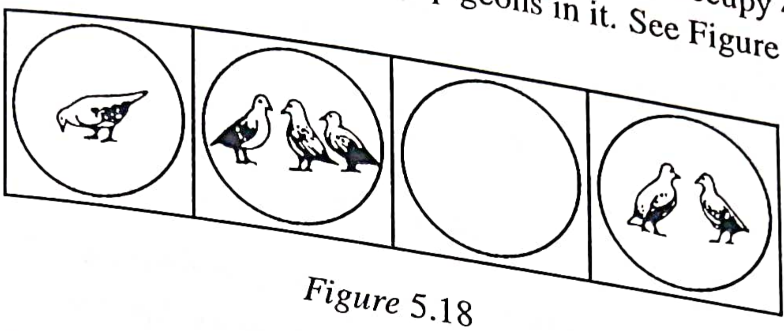

Figure 5.18

As a simple application of the principle, we may note that if 8 children are born in the same week, then two or more children are born on the same day of the week.

Generalization

The following is an extension/generalization of the pigeonhole principle.

If m pigeons occupy n pigeonholes, then at least one pigeonhole must contain $(p+1)$ *or more pigeons, where* $p = \lfloor (m-1)/n \rfloor$.*

Proof: We prove this principle by the method of contradiction.

Assume that the conclusion part of the principle is not true. Then, no pigeonhole contains $(p+1)$ or more pigeons. This means that every pigeonhole contains p or less number of pigeons. Then:

$$\text{Total number of pigeons} \leq np = n \times \lfloor (m-1)/n \rfloor \leq n\left(\frac{m-1}{n}\right) = (m-1).$$

This is a contradiction, because the total number of pigeons is m. Hence our assumption is wrong, and the principle is true.

Example 1 *ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that at least two of these points are such that the distance between them is less than 1/2 cm.*

▶ Consider the triangle DEF formed by the mid-points of the sides BC, CA and AB of the given triangle ABC; see Figure 5.19. Then the triangle ABC is partitioned into four small equilateral triangles (portions), each of which has sides equal to 1/2 cm. Treating each of these four portions as a pigeonhole and five points chosen inside the triangle as pigeons, we find by using the pigeonhole principle that at least one portion must contain two or more points. Evidently, the distance between such points is less than 1/2 cm. •

*For any real number x, the symbol $\lfloor x \rfloor$ represents the greatest integer less than or equal to x. Thus, $\lfloor x \rfloor \leq x$.

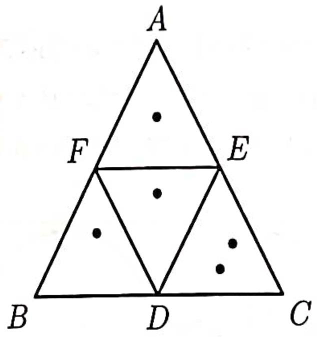

Figure 5.19

Example 2 *A bag contains* 12 *pairs of socks (each pair in different color). If a person draws the socks one by one at random, determine at most how many draws are required to get at least one pair of matched socks.*

▶ Let n denote the number of the draw. For $n \leq 12$, it is possible that the socks drawn are of different colors, because there are 12 colors. For $n = 13$, all socks cannot have different colors – at least two must have the same color (Treat 13 as the number of pigeons and 12 colors as 12 pigeonholes). Thus, at most 13 draws are required to have at least one pair of socks of the same color. ∎

Example 3 *A magnetic tape contains a collection of* 5 *lakh strings made up of four or fewer number of English letters. Can all the strings in the collection be distinct?*

▶ Each place in an n letter string can be filled in 26 ways. Therefore, the possible number of strings made up of n letters is 26^n. Consequently, the total number of different possible strings made up of four or fewer letters is

$$26^4 + 26^3 + 26^2 + 26 = 4,75,254.$$

Therefore, if there are 5 lakh strings in the tape, then at least one string is repeated. (Treat 4,75,254 strings as pigeonholes and 5 lakh strings as pigeons). Thus, all the strings in the collection cannot be distinct. ∎

Example 4 *Prove that if* 30 *dictionaries in a library contain a total of* 61,327 *pages, then at least one of the dictionaries must have at least* 2045 *pages.*

▶ Treating the pages as pigeons and dictionaries as pigeonholes, we find by using the generalized pigeonhole principle that at least one of the dictionaries must contain $p + 1$ or more pages where

$$p = \left\lfloor \frac{61327 - 1}{30} \right\rfloor = \lfloor 2044.2 \rfloor = 2044.$$

This proves the required result. ∎

$m = 4$. $5 = m$ $p = \frac{m-1}{m} = \frac{5-1}{4} = \frac{4}{4} = 1 : 2$

Example 5 *If 5 colours are used to paint 26 doors, prove that at least 6 doors will have the same colour.*

▸ Treating 26 doors as pigeons and 5 colours as pigeonholes, we find by using the generalized pigeonhole principle that at least one of the colours must be assigned to $\left(\frac{26-1}{5}\right)+1=6$ or more doors.

This proves the required result. ■

Example 6 *Prove that in any set of 29 persons at least five persons must have been born on the same day of the week.*

▸ Treating the seven days of a week as 7 pigeonholes and 29 persons as pigeons, we find by using the generalized pigeonhole principle that at least one day of the week is assigned to $\left(\frac{29-1}{7}\right)+1=5$ or more persons. In other words, at least 5 of any 29 persons must have been born on the same day of the week. ■

Example 7 *How many persons must be chosen in order that at least five of them will have birth days in the same calendar month?*

▸ Let n be the required number of persons. Since the number of months over which the birthdays are distributed is 12, the least number of persons who have their birthdays in the same month is, by the generalized pigeonhole principle, equal to $\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1$. This number is 5, if

$$\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1 = 5, \text{ or } n = 49.$$

Thus, the number of persons is 49 (at the least). ■

Example 8 *Find the least number of ways of choosing three different numbers from 1 to 10 so that all choices have the same sum.*

▸ From the numbers from 1 to 10, we can choose three different numbers in $C(10,3) = 120$ ways.

The smallest possible sum that we get from a choice is 1+2+3=6 and the largest sum is 8+9+10=27. Thus, the sums vary from 6 to 27 (both inclusive), and these sums are 22 in number.

Accordingly, here, there are 120 choices (pigeons) and 22 sums (pigeonholes). Therefore, the least number of choices assigned to the same sum is, by the generalized pigeonhole principle,

$$\left\lfloor \frac{120-1}{22} \right\rfloor + 1 = \lfloor 6.4 \rfloor \approx 6.$$ ■

Example 9 *Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. When any 3 of these students are chosen to be a debating team from the class, the sum of their shirt numbers is used as the code number of the team. Show that if any 8 of the 20 are selected, then from these 8 we may form at least two different teams having the same code number.*

▶ From the 8 of the 20 students selected, the number of teams of 3 students that can be formed is ${}^8C_3 = 56$.

According to the way in which the code number of a team is determined, we note that the smallest possible code number is $1 + 2 + 3 = 6$ and the largest possible code number is $18 + 19 + 20 = 57$. Thus, the code numbers vary from 6 to 57 (inclusive), and these are 52 in number. As such, only 52 code numbers (pigeon holes) are available for the 56 possible teams (pigeons). Consequently, by the pigeonhole principle, at least two different teams will have the same code number. ∎

Example 10 *Prove that every set of 37 positive integers contains at least two integers that leave the same remainder upon division by 36.*

▶ When a positive integer is divided by 36, the possible remainders are $0, 1, 2, \ldots 35$. Let A_r denote the set of all positive integers that leave the remainder r when divided by 36. Thus, every positive integer belongs to one or the other of the 36 sets: $A_0, A_1, A_2, \ldots, A_{35}$. Hence if we take any 37 positive integers, then at least two of them must belong to one of these A_r's (Treat A_r's as pigeonholes and 37 as the number of pigeons). This proves the required result. ∎

Example 11 *Show that if any $n+1$ numbers from 1 to $2n$ are chosen, then two of them will have their sum equal to $2n+1$.*

▶ Let us consider the following sets:

$$A_1 = \{1, 2n\}, \quad A_2 = \{2, 2n-1\}, \ldots A_{n-1} = \{n-1, n+2\}, \quad A_n = \{n, n+1\}$$

These are the only sets containing two numbers from 1 to $2n$ whose sum is $2n+1$.

Since every number from 1 to $2n$ belongs to one of the above sets, each of the $n+1$ numbers chosen must belong to one of the sets. Since there are only n sets, two of the $n+1$ chosen numbers have to belong to the same set (according to the pigeonhole principle). These two numbers have their sum equal to $2n+1$. ∎

Example 12 *Show that every set of seven distinct integers includes two integers x and y such that at least one of $x+y$ or $x-y$ is divisible by 10.*

▶ Let $X = \{x_1, x_2, \ldots, x_7\}$ be a set of seven distinct integers and let r_i be the remainder when x_i is divided by 10.

Consider the following subsets of X:

$$A_1 = \{x_i \in X \mid r_i = 0\}$$
$$A_2 = \{x_i \in X \mid r_i = 5\}$$

19. Show that in any set of $n \geq 2$ persons there will be at least two persons who know the same number of persons in the set.

20. Consider a tournament in which each player plays against every other player and each player wins at least once. Show that there are at least two players having the same number of wins.

5.5 Composition of functions

Consider three non-empty sets A, B, C (which are not necessarily distinct) and the functions $f : A \to B$ and $g : B \to C$. The ***composition*** (or *product*) of these two functions is *defined as* the function $g \circ f : A \to C$ with $(g \circ f)(a) = g\{f(a)\}$, for all $a \in A$.

The pictorial representation of $g \circ f$ is shown in Figure 5.20.

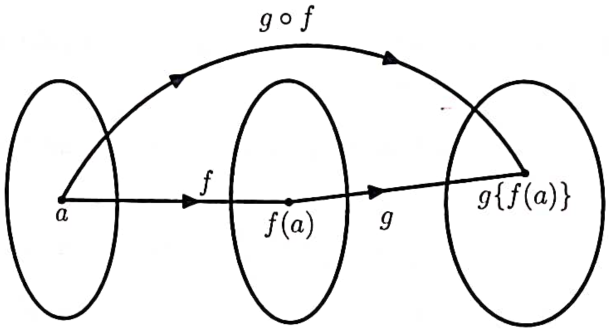

Figure 5.20

For a function $f : A \to A$, $f \circ f$ is denoted by f^2, $f \circ f^2$ is denoted by f^3, and so on. For any integer $n \geq 2$, the function $f^n : A \to A$ is defined *recursively* by

$$f^1 = f, \quad f^n = f \cdot f^{n-1}.$$

Example 1 *Let* $A = \{1,2,3,4\}$, $B = \{a,b,c\}$ *and* $C = \{w,x,y,z\}$ *with* $f : A \to B$ *and* $g : B \to C$ *given by*

$$f = \{(1,a),\ (2,a),\ (3,b),\ (4,c)\},\ \textit{and}\ g = \{(a,x),\ (b,y),\ (c,z)\}$$

Find $g \circ f$.

► We find, by using the definitions of f and g, that

$$(g \circ f)(1) = g\{f(1)\} = g(a) = x,$$
$$(g \circ f)(2) = g\{f(2)\} = g(a) = x,$$

$$(g \circ f)(3) = g\{f(3)\} = g(b) = y,$$

$$(g \circ f)(4) = g\{f(4)\} = g(c) = z.$$

Thus, $g \circ f = \{(1, x), (2, x), (3, y), (4, z)\}$. ■

Example 2 *Consider the functions f and g defined by $f(x) = x^3$ and $g(x) = x^2 + 1$, $\forall x \in$ R. Find $g \circ f$, $f \circ g$, f^2 and g^2.*

▸ Here, both f and g are defined on R. Therefore, all of the functions $g \circ f$, $f \circ g$, $f^2 = f \circ f$ and $g^2 = g \circ g$ are defined on R, and we find that

$$(g \circ f)(x) = g\{f(x)\} = g\left(x^3\right) = \left(x^3\right)^2 + 1 = x^6 + 1,$$

$$(f \circ g)(x) = f\{g(x)\} = f\left(x^2 + 1\right) = \left(x^2 + 1\right)^3,$$

$$f^2(x) = (f \circ f)(x) = f\{f(x)\} = f\left(x^3\right) = \left(x^3\right)^3 = x^9,$$

$$g^2(x) = (g \circ g)(x) = g\{g(x)\} = g\left(x^2 + 1\right) = \left(x^2 + 1\right)^2 + 1.$$

The above expressions define the function's $g \circ f$, $f \circ g$, f^2 and g^2 respectively.

Remark: This example illustrates the important fact that the functions $g \circ f$ and $f \circ g$ are *not* one and the same even when they exist. The composition of functions is thus *not commutative*. ■

Example 3 *Let I_A and I_B denote the identity functions on sets A and B respectively. For any function $f : A \to B$, prove that*

$$f \circ I_A = f = I_B \circ f.$$

▸ For any $a \in A$, we have

$$(f \circ I_A)(a) = f\{I_A(a)\} = f(a)$$

and

$$(I_B \circ f)(a) = I_B\{f(a)\} = f(a)$$

Therefore, $f \circ I_A = f$ and $I_B \circ f = f$. ■

Example 4 *Let f and g be functions from R to R defined by $f(x) = ax+b$ and $g(x) = 1-x+x^2$. If $(g \circ f)(x) = 9x^2 - 9x + 3$, determine a, b.*

▸ We have

$$\begin{aligned} 9x^2 - 9x + 3 &= (g \circ f)(x) \\ &= g\{f(x)\} = g(ax + b) \\ &= 1 - (ax + b) + (ax + b)^2 \\ &= a^2x^2 + (2ab - a)x + \left(1 - b + b^2\right). \end{aligned}$$

Comparing the corresponding coefficients, we get

$$9 = a^2,\ 9 = a - 2ab,\ 3 = 1 - b + b^2.$$

The first of these gives $a = \pm 3$. For $a = 3$, the second and third are satisfied if $b = -1$. For $a = -3$, we get $b = 2$.

Thus, $a = 3, b = -1$, and $a = -3, b = 2$ are the required values of a, b.

Example 5 *Let f and g be functions from* R *to* R *defined by $f(x) = ax + b$ and $g(x) = cx + d$. What relationship must be satisfied by a, b, c, d if $g \circ f = f \circ g$?*

► We have, for any $x \in$ R,

$$(g \circ f)(x) = g\{f(x)\} = g(ax + b)$$

$$= c(ax + b) + d = cax + (cb + d)$$

and $$(f \circ g)(x) = f\{g(x)\} = f(cx + d)$$

$$= a(cx + d) + b = cax + (ad + b)$$

Therefore, if $(g \circ f) = (f \circ g)$, we should have

$$cb + d = ad + b.$$

The following theorems contain some important results on composition of functions.

Theorem 1. *Let $f : A \to B$ and $g : B \to C$ be any two functions. Then the following are true:*

(1) *If f and g are one-to-one, so is $g \circ f$.*

(2) *If $g \circ f$ is one-to-one, then f is one-to-one.*

(3) *If f and g are onto, so is $g \circ f$.*

(4) *If $g \circ f$ is onto, then g is onto.*

Proof: First, we note that $g \circ f : A \to C$.

(1) Take any $a_1, a_2 \in A$. We find that

$$(g \circ f)(a_1) = (g \circ f)(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$$

$$\Rightarrow f(a_1) = f(a_2), \text{ because } g \text{ is one-to-one.}$$

$$\Rightarrow a_1 = a_2, \text{ because } f \text{ is one-to-one.}$$

Therefore, $g \circ f$ is one-to-one.