ATMS Chapter3 : Transformations of Given Statistics

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Table of Contents

1. 3.1 Functions of Asymptotically Normal Statistics: Univariate Case
2. 3.2 Examples and Applications
3. 3.3 Functions of Asymptotically Normal Vectors
4. 3.4 Further Examples and Applications
5. 3.5 Quadratic Forms in Asymptotically Multivariate

Table of Contents

1. 3.1 Functions of Asymptotically Normal Statistics: Univariate Case
2. 3.2 Examples and Applications
3. 3.3 Functions of Asymptotically Normal Vectors
4. 3.4 Further Examples and Applications
5. 3.5 Quadratic Forms in Asymptotically Multivariate

Theorem (3.1A)

Suppose that X_n *is* $AN(\mu, \sigma_n^2)$, *with* $\sigma_n \to 0$. *Let* g *be a real-valued function differentiable at* $x = \mu$, *with* $g'(\mu) \neq 0$. *Then*

$$g(X_n) \text{ is } AN(g(\mu), [g'(\mu)]^2 \sigma_n^2).$$

Theorem (3.1B)

Suppose that X_n is $AN(\mu, \sigma_n^2)$, with $\sigma_n \to 0$. Let g be a real-valued function differentiable $m(\geq 1)$ times at $x = \mu$, with $g^{(m)}(\mu) \neq 0$ but $g^{(j)}(\mu) = 0$ for $j < m$. Then

$$\frac{g(X_n) - g(\mu)}{\frac{1}{m!} g^{(m)}(\mu) \sigma_n^m} \xrightarrow{d} [N(0,1)]^m.$$

Table of Contents

1 3.1 Functions of Asymptotically Normal Statistics: Univariate Case

2 3.2 Examples and Applications

3 3.3 Functions of Asymptotically Normal Vectors

4 3.4 Further Examples and Applications

5 3.5 Quadratic Forms in Asymptotically Multivariate

Variance-Stabilizing Transformations

In the case that $\sigma_n^2(\theta)$ is the form $\sigma_n^2(\theta) = h^2(\theta)v_n$, where $v_n \to 0$, the appropriate choice of g may be found via Theorem 3.1A. For, if $Y_n = g(X_n)$ and $g'(\theta) \neq 0$, we have

$$Y_n \text{ is } AN(g(\theta), [g'(\theta)]^2 h^2(\theta) v_n).$$

Thus, in order to obtain that Y_n is $AN(g(\theta), c^2 v_n)$, where c is a constant independent of θ, we choose g to be the solution of the differential equation

$$\frac{dg}{d\theta} = \frac{c}{h(\theta)}.$$

Hanging Rootogram

Typically, $f_n(x)$ is asymptotically normal. For example, in the case of the simple $f_0(\cdot)$ considered in 2.1.8 and in Problems 2.P.3–5, we have that

$$f_n(x) \text{ is } AN(f(x), f(x)/2nh_n),$$

where $nh_n \to \infty$.
Take $g(x) = x^{1/2}$, then

$$f_n^{1/2}(x) \text{ is } AN(f^{1/2}(x), 1/8nb_n).$$

And $f_n^{1/2}(x) - f_0^{1/2}(x)$ are $AN(0, 4nb_n)$, each x.

Table of Contents

1. 3.1 Functions of Asymptotically Normal Statistics: Univariate Case
2. 3.2 Examples and Applications
3. 3.3 Functions of Asymptotically Normal Vectors
4. 3.4 Further Examples and Applications
5. 3.5 Quadratic Forms in Asymptotically Multivariate

Theorem (3.2A)

Suppose that $X_n = (X_{n1}, \ldots, X_{nk})$ *is* $AN(\mu, b_n^2\Sigma)$*, with* Σ *a covariance matrix and* $b_n \to 0$*. Let* $g(x) = (g_1(x), \ldots, g_m(x))$*,* $x = (x_1, \ldots, x_n)$*, be a vector-valued function for which each component function* $g_i(x)$ *is real-valued and has a nonzero differential* $g_i(\mu; t)$*,* $t = (t_1, \ldots, t_n)$*, at* $x = \mu$*. Put*

$$D = \left[\left. \frac{\partial g_i}{\partial x_j} \right|_{x=\mu} \right]_{m \times k}$$

Then

$$g(X_n) \text{ is } AN(g(\mu), b_n^2 D\Sigma D').$$

Theorem

Suppose that $X_n = (X_{n1}, \ldots, X_{nk})$ *is* $AN(\mu, n^{-1}\Sigma)$. *Let* $g(x)$ *be a real-valued function possessing continuous partials of order* $m(> 1)$ *in a neighborhood of* $x = \mu$, *with all the partials of order* j, $1 \leq j \leq m-1$, *vanishing at* $x = \mu$, *but with the mth order partials not all vanishing at* $x = \mu$. *Then*

$$n^{m/2}[g(X_n) - g(\mu)] \xrightarrow{d} \frac{1}{m!} \sum_{i_1=1}^{k} \cdots \sum_{i_m=1}^{k} \left. \frac{\partial^m g}{\partial x_{i_1} \cdots \partial x_{i_m}} \right|_{x=\mu} \cdot Z_{i_j},$$

where $Z = (Z_{ij}) = N(0, \Sigma)$.

Table of Contents

1. 3.1 Functions of Asymptotically Normal Statistics: Univariate Case
2. 3.2 Examples and Applications
3. 3.3 Functions of Asymptotically Normal Vectors
4. 3.4 Further Examples and Applications
5. 3.5 Quadratic Forms in Asymptotically Multivariate

Sample Correlation Coefficient

$\rho = \sigma_{xy}/\sigma_x\sigma_y$
$\sigma_{xy} = E\{(X_1 - \mu_x)(Y_1 - \mu_y)\}$
$\mu_x = E\{X_1\}$
$\mu_y = E\{Y_1\}$
$\sigma_x^2 = \text{Var}\{X_1\}$
$\sigma_y^2 = \text{Var}\{Y_1\}$
$\hat{\rho} = \frac{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\left[\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2\right]^{1/2}\left[\frac{1}{n}\sum_{i=1}^n (Y_i - \bar{Y})^2\right]^{1/2}}$
Let $\hat{\rho} = g(V)$
$V = \left(X, \bar{X}, \frac{1}{n}\sum_{i=1}^n X_i^2, \frac{1}{n}\sum_{i=1}^n Y_i^2, \frac{1}{n}\sum_{i=1}^n X_i Y_i\right)$
$g(z_1, z_2, z_3, z_4, z_5) = \frac{z_5 - z_1 z_2}{(z_3 - z_1^2)^{1/2}(z_4 - z_2^2)^{1/2}}$

V is $AN(E(V), n^{-1}\Sigma)$,
where $\Sigma_{5\times 5}$ is the covariance matrix of $(X_1, Y_1, X_1^2, Y_1^2, X_1 Y_1)$
Then$\hat{\rho}$ is $AN(\rho, n^{-1} d\Sigma d')$,
where $d = \left(\frac{\partial g}{\partial z_1}\Big|_{x=E(V)}, \ldots, \frac{\partial g}{\partial z_5}\Big|_{x=E(V)} \right)$

$\hat{\theta}_{n1}, \ldots, \hat{\theta}_{nk}$
$X_n = (\hat{\theta}_{n1}, \ldots, \hat{\theta}_{nk})$ is $AN((\theta, \ldots, \theta), n^{-1}\Sigma)$
$\hat{\theta}_n = \sum_{i=1}^{k} \beta_i \hat{\theta}_{ni}$
$\hat{\theta}_n$ is $AN(\theta, n^{-1}\beta\Sigma\beta')$
The solution may be obtained as a special case of useful results given by Rao (1973), Section 1.f, $\inf_{\sum_{i=1}^{k}\beta_i=1} \beta\Sigma\beta' = \frac{1}{\sum_{i=1}^{k}\sum_{j=1}^{k}\sigma_{ij}^*}$, where $\Sigma^* = \Sigma^{-1} = (\sigma_{ij}^*)$.
So,

$$\beta_0 = (\beta_{01}, \ldots, \beta_{0k}) = \left(\frac{\sum_{j=1}^{k} \sigma_{1j}^*}{\sum_{i=1}^{k}\sum_{j=1}^{k}\sigma_{ij}^*}, \ldots, \frac{\sum_{j=1}^{k} \sigma_{kj}^*}{\sum_{i=1}^{k}\sum_{j=1}^{k}\sigma_{ij}^*} \right)$$

Table of Contents

1. 3.1 Functions of Asymptotically Normal Statistics: Univariate Case
2. 3.2 Examples and Applications
3. 3.3 Functions of Asymptotically Normal Vectors
4. 3.4 Further Examples and Applications
5. 3.5 Quadratic Forms in Asymptotically Multivariate

A lemma proved in Rao (1973), Section 3.8.4

Lemma

Let $X = (X_1, \ldots, X_k)$ *be* $N(\mu, I_k)$, I_k *the identity matrix, and let* $C_{k\times k}$ *be a symmetric matrix. Then the quadratic form* $X'CX$ *has a (possibly noncentral) chi-squared distribution if and only if* C *is idempotent, that is,* $C^2 = C$, *in which case the degrees of freedom is* $rank(C) = trace(C)$ *and the noncentrality parameter is* $\mu' C \mu'$.

Theorem

Let $X = (X_1, \dots, X_k)$ *be* $N(\mu, \Sigma)$*, and let* $C_{k\times k}$ *be a symmetric matrix. Assume that, for* $\eta = (\eta_1, \dots, \eta_k)$*,*

$$\eta\Sigma = 0 \Rightarrow \eta\mu' = 0.$$

Then XCX' *has a (possibly noncentral) chi-squared distribution if and only if*

$$\Sigma C\Sigma C\Sigma = \Sigma C\Sigma,$$

in which case the degrees of freedom is trace$(C\Sigma)$ *and the noncentrality parameter is* $\mu C\mu'$*.*