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COURSE	**MATHEMATICAL STRUCTURES**
COURSE CODE	**21MAT41A**
MODULE	**3**
MODULE NAME	**FUNCTIONS**
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Cartesian product

The Cartesian product $X \times Y$ between two sets X and Y is the set of all possible ordered pairs with first element from X and second element from Y:

$$X \times Y = \{(x , y) : x \in X \text{ and } y \in Y\}$$

One example is the standard Cartesian coordinates of the plane, where X is the set of points on the x-axis, Y is the set of points on the y-axis, and $X \times Y$ is the XY - plane.

If X=Y, we can denote the Cartesian product of X with itself as $X \times X = X^2$. For examples, since we can represent both the X-axis and the Y-axis as the set of real numbers R, we can write the XY-plane as

$R \times R = R^2$.

Functions

Let A and B be two non-empty sets. Then ***a function*** f from A to B is a relation from A to B such that for each a in A, there is a unique b in B such that (a,b) ϵ f written as b=f(a). Here b is called the ***image of a*** and a is called a ***pre-image of b*** under f. Also, the element a is called ***an argument*** of the function f and b=f(a) is called the value of the function f for the argument a.

Note: Every function is a relation, but a relation need not be a function since under a function an element of A can be related to only one element of B.

For the function $f: A \to B$, A is called the ***domain*** of f and B is called the ***codomain*** of f. The subset of B consisting of the images of all elements of A under f is caked the ***range*** of f and is denoted by f(A).

Ex: Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$ and R and S be relations from A to B such that $R = \{(1, -1), (1, 0), (2, -1), (3,0)\}$ and $S = \{(1, -1), (2, -1), (3, 0)\}$. Verify if R and S are functions from A to B?

Solution: We observe that under R, the element 1 of A is related to two different elements -1 and 0 of B. Therefore, R is not a function. Whereas under relation S, every element of A is related to a unique element in B. Therefore, S is a function.

Ex: Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{6, 7, 8, 9, 10\}$. If a function $f: A \to B$ is defined by $f = \{(1,7), (2,7), (3,8), (4,6), (5,9), (6,9)\}$ determine $f^{-1}(6)$ and $f^{-1}(9)$. If $B_1 = \{7,8\}$ and $B_2 = \{8, 9, 10\}$, find $f^{-1}(B_1)$ and $f^{-1}(B_2)$.

Solution: Note that $f^{-1}(6) = \{x \in A \,|f(x) = 6\} = \{4\}$ and

$$f^{-1}(9) = \{x \in A \,|f(x) = 9\} = \{5,6\}$$

Here B_1 and B_2 are subsets of B and thus $f^{-1}(B_1) = \{x \in A \,|f(x) \in B_1\}$ and

$$f^{-1}(B_2) = \{x \in A \,|f(x) \in B_2\}$$

From the definition of we note that

$$f(x0 = 7 \; when \; x = 1 \; and \; x = 2, f(x) = 8 \; when \; x = 3, f(x) = 9 \; when \; x = 5,6$$

Therefore $f^{-1}(B_1) = \{1, 2, 3\}$ and $f^{-1}(B_2) = \{3, 5, 6\}$

Consequences

1. Every element if A has an image in B (under f) and if an element a of A has two images in B, then the two images cannot be different.
2. An element b ϵ B need not have a preimage in A, under f.
3. Two different elements of A can have the same image in B under f.
4. If g is a function from A to B then f=g iff f(a)=g(a) for every aϵA

Types of functions

Identity Function: A function $f: A \to A$ such that f(a)=a for every a ϵ A is called the identity function or mapping on A.
In other words, a function f on a set A is an identity function if the image of every element of A is itself usually denoted by I_A.

Constant Function: A function $f : A \to B$ such that f(a) = c for every a ϵ A, where c is a fixed element of B is called a constant function.
In other words, a function f from A to B is a constant function if all elements of A have the same image in B.

Onto Function: A function $f : A \to B$ is said to be an onto function if for every element b of B there is an element a of A such that f(a) = b.
In other words, f is an onto function from A to B if every element of B has a preimage in A that is to say that the range of f is equal to B.
An onto function is also called a Surjective Function.

One-to-One Function: A function $f : A \to B$ is said to be a one-to-one function if different elements of A have different images in B under f that is whenever $f(a_1)=f(a_2)$ for a_1, a_2 ϵ A then $a_1 = a_2$.
Thus if $f : A \to B$ is a one-to-one function, then every element of A has a unique image in B and every element of f(A) has a unique preimage in A.
A one-to-one function is also called an injective function.

Ex: The functions $f: R \to R \;\; and \;\; g: R \to R$ are defined by
$f(x) = 3x + 7 \; and \; g(x) = x(x^3 - 1)$ for all $x \in R$. Verify that f is one-to-one but g is not?

Solution: If $f(x_1) = f(x_2)$ we have $3x_1 + 7 = 3x_2 + 7$ so that $x_1 = x_2$. Therefore, f is a one-to-one function.
We may see that $g(0) = 0 = g(1)$. Thus for $x_1 = 0 \; and \; x_2 = 1$ we have $g(x_1) = g(x_2)$. Thus g is not a one-to-one function.

Stirling numbers of the second kind

Let A and B be finite sets with |A| = m and |B| = n, where m ≥ n. Then the number of onto function from A to B is given by the formula

$p\,(m, n) = \sum_{k=0}^{n} (-1)^k \, ({}^{n}C_{n-k}) \; (n-k)^m$

The Stirling number of the second kind is given by $S(m, n\,) = \frac{p(m,n)}{n!}$
This number represents the number of ways in which it is possible to assign m distinct objects into n identical places with no place left empty.

Note:
1. A function f from A to B with |A| = |B| is bijective iff f is one-to-one or onto.
2. If |A| = m, and |B| = n, then
    (a) The number of functions possible from A to B is n^m.
    (b) The number of functions possible from B to A is m^n.
    (c) The number of one-to-one functions possible from A to B is $\frac{n!}{(n-m)!}$
3. $S(m+1, n) = S(m, n-1) + n\,S(m, n)$

Ex: Find the number of ways of distributing four distinct objects among three identical containers, with some containers possibly empty.

Ans: Here, the number of objects is m=4 and the number of containers is n=3. Therefore, the required number is

$$p(4) = \sum_{i=1}^{3} S(4, i) = S(4,1) + S(4,2) + S(4,3)$$
$$S(4,1) = 1, \quad S(4,2) = 7, \quad S(4,3) = 6$$

Therefore, $p(4) = 14$

PIGEONHOLE PRINCIPLE:

If n pigeonholes are occupied by n+1 or more pigeons, then at least one pigeonhole is occupied by greater than one pigeon. OR This can be stated as follows:

Theorem - (Pigeonhole Principle) Suppose that n+1 (or more) objects are put into n boxes. Then some box contains at least two objects.

The validation for the above statement is given by, suppose each box contains at most one object. Then the total number of objects is at most 1+1+⋯+1=n, a contradiction.

An example for this is, among any 13 people, at least two share a birth month.

Label 12 boxes with the names of the months. Put each person in the box labeled with his or her birth month. Some box will contain at least two people, who share a birth month. □

Another example is, suppose 5 pairs of socks are in a drawer. Picking 6 socks guarantees that at least one pair is chosen.

Label the boxes by "the pairs" (e.g., the red pair, the blue pair, the argyle pair,…). Put the 6 socks into the boxes according to description.

Composition of functions

For any three non-empty sets A, B, C and the functions $f : A \rightarrow B$ and $g : B \rightarrow C$, the composition of f and g is defined as the function $g^\circ f : A \rightarrow C$ with $(\boldsymbol{g}^\circ \boldsymbol{f})(\boldsymbol{a}) = \boldsymbol{g}\{\boldsymbol{f}(\boldsymbol{a})\}$ for all a ϵ A.

Note:
1. For any integer n≥2, the function $f^n : A \rightarrow A$ is defined as $f^n = f \,.\, f^{n-1}$.
2. If I_A and I_B denote the identity functions on sets A and B respectively. Then for any function $f : A \rightarrow B$ prove that $^\circ I_A = f = I_B {}^\circ f$.
3. For any three functions $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$
(i) If f and g are one-to-one so is $(g^\circ f)$
(ii) If $(g^\circ f)$ is one-to-one then f is one-to-one.
(iii) If f and g are onto so is $(g^\circ f)$
(iv) If $(g^\circ f)$ is onto, then g is onto.
(v) $(h^\circ g)^\circ f = h^\circ(g^\circ f)$

Ex: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{w, x, y, z\}$ $with\ f: A \rightarrow B$ $and\ g: B \rightarrow C$ given by $f = \{(1, a), (2,1), (3, b), (4, c)\}$ and $g = \{(a, x), (b, y), (c, z)\}$. Find $g^\circ f$.

Ans: Using the definitions of f and g, we see that

$$(g^\circ f)(1) = g[f(1)] = g(a) = x$$
$$(g^\circ f)(2) = g[f(2)] = g(a) = x$$
$$(g^\circ f)(3) = g[f(3)] = g(b) = y$$

$$(g∘f)(4) = g[f(4)] = g(c) = z$$

Thus $g∘f = \{(1, x), (2, x), (3, y), (4, z)\}$

Exercise:

Let X = {x, y, z, k} and Y = {1, 2, 3, 4}. Determine which of the following functions. Give reasons if it is not. Find range if it is a function.

a. f = {(x, 1), (y, 2), (z, 3), (k, 4)
b. g = {(x, 1), (y, 1), (k, 4)
c. h = {(x, 1), (x, 2), (x, 3), (x, 4)
d. l = {(x, 1), (y, 1), (z, 1), (k, 1)}
e. d = {(x, 1), (y, 2), (y, 3), (z, 4), (z, 4)}.

Solution:

1. It is a function. Range (f) = {1, 2, 3, 4}
2. It is not a function because every element of X does not relate with some element of Y i.e., Z is not related with any element of Y.
3. h is not a function because h (x) = {1, 2, 3, 4} i.e., element x has more than one image in set Y.
4. d is not a function because d (y) = {2, 3} i.e., element y has more than image in set Y.