DAYANANDA SAGAR COLLEGE OF ENGINEERING

(An Autonomous Institute Affiliated to VTU, Belagavi)

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DEPARTMENT OF MATHEMATICS

COURSE	**MATHEMATICAL STRUCTURES**
COURSE CODE	**21MAT41A**
MODULE	**4**
MODULE NAME	**RELATIONS**
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Objectives:

After studying this module, student will be able to

- Define a relation, types of relations
- Verify the condition of equivalence relation
- Understand Partial orders, Poset definition
- Do problems on Relation matrix and digraphs plotting

Relations

Let A and B be two sets. Then the set of all ordered pairs (a, b) where a ϵ A and b ϵ B is called the ***Cartesian Product*** or ***Cross Product*** or Product set of A and B and is denoted by **A X B**.

Hence, $$A\ X\ B = \{(a, b) | a \in A \ and\ b \in B\}$$

Ex: If $A = \{1, 0, 2\}$ and $B = \{2, 3\}$ then

$$A\ X\ B = \{(1, 2), (1, 3), (0, 2), (0, 3), (2,2), (2, 3)\}$$

and $$B\ X\ A = \{\ (2, 1), (2, 0), (2, 2), (3, 1), (3, 0), (3, 2)\}$$

evidently, $$A\ X\ B\ \neq B\ X\ A$$

Any subset of A X B is called a ***(binary) relation*** from A to B denoted by R which consists of ordered pairs (a, b) where a ϵ A and b ϵ B and said "***a is related to b by R***" denoted by aRb .

Note:

1. If A is a set with *m* elements and B is a set with *n* elements then the number of relations from A to B are 2^{mn}.
2. If R is a relation from A to A, we say that R is a binary relation on A.

Ex: Let $A = \{\ 1, 2, 3\ \}$ and $B = \{\ 2, 4, 5\ \}$. Then,

(i) $|A\ X\ B| = 3\ X\ 3 = 9$

(ii) Number of relations from A to B are 2^9=512

(iii) Number of relations on A are 2^{3X3} = 512.

Properties of Relations

Reflexive Relation: A relation on a set A is said to be reflexive if (a,a)ϵR for all a ϵ A .

Ex: For a set $A = \{1, 2, 3\ \}$ the relation $R = \{(1,1), (2,2), (3,3)\}$ is a reflexive relation.

Irreflexive Relation: A relation on a set A is said to be irreflexive if (a,a)¢R for any aϵA.

Ex: In the above example if $R = \{(1,1), (2,2)\}$ then R is irreflexive.

Symmetric Relation: A relation R on a set is said to be symmetric if (b,a) ϵ R whenever (a,b) ϵ R for all a,b ϵ A.
Ex: For a set $A = \{1, 2, 3\}$ the relation $R = \{(1,2), (2,1), (3,3)\}$ is a symmetric relation.

A relation which is not symmetric is called an ***asymmetric relation***.
Ex: In the above example the relation $R = \{(1,2), (2,1), (1,3)\}$ is an asymmetric relation.

A relation R on a set A is said to be ***antisymmetric*** if whenever (a,b) ϵ R and (b,a) ϵ R then a=b.

Transitive Relation: A relation R on a set A is said to be Transitive if whenever (a,b) ϵ R and (b,c) ϵ R then (a,c) ϵ R for all a,b,c ϵ R.
Ex: For a set $A = \{1, 2, 3\}$ the relation $R = \{(1,1), (1,2), (2,3), (1,3), (3,1), (3,2)\}$ is a transitive relation.

Equivalence Relation: A relation R on a set A is said to be an equivalence relation on A if R is reflexive, symmetric as well as transitive on A.
Ex: Let $R = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (3,1), (3,3), (4,1), (4,4)\}$ be a relation on a set $A = \{1, 2, 3, 4\}$. Then we see that

(i) R is reflexive as $(a, a) \in R$ for every $a \in A$
(ii) R is symmetric as for every $a, b \in A$, we have $(b, a) \in R$ whenever $(a, b) \in R$.
(iii) R is transitive since for all $a, b, c, \in R$ we have $(a, c) \in R$ whenever $(a, b) \in R$ and $(b, c) \in R$

Hence R is an equivalence relation.

Let A be a nonempty set. Suppose there exist nonempty subsets $A_1, A_2, A_3, \ldots, A_k$ of A such that the following two conditions hold:
(i) A is the union of $A_1, A_2, A_3, \ldots, A_k$; that is $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k$
(ii) Any two of the subsets $A_1, A_2, A_3, \ldots, A_k$ are disjoint, that is, $A_i \cap A_j = \Phi$ for $i \neq j$.
Then the set $P = \{A_1, A_2, A_3, \ldots, A_k\}$ is called ***partition of A***.
Also, $A_1, A_2, A_3, \ldots, A_k$ are called the ***blocks or cells*** of the partition.

Ex: Let $A_1 = \{1, 3, 5, 7\}, A_2 = \{2,4\}, A_3 = \{6,8\}$ be subsets of a set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
Then we see that A is the union of A_1, A_2, A_3. Also, any of these subsets is disjoint. Hence $P = \{A_1, A_2, A_3\}$ is a partition of A with A_1, A_2, A_3 as cells of the partition.

A relation R on set A is said to be a partial ordering relation or a ***partial order*** on A if
(i) R is Reflexive (ii) R is antisymmetric and (ii) R is transitive on A.

A set A with a partial order R defined on it is called a partially ordered set or an order set or a ***poset*** and is denoted by the pair (A,R).

Ex: $(Z,\leq)\ and\ (Z,\geq)$ that is the relations "less than or equal to" and "greater than or equal to" on the "set of integers" are Posets as the two relations are reflexive, antisymmetric and transitive.

Let R be a relation from A to B so that R is a subset of A X B.

Let $m_{ij} = (a_i, b_j)$ and assign the values 1 or 0 to m_{ij} according to the following rule:

$$m_{ij} = \begin{cases} 1\ if\ (a_i, b_j) \epsilon\ R \\ 0\ if\ (a_i, b_j)\ ¢R \end{cases}$$

Then m x n matrix formed by these m_{ij}'s is called the ***relation matrix for R***, and is denoted by M_R or M(R). Since it contains only 0 and 1 as its elements, M(R) is also called the ***Zero-one matrix for R.***

It is to be noted that the rows of M_R correspond to the elements of A and the columns to those of B.
When B = A, the matrix M_R becomes an n x n matrix whose elements are

$$m_{ij} = \begin{cases} 1\ \ if\ (a_i, a_j) \in R \\ 0\ \ if\ (a_i, a_j) ¢\ R \end{cases}$$

Note:
1. For any relation R from a set A to B, M_R is zero matrix iff R=Φ.
2. $[M(R)]^n = M(R^n)$
3. $M(R_1 \circ R_2) = M(R_1) \circ M(R_2)$

Ex: Consider set $A = \{1, 2, 3, 4\}$ and a relation R defined on A by $R\{(1,2), (1,3), (2,4), (3,2)\}$

Here $A = \{a_1, a_2, a_3, a_4\} = B$ where $a_1 = 1,\ \ a_2 = 2,\ \ a_3 = 3,\ \ a_4 = 4.$
Accordingly, $m_{ij} = (a_i, a_j) = (i, j),\qquad i, j = 1, 2, 3, 4.$ Thus

$m_{11} = 0$ because $(1,1) \notin R$
$m_{12} = 1$ because $(1,2) \notin R$
$m_{13} = 1, \quad m_{14} = 0, \quad m_{21} = 0, \quad m_{22} = 0, \quad m_{23} = 0, \quad m_{24} = 1,$
$m_{31} = 0, \quad m_{32} = 1, \quad m_{33} = 0, \quad m_{34} = 0,$
$m_{41} = 0, \quad m_{42} = 0, \quad m_{43} = 0, \quad m_{44} = 0$

Thus, the matrix of R is

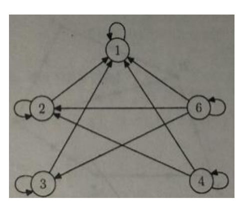
$$M_R = [m_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let R be a relation on a finite set A. Then R can be represented pictorially as described below: Draw a small circle for each element of A and label the circle with the corresponding element of A. These circles are called ***vertices or nodes***. Draw an arrow called an ***edge***, from a vertex x to a vertex y iff (x,y) ϵ R. The resulting representation of R is called ***a directed graph or digraph of R***.

If a relation is pictorially represented by a digraph, a vertex from which an edge leaves is called the ***origin or the source*** for that edge, and a vertex where an edge ends is called the ***terminus*** for that edge. A vertex which is neither a source nor a terminus of any edge is called ***an isolated vertex***. An edge for which the source and terminus vertex are one and the same is called a ***loop***. The number of edges terminating at a vertex is called the ***in-degree of that vertex*** and the number of edges leaving a vertex is called the ***out-degree of that vertex.***

Ex: Let $A = \{1, 2, 3, 4, 5\}$ and R be a relation on A defined by aR_b if and only if $a\ isa\ multiple\ of\ b$. Represent the relation R as a matrix and draw its digraph.
From the given definition we note that,

$$R = \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), (4,4), (5,1), (5,2), (5,3), (5,6)\}$$



Then $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

Operations on Relations

Given the relations R_1 and R_2 from set A to a set B, ***the union*** of R_1 and R_2 denoted by $R_1 \cup R_2$ is defined as a relation from A to B with the property that (a, b) $\in R_1 \cup R_2$ iff (a, b) $\in R_1$ or (a, b) $\in R_2$.

The ***intersection*** of R_1 and R_2 denoted by $R_1 \cap R_2$ is defined as a relation from A to B with the property that (a, b) $\in R_1 \cap R_2$ iff (a, b) $\in R_1$ and (a, b) $\in R_2$.

Given a relation R from set A to a set B, ***the complement of*** R, denoted by $\bar{R}$, is defined as a relation from A to B with the property that (a, b) $\in \bar{R}$ iff (a, b) ¢ R.

Given a relation R from set A to a set B, ***the converse*** of R, denoted by R^c, is defined as a relation from B to A with the property that (a, b) $\in R^c$ iff (b, a) $\in$ R.

Note: If M_R is the matrix of R, the transpose of $(M_{R)})^T$, the transpose of M_R, is the matrix of R^c and $(R^c)^c=R$

Consider a relation R from set A to set B and a relation S from set B to a set C. Then the product or ***composition*** of R and S from set A to set C is defined as

$$R°S = \{(a,c)|a\epsilon A, c\epsilon C and\ there\ exists\ b\epsilon B\ with\ (a,b)\epsilon A and\ (b,c)\epsilon B\}$$

That is If a is in A and c is in C, then (a, c) $\in R°S$ iff there is some b in B such that (a, b) $\in$ R and (b, c) $\in$ S.

Note: 1. M(R) X M(S) = M($R°S$) 2. $R°(S°T) = (R°S)\ °T$

Ex: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations R and S from A to B are represented by the following matrices. Determine the relations $\bar{R}$, $R \cup S$, $R \cap S$, S^c, $R \circ S$.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

From the data we can write

$$R = \{(1,1),(1,3),(2,4),(3,1),(3,2),(3,3)\}$$
$$S = \{(1,1),(1,2),(1,3),(1,4),(2,4),(3,2),(3,4)\}$$

Then $\bar{R} = \{(1,2),(1,4),(2,1),(2,2),(2,3),(3,4)\}$

$$R \cup S = \{(1,1),(1,2),(1,3),(2,4),(3,1),(3,2),(3,3),(3,4)\}$$
$$R \cap S = \{(1,1),(1,3),(2,4),(3,2)\}$$
$$S^c = \{(1,1),(2,1),(3,1),(4,1),(4,2),(2,3),(4,3)\}$$
$$R \circ S = \{(1,1),(1,2),(3,1),(3,4),(3,2),(3,3),(3,4)\}$$

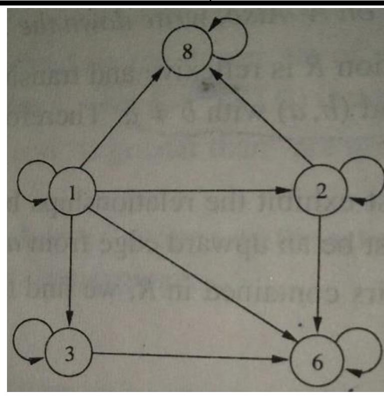
Hasse diagram

The digraph of a partial order drawn by adopting the following conventions is called a ***Poset diagram*** or Hasse diagram.

1. Since a partial order is reflexive, at every vertex in the digraph of a partial order, there would be a cycle or length 1. Hence such cycles need not be exhibited explicitly.
2. If in the digraph of a partial order, there is an edge from a vertex a to vertex b and there is an edge from vertex b to a vertex c, then there should be an edge from a to c since it is transitive. Hence there is no need to exhibit an edge from a to c explicitly.
3. To simplify the format of the digraph of a partial order, the vertices are represented by dots and all edges are pointed upwards.

Note:
1. The matrix of a reflexive relation must have 1's on its main diagonal.
2. The matrix of an irreflexive relation must have 0's on its main diagonal.
3. The matrix of a symmetric relation is a symmetric matrix.
4. A relation R on a set A is transitive iff its matrix $M_R=[m_{ij}]$ is such that
   When $m_{ik}=1$ and $m_{kj}=1$ then $m_{ij}=1$
5. If R is an equivalence relation on a set A, then so is R^c and for an equivalence relation S, R∩S is also an equivalence relation.

Ex: The digraph for a relation on the set $A = \{1, 2, 3, 6, 8\}$ is as shown below. Verify that (A, R) is a poset and write its Hasse diagram.



Looking at the digraph, we see that

$$R=\{(1,1),(1,2),(1,3),(1,6),(1,8),(2,2),(2,6),(2,8),(3,3),(3,6),(6,6),(8,8)\}$$

We see that R is reflexive, transitive and antisymmetric. Thus R, is a partial order on A, that is (A, R) is a poset. The Hasse diagram for R is thus given below.