	DAYANANDA SAGAR COLLEGE OF ENGINEERING *(An Autonomous Institute Affiliated to VTU, Belagavi)* Shavige Malleshwara Hills, Kumaraswamy Layout, Bengaluru-560078 **DEPARTMENT OF MATHEMATICS**

COURSE MATERIAL

COURSE	**MATHEMATICAL STRUCTURES**
COURSE CODE	**21MAT41A**
MODULE	**2**
MODULE NAME	**Fundamentals of Logic**
STAFF INCHARGE	**Dr. Rose Bindu Joseph P**

OBJECTIVES:

After going through this unit, students will be able to:

- Define Basic Connectives and Truth Tables
- Describe the Logical Equivalence – The Laws of Logic,
- Describe the Logical Implication– Rules of Inference,
- Quantifiers with one variable.

INTRODUCTION:

Mathematics is assumed to be an exact science. Every statement in Mathematics must be precise. Also there can't be Mathematics without proofs and each proof needs proper reasoning. Proper reasoning involves logic. The dictionary meaning of 'Logic' is the science of reasoning. The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid & invalid mathematical arguments.

In addition to its importance in mathematical reasoning, logic has numerous applications in computer science to verify the correctness of programs & to prove the theorems in natural & physical sciences to draw conclusion from experiments, in social sciences & in our daily lives to solve a multitude of problems.

The area of logic that deals with propositions is called the propositional calculus or propositional logic. The mathematical approach to logic was first discussed by British mathematician George Boole; hence the mathematical logic is also called as Boolean logic.
In this chapter we will discuss a few basic ideas.

PROPOSITION (OR STATEMENT):

A proposition (or a statement) is a declarative sentence that is either true or false, but not both.

Imperative, exclamatory, interrogative or open sentences are not statements in logic.

Example: For Example consider the following sentences.

(i) Bangalore is the capital city of Karnataka.
(ii) $2 + 3 = 5$
(iii) The Sun rises in the east.

(iv) Does your homework.
(v) What are you doing?
(vi) $2 + 4 = 8$
(vii) $5 < 4$
(viii) The square of 5 is 15.
(ix) $x + 3 = 2$
(x) May God Bless you!

All of them are propositions except (iv), (v), (ix) & (x), sentences (i), (ii) & (iii) are true, whereas (vi), (vii) & (viii) are false. Sentence (iv) is command, hence not a proposition. (v) is a question so not a statement. (ix) is a declarative sentence but not a statement, since it is true or false depending on the value of x. (x) is a exclamatory sentence and so it is not a statement.
Mathematical identities are considered to be statements. Statements which are imperative, exclamatory, interrogative or open are not statements in logic.
Prepositions are usually represented by small letters such as p, q, r……….. The truth or the falsity of a proposition is called its truth value. If a proposition is true, we will indicate its truth value by the symbol **1** and if it if false by the symbol **0**.

COMPOUND STATEMENTS:

Many propositions are composites that are, composed of sub propositions and various connectives discussed subsequently. Such composite propositions are called compound propositions.

A proposition is said to be primitive if it cannot be broken down into simpler propositions, that is, if it is not composite.

Example: Consider, for example following sentences.

a. "The sun is shining today and it is colder than yesterday"
b. "Sita is intelligent and she studies every night."

Also the propositions in above Example are primitive propositions.

LOGICAL OPERATORS OR LOGICAL CONNECTIVES:

The phrases or words which combine simple statements are called logical connectives. There are five types of connectives. Namely, 'not', 'and', 'or', 'if…then', iff etc. The first one is a unitary operator whereas the other four are binary operators.
In the following table we list some possible connectives, their symbols & the nature of the compound statement formed by them.
Now we shall study each of basic logical connectives in details.

Sr. No.	Connective	Symbol	Compound statement
1	AND	$\wedge$	Conjunction
2	OR	$\vee$	Disjunction
3	NOT	$\neg$	Negation
4	IF …. THEN	$\rightarrow$	Conditional or implication
5	IF AND ONLY IF (IFF)	$\leftrightarrow$	Bi-conditional

Basic Logical Connectives:

Conjunction (AND):

If two statements are combined by the word "and" to form a compound proposition (statement) then the resulting proposition is called the conjunction of two propositions.

Symbolically, if P & Q are two simple statements, then '$P \wedge Q$' denotes the conjunction of P and Q and is read as 'P and Q.

Since, $P \wedge Q$ is a proposition it has a truth value and this truth value depends only on the truth values of P and Q. Specifically, if P & Q are true then $P \wedge Q$ is true; otherwise $P \wedge Q$ is false.

The truth table for conjunction is as follows.

P	Q	$P\wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Example:

Let P: In this year monsoon is very good.
Q: The Rivers are flooded.

Then, $P \wedge Q$: In this year monsoon is very good and the rivers are flooded.

Disjunction (OR):

Any two statements can be connected by the word 'or' to form a compound statement called disjunction.

Symbolically, if P and Q are two simple statements, then $P \vee Q$ denotes the disjunction of P and Q and read as ' P or Q '.

The truth value of $P \vee Q$ depends only on the truth values of P and Q. Specifically if P and Q are false then $P\vee Q$ is false, otherwise $P \vee Q$ is true.

The truth table for disjunction is as follows.

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Example:

P: Paris is in France
Q $2 + 3 = 6$
Then $P \vee Q$: Paris is in France or $2 + 3 = 6$.
Here, $P \vee Q$ is true since P is true & Q is False.

Thus, the disjunction $P \vee Q$ is false only when P and Q are both false.

Negation (NOT)

Given any proposition P, another proposition, called negation of P, can be formed by modifying it by "not". Also by using the phrase "It is not the case that or" "It is false that" before P we will able to find the negation.

Symbolically, $\neg$ P Read as "not P" denotes the negation of P. the truth value of $\neg$ P depends on the truth value of P

If P is true then $\neg$ P is false and if P is false then $\neg$ P is true. The truth table for Negation is as follows:

P	$\neg$ P
1	0
0	1

Example:

Let P: 3 is a factor of 12.
Then Q = $\neg$ P: 3 is not a factor of 12. Here P is true & $\neg$ P is false.

Conditional or Implication: (If…then)

If two statements are combined by using the logical connective 'if…then' then the resulting statement is called a conditional statement

If P and Q are two statements forming the implication "if P then Q" then we denotes this implication $P \to Q$ In the implication $P \to Q$,P is called antecedent or hypothesis Q is called consequent or conclusion.

The statement $P \to Q$ is true in all cases except when P is true and Q is false.

The truth table for implication is as follows.

P	Q	$P \to Q$
1	1	1
1	0	0
0	1	1
0	0	1

Since conditional statement play an essential role in mathematical reasoning a variety of terminology is used to express $P \to Q$.

i) If P then Q
ii) P implies Q
iii) P only if Q
iv) Q if P
v) P is sufficient condition for Q
vi) Q when P
vii) Q is necessary for P
viii) Q follows from P
ix) if P, Q
x) Q unless $\neg$P

Bi-conditional Statement:

Let P and Q be propositions. The bi-conditional statement $P \leftrightarrow Q$ is the proposition "P if and only if Q". The bi-conditional statement is true when P and Q have same truth values and is false otherwise.

Bi-conditional statements are also called bi-implications. It is also read as p is necessary and sufficient condition for Q.

The truth table for bi-conditional statement is as follows.

P	Q	$P \leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Example : Let P : Ram can take the flight.
Q: Ram buy a ticket.
Then $P \leftrightarrow Q$ is the statement.
"Ram can take the flight iff Ram buy a ticket"

Tautology:

A tautology or universally true formula is a well formed formula, whose truth value is T for all possible assignments of truth values to the propositional variables.

Example: Consider $P \vee \neg P$, the truth table is as follows.

P	$\neg P$	$P \vee \neg P$
1	0	1
0	1	1

$P \vee \neg\ P$ always takes value T for all possible truth value of P, it is a tautology.

Contradiction:

A contradiction or (absurdity) is a well formed formula whose truth value is false (F) for all possible assignments of truth values to the propositional variables.
Thus, in short a compound statement that is always false is a contradiction.

Example: Consider the truth table for $P \wedge \neg P$.

P	$\neg P$	$P \wedge \neg P$
1	0	0
0	1	0

$\therefore\ P \wedge \neg\ P$ always takes value F for all possible truth values of P, it is a Contradiction.

Contingency:

A well-formed formula which is neither a tautology nor a contradiction is called a contingency.
Thus, contingency is a statement pattern which is either true or false depending on the truth values of its component statement.

Example: Consider the truth table for $\neg P \vee Q$.

P	Q	$\neg P$	$\neg P \vee Q$
1	1	1	1
1	0	0	0
0	1	0	1
0	0	1	1

$\neg P \vee Q$ can take the truth value either 1 or 0. $\therefore$it is an example for contingency.

LOGICAL EQUIVALANCE:

Compound propositions that have the same truth values in all possible cases are called logically equivalent.

Definition: The compound propositions P and Q are said to be logically equivalent if $P \leftrightarrow Q$ is a tautology. The notation $P \equiv Q$ or $P \Leftrightarrow Q$ denotes that P and Q are logically equivalent.
Some equivalence statements are useful for deducing other equivalence statements. The following

table shows some important equivalence.

Logical Identities or Laws of Logic:

	Name	Equivalence
1	IDENTITY LAWS	$P \wedge T \Leftrightarrow P$ $P \vee F \Leftrightarrow P$
2	DOMINATION LAWS	$P \vee T \Leftrightarrow T$ $P \wedge F \Leftrightarrow F$
3	DOUBLE NEGATION	$\neg(\neg P) \Leftrightarrow P$
4	IDEMPOTENT LAWS	$(P \vee P) \Leftrightarrow P$ $(P \wedge P) \Leftrightarrow P$
5	COMMUTATIVE LAWS	$(P \wedge Q) \Leftrightarrow (Q \wedge P)$ $(P \vee Q) \Leftrightarrow (Q \vee P)$
6	ASSOCIATIVE LAWS	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$
7	DISTRIBUTIVE LAWS	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
8	DE MORGAN'S LAWS	$\neg (P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q$ $\neg (P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$
9	ABSORPTION LAWS	$P \vee (P \wedge Q) \Leftrightarrow P$ $P \wedge (P \vee Q) \Leftrightarrow P$
10	NEGATION LAWS (INVERSE/ COMPLEMENT)	$(P \vee \neg P) \Leftrightarrow T$ $(P \wedge \neg P) \Leftrightarrow T$
11	IMPLICATION LAW	$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$
12	BI-CONDITIONAL PROPERTY	$(P \leftrightarrow Q) \Leftrightarrow (P \wedge Q) \wedge (P \wedge \neg Q)$

Prove the Implication law using truth table

P	$\neg P$	Q	P$\rightarrow Q$	$\neg P \vee Q$
1	0	1	1	1
1	0	0	0	0
0	1	1	1	1
0	1	0	1	1

Example: If P: "This book is good." and Q: "This book is costly."

Write the following statements in symbolic form.

a) This book is good & costly.
b) This book is not good but costly.
c) This book is cheap but good.
d) This book is neither good nor costly.
e) If this book is good then it is costly.

Answers:

a) $P \wedge Q$
b) $\neg P \wedge Q$
c) $\neg Q \wedge P$
d) $\neg P \wedge \neg Q$
e) $P \rightarrow Q$

Logical Equivalence Involving Implications:

Let P & Q be two statements.

The following table displays some useful equivalence for implications involving conditional and bi-conditional statements.

Sr. No.	Logical Equivalence involving implications
1	$P \rightarrow Q \equiv \neg P \vee Q$
2	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
3	$P \vee Q \equiv \neg P \rightarrow Q$
4	$P \wedge Q \equiv \neg(P \rightarrow \neg Q)$
5	$\neg (P \rightarrow Q) \equiv P \wedge \neg Q$
6	$(P \rightarrow Q) \wedge (P \rightarrow r) \equiv P \rightarrow (Q \wedge r)$
7	$(P \rightarrow r) \wedge (Q \rightarrow r) \equiv (P \vee Q) \rightarrow r$
8	$(P \rightarrow Q) \vee (P \rightarrow r) \equiv P \rightarrow (Q \vee r)$
9	$(P \rightarrow r) \vee (Q \rightarrow r) \equiv (P \wedge Q) r$
10	$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
11	$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$
12	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$
13	$\neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$

All these identities can be proved by using truth tables.

Example: Obtain truth value for $\alpha = (P \to Q) \wedge (Q \to P)$.

Solution: The truth table for the given well-formed formula is given below.

P	Q	$P \to Q$	$Q \to P$	α
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	0	0

Example: Construct the truth table for the compound propositions $Q \leftrightarrow (\neg P \vee \neg Q)$,

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$Q \leftrightarrow (\neg P \vee \neg Q)$
1	1	0	0	0	0
1	0	0	1	1	0
0	1	1	0	1	1
0	0	1	1	1	0

Example: Determine whether each of the following form is a tautology or a contradiction or neither:

i) $(P \wedge Q) \to (P \vee Q)$

ii) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$

iii) $(\neg P \wedge \neg Q) \to (P \to Q)$

iv) $\{p \to (q \longrightarrow r)\} \to \{(p \longrightarrow q) \longrightarrow (p \longrightarrow r)\}$

v) $(P \to Q) \wedge (P \wedge \neg Q)$

vi) $[P \wedge (P \to \neg Q) \to Q]$

Solution:

i) The truth table for $(P \wedge Q) \to (P \vee Q)$

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \to (P \vee Q)$
1	1	1	1	1
1	0	0	1	1
0	1	0	1	1

0	0	0	0	1

The entries in the last column are 'T'. Hence $(P \wedge Q) \to (P \vee Q)$ is a tautology

ii) The truth table for $(P \vee Q) \wedge (\neg P \wedge \neg Q)$ is

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg P \wedge \neg Q$	$(P \vee Q) \wedge (\neg P \wedge \neg Q)$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	0

The entries in the last column are 'F'. Hence $(P \vee Q) \wedge (\neg P \wedge \neg Q)$ is a contradiction

iii) The truth table for $(\neg P \wedge \neg Q) \to (P \to Q)$

.

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$P \to Q$	$(\neg P \wedge \neg Q) \to (P \to Q)$
1	1	0	0	0	1	1
1	0	0	1	0	0	1
0	1	1	0	0	1	1
0	0	1	1	1	1	1

Here all entries in last column are 'T', $\therefore$ $(\neg P \wedge \neg Q) \to (P \to Q)$ is a tautology.

(iv) The truth table for $\{p \rightarrow (q \longrightarrow r)\} \rightarrow \{(p \longrightarrow q) \longrightarrow (p \longrightarrow r)\}$

P	Q	r	p→ q	p → r	q → r	$\{p \rightarrow (q \longrightarrow r)\}$ (1)	$\{(p \longrightarrow q) \longrightarrow (p \longrightarrow r)\}$ (2)	(1)→ (2)
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	1	1	0	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

The last entries are all ‘T, ∴ $\{p \rightarrow (q \longrightarrow r)\} \rightarrow \{(p \longrightarrow q) \longrightarrow (p \longrightarrow r)\}$ is tautology.

v) The truth table for $(P \rightarrow Q) \wedge (P \wedge Q$ is as follows.

P	Q	$\neg Q$	$P \wedge \neg Q$	$P \rightarrow Q$	$(P \rightarrow Q) \wedge (P \wedge Q)$
1	1	0	0	1	0
1	0	1	1	0	0
0	1	0	0	1	0
0	0	1	0	1	0

All the entries in the last column are ‘F’. Hence it is contradiction.

vi) The truth table for $[P \wedge (P \to \neg Q) \to Q]$

P	Q	$\neg Q$	$P \to \neg Q$	$P \wedge (P \to \neg Q)$	$[P \wedge (P \to \neg Q) \to Q]$
1	1	0	0	0	1
1	0	1	1	1	0
0	1	0	1	0	1
0	0	1	1	0	1

The last entries are neither all 'T' nor all 'F'.

$\therefore$ $[P \wedge (P \to \neg Q) \to Q]$ is a neither tautology nor contradiction. It is a Contingency.

Example: Show that $\neg (P \vee Q)$ and $(\neg P \wedge \neg Q)$ are logically equivalent

Solution: The truth table for this compound proposition is as follows.

P	Q	$\neg$ P	$\neg$ Q	P $\vee$ Q	$\neg$ (P $\vee$ Q)	$\neg$ P $\wedge$ $\neg$ Q
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

We can observe that the truth values of $\neg (P \vee Q)$ and $\neg P \wedge \neg Q)$ agree for all possible combinations of the truth values of P and Q.

It follows that $\neg$ (P $\vee$ Q) $\leftrightarrow$ $(\neg P \wedge \neg Q)$ is a tautology; therefore the given compound propositions are logically equivalent.

Example: Show that $P \to Q$ and $\neg P \vee Q$ are logically equivalent.

Solution: The truth tables for these compound proposition as follows.

P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

As the truth values of p $\to$ q and $\neg$ p $\vee$ q are logically equivalent

Problems.

(Q1) Let p, q, r denote the following statements about a triangle ABC.
p: Triangle ABC is isosceles
q: Triangle ABC is equilateral
r: Triangle ABC is equiangular. Translate each of the following into an English sentence.

1. $q \rightarrow p$
2. $\neg p \rightarrow \neg q$
3. $q \leftrightarrow r$
4. $p \wedge \neg q$
5. $r \rightarrow p$

Solution:

1. $q \rightarrow p$: If triangle ABC is equilateral, then it is isosceles.
2. $\neg p \rightarrow \neg q$: If triangle ABC is not isosceles, then it is not equilateral.
3. $q \leftrightarrow r$: Triangle ABC is equilateral if and only if it is equiangular.
4. $p \wedge \neg q$: Triangle ABC is isosceles, but not equilateral.
5. $r \rightarrow p$: If triangle ABC is equiangular, then it is isosceles.

(Q2) Find the possible truth values for p, q and r if
(i) $p \rightarrow (q \vee r)$ is false
(ii) $p \wedge (q \rightarrow r)$ is true

Solution: (i) $p \rightarrow (q \vee r)$ is false

P is true and $(q \vee r)$ is false

$(q \vee r)$ is false $\Rightarrow$ q is false and r is false

$\therefore$ p is true,q is false and r is false

(ii) $p \wedge (q \rightarrow r)$ is true

P is true and $(q \rightarrow r)$ is true

$(q \rightarrow r)$ is true when q is true and r is true or q is false and r is true

$\therefore$ p is true, q is true and r is true

Or p is true, q is false and r is true

Or p is true, q is false and r is false

Prove following using laws of logic

1) Prove: $(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$	Left-Hand Statement
$\Leftrightarrow q \vee (p \wedge \neg q)$	Commutative
$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$	Distributive
$\Leftrightarrow (q \vee p) \wedge T$	Or Tautology
$\Leftrightarrow q \vee p$	Identity

$\Leftrightarrow p \vee q$	Commutative

2) Prove: $p \to (p \vee q)$ is a tautology

$p \to (p \vee q)$	
$\Leftrightarrow \neg p \vee (p \vee q)$	Implication Equivalence
$\Leftrightarrow (\neg p \vee p) \vee q$	Associative
$\Leftrightarrow (p \vee \neg p) \vee q$	Commutative
$\Leftrightarrow T \vee q$	Or Tautology
$\Leftrightarrow q \vee T$	Commutative
$\Leftrightarrow T$	Domination

3) Prove: $\neg p \leftrightarrow q \Leftrightarrow p \leftrightarrow \neg q$

$\neg p \leftrightarrow q$	
$\Leftrightarrow (\neg p \to q) \wedge (q \to \neg p)$	Bi-conditional Equivalence
$\Leftrightarrow (\neg\neg p \vee q) \wedge (\neg q \vee \neg p)$	Implication Equivalence
$\Leftrightarrow (p \vee q) \wedge (\neg q \vee \neg p)$	Double Negation
$\Leftrightarrow (q \vee p) \wedge (\neg p \vee \neg q)$	Commutative
$\Leftrightarrow (\neg\neg q \vee p) \wedge (\neg p \vee \neg q)$	Double Negation
$\Leftrightarrow (\neg q \to p) \wedge (p \to \neg q)$	Implication Equivalence
$\Leftrightarrow p \leftrightarrow \neg q$	Bi-conditional Equivalence

4) **Prove**: $\neg(q \to p) \vee (p \wedge q) \Leftrightarrow q$

$\neg(q \to p) \vee (p \wedge q)$	
$\Leftrightarrow \neg(\neg q \vee p) \vee (p \wedge q)$	Implication law
$\Leftrightarrow (q \wedge \neg p) \vee (p \wedge q)$	Demorgan's law and double negation
$\Leftrightarrow (q \wedge \neg p) \vee (q \wedge p)$	Commutative law
$\Leftrightarrow q \wedge (\neg p \vee p)$	Distributive law
$\Leftrightarrow q \wedge (T)$	Identity law
$\Leftrightarrow q$	Identity law

5) Prove: $\neg[p \vee (\neg p \wedge q)] \Leftrightarrow (\neg p \wedge \neg q)$

$:\neg[p \vee (\neg p \wedge q)] \Leftrightarrow \neg p \wedge \neg(\neg p \wedge q)$	Demorgan's law
$\Leftrightarrow \neg p \wedge (p \vee \neg q)$	Double negation and Demorgan's law
$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive law
$\Leftrightarrow F \vee (\neg p \wedge \neg q)$	Inverse law
$\Leftrightarrow (\neg p \wedge \neg q) \vee F$	Commutative law
$\Leftrightarrow (\neg p \wedge \neg q)$	Identity law

6) $(p \wedge q) \wedge [(q \wedge \neg r) \vee (p \wedge r)] \Leftrightarrow \neg(p \rightarrow \neg q)$. Use LHS to denote the expression on the left hand side.

Then LHS $\Leftrightarrow [(p \wedge q) \wedge (q \wedge \neg r)] \vee (p \wedge q) \wedge (p \wedge r)]$ Distributive
$\Leftrightarrow [((p \wedge q) \wedge q) \wedge \neg r) \vee [((p \wedge q) \wedge p) \wedge r)]$ Associative
$\Leftrightarrow [(p \wedge (q \wedge q)) \wedge \neg r) \vee [((p \wedge p) \wedge q) \wedge r)]$ Commutative, Associative
$\Leftrightarrow [(p \wedge q) \wedge \neg r) \vee [(p \wedge q) \wedge r)]$ Idempotence
$\Leftrightarrow (p \wedge q) \wedge (\neg r \vee r)$ Distributive
$\Leftrightarrow (p \wedge q) \wedge 1$ Known tautology
$\Leftrightarrow (p \wedge q)$ Absorbtion
$\Leftrightarrow \neg\neg(p \wedge q)$ Double Negation
$\Leftrightarrow \neg(\neg p \vee \neg q)$ DeMorgan
$\Leftrightarrow \neg(p \rightarrow \neg q)$ Known L.E.

7). Let p, q, r denote primitive statements. Use the laws of logic to show that
$[p \rightarrow (q \vee r)] \Longleftrightarrow [(p \wedge \neg q) \rightarrow r]$

Using the laws of logic we obtain:
$p \rightarrow (q \vee r) \Leftrightarrow \neg p \vee (q \vee r)$ Logical equivalence
$\Leftrightarrow (\neg p \vee q) \vee r$ Associative law
$\Leftrightarrow \neg\neg(\neg p \vee q) \vee r$ Double negation law
$\Leftrightarrow \neg(\neg\neg p \wedge \neg q) \vee r$ De Morgan’s law
$\Leftrightarrow \neg(p \wedge \neg q) \vee r$ Double negation law
$\Leftrightarrow (p \wedge \neg q) \rightarrow r$ Logical equivalence

Therefore, $[p \rightarrow (q \vee r)] \Leftrightarrow [(p \wedge \neg q) \rightarrow r]$.

(8) Let p, q, r denote primitive statements. Use the laws of logic to show that

$$[p \rightarrow (q \vee r)] \Leftrightarrow [(p \wedge \neg q) \rightarrow r].$$

Using the laws of logic we obtain:

$p \rightarrow (q \vee r) \Longleftrightarrow \neg p \vee (q \vee r)$ Logical equivalence

$\Longleftrightarrow (\neg p \vee q) \vee r$ Associative law

$\Longleftrightarrow \neg(\neg p \vee q) \vee r$ Double negation law

$\Longleftrightarrow \neg(\neg\neg p \wedge \neg q) \vee r$ De Morgan’s law

$\Longleftrightarrow \neg(p \wedge \neg q) \vee r$ Double negation law

$\Longleftrightarrow (p \wedge \neg q) \rightarrow r$ Logical equivalence

Therefore, $[p \rightarrow (q \vee r)] \Longleftrightarrow [(p \wedge \neg q) \rightarrow r]$.

Converse, Inverse and Contra positive of a conditional statement:

We can form some new conditional statements starting with a conditional statement $P \rightarrow Q$ that occur so often. Namely converse, inverse, contrapositive. Which are as follows?

1. **Converse:** If $P \rightarrow Q$ is an implication then $Q \rightarrow P$ is called the converse of $P \rightarrow Q$

2. **Inverse:** If If $P \rightarrow Q$ is an implication then $\neg P \rightarrow \neg Q$ is called inverse of $P \rightarrow Q$

3. **Contrapositive:** If $P \rightarrow Q$ is an implication then the implication $\neg Q \rightarrow \neg P$ is called it's contrapositive.

P	Q	$\neg P$	$\neg Q$	$(P \rightarrow Q)$	$(Q \rightarrow P)$	$(\neg P \rightarrow \neg Q)$	$(\neg Q \rightarrow \neg P)$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

From table , it is evident that

$(P \rightarrow Q) \Leftrightarrow (\neg Q \rightarrow \neg P)$ and $(Q \rightarrow P) \Leftrightarrow (\neg P \rightarrow \neg Q$

Problems:

(1) Let P: You are good in Mathematics.

Q: You are good in Logic.

Then, $P \rightarrow Q$: If you are good in Mathematics then you are good in Logic.

Converse: $(Q \rightarrow P)$
If you are good in Logic then you are good in Mathematics.

Contra positive: $\neg Q \rightarrow \neg P$
If you are not good in Logic then you are not good in Mathematics.

Inverse: $(\neg P \rightarrow \neg Q)$
If you are not good in Mathematics then you are not good in Logic.

(2)Write the converse, inverse, contrapositive, and negation of the following statement.
"If Sandra finishes her work, she will go to the basketball game."
Let p: Sandra finishes her work.
q: Sandra goes to the basketball game.
Implication: $(p \rightarrow q)$
If Sandra finishes her work, she will go to the basketball game.
Converse: $(q \rightarrow p)$
If Sandra goes to the basketball game, she will finish her work.

Inverse: $(\neg p \to \neg q)$
If Sandra does not finish her work, she will not go to the basketball game.

Contrapositive: $(\neg q \to \neg p)$
If Sandra does not go to the basketball game, she does not finish her work.

Negation: $(p \land \neg q)$
Sandra finishes her work, and she does not go to the basketball game.

(3) Write the converse, inverse and contrapositive of the following statements:
"If today is Saturday, then I will go for a 10km run."

Solution. Let p and q be the following statements:
p: Today is Saturday.
q: I will go for a 10km run.
Then the given statement can be written as $p \to q$.

The converse of this implication is $q \to p$, which is $q \to p$: If I go for a 10km run, then today is Saturday.

The inverse of this implication is $\sim p \to \sim q$, which is $\sim p \to \sim q$: If today is not Saturday, then I will not go for a 10km run.

The contrapositive of this implication is $\sim q \to \sim p$, which is $\sim q \to \sim p$: If I do not go for a 10km run, then today is not Saturday

Electrical Circuits and Logic (Application to switching networks):

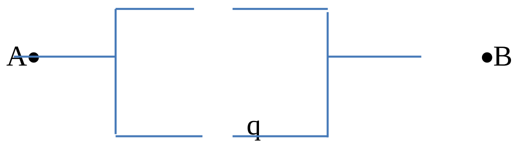
Proposition – Electrical Switch
A proposition is true or false, not both.
An electric switch is on or off, not both.
If a switch is open (off) the power cannot pass through.
If a switch is closed (on) the power will pass through. Series

A switching network is made up of wires and switches connecting two terminals, say A and B. In such a network, each switch is open (no current flows through it) or closed (current flows through it).If a switch p is open, we assign the symbol 0 to it and if p is closed , we assign the symbol 1 to it. There is a close analogy between switches and their states and propositions and their truth values.

A• —— p —— • **B**

The above figure shows a network having only one switch p. Current flows from the terminal A to B If the switch is closed, i.e. if p is assigned the symbol 1. This network is represented by the symbol p.

p



The above figure shows a parallel network consisting of two switches p and q in which the Current flows from the terminal A to B if p or q are closed, i.e. if p or q are assigned the symbol 1. This network is represented by the symbol p∨q.

The above figure shows a series network consisting of two switches p and q in which the Current flows from the terminal A to B if both p and q are closed, i.e. if both p and q are assigned the symbol 1. This network is represented by the symbol p ∧ q.

Duality:

Suppose u is a compound proposition that contains the connectives ∧ and ∨. Suppose we replace each occurrence of ∧ and ∨ in u by ∨ and∧ respectively. Also , if u contains T_0 and F_0 as components, suppose we replace each occurrence of T_0 and F_0 by $F_0$and T_0 respectively. Then the resulting compound proposition is called the dual of u and it is denoted by u^d.

For example, if u: $p\wedge(q\vee\neg r)\vee(s\wedge T_0)$ then u^d:$p\vee(q\wedge\neg r)\wedge(s\vee F_0)$
Note:

(1) $\left(u^d\right)^d \Leftrightarrow u$
(2) For any two propositions u and v, if u⇔v then $u^d \Leftrightarrow v^d$ known as **Principle of Duality**

The connectives NAND and NOR:

For any two propositions p and q , we have

$$\neg(p\wedge q) \Leftrightarrow \neg p\vee\neg q$$
$$\neg(p\vee q) \Leftrightarrow \neg p\wedge\neg q$$

The compound proposition $\neg(p\wedge q)$ is read as “Not p and q” and is also denoted by (p $\uparrow q$). The symbol ↑ is called NAND connective. Here NAND is a combination of “Not and And”

The compound proposition $\neg(p\vee q)$ is read as “Not p orq” and is also denoted by (p $\downarrow q$). The symbol ↓ is called NOR connective. Here NOR is a combination of “Not and Or”. Thus

$$(\mathbf{p}\uparrow \boldsymbol{q}) \Leftrightarrow \neg(\boldsymbol{p}\wedge\boldsymbol{q}) \Leftrightarrow \neg\boldsymbol{p}\vee\neg\boldsymbol{q}$$
$$(\mathbf{p}\downarrow \boldsymbol{q}) \Leftrightarrow \neg(\boldsymbol{p}\vee\boldsymbol{q}) \Leftrightarrow \neg\boldsymbol{p}\wedge\neg\boldsymbol{q}$$

(3) Let p, q, and r be primitive statements. Write the dual for the following statements.
1. q → p
2. p → (q ∧ r)
3. p ↔q
4.$p \to (q \to r)$

From the definition of duality it is not possible to give the dual of a logical statement that contains “→” or “↔”. We have to find its logical equivalent statements that contain no logical connectives other than “∧” and “∨”.

1. Since $q \rightarrow p \equiv \neg q \vee p$, hence the dual of $q \rightarrow p$ is $\neg q \wedge p$.
2. Since $p \rightarrow (q \wedge r) \equiv \neg p \vee (q \wedge r)$ its dual is $\neg p \wedge (q \vee r)$.
3. Reduction of $p \leftrightarrow q$ to a formula that contains connectives only $\wedge$, $\vee$, and $\neg$ is given below.

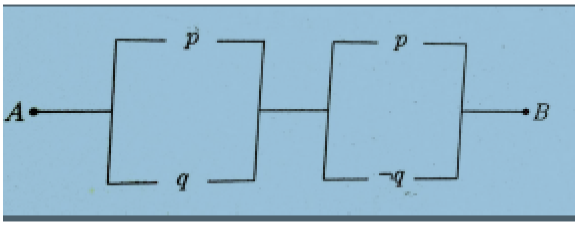
$$\begin{aligned} p \leftrightarrow q &\equiv (p \rightarrow q) \wedge (q \rightarrow p) \\ &\equiv (\neg p \vee q) \wedge (\neg q \vee p) \\ &\equiv [(\neg p \vee q) \wedge \neg q] \vee [(\neg p \vee q) \wedge p] \\ &\equiv (\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge p) \\ &\equiv (\neg p \wedge \neg q) \vee F \vee F \vee (q \wedge p) \\ &\equiv (\neg p \wedge \neg q) \vee (q \wedge p). \end{aligned}$$

Thus, $(p \leftrightarrow q)^d$ is $(\neg p \vee \neg q) \wedge (q \vee p)$

4. $p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\neg q \vee r) \Leftrightarrow \{\neg p \vee (\neg q \vee r)\}$
$(p \rightarrow (q \rightarrow r))^d \Leftrightarrow \{\neg p \wedge (\neg q \wedge r)\}$

Exercise

(1) Simplify the switching network using laws of logic

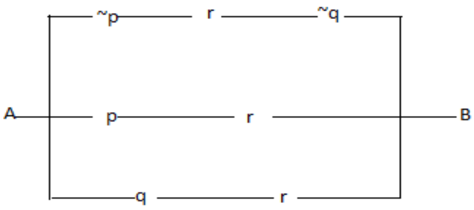


The switches p and q are in parallel and switches p and $\neg q$ are in parallel ,these two parallel connections are in series. The network connection is represented as $(p \vee q) \wedge (p \vee \neg q)$

$(p \vee q) \wedge (p \vee \neg q) \Leftrightarrow p \vee (q \wedge \neg q)$	Distributive law
$\Leftrightarrow (p \vee F)$	Inverse law
$\Leftrightarrow p$	Identity law

$\therefore$ The network can be represented as A• ——— p ——— • **B**

(3) Simplify the switching network using laws of logic



The given network can be represented as $(\neg p \wedge r \wedge \neg q) \vee (p \wedge r) \vee (q \wedge r)$

$(\neg p \wedge r \wedge \neg q) \vee \{(p \wedge r) \vee (q \wedge r\} \Leftrightarrow (\neg p \wedge r \wedge \neg q) \vee \{(p \vee q) \wedge r\}$	Distributive law
$\Leftrightarrow \{\neg (p \vee q) \wedge r\} \vee \{(p \vee q) \wedge r\}$	Demorgan's law

$\Leftrightarrow\{(p\vee q)\wedge r\} \vee \{\neg(p\vee q)\wedge r\}$ associative law
$\Leftrightarrow \{(p\vee q)\vee\neg(p\vee q)\}\wedge r$ Distributive law and
$\Leftrightarrow (T\wedge r)$ Inverse law
$\Leftrightarrow r$ Identity law

∴ The network can be represented as A•—— r ——• **B**

(3) Verify the principle of duality for the logical equivalence $[\neg(p\wedge q) \to \neg p\vee(\neg p\vee q)] \Leftrightarrow (\neg p\vee q)$

Solution: (i) Let $u \Leftrightarrow[\neg(p\wedge q) \to \neg p\vee(\neg p\vee q)]$ and $v \Leftrightarrow (\neg p\vee q)$
$\Leftrightarrow[\neg\neg(p\wedge q) \vee \neg p\vee(\neg p\vee q)$
$\Leftrightarrow[(p\wedge q) \vee [\neg p\vee(\neg p\vee q)]$

$u^d \Leftrightarrow[(p \vee q)\wedge[\neg p \wedge (\neg p\wedge q)]$
$\Leftrightarrow[(p \vee q)\wedge(\neg p\wedge q)$
$\Leftrightarrow [p\wedge(\neg p\wedge q)] \vee [q\wedge(\neg p\wedge q)]$
$\Leftrightarrow(F\wedge q) \vee (q\wedge\neg p)$
$\Leftrightarrow F) \vee (q\wedge\neg p)$
$\Leftrightarrow(q\wedge\neg p)$
$v^d \Leftrightarrow (\neg p\wedge q)$

(4) Express the following proposition in terms of only NAND and only NOR connectives
(i) $p\to q$ (ii) $p \leftrightarrow q$

Solution: (i) $(p\to q)\Leftrightarrow(\neg p\vee q) \Leftrightarrow\neg(p\wedge\neg q) \Leftrightarrow (p \uparrow \neg q) \Leftrightarrow [p \uparrow \neg(q\wedge q)] \Leftrightarrow [p \uparrow (q \uparrow q)]$
$(p \to q) \Leftrightarrow \neg\neg(p \to q) \Leftrightarrow \neg\{\neg(\neg p\vee q)\} \Leftrightarrow (\neg p \downarrow q) \Leftrightarrow \neg(p\vee p)\vee q \Leftrightarrow [(p \downarrow p) \downarrow q]$

(ii) $(p \leftrightarrow q) \Leftrightarrow (p \to q)\wedge(q \to p) \Leftrightarrow (r\wedge s) \Leftrightarrow \neg\neg(r\wedge s)$,
where $r = (p \to q)$ and $s = (q \to p)$
$\Leftrightarrow \neg(\neg(r\wedge s))$
$\Leftrightarrow \neg[\neg(r\wedge s)\wedge\neg(r\wedge s)]$
$\Leftrightarrow\neg[(r \uparrow s)\wedge(r \uparrow s)]$
$\Leftrightarrow(r \uparrow s) \uparrow (r \uparrow s)$
$r = (p \to q) \Leftrightarrow [p \uparrow (q \uparrow q)]$ and $s = (q \to p) \Leftrightarrow [q \uparrow (p \uparrow p)]$

$(p \leftrightarrow q) \Leftrightarrow (p \to q)\wedge(q \to p) \Leftrightarrow (r\wedge s) \Leftrightarrow \neg\neg(r\wedge s) \Leftrightarrow \neg(\neg r\vee\neg s)$
$\Leftrightarrow (\neg r) \downarrow (\neg s)\Leftrightarrow\neg(r\vee r) \downarrow \neg(s\vee s)\Leftrightarrow(r\downarrow r) \downarrow (s \downarrow s)$

$r = (p \to q) \Leftrightarrow [(p \downarrow p) \downarrow q]$ and $s = (q \to p) \Leftrightarrow [(q \downarrow q) \downarrow p]$

(5) P.T $[p\to (\neg p \to q)] \Leftrightarrow [p \uparrow (p \downarrow q)]$
Solution: $[p\to (\neg p \to q)] \Leftrightarrow \neg p\vee(p\vee q)$
$\Leftrightarrow \neg[p\wedge\neg(p\vee q)]$
$\Leftrightarrow \neg[p\wedge(p \downarrow q)]$
$\Leftrightarrow [p \uparrow \wedge(p \downarrow q)]$

THEORY OF INFERENCE FOR THE PREDICAT CALCULAS

If an implication $P \Rightarrow Q$ is a tautology where P and Q may be compound statements involving any number of propositional variables we say that Q logically follows from P. Suppose $P(P_1, P_2 ....P_n) \rightarrow Q$. Then this implication is true regardless of the truth values of any of its components. In this case, we say that Q logically follows from $P_1, P_2 \ldots..., P_n$.
Proofs in mathematics are valid arguments that establish the truth of mathematical statements. To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments. Rules of inference are our basic tools for establishing the truth of statements. The rules of inference for statements involving existential and universal quantifiers play an important role in proofs in Computer Science and Mathematics, although they are often used without being explicitly mentioned.

Rules of Inference

Argument:

An argument in propositional logic is a sequence of propositions. All propositions in the argument are called **hypothesis** or **Premises**.The final proposition is called the **conclusion**. An argument form in propositional logic is a sequence of compound propositions - involving propositional variables.

An argument form is valid if no matter which particular propositions are substituted for the Propositional variables in its premises, the conclusion is true if the premises are all true. Thus we say the conclusion C can be drawn from a given set of premises or the argument is valid if the conjunction of all the premises implies the conclusion is a tautology.

Consider a set of propositions $P_1, P_2 \ldots..., P_n$ and a proposition Q. Then a compound proposition of the form $(\mathbf{P_1} \wedge \mathbf{P_2} \wedge \ldots.. \wedge \mathbf{P_n}) \rightarrow \mathbf{Q}$ is called an argument. Here, $P_1, P_2 \ldots..., P_n$ are all called the **premises** of the argument and Q is called a **conclusion** of the argument. We write it in the form

$$\begin{array}{c} \boldsymbol{P_1} \\ \boldsymbol{P_2} \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{P_n} \\ \hline \therefore \boldsymbol{Q} \end{array}$$

Rules of Inference for Propositional logic:

We can always use a truth table to show that an argument form is valid. Arguments based on tautologies represent universally correct method of reasoning. Their validity depends only on the form of statements involved and not on the truth values of the variables they contain such

arguments are called **rules of inference**.
These rules of inference can be used as building blocks to construct more complicated valid argument forms
e.g
Let P: "You have a current password"
Q: "You can log onto the network".

Then, the argument involving the propositions,
"If you have a current password, then you can log onto the network".

"You have a current password" therefore: You can log onto the network" has the form …
.

$$\begin{array}{l} P \rightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

where ∴ is the symbol that denotes 'therefore we know that when P & Q are proposition variables, the statement $((P \rightarrow Q) \wedge P) \rightarrow Q$ is a tautology
.
So, this is valid argument and hence is a rule of inference, called modus ponens or the law of detachment.

(Modus ponens is Latin for mode that affirms)
The most important rules of inference for propositional logic are as follows…..

	Rule of Inference	Tautology	Name
1)	P, $\underline{P \rightarrow Q}$, $\therefore Q$	$(P \wedge (P \rightarrow Q)) \rightarrow Q$	Modus ponens
2)	$\neg Q$, $\underline{P \rightarrow Q}$, $\therefore \neg P$	$[\neg Q \wedge (P \rightarrow Q)] \rightarrow \neg P$	Modus tollens
3)	$P \rightarrow Q$, $\underline{Q \rightarrow R}$, $\therefore P \rightarrow R$	$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow$	Hypothetical syllogism
4)	$P \vee Q$, $\underline{\neg P}$, $\therefore Q$	$[(P \vee Q) \wedge \neg P] \rightarrow Q$	Disjunctive syllogism
5)	$\underline{P}$, $\therefore PVQ$	$P \rightarrow (P \vee Q)$	Addition
6)	$\underline{P \wedge Q}$, $\therefore P$	$(P \wedge Q) \rightarrow P$	Simplification
7)	P, $\underline{Q}$, $\therefore P \wedge Q$	$((P) \wedge (Q)) \rightarrow P \wedge Q$	Conjunction
8)	$P \vee Q$, $\underline{\neg P \vee R}$, $\therefore Q \vee R$	$[(P \vee Q) \wedge (\neg P \vee R)] \rightarrow (Q \vee R$	Resolution

Examples using Rules of Inference

1) Given the premises:
p
$p \to q$
$s \lor r$
$r \to \neg q$
Arrive at the conclusion:
$s \lor t$

<u>Step</u>	<u>Reason</u>
1) p	Premise
2) $p \to q$	Premise
3) q	Modus Ponens (using 1, 2)
4) $r \to \neg q$	Premise
5) $q \to \neg r$	Contrapositive statement of 4
6) $\neg r$	Modus Ponenes (using 3, 5)
7) $s \lor r$	Premise
8) s	Disjunctive Syllogism (using 6, 7)
9) $s \lor t$	Disjunctive Amplification (using 8)

2) Write the following arguments in symbolic form. (Assign p, q, r, etc. to simple statements and then translate the argument into multiple complex statements using p, q and r...) Then establish the validity of the argument or give a counterexample to show that it is invalid.

If get my Christmas bonus and my friends are free, I will take a road trip with my friends..
If my friends don't find a job after Christmas, then they will be free.
I got my Christmas bonus
My friends did not find a job after Christmas.
Therefore, I will take a road trip with my friends!

Assign the statements as follows:

p = "I get my Christmas bonus."
q = "My friends are free."
r = "I will take a road trip with my friends."
s = "My friends find a job after Christmas."

The argument, symbolically is as follows:
Premises:
$(p \land q) \to r$
$\neg s \to q$
p

$\neg s$

Conclusion: r

Steps	Reasons
1) $\neg s \to p$	Premise
2) $\neg s$	Premise
3) p	Modus Ponens (using 1, 2)
4) q	Premise
5) $p \wedge q$	Conjunction
6) $(p \wedge q) \to r$	Premise
7) r	Modus Ponens (using 5, 6)

3) Consider the following propositions:
p: It's sunny this afternoon.
q: It's colder than yesterday.
r: We will go swimming.
s: We will take a trip.
t: We will be home by sunset.

Premises
a. "It's not sunny and it's colder than yesterday" $\neg p \wedge q$
b ."We will go swimming only if it's sunny." $r \to p$
c. "If we don't go swimming then we will take e trip." $\neg r \to s$
d. "If we take a trip, then we will be home by sunset." $s \to t$

Conclusion: "We will be home by sunset." t.

(1) $\neg p \wedge q$ Premise
(2) $\neg p$ Simplification rule using (1)
(3) $r \to p$ Premise
(4) $\neg p \to \neg r$ Contrapositive
(5) $\neg r$ MP using (2), (4)
(6) $\neg r \to s$ Premise
(7) s MP using (5) ,(6)
(8) $s \to t$ Premise
(9) t MP using (6) (7)

This is a valid argument showing that from the premises (a), (b), (c) and (d), we can prove the conclusion t.

4)Suppose $p \to q$; $\neg p \to r$; $q \to s$. Prove that $\neg r \to s$.

(1) $p \to q$ Premise
(2) $\neg p \vee q$ Logically equivalent to (1)

(3) $\neg p \rightarrow r$	Premise
(4) $p \lor r$	Logically equivalent to (3)
(5) $q \lor r$	Apply resolution rule to (2),(4)
(6) $\neg r \rightarrow q$	logically equivalent to (5)
(7) $q \rightarrow s$	Premise
(8) $\neg r \rightarrow s$	Apply HS rule to (6)(7)

5)

(1) If it is Saturday today, then we play soccer or basketball.
(2) If the soccer field is occupied, we don't play soccer.
(3) It is Saturday today, and the soccer field is occupied.

Prove: "we play basketball or volleyball".

First we formalize the problem:
p: It is Saturday today.
q: We play soccer.
r: We play basketball.
s: The soccer field is occupied.
t: We play volleyball.

Premise: $p \rightarrow (q \lor r)$, $s \rightarrow \neg q$, p, s Need to prove: $r \lor t$.

(1) $p \rightarrow (q \lor r)$	Premise
(2) p	Premise
(3) $q \lor r \Leftrightarrow (\neg q \rightarrow r)$	Apply MP rule to (1)(2)
(4) $s \rightarrow \neg q$	Premise
(5) s	Premise
(6) $\neg q$	Apply MP rule to (4)(5)
(7) r	Apply DS rule to (3)(6)
(8) $r \lor t$	Apply Addition rule to (7)

6) Test the validity of the following arguments :

1. If milk is black then every crow is white.
2. If every crow is white then it has 4 legs.
3. If every crow has 4 legs then every Buffalo is white and brisk.
4. The milk is black.
5. So, every Buffalo is white.

Solution:

Let P: The milk is black
Q: Every crow is white
R: Every crow has four

legs.

S : Every Buffalo is white

T: Every Buffalo is

brisk The given premises are

(i) $P \to Q$

(ii) $Q \to R$

(iii) $R \to (S \wedge T)$

(iv) P

The conclusion is S. The following steps checks the validity of argument.

1.	$P \to Q$	premise (1)
2.	$Q \to R$	Premise (2)
3.	$P \to R$	line 1. and 2. Hypothetical syllogism (H.S.)
4.	$R \to (S \wedge T)$	Premise (iii)
5.	$P \to (S \wedge T)$	Line3 and 4.H.S
6.	P	Premise (iv)
7.	$S \wedge T$	Line 5, 6 modus ponens
8.	S	Line 7, simplification
$\therefore$	The argument is valid	

PREDICATES AND QUANTIFIERS

Predicates: A Predicate is a declarative sentence whose true/false value depends on one or more variables.

Definition: A predicate or propositional function or an open statement is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The domain of a predicate variable is the set of all values that may be substituted in place of the variables.

The statement "x is greater than 3" has two parts: the subject: x is the subject of the statement the predicate: "is greater than 3" (a property that the subject can have).

We denote the statement "x is greater than 3" by P(x), where P is the predicate "is greater than 3" and x is the variable. The statement P(x) is also called the value of propositional function P at x. Assign a value to x, so P(x) becomes a proposition and has a truth value: P(5) is the statement "5 is greater than 3", so P(5) is true. P(2) is the statement "2 is greater than 3", so P(2) is false

An open sentence is a statement involving one or more variables.

For example: (1) x+3=6
(2) $x < 9$
(3) $x^2 < 10$

An open sentence is neither true nor false. An open sentence becomes a proposition only after the variables are replaced by some particular values. Open statements containing a variable are denoted by p(x),q(x) etc.

If x belongs to the set of real numbers R, in p(x) then R is called the universe for variable x in an open statement p(x)

$\neg p(x)$ is the negation of an open statement p(x). also p(x) ∧ q(x) is the conjunction, p(x) ∨ q(x) is the disjunction , p(x)→q(x) is a conditional and p(x) ↔ q(x) is the bi-conditional of open statements p(x) and q(x)

e.g. Let A = {x / x is an integer < 8}

Here P(x) is the sentence "x is an integer less than 8".

The common property is "an integer less than 8".

∴ P(1) is the statement "1 is an integer less than 8".

∴ P(1) is true.

.**Example:** Let p(x) denote the statement "x > 10″. What are the truth values of p(11) and p(5) ?
p(11) is equivalent to the statement 11 > 10, which is True.
p(5) is equivalent to the statement 5 > 10, which is False.

Quantifiers: consider the following propositions

(1) All squares are rectangles
(2) Some determinants are equal to zero
(3) There exists a real number whose square is equal to itself

In these propositions, the words "all", "some", " there exists" are associated with the idea of a quantity. Such words are called Quantifiers.

A proposition involving the quantifiers is called quantified statement.

Let S be the set of all squares, the proposition (1) may be written as, For all $x \in S, x\ is\ a\ rectangle$.

Symbolically, this is written as $\forall x \in S, p(x)$

The phrases "for all", "for each", "for every", denoted by the symbol ∀ called **Universal quantifier**.

Let us consider the proposition (2) .If D denotes the set of all determinants, then the proposition may be rewritten as : For some x$\in D, xis\ equal\ to\ zero$.

Symbolically, this is written as: $\exists x \in D, p(x)$

The phrases "for some" , " there exists", "for any" denoted by symbol ∃ called **Existential quantifier.**

Universal quantification: ∀x, P(x) says "the predicate P is true for every element under consideration." Under the domain of natural numbers.

Existential quantification: ∃x, P(x) says "there is one or more element under consideration for which the predicate P is true" Under the domain of natural numbers.

Before deciding on the truth value of a quantified predicate, it is mandatory to specify the domain (also called domain of discourse or universe of discourse). P(x) ="x is an odd number" ∀x,P(x) is false for the domain of integer numbers; but ∀x, P(x) is true for the domain of prime numbers greater than 2.

Truth values of Quantified statement:

STATEMENT	TRUE	FALSE
$\forall x \in D, P(x)$	$P(x)$ is true for every x.	There is one x for which $P(x)$ is false.
$\exists x \in D, P(x)$	There is one x for which P(x) is true.	$P(x)$ is false for every x.

$Assume\ that\ D\ consists\ of\ x_1, x_2, \ldots, x_n$

♣ $\forall x \in D, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
♣ $\exists x \in D, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Logical Equivalences Involving Quantifiers:

Definition: Two statements S and T involving predicates and quantifiers are logically equivalent if and only if they have the same truth value regardless of the interpretation, i.e. regardless of the meaning that is attributed to each propositional function, the domain of discourse. We denote $S \equiv T$.

$(i)\ \forall x, (P(x) \wedge Q(x))\ logically\ equivalent\ to\ \forall x, P(x) \wedge \forall x, Q(x)$

$(ii) \forall x, \big(P(x) \vee Q(x)\big) logically\ equivalent\ to\ \forall x, P(x) \vee \forall x, Q(x)$

$(iii)\ \exists x, (P(x) \vee Q(x))\ logically\ equivalent\ to\ \exists x, P(x) \vee \exists x, Q(x)$

$(iv)\ \exists x, [p(x) \to q(x)] \Leftrightarrow \exists x, [\neg p(x) \vee q(x)]$

$(v))\ \forall x, \neg(P(x) \Leftrightarrow For\ no\ x, p(x)$

Inference Rules for Quantified Predicates:

Let $P(x)$ be predicates in one variable respectively.

Following are some basic inference rules for quantified predicates.

1. $Negation$: $\neg\{\forall x, P(x) \Leftrightarrow \exists x, \neg P(x)\}$ and $\neg\{\exists x, P(x)\} \Leftrightarrow \forall x, \neg p(x)$.

2. $Universal\ Specification$: $\forall x \in S, P(x) \Rightarrow P(a), any\ a \in S$.

3. $Existential\ Specification$: $\exists x, P(x) \Rightarrow P(a), some\ a \in S$.

4. $Universal\ Generalization$: $(any\ a \in S, P(a) = T) \Rightarrow \forall x \in S, P(x)$.

5. $Existential\ Generalization$: $(some\ a \in Dx, P(a) = T) \Rightarrow \exists x, P(\mathrm{x})$.

1) Let P(x) be the statement "$x = x^2$." If the domain consists of all the integers, what are these truth values?
Here $p(x)$: $x = x^2$
a) P(0) is True { for $x = 0, p(0){:}\, 0 = 0$}

b) P(2) is False { for $x = 2, p(2){:}\, 2 \neq 4$}

c) $\exists$x, P(x) is True { for $x = 1, p(1){:}\, 1 = 1$}

d) $\forall$x, P(x) is False { P(x) is not true for all x}

2) Let $P(x)$ be "$x^2 > 10$. What is the truth value of $\forall x, P(x)$ for each of the following domains?

1: The set of real numbers: R
False. $p(3)$ is a counter example.

2. The set of positive integers not exceeding 4: {1, 2, 3, 4}
False. 3 is a counter example

3. The set of real numbers in the interval [10, 39.5]
True. It takes a bit longer to verify than in false statements. Let x ∈ [10, 39.5]. Then x ≥ 10 which implies
$x^2 \geq 10^2 = 100 > 10$, and so $x^2 > 10$.

3) Let P(x) be "$x^2 > 10$. What is the truth value of $\exists x, P(x)$ for each of the following domains:
I: the set of real numbers: R
True. 10 is a witness.
II: the set of positive integers not exceeding 4: {1, 2, 3, 4}
True. 4 is a witness.
Also note that here $\exists x, P(x)$ is $P(1) \vee P(2) \vee P(3) \vee P(4)$, so its enough to observe that $P(4)$ is true.
III: the set of real numbers in the interval [0, √ 9.8]
False. It takes a bit longer to conclude than in true statements. Let x ∈ [0, √ 9.8]. Then $0 \leq x \leq \sqrt{9.8}$ which implies $x^2 \leq (\sqrt{9.8})^2 = 9.8 < 10$, and so $x^2 < 10$. What we have shown is that

$\forall x, \neg P(x)$, which (we will see) is equivalent to $\neg\exists x, P(x)$.

4) Let $P(x)$ be the predicate "$P(x): \ x^2 \geq x$." Determine whether the following universal statements are true or false.
(a) $\forall x \in R, P(x)$
(b) $\forall x \in Z, P(x)$
(c) $\exists x \in R, P(x)$
(d) $\exists x \in Z, P(x)$

(a) Let x = 1/ 2 ∈ R. Then, $(1/2)^2 = 1/4 < 1/2$, and so P(1/ 2) is false. Therefore, "$\forall x \in R, P(x)$" is false.
(b) For all integers x, $x^2 \geq x$ is true, and so P(x) is true for all $x \in Z$. Hence, "$\forall x \in Z, P(x)$" is true.
(c) Let x = 1/ 2 ∈ R. Then, $(1/2)^2 = 1/4 < 1/2$, and so P(1 /2) is true. Therefore, "$\exists x \in R, P(x)$" is true.
(d) For all integers x, $x^2 \geq x$ is true, and so there is no integer x such that P(x) is true. Hence, "$\exists x \in Z, P(x)$" is false.

(5) Let the set Z all integers be the universe. Consider the statements P(x): 2x+1=5 and q(x): $x^2=9$
Obtain the negation of the quantified statement $\exists x \in Z, [p(x) \wedge q(x)]$ and express it in words

Solution: $\neg\{\exists x \in Z, [p(x) \wedge q(x)]\} \Leftrightarrow \forall x \in Z, [\neg p(x) \vee \neg q(x)]$
In words, For all integers x, $2x+1 \neq 5$ *or* $x^2 \neq 9$

(6) Write down the following proposition in symbolic form, and find its negation: " If all triangle are right-angled, then no triangle is equiangular"
Solution:
Let T denote the set of all triangles. Also, let p(x): x is right angled, q(x): x is equiangular
The symbolic form of the proposition is $\{\forall x \in T, p(x)\} \rightarrow \{\forall x \in T, \neg q(x)\}$
The negation of this proposition is
$$\neg[\{\forall x \in T, p(x)\} \rightarrow \{\forall x \in T, \neg q(x)\}] \Leftrightarrow \neg[\neg\{\forall x \in T, p(x)\} \vee \{\forall x \in T, \neg q(x)\}]$$
$$\Leftrightarrow [\forall x \in T, p(x)] \wedge [\exists x \in T, q(x)]$$
In other words, this reads "All triangles are right –angled and some triangles are equiangular".

(7) Prove that the following argument is not valid.

All squares have four sides
The quadrilateral ABCD has four sides
∴ABCD is a square

Let the set of all quadrilaterals be the universe
$P(x)$: *x is a square*
$q(x)$: *x has four sides*
$r(x)$: *the quadrilateral ABCD*
We write the given argument as

$\forall x,[p(x) \rightarrow q(x)]$
q(a)
$\therefore$p(a)

$\forall x,[p(x) \rightarrow q(x)] \Rightarrow p(a) \rightarrow q(a)$, by the rule of universal specification
Therefore, $\forall x,[p(x) \rightarrow q(x)] \wedge q(a) \Rightarrow [p(a) \rightarrow q(a)] \wedge q(a) \not\Rightarrow p(a)$
Because p(a) can be false when both of $p(a) \rightarrow q(a)$ and q(a) are true
As such, the given argument is not valid

(8) Let the domain range over all people in the USA. Find a possible premise for the following inference.
Premises:
All librarians know the Library of Congress Classification System.
(unknown premise)
Conclusion:
Margaret knows the Library of Congress Classification System.
Let *m* denote Margaret. Define
P(x) : x is a librarian.
Q(x) : x knows the system.
We want to use the rule of universal specification and Modus Ponens. If the unknown premise is "Margaret is a librarian," then we can have the following

inference.
P1 : $\forall x,[P(x) \rightarrow Q(x)]$
P2 : P(m)

C : Q(m)

Steps	reasons
1: $\forall x, (P(x) \rightarrow Q(x))$	P1
2: $P(m)$	P2
3: $P(m) \rightarrow Q(m)$	1; Universal Specification
4: $Q(m)$	2; 3;Modus Ponens

Where P(m) means: Margaret is a librarian.

(9) Are the following arguments logically correct?
Premises:
There are men who are soldiers.
All soldiers are strong.
All soldiers are brave.
Conclusion:
Therefore some strong men are brave.

Let the domain *Dx* be the set of all people. Define the following predicates with variables over *Dx*.
$P(x)$: x *is a man*.
$Q(x)$: x *is a soldier*.
$R(x)$: x *is strong*.
$S(x)$: x *is brave*.
The premises and the conclusion in terms of the predicates de¯ned above
are:

$$P1 : \exists x, [P(x) \land Q(x)]$$
$$P2 : \forall x, [Q(x) \rightarrow R(x)]$$
$$P3 : \forall x, [Q(x) \rightarrow S(x)]$$

$$C : \exists x, [P(x) \land R(x) \land S(x)]$$

Proof by using the laws of logic and inference rules for quantified predicate calculus. *steps reasons*

1: $\exists x, [P(x) \land Q(x)]$	$P1$
2: $P(a) \land Q(a)$	1; *Existential Specification*
3: $\forall x, [Q(x) \rightarrow R(x)]$	$P2$
4: $Q(a) \rightarrow R(a)$	3; *Universal Speci¯cation*
5: $Q(a)$	2; *Conjunctive Simpli¯cation*
6: $R(a)$	4; 5; *Modus Ponens*
7: $\forall x[Q(x) \rightarrow S(x)]$	$P3$
8: $Q(a) \rightarrow S(a)$	7; *Universal Specification*
9: $S(a)$	5; 8*Modus Ponens*
10: $P(a)$	2; *Conjunctive Simpli¯cation*
11: $P(a) \land R(a) \land S(a)$	6; 9; 10; *conjunction*
12: $\exists x, (P(x) \land R(x) \land S(x))$	11; *Existential Generalization*

(10) Let the domain range over all real numbers. Find a possible conclusion from the given premises.
Premises:
All integers are rational numbers.
The real number π is not a rational number.

Premises:
P1: $\forall x, [P(x) \rightarrow Q(x)]$
P2 : $R(\pi) \land \neg Q(\pi)$

steps	reasons
1: $\forall x, [P(x) \rightarrow Q(x)]$	P1
2: $R(\pi) \land \neg Q(\pi)$	P2
3: $P(\pi) \rightarrow Q(\pi)$	1; Universal Specification
4: $\neg Q(\pi) \rightarrow \neg P(\pi)$	3; Contrapositive
5: $\neg Q(\pi)$	2; Conjunctive Simplification
6: $\neg P(\pi)$	4; 5;Modus Ponens
7: $R(\pi)$	2; Conjunctive Simplification
8: $R(\pi) \land \neg P(\pi)$	6; 7; Conjunction
9: $\exists x,(R(x) \land \neg P(x))$	8; Existential Generalization

10: $\neg P(\pi) \vee \neg R(\pi)$	6; Disjunctive Amplification
11: $\neg [P(\pi) \wedge R(\pi)]$	10; De Morgan's Law
12: $\exists x, \neg (P(x) \wedge R(x))$	11; Existential Generalization
13: $\neg\forall x,(P(x) \wedge R(x))$	12; Logical Equivalence

In step 6, π is not an integer.
In step 9, there is a real but not rational number.
In step 13, not all numbers are both integer and real.

Method of Proof and Disproof:

The process of establishing the conditional is true by using the rules/laws of logic and other known facts constitutes a proof of the conditional.
The process of establishing the conditional is false by using the rules/laws of logic and other known facts constitutes a disproof of the conditional.

Direct Proof:

The direct method of proving a conditional $p\rightarrow q$ has the following lines of arguments
(a) Hypothesis: First assume that p is true.
(b) Analysis: starting with the hypothesis and employing the rules/ laws of logic and other known facts, infer that q is true.
(c) $p\rightarrow q$ is true.

Indirect Proof:

The conditional $p\rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ are logically equivalent. In some situation, given a conditional $p\rightarrow q$, a direct proof of the contrapositive $\neg q \rightarrow \neg p$ is easier. On the basis of this proof, we infer that the conditional $p\rightarrow q$ is true. This method of proving a conditional is called an indirect method of proof.

Proof by Contradiction:

The indirect method of proof is equivalent to what is known as proof by Contradiction. The method of proof of the statement $p\rightarrow q$ are as follows.
(a) Hypothesis: Assume that $p\rightarrow q$ is false, That is, assume that p is true and q is false.
(b) Analysis: Starting with the hypothesis that q is false and employing the rules of logic and other known facts, infers that p is false. This contradicts the assumption that p is true.
(c) Conclusion: Because of the contradiction arrived in the analysis, we infer that $p \rightarrow q$ is true.

Problems

(1) Give (i) direct proof, (ii) indirect proof, and (iii) proof by contradiction for the following statement
"If n is an odd integer, then n+11 is an even integer"

(i) Direct proof : Assume that n is an odd integer. Then $n = 2k + 1$ for some integer k. This gives $n + 11 = (2k + 1) + 11 = 2k + 12 = 2(k + 6)$ from which it is evident that $n + 11$ is even. This establishes the given statement by direct proof.

(ii) Indirect proof: Assume that $n + 11$ is not an even integer. Then $n + 11 = 2k + 1$ for some integer k. This gives $n = 2k + 1 - 11 = 2k - 10 = 2(k - 5)$, which shows that n is even. Thus, if $n + 11$ is not even, then n is not odd. This proves the contrapositive serves as an indirect proof of the given statement.

(iii) Proof of Contrapositive: Assume that the given statement is false. That is, assume that n is odd and $n + 11$ is odd. Since $n + 11$ is odd, $n + 11 = 2k + 1$ for some integer k so that $n = (2k + 1) - 11 = 2(k - 5)$ which shows that n is even. This contradicts the assumption that n is odd. Hence the given statement must be true.

(2) Give direct Proof to Prove the statement: For all integers$m\ and\ n$, if m and n are odd integers, then $m + n$ is an even integer.
Proof. Assume m and n are arbitrary odd integers. Then m and n can be written in the form $m = 2a + 1\ and\ n = 2b + 1$, where $a\ and\ b$ are also integers. Then $m + n = (2a + 1) + (2b + 1)$ (substitution) $= 2a + 2b + 2$ (associative and commutative laws of addition) $= 2(a + b + 1)$ (distributive law) since $m + n$ is twice another integer, namely, $a + b + 1, m + n$ is an even integer.

(3) Proof by Contrapositive. Prove the statement: For all integers m and n, if the product of m and n is even, then m is even or n is even.
We prove the contrapositive of the statement: If m and n are both odd integers, then mn is odd.
Proof: Suppose that m and n are arbitrary odd integers. Then$m = 2a + 1\ and\ n = 2b + 1$, where a and b are integers. Then $mn = (2a + 1)(2b + 1)$ (substitution) $= 4ab + 2a + 2b + 1$ (associative, commutative, and distributive laws) $= 2(2ab + a + b) + 1$ (distributive law) since mn is twice an integer (namely, 2ab + a + b) plus 1, mn is odd.

Discussion If a direct proof of an assertion appears problematic; the next most natural strategy to try is a proof of the contrapositive. We use this method to prove that if the product of two integers, m and n, is even, then m or n is even. This statement has the form $p \rightarrow (r \vee s)$. If you take our advice above, you will first try to give a direct proof of this statement: assume mn is even and try to prove m is even or n is even. Next, you would use the definition of "even" to write $mn = 2k$, where k is an integer. You would now like to conclude that m or n has the factor 2. This can, in fact, be proved directly, but it requires more knowledge of number theory than we have available at this point. Thus, we seem to have reached a dead-end with the direct approach, and we decide to try an indirect approach instead. The contrapositive of $p \rightarrow (r \vee s)\ is\ \neg(r \vee s) \rightarrow \neg p$, or, by De Morgan's Law, $(\neg r \wedge \neg s) \rightarrow \neg p$. This translates into the statement "If m and n are odd, then mn is odd" (where "not even" translates to "odd"). This is a good illustration of how the symbolic form of a proposition can be helpful in finding the correct statement we wish to prove. In this particular example, the necessity of De Morgan's Law may be more evident in the symbolic form than in the "English version." Now we give a direct proof of the contrapositive: we assume m and n are arbitrary odd integers and deduce mn is odd. other strategy.

(4) Prove: For any integer n, n is odd if and only if n^2 is odd. In order to prove this statement, we must prove two implications: (a) If n is odd, then n^2 is odd. (b) If n^2 is odd, then n is odd.

Proof of (a): We give a direct proof of this statement. Assume n is an odd integer. Then n = 2a + 1 for some integer a. Then $n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$, which is twice an integer plus 1. Thus, n^2 is odd.

Proof of (b): We give a proof of the contrapositive of this statement: "If n is even (not odd), then n^2 is even (not odd). Assume n is an even integer. Then n = 2a for some integer a. Then $n^2 = (2a)^2 = 4a^2 = 2(2a^2)$, which is an even integer.

(5) If n is an even integer, then $3n + 5$ is an odd integer.

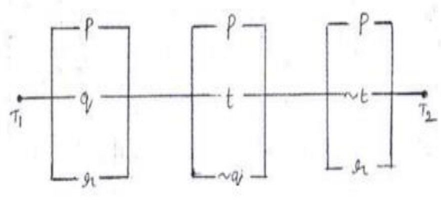
Proof. If we let "P(n): n is even" and "Q(n): 3n + 5 is odd", then we need to show that the universal statement $\forall n \in Z, P(n) \rightarrow Q(n)$ is true. To do this, we assume P(n) is true for some particular but arbitrary element $n \in Z$ and show that $Q(n)$ is true for this element n. Since $P(n)$ is true, $n = 2k$ for some integer k. Hence, $3n + 5 = 3(2k) + 5 = 6k + 5 = 2(3k+2) + 1 = 2m + 1, where\, m = 3k + 2$. Since $k \in Z$, we must have $m \in Z$ (since the product of two integers is an integer, and the sum and difference of two integers is an integer). Hence, $3n + 5 = 2m + 1$ for some integer m, whence $Q(n)$ is true. Thus by the method of direct proof, we have proven our desired result.

(6) Prove by contrapositive, $Let\ n \in Z$. If $n^2 + 5$ is odd, then n is even.
Proof: Let $P(n)$ be the statement "$n^2 + 5$ is odd" and let Q(n) be the statement "n is even". Then we need to show that the universal statement $\forall n \in Z, P(n) \rightarrow Q(n)$ is true. To do this, we use a proof by contrapositive. We give a direct proof to show that $\sim Q(n) \rightarrow \sim P(n)$. Hence we assume that $\sim Q(n)$ is true for some particular but arbitrary element $n \in Z$ and show that $\sim P(n)$ is true for this element n. Since $\sim Q(n)$ is true, n is not even. Thus, n is odd, and so $n = 2k + 1$ for some integer k.

Hence, $n^2 + 5 = (2k+1)^2 + 5 = 4k^2 + 4k + 6 = 2(k^2 + 2k + 3) = 2m$, where $m = k^2 + 2k + 3$. Since $k \in Z$, we must have $m \in Z$. Hence, $n^2 + 5 = 2m$ for some integer m, and so $n^2 + 5$ is even, i.e., $n^2 + 5$ is not an odd integer. Thus, ~P(n) is true. Therefore by the method of direct proof, we have proven that ~Q(n) → ~P(n) is true. Hence, P(n) → Q(n) is true. Therefore, ∀n ∈ Z, P(n) → Q(n).

Exercise
(1) Examine whether $[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$ is a tautology.
Ans: Contigency

(2) Apply laws of logic and simplify the following switching network.



(3) Verify the principle of duality for the logical equivalence $(p \vee q) \wedge (\neg p \wedge (\neg p \wedge q)) \Leftrightarrow (\neg p \vee q)$.

(4). Express $(p \vee q)$ and $(p \wedge q)$ in terms of only NAND and only NOR connectives.

Ans: $(p \vee q) \Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)$, $(p \vee q) \Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$
$(p \wedge q) \Leftrightarrow) \Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q,) \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$

(5) Test whether the following is a valid argument
If I study, then I do not fail in the examination.
If I do not fail in the examination, my father gifts a two-wheeler to me.

$\therefore$, if I study then my father gifts a -wheeler to me.

Ans: It is a valid argument

(6) Prove that $[\{\forall\, x, p(x)\} \vee \{\forall\, x, q(x)\}] \rightarrow \forall\, x, \{p(x) \vee q(x)\}$ is a tautology.