Chapter
Six

Graph Theory

1. GRAPHS

A graph G is a pair (V, E), where V and E are finite sets and the elements of V are called vertices or points or nodes and the elements of set E are called edges or lines or arcs connecting pair of vertices.

In other words, in a graph there is a mapping from the set of edges E to the set of vertices V such that each edge in E is associated with ordered or unordered pair of vertices of V.

Each pair of vertices connecting with an edge is called the end points of the edge. If we have two vertices v_i and v_j associated with edge e then we write $e = (v_i, v_j)$.

In a graph a vertex is represented by a dot or a small circle and an edge is represented by a line segment joining its end vertices.

Example 1. If $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ are such that $e_1 = (v_1, v_2)$, $e_2 = (v_1, v_3)$, $e_3 = (v_2, v_3)$, $e_4 = (v_3, v_5)$, $e_5 = (v_2, v_5)$, $e_6 = (v_2, v_4)$. Then $G = (V, E)$ is a graph, represented as Fig. 1.

2. DIRECTED GRAPH

A graph $G = (V, E)$ is a said to be a directed graph if it is required to associate a direction with each edge of the graph G *i.e.*, edges are ordered pairs of distinct vertices. A directed arc (i, j) permits flow only from i to j not from j to i.

Example 2 : Figure (2) gives an example of a directed graph on $V = \{v_1, v_2, v_3, v_4\}$ with

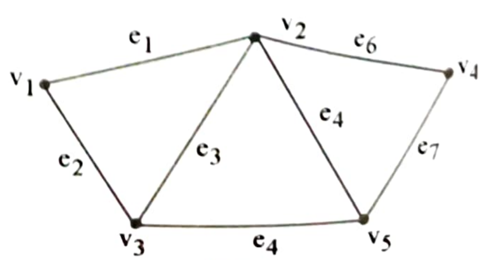


Fig. 1.

$E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_4), (v_4, v_2)\}$. The direction of an edge is shown by placing a directed arrow on the edge.

3. INDEGREE AND OUTDEGREE OF A VERTEX

The indegree of a vertex v in a graph is the number of edges ending at it and is denoted by indeg (g). Similarly the outdegree of a vertex v in a graph is the number of edges beginning from it and is denoted by outdeg (v).

Example 3 : Find the outdegrees and indegrees of each vertex of the following directed graph.

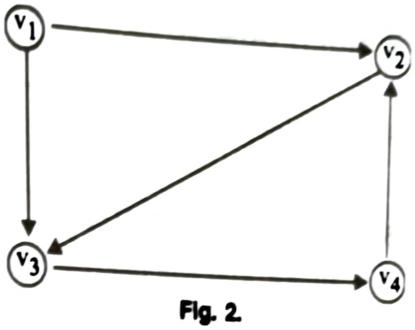


Fig. 2.

Solution :

Vertex :	v_1	v_2	v_3	v_4	v_5	v_6
Indegree :	0	1	2	2	2	2
Outdegree :	2	3	0	2	2	0

4. PARALLEL EDGES IN A GRAPH

In a graph, the edges, having the same pair-of vertices are called parallel edges. In the figure (4) e_1 and e_2 are parallel edges as both have the same end vertices.

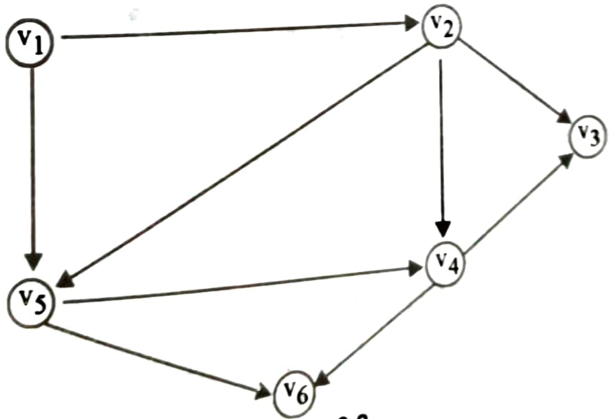


Fig. 5.3.

5. SELF-LOOP IN A GRAPH

An edge which has the same end vertices, is called a self-loop. Edge e_8 in Fig. 4 is a self loop.

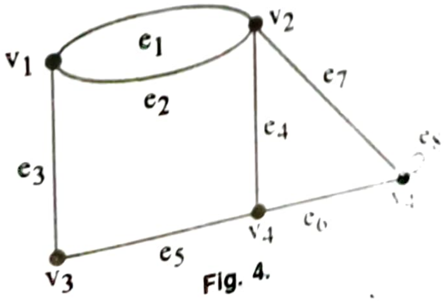


Fig. 4.

6. SIMPLE GRAPH

A graph which has neither parallel edges nor self-loops is called a simple graph.

7. MULTI GRAPH

A graph which contains parallel edges as well as self-loops is called a multi graph. The graph a in Fig. 5 is the example of a multi graph.

8. DEGREE OF A VERTEX

In a graph the degree of a vertex v is the number of edges connected with it. The degree of a vertex v is denoted by $d(v)$ or deg (v).

Example 4 : Find the degree of each vertex of the graph in Figure (5).

Solution : $d(v_1) = 1$, $d(v_2) = 3$, $d(v_3) = 3$,

$d(v_4) = 3$, $d(v_5) = 4$, $d(v_6) = 0$.

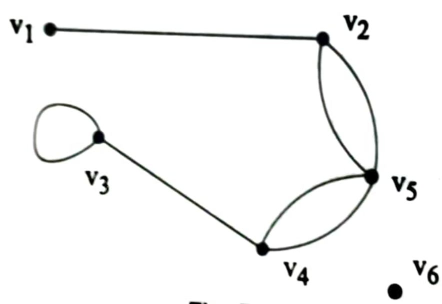


Fig. 5.

9. EVEN AND ODD VERTEX

A vertex is said to be an even vertex when its degree is even otherwise odd vertex.

10. ISOLATED VERTEX, PENDENT VERTEX

A vertex is said to be an isolated vertex if its degree is zero. For example in Figure (5) the vertex v_6 is an isolated vertex.

A vertex is said to be a pendant vertex or an end vertex if its degree is one. For example in figure (5) the vertex v_1 is a pendant vertex.

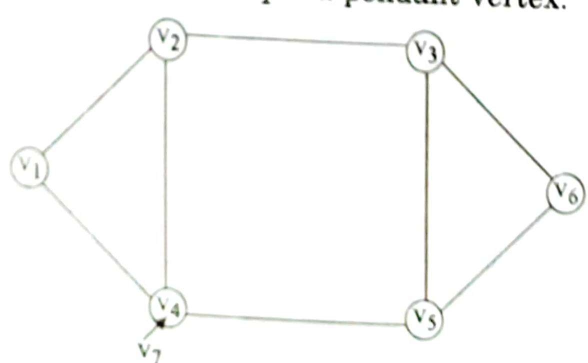


Fig. 6.

11. FINITE AND INFINITE GRAPH

A graph G is said to be finite if both sets V and E are finite otherwise it is called infinite graph. The graph as we have discussed are finite graphs. In this book we study only finite graphs.

Theorem 1 : Prove that the number of vertices of odd degree in a graph is always even.

Proof : Let G be a graph, having vertices of even and odd degree. Then the sum of degrees of all vertices can be expressed as the sum of even degree vertices and odd degree vertices *i.e.*,

$$\sum_{i=1}^{n} \deg(v_i) = \sum_{\text{even}} \deg(v_j) + \sum_{\text{add}} \deg(v_k)$$

Since the sum of degrees of all vertices is even, the left side of above equation is even and in right side the first sum of j vertices is even. Thus for the second sum to be even, the number k must be even. Hence the result is proved.

Illustrations : In figure (6),

1. The sum of degrees of all vertices is even *i.e.*,

$$\begin{aligned} d(v) &= d(v_1) + d(v_2) + d(v_3) + d(v_4) \\ &\quad + d(v_5) + d(v_6) \\ &= 2 + 3 + 4 + 4 + 3 + 2 = 18 \text{ (even)} \end{aligned}$$

2. The number of vertices of odd degrees 3, and 3 are 2, which is even. Hence we have even number of vertices of odd degree.

12. ADJACENT VERTICES OR NODES

Two vertices are said to be adjacent if they are connected by an edge. If there is an edge $e = (v_1, v_2)$ connecting vertices v_1 and v_2 then v_1 and v_2 are called adjacent to each other and also called end points of the edge e. Also the edge e is called incident with the vertices v_1 and v_2. It connects its end points.

In Figure (6) the vertices v_1 and v_2 are adjacent vertices while v_2 and v_5 are not adjacent.

13. UNDIRECTED GRAPH

An undirected graph G consists of a set of vertices V and a set of edges E. The edge set contains the unordered pair of vertices. If $(v_1, v_2) \in E$ then we say v_1 and v_2 are connected by an edge where v_1 and v_2 are the vertices in the set V.

The graph in Fi. (6) is an undirected graph.

Theorem (Handshaking Lemma) 2 : The sum of the degrees of all vertices in a graph G is equal to twice the number of edges in G.

Proof : Let G be a graph with e edges and n vertices $v_1, v_2, \ldots\ldots v_n$. Since each edge is incident on two vertices, each edge contributes two degrees in graph G. Thus the sum of degrees of all the vertices in G is twice the number of edges in G. *i.e.*,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

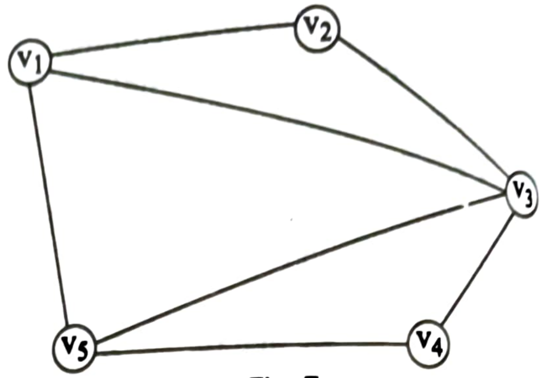


Fig. 7.

To verify the above theorem taking the graph as in Fig. 7 :

$d(v_1) = 3, d(v_2) = 2,$
$d(v_3) = 4, d(v_4) = 2$
$d(v_5) = 3$, therefore

$$\sum_{i=1}^{6} d(v_i) = 3 + 2 + 4 + 2 + 3$$
$$= 14$$
$$= 2 \times \text{number of edges}$$
$$= 2e$$

14. WALK

Let $G = (V, E)$ be a graph. Then a walk in G is a loop-free finite alternating sequence $v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_{n-1}, e_{n-1}, e_n, v_n$ of vertices and edges, begining and ending with the vertices such that v_{i-1} and v_i are the end vertices of the edge e_i where $1 \le i \le n$, and all the edges are distinct. The vertices v_0 and v_n are called the end or terminal vertices of the walk. In a walk, vertices can appear more than once.

Open and Closed Walk

A walk is said to be open if its end vertices are different.

A walk is said to be closed walk if its end vertices are same.

Length of a walk : Length of a walk is the number of edges it contains, with repeated edges counted.

Illustrations : In the graph Fig. 8 with 5 vertices and 7 edges, a walk of length 4 is $W_1 = \{v_1, e_1, v_2, e_6, v_4, e_3, v_3, e_5, v_5\}$

Similarly a walk of length 8 in the same graph is $W_2 = \{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_6, v_2,$
$e_7, v_5, e_5, v_3, e_3, v_4, e_4, v_5\}$.

Here e_3 has been repeated.

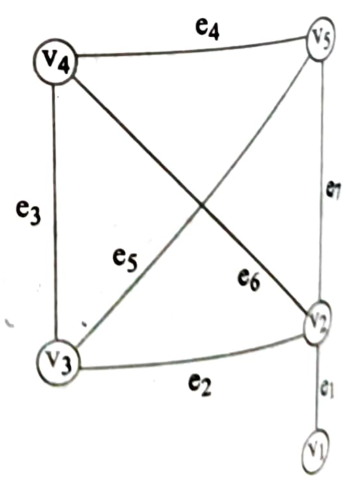


Fig. 8.

If the initial vertex is u and end vertex is then the walk is called $u - v$.

So W_1 and W_2 are $v_1 - v_5$ walk.

Both the above walks are open as their initial and end vertices are distinct. Now, taking

$$W_3 = \{v_2, e_7, v_5, e_5, v_3, e_2, v_2\}$$

$W_4 = \{v_1, e_1, v_2, e_7, v_5, e_5, v_3, e_3, v_4, e_4, v_5, e_7, v_2, e_1, v_1\}$

Both the walks W_3 and W_4 are closed walks as their end vertices are same.

15. TRIAL

An open walk is said to be a trial if the edges are not repeated vertices can be repeated in a trial.

In the graph given in figure (8), the sequence $\{v_1, e_1, v_2, e_7, v_5, e_5, v_3, e_2, v_2\}$ is a trial in which vertex e_2 has been repeated but edge has not been repeated.

16. PATH

An open walk in which no vertex appears more than once is called a path. If there exists a path from vertex v_0 to v_n in an undirected graph, then there always exists a path from v_n to v_o too, but in a directed graph it is not necessary.

In the figure (9), the open walk $\{v_1, e_6, v_2, e_5, v_4, e_9, v_5, e_3, v_6\}$ is a path. But the open walk $\{v_2, e_6, v_1, e_1, v_3, e_8, v_2, e_5, v_4\}$ is not a path because the vertex v_2 is repeated twice in it.

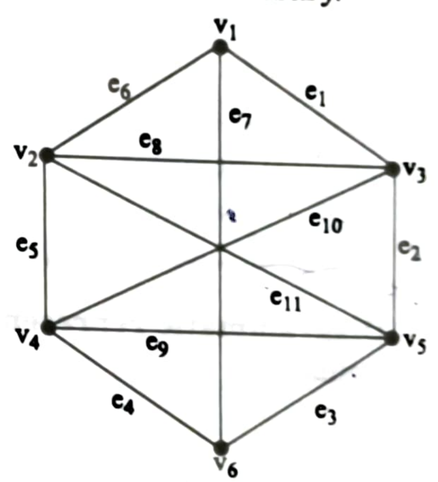


Fig. 9.

17. SIMPLE PATH

A simple path from the vertex u to vertex v is the one which no edge is repeated in the path. That is, a simple path is also a trial where all the vertices and all edges are different except the first vertex is equal to last vertex.

A path is called elementary path if no vertex repeated more than once in the path.

The above example of a path in figure 9, is also an example of a simple path.

18. LENGTH OF A PATH

The number of edges in a path is called the length of the path. The sequence of $n + 1$ vertices of a graph in which each pair of vertices is an edge of the graph, is of length n.

An edge which is not a loop is called a path of length one. A loop is considered as a walk but not as a path.

19. CIRCUIT

A circuit is a closed trial or a closed walk in which no edge is repeated. Vertices may however be repeated. *i.e.* the sequence $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$ is a circuit if $v_0 = v_n$ and all e_i's are distinct.

20. CYCLE

A cycle is a circuit in which no vertex is repeated except the initial vertex, which is also the end vertex of the sequence.

In other words a cycle is a closed walk in which neither any edge nor any vertex except the initial vertex, is repeated.

Note : Every cycle is a circuit but converse is not always true.

Fig. 10.

The graph in figure (10) is the example of a circuit as well as a cycle.

The walk $\{v_1, e_1, v_2, e_2, v_3, e_4, v_4, e_6, v_5, e_5, v_3, e_3, e_1\}$ in the graph in figure (11) is a circuit because this walk is a closed walk in which no edges are repeated. But this walk is not a cycle as the vertex v_3 has been repeated.

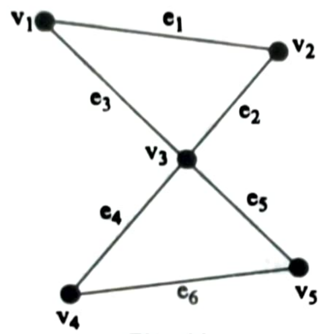


Fig. 11.

Diagrammatic representation of walk, trial, circuit, path and cycle-relationship :

WALK
(Edges and vertices may be repeated)

OPEN WALK
Initial and final vertices different

TRIAL
(Edges are not repeated)

PATH
Vertices also not repeated

CLOSED WALK
Initial and final vertices same

CIRCUIT
(Edges are not repeated)

CYCLE
Vertices also not repeated except initial vertex

21. SUBGRAPH

If $G = (V, E)$ is a directed or undirected graph, then a graph $H = (V', E')$ where $V' \le V$ and $E' \le E$ is called a subgraph of G, provided each of H has the same end vertices in H as in graph G.

For example the graph in Fig. (12) is the subgraph of the graph in Fig. (11).

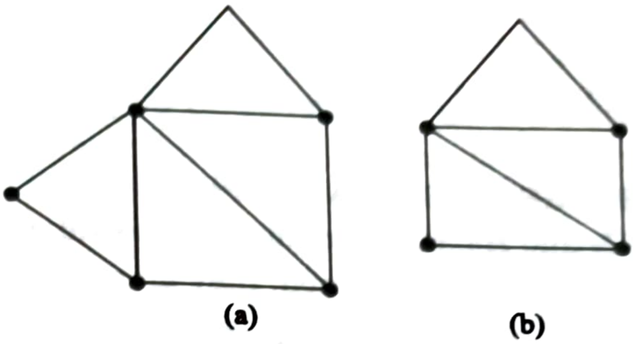


Fig. 12.

22. SPANNING SUBGRAPHS

A subgraph H of a graph G is said to be spanning subgraph if H has all the vertices of G. If $H = (V', E')$ is a spanning subgraph of graph $G = (V, E)$ then $E' \leq E$ and $V' = V$. In the figure given below H_1, H_2 and H_3 are the spanning subgraph of the graph G.

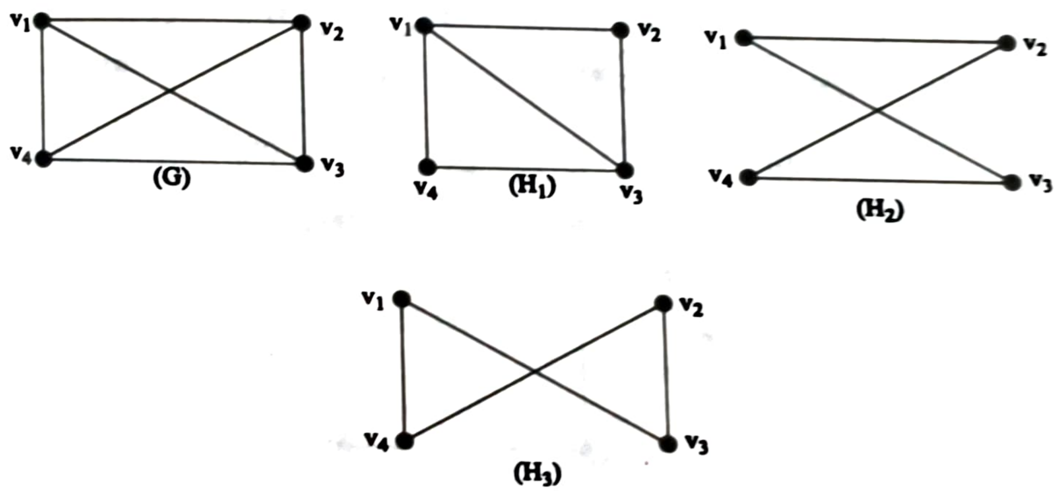


Fig. 13.

23. SUBGRAPH OF A DIAGRAPH

Let $G = (V, E)$ be a directed graph. Then a graph $H = (V', E')$ is called the subgraph of G if

(i) All the vertices of H are in G *i.e.*, $V' \leq V$.

(ii) All the directed edges of H are in G *i.e.*, $E' \leq E$.

(iii) Each directed edge of H has the same pair of ordered end vertices in H as in G.

In the figure given below. The graph H is the subgraph of the graph G.

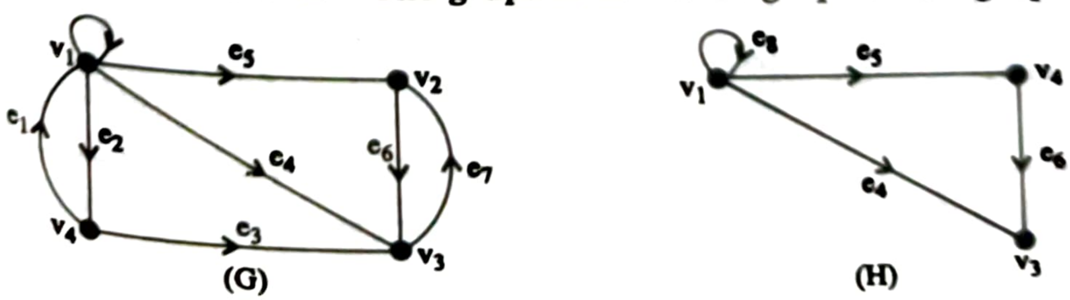


Fig. 14.

Here $V = \{v_1, v_2, v_3, v_4\}$ and

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

$V' = \{v_1, v_3, v_4\} \leq V$

$E' = \{e_4, e_5, e_6, e_8\} \leq E$

24. EDGE-DISJOINT SUBGRAPHS OF A GRAPH

Two subgraphs H_1 and H_2 of a graph G are said to be edge-disjoint if H_1 and H_2 do not have any edge in common, they may have vertices in common.

For example, in the following figure subgraphs H_1 and H_2 are edge-disjoint subgraphs of a graph G.

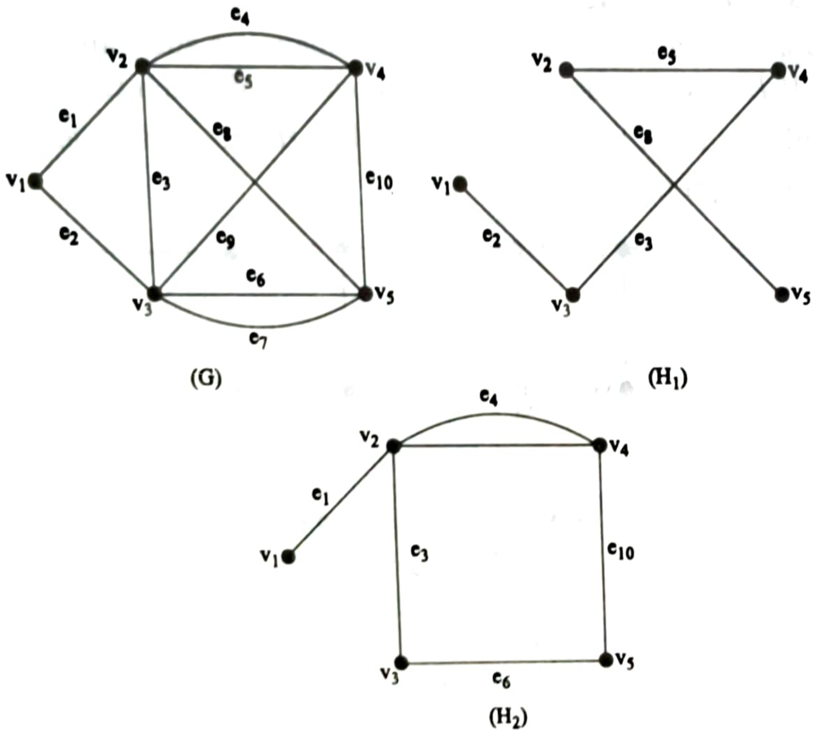


Fig. 15.

25. VERTEX-DISJOINT SUBGRAPHS

Two subgraphs H_1 and H_2 of a graph G are said to be vertex-disjoint if H_1 and H_2 do not have any vertices in common. Such graphs can not possibly have edges in common.

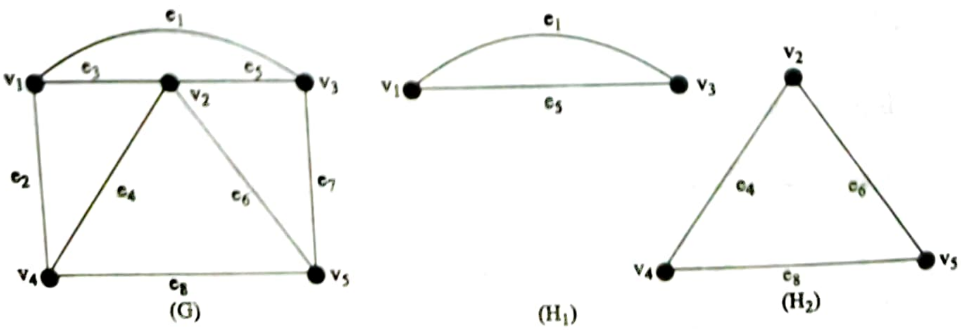


Fig. 16.

In the figure given below the graphs H_1 and H_2 are vertex-disjoint sub graphs of graph G.

26. OPERATIONS ON GRAPHS COMPLEMENT OF A SUBGRAPH

Let $H = (V', E')$ be the subgraph of a graph $G = (V, E)$. The complement of graph with respect to the graph G is the subgraph $\overline{H} = (V; E - E')$.

In the figure given below $\overline{H}$ is the complement of the subgraph H of graph G.

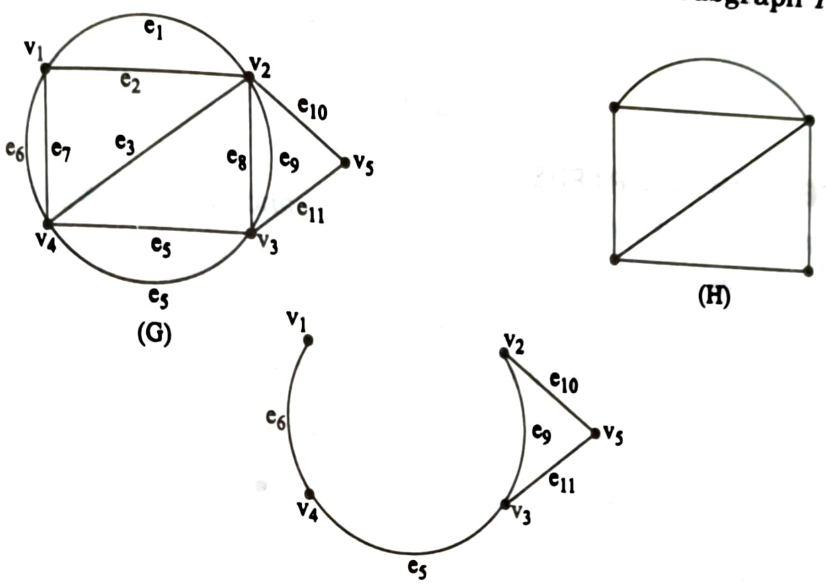


Fig. 17.

27. COMPLEMENT OF A SIMPLE GRAPH

Let $G = (V, E)$ be a simple graph. Then the complement of G is the graph. $\overline{G} = (V, E')$ such that $e \in E'$ if and only if $e \notin E$.

In the figure given below the graph $\overline{G}$ is the complement of the simple graph G.

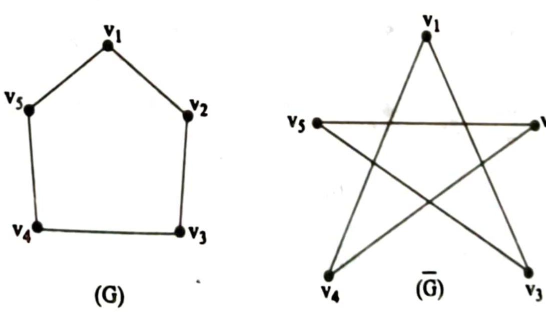


Fig. 18.

28. UNION OF GRAPHS

Let $G = (V_1, E_1)$ and $G = (V_2 - E_2)$ be two graphs. Then the union of G_1 and G_2 is also a graph G_3 and is defined as follows :

$$G_3 = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

In the figure given below graph G_3 is the union of graphs G_1 and G_2

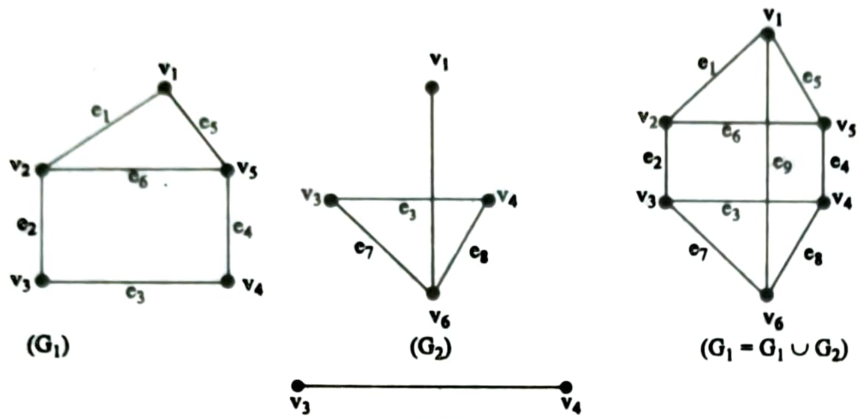


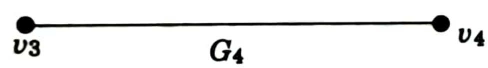
Fig. 19.

29. INTERSECTION OF GRAPHS

The intersection of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph G_3 defined as below :

$$G_3 = G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$$

For example the intersection of the graphs G_1 and G_2 given as in figure (19) is a graph G_3, *i.e.*,



30. RING SUM OF GRAPHS

The ring sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is denoted by $G_1 \oplus G_2$ and it is also a graph, defined as

$$G \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2 - E_1 \cap E_2)$$

Example : If G_1 and G_2 are two graphs, given as below :

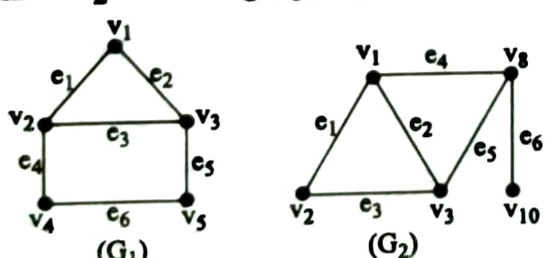


Fig. 20.

Find $\quad G_1 \cup G_2, G \cap G_2, G_1 \oplus G_2$

Solution :

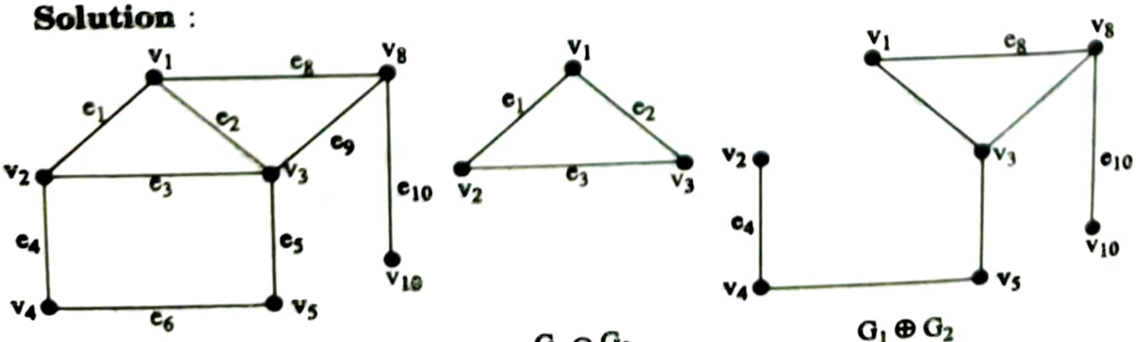
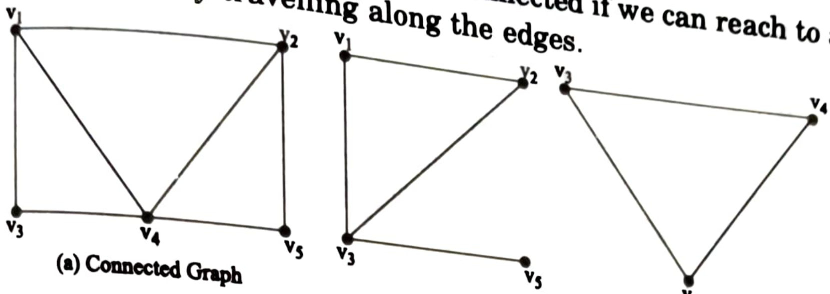


Fig. 21.

31. CONNECTED AND DISCONNECTED GRAPH

A graph G is said to be connected if there exists at least one path between every pair of vertices, otherwise it is disconnected.

In other words a graph G is said to be connected if we can reach to any vertex from any other vertex by travelling along the edges.



(a) Connected Graph

(b) Discontented Graph

Fig. 22

32. COMPONENTS

A disconnected graph G has two or more connected graphs, These connected graphs are called components of the graph G.

For example the graph in figure (22-a) is connected whereas the graph in figure (22-b) is not connected.

The graph in fig. (22-b) is disconnected but it has two components whose vertex sets are $\{v_1, v_2, v_6, v_7\}$ and $\{v_3, v_4, v_5\}$.

Theorem : A graph $G = (V, E)$ is disconnected if and only if the vertex set V of G can be partitioned into two non-empty, disjoint subsets V_1 and V_2 such that there does not exist edge in G whose one end vertex is in V_1 and the other in V_2.

Proof : Let $G = (V, E)$ be a disconnected graph. Let $u \in V$. Let V_1 be the set of all those vertices which are joined by paths to u. Since G is disconnected therefore V_1 has not all vertices of G. Let V_2 be the set of all vertices of G, which are not in V_1 *i.e.*, $V_1 \cap V_2 = \phi$. Then V_2 is non-empty and no vertex in V_1 is joined to any vertex in V_2. Hence the set V can be partitioned into two subsets V_1 and V_2 such $V_1 \cap V_2 = \phi$.

Conversely, suppose that such a partition of vertex set V_1 exists. Consider two arbitrary vertices u and v of V in G such that $u \in V_1$ and $v \in V_2$. Now it is clear that there is no path between u and v as there exists no edge whose one end vertex is in V_1 and the other in V_2. Thus G is not connected *i.e.* G is disconnected.

Theorem : A simple graph with n vertices and k components can have at most $\dfrac{(n-k)(n-k+1)}{2}$ edges.

Proof : Let $g = (V, E)$ be a simple graph with n vertices and k components. Let n_i represent the number of vertices in a component i, for $1 \le i \le k$ of a graph G. Then we have

$$\sum_{i=1}^{k} n_i = n.$$

We shall use the following inequality to prove the theorem.

$$\sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k). \qquad \ldots (1)$$

Since we know that the maximum number of edges in simple graph with n vertices is $\frac{n(n-1)}{2}$, therefore the maximum number of edges in the ith component of G is $\frac{1}{2} n_i (n_i - 1)$.

Thus the maximum number of edges in G is

$$\sum_{i=1}^{k} \frac{1}{2} n_i (n_i - 1) = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right] = \frac{1}{2}[n^2 - (k-1)(2n-k) - n]$$

$$= \frac{1}{2}(n-k)(n-k+1), \text{ proved.}$$

Corollary : If a simple graph G, with n vertices has more than $\frac{(n-1)(n-2)}{2}$ edges then G is connected.

Proof : Let G be the graph as given. Suppose G has k components, then by the previous theorem, G has at most $(n-k)(n-k+1)/2$ edges. We must have

$$\frac{(n-k)(n-k+1)}{2} > \frac{(n-1)(n-2)}{2}$$

which is possible only when $k = 1$. Thus G is connected.

Example : Let G be a simple graph with 6 vertices and 11 edges. Check whether the graph G is connected or not.

Solution : Above corollary states that a simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges is connected.

In the question given that $n = 6$. Therefore

$$\frac{(n-1)(n-2)}{2} = \frac{(6-1)(6-2)}{2} = 10$$

which is less than 11.

Hence G is connected.

Example : Find the connected components of the graph given in fig. 23.

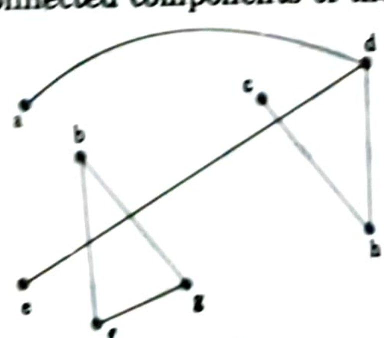


Fig. 23.

Solution : Take any vertex say a, and find all vertices connected to it. Thus the set of connected components is $\{a, d, e, h, c\}$.

Now select a vertex not included in this component and repeat the process to obtain the other component.

Thus the another sets of the connected components is $\{b, f, g\}$.

33. ISOMORPHIC GRAPHS

Let $G_1(V_1, E_1)$ and $G_2 = (E_2, V_2)$ be two undirected graphs. A function $f : V_1 \to V_2$ is called a graph isomorphism if (i) f is one-one and no to function *i.e.* there exists a correspondence between their vertex sets (V_1 & V_2) and as well as their edges sets (E_1 & E_2).

If such a function exists then the graphs G_1 and G_2 are called isomorphic graphs.

It means that if in graph G_1 an edge e_k is incident with vertices v_i and v_j then in graph G_2 its corresponding e_k must be incident with the vertices v_i and v_j that correspondent to the vertices v_i and v_0 respectively.

Remark : By definition of two isomorphic graphs we must have

(i) equal number of vertices and edges.

(ii) equal number of vertices of same degree.

(iii) It should be possible to start from any vertex in two graphs and find a circuit that includes every edge of the graph.

The following graphs G_1 and G_2 are isomorphic graphs.

Vertices v_1, v_2, v_3, v_4 and v_5 in G_1, corresponds to v_1, v_2, v_3, v_4 and v_5 respectively. Edges e_1, e_2, e_3, e_4, e_5 and e_6 in G_1 corresponds to $e_1', e_2', e_3', e_4', e_5'$ and e_6' respectively in G_2.

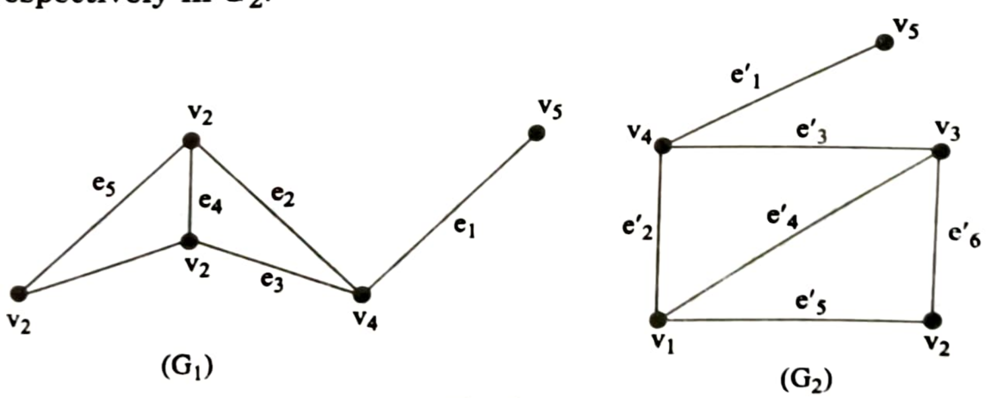


Fig. 24

Example : Show that the graphs as given below are isomorphic.

Solution : To show that two graphs are isomorphic. We can arrange vertices from both graphs having same degrees in decreasing order of degrees. If both the graphs have vertices of the same degree, then they are isomorphic, otherwise not. Note that no vertex is left out in any of the graphs which is not matching in terms of degree with the vertex in other graph.

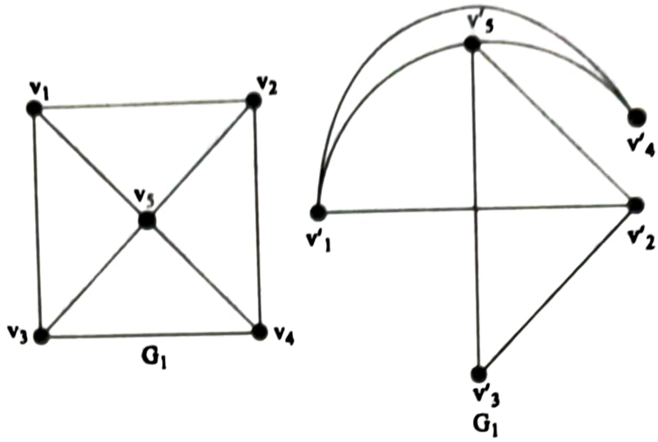


Fig. 25.

In graph G_1 and G_2 pairs of vertices in decreasing order of their degrees are as follows :

Degree 3 : $d\,(a) = d\,(V_3), d\,(a) = d\,(v_2)$

Degree 2 : $d\,(b) = d\,(v_1), d\,(e) = d\,(v_4), d\,(c) = d\,(v_5)$

Since both the graphs contain vertices having same degrees, therefore they are isomorphic.

Example : Show that the graphs G_1 and G_2 as given below, are not isomorphic.

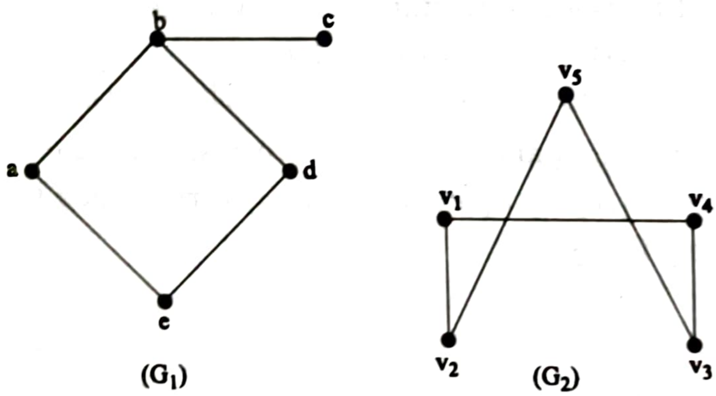


Fig. 26

Solution : In the graph G_1 vertex b has degree 3, whereas graph G_2 has no vertex of degree 3. Hence graphs are not isomorphic.

Example : Show that the graphs given below are isomorphic.

Solution : The above two graphs are isomorphic can be seen as follows.

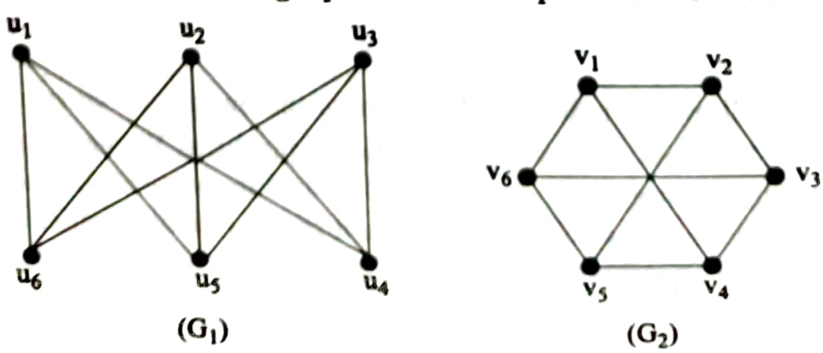


Fig. 27

From the graphs G_1 and G_2 it is clear that both the graphs have same number of vertices and each vertex in each graph has the same degree as 3. Thus the graphs are isomorphic.

34. TYPES OF GRAPHS

A graph is said to be a null graph if there are no edges or every vertex is an isolated vertex.

A null graph is also called vertex graph. Mathematically a graph $G = (V, E)$ is said to be null graph if $E = \phi$ and $V \neq \phi$.

A null graph with six vertices is given below :

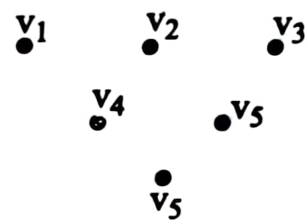


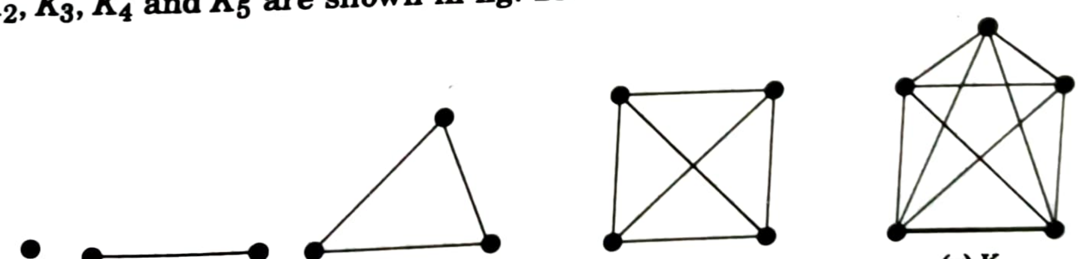
Fig. 28. Null graph.

35. COMPLETE GRAPH

A complete graph is a simple graph $G = (V, E)$ where for all vertices $v_i, v_j \in V$, $v_i \neq v_j$, there exists an edge (v_i, v_j).

In other words, in a complete graph every vertex is connected to every other vertex *i.e.* every pair of different vertices are adjacent.

If a complete graph G has n vertices, then it will be denoted by K_n and has $n(n-1)/2$ edges and degree of each vertex is $(n-1)$, Graphs K_1, K_2, K_3, K_4 and K_5 are shown in fig. 28.



(a) K_1 (b) K_3 (c) K_4 (d) K_4 (e) K_5

Fig. 28. Examples of undirected complete graphs.

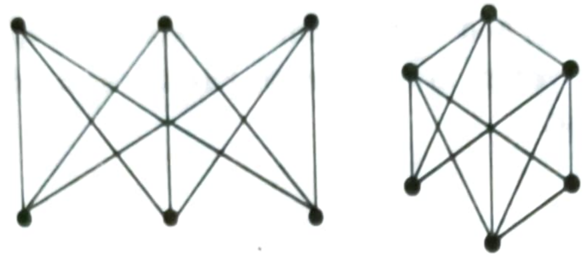
36. REGULAR GRAPH

A graph $G = (V, E)$ is said to be regular if each vertex in G has some degree. If a graph G is regular with $d(v) = r$ for each vertex v in G, then G is called r-regular.

Every null graph is regular of degree zero and a complete graph K_n is regular of degree $n - 1$.

If a graph has n vertices and it is regular of degree r, then it has $\frac{rn}{2}$ edges.

For example in Figure-29 (a) and 29(b) are 3-regular and 4-regular.



(a) 3-regular graph (b) 4-regular graph

Fig. 29..

Example : Draw the connected regular graphs of degree 0, 1 and 2.

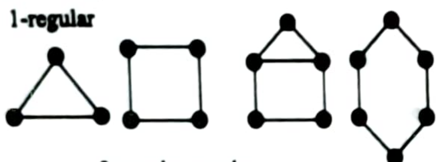
Solution :

(i) 

0-regular

(ii) 

1-regular

(iii) 

2-regular graphs

Fig. 30.

37. BIPARTITE GRAPH

A loop-free graph $G = (V - E)$ is said to be a Bipartite graph, if it is possible that the vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge (v_i, v_j) connects a vertex in V_1 and a vertex in V_2 *i.e.* there is no edge joining two vertices in set V_1 or V_2.

In other words a graph G is called Bipartite graph of $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$ and every edge of G is of the form (v_i, v_j) with $v_i \in V_1$ and $v_j \in V_2$. Following graph are the examples of bipartite graph.

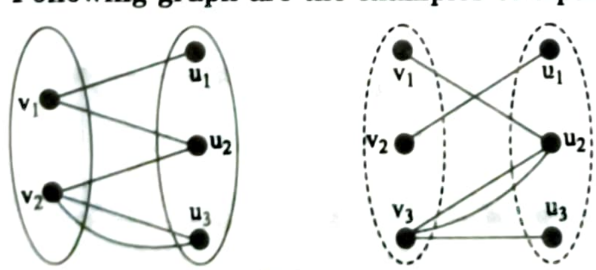
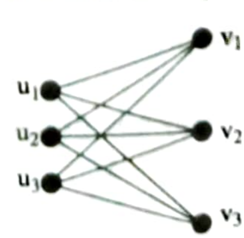


Fig. 31.

Complete Bipartite Graph : A graph $G = (V, E)$ is called a complete bipartite graph if its vertex set V can be decomposed into two subsets V_1 and V_2 such that each vertex of V_1 is connected to each vertex of V_2 *i.e.*, there is an edge between each pair of vertices in V_1 and V_2. If number of vertices in V_1 is m and number of vertices in V_2 is n then such a graph is denoted by K_{mn}. A complete poipatite graph $K_{m,n}$ has $m + n$ vertices and mn edges. An example of a complete bipartite graph $K_{3,3}$ is given as below :

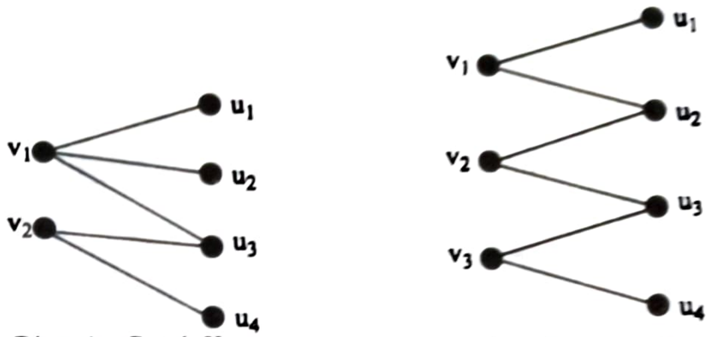


$K_{3,3}$ (Complete bipartite graph)

Fig. 32.

Example : Draw the bipatite graph $K_{2,4}$ and $K_{3,4}$. Asuming any number of edges.

Solution :

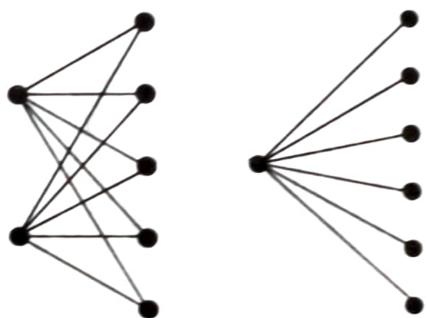


Bipartite Graph $K_{2,4}$ Bipartite Graph $K_{3,4}$

Fig. 33.

Example : Draw the complete bipartite graph $K_{2,5}$ and $K_{1,6}$.

Solution :



$K_{2,5}$ $K_{1,6}$

Fig. 34. Complete bipartite graphs.

Example : Show that the maximum number of edges in a complete bipartite graph of n vertices is $n^2/4$.

Solution : Let $G = (V, E)$ be a complete bipartite graph with n vertices. Let V_1 and V_2 be the partition of the vertex set V of G. Let n_1 and n_2 be the number of vertices in the portions V_1 and V_2 respectively .

Since G is complete bipartite, each vertex in V_1 is joined to each vertex of V_2 by exactly one edge. Thus G has $n_1 . n_2$ edges where $n_1 + n_2 = n$. But the maximum value of $n_1 n_2$ subject to $n_1 + n_2 = n$ is $\frac{n^2}{4}$. Hence the maximum number of edges is G is $\frac{n^2}{4}$.

Example : If the intersection of two paths in a graph G is disconnected, then their union has at least one circuit.

Solution : Let P_1 and P_2 be two paths in a graph $G = (V, E)$ such that $P_1 \cap P_2$ is disconnected graph. We have to show that $P_1 \cup P_2$ contains a circuit. Since $P_1 \cap P_2$ is disconnected therefore it has a pair of vertices v_i, v_j such that there is no path between them in $P_1 \cap P_2$. Since $v_i, v_j \in P_1 \cap P_2$ implied that

v_i, v_j belong to both P_1 and P_2 and so there is a path in P_1 and P_2 which connects vertices v_i and v_j. Since there is no path in $P_1 \cap P_2$ connecting v_i and v_j, therefore we may assume that the paths in P_1 and P_2 connecting v_i and v_j are edge disjoint.

Thus union of these two paths connecting v_i and v_j forms a circuit in $P_1 \cup P_2$ as shown below.

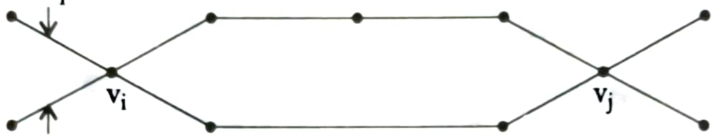
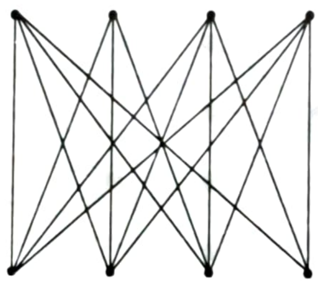


Fig. 35.

Example : Draw a graph which is both regular and bipartite.

Solution : An example of the graph $K_{4,\,4}$, is shown in the graph given below, which is 3-regular and bipartite.



$K_{1,4}$

Fig. 36.

39. Traversable Multigraph

A graph G which contains parallel edges as well as loops is called a multigraph.

A multigraph G is said to be traversable if it can be drawn without any break in the curve and without repeating any edge *i.e.*, if there is a path which has all vertices and each edge exactly once. A traversable multigraph is always connected and has either zero or two vertices of odd degree.

In the figure given below the graph G' is the traversable of the multigraph G.

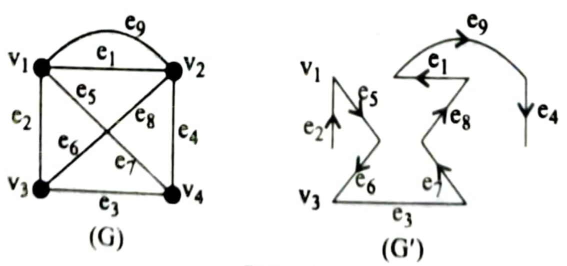


Fig. 37.

Remark : To prove a graph to be traversable, find the degree of each vertex and determine whether all the vertices are of even degree or exactly two are of odd degree. If either condition is satisfied then the graph will be traversable.

Example : Find the traversable trail for the graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)$.

Solution : The graph $G = (V, E)$ is shown in fig. 38(a). The traversable trail should start from vertices of odd degree and end at the other vertices of odd degree.

In figure 38(a) vertices v_3 and v_4 are of odd degree. Figure 38(b) gives one such solution as $\{v_3, v_1, v_4, v_2, v_3, v_4\}$.

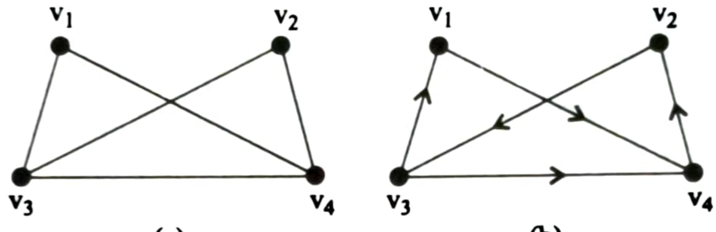


Fig. 38.

Example : Find which of the graph shown as below are traversable.

Solution :

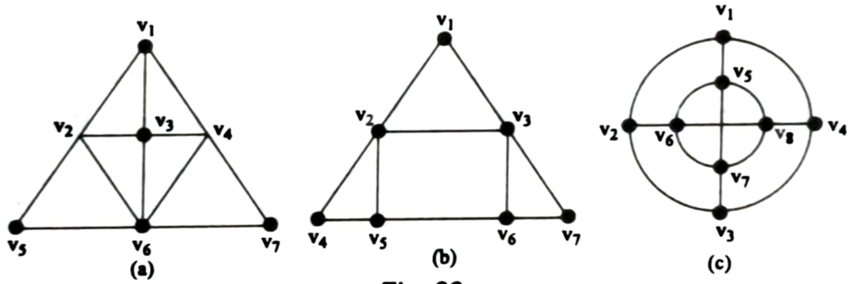


Fig. 39.

(i) Graph (a) is traversable as two of vertices v_1 and v_6 are of odd degree.

(ii) Graph (b) is traversable as two of vertices v_2 and v_3 are of odd degree.

(iii) Graph (c) is not traversable as four vertices v_1, v_2, v_3 and v_4 are of odd degree.

40. PETERSEN GRAPH

A 3-regular graph shown below is known as Petersen graphs.

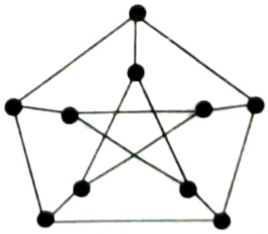


Fig. 40. Petersen graph.

41. CYCLE GRAPHS

A connected 2-regular graph is called a cycle graph. A cycle graph with n_1, vertices is denoted by C_n. Cycle graph is also called circuit graph. A cycle graph with six vertices is given below.

Fig. 41. Cycle graph-C_6

42. PATH GRAPHS

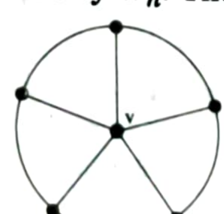
The graph obtained from a cycle graph C_n by removing an edge is called a path graph. The path graph with n vertices is denoted by P_n. The path graph P_6 is given in fig. 42.

Path graph P_6

Fig. 42.

43. WHEELS

The graph obtained from C_{n-1} (Cycle graph with $n-1$ vertices) by joining each vertex to a new vertex v is called the wheel with n vertices. The wheel with n vertices is denoted by W_n. The wheel W_6 is given below.



Wheel – W_6

Fig. 43.

Exercise

1. Describe formally the graph shown in Figs. 44 (a) and (b).

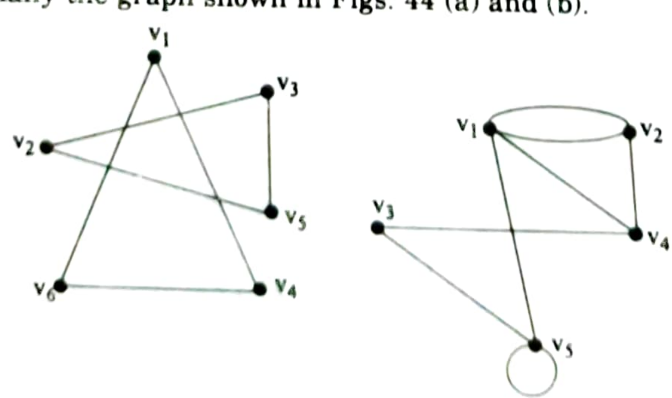


Fig. 44.

2. Find the degree of each vertex in the graphs shown in fig. 44 (a) and (b).
3. Let G be a graph shown in figure 46. Check which of the following sequences of edges form a path.

(a) $\{e_2, e_6, e_7, e_3\}$

(b) $\{e_2, e_3, (v_4, v_5), e_5\}$

(c) $\{e_6, e_4, e_7\}$

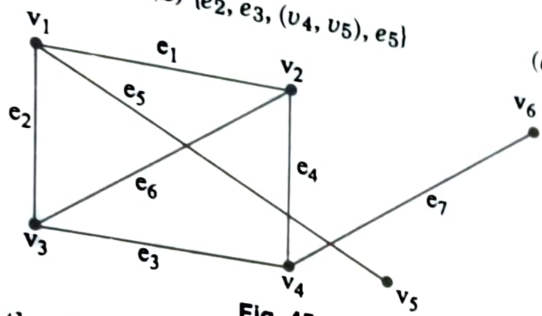


Fig. 45.

4. Find all cycles in the fig. 45.
5. Find $G_1 \cup G_2$, $G_1 \cap G_2$ and $G_1 \oplus G_2$ from the graphs G_1 and G_2, given as below :

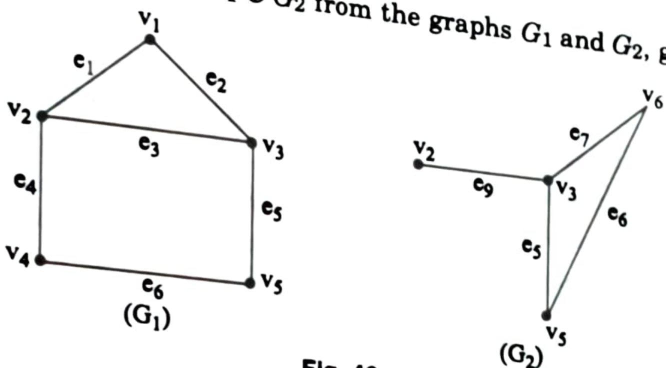
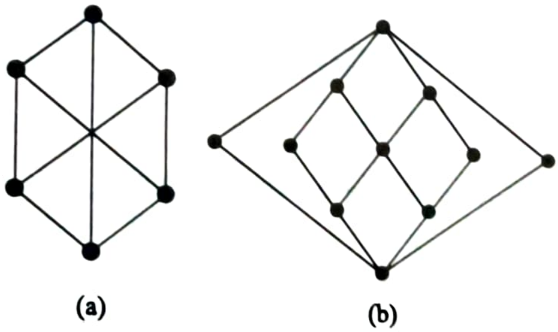


Fig. 46.

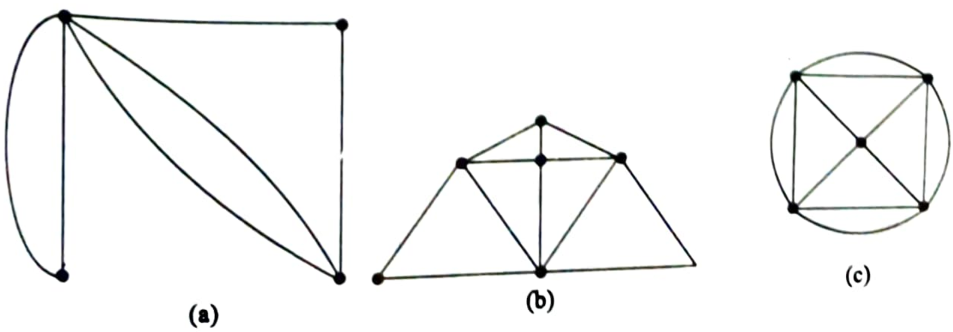
6. Let G be a 3-regular graph with n-vertices. What is the sum of the degrees of the vertices? Show that in such a graph n must be even.
7. Prove that a graph is bipartite if and only if it contains no circuit of odd length.
8. Which of the following are bipartite graphs?



(a) (b)

Fig. 47.

9. Prove that 9 simple graph with n vertices is not bipartite if it has more than $\frac{n^2}{4}$ edges.
10. Determine which of the graphs shown in fig. 48 is traversable?

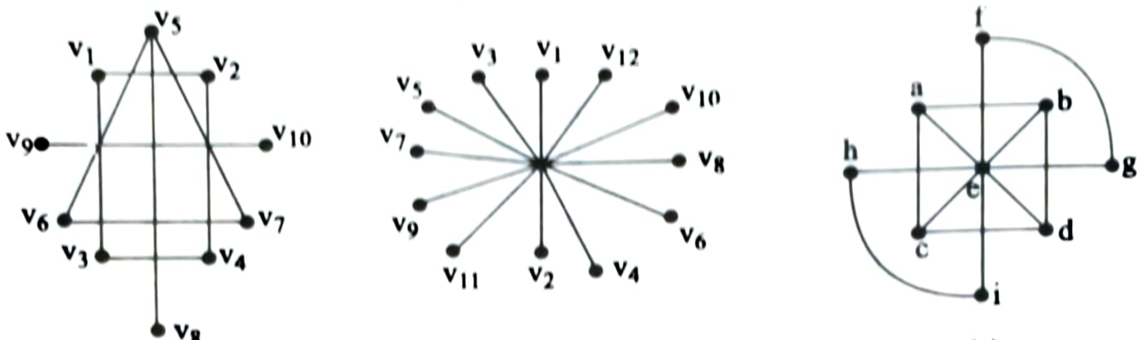


(a) (b) (c)

Fig. 48.

11. Find the number of edges in a complete graph, K_n:

12. Consider the graphs shown in figure 49 (a), (b) and (c). Determine whether the graphs are connected or disconnected graphs. Also write their connected components.



(a) (b) (c)

Fig. 49.

13. Suppose G_1 and G_2 are isomorphic graphs. Find the number of connected components of G_2 if G_1 has eight connected components.

14. It is possible for wheel W_n $(n \geq 3)$ to be bipartite?

15. Prove that a connected graph G remains connected after removing an edge e from a circuit.

16. Show that the graphs given below are isomorphic :

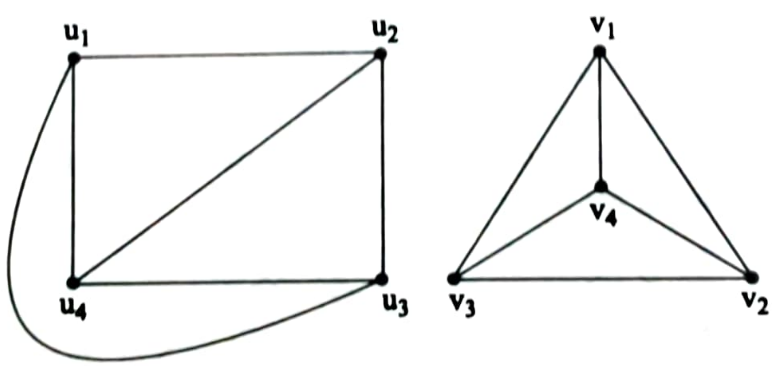


Fig. 50.

17. Show that the graphs given below are not isomorphic :

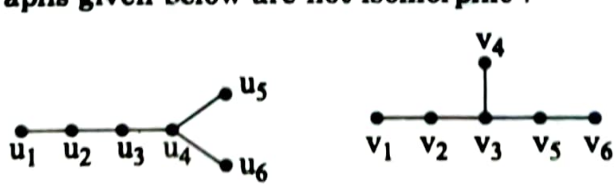


Fig. 51.

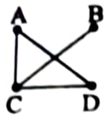
18. Show that the graphs given below are isomorphic.

Fig. 52.

19. Show that the maximum number of edge in a simple graph with n, vertices is $\underline{n(n-1)}$

20. Can a graph with seven vertices be isomorphic to its complement.

21. Find the adjacency matrix of the following graphs.

(b) 

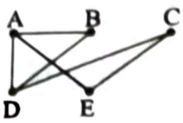
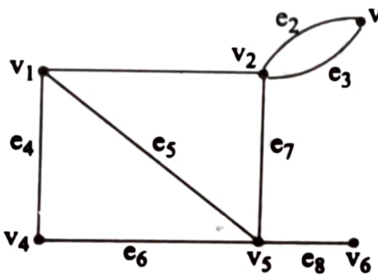
(d) 

Fig. 53.

22. Draw a graph with the following adjacency matrices :

(a) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

23. Find the incidence matrix of the following graphs :

(a) 

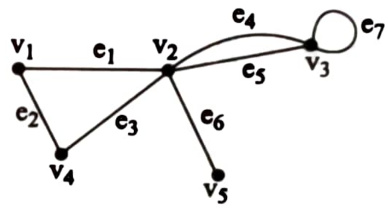
(b) 

Fig. 54.

24. Draw a graph with the help of the following incidence matrix :

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Answers

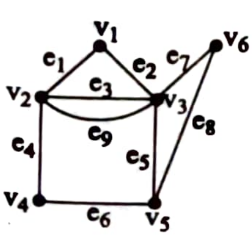
2. (a) Degree of each vertex is 2.

(b) $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 2$, $d(v_4) = 3$, $d(v_5) = 4$.

3. (c)

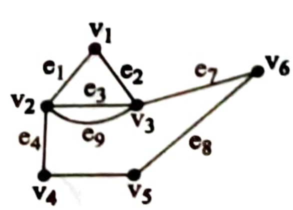
4. Three is only one cycle in the graph *i.e.* 3-cycle (v_2, v_3, v_4, v_2).

5.



$G_1 \cup G_2$

$G_1 \cap G_2$



$G_1 \oplus G_2$

8. Both are bipartite.

10. (a) Yes, since exactly two of the vertices are of odd degree.

(b) Yes, since five vertices are of even degree and two are of odd degree.

(c) Yes, since all the vertices are even.

11. ${}^{n}C_2$

12. (a) It is a disconnected graph and its connected components are : $\{v_1, v_2, v_3, v_4\}$, $\{v_5, v_6, v_7, v_8\}$ and $\{v_9, v_{10}\}$.

(b) It is a disconnected graph and its connected components are $\{v_1, v_2\}$, $\{v_3, v_4\}$, $\{v_5, v_6\}$, $\{v_7, v_8\}$, $\{v_9, v_{10}\}$ and $\{v_{11}, v_{12}\}$.

(c) It is a connected graph.

13. The graph G_2 must also have eight connected components.

14. No. because W_n $(n \geq 3)$ contains triangles.

20. No

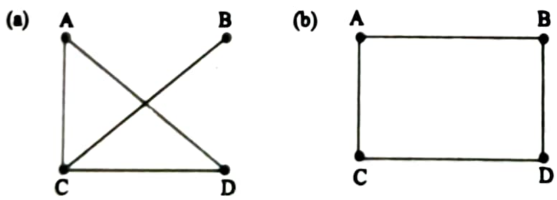
21. (a)
$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(c)
$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

22.

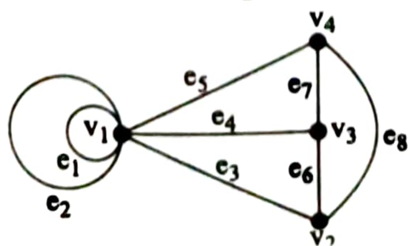


23.

(a)
$$\begin{array}{cc} & \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{cc} & \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

24.



44. MATRIX REPRESENTATION OF A GRAPH

The graphical representation of a graph does not provide input data in the form of numbers say 0 and 1. Therefore, we can not use computers for the study of graphs that require the input data in the form of numbers. In order to make the use of computers in the study of graph theory. We have to represent the grahps in such a form that provides input data of graphs in the form of numbers. We solve this problem by matrix representation of graphs. The follwoing four representations of the graph are most common.

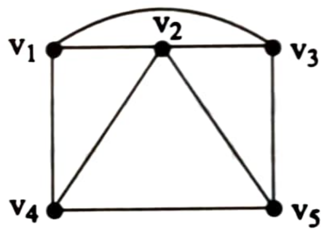
1. Adjacency matrix or vertex-vertex adjacency matrix
2. Incidence matrix or vertex-edge incidence matrix.
3. Adjacency lists
4. Forward and reverse star representation.

45. REPRESENTATION OF UNDIRECTED GRAPHS

Adjacency matrix : Adjacency matrix of a graph G, consists of n vertices and no parallel edges in the order $v_1, v_2, \ldots v_n$ as an $n \times n$ matrix $A = [x_{ij}]$ defined as

$$x_{ij} = \begin{cases} 1, & \text{if there is an edge between } v_i \text{ and } v_j \\ 0, & \text{otherwise} \end{cases}$$

Example : Write the adjacency matrix of the following graph :



Solution : The adjacency matrix is given below :

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

46. PROPERTIES OF ADJACENCY MATRIX

1. The Adjacency matrix A is symmetric *i.e.*, $a_{ij} = a_{ji}$ for all i and j.

2. If graph has no self-loops then the diagonal entries of A are zero and vice versa.

3. A graph G is connected graph iff the matrix $B = A + A^2 + A^3 + \ldots\ldots\ldots A^{n-1}$, has no zero entries of the main diagonal.

4. Adjacency matrices are useful for determination of paths in graphs.

5. The number of non-zero elements in the matrix is equal to the sum of degrees of all vertices of the graph.

6. If G has two components G_1 and G_2 then adjacency matrix A of G is

$$A = \begin{bmatrix} A(G_1) & O \\ O & A(G_2) \end{bmatrix},$$

where $A(G_1)$ and $A(G_2)$ denote the adjacency matrices of G_1 and G_2 respectively.

47. INCIDENCE MATRIX

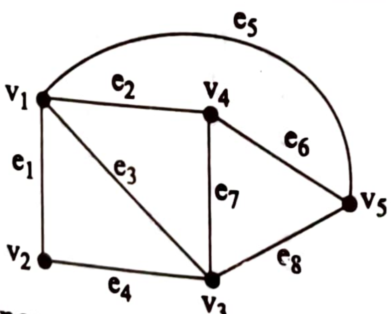
Let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, v_3, \dots v_n\}$ and $E = \{e_1, e_2, \dots e_m\}$ and has no self-loops. Then we define an $n \times m$ matrix $M = [x_{ij}]$, whose n rows correspond to the n vertices and m columns correspond to the m edges as follows :

$$x_{ij} = \begin{cases} 1, & \text{if vertex } v_i \text{ is incident on edge } e_j \\ 0, & \text{otherwise.} \end{cases}$$

48. PROPERTIES OF INCIDENCE MATRIX

1. Each column contains exactly two unit elements.
2. A row with all non-zero elements corresponds to an isolated vertex.
3. If graph G is connected with m vertices, then rank of incidence matrix m is $n - 1$.
4. The sum of each row gives the degree of corresponding vertex.
5. If a graph has parallel edges then the corresponding columns in incidence matrix M are identical.

Example : Write the incidence matrix of the following graph.



Solution : The incidence matrix is given below :

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence matrix

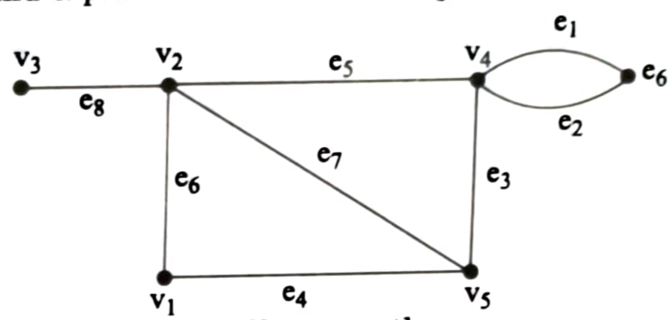
49. PATH MATRIX

Let $G = (V, E)$ be a graph with n vertices and m edges. Let $v_i, v_j \in V$. Let K different paths between two vertices v_i and v_j be denoted by $P_1, P_2, P_3, \ldots P_K, K \geq 1$. Then the path matrix between the vertices v_i and v_j is given by :

$$P(v_i, v_j) = [p_{ij}], \text{ where}$$

$$p_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ path contains } j^{th} \text{ edge} \\ 0, & \text{otherwise} \end{cases}$$

Example : Find a path matrix between v_3 and v_4 in the following graph.



Solution : There are three different paths

$$\{e_8, e_5\}, \{e_8, e_7, e_3\}, \{e_8, e_6, e_4, e_3\}$$

between v_3 and v_4. Since there are 8 edges and 3 paths say p_1, p_2 and p_3 So, we have a 3×8 path matrix $P(v_3, v_4)$.

$$P(v_3, v_4) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

50. CIRCUIT MATRIX

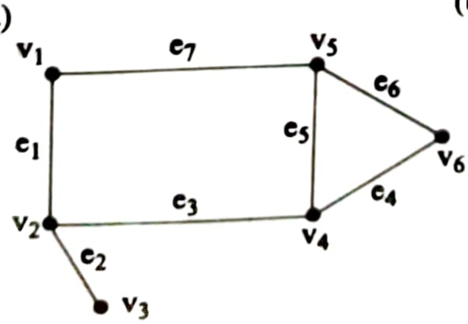
Let G be graph with n vertices and m edges. Suppose that the graph G has C circuits. Then the circuit matrix of graph G is denoted by $C(G)$ and is defined as

$$C(G) = [c_{ij}]_{c \times m}, \text{ where}$$

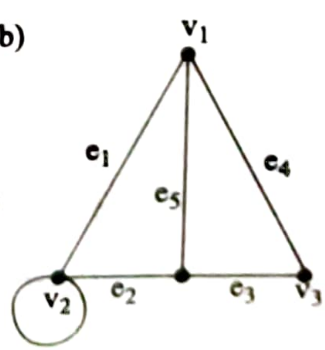
$$x_{ij} = \begin{cases} 1, & \text{if i}^{\text{th}} \text{ circuit contains j}^{\text{th}} \text{ edge} \\ 0, & \text{otherwise} \end{cases}$$

Example : Find the circuit matrix of the following graphs.

(a)



(b)



Solution : (a) The given graph (a) has three circuits $C_1 = (e_1, e_3, e_4, e_6, e_7)$, $C_2 = (e_1, e_3, e_5, e_7)$, $C_3 = (e_4, e_5, e_6)$. Thus the corresponding circuit matrix shall be as given below :

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(b) The given graph (b) has three circuits $C_1 = (e_6)$, $C_2 = (e_1, e_2, e_5)$, $C_3 = (e_3, e_4, e_5)$.

Thus the corresponding circuit matrix shall be as given below :

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

51. MATRICES IN DIAGRAPHS

Adjacency matrix of a Diagraph : Let G be a diagraph with n vertices and m edges, containing no parallel edges. Then the adjacency matrix A is defined as

$$A = [a_{ij}]_{n \times n},$$

where
$$a_{ij} = \begin{cases} 1, & \text{if there is an edge directed } i^{th} \text{ vertex } j^{th} \text{ vertex} \\ 0, & \text{otherwise} \end{cases}$$

It has different names *e.g.*, transition matrix, relation matrix, connection matrix, precedence matrix or preference matrix, predecessor matrix in different displines where it is used.

52. INCIDENCE MATRIX OF A DIAGRAPH

Let G be a diagraph with n vertices and m edges and with no self-loops. Then the incidence matrix M of graph G is defined as

$$M = [a_{ij}]_{n \times m}, \text{ where}$$

$$\begin{aligned} a_{ij} &= 1, \quad \text{if the } j^{th} \text{ edge is incident out of } i^{th} \text{ vertex} \\ &= -1, \text{ if the } j^{th} \text{ edge is incident into } i^{th} \text{ vertex} \\ &= 0, \quad \text{if } j^{th} \text{ edge is not incident.} \end{aligned}$$

Example : Find the adjacency matrix of the following diagraph :