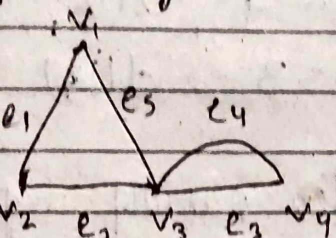
Maths Question Bank

Module - 1

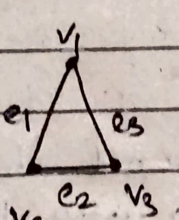
1) Define with an example

(i) Graph - A graph is a mathematical structure consisting of two sets V and E where V is a non-empty set of vertices and E is set of edges Eg. 

(ii) Multigraph - A graph is said to be a multiple edged graph or a multigraph if it contains multiple edges and no self loops

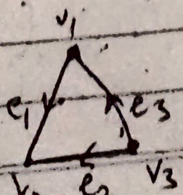
Eg. 

(iii) Simple graph - a graph which doesn't contain multiple edges or self loops

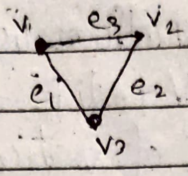
Eg. 

b) Define with example

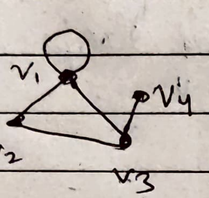
(i) Directed graph - If a graph contains edges with directions, it is called a directed graph



(ii) Undirected graph - graph which is not directed is called undirected graph



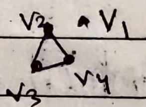
2(a) Define (i) Degree of vertex - It is equal to the no. of edges which are incident on the vertex with self loop counted twice.



$d(v_1) = 4$, $d(v_2) = 2$

$d(v_3) = 3$, $d(v_4) = 1$

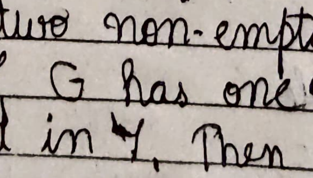
(ii) Isolated vertex - a vertex having degree 0 is called isolated vertex.



Eg. V_1

2(b) Define with example

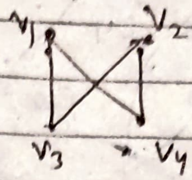
(i) Bipartite graph - If the vertex set V of a graph can be divided into two non-empty disjoint subsets in such a way that edge of G has one end in X and one end in Y, Then G is called bipartite.



$X = \{v_1, v_2, v_3\}$

$Y = \{v_4, v_5, v_6\}$

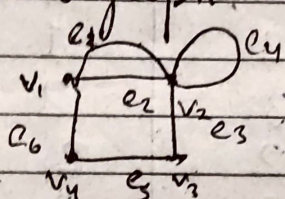
(ii) Complete bipartite graph - If every vertex in X is disjoint with every vertex in Y, then it is called a complete bipartite graph. If X and Y contain m and n vertices then this graph is denoted as $G_{m,n}$



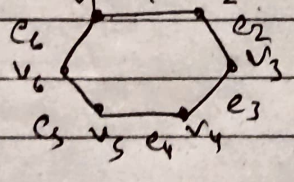
$X = \{v_1, v_2\}$
$Y = \{v_3, v_4\}$

Q. Define (a)(i) pseudograph - A graph which contains both self loop and multi-edge is called pseudograph.

Eg.

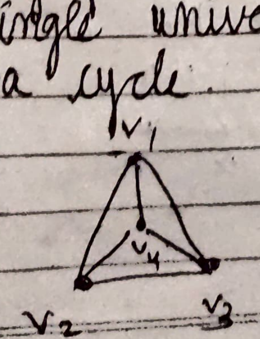


(ii) Cycle graph - graph which ~~contains~~ consists of a single cycle or some no. of vertices in a closed chain.

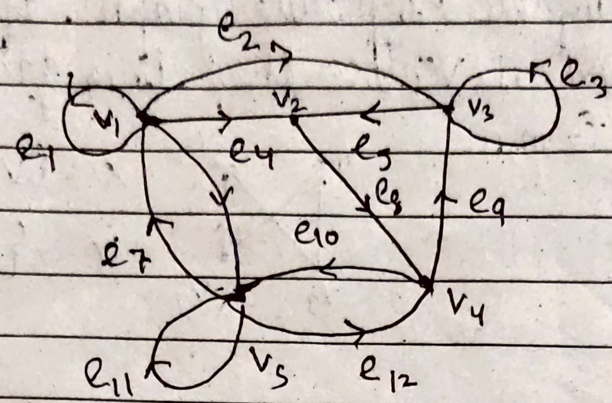


(iii) Wheel graph - A graph formed by connecting a single universal vertex to all vertices of a cycle.

Eg



b) Verify $\sum_{i=1}^{n} deg^+(v_i) = \sum_{i=1}^{n} deg^-(v_i) = |E|$



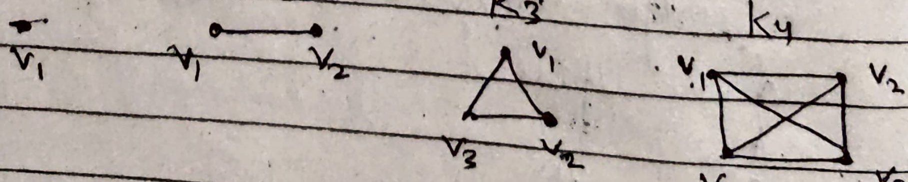
	$deg^+(v_i)$	$deg^-(v_i)$	Total
v_1	2	4	~~4~~ 6
v_2	2	1	3
v_3	3	2	6
v_4	2	2	4
v_5	3	3	6
	12	12	24

$|E| = 12$

4. (a) Draw the graphs

(i) K_n for $1 \le n \le 4$

sol) K_1 K_2 K_3 K_4

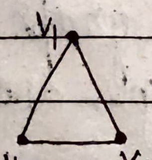


• mv f2.sh tt4.sh

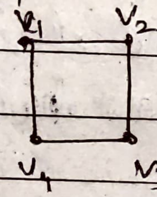
To count lines   wc -l filename
line counts

Chmod , grep { grep -v ~~file~~ "word" filename
displays lines which don't
contain word"

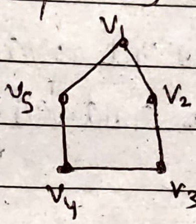
(ii) a) C_3



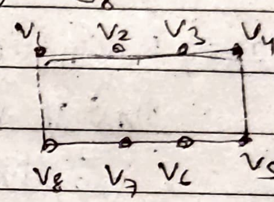
b) C_4



c) C_5

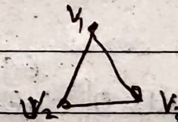


d) C_8

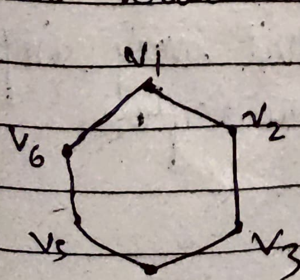


b) Is C_3 and C_6 bipartite? Explain

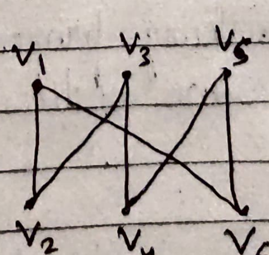
C_3 is not a bipartite graph as every vertex in C_3 is connected to the other vertex



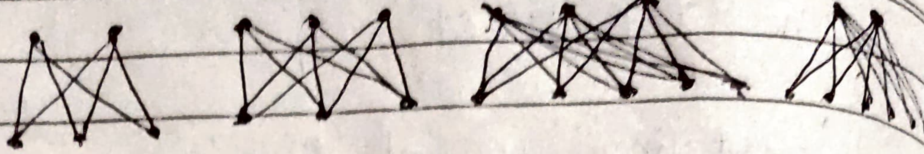
C_6 ~~C8~~ is a bipartite graph as we can partition its vertices into 2 sets $X = \{V_1, V_3, V_5\}$ & $Y = \{V_2, V_4, V_6\}$, the vertices of set are not connected to each other while connected to the other vertices in the other set.



$\cong$

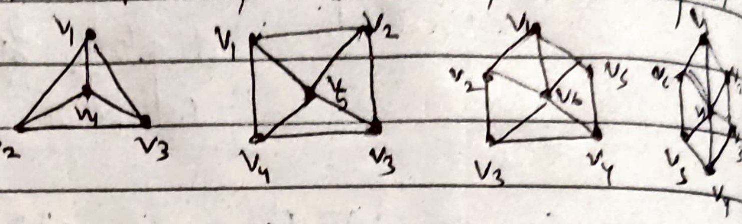


5(a) ~~$K_{2,3}$~~



$K_{2,3}$ $K_{3,3}$ $K_{3,5}$ $K_{2,6}$

5(b) a) W_3 not possible b) W_4 c) W_5 d) W_6 e) W_7



6(a) Define compliment graph and if simple graph G has v vertices and e edges, how many edges does $\bar{G}$ have?

Given a graph G, it's complement $\bar{G}$ is defined on the same set of vertices, two set of vertices are adjacent in $\bar{G}$ if they're non-adjacent in G

Total no. of ~~vertices~~ edges in a complete graph $= \frac{v(v-1)}{2}$

Vertices in $\bar{G} = v$

Edges in $\bar{G} = \frac{v(v-1)}{2} - E(G)$

= Total − edges in G

b) State Handshaking property. Show that every simple graph has two vertices of the same degree.

Ans Handshaking lemma states that in any graph, the sum of degree of all vertices is twice the no. of edges contained in it.

$$\sum_{i=1}^{n} d(v_i) = 2|E|$$

Assume that a graph G has n vertices it can't have one vertex of degree 0 and one vertex of degree n-1, as if n=2 we would have both vertex of degree 1 i.e. equal degree. to satisfy degree n-1.

7(a) Define Adjacency matrix and incidence matrix

Let a_{ij} denote the no. of edges b/w v_i and v_j then $A = [a_{ij}]_{m \times m}$ is called adjacency matrix of G if

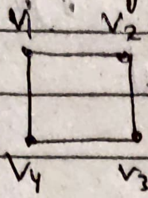
$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

(b) Incidence Matrix - Let G be a graph with m vertices $v_1, v_2, \ldots v_m$ and n edges $e_1, e_2, \ldots e_n$ matrix $M = [m_{ij}]_{m \times n}$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident on } e_j \\ 0 & \text{if } v_i \text{ is not incident on } e_j \\ 2 & \text{if } v_i \text{ is an end of the loop } e_j \end{cases}$$

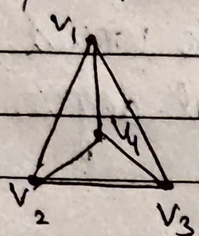
7(a) Write the adjacency matrix for C_4 and W_4

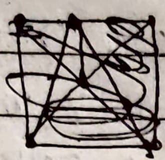
(i) $C_4 \Rightarrow$



$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$$

(ii) W_4

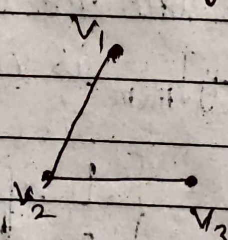




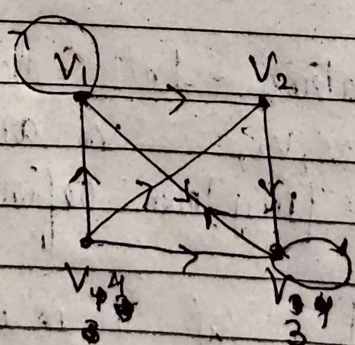
$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 1 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 1 & 1 & 0 \end{array}$$

8(a) Draw a graph of the given adjacency matrix

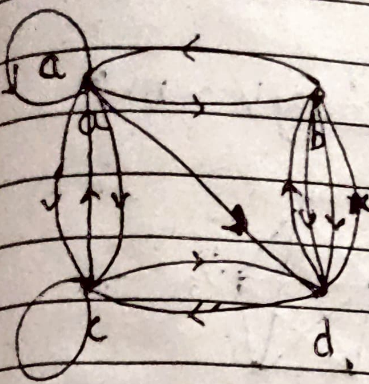
(i) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$



(ii) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

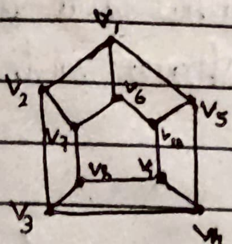


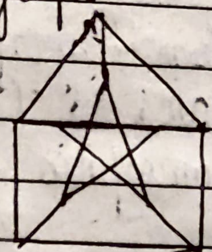
b) Find the adjacency matrix of the following directed multigraph



$$\begin{array}{c|cccc} & a & b & c & d \\ a & 1 & 1 & 2 & 1 \\ b & 1 & 0 & 0 & 2 \\ c & 1 & 0 & 1 & 1 \\ d & 0 & 2 & 1 & 0 \end{array}$$

9(a) Draw Petersen graph





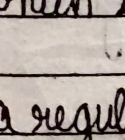
(b) Is there a simple graph with 1,1,3,3,3,4,6,7 as the degree of its vertices

Ans It is not realisable as there are two vertices with degree 7 and degree 7, which means all other vertices need to have degree 2 or greater than 2 which is not the case given.

10 (a). Explain a regular graph with example

A regular graph is one where each vertex has the same no. of neighbour i.e. every vertex has the same degree.

Eg.



b) How many vertices does a regular graph of degree 4 with 10 ~~vertices~~ edges have?

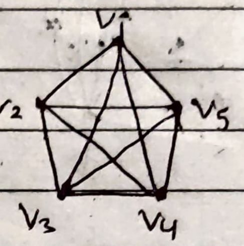
Sol W.K.T $\sum \deg v = 2|E|$

As degree of all vertices is 4, $\sum \deg v = n \times 4$

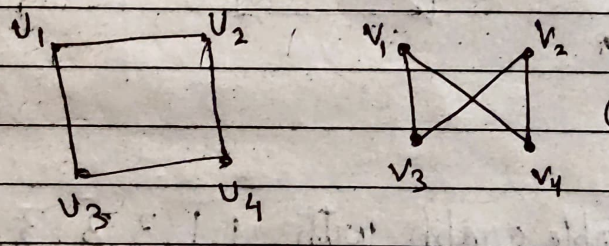
$n \times 4 = 2|E|$

$n \times 4 = 2 \times 10$

$\boxed{n = 5}$



1) a) Define isomorphism of graphs. Show that the graphs $G = (V, E)$ and $H = (W, F)$ shown in the figures are isomorphic.



$V_1 = U_1$

$V_3 = U_3$

$V_4 = U_2$

Sol Two graphs G_1 and G_2 are said to be isomorphic if

→ The no. of vertices are same

→ The no. of edges are same

→ An equal no. of vertices with given degree

→ ~~vector~~ vertice correspondance & edge correspondance valid

$f(U_1) = V_1, \ f(U_2) = V_2, \ f(U_3) = V_3, \ f(U_4) = V_4$

$U_1 \to V_1$

$U_2 \to V_4$

$U_3 \to V_3$

$U_4 \to V_2$

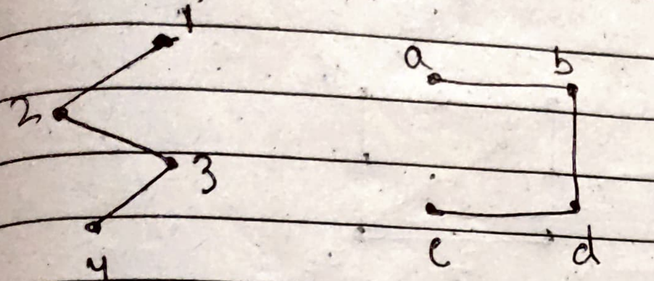
$f(U_1) f(U_2) = V_1 V_4$

$f(U_1) f(U_3) = V_1 V_3$

$f(U_2) f(U_4) = V_4 V_3$

$f(U_3) f(U_4) = V_2 V_3$

b) Show that the two graphs shown in the figure are isomorphic

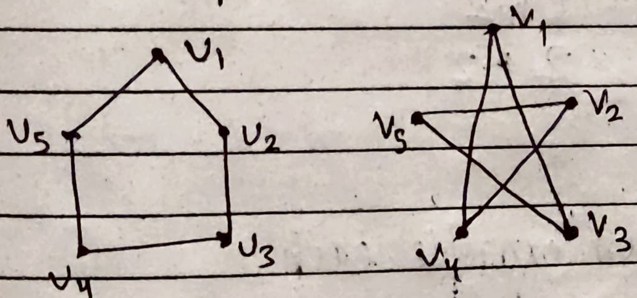


vertices = 4
Edges = 4.

Correspondance $1 \to a$, $2 \to b$, $3 \to d$, $4 \to c$

$f(1)f(2) = ab$
$f(2)f(3) = bd$
$f(3)f(4) = dc$

12. Show that the graphs G and $\bar{G}$ shown in the figure are isomorphic.

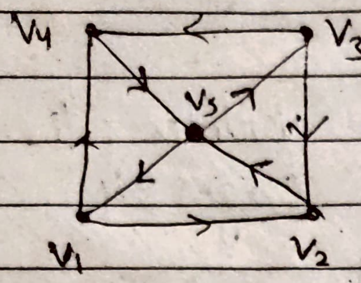


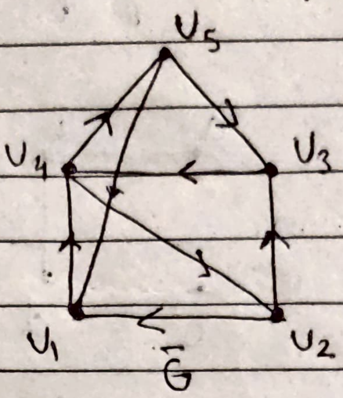
Vertices = 5
Edges = 5

$U_1 \to V_1$
$U_2 \to V_3$
$U_3 \to V_5$
$U_4 \to V_2$
$U_5 \to V_4$

f
$U_1U_2 = V_1V_3$
$U_2U_3 = V_3V_5$
$U_3U_4 = V_5V_2$
$U_4U_5 = V_2V_4$
$U_5U_1 = V_4V_1$

b) Show that the diagrams are isomorphic





Equal vertices = 5
edges = 8

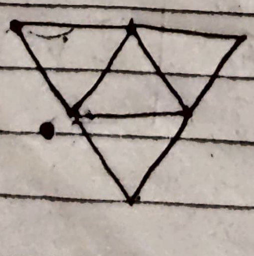
Degree of vertices

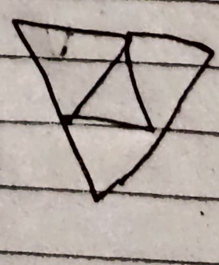
	O + I		O + I
V_1	= 2 + 1 →	U_2	= 2 + 1
V_2	= 1 + 2 →	U_3	= 1 + 2
V_3	= 2 + 1 →	U_5	= 2 + 1
V_4	= 1 + 2 →	U_1	= 1 + 2
V_5	= 2 + 2 →	U_4	= 2 + 2

13 (a) Define with an example each

(i) Euler circuit : A circuit in a graph is said to be an Eulerian circuit if it traverses each edge in the graph only once.

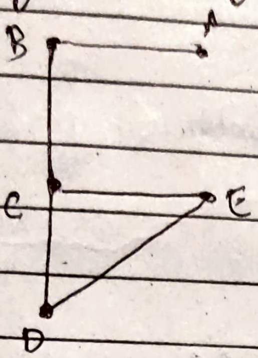
Eg.





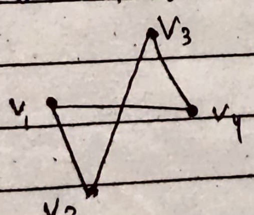
(ii) Eulerian path - A path in a graph is said to be Eulerian if it traversed each edge in the graph only once

Eg.

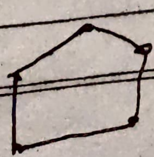


(iii) Euler Trail - an open walk which visits each edge of graph only once is called eulerian ~~walk (strict~~ trail

13(b) (i) Hamiltonian graph - A graph G is called Hamiltonian graph if it has a spanning cycle. A spanning path is called a Hamiltonian path — path which visits every vertex exactly once

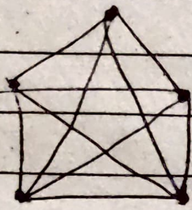


(ii) Hamiltonian circuit - Let $V_1, V_2, \text{——} V_n$ be any way of listing the vertices in order. Then $V_1 - V_2 - \text{—} V_n - V_1$ is a Hamiltonian circuit covering up all the edges on the periphery and vertices



15) Which of the following graphs are Eulerian

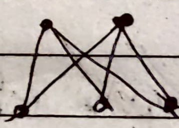
~~(i)~~ The complete graph K_5



K_5

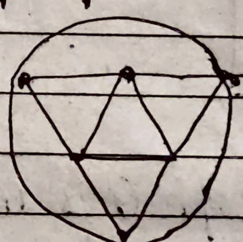
Yes, it is Eulerian

√ ~~(ii)~~ $K_{2,3}$



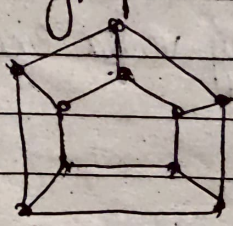
(Two vertices odd degree

~~(iii)~~ Graph of octahedron



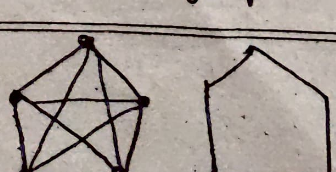
All vertices have even degree

~~(iv)~~ Peterson graph

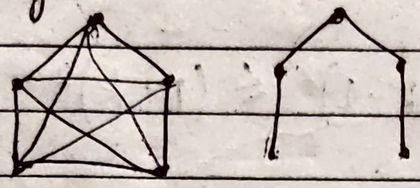


14(a) Define : (a) Subgraph of a graph

Let $G(V,E)$ be a graph, let V' be a subset of V and E' be a subset of E whose end points belong to V'. Then $G(V',E')$ is a graph and called a subgraph of $G(V,E)$

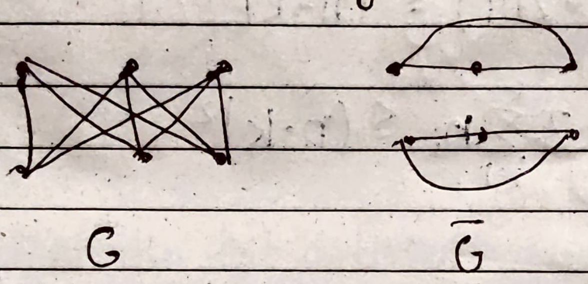


(b) Spanning subgraph - A spanning subgraph is a subgraph which contains all the vertices in G. A spanning subgraph need not contain all the edges in G.



14 (b) Show that the component of a bipartite graph need not be a bipartite graph

Sol.



16. Prove that a simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

Sol. Max. no. of edges in a simple graph $= \dfrac{n(n-1)}{2}$

where n = no. of vertices.

Let $n_1, n_2 - - n_k$ be no. of vertices in each of k components of graph G

$\Rightarrow n_1 + n_2 + \ldots + n_k = n$ —— (a)

Subtracting 1 from each term in LHS.

$\Rightarrow (n_1-1)+(n_2-1)+ \text{——} (n_k-1) = n-k$

Squaring both sides

$$\Rightarrow (n_1-1)^2+(n_2-1)^2+ \text{——} (n_k-1)^2 + 2 \ldots = (n-k)^2$$

$$\Rightarrow (n_1-1)^2+(n_2-1)^2+ -- (n_k-1)^2 \leq (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 \leq (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k}(n_i^2+1-2n_i) \leq (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2\sum_{i=1}^{k} n_i \leq (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2n \leq (n^2+k^2-2nk) \text{ —— from ⓐ}$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2+k^2-2nk-k+2n \text{ — ①}$$

Max no. of edges in i^{th} component of G is $\frac{n_i(n_i-1)}{2}$

$$\text{in G} = \sum_{i=1}^{k}\frac{n_i(n_i-1)}{2}$$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{k} n_i^2 - n_i = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[(n^2+k^2-2nk-k+2n)-n\right] \text{ from ① \& ⓐ}$$

$$= \frac{1}{2}\left[n^2-nk-nk+k^2-k+n\right]$$

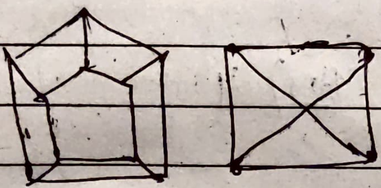
$$= \frac{1}{2}\left[n(n-k) - k(n-k) + (n-k)\right]$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

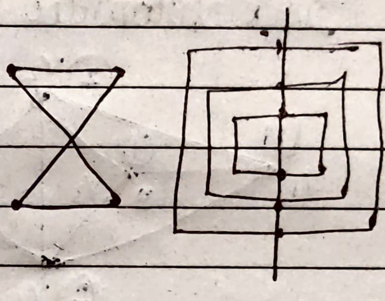
17 (a) (i) Give an example of a graph which has a Hamiltonian circuit but not an Euler circuit

(ii) Euler circuit but not Euler

(i)

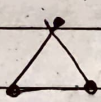


(ii)



(b) Show that K_n has a Hamilton circuit whenever $n \geq 3$

Consider graph K_3

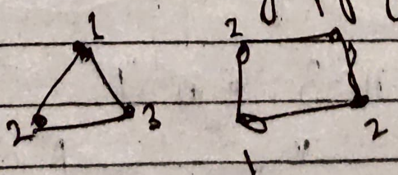


It is a spanning cycle i.e. it covers all vertices

Thus it has a hamiltonian circuit.

18 (a) Define graph coloring with an example.

Painting all the vertices of a graph with colors such that no two adjacent vertices have same color is called coloring of graph.

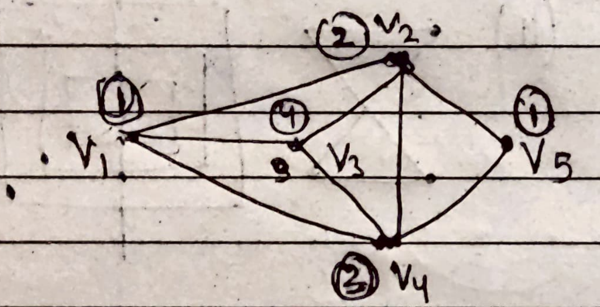


18(b) ~~Show that~~ Define chromatic number of a graph with example

The least no. of colors required for coloring ~~an~~ graph

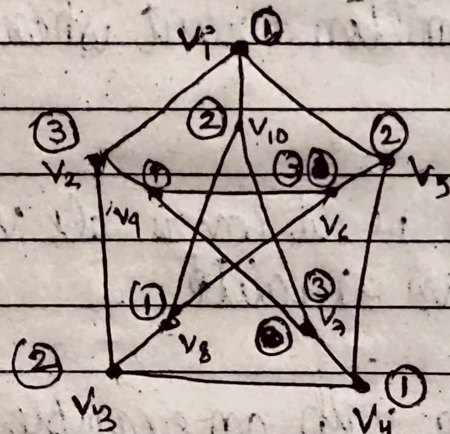
C.No = 2

19 a) Find chromatic no.



Ans) 4

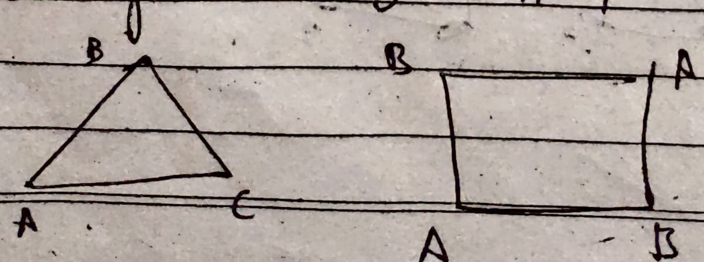
b)



Ans) 3

20 a) Prove that a graph of order $(n \geq 2)$ consisting of a single cycle is 2-chromatic if n is even and 3-chromatic if n is odd.

Consider cycles $n = 3$ & $n = 4$



In C_3, once any two vertices are colored, the remaining vertex must be differently colored ~~to~~ so as to follow the rules of graph coloring
In C_4, once two are colored, the other two vertices can be alternatively colored avoiding similar adjacent colors
Then n ?

20.(b) Prove that every connected planar graph G is 6-colorable

Ans We prove the result by induction on the no. of vertices. Suppose we have a graph such that $V \leq 6$. For $V \leq 6$, we can give each vertex a different color and use ≤ 6 colors.

Now assume that any simple planar graph with $V = n$ vertices can be properly colored with six colors

Let G be any simple planar graph on $V = n+1$ vertices. From our lemma above, we know that G must have some vertex W of degree ≤ 5. Remove W from G to form G'. G' has $V = n$ vertices and we may apply our induction hypothesis to know it can be properly colored with 6 colors. Now, we can think of this as coloring all of G except W.
But since W has degree at most 5, one of the 6 colors won't be used for any of the neighbours of W and we can finish coloring G.