Number Theory:

Division Algorithm:

If a and b are two integers with $b > 0$, then there exists unique integer q and r such that $a = bq + r$ where $0 \le r < |b|$

Note:

* When $a = bq + r$. q is called quotient and r is called the remainder.

* $a = qb + r$

$$\frac{a}{b} = q + \frac{r}{b}, \quad 0 \le \frac{r}{b} < 1$$

* $q = [a/b]$
  - integer part of a/b
  - greatest integer $\le a/b$.

* $r = a - bq$.

Ex: ① $a = 46$, $b = 13$

$$13 \overline{)46}\ \ 3$$
$$\underline{39}$$
$$7$$

$q = 3$, $r = 7$.

② $a = 46$, $b = -13$

$$-13 \overline{)46}\ \ -3$$
$$\underline{39}$$
$$07$$

$q = -3$ $r = 7$

(3) $a = -46. \quad b = 13.$

$$\begin{array}{r} -3 \\ 13\overline{)-46} \\ -39 \\ \hline -7 \end{array}$$

$q = \underline{-3} \quad r = \underline{-7}$

(4) $a = -46 \quad b = -13$

$$\begin{array}{r} 3 \\ -13\overline{)-46} \\ 39 \\ \hline -7 \end{array}$$

$q = 3 \quad r = -7$

(1). How many integers are between 1 and 200 which are divisible by any one of the integers 2, 3 and 5 ?

Let A = set of Z b/w 1 to 200 ÷ by 2

B = —"— —"— ÷ by 3

C = —"— —"— ÷ by 5

$|A| = \left[\frac{200}{2}\right]$ = integer part of $\frac{200}{2}$ = $\underline{100}$

$|B| = \left[\frac{200}{3}\right]$ = Integer part of $\frac{200}{3}$ = $[66.66] = \underline{66}$

$|C| = \left[\frac{200}{5}\right] = \underline{40}$

$|A \cap B| = \left[\frac{\cancel{200}^{100}}{\cancel{2} \times 3}\right] = [33.33] = \underline{33}$

$|B \cap C| = \left[\frac{200}{3 \times 5}\right] = [13.33] = \underline{13}$

$|A \cap C| = \left[\frac{200}{2 \times 5}\right] = \underline{20}$

$|A \cap B \cap C| = \left[\frac{200}{2 \times 3 \times 5}\right] = [6.66] = \underline{6}$

$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 100 + 66 + 40 - 33 - 13 - 20 + 6 = \underline{146}$

③ How many integers are b/w 1 and 250 which are divisible by any one of the integers 3, 5 and 7 ?

Let A = set of z b/w 1 and 250 ÷ by 3

B = —"— —"— —"— 5

C = —"— —"— —"— 7

$|A| = \left[\frac{250}{3}\right] = [83.33] = \underline{83}$

$|B| = \left[\frac{250}{5}\right] = \underline{50}$

$|C| = \left[\frac{250}{7}\right] = [35.714] = \underline{35}$

$|A \cap B| = \left[\frac{250}{3 \times 5}\right] = [16.66] = \underline{16}$

$|B \cap C| = \left[\frac{250}{5 \times 7}\right] = [7.1428] = \underline{7}$

$|A \cap C| = \left[\frac{250}{3 \times 7}\right] = [11.904] = \underline{11}$

$|A \cap B \cap C| = \left[\frac{250}{3 \times 5 \times 7}\right] = [2.3809] = \underline{2}$

$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 83 + 50 + 35 - 16 - 7 - 11 + 2$

$= \underline{\underline{136}}$

③ Find the no/. of +ve z ≤ 3000 and divisible 3, 5 or 7.

Let A, B and C are the set of z b/w 1 to 3000 which are ÷ by 3, 5 and 7 respectively.

$|A| = \left[\frac{3000}{3}\right] = \underline{\underline{1000}}$

$|B| = \frac{3000}{5} = \underline{\underline{600}}$

$$|C| = \frac{3000}{7} = \lfloor 428.57 \rfloor = \underline{428}$$

$$|A \cap B| = \left\lfloor \frac{3000}{3 \times 5} \right\rfloor = \underline{200}$$

$$|B \cap C| = \left\lfloor \frac{3000}{5 \times 7} \right\rfloor = \lfloor 85.714 \rfloor = \underline{85}$$

$$|A \cap C| = \left\lfloor \frac{3000}{3 \times 7} \right\rfloor = \lfloor 142.857 \rfloor = \underline{142}$$

$$|A \cap B \cap C| = \left\lfloor \frac{3000}{3 \times 5 \times 7} \right\rfloor = \lfloor 28.5714 \rfloor = \underline{28}$$

$$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= 1000 + 600 + 428 - 200 - 85 - 142 + 28$$

$$= \underline{1629}$$

(4) Find the no. of $+^{ve}$ $\mathbb{Z} \leq 2076$ and divisible by neither 4 nor 5.

Let A and B be the set of $\mathbb{Z} \leq 2076$ and $\div$ by 4 and 5 respectively.

$$|A| = \left\lfloor \frac{2076}{4} \right\rfloor = \underline{519}$$

$$|B| = \left\lfloor \frac{2076}{5} \right\rfloor = \lfloor 415.2 \rfloor = \underline{415}$$

$$|A \cap B| = \left\lfloor \frac{2076}{4 \times 5} \right\rfloor = \lfloor 103.8 \rfloor = \underline{103}$$

$$\therefore |A \cup B| = |A| + |B| - |A \cap B|$$

$$= \underline{831}$$

$$\therefore |\overline{A \cup B}| = 2076 - 831$$

$$= \underline{1245}$$

Base b representation of Integers:

Integers are usually represented as finite strings of decimal digits. Ex: 7302.

This familiar place value notation is actually shorthand for a sum involving powers of the base 10.

Ex. $7302 = 7\times10^3 + 3\times10^2 + 0\times10^1 + 2\times10^0$

Division algorithm can be used to convert decimal integer to division any other base. Any +ve integer $b>1$ is a valid choice for base.

Theorem [Base b representation].

Let $b>1$ be an integer. Every $n\in N$ can be written uniquely in the form $n = a_m b^m + a_{m-1} b^{m-1} + \dots + a_2 b^2 + a_1 b^1 + a_0$ with $a_i \in \{0,1,2 \dots b-1\}$ $\forall$ i and $a_m \neq 0$

Note:

★ In base b representation of n. $n = (a_m a_{m-1} \dots a_1 a_0)$

★ The no/- system with base 10 is the decimal system which uses the no/.s 0, 1, 2 .... 9 to represent any no/.

★ The no/. system with base 2 is the Binary no/. system which uses the no/.s 0, 1 to represent any no/..

★ The no/- system with base 8 is the octal system which uses the no/.s 0, 1, 2 ... 7 to represent any no/

★ When the base is greater than 10, the capital letters in English alphabets are used i.e A, B, C ... are used to represent eleven, twelve respectively

★ The no/. system with base 16 is the hexadecimal system which uses {0, 1, 2 .... 9, A, B, C, D, E, F} to represent any no/.

① Express 10110_{two} in base ten.

$$\overset{4}{1}\overset{3}{0}\overset{2}{1}\overset{1}{1}\overset{0}{0}_{two} = 1\times2^4+0\times2^3+1\times2^2+1\times2^1+0\times2^0$$

$$= 16+4+2 = \underline{22}_{10}$$

② Express 3014 in base eight

$$3014_{ten} = 8\times376+6$$

$8\overline{)3014}$ — quotient 376; 3008; remainder 6

$8\overline{)376}$ — quotient 47; 376; remainder 00

$$376 = 8\times47+0$$

$8)\,47\,(5$; 40; 7

$$47 = 8\times5+7$$

$$5 = 8\times0+5$$

$$(3014)_{ten} = (5706)_{eight}$$

③ Express 15036 in base sixteen.

$16)\,15036\,(939$; 15024; 12

$$15036 = 16\times939+12\ C$$

$16)\,939\,(58$; 928; 11

$$939 = 16\times58+11\ B$$

$16)\,58\,(3$; 48; 10

$$58 = 16\times3+10\ A$$

$$3 = 16\times0+3$$

$$(15036)_{10} = (3ABC)_{16}$$

(4) Establish the validity of the number pattern

$$1 \times 9 + 2 = 11$$
$$12 \times 9 + 3 = 111$$
$$123 \times 9 + 4 = 1111$$
$$1234 \times 9 + 5 = 11111$$
$$12345 \times 9 + 6 = 111111$$
$$123456 \times 9 + 7 = 1111111$$

From the pattern we can see that

$$123 \ldots n \times 9 + (n+1) = 111 \ldots 1 \quad (n+1 \text{ ones})$$

$$LHS = 123 \ldots n \times 9 + (n+1)$$
$$= 9 [1 \times 10^{n-1} + 2 \times 10^{n-2} + 3 \times 10^{n-3} + \ldots + n] + n + 1$$
$$= (10 - 1) [1 \times 10^{n-1} + 2 \times 10^{n-2} + \ldots + n] + (n-1)$$
$$= [10^n + 2 \times 10^{n-1} + \ldots + n \times 10] - [1 \times 10^{n-1} + 2 \times 10^{n-2} + \ldots + n] + (n-1)$$
$$= 10^n + 10^{n-1} + 10^{n-2} + \ldots + 10 + 1$$
$$= \underbrace{111 \ldots 1}_{(n+1) \text{ ones}}$$

Prime and Composite Numbers

* A +ve Z $p > 1$ is called prime no/. if the only +ve factors of p is 1 and p
* A +ve $Z > 1$ i.e not a prime is called composite no/.

Note:

* +ve Z 1 is neither a prime nor a composite.
* +ve Z n is composite if there exists a +ve Z a and b such that $ab = n$
* 2 is the only even prime no/.
* Except 2 and 3 all the other prime no/.s can be expressed in general form as $6n+1$ or $6n-1$ where n is a natural no/.

Prime Counting Function:

$\pi(x)$ = No/. of primes $\leq x$.

Theorem:

Let $p_1 . p_2 \dots p_t$ be primes $\leq \sqrt{n}$

$$\pi(n) = n - 1 + \sum_i \left\lfloor \frac{n}{p_i} \right\rfloor + \sum_{i<j} \left\lfloor \frac{n}{p_i p_j} \right\rfloor - \sum_{i<j<k} \left\lfloor \frac{n}{p_i p_j p_k} \right\rfloor + \dots + (-1)^k \left\lfloor \frac{n}{p_1 p_2 \dots p_k} \right\rfloor$$

Integer part • $[x]$ = Integer part of x

Floor fu^n $\lfloor x \rfloor$ = greatest integer $\leq x$

Ceiling fu^n $\lceil x \rceil$ = least integer $\geq x$

Ex: $[1.5] = 1$ $\quad \lfloor -1.5 \rfloor = -2$ $\quad \lfloor 1.5 \rfloor = 1$

$[-1.5] = -1$ $\quad \lceil 1.5 \rceil = -1$ $\quad \lceil 1.5 \rceil = 2$

① Find the no/ of primes ≤ 100

$n = 100$

$\sqrt{n} = \sqrt{100} = \underline{10}$

$\pi(\sqrt{n}) = \pi(10) = \underline{\underline{4}}$

$$\pi(100) = 99 + 4 - \left(\left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{3}\right\rfloor + \left\lfloor\frac{100}{5}\right\rfloor + \left\lfloor\frac{100}{7}\right\rfloor\right) + \left(\left\lfloor\frac{100}{2\times3}\right\rfloor + \left\lfloor\frac{100}{2\times5}\right\rfloor\right.$$

$$\left. + \left\lfloor\frac{100}{2\times7}\right\rfloor + \left\lfloor\frac{100}{3\times5}\right\rfloor + \left\lfloor\frac{100}{3\times7}\right\rfloor + \left\lfloor\frac{100}{5\times7}\right\rfloor\right) - \left(\left\lfloor\frac{100}{2\times3\times5}\right\rfloor + \right.$$

$$\left. \left\lfloor\frac{100}{2\times3\times7}\right\rfloor + \left\lfloor\frac{100}{3\times5\times7}\right\rfloor\right) + \left\lfloor\frac{100}{2\times3\times5\times7}\right\rfloor$$

$$= 99 + 4 - (\lfloor 50\rfloor + \lfloor 33.33\rfloor + \lfloor 20\rfloor + \lfloor 14.28\rfloor) + (\lfloor 16.66\rfloor$$

$$+ \lfloor 10\rfloor + \lfloor 7.142\rfloor + \lfloor 6.66\rfloor + \lfloor 4.76\rfloor + \lfloor 2.85\rfloor) - (\lfloor 3.33\rfloor$$

$$+ \lfloor 2.38\rfloor + \lfloor 0.95\rfloor) + \lfloor 0.47\rfloor$$

$$= 99+4 - (50+33+20+14) + (16+10+7+6+4+2) - (3+2+0) + 0$$

$$= \underline{\underline{26}}$$

Note

★ For each natural no/ n $\exists$ a set of n consecutive $\mathbb{Z}$ such that none of them are prime

★ These n consecutive composite numbers are

$(n+1)! + 2,\ (n+1)! + 3, \ldots\ (n+1)! + (n+1)$

② Find six consecutive $\mathbb{Z}$ that are composite.

$n = 6$

$\Rightarrow (6+1)! + 2,\ (6+1)! + 3, \ldots\ (6+1)! + (6+1)$

$\Rightarrow 5040 + 2,\ 5040 + 3,\ 5040 + 4,\ 5040 + 5,\ 5040 + 6,\ 5040 + 7$