Module - 04

Relations

Relations:

Let A and B be sets. A relⁿ (or binary relⁿ) from A to B is a subset of $A \times B$

Note:

* If $(a,b) \in R$ then, 'a related to b' written as aRb
* If $(a,b) \notin R$ then, 'a is not related to b' denoted by $a \not R b$
* If R is relation from A to A then R is said to be a relⁿ on A.
* Let A be any set, then $A \times A$, ϕ are relⁿˢ on A known as universal relⁿ and empty relⁿ respectively
* No. of relⁿˢ on a set with n elements = 2^{n}

Inverse relation:

Let R be any relⁿ from A to B. The inverse of R denoted by R^{-1} is the relⁿ from B to A given by

$R^{-1} = \{(b,a) / (a,b) \in R\}$

Ex: ① Consider the relⁿ $R = \{(1,y), (1,z), (3,y)\}$ from $A = \{1,2,3\}$ and $B = \{x,y,z\}$

$R^{-1} = \{(y,1), (z,1), (y,3)\}$

② Let $A = \{1,2,3\}$ be a set $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Let $R = \{(1,1), (3,2), (2,3), (2,1)\}$ be a relⁿ on A

$R^{-1} = \{(1,1), (2,3), (3,2), (1,2)\}$

$D(R) = \{1,2,3\}$

$R(R) = \{1,2,3\}$

Equivalence relation:

A relⁿ R on a set A is said to be an equivalence relⁿ if it satisfies the following properties:

i> Reflexivity $(a,a) \in R$, $\forall a \in A$

ii> Symmetry $(a,b) \in R \Rightarrow (b,a) \in R$, $\forall a,b \in A$

iii> Transitivity $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$ $\forall a,b,c \in A$.

Ex: Consider $A = \{1, 2, 3\}$

i) $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

It is an equivalence rel^n

ii) $R_2 = \{(1,2), (2,1), (2,2), (3,1), (3,3)\}$

$\therefore$ It is not equivalence $\because$ it is not reflexive, not symmetric nor transitive.

iii) $R_3 = \{(1,1), (2,2), (3,3), (1,2)\}$

$\therefore$ It is reflexive.

It is not symmetric

It is transitive.

$\therefore$ It is not an equivalence rel^n

Matrix representation of a relation:

Let $A = \{a_1, a_2 \dots a_m\}$ and $B = \{b_1, b_2 \dots b_n\}$. Let R be a rel^n from A to B then R can be represented by an $m \times n$ matrix.

$M_R = M(R) = [m_{ij}]_{m \times n}$

where $m_{ij} = 1$ if $(a_i, b_j) \in R$

0 if $(a_i, b_j) \notin R$

The matrix M_R is known as the matrix of rel^n R or adjacency matrix or Boolean matrix.

① Let $A = \{1, 2\}$ $B = \{p, q, r, s\}$ and let the relation R from A to B defined by $R = \{(1,q), (1,r), (2,p), (2,s)\}$ Find the matrix of R.

$$M_R = \begin{array}{c c} & \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

② Show that the identity rel^n is an equivalence rel^n

Consider the identity rel^n on a set A is given by, $R = \{(a,a) / a \in A\}$

i) Clearly $(a,a) \in R$, $a \in A$

$\therefore$ R is reflexive

ii> Let $(a,a) \in R$ then $(a,a) \in R$

$\therefore$ R is symmetric

iii> Let $(a,a), (a,a) \in R$ then $(a,a) \in R$

$\therefore$ R is transitive

Hence R is an equivalence rel^n.

③ On the set of all integers z defined by the rel^n R by aRb if and only if $ab \geq 0$. Show that R is an equivalence rel^n.

Let $R = \{(a,b) / ab \geq 0\}$, a, b are z

i> Let $a \in z \Rightarrow a \cdot a = a^2 \geq 0$

$\Rightarrow aRa \ \forall a \in z$

$\therefore$ R is reflexive

ii> Let $(a,b) \in R\ z \Rightarrow ab \geq 0$

$\Rightarrow ba \geq 0$

$\Rightarrow (b,a) \in z$

$\therefore$ R is symmetric

iii> Let $(a,b) \in z$, $(b,c) \in z \Rightarrow ab \geq 0$, $bc \geq 0$

$\Rightarrow (ab)(bc) \geq 0$

$\Rightarrow ab^2c \geq 0$

$\Rightarrow ac \geq 0$

$\Rightarrow ac \geq 0$, b^2 is always positive.

$\Rightarrow (a,c) \in z$

$\therefore$ R is transitive.

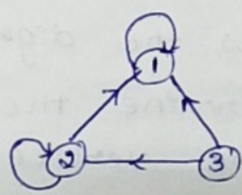
Hence R is an equivalence rel^n

Digraph of a relation:

The pictorial representation of the relation R of a set A is called directed graph or digraph of R.

Ex: If $A = \{1, 2, 3\}$

$R = \{(1,1), (2,1), (3,1), (2,2), (3,2)\}$



Vertex - domain elements

* Node (vertex) → The each element of A

* Edge → Draw an arrow called as edge from one vertex to another vertex.

* loop → The initial and terminal points are same for the vertex.

→ The initial and terminal vertexs are connecting same vertex is called loop

* Indegree - The no. of edges entering to a vertex is called indegree of that vertex.

* Out degree - The no. of edges leaving a vertex is called outdegree of that vertex.

① Let $A = \{1, 2, 3, 4\}$ and R be the reln on A defn by xRy iff $y = 2x$

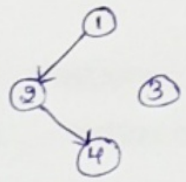
i> write down R.

ii> Draw the digraph of R.

iii> Determine the indegrees and outdegrees of vertices in the graph.

i> $R = \{(1,2), (2,4)\}$

ii>



iii>

Vertices	1	2	3	4
Indegree	0	1	0	1
outdegree	1	1	0	0

② Let $A = \{1, 2, 3, 4\}$ and let R be the reln on A defn by xRy iff "x divides y" written $x|y$

a> Write down R as a set of ordered pairs

b> Draw the digraphs of R.

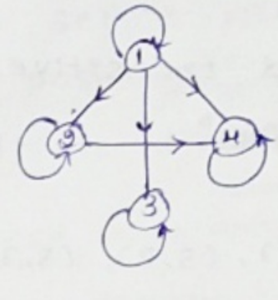
c> Determine the indegrees and out degrees of the vertices in the diagraph.

a> "x divides y", x/y

$1/1$, $1/2$, $1/3$, $1/4$, $2/2$, $2/4$, $3/3$, $4/4$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

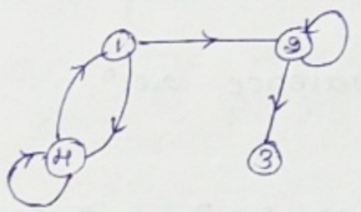
b>



c>

Vertices	1	2	3	4
Indegree	1	2	2	3
Outdegree	4	2	01	01

③ Find the rel^n represented by the digraph given below. Also write down its matrix.



$R = \{(1,2), (1,4), (2,2), (2,3), (4,1), (4,4)\}$

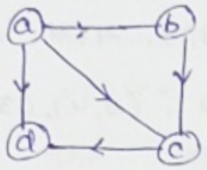
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

④ Let $A = \{1, 2, 3, 4\}$ Determine the nature of the following relations on A

i> $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

ii> $$M(R_2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

iii> R_3 is represented by the digraph



i> R_1 is reflexive, symmetric and transitive.

$\therefore$ R_1 is an <u>equivalence</u> rel^n

ii> $R_2 = \{(1,1), (1,2), (1,3), (1,4), (3,1), (3,2), (3,3), (3,4), (4,2)\}$

$\therefore$ R_2 is not reflexive, not symmetric and is transitive.

$\therefore$ R_2 is <u>not</u> an equivalence rel^n

iii> $R_3 = \{(a,b), (a,d), (a,c), (b,c)\}$

R_3 is <u>transitive</u> but neither reflexive nor symmetric

$\therefore$ R_3 is <u>not</u> an equivalence rel^n

<u>Equivalence</u> <u>class</u>:

Let R be an equivalence rel^n on a set A and $a \in A$. Then the set of all those elements x of A which are related by 'a' is called the equivalence class of a and it is denoted by $[a]$ or $R(A)$

i.e $[a] = \{x \in A \,/\, (a,x) \in R\}$

<u>Ex</u>: $A = \{1, 2, 3\}$

$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

$[1] = \{1, 3\}$

$[2] = \{2\}$

$[3] = \{1, 3\}$

⊛ <u>Partition of a set</u>:

⊛ <u>Partial order (Posets)</u>:

A rel^n R on a set A is said to be partially order in rel^n on set A if

i) R is reflexive

ii) R is anti-symmetric

iii) R is transitive on A.

* A set A with partial order R on it is called a poset.

* It is denoted by P(A, R) (or) (A, R)

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Hasse Diagrams:

The digraph of a partial order is called poset diagrams (or) Hasse diagrams

Working Rule:

* Eliminate reflexive edges (eliminate loops)
* Suppose in a digraph edges from a to b & edges from b to c is given then eliminate the edge from a to c.
* Now draw the digraph, all edges point for upward direct^n, so no need to put arrows across in the edges

① Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$. verify that R is a partial order on A and also write down Hasse diagram of R.

Given $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$

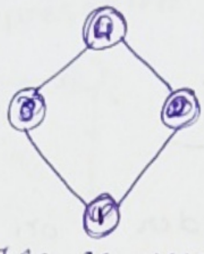
Here R is reflexive, antisymmetric and it is transitive.

∴ R is a partial order.

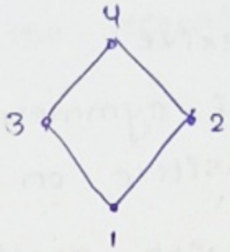
⇒ A is poset under R.

Hasse diagram:

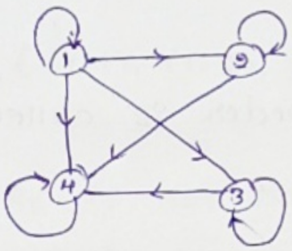
$R = \{\overset{\times}{(1,1)}, (1,2), \overset{\times}{(2,2)}, (2,4), (1,3), \overset{\times}{(3,3)}, (3,4), \overset{\times}{(1,4)}, \overset{\times}{(4,4)}\}$



This is wrong



Digraph of R:



② Let $A = \{1, 2, 3, 4\}$ and R be defined as xRy if $x|y$. P.T (A, R) is a poset. Write its Hasse diagram.

Given $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

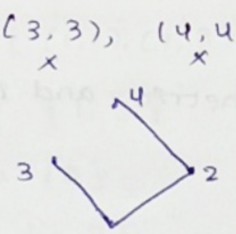
$\therefore$ R is reflexive, anti-symmetric and transitive

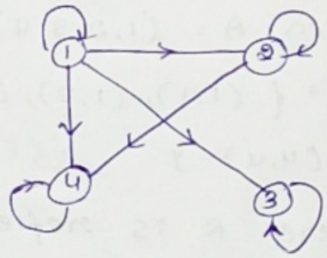
$\therefore$ R is a partial order.

Hence A is a poset on R

Hasse diagram: **Digraph of R**

$R = \{\cancel{(1,1)}, (1,2), (1,3),$
$\cancel{(1,4)}, \cancel{(2,2)}, (2,4)$
$\cancel{(3,3)}, \cancel{(4,4)}\}$





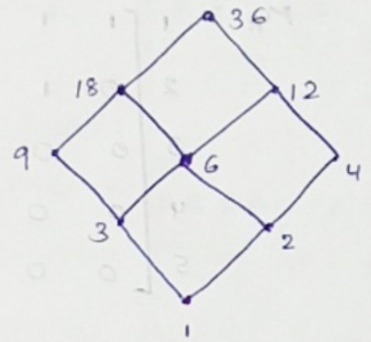
vimpt.

③ Draw the hasse diagram representing the +ve divosors of 36 with the relation defined as aRb if a|b.

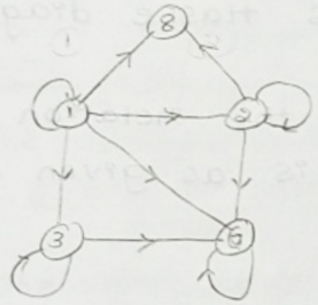
A = {1, 2, 3, 4, 6, 9, 12, 18, 36}

R = { (1,1), (1,2), (1,3), (1,4), (1,6), (1,9), (1,12), (1,18), (1,36), (2,2), (2,4), (2,6), (2,12), (2,18), (2,36), (3,3), (3,6), (3,9), (3,12), (3,18), (3,36), (6,6), (6,12), (6,18), (6,36), (9,9), (9,18), (9,36), (12,12), (12,36), (18,18), (18,36), (36,36), (4,12), (4,4), (4,36) }

Hasse diagram:



④ The digraph for a relⁿ on the set A = {1, 2, 3, 6, 8} is shown below



verify that (A,R) is a poset and write down its Hasse diagram.

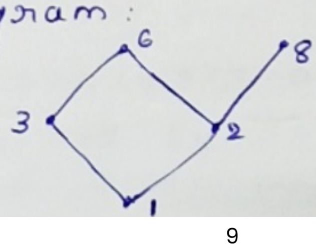
R = { (1,1), (1,2), (1,3), (1,6), (1,8), (2,2), (2,6), (2,8), (3,3), (3,6), (6,6), (8,8) }

∴ R is reflexive, anti-symmetric and transitive

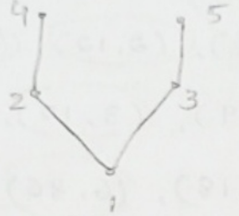
∴ R is a partial order on A.

∴ A is a poset on R.

Hasse diagram:



⑤ Determine the matrix of the partial order whose Hasse diagram is

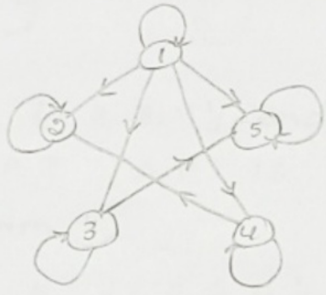


and draw its diagraph and matrix

(write reflexivity and transitive order pairs also in relⁿ)

R = { ~~(1,1), (1,2), (1,3), (2,2), (2,4)~~

R = { (1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (1,3), (1,4)
(1,5), (2,4), (3,5) }



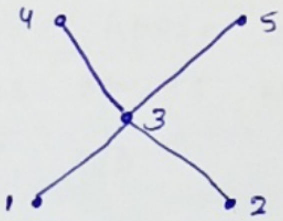
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

⑥ Let A = {1, 2, 3, 4, 6, 12} on A, define the relⁿ R by aRb, iff a divides b. P.T R is a partial order on A. Draw its Hasse diagram.

⑦ Draw the hasse diagram of the relation R on A = {1, 2, 3, 4, 5} whose matrix is as given below

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

∴ R = { (1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4)
(2,5), (3,3), (3,4), (3,5), (4,4), (5,5) }



Extremal elements in posets:

Consider a poset (A, R)

1) **Maximal element:**

An element $a \in A$ is called maximal element of A if there exists no element $x \neq a$ in A such that aRx.

2) **Minimal element:**

An element $a \in A$ is called minimal element of A if there exists no element $x \neq a$ in A such that xRa.

3) **Upper bound:**

An element $a \in A$ is said to be upper bound of a subset B of A. If xRa for all $x \in B$.

4) **Lower bound:**

An element $a \in A$ is said to be lower bound of a subset B of A. if aRx for all $x \in B$.

Some operations on relns:

1. The intersection of R and S denoted by $R \cap S$, defined by $a(R \cap S)b = aRb \wedge aSb$
2. The union of R and S, denoted by $R \cup S$, defined by $a(R \cup S)b = aRb \vee aSb$
3. The difference of R and S, denoted by $R - S$, defined by $a(R-S)b = aRb \wedge a \not S b$
4. The complement of R, denoted by R' or $\sim R$, defined by $a(R')b = a \not R b$

1. Let $A = \{x, y, z\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$, $D = \{2, 3\}$. Let R be a reln from A to B defined by $R = \{(x,1), (x,2), (y,3)\}$ and let S be a reln from C to D defined by $S = \{(x,2), (y,3)\}$

Then $R \cap S = \{(x,2), (y,3)\}$

Note: $R' = (A \times B) - R$

$R \cup S = \{(x,1), (x,2), (y,3)\} = R$

$R - S = \{(x,1)\}$

$R' = \{(x,3), (y,1), (y,2), (z,1), (z,2), (z,3)\}$

Composition of relns:

$R: A \to B$ and $S: B \to C$ then $R \cdot S: A \to C$

~~The~~ R is a subset of $A \times B$ and S is a subset of $B \times C$, then the composition of R and S, denoted by $R \cdot S$ is defined by

$R \cdot S = \{(a,c) \mid \exists$ some $b \in B$ for which $(a,b) \in R$ and $(b,c) \in S\}$

Ex: Let $R = \{(1,1), (1,3), (3,2), (3,4), (4,2)\}$ and $S = \{(2,1), (3,3), (3,4), (4,1)\}$

$\therefore R \cdot S = \{(1,3), (1,4), (3,1), (4,1)\}$

Note:

- $R \cdot S$ is empty, if the intersection of the range of R and the domain of S is empty.
- If R is a reln on a set A, then $R \cdot R$, the composition of R with itself is always defined and sometimes denoted as R^2.

$S \circ R = \{(2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}$

$R \cdot R = \{(1,1), (1,3), (1,2), (1,4), (3,2)\}$

$S \cdot S = \{(3,4), (3,1), (3,3)\}$

$(R \cdot S) \cdot R = \{(1,2), (1,4), (3,1), (3,3), (4,1), (4,3)\}$

$R \cdot (S \cdot R) = \{(1,2), (1,4), (3,1), (3,3), (4,1), (4,3)\}$

$R^3 = R \cdot R \cdot R = (R \cdot R) \cdot R = R \cdot (R \cdot R)$

$= \{(1,1), (1,3), (1,2), (1,4)\}$

Matrix representation of relns:

* If R and S are relns on set A, represented by M_R and M_S, respectively, then $R \cup S$ is the join

of M_R and M_S.

$\therefore M_{R \cup S} = M_R \vee M_S$

* $R \cap S$ is the meet of M_R and M_S.

$M_{R \cap S} = M_R \wedge M_S$

Ex: ① If R and S are relⁿˢ on set A represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ respectively.}$$

then $M_{R \cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 1 \vee 1 & 0 \vee 0 & 1 \vee 1 \\ 0 \vee 1 & 1 \vee 0 & 1 \vee 0 \\ 1 \vee 0 & 0 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$M_{R \cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 1 \wedge 1 & 0 \wedge 0 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 0 & 1 \wedge 0 \\ 1 \wedge 0 & 0 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

② If R ~~is a relⁿ from a~~ set

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R = \{(1,2), (2,2), (1,3)\}$ then $R^{-1} = \{(2,1), (2,2), (3,1)\}$

Note:

In $M_R \cdot M_S \Rightarrow$ multiplication is replaced by $\wedge$
addition is replaced by $\vee$.

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_R \cdot M_S = \begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$