Functions :-

→ cartesian product
→ Definitions + Examples
→ Types of functions

* Identity function
* constant function
* Onto function
* One - to - one function
* one - to - one correspondence

→ properties of functions
→ Stirling Numbers of the second Kind
→ The pigeon-hole principle
→ Function composition - case study.

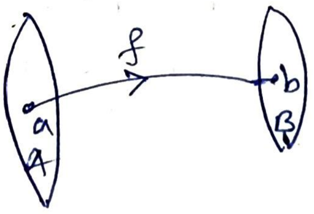
Definition :-

Let A and B be two non-empty sets. Then a function (or mapping) f from A to B is a relation from A to B such that for each a in A there is a unique b in B such that $(a.b) \in f$.

⇒ we write $b = f(a)$
'b' is called 'image' of 'a'
and 'a' is called 'pre-image of b under f.

⇒ 'a' is called argument of the function f
$b = f(a)$ is then called the value of the funᶜ f

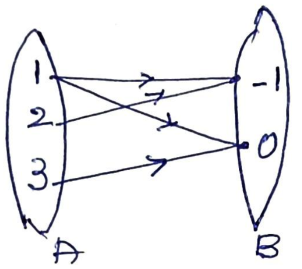
Pictorial representation :-



* Every function is a relation, but a relation need not be a function.

→ $f: A \to B$, 'A' is called <u>domain</u> of f and B is called codomain or range of f. ②

1. Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$ and R be a relation from A to B defined by
$$R = \{(1,-1)\ (1,0)\ (2,-1)\ (3,0)\}$$
Is R is a relation from A to B



we observe that, under R, the element 1 of A is related to two different elements, -1 & 0 of B
∴ R is not a function.

* Every a in A belongs to some pair $(a, b) \in f$, and if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.
This means that every element of A has an image in B under f if an $a \in A$ has two images in B, then the two images can not be different.

* An element $b \in B$ need not have a preimage in A under f.

* If an element $b \in B$ has a preimage $a \in A$ under f
i.e two different elements of A can have the same images in B under f.

(1) Let $f: R \to R$ be defined by

$$f(x) = \begin{cases} 3x - 5 & \text{for } x > 0 \\ -3x + 1 & \text{for } x \leq 0 \end{cases}$$

determine $f(0)$, $f(-1)$, $f(5/3)$, $f(-5/3)$

$f(0) = (-3 \times 0) + 1 = 1 = (-3x+1)$

$f(5/3) = 3x - 5 = 3(5/3) - 5 = 0$

$f(-5/3) = -3x + 1 = -3(-5/3) + 1 = 6$

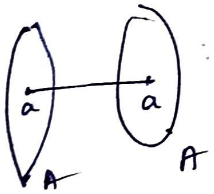
$f(-1) = -3x + 1 = -3(-1) + 1 = 4.$

Types of Functions:-

1. Identity function:

A function $f: A \to A$ such that $f(a) = a$ for every $a \in A$ is called the Identity function (or identity mapping)

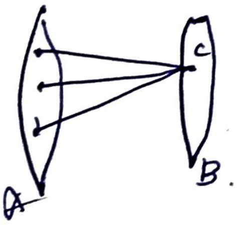
It is denoted as I_A or 1_A (every element is image of itself)



2. Constant function:

A function $f: A \to B$ such that $f(a) = C$ for every $a \in A$ where C is a fixed element of B, is called a Constant function

In other words, a function f from A to B is constant funⁿ if all elements of A have the same image (say c) in B. i.e $f(A) = C$

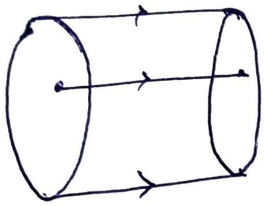


Onto function:

A function $f: A \to B$ is said to be an onto function if for every element b of B there is an element a of A such that $f(a) = b$

In other words, f is an onto function from A to B if every element of B has a preimage in A:

This amount to saying that f is an onto function of the range of f is equal to B. [Surjective fun^c]



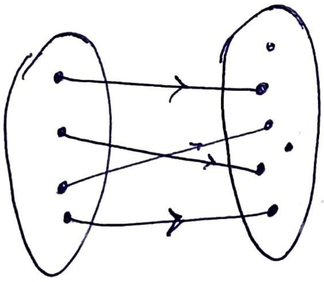
4) One - to - one function:-

A function $f: A \to B$ is said to a one-to-one fun^c (1-1). if different elements of A have different images in B. under f:

i.e. $a_1 . a_2 \in A$ with $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$

whenever. $f(a_1) = f(a_2)$ for $a_1 . a_2 \in A$ then $a_1 = a_2$

[Injective fun^c]



5) One-to-one correspondence:

A function which is both one-to-one and onto is called a one-to-one correspondence or Bijective fun^c

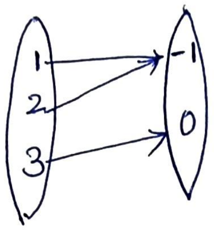
* . If $f: A \to B$ is such a fun^c then every element of A has a unique image in B and every element in B has a unique preimage in A

Properties of functions:-

(2) $A = \{1, 2, 3\}$ $B = \{-1, 0\}$ S be a ~~fun~~ relation from A to B

$S = \{(1,-1)\ (2,-1)\ (3,0)\}$. Is S a function?

Yes S is a function. (5)



(3) Let $A = \{1, 2, 3, 4\}$. Determine whether or not the following relation on A are functions.

(i) $f = \{(2,3)\ (1,4)\ (2,1)\ (3,2)\ (4,4)\}$ → not a func

(ii) $g = \{(3,1)\ (4,2)\ (1,1)\}$ → not a func

(iii) $h = \{(2,1)(3,4)(1,4)\ (2,1)(4,4)\}$ → yes func.

(4) Let $A = \{0, \pm 1, \pm 2, 3\}$ consider the function $f: A \to R$ (Where R is the set of all real no/.)

defined by $f(x) = x^3 - 2x^2 + 3x + 1$ for all $x \in A$.

Find the range of f.

$f(0) = 1$. $f(1) = 3$. $f(-1) = -5$. $f(2) = 7$, $f(-2) = -21$, $f(3) = 19$.

$f(A) = B = \{1, 3, -5, 7, -21, 19\}$

5) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{6, 7, 8, 9, 10\}$.

If a function $f: A \to B$ is defined by

$f: \{(1,7)(2,7)\ (3,8)\ (4,6)\ (5,9)\ (6,9)\}$

determine. $f^{-1}(6)$ & $f^{-1}(9)$.

$f^{-1}(6) = \{x \in A \mid f(x) = 6\} = \{4\}$

$f^{-1}(9) = \{x \in A \mid f(x) = 9\} = \{5, 6\}$

(4)

If $B_1 = \{7, 8\}$. $B_2 = \{8, 9, 10\}$. find $f^{-1}(B_1)$ & $f^{-1}(B_2)$

for $B_1 = \{7, 8\}$, $f(x) \in B_1$ when $f(x) = 7$ and $f(x) = 8$

from definition f, $f(x) = 7$ when $x = 1$ & $x = 2$

$f(x) = 8$ when $x = 3$

(6)

$\therefore f^{-1}(B_1) = \{1, 2, 3\}$

$||^{ly}$. $f^{-1}(B_2) = \{x \in A \mid f(x) \in B_2\} = \{3, 5, 6\}$

Properties of functions:-

Theorem: Let $X \to Y$ be a function and A and B be arbitrary non-empty of X. Then

(i) If $A \subseteq B$ then $f(A) \subseteq f(B)$

(ii) $f(A \cup B) = f(A) \cup f(B)$

(iii) $f(A \cap B) \subseteq f(A) \cap f(B)$ the equality holds if f is one-to-one

Theorem 2:- Let A and B are finite sets and f be a function from A to B. Then the following are true.

1. If f is $1-1$ then $|A| \leq |B|$
2. If f is onto then $|B| \leq |A|$
3. If is one-to-one corres then $|A| = |B|$
4. If $|A| > |B|$, then at least two different elements of A have the same image under f.

Theorem:- Suppose A and B are finite sets having the same number of elements, and f is a function from A to B. Then f is one-to-one if and if f is onto.

Find the nature of the function defined on $A = \{1.2.3\}$

I! $f: \{(1.1), (2,2)(3,3)\}$ Identity

fo $g: \{(1,2)(2,2)(3,2)\}$ constant

for $h = \{(1.2)(2.2)(3.1)\}$ neither Identity nor constant

not onto. not one-to-one

$p = \{(1,2)(2,3)(3,1)\}$ one-to-one correspondance

(8)

(1) The function $f: (Z \times Z) \to Z$ is defined by $f(x.y) = 2x + 3y$. Verify that f is onto but not one-to-one

Let $f: Z \to Z$ be fined by $f(a) = a+1$ for $a \in Z$.
Find whether f is one-to-one & onto (& both & neither)

$\Rightarrow$ Take any $a_1 . a_2 \in Z$. with $a_1 \neq a_2$. then $f(a_1) = a+1$ & $f(a_2) = a_2 + 1$.

Since $a_1 \neq a_2$. it is evident that $f(a_1) \neq f(a_2)$.
Thus, different elements of Z have different images under f. Therefore f is One-to-one.

Take any $b \in Z$. we check that b has $b-1$ as it's image under f;
because $f(b-1) = (b-1)+1 = b$. Thus every element of Z has a preimage under f.

$\therefore f$ is Onto.

f is Bijective.

1. Let A and B be finite sets with $|A| = m$. $|B| = n$.
① Find how many one-to-one fun^c are possible from A to B
② If there are 60 1-1 fun^c from $A \to B$ & $|A| = 3$ what is $|B|$?

Soln: If $m > n$ there exist no 1-1 fun^c from A to B
let $A = \{a_1, a_2, \ldots a_m\}$ $B = \{b_1, b_2, \ldots b_n\}$ where $m \leq n$.

Then a 1-1 function $f: A \to B$ is of the form.

$$f = \{(a_1, x), (a_2, x), \ldots (a_m, x)\}$$

Where x stands for b_j for some j. Since there are n number of b_j's. there are n choices for x in the pair (a_1, x). Since f is 1-1, the same x can not appear in (a_1, x) & (a_2, x);
as such there are $(n-1)$ choices for x in (a_2, x).
For a similar reason, there are $(n-2)$ choices for x in (a_3, x). proceeding like this, we find there are $n-(m-1)$ choices for x in (a_m, x).

Therefore, the total no/ of possible choices for x

$$n(n-1)(n-2)\ldots(n-(m-1)) = \frac{n!}{(n-m)!}$$

Thus if $m \leq n$ there are $\frac{n!}{(n-m)!}$ number of 1-1 fun^c $A \to B$. This number denoted by $P(n, m)$

② Here $m = 3$. $\frac{n!}{(n-m)!} = 60$. thus. $\frac{n(n-1)\cancel{(n-3)!}^{(n-2)}}{\cancel{(n-3)!}} = 60$

$n(n-1)^{(n-2)} = 60$

$n = 5$ $5 \times 4 \times 3 = 60$

$|B| = 5$.