

# The Optimal Sequence of Prices and Auctions

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## Abstract

A seller chooses to either post a price or run a reserve-price auction each period to sell a good before a deadline. Buyers with independent private values arrive over time. Assume that an auction costs more to the seller than a posted price. For a wide range of auction costs, the profit-maximizing mechanism sequence is to post prices first and then to run auctions. The optimality of the prices-then-auctions mechanism sequence provides a new justification for the use of the buy-it-now selling format on eBay.

**Keywords:** posted price, reserve price auction, eBay, buy-it-now

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# 1 Introduction

Although in theory a second-price auction with a carefully chosen reserve price can maximize the expected revenue when buyers have independent private values (Myerson, 1981), in practice it can be very costly and complicated to run. Especially in comparison with a simple posted price, an auction has operational costs and mental costs for sellers and buyers.

This paper takes into account auction costs for sellers and buyers and investigates how a seller repeatedly chooses between prices and auctions to maximize her expected profit. Specifically, a seller must sell one unit of an indivisible good within  $T$  periods and in each period, she either runs a reserve-price auction incurring a per-period auction cost or posts a price for free. The deadline  $T$  can be one, finite, infinite, or stochastic. Buyers with independent private values enter the market each period and pay a cost to bid in an auction. Buyers can be short-lived or long-lived, myopic or forward-looking.

Most interestingly, when the good has to be sold before a deadline and when the auction cost is within the most economically relevant range, the optimal mechanism sequence takes a neat form: prices then auctions. No other combination of prices and auctions though feasible is optimal; it is never optimal to run auctions then to post prices, or to alternate between prices and auctions. The prices-then-auctions mechanism sequence in particular resembles the buy-it-now selling mechanism on eBay: the seller posts a price at which any buyer can snatch the good before an auction starts.

The main result, the optimality of the prices-then-auctions sequence, relies on the endogenously higher opportunity cost of an auction in the dynamic setting. In the static setting, the seller only faces the trade-off between (i) a higher revenue for the optimal auction than for the optimal posted price and (ii) a higher cost for an auction than for a posted price. In the dynamic setting, the seller faces an additional cost when she uses an auction. The good not sold today is worth the expected revenue it generates in the next period. Since the probability of sale is higher using the optimal auction than using the optimal posted price (the optimal posted price is always higher than the optimal reserve price), using an auction in a dynamic setting incurs not only an operational cost but also a higher opportunity cost associated with selling the good early. The operational cost stays constant but the opportunity cost decreases over time. Therefore, running an auction in a later period incurs lower combined operational and opportunity cost than doing so in an earlier period. If an auction is ever used, it is used in later periods.

Having understood the auction's endogenous declining opportunity cost, we can more easily see that the optimal prices-then-auctions sequence persists in more general settings. The optimality of the prices-then-auctions sequence holds even when the sale deadline is stochastic, the seller becomes increasingly impatient, the seller faces declining auction costs, buyers arrive stochasti-

cally and have sequential outside options, the seller faces separate markets for prices and auctions, the mechanism designer is procuring a contract rather than selling, and the seller sequentially sells multiple goods. Furthermore, with more effort, the price-then-auction sequence is also shown to be optimal in two-period settings in which the buyers are long-lived. The key complication with long-lived buyers is that the seller's dynamic problem cannot be dissected into per-period problems because each period's mechanism affects buyers' strategies in each period.

When the horizon is infinite so that there is no exogenous deadline to sell the good, the problem the seller faces in each period is stationary, so the seller runs the same mechanism repeatedly in each period. A high-cost seller posts a constant price and a low-cost seller runs auctions with a constant reserve price. An interesting comparative statics result is derived: more patient sellers are more likely to post prices. This comparative statics sheds light on the issue about the declining use of auctions on eBay ([Bajari and Hortacsu, 2004](#); [Einav et al., 2018](#)). It shows that changes in market size and market transaction alone, without changes in seller or buyer preferences, can shift the sellers' choice of mechanisms.

The setup intends to capture an individual seller's problem in a large market such as eBay. For example, a person who has bought a lyric opera ticket but could no longer attend the event scheduled in two weeks chooses between posting a fixed price for the ticket and auctioning off the ticket before it loses its value. The item's posting can last for a week (e.g. it stays as a new item for a week on the front page of the website, the much more likely place for buyers to search). Potential buyers browse the website and encounter the posting for the item on sale, and decide whether or not to buy the ticket immediately. They have idiosyncratic values for the ticket. Although their individual values are unknown, their aggregate demand curve is known to the seller. Buyers are anonymous to the seller so the seller is restricted to use either prices or auctions.

In practice, there are additional fees associated with auction-style listings. Additional fees associated with the auction-style listing on eBay include a *reserve price fee*, a *special duration fee*, and fees if you list duplicate, identical auction-style listings. The reserve price fee is applied if a reserve price is set. This fee is equal to \$3 if the reserve price is less than \$150 but is 2% of the reserve price (with a maximum of \$100) for reserve prices greater than \$150. The special duration fee is \$1 to have your auction last only one or three days. This would be a preferred auction length because buyers search auctions that are about to end more frequently than ones with a lot of time left. Finally, the fee associated with listing duplicate, identical auction style-listings varies on the item but is used to allow the buyer to have different buying options, whether it be different types of auctions or fixed price listings. This fee prevents a single seller from having complete control over the types of listings on eBay and giving buyers a variety of buying options. From a close look at the types of fees associated with the two types of listings on eBay, it can be concluded that the auction style listing comes at a higher cost to the seller.

The paper provides a new justification for the use of buy-it-now options on eBay. Previous explanations include reasons for both the seller and the buyer. Reasons for the buyer include impatience in waiting for an auction to end (Mathews, 2004), risk aversion of the buyer (Reynolds and Wooders, 2009), heterogeneous preferences in listing styles (Bauner, 2015), and experienced buyers' ability to recognize buy-it-now prices below market price (Standifird et al., 2005). Reasons for the seller include increased competition forcing prices down using the buy-it-now feature (Anwar and Zheng, 2015), the increased revenue due to risk averse buyers (Budish and Takeyama, 2001), and seller's impatience and risk aversion (Hammond, 2013).

The rest of the paper is organized as follows. Section 2 introduces the basic setup. Section 3 solves the seller's static problem. Section 4 demonstrates the key insight of the paper with a simple two-period example. Section 5 solves the seller's problem in the finite horizon. Section 6 demonstrates the robustness of the results under extensions of the basic setup. Section 7 further demonstrates the robustness of the results when buyers are long-lived. Section 8 solves and discusses the seller's problem without deadline. Section 9 concludes. The appendix contains omitted proofs and calculations.

## 2 Basic Setup

Let me first introduce the basic setup and characterize the seller's static problem. Many components of the basic setup (e.g. constant discounting, constant fixed buyer arrival, fixed auction cost, short-lived buyers) will be relaxed in Sections 6 and 7.

A (female) seller wants to sell one unit of an indivisible good for which she has zero consumption value. She must sell the good within  $T$  periods, where  $T$  can be one, finite, stochastic, or infinite. She discounts each period by the same discount factor  $\delta \in [0, 1]$ . In each period  $t$ ,  $n$  one-period-lived (male) buyers enter the market. Each buyer has a private value  $v$  independently drawn from the identical value distribution  $F$  with positive density  $f$  on the entire support  $[0, 1]$ . All the agents are risk-neutral and have quasi-linear preferences in transfers. Everything is common knowledge except for the private value each buyer is born with.

Throughout the paper we maintain the assumption that the virtual value is increasing. The sole purpose of the assumption is to guarantee that the optimal reserve price and the optimal posted price are uniquely determined so that we do not have to deal with the complication that the optimal mechanism involves ironing.

**Assumption 1.** *The virtual utility  $\alpha(v) \equiv v - (1 - F(v))/f(v)$  is increasing in  $v$ .*

At the beginning of each period  $t$ , the seller chooses a mechanism  $m_t$ , either a reserve price second-price auction or a posted price. A seller running a second-price auction  $A_r$  with reserve

price  $r$  asks each buyer for a sealed bid. At the end of the period the seller assigns the good to the buyer with the highest bid if it is above  $r$ , and the winning buyer pays the bigger of the reserve price and the second highest bid. It is a dominant strategy for a buyer to bid his value  $v$ . A reserve price auction costs  $c \geq 0$  to run. On the other hand, it is free to post a price. In a posted price mechanism  $P_p$ , the seller posts a fixed price  $p$  at the beginning of a period and the buyers with values higher than  $p$  have equal chances of receiving the good.

Let  $(m_\tau, \dots, m_T)$  denote the *mechanism sequence* of the seller who runs mechanism  $m_t$  in period  $t$  if the good has not been sold by the end of period  $t - 1$ . The seller's problem is to choose the optimal mechanism sequence  $\mathbf{m}^* \equiv (m_1, \dots, m_T)$  so that the expected profit from running  $(m_\tau, \dots, m_T)$  is maximized for any period  $\tau$ . Since the buyer arrival process is known and there is no learning by the seller, the seller essentially chooses a sequence of mechanisms at the beginning of the first period, to be executed in each period if the good has not been sold.

### 3 Static Problem

I solve the seller's one-period problem as a building block for the subsequent multi-period problem. The optimal reserve price auction is restating [Myerson \(1981\)](#). I state the characterization of the optimal posted price in a parallel way. The introduction of posted price marginal revenue curve facilitates the exposition as well as the solution of the seller's problem in the dynamic setting.

The realized revenue of an auction  $A_r$  is  $r$  if the second highest bid is lower than  $r$ , and is  $v$  if the second highest bid  $v$  is greater than  $r$ . Therefore, the expected revenue of an auction is

$$R(A_r) = rn[1 - F(r)]F^{n-1}(r) + \int_r^1 vn(n-1)[1 - F(v)]F^{n-2}(v)f(v)dv.$$

It can be rearranged to

$$R(A_r) = \int_r^1 \alpha(v) dF^n(v)$$

where

$$\alpha(v) = v - \frac{1 - F(v)}{f(v)}.$$

[Myerson \(1981\)](#) calls  $\alpha(v)$  the virtual utility curve and we follow [Bulow and Roberts \(1989\)](#) to call  $\alpha(v)$  an auction's marginal revenue curve.<sup>1</sup> The expected revenue maximizing mechanism

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<sup>1</sup>[Bulow and Roberts \(1989\)](#) have named it so because of the following interpretation of the problem. The probability that a buyer buys the good at price  $v$  is  $q(v) = 1 - F(v)$ . An inverse demand curve can be therefore constructed as  $v(q) = F^{-1}(1 - q)$ . The expected revenue from selling quantity  $q$  is  $R(q) = q \cdot v(q) = q \cdot F^{-1}(1 - q)$ , so the marginal revenue is  $MR(q) = F^{-1}(1 - q) - q/F'(F^{-1}(1 - q))$ . Substituting in  $v(q)$ ,  $MR(q(v)) = v - [1 - F(v)]/f(v) \equiv \alpha(v)$ . The reserve price auction  $A_r$  asks each buyer to submit  $v$  and calculates the marginal revenue  $\alpha(v)$  accordingly. The good is assigned to the buyer with the highest marginal revenue  $\alpha(v)$  if it is positive.

among all direct revelation mechanisms is a reserve price auction with the optimal reserve price  $r^*$  uniquely determined by  $\alpha(r^*) = 0$ .<sup>2</sup>

Posting price  $p$  results in revenue  $p$  if there is a buyer who values it more than  $p$  and 0 if no buyer values it more than  $p$ . Its expected revenue can be written as  $R(P_p) = p[1 - F^n(p)]$ . As Bulow and Roberts (1989) construct an auction marginal revenue curve  $\alpha(v)$ , I construct here a posted price marginal revenue curve. In contrast to the auction marginal revenue curve that is constructed for each buyer with value drawn from distribution  $F(v)$ , the posted price marginal revenue curve is constructed for the highest value buyer out of the  $n$  buyers. Recognize that the seller generates the same revenue from posting a price to all buyers and from posting the same price to the highest value buyer<sup>3</sup>, the posted price marginal revenue curve is the auction marginal revenue curve with respect to a buyer who draws his value from the first-order distribution  $F^n$ ,

$$\rho(v) = v - \frac{1 - F^n(v)}{[F^n(v)]'}.$$

We can similarly write posted price  $P_p$ 's expected revenue as

$$R(P_p) = \int_p^1 \rho(v) dF^n(v).$$

The optimal posted price  $p^*$  is uniquely determined by  $\rho(p^*) = 0$ .<sup>4</sup>

Such a representation helps us see the similarities between the two classes of mechanisms. More importantly, the representation facilitates the exposition and eases our understanding of the optimal price determination in the dynamic setting. Figure 1 provides an illustration of the two marginal revenue curves. The optimal reserve price and the optimal posted price equate the marginal revenue to zero. The areas under the curves (weighted with respect to  $dF^n(v)$ ) depict the expected revenues of the two mechanisms.

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<sup>2</sup>Since  $\alpha(v)$  is continuous,  $\alpha(0) < 0$  and  $\alpha(1) = 1$ ,  $r^*$  exists. Since  $F$  satisfies the monotone hazard rate condition,  $\alpha(v)$  is strictly increasing, and  $r^*$  is uniquely determined. The probability the good is not sold is  $k(r^*) = F^n(r^*)$ .

<sup>3</sup>At least one buyer is willing to pay price  $p$  if and only if the highest value buyer is willing to pay price  $p$ . The highest value is drawn from the first-order distribution  $F^n(v)$ . The highest value buyer's inverse demand curve is  $v(q) = (F^n)^{-1}(1 - q)$ , and the marginal revenue is derived from  $d[q \cdot (F^n)^{-1}(1 - q)]/dq$ .

<sup>4</sup>The optimal price  $p^*$  exists because  $\rho(v)$  is continuous,  $\rho(0) < 0$  and  $\rho(1) = 1$ . The optimal price  $p^*$  is unique because

$$\rho(v) = v - \frac{1 - F^n(v)}{[F^n(v)]'} = v - \frac{F^{n-1}(v) + \dots + 1}{nF^{n-1}(v)} \cdot \frac{1 - F(v)}{f(v)}$$

is strictly increasing in  $v$ , as  $[F^{n-1}(v) + \dots + 1]/[nF^{n-1}(v)]$  is strictly decreasing in  $v$ , and  $[1 - F(v)]/f(v)$  is decreasing in  $v$  by Assumption 1. The probability the good is not sold is  $k(p^*) = F^n(p^*)$ .

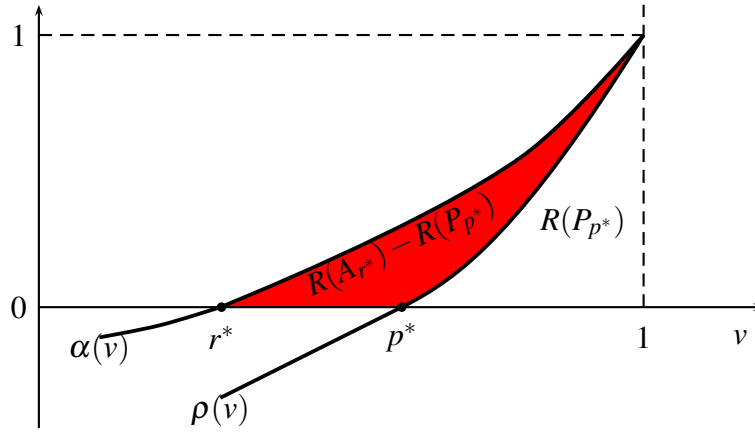


Figure 1: The marginal revenues and expected revenues in the static problem. An auction marginal revenue curve is  $\alpha(v) = v - [1 - F(v)]/f(v)$  and a posted price marginal revenue curve is  $\rho(v) = v - [1 - F^n(v)]/[F^n(v)]'$ . The optimal reserve price is  $r^* = \alpha^{-1}(0)$  and the optimal posted price is  $p^* = \rho^{-1}(0)$ . The optimal auction's expected revenue is  $R(A_{r^*}) = \int_{r^*}^1 \alpha(v) dF^n(v)$  and the optimal posted price's expected revenue is  $R(P_{p^*}) = \int_{p^*}^1 \rho(v) dF^n(v)$ . The difference between the optimal revenues can be written as  $R(A_{r^*}) - R(P_{p^*}) = \int_0^1 x d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]$ .

Since

$$\rho(v) = v - \frac{F^{n-1}(v) + \dots + 1}{nF^{n-1}(v)} \cdot \frac{1 - F(v)}{f(v)} = \alpha(v) + \left[1 - \frac{F^{n-1}(v) + \dots + 1}{nF^{n-1}(v)}\right] \frac{1 - F(v)}{f(v)},$$

$\rho(v)$  is always smaller than  $\alpha(v)$ . Since  $\alpha(r^*) = \rho(p^*) = 0$ , the optimal posted price is always higher than the optimal reserve price. Clearly from Figure 1, the optimal posted price's expected revenue is smaller. Since  $\alpha(r^*) = \rho(p^*) = 0$  and  $\alpha(1) = \rho(1) = 1$ , by changes of variables, the optimal revenue difference can be expressed as

$$R(A_{r^*}) - R(P_{p^*}) = \int_0^1 x d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]. \quad (1)$$

Although the optimal reserve price auction generates higher revenue, the optimal posted price can generate higher profit if there is a sufficiently high cost of running the auction. In general, running an auction is more appealing for the seller if her cost is lower than a cutoff cost  $c^*$  and posting a price is more appealing if her cost is higher than the cutoff cost. The cutoff cost equals the optimal revenue difference expressed in Equation (1). Proposition 1 characterizes the seller's optimal mechanism in the static setting.

**Proposition 1.** *Suppose  $T = 1$ . Let  $r^*$  and  $p^*$  be the unique solutions to  $\alpha(r^*) = \rho(p^*) = 0$  and  $c^* = R(A_{r^*}) - R(P_{p^*})$  in Equation (1). The seller's optimal mechanism is  $A_{r^*}$  if  $c < c^*$ ; is  $P_{p^*}$  if*

$c > c^*$ ; and is  $A_{r^*}$  and  $P_{p^*}$  if  $c = c^*$ .

## 4 A Two-Period Example

Let me use a simple two-period example to demonstrate the key insights of the seller's multi-period problem. Suppose that there are two buyers in each of the two periods. Their values are independently drawn from the uniform  $[0, 1]$  distribution ( $F(v) = v$  and  $f(v) = 1$ ). For simplicity, suppose that the seller does not discount ( $\delta = 1$ ). Her objective is to choose a selling mechanism  $m_1$  in the first period and a selling mechanism  $m_2$  in the second period to maximize her expected profit  $\pi(m_1, m_2) = \pi(m_1) + k(m_1)\pi(m_2)$ , where  $k(m_1)$  is the probability that the good is kept to the second period (i.e., not sold after mechanism  $m_1$  in the first period).

We characterize the seller's profit-maximizing mechanism sequence  $(m_1^*, m_2^*)$ . We can solve for the profit-maximizing mechanism choice problem by backward induction. We first solve for the optimal mechanism in the second period. The seller's problem in the last period is essentially a static problem. We thus directly apply the solution of the static problem described in the previous section. The optimal price to post in the second period is  $p_2^* = \sqrt{3}/3 \approx 0.577$  determined by  $\rho(p_2^*) = p_2^* - \frac{1-(p_2^*)^2}{2p_2^*} = 0$ , and the revenue is  $R(P_{\sqrt{3}/3}) = \pi(P_{\sqrt{3}/3}) = 2\sqrt{3}/9 \approx 0.385$ . The optimal reserve price is  $r_2^* = 1/2$  determined by  $\alpha(r_2^*) = 2r_2^* - 1 = 0$ , and the revenue is  $R(A_{r_2^*}) = \int_{r_2^*}^1 (2v - 1)dv = 5/12 \approx 0.417$ . By Equation (1), the optimal revenue difference is  $R(A_{r_2^*}) - R(P_{p_2^*})$ , which is

$$c_2^* \equiv \int_0^1 x d \left[ (\alpha^{-1}(x))^2 - (\rho^{-1}(x))^2 \right] = \frac{5}{12} - \frac{2\sqrt{3}}{9} \approx 0.032. \quad (2)$$

By Proposition 1, if  $c < c_2^*$ , then the optimal mechanism is  $A_{r_2^*} = A_{0.5}$  and  $P_{p^*} = P_{\sqrt{3}/3}$  if  $c > c_2^*$ .

Having solved the second period's optimal mechanism, we can solve for the first period's optimal mechanism. Let  $\pi_2^* = \pi(m_2^*) > 0$  denote the optimal second period profit. The total expected profit from posting price  $p_1$  in the first period and using  $m_2^*$  in the second period is the expected profit posting price  $p_1$  plus the expected profit using  $m_2^*$  in case the good is not sold. Since the probability that the good is not sold in the first period is  $p_1^2$ ,  $\pi(P_{p_1}, m_2^*) = R(P_{p_1}) + p_1^2 \pi_2^*$ , which can be written as

$$\pi(P_{p_1}, m_2^*) = \int_{p_1}^1 \rho(v) dv^2 + \pi_2^* - \int_{p_1}^1 \pi_2^* dv^2 = \int_{p_1}^1 [\rho(v) - \pi_2^*] dv^2 + \pi_2^*.$$

The optimal price is thus determined by  $\rho(p_1^*) = p_1^* - [1 - (p_1^*)^2]/2p_1^* = \pi_2^*$ ; the optimal posted price is set so that the posted price's marginal revenue equates the opportunity cost of selling the good. Solving for the optimal posted price,  $p_1^* = (\pi_2^* + \sqrt{\pi_2^{*2} + 3})/3$ . The total expected profit



from running an auction  $A_{r_1}$  in the first period is

$$\pi(A_{r_1}, m_2^*) = R(A_{r_1}) - c + r_1^2 \pi_2^* = \int_{r_1}^1 [\alpha(v) - \pi_2^*] dv^2 + \pi_2^* - c.$$

The optimal reserve price is determined by  $\alpha(r_1^*) = r_1^* - (1 - r_1^*) = \pi_2^*$ ; the optimal reserve price is set so that the auction's marginal revenue is equated to the opportunity cost of selling the good. Solving for the optimal reserve price,  $r_1^* = (\pi_2^* + 1)/2$ . Let  $c_1^*$  satisfy  $\pi(P_{p_1^*}, m_2^*) = \pi(A_{r_1^*}, m_2^*)$ ; a cost  $c_1^*$  seller is indifferent between  $P_{p_1^*}$  and  $A_{r_1^*}$  in the first period.

$$\begin{aligned} c_1^* &= \left[ \int_{r_1^*}^1 [\alpha(v) - \pi_2^*] dv^2 + \pi_2^* \right] - \left[ \int_{p_1^*}^1 [\rho(v) - \pi_2^*] dv^2 + \pi_2^* \right] \\ &= \int_{r_1^*}^1 [\alpha(v) - \pi_2^*] dv^2 - \int_{p_1^*}^1 [\rho(v) - \pi_2^*] dv^2. \end{aligned}$$

But remember that  $\alpha(r_1^*) = \rho(p_1^*) = \pi_2^*$ . With the same change of variable performed for Equation (1),

$$c_1^* = \int_{\pi_2^*}^1 x d \left[ (\alpha^{-1}(x))^2 - (\rho^{-1}(x))^2 \right] - \int_{\pi_2^*}^1 \pi_2^* d \left[ (\alpha^{-1}(x))^2 - (\rho^{-1}(x))^2 \right]. \quad (3)$$

$c_1^* \approx 0.0075$ . When  $c < c_1^*$ , auction  $A_{r_1^*}$  is used, and when  $c \geq c_1^*$ , posted price  $P_{p_1^*}$  is used.

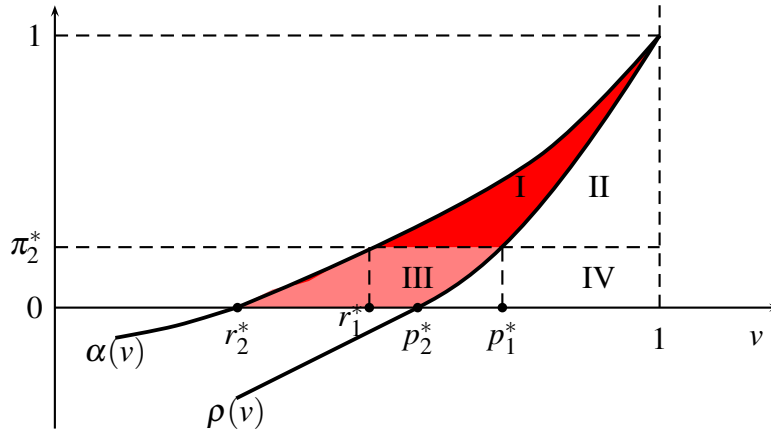


Figure 2: The marginal revenue curves and the expected revenues in the two-period setting. The revenue difference between the optimal auction and the optimal price is larger in the second period (the dark red region plus the light red region) than in the first period (the light red region).

Comparing Equations (2) and (3), we can easily see that  $c_1^* < c_2^*$ . Therefore, the optimal mechanism sequence can be characterized as follows. When  $c < c_1^*$ , the optimal mechanism sequence is an auction in the first period followed by another in the second period. If  $c_1^* \leq c < c_2^*$ , the optimal mechanism sequence is a price in the first period followed by an auction in the second period. If

$c \geq c_2^*$ , the optimal mechanism sequence is a price in the first period followed by a lower price in the second period. The optimal mechanism's optimal reserve price and optimal posted price are as illustrated in Figure 3. Although auction-auction, auction-price, price-price, and price-auction sequences are all feasible, an auction-price sequence is never optimal.

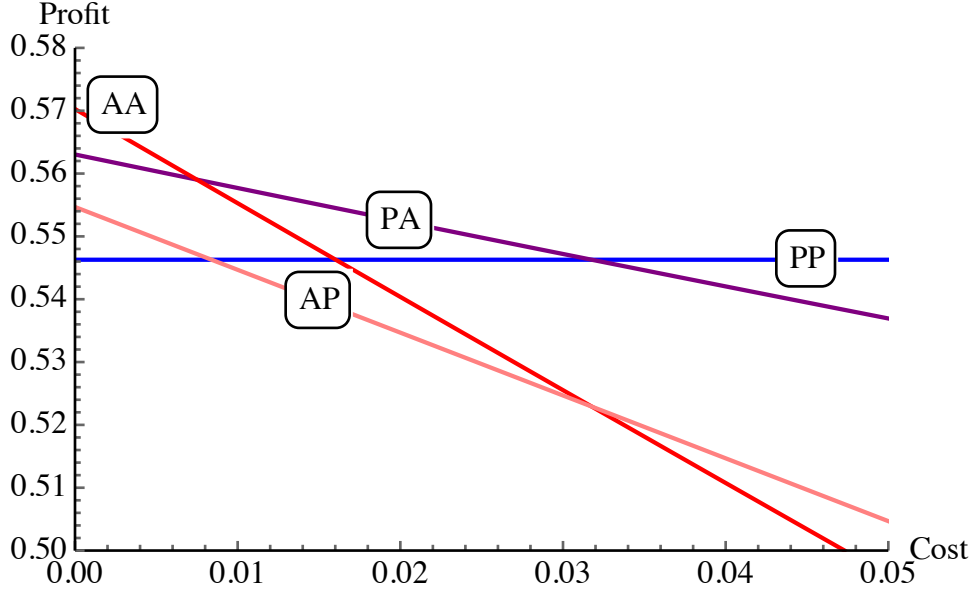


Figure 3: The optimal mechanism sequence's revenue as the auction cost varies. When the auction cost is smaller than 0.0075, the optimal mechanism sequence is auction-auction. When the auction cost is between 0.0075 and 0.032, the optimal mechanism sequence is price-auction. When the auction cost is bigger than 0.032, the optimal mechanism sequence is price-price.

The key to understand the result lies in the consideration of the opportunity cost, the retention value of the good in the two-period setting. Let's take a closer look at the cutoff costs  $c_1^*$  and  $c_2^*$  to understand the economics behind the result. The cutoff cost  $c_2^*$ , on one hand, is simply the revenue difference between the optimal static auction and the optimal static posted price; a seller chooses price when the cost exceeds the revenue advantage  $c_2^*$ . The cutoff cost  $c_1^*$ , on the other hand, consists of two terms. The first term,  $\int_{\pi_2^*}^1 x d[(\alpha^{-1}(x))^2 - (\rho^{-1}(x))^2]$ , is the difference in the first-period expected revenue between the optimal first-period auction and the optimal first-period posted price. In Figure 2, the optimal first-period auction's revenue is areas  $I + II + III + IV$ , and the optimal first-period posted price's revenue is areas  $II + IV$ . The revenue difference is thus areas  $I + III$ . The second term,  $\int_{\pi_2^*}^1 \pi_2^* d[(\alpha^{-1}(x))^2 - (\rho^{-1}(x))^2]$ , is the difference in the opportunity costs between the optimal first-period auction and the optimal first-period posted price. The opportunity cost of an optimal auction is the probability of no sale in the first period times the expected profit from retaining the good for another period, which generates the expected profit of  $F^n(r_1^*)\pi_2^*$ , areas  $III + IV$ . The opportunity cost of selling using optimal posted price is also the probability of no sale times expected profit generated in the second period, so is  $F^n(p_1^*)\pi_2^*$ , area

IV. Together, the opportunity cost difference is IV. On net, the total expected revenue difference between the optimal first-period auction and the optimal first-period price is only area I, the dark red region. It is smaller than the revenue difference between the second-period auction and the second-period price, the light red region plus the dark red region.

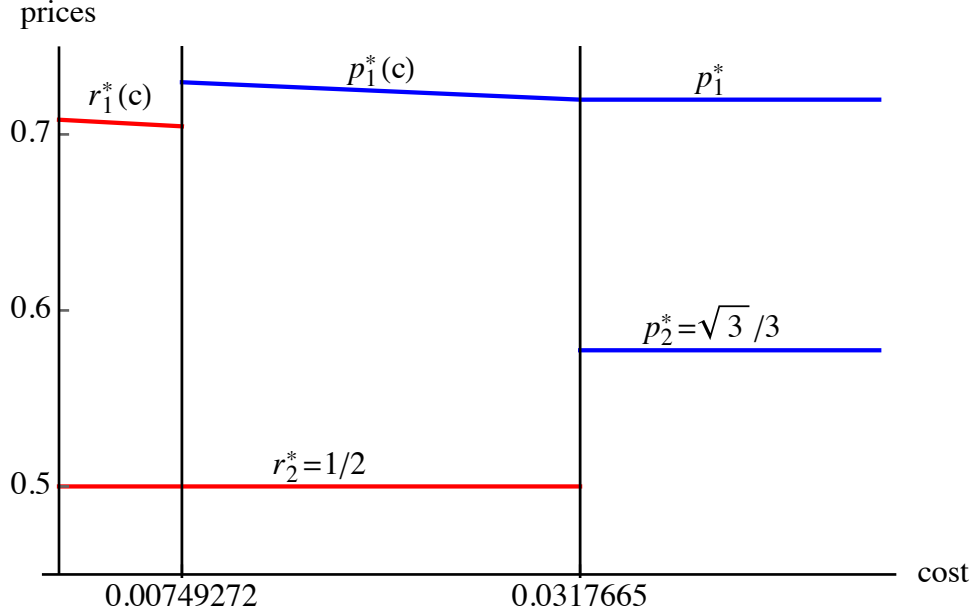


Figure 4: The optimal reserve price and posted price as the auction cost varies. When the auction cost is smaller than 0.0075, the optimal mechanism sequence is auction with reserve price  $r_1^*(c)$  and auction with reserve price  $1/2$ . When the auction cost is between 0.0075 and 0.032, the optimal mechanism sequence is price  $p_1^*(c)$  and auction with reserve price  $1/2$ . When the auction cost is bigger than 0.032, the optimal mechanism sequence is price  $p_1^*$  in the first period and price  $\sqrt{3}/3$  in the second period.

Although the auction's revenue is always higher, for an economically relevant very small auction cost, the optimal price dominates the optimal auction. In the current example, only when  $c < 0.007$ , in other words, the cost is smaller than 1.5% of the average value of the good, auctions are used in both periods. When the cost is between 0.007 and 0.03, between 1.5% and 6% of the average value of the good, a price followed by an auction is optimal.<sup>5</sup>

The optimal prices adjust downward over time. Mathematically, we can easily see from the optimal price determination: the optimal reserve prices are determined by  $r_2^* = \alpha^{-1}(0)$  and  $r_1^* = \alpha^{-1}(\pi_2^*)$  and the optimal posted prices are determined by  $p_2^* = \rho^{-1}(0)$  and  $p_1^* = \rho^{-1}(\pi_2^*)$ . Economically, the optimal prices equate the marginal revenue to the opportunity cost of selling the

<sup>5</sup>If the costs of the sellers are uniformly drawn between 0 and 0.1, we would see roughly that 7% of the sellers use sequential auctions, 25% of the sellers use buy-it-now options, and the majority of the sellers will simply adjust prices. These percentage roughly match the current distribution of selling mechanisms on eBay.

good. Since the opportunity cost of selling the good decreases from  $\pi_2^*$  in the first period to 0 in the second period, the optimal reserve and posted prices also decrease accordingly.

The rest of the paper is geared towards showing the robustness of the optimal mechanisms sequence in more general settings: for sufficiently low auction costs, a sequence of auctions with declining reserve prices is optimal; for sufficiently high auction costs, a sequence of declining prices is optimal; and for intermediate auction costs, a sequence of declining prices followed by a sequence of auctions with declining reserve prices is optimal. Any mechanism sequence of auctions followed by prices is never optimal. The optimal sequence of auctions for sufficiently low costs and the optimal sequence of prices for sufficiently high costs are not surprising as they arise almost trivially from the assumption of that auction costs more. However, for intermediate auctioning costs, it is not straightforward to arrive at the conclusion that such a nice sequence of mechanisms is the only possible optimal sequence, as any combination of prices and auctions in any order is feasible.

## 5 Finite-Horizon Problem

In this section, we solve the seller's profit maximization problem when  $T$  is finite and fixed. The seller's discounted sum of payoffs at period  $t < T$  is her expected profit in the current period plus her discounted payoff if the good is not sold in the current period,

$$\pi(m_t, m_{t+1}, \dots, m_T) = \pi(m_t) + k(m_t)\delta\pi(m_{t+1}, \dots, m_T).$$

By the Principle of Optimality, we can solve the problem backwards. We have already solved the period  $T$  problem, as it has the same solution as the one-period problem. We restate Proposition 1 in the  $T$ -period problem.

**Lemma 1.** *Suppose  $T$  is finite. Let  $r_T^* = \alpha^{-1}(0)$ ,  $p_T^* = \rho^{-1}(0)$ , and  $c_T^* = R(A_{r_T^*}) - R(P_{p_T^*})$ . The seller's optimal mechanism in period  $T$  is  $m_T^*(c) = A_{r_T^*}$  if  $c < c_T^*$ ,  $m_T^*(c) = P_{p_T^*}$  if  $c > c_T^*$ , and  $m_T^*(c) = A_{r_T^*} = P_{p_T^*}$  if  $c = c_T^*$ .*

The maximized expected profit is the larger of the expected profit of running the optimal auction and that of posting the optimal price,

$$\pi_T^*(c) = \max \left\{ R(A_{r_T^*}) - c, R(P_{p_T^*}) \right\}. \quad (4)$$

Given the optimal solution in period  $T$ , we can solve for the seller's problem in period  $T - 1$ .

Her problem in period  $T - 1$  is

$$\max_{m_{T-1}} \pi(m_{T-1}) + \delta k(m_{T-1}) \pi_T^*(c).$$

If she chooses an auction  $A_r$  in period  $T - 1$ , then her expected profit is

$$\begin{aligned} \pi(A_r, m_T^*(c)) &= R(A_r) - c + k(r) \delta \pi_T^*(c) \\ &= \int_r^1 \alpha(v) dF^n(v) - c + F^n(r) \delta \pi_T^*(c) \\ &= \delta \pi_T^*(c) + \int_r^1 [\alpha(v) - \delta \pi_T^*(c)] dF^n(v) - c. \end{aligned}$$

If she chooses a posted price  $P_p$  in period  $T - 1$ , then her expected profit is

$$\begin{aligned} \pi(P_p, m_T^*(c)) &= R(P_p) + k(p) \delta \pi_T^*(c) \\ &= \int_p^1 \rho(v) dF^n(v) + F^n(p) \delta \pi_T^*(c) \\ &= \delta \pi_T^*(c) + \int_p^1 [\rho(v) - \delta \pi_T^*(c)] dF^n(v). \end{aligned}$$

Her optimal reserve price is  $r_{T-1}^*(c) = \alpha^{-1}(\delta \pi_T^*(c))$  and her optimal posted price is  $p_{T-1}^*(c) = \rho^{-1}(\delta \pi_T^*(c))$ . The seller's optimal profit in period  $T - 1$  is

$$\begin{aligned} \pi_{T-1}^*(c) &= \max \left\{ R(A_{r_{T-1}^*(c)}) - c + k(r_{T-1}^*(c)) \delta \pi_T^*(c), R(P_{p_{T-1}^*(c)}) + k(p_{T-1}^*(c)) \delta \pi_T^*(c) \right\} \\ &= \delta \pi_T^*(c) \\ &\quad + \max \left\{ \int_{r_{T-1}^*(c)}^1 [\alpha(v) - \delta \pi_T^*(c)] dF^n(v) - c, \int_{p_{T-1}^*(c)}^1 [\rho(v) - \delta \pi_T^*(c)] dF^n(v) \right\}. \end{aligned}$$

There is a cutoff cost  $c_{T-1}^*$  such that the seller runs  $A_{r_{T-1}^*(c)}$  if her cost is lower than it and runs  $P_{p_{T-1}^*(c)}$  otherwise.  $c_{T-1}^*$  is determined by

$$\begin{aligned} c_{T-1}^* &= \int_{r_{T-1}^*(c_{T-1}^*)}^1 [\alpha(v) - \delta \pi_T^*(c_{T-1}^*)] dF^n(v) - \int_{p_{T-1}^*(c_{T-1}^*)}^1 [\rho(v) - \delta \pi_T^*(c)] dF^n(v) \\ &= \int_{\delta \pi_T^*(c_{T-1}^*)}^1 [x - \delta \pi_T^*(c_{T-1}^*)] d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]. \end{aligned}$$

The entire optimal mechanism sequence can be solved by iterating this procedure over all periods  $t \leq T - 1$ . The optimal mechanism sequence is summarized in Lemma 2.

**Lemma 2.** Suppose  $T$  is finite. For any  $t \leq T - 1$ , let  $\pi_{t+1}^*(c)$  denote the optimal expected profit of a cost  $c$  seller in period  $t + 1$ . Let  $r_t^*(c) = \alpha^{-1}(\delta\pi_{t+1}^*(c))$  and  $p_t^*(c) = \rho^{-1}(\delta\pi_{t+1}^*(c))$ , and let  $c_t^*$  be the unique solution to

$$c_t^* = \int_{\delta\pi_{t+1}^*(c_t^*)}^1 [x - \delta\pi_{t+1}^*(c_t^*)] d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))] . \quad (5)$$

A cost  $c$  seller's optimal mechanism  $m_t^*(c)$  in period  $t \leq T - 1$  is  $A_{r_t^*(c)}$  if  $c < c_t^*(c)$ , is  $P_{p_t^*(c)}$  if  $c > c_t^*(c)$ , and is  $A_{r_t^*(c)} = P_{p_t^*(c)}$  if  $c = c_t^*(c)$ .

**Proof of Lemma 2.** We need to show that Equation (5) rearranged as below has a unique solution  $c_t^*$  for each  $t$ ,

$$\gamma(c_t^*) \equiv c_t^* - \int_{\delta\pi_{t+1}^*(c_t^*)}^1 [x - \delta\pi_{t+1}^*(c_t^*)] d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))] = 0. \quad (6)$$

It suffices to show that  $\gamma(c)$  is continuous and increasing in  $c$ ,  $\gamma(0) < 0$  and  $\gamma(1) > 0$ .  $\gamma(c)$  is differentiable:

$$\begin{aligned} \gamma'(c) &= 1 + \delta\pi_{t+1}'(c) \int_{\delta\pi_{t+1}^*(c)}^1 d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))] \\ &= 1 - \delta\pi_{t+1}'(c) [F^n(\alpha^{-1}(\delta\pi_{t+1}^*(c))) - F^n(\rho^{-1}(\delta\pi_{t+1}^*(c)))] \\ &= 1 - \delta\pi_{t+1}'(c) [k(r_t^*) - k(p_t^*)] \end{aligned}$$

The maximal profit is

$$\pi_{t+1}^*(c) = \max\{R(A_{r_{t+1}^*}) - c + \delta k(r_{t+1}^*)\pi_{t+2}^*(c), R(P_{p_{t+1}^*}) + \delta k(p_{t+1}^*)\pi_{t+2}^*(c)\}$$

Therefore, for any  $t$ ,

$$\pi_{t+1}'(c) \geq -1 + \delta k(r_{t+1}^*)\pi_{t+2}'(c),$$

and

$$\pi_{t+2}'(c) \geq -1 + \delta k(r_{t+2}^*)\pi_{t+3}'(c),$$

and so on for  $\pi_{\tau}'(c)$  through  $T$ . Altogether, the inequalities, coupled with the inequalities that

$r_t^* > r_{t+1}^*$  for all  $t$ , imply that

$$\begin{aligned}\pi_{t+1}^{*'}(c) &\geq -[1 + \delta k(r_{t+1}^*) + \delta^2 k(r_{t+1}^*)k(r_{t+2}^*) + \dots] \\ &\geq -[1 + \delta k(r_{t+1}^*) + \delta^2 k^2(r_{t+1}^*) + \dots] \\ &= -\frac{1}{1 - \delta k(r_{t+1}^*)} \geq -\frac{1}{1 - \delta k(r_t^*)}\end{aligned}$$

Therefore,

$$\gamma'(c) \geq 1 - \frac{\delta k(p_t^*) - \delta k(r_t^*)}{1 - \delta k(r_t^*)} = \frac{1 - \delta k(p_t^*)}{1 - \delta k(r_t^*)} > 0.$$

Finally, since  $\int_{\delta\pi_{t+1}^*(c)}^1 [x - \delta\pi_{t+1}^*(c)] d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]$  is the revenue difference between the optimal auction and the optimal price in a period, it is between 0 and 1 for any  $c$ .  $\gamma(0) < 0$  and  $\gamma(1) > 0$  follow directly.  $\square$

The optimal profit in period  $T$  is determined by Equation (4), and the optimal profit in period  $t \leq T - 1$  is

$$\pi_t^*(c) = \max \{ R(A_{r_t^*(c)}) - c + k(r_t^*(c))\delta\pi_{t+1}^*(c), R(P_{p_t^*(c)}) + k(p_t^*(c))\delta\pi_{t+1}^*(c) \}. \quad (7)$$

The optimal mechanism sequence is thus completely characterized by the two lemmas and each period's optimal profit function.

**Proposition 2.** *Suppose  $T$  is finite. The optimal mechanism sequence  $(m_1^*(c), \dots, m_T^*(c))$  is characterized by Lemmas 1 and 2, with  $\pi_t^*(c)$  defined by Equations (4) and (7), and  $c_t^*$  determined by Equation (5).*

Although Proposition 2 completely characterizes the seller's problem for any cost  $c$  seller, the solution is not very informative. We only know that there is a cutoff cost  $c_t^*$  in each period  $t$  such that a seller chooses a reserve price auction if her cost is smaller than  $c_t^*$  and posts a price if her cost is bigger than  $c_t^*$ . In other words, all we know so far is that in each period if the auction cost is low, use an auction, and if the auction cost is high, post a price.

A little bit more work gives us a much neater result. We can show that the cutoff costs increase over the periods. In other words, it is more and more likely a seller will run an auction in a later period.

**Proposition 3.** *Suppose  $T$  is finite. The cutoff costs increase over time:  $c_1^* < c_2^* < \dots < c_T^*$ . That is, the optimal mechanism sequence is a sequence of auctions when the auction cost is smaller than  $c_1^*$ , is a sequence of prices then a sequence of auctions when the auction cost is between  $c_1^*$  and  $c_T^*$ , and is a sequence of prices when the auction cost is bigger than  $c_T^*$ .*

**Proof of Proposition 3.** Equation (5) determines the cutoff cost  $c_t^*$  in each period. In Equation (5), the cutoff cost  $c_t^*$  is decreasing in  $\delta\pi_{t+1}^*(c)$ . Since  $\pi_{t+1}^*(c) > \pi_t^*(c)$  for any  $t$ ,  $c_t^* < c_{t+1}^*$  for any  $t$ .  $\square$

It is more profitable paying the auction cost later than earlier. In the earlier periods, there are still many more periods left and many opportunities to sell the good. It is not worth paying an auction cost yet because the good has a retention value. However, as time passes on and the sale opportunities diminish, the auction cost, though constant in absolute terms, is deemed more attractive relative to the risk of not selling the good.

As a corollary of Proposition 3, it is impossible for any mechanism sequence that has an auction before a posted price to be optimal. In other words, all mechanism sequences consisting of an auction and a posted price following it are always dominated strictly by at least one alternative mechanism sequence - either dynamic pricing, sequential auctions, or posted prices followed by auctions.

**Proposition 4.** *Suppose  $T$  is finite. An auctions-then-prices sequence is never optimal when buyers are short-lived. Consequently, any mechanism sequence with an auctions-then-prices sequence is not optimal.*

We present a proof independent from the previous proof. It directly constructs the mechanisms that dominate the profit-maximizing auction-price sequence. The marginal revenue curve representations of the expected revenues and the optimal prices are useful for the proof. This proof will be adapted in Section 7 to show that the auction-price sequence is suboptimal even when buyers are long-lived.

**An Alternative Proof of Proposition 4.** Because the mechanisms chosen before period  $t$  do not affect the optimal mechanism sequence after period  $t$ , without loss of generality, it suffices to show that the mechanism sequence of an auction in the first period and then a posted price in the second period is never optimal. Suppose the seller runs the mechanism sequence  $\mathbf{m}$  in periods 3 through  $T$  and generates expected profit  $\pi(\mathbf{m})$ . It suffices to show that the optimal sequence  $(A_{r_1}, P_{p_2}, \mathbf{m})$  is dominated by at least one other mechanism sequence not consisting of the auction-price sequence.

Suppose that  $(A_{r_1}, P_{p_2}, \mathbf{m})$  is optimal and  $r_1 > p_2$ . The optimal  $r_1$  and  $p_2$  are determined respectively by  $\rho(p_2) = \delta\pi(\mathbf{m})$  and  $\alpha(r_1) = \delta\pi(P_{p_2}, \mathbf{m}) = \delta R(P_{p_2}) + \delta^2 k(p_2) \pi(\mathbf{m})$ .  $(A_{r_1}, P_{p_2}, \mathbf{m})$  dominates both  $(A_{r_1}, A_{p_2}, \mathbf{m})$  and  $(P_{r_1}, P_{p_2}, \mathbf{m})$ . That is,  $\pi(A_{r_1}, P_{p_2}, \mathbf{m}) \geq \pi(A_{r_1}, A_{p_2}, \mathbf{m})$  and  $\pi(A_{r_1}, P_{p_2}, \mathbf{m}) \geq \pi(P_{r_1}, P_{p_2}, \mathbf{m})$  where

$$\begin{aligned}\pi(A_{r_1}, P_{p_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1) R(P_{p_2}) + \delta^2 k(r_1) k(p_2) \pi(\mathbf{m}), \\ \pi(A_{r_1}, A_{p_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1) [R(A_{p_2}) - c] + \delta^2 k(r_1) k(p_2) \pi(\mathbf{m}), \\ \pi(P_{r_1}, P_{p_2}, \mathbf{m}) &= R(P_{r_1}) + \delta k(r_1) R(P_{p_2}) + \delta^2 k(r_1) k(p_2) \pi(\mathbf{m}).\end{aligned}$$



The two inequalities become

$$\begin{aligned} R(P_{p_2}) - [R(A_{p_2}) - c] &\geq 0, \\ [R(A_{r_1}) - c] - R(P_{r_1}) &\geq 0. \end{aligned}$$

Adding the two inequalities up,

$$R(A_{r_1}) - R(P_{r_1}) \geq c \geq R(P_{p_2}) - R(A_{p_2}).$$

Since  $R(A_r) - R(P_r) = \int_r^1 [\alpha(v) - \rho(v)] dF^n(v)$  is strictly decreasing in  $r$ , and since  $r_1 > p_2$ ,

$$R(A_{r_1}) - R(P_{r_1}) < R(P_{p_2}) - R(A_{p_2}),$$

a contradiction with the inequality above and the premise that  $(A_{r_1}, P_{p_2}, \mathbf{m})$  is optimal.

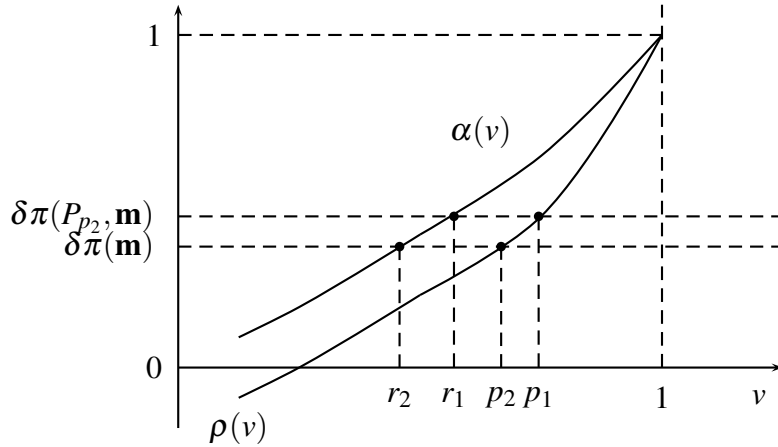


Figure 5:  $(A_{r_1}, P_{p_2}, \mathbf{m})$  when  $r_1 \leq p_2$  cannot simultaneously dominate  $(P_{p_2}, P_{p_2}, \mathbf{m})$  and  $(A_{r_1}, A_{r_2}, \mathbf{m})$ .

Suppose that  $(A_{r_1}, P_{p_2}, \mathbf{m})$  is optimal and  $r_1 \leq p_2$ . It should dominate all other alternative mechanism sequences, in particular both  $(P_{p_2}, P_{p_2}, \mathbf{m})$  and  $(A_{r_1}, A_{r_2}, \mathbf{m})$  where  $r_2 = \alpha^{-1}(\delta\pi(\mathbf{m}))$ . That is,  $\pi(A_{r_1}, P_{p_2}, \mathbf{m}) \geq \pi(P_{p_2}, P_{p_2}, \mathbf{m})$  and  $\pi(A_{r_1}, P_{p_2}, \mathbf{m}) \geq \pi(A_{r_1}, A_{r_2}, \mathbf{m})$ , where

$$\begin{aligned} \pi(A_{r_1}, P_{p_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1)R(P_{p_2}) + \delta^2 k(r_1)k(p_2)\pi(\mathbf{m}), \\ \pi(P_{p_2}, P_{p_2}, \mathbf{m}) &= R(P_{p_2}) + \delta k(p_2)R(P_{p_2}) + \delta^2 k(p_2)k(p_2)\pi(\mathbf{m}), \\ \pi(A_{r_1}, A_{r_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1)[R(A_{r_2}) - c] + \delta^2 k(p_1)k(r_2)\pi(\mathbf{m}). \end{aligned}$$

We can plug in and rewrite the two inequalities as

$$\begin{aligned} R(A_{r_1}) - c - R(P_{p_2}) - \delta [k(p_2) - k(r_1)] \pi(R_{p_2}, \mathbf{m}) &\geq 0, \\ R(P_{p_2}) - R(A_{r_2}) + c - \delta [k(p_2) - k(r_2)] \pi(\mathbf{m}) &\geq 0. \end{aligned}$$

Summing the two inequalities,

$$R(A_{r_1}) - R(A_{r_2}) - \delta [k(p_2) - k(r_1)] \pi(R_{p_2}, \mathbf{m}) - \delta [k(p_2) - k(r_2)] \pi(\mathbf{m}) \geq 0. \quad (8)$$

$r_1$  and  $r_2$  satisfy  $\alpha(r_1) = \delta \pi(P_{p_2}, \mathbf{m})$  and  $\alpha(r_2) = \delta \pi(\mathbf{m})$ . By the optimality of  $(A_{r_1}, P_{p_2}, \mathbf{m})$ ,  $\pi(P_{p_2}, \mathbf{m}) > \pi(\mathbf{m})$ . As illustrated by Figure 1,  $r_T^* \leq r_2 < r_1$ , so  $R(A_{r_1}) < R(A_{r_2})$ . Coupled with the inequality  $r_2 < r_1 \leq p_2$ ,

$$[R(A_{r_1}) - R(A_{r_2})] - \delta [k(p_2) - k(r_1)] \pi(R_{p_2}, \mathbf{m}) - \delta [k(p_2) - k(r_2)] \pi(\mathbf{m}) < 0,$$

contradicting inequality (8) above. □

## 6 Finite-Horizon Problem Extended

The previous section shows two main results: the optimality of the prices-then-auctions sequence in Proposition 3 and the sub-optimality of the auctions-then-prices sequence in Proposition 4. This section extends the benchmark setting and shows that both results continue to hold. We show the robustness of the result when the seller has stochastic sale deadline, increasing impatience, or declining auction cost, when the buyers arrive stochastically, have outside options, incur a bidding cost, when markets are separate for auctions and posted prices, when the mechanism designer is procuring a contract from potential contractors rather than selling a good to buyers, and when multiple goods are sold sequentially.

### 6.1 Stochastic Sale Deadline

Suppose the seller survives the market with probability  $\mu < 1$  to the next period. The stochastic deadline may result from the seller's uncertainty about when or whether her good will be out of favor, or when it may be perished or banned to be sold. It has a similar effect as time discounting. The expected payoff of the future periods becomes  $\mu \delta \pi_{T+1}$ ; by redefining  $\delta' = \mu \delta$ , the problem is the same as before.

## 6.2 Increasingly Impatient Seller

Even when the seller survives the market less likely over time ( $\mu_t$  decreases) or becomes increasingly impatient ( $\delta_t$  decreases), the main results are unchanged. Because the assertion that discounted payoff in a period is still greater than that in a later period,  $\mu_t \delta_t \pi_t^*(c) > \mu_{t+1} \delta_t \pi_{t+1}^*(c)$  for any  $c$ , the result that the optimal cutoff  $c_t^*$  increases over time is unaltered.

## 6.3 Decreasing Auction Costs

The result continues to hold even when the auction cost is not constant. Instead, it either declines over time, or declines whenever the seller uses an auction an additional time. To understand the result, remember the key trade-off between an auction and a posted price. The revenue advantage of an auction over a posted price increases in later periods, and the opportunity cost of selling a good by an auction over that by a posted price decreases in later periods. If the originally constant auction cost becomes smaller in later periods, the result that an auction is more profitable to be used in a later period than in an earlier period is reinforced.

## 6.4 Stochastic Buyer Arrival

If in each period, the buyer arrival process is stochastic instead of deterministic, the results do not change. Suppose that the probability that the number of buyers is  $n$  is  $q(n)$ , then the probability of selling is  $s(p) = 1 - \sum q(n) F^n(p)$ . The optimal posted price is then determined by

$$\rho(p_t^*) = p_t^* - \frac{1 - F^n(v)}{[F^n(v)]'} = p_t^* + \frac{s(p_t^*)}{s'(p_t^*)} = \delta \pi_{t+1}^*(c).$$

A posted price's expected revenue is  $R(P_p) = ps(p)$  and an auction's expected revenue is

$$R(A_r) = \int_r^1 \alpha(v) d \left[ \sum_n q(n) F^n(v) \right].$$

Since the determination of  $c_t^*$  is not changed from Equation (5), the results from the previous section on the form of mechanism sequence carry over.

## 6.5 Buyers with Outside Options

Suppose that buyers have other options to buy the good after the current period (e.g. [Satterthwaite and Shneyerov \(2007, 2008\)](#)). The buyers' willingnesses-to-pay in the current period that determine their bids in the auction and their purchasing decision in the posted price are depressed. The

seller solves the problem with respect to the buyers' willingness-to-pay instead of their values. As long as the willingness-to-pay  $w(v)$  is a concave transformation of the value  $v$ , the new marginal revenue curves become  $\tilde{\alpha}(v) = v - \frac{1-F(v)}{f(v)}w'(v)$  and  $\tilde{\rho}(v) = v - \frac{1-F^n(v)}{F^n(v)}w'(v)$ .  $\tilde{\alpha}(v)$  and  $\tilde{\rho}(v)$  are still increasing and the main results continue to hold.

## 6.6 Buyers with Bidding Costs

Suppose that in addition to the seller who incurs the auction cost  $c$ , each buyer also incurs a bidding cost  $c_b \geq 0$  when bidding. (Similar to the seller's auction cost, the buyer's cost is not necessarily an actual physical cost he pays, but could also be a mental cost or an opportunity cost of participating in the current auction.) The seller who runs an auction  $A_r$  is effectively setting a higher reserve price than the actual reserve price  $r$  because only the buyers with values above some  $\tilde{r} > r$  bid.  $\tilde{r}$  satisfies  $F^{n-1}(\tilde{r})(\tilde{r} - r) = c_b$ . The expected profit of  $A_r$  followed by a mechanism sequence which generates profit  $\pi$  is<sup>6</sup>

$$\int_{\tilde{r}}^1 \left[ \alpha(v) - \frac{c_b}{F^{n-1}(v)} - \delta\pi \right] dF^n(v) + \delta\pi.$$

The bidding cost  $c_b$  generates a new effective marginal revenue curve  $\tilde{\alpha}(v) = \alpha(v) - \frac{c_b}{F^{n-1}(v)}$ , lower than the original marginal revenue curve  $\alpha(v)$ . The bidding cost  $c_b$  lowers the seller's revenue by  $c_b/F^{n-1}(v)$ , even more than  $c_b$ . The optimal effective reserve price  $\tilde{r}$  is determined by  $\tilde{\alpha}(\tilde{r}) = \delta\pi$  (the actual reserve price is  $r = \tilde{r} - c_b/F^{n-1}(\tilde{r})$ ). Figure 6 depicts the effective marginal revenues and the optimal expected revenues. The reformulation of the problem with the effective marginal revenue curves easily shows that Propositions 3 and 4 continue to hold.

## 6.7 Separate Markets for Prices and Auctions

Suppose that the auction market and the posted price market are separate:  $n_A$  buyers will show up for an auction and  $n_P$  buyers will show up for a posted price. The revenue difference between

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<sup>6</sup>The profit is calculated as follows

$$\begin{aligned} & rn(1 - F(\tilde{r}))F^{n-1}(\tilde{r}) + \int_{\tilde{r}}^1 vdn(1 - F(v))F^{n-1}(v) + F^n(\tilde{r})\delta\pi \\ &= -(\tilde{r} - r)n(1 - F(\tilde{r}))F^{n-1}(\tilde{r}) + \int_{\tilde{r}}^1 \alpha(v)dF^n(v) + F^n(\tilde{r})\delta\pi \\ &= -(\tilde{r} - r)n(1 - F(\tilde{r}))F^{n-1}(\tilde{r}) + \int_{\tilde{r}}^1 [\alpha(v) - \delta\pi]dF^n(v) + \delta\pi \\ &= \int_{\tilde{r}}^1 [\alpha(v) - \delta\pi]dF^n(v) - c_b n(1 - F(\tilde{r})) + \delta\pi \end{aligned}$$

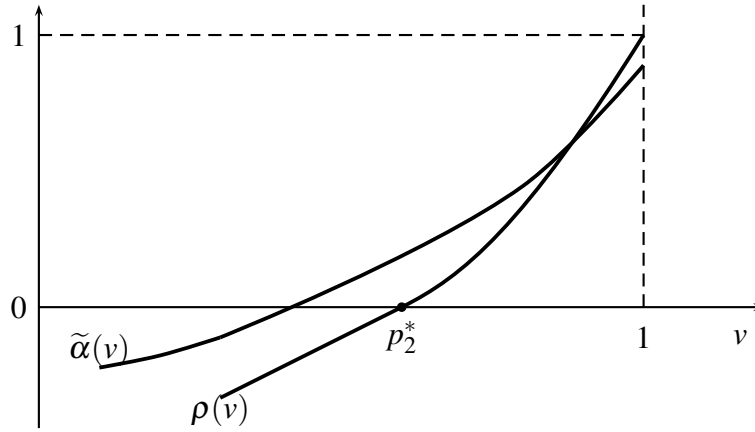


Figure 6: The effective marginal revenue curve when buyers pay a bidding cost  $c_b$ . The auction's marginal revenue curve shifts down to  $\tilde{\alpha}(v)$ .

running an optimal auction and an optimal price becomes

$$R(A_{r_t^*}, \mathbf{m}) - R(P_{p_t^*}, \mathbf{m}) = \int_{\delta\pi(\mathbf{m})}^1 [x - \delta\pi(\mathbf{m})] d [F^{n_A}(\alpha^{-1}(x)) - F^{n_P}(\rho^{-1}(x))] \equiv \Delta(\delta\pi(\mathbf{m})).$$

$\Delta(0) > 0$  and  $\Delta(1) = 0$ , and

$$\Delta'(\delta\pi(\mathbf{m})) = -\delta \int_{\pi}^1 d [F^{n_A}(\alpha^{-1}(x)) - F^{n_P}(\rho^{-1}(x))] = \delta [F^{n_A}(\alpha^{-1}(\pi)) - F^{n_P}(\rho^{-1}(\pi))].$$

The characterization of the optimal mechanism sequence remains.

In general, if the buyers arrive stochastically, let  $q_A(n)$  denote the probability that  $n$  buyers arrive when an auction is run, and  $q_P(n)$  the probability that  $n$  buyers arrive when a posted price is run. The difference between the optimal revenues is then

$$R(A_{r_t^*}, \mathbf{m}) - R(P_{p_t^*}, \mathbf{m}) = \int_{\delta\pi(\mathbf{m})}^1 [x - \delta\pi(\mathbf{m})] d \left[ \sum q_A(n) F^n(\alpha^{-1}(x)) - \sum q_P(n) F^n(\rho^{-1}(x)) \right].$$

We say that an auction faces a higher demand than a posted price if  $\sum_{i=1}^n q_A(i) \leq \sum_{i=1}^n q_P(i)$  for all  $n$ . Propositions 3 and 4 continue to hold.

## 6.8 Procurement Contracts

Throughout, we have considered the setting in which a monopolist is selling a good. We now turn our attention and consider the problem of a procurement contract in which the monopolist is looking for contracts to complete a project with the lowest cost. We show that the optimal mechanism

sequence takes the same form. McAfee and McMillan (1988) consider the procurement problem when the communication cost is high and similar to our argument, find that the principal may search sequentially for contractors if the communication is too costly. A principal who needs a cost of 1 to complete the task, finds a contractor to finish a task for a minimum compensation. Suppose  $n$  contractors whose true costs  $c$  for the project are independently and identically distributed according to  $G$  arrive each period; assume  $G$  has increasing hazard rate  $G/g$ . The principal's choice is the same as the seller in the previous sections. Her auction's expected profit with a requirement of a maximum bid  $r$  is

$$\pi(A_r) = \int_0^r \left[ 1 - c - \frac{G(c)}{g(c)} \right] dG^n(c) - c.$$

The probability of sale becomes  $s(r) = G^n(r)$ . A posted price  $P_p$  yields expected revenue  $(1 - p) G^n(p)$ . The optimal reserve price and the optimal posted price are determined by

$$r_t^* + G(r_t^*)/g(r_t^*) = p_t^* + G^n(p_t^*)/(G^n(p_t^*))' = \delta \pi_{t+1}^*(c). \quad (9)$$

In the static setting, the optimal auction yields more expected revenue than the optimal posted price. In a finite horizon, the counterpart of Equation (5) is

$$c_t^* = R(A_{r_t^*(c_t^*)}) - R(P_{p_t^*(c_t^*)}) - [k(p_t^*(c_t)) - k(r_t^*(c_t))] \delta \pi_{t+1}^*(c_t). \quad (10)$$

Therefore, the optimal sequence of mechanisms in the procurement problem depends on the auction cost in the same way as the seller's problem in the benchmark setting.

## 6.9 Multiple Goods

Suppose that a seller has multiple not necessarily identical goods (e.g. movie tickets of different seats) and must sell them sequentially by a certain deadline. The revenue from selling these goods is the sum of the individual revenues. Therefore, each good is sold by the optimal sequence of mechanisms as characterized. The revenue from the remaining good enters as an additional continuation payoff in the seller's sale decision. The existence of additional subsequently sold goods lowers the optimal reserve price or posted price of each good.

## 7 Two-Period Problem with Long-Lived Buyers

Thus far we have assumed that the buyers are short-lived. In this section, we consider the more complicated environment in which the buyers are long-lived. We consider both the case that buyers are long-lived and myopic and the case that buyers are long-lived and forward-looking. The results

continue to hold but the proofs are different from the previous sections. The proofs leverage the alternative proof of Proposition 4.

## 7.1 Long-Lived Myopic Buyers

Now we consider long-lived buyers who are myopic. They bid or buy as if they only live for one period. The numeric illustration with two buyers shows that the results continue to hold. The proposition below shows the general result when there are two periods and an arbitrary number of buyers.

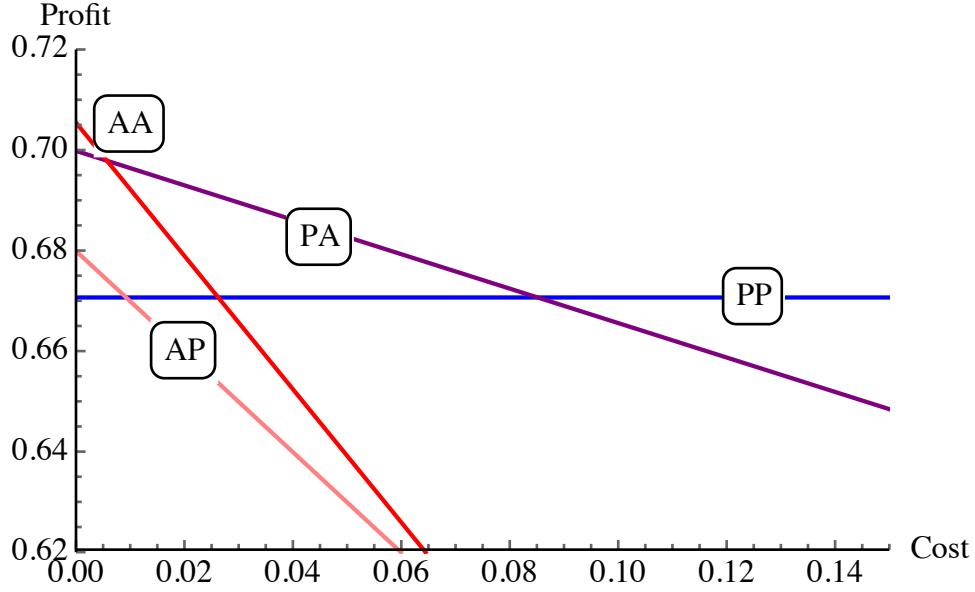


Figure 7: The optimal profits of different mechanism sequences for different auction costs when buyers are long-lived and myopic. When the auction cost is smaller than 0.007, the optimal mechanism sequence is auction-auction. When the auction cost is between 0.007 and 0.08, the optimal mechanism sequence is price-auction. When the auction cost is bigger than 0.08, the optimal mechanism sequence is price-price.

**Proposition 5.** *Suppose there are two periods and buyers are long-lived and myopic. An auction-price sequence is never optimal. The optimal sequence is an auction-auction sequence when the auction cost is sufficiently small, is a price-price sequence when the auction cost is sufficiently large; and is a price-auction sequence otherwise.*

**Proof of Proposition 5.** Suppose that  $(A_{r_1^*}, P_{p_2^*})$  is optimal.

If  $r_1^* < p_2^*$ , the first period buyers do not buy in the second period, so the mechanism sequence  $(A_{r_1^*}, P_{p_2^*})$  generates the same profit as in the setting where all the sellers are short-lived. As the proof of Proposition 4 has shown,  $(A_{r_1^*}, A_{p_2^*})$  cannot simultaneously dominate  $(A_{r_1^*}, A_{r_2^* = \alpha^{-1}(0)})$  and

$(P_{p_2}^*, P_{p_2}^*)$  in short-lived buyers setting.  $(A_{r_1}^*, A_{r_2}^*)$  generates higher payoff in the current long-lived myopic buyers setting than in the short-lived buyers setting since some buyers born in the first period buy in the second period.

If  $r_1^* \geq p_2^*$ , then the following contradiction shows that  $(A_{r_1}^*, P_{p_2}^*)$  cannot be optimal. The optimal mechanism sequence  $(A_{r_1}^*, P_{p_2}^*)$  must dominate  $(A_{r_1}^*, A_{p_2}^*)$ ,

$$\pi(A_{r_1}^*, P_{p_2}^*) - \pi(A_{r_1}^*, A_{p_2}^*) = -F^n(r_1^*)[R_2(P_{r_1}^*, A_{p_2}^*) - R_2(A_{r_1}^*, P_{p_2}^*) - c] \geq 0$$

where  $R_2(P_{r_1}^*, A_{p_2}^*)$  and  $R_2(A_{r_1}^*, P_{p_2}^*)$  denote the second-period revenue with the mechanism sequence  $(P_{r_1}^*, A_{p_2}^*)$  and  $(A_{r_1}^*, P_{p_2}^*)$ .  $(A_{r_1}^*, P_{p_2}^*)$  must also dominate  $(P_{r_1}^*, P_{p_2}^*)$ ,

$$\begin{aligned} & \pi(A_{r_1}^*, P_{p_2}^*) - \pi(P_{r_1}^*, P_{p_2}^*) \\ &= [R(A_{r_1}^*) - c + F^n(r_1^*)R_2(A_{r_1}^*, P_{p_2}^*)] - [R(P_{r_1}^*) + F^n(r_1^*)R_2(P_{r_1}^*, P_{p_2}^*)] \\ &= R(A_{r_1}^*) - R(P_{r_1}^*) - c \geq 0 \end{aligned}$$

Therefore, we must have

$$R(A_{r_1}^*) - R(P_{r_1}^*) \geq c \geq R_2(P_{r_1}^*, A_{p_2}^*) - R_2(P_{r_1}^*, P_{p_2}^*). \quad (11)$$

$R(A_{r_1}^*) - R(P_{r_1}^*) \leq R(A_{p_2}^*) - R(P_{p_2}^*) \leq R_2(P_{r_1}^*, A_{p_2}^*) - R_2(P_{r_1}^*, P_{p_2}^*)$ , a contradiction. The second inequality holds because of the extra buyers in the second period (for any  $x$ , the revenue difference between the reserve price  $x$  auction and the posted price  $x$  increases in the number of buyers).

In summary, the optimal auction-price sequence is shown to be dominated by at least one other mechanism. Therefore, such a sequence can never be optimal.

Let  $R_{AA}^*(c)$  denote the revenue of the optimal auction-auction sequence when the auction cost is  $c$ . Let  $R_{PA}^*(c)$  and  $R_{PP}^*(c)$  be similarly defined. We know that  $R_{AA}^*(0) > R_{PA}^*(0) > R_{PP}^*(0)$ . In addition,  $dR_{AA}^*(c)/dc < -1$ ,  $-1 < dR_{PA}^*(c)/dc < 0$ , and  $R_{PP}^*(c) = 0$ . The unambiguous ordering in the decreases in the slopes guarantees that there exist  $c_1^*$  and  $c_2^*$  such that for cost  $c \leq c_1^*$ ,  $R_{AA}^*(c) \geq R_{PA}^*(c), R_{PP}^*(c)$ ; for cost  $c_1^* < c \leq c_2^*$ ,  $R_{PA}^*(c) \geq R_{AA}^*(c), R_{PP}^*(c)$ ; and for  $c > c_2^*$ ,  $R_{PP}^*(c) \geq R_{AA}^*(c), R_{PA}^*(c)$ .  $\square$

## 7.2 Long-Lived Forward-Looking Buyers

Now we consider the case with forward-looking buyers. Assume that the buyers born in the first period continue to be around in the second period and have a chance to buy the good in the second period if the good has not been sold. Furthermore, the seller and the buyers discount by  $\delta < 1$  (if the seller does not discount, the optimal mechanism is trivial: wait until the last period to run



the static optimal auction or post the static optimal price). Because buyers are forward-looking, we can no longer solve the seller's problem backwards like we have done when buyers are short-lived. The mechanisms and prices chosen in the latter periods affect buyers' behavior in earlier periods and in turn the seller's profit in the earlier periods. The optimal prices are characterized by a system of equations. In the appendix, we solve one by one the optimal price-price, price-auction, auction-auction, and auction-price sequences when there are two buyers.

Despite of more complication, the main result remains: the optimal mechanism sequence is auction-auction when the auction cost is sufficiently low, price-auction when the auction cost is intermediate, and price-price when the auction cost is sufficiently high, and an auction-price sequence is never optimal. Figure 8 illustrates each mechanism sequence's optimal revenue for different auction costs in the setting of Section 4. Note that for only very small auction costs ( $c < 0.007$ ) the auction-auction sequence is optimal, and for only very large auction cost ( $c > 0.15$ ) the price-price sequence is optimal. For the most economically relevant range, the price-auction sequence is optimal.

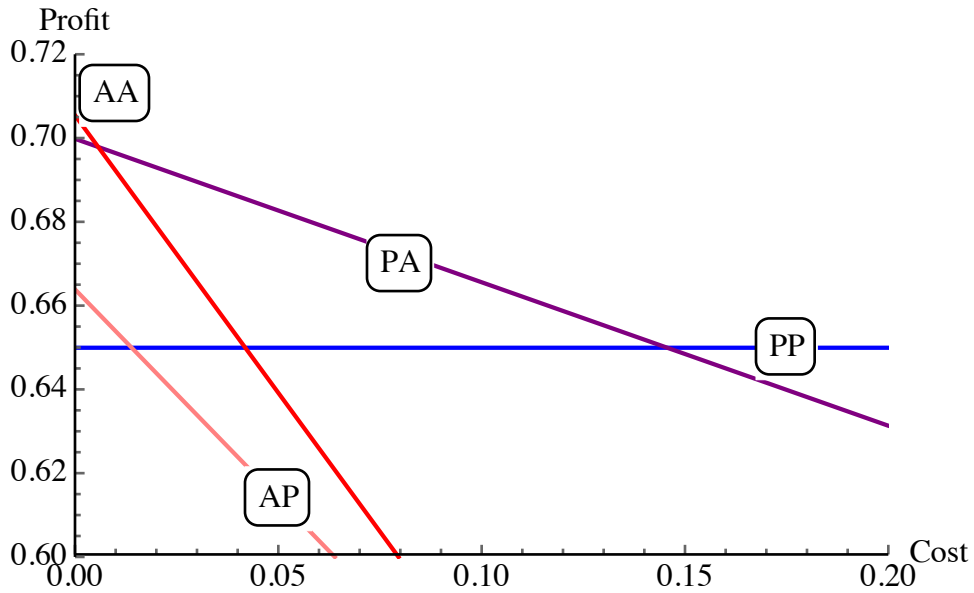


Figure 8: The optimal profits of different mechanism sequences for different auction costs when buyers are long-lived and forward-looking. When the auction cost is smaller than 0.007, the optimal mechanism sequence is auction-auction. When the auction cost is between 0.007 and 0.15, the optimal mechanism sequence is price-auction. When the auction cost is bigger than 0.15, the optimal mechanism sequence is price-price.

**Proposition 6.** *Suppose there are two periods and the buyers are long-lived and forward-looking. An auction-price sequence is never optimal. The optimal sequence is an auction-auction sequence when the auction cost is sufficiently small, is a price-price sequence when the auction cost is sufficiently large; and is a price-auction sequence otherwise.*

**Proof of Proposition 6.** Suppose that  $(A_{r_1^*}, P_{p_2^*})$  is optimal.

If  $r_1^* \leq p_2^*$ , the profit generated is the same as in the case when buyers are long-lived and myopic, and also the same as in the case when buyers are short-lived. The proof that  $(A_{r_1^*}, P_{p_2^*})$  cannot be optimal follows the proof of Proposition 5.

If  $r_1^* > p_2^*$ , let  $\tilde{v} \geq r_1^*$  denote the value of the buyer who is indifferent between buying and not buying in the first period in the mechanism sequence  $(A_{r_1^*}, P_{p_2^*})$ . Let  $(P_{p_1}, P_{p_2^*})$  be the price-price mechanism sequence where the buyer of value  $\tilde{v}$  is indifferent between buying and not buying. The optimality of  $(A_{r_1^*}, P_{p_2^*})$  implies that  $(A_{r_1^*}, P_{p_2^*})$  dominates  $(P_{p_1}, P_{p_2^*})$ , i.e.

$$\pi(A_{r_1^*}, P_{p_2^*}) - \pi(P_{p_1}, P_{p_2^*}) = R(A_{\tilde{v}}) - R(P_{\tilde{v}}) - c > 0.$$

Let  $(A_{r_1^*}, A_{r_2})$  be the auction-auction mechanism sequence in which the buyer of value  $\tilde{v}$  is indifferent between buying and not buying in the first period.

$$\pi(A_{r_1^*}, P_{p_2^*}) - \pi(A_{r_1^*}, A_{r_2}) = -F''(\tilde{v})[R_2(A_{r_1^*}, A_{r_2}) - R_2(A_{r_1^*}, P_{p_2^*}) - c] > 0$$

where  $R_2(A_{r_1^*}, A_{r_2})$  and  $R_2(A_{r_1^*}, P_{p_2^*})$  are the revenues. The two inequalities together imply

$$R(A_{\tilde{v}}) - R(P_{\tilde{v}}) > c > R_2(A_{r_1^*}, A_{r_2}) - R_2(A_{r_1^*}, P_{p_2^*}) > 0.$$

Since  $(A_{r_1^*}, P_{p_2^*})$  is optimal,  $R_2(A_{r_1^*}, P_{p_2^*})$  is bigger than the revenue  $\tilde{R}$  generated by  $P_{r_2}$  when there are  $n$  buyers with values smaller than  $\tilde{v}$  and  $n$  new born buyers with values drawn from distribution  $F$ .  $R_2(A_{r_1^*}, A_{r_2}) - R_2(A_{r_1^*}, P_{p_2^*}) > R_2(A_{r_1^*}, A_{r_2}) - \tilde{R}$ . Since  $\tilde{v} > r_2$ ,  $R_2(A_{r_1^*}, A_{r_2}) - \tilde{R} > R(A_{\tilde{v}}) - R(P_{\tilde{v}})$ , contradicting the inequality above. Therefore, an auction-price sequence cannot be optimal.

Let  $R_{AA}^*(c)$  denote the revenue of the optimal auction-auction sequence when the auction cost is  $c$ . Let  $R_{PA}^*(c)$  and  $R_{PP}^*(c)$  be similarly defined. We know that  $R_{AA}^*(0) > R_{PA}^*(0) > R_{PP}^*(0)$ . In addition,  $dR_{AA}^*(c)/dc < -1$ ,  $-1 < dR_{PA}^*(c)/dc < 0$ , and  $R_{PP}^*(c) = 0$ . The unambiguous ordering in the decreases in the slopes guarantees that there exist  $c_1^*$  and  $c_2^*$  such that for cost  $c \leq c_1^*$ ,  $R_{AA}^*(c) \geq R_{PA}^*(c), R_{PP}^*(c)$ ; for cost  $c_1^* < c \leq c_2^*$ ,  $R_{PA}^*(c) \geq R_{AA}^*(c), R_{PP}^*(c)$ ; and for  $c > c_2^*$ ,  $R_{PP}^*(c) \geq R_{AA}^*(c), R_{PA}^*(c)$ .  $\square$

## 8 Infinite-Horizon Problem

Unlike goods with expirations such as airline tickets and hotel rooms or seasonal clothes that might fall out of favor in three months, many goods such as cellphones, stamps and books do not lose value and do not have fixed deadlines to be sold, but nonetheless, the seller has incentive to sell the good and realize the payment as soon as possible. In this section, the optimal sequence

of mechanisms in the infinite-horizon is characterized. The solution turns out to be relatively simple and straightforward: the optimal mechanism sequence is an infinite repetition of the same mechanism.

Although it is an infinite-horizon problem, the seller faces the same problem in each period. Therefore, she only needs to solve a static problem in a stationary setting, and her optimal mechanism sequence is either an infinitely repeated sequence of auctions or an infinitely repeated sequence of posted prices.

**Proposition 7.** *Suppose  $T = \infty$ . Let  $p_\infty^*$  and  $\pi_\infty^P$  be the unique solution to*

$$\begin{aligned}\rho(p_\infty^*) &= \delta \pi_\infty^P \\ \pi_\infty^P &= \pi(P_{p_\infty^*}) + \delta k(p_\infty^*) \pi_\infty^P.\end{aligned}$$

*Let  $c_\infty^* = R(A_{\alpha^{-1}(\delta \pi_\infty^P)}) - [1 - \delta k(\alpha^{-1}(\delta \pi_\infty^P))] \pi_\infty^P$ . For any  $c \leq R(A_0)$ , let  $r_\infty^*(c)$  be the unique solution to*

$$\frac{1 - \delta}{1 - \delta k(r_\infty^*(c))} r_\infty^*(c) = \frac{1 - F(r_\infty^*(c))}{f(r_\infty^*(c))} + \frac{\delta [R(A_{r_\infty^*(c)}) - R(P_{r_\infty^*(c)}) - c]}{1 - \delta k(r_\infty^*(c))}.$$

*A cost  $c$  seller's optimal mechanism sequence is an infinitely repeated sequence of  $P_{p_\infty^*}$  if  $c > c_\infty^*$ , of  $A_{r_\infty^*(c)}$  if  $c < c_\infty^*$ , and of  $P_{p_\infty^*}$  or  $A_{r_\infty^*(c_\infty^*)}$  if  $c = c_\infty^*$ .*

We also have the following comparative statics result.

**Proposition 8.** *Suppose  $T = \infty$ . When the seller is more patient, the cutoff cost  $c_\infty^*$  decreases.*

The intuition of the result is as follows. Take the extreme case that the seller is infinitely patient ( $\delta = 1$ ). As there is an infinite number of periods, essentially there is an infinite stream of buyers. Then posting a price very close to 1 generates as much revenue as running the optimal auction, which also gives an expected revenue close to 1.

The result sheds light on the declining use of auctions on eBay. As eBay became a more popular platform, more and more sellers switched to simple prices from auctions. In January 2002, more than 90% of the active listings on eBay were auctions, but by late 2012, only 10% of the active listings were auctions, and the rest were simple posted prices and the hybrid buy-it-now formats (Einav et al., 2018). An increase in seller's patience can be thought of as an increase in the transaction speed or a reduction in the transaction period. The result implies that as the market becomes thicker, even without a change in sellers' cost composition, more sellers will choose to post prices. Simply an increase in the size of the market can alter the seller's choice of mechanism.

## 9 Conclusion

This paper studies a monopolist seller's expected profit maximizing sequence of costless posted prices and costly auctions when buyers arrive to the market over time. Most interestingly, we find that in a variety of settings, posting a price before auctioning is optimal. An optimal auction has a higher sale probability than an optimal posted price, so the use of an auction in an earlier period is associated with a higher probability of giving up the good. An auction incurs not only a constant fixed operational cost, but also an endogenously higher opportunity cost associated with the retention value of the good. A prices-then-auctions sequence is more desirable than an auctions-then-prices sequence because the endogenous opportunity cost of selling through an auction is decreasing over time. On eBay in particular, a buy-it-now option that allows a buyer to snatch a good for a fixed price before the seller starts an auction resembles the optimal prices-then-auctions mechanism sequence presented in this paper. Finally, a constant price can be optimal when the seller does not have a deadline to sell the good. When the market transaction speeds up, it is more likely to use posted prices, as we have observed on eBay.

Although we have explored the following extensions to different degrees, they are still interesting and worthwhile to be explored to fuller extent.

First, the seller may have multiple copies of the identical goods. We have explored the possibility that the seller sells the good sequentially, but it is worthwhile to consider other forms of auctions in which the seller can sell more than one good at a time. The dynamic programming problem carries one more state variable besides the seller's age - the number of remaining objects she has. Given the complexity of the problem with even one good, the multiple-goods extension should be treated as a separate pursuit complementary to the current one.

Another nontrivial and interesting extension is to allow the buyers to be long-lived and strategic in choosing their purchase times. It is important to explore how the result holds in general, when there are more than two periods. I suspect that the optimality of prices-auctions sequence and the sub-optimality of the auction-prices sequence remains. The main difficulty is that we cannot solve the problem period-by-period, and the optimal profit does not necessarily decrease over time because new buyers arrive and old buyers stay. [Deb and Pai \(2012\)](#) and [Pai and Vohra \(2013\)](#) study this problem in a costless environment and show that simple index rules and ironing can be optimal. [Said \(2012\)](#) and [Li \(2009\)](#) show that open ascending auctions are suitable for perishable and storable goods, respectively, when the buyers arrive over time and stay and strategically time their purchases. They have considered this problem from a mechanism design perspective, but none has considered from the practical perspective with a possible auctioning cost.

Finally, the seller has been assumed to commit to a particular sequence of mechanisms and the commitment as well as the mechanism are public knowledge. This assumption can be relaxed.

[Dilme and Li \(2014\)](#) consider the equilibrium non-committal price paths of the seller. [Skreta \(2006\)](#) shows that simple prices can be optimal when the seller does not have commitment power.

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# Appendix

## A Proofs of Propositions 7 and 8

**Proof of Proposition 7.** We will first calculate respectively the optimal constant price and the optimal constant reserve price and then compare to see which infinite mechanism sequence generates more expected profit. First, we solve the optimal posted price  $p_\infty^*$ . Let  $\pi_\infty^P$  denote the expected profit of posting  $p_\infty^*$  in each period. Together,  $p_\infty^*$  and  $\pi_\infty^P$  must simultaneously satisfy the following equations,

$$\rho(p_\infty^*) = \delta \pi_\infty^P, \quad (12)$$

$$\pi_\infty^P = \pi(P_{p_\infty^*}) + \delta k(p_\infty^*) \pi_\infty^P, \quad (13)$$

where Equation (12) equates the posted price marginal revenue to the expected profit, and Equation (13) equates the optimal expected profit to the profit of using  $P_{p_\infty^*}$  plus the expected profit if it is not sold. Rearranging Equation (13) and plugging it into Equation (12),  $p_\infty^*$  is uniquely determined by<sup>7</sup>

$$\frac{1 - \delta}{1 - \delta[1 - F^n(p_\infty^*)]} p_\infty^* - \frac{1 - F^n(p_\infty^*)}{[F^n(p_\infty^*)]'} = 0. \quad (14)$$

Similarly, we can calculate the optimal reserve price  $r_\infty^*(c)$  for a cost  $c$  seller. Let  $\pi_\infty^A(c)$  denote the continuation value of running  $A_{r_\infty^*(c)}$  each period. Together,  $r_\infty^*(c)$  and  $\pi_\infty^A(c)$  must simultaneously satisfy the following two equations,

$$\alpha(r_\infty^*(c)) = \delta \pi_\infty^A(c), \quad (15)$$

$$\pi_\infty^A(c) = \pi(A_{r_\infty^*(c)}) + \delta k(r_\infty^*(c)) \pi_\infty^A(c). \quad (16)$$

Substitute Equation (16) into Equation (15) and rearrange,

$$\frac{1 - \delta}{1 - \delta k(r_\infty^*(c))} r_\infty^*(c) = \frac{1 - F(r_\infty^*(c))}{f(r_\infty^*(c))} + \frac{\delta [R(A_{r_\infty^*(c)}) - R(P_{r_\infty^*(c)}) - c]}{1 - \delta k(r_\infty^*(c))}. \quad (17)$$

The LHS is increasing and the RHS is decreasing in  $r_\infty^*(c)$ ; the LHS is 0 and the RHS is  $1/f(0) + \delta[R(A_0) - c]/(1 - \delta)$  when  $r_\infty^*(c) = 0$  and the LHS is 1 and the RHS is  $-\delta c$  when  $r_\infty^*(c) = 1$ . Therefore, if  $c \leq R(A_0)$ ,  $r_\infty^*(c)$  is uniquely determined by the system of equations. If  $c > R(A_0)$ , the seller chooses a posted price for sure, because for any  $r$ ,  $A_r$  generates strictly less expected

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<sup>7</sup>The first term of Equation 14 is strictly increasing in and the second term is strictly decreasing in  $p_\infty^*$ , and when  $p_\infty^* = 0$ , the LHS is negative but when  $p_\infty^* = 1$  the LHS is positive.

profit than  $P_r$ , as

$$\pi(A_r) - \pi(P_r) \leq \pi(A_0) - \pi(P_0) = R(A_r) - c < 0.$$

Therefore, there is a cutoff cost  $c_\infty^*$  such that the seller runs  $A_{r_\infty^*(c)}$  if  $c > c_\infty^*$ , runs  $P_{p_\infty^*}$  if  $c < c_\infty^*$  and is indifferent between the two if  $c = c_\infty^*$ .  $c_\infty^*$  must satisfy  $\pi_\infty^A(c_\infty^*) = \pi_\infty^P$ . There is an easy way to calculate  $c_\infty^*$  than solving Equations (16) and (17).  $r_\infty^*(c_\infty^*)$  is determined by Equation (16),

$$\alpha(r_\infty^*(c_\infty^*)) = \delta \pi_\infty^P, \quad (18)$$

but  $c_\infty^*$  must satisfy that in any period, the seller is indifferent between posting the optimal price and running the optimal auction instead, so  $\pi_\infty^P = R(A_{r_\infty^*(c_\infty^*)}) - c_\infty^* + \delta k(r_\infty^*(c_\infty^*)) \pi_\infty^P$ , or

$$c_\infty^* = R(A_{\alpha^{-1}(\delta \pi_\infty^P)}) - [1 - \delta k(\alpha^{-1}(\delta \pi_\infty^P))] \pi_\infty^P. \quad (19)$$

□

**Proof of Proposition 8.** Let  $r_\infty^* \equiv r_\infty^*(c_\infty^*)$  throughout the proof. Differentiate  $c_\infty^*$  with respect to  $\delta$  in Equation (19),

$$\begin{aligned} \frac{dc_\infty^*}{d\delta} &= -\alpha(r_\infty^*)k'(r_\infty^*) - [1 - \delta k(r_\infty^*)] \frac{d\pi_\infty^P}{d\delta} - [-k(r_\infty^*) - \delta k'(r_\infty^*)] \pi_\infty^P \\ &= k(r_\infty^*) [-\alpha(r_\infty^*) + \delta \pi_\infty^P] + k(r_\infty^*) \pi_\infty^P - [1 - \delta k(r_\infty^*)] \frac{d\pi_\infty^P}{d\delta}. \end{aligned}$$

The first term  $k(r_\infty^*) [-\alpha(r_\infty^*) + \delta \pi_\infty^P] = 0$  by Equation (18), so

$$\frac{dc_\infty^*}{d\delta} = k(r_\infty^*) \left[ \pi_\infty^P + \delta \frac{d\pi_\infty^P}{d\delta} \right] - \frac{d\pi_\infty^P}{d\delta}. \quad (20)$$

By Equations (12) and (18), and the facts that  $\alpha(\cdot) \geq \rho(\cdot)$ , we have  $r_\infty^* \leq p_\infty^*$  and consequently  $k(r_\infty^*) \leq k(p_\infty^*)$ . Therefore,

$$\frac{dc_\infty^*}{d\delta} \leq k(p_\infty^*) \left[ \pi_\infty^P + \delta \frac{d\pi_\infty^P}{d\delta} \right] - \frac{d\pi_\infty^P}{d\delta}.$$

It suffices to show that the RHS is negative.

Rearrange Equation (13),

$$[1 - \delta k(p_\infty^*)] \pi_\infty^P = p_\infty^* [1 - k(p_\infty^*)],$$



and differentiate it with respect to  $\delta$ ,

$$\frac{d\pi_\infty^P}{d\delta} [1 - \delta k(p_\infty^*)] - \pi_\infty^P [k(p_\infty^*) + k'(p_\infty^*)] = -p_\infty^* k'(p_\infty^*) + [1 - k(p_\infty^*)] \frac{dp_\infty^*}{d\delta}.$$

Rearrange, we obtain

$$k(p_\infty^*) \left[ \pi_\infty^P + \delta \frac{d\pi_\infty^P}{d\delta} \right] - \frac{d\pi_\infty^P}{d\delta} = [\rho(p_\infty^*) - \pi_\infty^P] k'(p_\infty^*) \frac{dp_\infty^*}{d\delta}.$$

The RHS by Equation (12) equals  $(\delta - 1) \pi_\infty^P k'(p_\infty^*) \frac{dp_\infty^*}{d\delta}$ . Differentiate Equation (12) with respect to  $\delta$ ,

$$\pi_\infty^P + \delta \frac{d\pi_\infty^P}{d\delta} = \frac{\partial \rho(p_\infty^*)}{\partial p_\infty^*} \frac{dp_\infty^*}{d\delta} > 0.$$

$\rho' > 0$  implies  $dp_\infty^*/d\delta > 0$ , which implies  $(\delta - 1) \pi_\infty^P k'(p_\infty^*) \frac{dp_\infty^*}{d\delta} < 0$  as  $\delta < 1$ .  $\square$

## B Construction of Figure 8

Figure 8 in Section 7 shows the optimal revenues of different mechanism sequences when two long-lived, forward-looking buyers arrive in each of the two periods with values drawn from the uniform  $[0, 1]$  distribution, and the common discount factor is  $\delta = 0.8$ . This appendix shows the details in the determination of the revenues of the optimal mechanism sequences - price-price, price-auction, auction-auction, auction-price. We characterize the optimal price determinations and the revenues when there are two buyers in each period, with general value distributions and an arbitrary discount factor  $\delta < 1$ .

### B.1 Price-Price Sequence

Suppose the seller posts the price sequence  $(P_{p_1}, P_{p_2})$ . We can restrict our attention to the sequence with declining prices, because  $(P_{p_1}, P_{p_2})$  with  $p_1 \leq p_2$  is never optimal. An upward price path cannot capture the benefits of buyers living longer.

**Lemma 3.**  $(P_{p_1}, P_{p_2})$  with  $p_1 \leq p_2$  is never optimal.

*Proof.* When  $p_1 \leq p_2$ , buyers who are born with values above  $p_1$  in the first period do not wait until the second period to buy. They will buy in the first period. The expected revenue for the seller is then

$$\pi(P_{p_1}, P_{p_2}) = \int_{p_1}^1 \rho(v) dF^n(v) + \delta F^n(p_1) \int_{p_2}^1 \rho(v) dF^n(v).$$

The optimal prices are set by  $\rho(p_1^*) = R(P_{p_2^*})$  and  $\rho(p_2^*) = 0$ . But  $\rho(\cdot)$  is strictly increasing,  $\rho^{-1}(\cdot)$  is strictly decreasing,  $p_1^* = \rho^{-1}(R(P_{p_2^*})) > \rho^{-1}(0) = p_2^*$ , contradicting the assumption that  $p_1^* \leq p_2^*$ .  $\square$

Regardless of the price, there is a cutoff value  $\tilde{v} > p_1$  such that first-period buyers with values above  $\tilde{v}$  will buy in the first period, and buyers with value below  $\tilde{v}$  will wait to buy in the second period. Waiting lowers the probability the buyer gets the good but increases the buyer's utility when he gets the good because of the lower price in the second period. Some buyers with values just above  $p_1$  are willing to wait.

**Lemma 4.** *Suppose that the seller runs the mechanism sequence  $(P_{p_1}, P_{p_2})$  with  $p_1 > p_2$ . There is a cutoff value  $\tilde{v}$  such that any buyer with value above  $\tilde{v} > p_1$  will buy in the first period, and any buyer with value below  $\tilde{v}$  will wait to buy in the second period.*

*Proof.* Fix a value  $v$  buyer. Suppose that his opponent uses a strategy such that he will buy if his value falls into the (compact, possibly disconnected) set  $V \subseteq [p_1, 1]$ . Let  $F(V) \equiv \int_{v \in V} f(v)dv$  be the probability that the opponent buys. Let  $V_2 \equiv [p_2, 1] \setminus V$  be the set of the values  $v$  such that the value  $v$  opponent will buy in the second period instead of the first period. Then a value  $v > p_1$  buyer's utility of buying in the first period is

$$u_B(v|P_{p_1}, P_{p_2}) = (v - p_1)[1 - F(V) + \frac{1}{2}F(V)].$$

The utility  $u_{NB}(v|P_{p_1}, P_{p_2})$  of not buying in the first period and waiting for the second period is  $\delta \times$

$$\begin{aligned} (1 - F(V))(v - p_2) & \left[ \frac{F^2(p_2)}{1 - F(V)} F^2(p_2) + \frac{1}{2} \frac{F(p_2)}{1 - F(V)} 2F(p_2)(1 - F(p_2)) \right. \\ & + \frac{1}{2}(F(V) - F(p_2))F^2(p_2) + \frac{1}{3} \frac{F(p_2)}{1 - F(V)} (1 - F(p_2))^2 \\ & \left. + \frac{1}{3} \frac{F(V) - F(p_2)}{1 - F(V)} 2F(p_2)(1 - F(p_2)) + \frac{1}{4} \frac{F(V) - F(p_2)}{1 - F(V)} (1 - F^2(p_2)) \right]. \end{aligned}$$

Both utilities are linear in  $v$ , so the two lines  $u_B(v)$  and  $u_{NB}(v)$  intersect at most at one point. Furthermore,  $u'_B > 1 - F(V)$  and  $u_{NB} < 1 - F(V)$ . Since  $u_B(p_1) = 0 < u_{NB}(p_1)$ , the intersection is either in  $[1, \infty)$ , or  $(p_1, 1)$ . In the first case,  $u_{NB}(v) > u_B(v)$  for all  $v \in [p_1, 1]$ , so no one buys in the first period: we let  $\tilde{v} = 1$ . In the second case, the intersection  $\tilde{v} \in (p_1, 1)$ , so a buyer will buy in the first period if his value exceeds  $\tilde{v}$ .  $\square$

The two buyers will play symmetric strategies: he will buy if his value exceeds  $\tilde{v}$ . A value  $\tilde{v}$

buyer is indifferent between buying and not buying in the first period. The utility from buying is

$$u_B(v|P_{p_1}, P_{p_2}) = (\tilde{v} - p_1)[F(\tilde{v}) + \frac{1}{2}(1 - F(\tilde{v}))].$$

The utility  $u_{NB}(v|P_{p_1}, P_{p_2})$  of not buying in the first period is  $\delta \times$

$$\begin{aligned} & (\tilde{v} - p_2)[F^3(p_2) + \frac{1}{2}2F^2(p_2)(1 - F(p_2)) + \frac{1}{2}(F(\tilde{v}) - F(p_2))F^2(p_2) \\ & + \frac{1}{3}F(p_2)(1 - F(p_2))^2 + \frac{1}{3}(F(\tilde{v}) - F(p_2))2F(p_2)(1 - F(p_2)) \\ & + \frac{1}{4}(F(\tilde{v}) - F(p_2))(1 - F^2(p_2))]. \end{aligned}$$

$\tilde{v}$  satisfies  $u_B(\tilde{v}|P_{p_1}, P_{p_2}) = u_{NB}(\tilde{v}|P_{p_1}, P_{p_2})$ . When  $F(v) = v$ ,

$$\begin{aligned} (\tilde{v} - p_1)\frac{1 + \tilde{v}}{2} &= \delta(\tilde{v} - p_2)[p_2^3 + \frac{1}{2}2p_2^2(1 - p_2) + \frac{1}{2}(\tilde{v} - p_2)p_2^2 \\ &+ \frac{1}{3}p_2(1 - p_2)^2 + \frac{1}{3}2(\tilde{v} - p_2)p_2(1 - p_2) + \frac{1}{4}(\tilde{v} - p_2)(1 - p_2^2)] \end{aligned} \quad (21)$$

It is possible that the price in period two is so low that no one is willing to buy in the first period.

In such a case, let  $\tilde{v} = 1$ . The seller's expected profit is

$$\pi(P_{p_1}, P_{p_2}|p_1 > p_2) = [1 - F^2(\tilde{v})]p_1 + [F^2(\tilde{v}) - F^4(p_2)]p_2. \quad (22)$$

## B.2 Price-Auction Sequence

Suppose the seller's mechanism sequence is  $(P_{p_1}, A_{r_2})$  with  $p_1 > r_2$ . Similarly, we can characterize the buyers' strategies by cutoffs. The utility of buying in the first period is

$$(\tilde{v} - p_1) \left[ F(\tilde{v}) + \frac{1 - F(\tilde{v})}{2} \right].$$

The utility not buying in the first period is  $\delta \times$

$$\begin{aligned}
& F^3(\tilde{v}) \left[ \tilde{v} - \int_0^{\tilde{v}} \max\{r_2, v\} dF^3(v)/F^3(\tilde{v}) \right] \\
&= F^3(\tilde{v}) \left[ \tilde{v} - \int_{r_2}^{\tilde{v}} v dF^3(v)/F^3(\tilde{v}) - \int_0^{r_2} r_2 dF^3(v)/F^3(\tilde{v}) \right] \\
&= F^3(\tilde{v}) \left[ \tilde{v} - (\tilde{v} - \int_0^{r_2} r_2 dF^3(v)/F^3(\tilde{v}) - \int_{r_2}^{\tilde{v}} [F^3(v)/F^3(\tilde{v})] dv) - \int_0^{r_2} r_2 dF^3(v)/F^3(\tilde{v}) \right] \\
&= \int_{r_2}^{\tilde{v}} F^3(v) dv.
\end{aligned}$$

Therefore,  $\tilde{v}$  satisfies

$$(\tilde{v} - p_1) \frac{1 + F(\tilde{v})}{2} = \delta \int_{r_2}^{\tilde{v}} F^3(v) dv. \quad (23)$$

When  $F(v) = v$ ,  $(\tilde{v} - p_1)(\tilde{v} + 1)/2 = \int_{r_2}^{\tilde{v}} v^3 dv$ . The expected profit is

$$\begin{aligned}
\pi(P_{p_1}, A_{r_2} | p_1 > r_2) &= [1 - F^2(\tilde{v})] p_1 - \delta F^2(\tilde{v}) c - \delta F^4(r_2) r_2 \\
&+ \delta \int_0^{r_2} r_2 [6(1 - F(v)) + 6(F(\tilde{v}) - F(v))] f(v) F^2(v) dv \\
&+ \delta \int_{r_2}^{\tilde{v}} v [6(1 - F(v)) + 6(F(\tilde{v}) - F(v))] f(v) F^2(v) dv \\
&+ \delta \int_{\tilde{v}}^1 v 2(1 - F(v)) f(v) F^2(\tilde{v}) dv.
\end{aligned} \quad (24)$$

When the seller's mechanism sequence is  $(P_{p_1}, A_{r_2})$  with  $p_1 \leq r_2$ , the forward-looking buyers will not wait until next period and pay a higher price. The expected profit is

$$\pi(P_{p_1}, A_{r_2} | p_1 \leq r_2) = p_1 [1 - F^2(p_1)] + \delta F^2(p_1) \left[ \int_{r_1}^1 \alpha(v) dF^2(v) - c \right].$$

### B.3 Auction-Auction Sequence

Suppose the seller's mechanism sequence is  $(A_{r_1}, A_{r_2})$  with  $r_1 > r_2$ . The utility of buying in the first period is

$$F(\tilde{v})\tilde{v} - F(r_1)r_1 - \int_{r_1}^{\tilde{v}} v dF(v) = \int_{r_1}^{\tilde{v}} F(v) dv.$$

The utility of not buying in the first period is  $\delta \int_{r_2}^{\tilde{v}} F^3(v)dv$ .  $\tilde{v}$  satisfies

$$\int_{r_1}^{\tilde{v}} F(v)dv = \delta \int_{r_2}^{\tilde{v}} F^3(v)dv. \quad (25)$$

The expected profit  $\pi(A_{r_1}, A_{r_2} | r_1 > r_2)$  is

$$\begin{aligned} & 2[1 - F(\tilde{v})]F(\tilde{v})r_1 + \int_{\tilde{v}}^1 v2(1 - F(v))f(v)dv - c - \delta F^2(\tilde{v})c - \delta F^4(r_2)r_2 \\ & + \delta \int_0^{r_2} r_2[6(1 - F(v)) + 6(F(\tilde{v}) - F(v))]f(v)F^2(v)dv \\ & + \delta \int_{r_2}^{\tilde{v}} v[6(1 - F(v)) + 6(F(\tilde{v}) - F(v))]f(v)F^2(v)dv \\ & + \delta \int_{\tilde{v}}^1 v2(1 - F(v))f(v)F^2(\tilde{v})dv. \end{aligned} \quad (26)$$

If  $r_1 \leq r_2$ , then the first-period buyers will participate in the auction in the first period for its lower reserve price and reduced competition. The expected profit is

$$\pi(A_{r_1}, A_{r_2} | r_1 \leq r_2) = \int_{r_1}^1 \alpha(v)dF^2(v) - c + \delta F^4(r_1) \left[ \int_{r_2}^1 \alpha(v)dF^2(v) - c \right].$$

## B.4 Auction-Price Sequence

Suppose the seller's mechanism sequence is  $(A_{r_1}, P_{p_2})$  with  $r_1 > p_2$ . The utility from buying is  $\int_{r_1}^{\tilde{v}} F(v)dv$ . The utility not buying is

$$\begin{aligned} & \delta(\tilde{v} - p_2)[F^3(p_2) + F^2(p_2)(1 - F(p_2)) + \frac{1}{2}(F(\tilde{v}) - F(p_2))F^2(p_2) + \frac{1}{3}F(p_2)(1 - F(p_2))^2 \\ & + \frac{1}{3}(F(\tilde{v}) - F(p_2))2F(p_2)(1 - F(p_2)) + \frac{1}{4}(F(\tilde{v}) - F(p_2))(1 - F^2(p_2))]. \end{aligned}$$

When  $F(v) = v$ ,  $\tilde{v}$  satisfies

$$\begin{aligned} \int_{r_1}^{\tilde{v}} vdv &= \delta(\tilde{v} - p_2)[p_2^3 + \frac{1}{2}2p_2^2(1 - p_2) + \frac{1}{2}(\tilde{v} - p_2)p_2^2 \\ &+ \frac{1}{3}p_2(1 - p_2)^2 + \frac{1}{3}2(\tilde{v} - p_2)p_2(1 - p_2) + \frac{1}{4}(\tilde{v} - p_2)(1 - p_2^2)] \end{aligned} \quad (27)$$

The expected profit is

$$\pi(A_{r_1}, P_{p_2}) = 2[1 - F(\tilde{v})]F(\tilde{v})r_1 + \int_{\tilde{v}}^1 2f(v)(1 - F(v))dv - c + \delta[F^2(\tilde{v}) - F^4(p_2)]p_2. \quad (28)$$

When  $r_1 < p_2$ , all buyers buy in the first period and the expected profit is

$$\pi(A_{r_1}, P_{p_2}) = \int_{r_1}^1 \alpha(v) dF^2(v) - c + \delta F^2(r_1)[1 - F^2(p_2)]p_2.$$