Prices versus Auctions in Large Markets

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Abstract

This paper studies the use of posted prices versus auctions in large markets with many short-lived sellers and long-lived buyers. Although a reserve-price auction maximizes expected revenue, the optimal revenue decreases when the market becomes more buyer-friendly, namely, when the buyers survive longer, face fewer bidding competitors, and become more patient. In particular, as the market becomes more buyer-friendly, the revenue advantage from a reserve price auction over simply posting a price reduces; this helps explain the drastically declining use of auctions on eBay over the past decade. However, the increasing prevalence of posted prices results in market-wide externalities and inefficiencies; this helps explain the decline of eBay overall.

Keywords: optimal mechanism, reserve price auction, posted price, auction premium

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1 Introduction

In theory and practice, a standard auction with a well-chosen reserve price has shown to be desirable revenue-enhancing choice for unit-supply sellers facing buyers with independent private valuations. Although this paper confirms theoretically the revenue optimality of the reserve price auction in dynamic markets, the optimal revenue decreases when the market becomes more competitive for the sellers. More importantly, the revenue advantage relative to a simple posted price diminishes in such competitive environment.

We study steady states of the dynamic markets where infinitely many short-lived sellers and long-lived buyers who have possibly many chances to obtain a good are matched uniformly randomly. When the sellers have no fixed cost of running any mechanism, standard reserve price auctions are expected revenue maximizing with the strictly positive reserve prices indicating market competitiveness. The market becomes more consumer friendly when the buyers survive longer, face fewer competitors and become more patient, and the equilibrium reserve price and optimal auction revenue are lower in consumer friendly markets where the sellers have less market power. We generalize the monopolist's problem of Myerson (1981) by introducing a sequential market in which buyers may have opportunities to buy a perfect substitute of the good, and generalize the auction marginal revenue of Bulow and Roberts (1989) in this setting.

Although an auction with an optimally chosen reserve price generates the most expected revenue, we witness the prevalent and increasing uses of alternative mechanisms in many markets of goods with close substitutes because they have other operational advantages while achieving revenues sufficiently close to be optimal. For example used auto dealers use sequential bargaining followed by secret reserve price auctions (Larsen, 2012), treasury bill sales implement multi-unit auctions and price menus, and alternating bargaining are prevalent in housing markets. The effects of sequential competition offer a particularly plausible explanation to the rises of alternative mechanisms, in particular, the posted prices which rapidly take over eBay market (Einav et al., 2016).

Posted price mechanisms have the desirable properties of immediacy and simplicity compared with auctions which involve active organization of sellers and participation of buyers over a longer period of time. The gain in revenue from auction caused by buyer competition is depressed when the market becomes consumer friendly; more precisely, the difference between the expected revenue of the reserve price auction and that of simply posting the reserve price as the price, decreases as distribution of the buyers' potential price offers improves first-order stochastically in favor of the buyers. This result resonates that the choice between an auction and posted price depends on steepness of the marginal revenue curve in Wang (1993). When a buyer has multiple sequential opportunities to purchase a good, her willingness-to-pay to each seller is driven below her intrinsic value for the good, consequently affecting expected revenues of different sales mechanisms and possibly altering a seller's profit-maximizing choice if there is higher fixed cost associated with running an auction.

We also consider the dynamic markets with simultaneous existence of posted prices and auctions. When sellers have heterogeneous costs of running auctions, the sellers may switch to simpler mechanisms involving posting prices. The sellers who live for one period post prices, and those who can live longer use Buy-It-Now options and dynamically adjust prices, while auction followed by a posted price if it is not sold is never profit-maximizing. There are inefficiencies associated with posted prices, however, as it is less likely to be sold by seller, and it is not guaranteed that the buyer with the highest value receives the good. These inefficiencies result in market-wide externalities to future sellers and buyers.

The frictional matching process distinguishes the current work from the simultaneous auctions literature which studies competing sellers who simultaneously announce mechanisms to attract buyers to participate. In equilibrium, with mild technical conditions, each seller runs auction with reserve price lower than that in the monopoly case (Burguet and Sákovics, 1999; Pai, 2009), and as the number of sellers approaches infinity, the equilibrium mechanism is the efficient auction (McAfee, 1993; Peters and Severinov, 1997). Although the literature yields beautiful technical results, it is hardly applicable to real world: not all the agents can meet one another because of geographic or time constraints, and lower reserve price is a result of buyers' sequential opportunities rather than a tool to attract buyer participation.

The sale inefficiency that buyers who value the goods more than the sellers do not receive the goods is a result of the frictional matching process. Similar frictional matching models explain nontrivial prices in bargaining (Rubinstein and Wolinsky, 1985) and auctions (Wolinsky, 1988), as well as inevitable natural unemployment (Diamond and Maskin, 1979) despite of excess demand or supply. Another related paper, Satterthwaite and Shneyerov (2008), shows that the transaction price converges to the competitive Walrasian price.

The idea that sequential opportunities reduce buyers' willingnesses-to-pay is explored in Said (2011). It considers a dynamic setting where bidders randomly arrive and incorporate beliefs of current and future dynamics in their bids shading in efficient second price auctions. However, it takes a partial equilibrium approach by assuming that the sellers are restricted to run efficient auctions in addition to that the buyers have value independent outside options as they receive new value draws for each imperfect substitute, an assumption maintained in Wolinsky (1988) as well. This paper studies a general equilibrium when buyers have value dependent outside options so that in equilibrium the buyers with higher valuations have better opportunities and exit the market faster.

In summary, the main contributions of the paper are three-fold. First, it characterizes the revenue-maximizing mechanism when buyers have sequential outside options. Second, equilibrium of the dynamic settings where buyers have value dependent outside options, previously deemed infeasible to solve, is uniquely characterized with definitive comparative statics results which offer guidance to auction platform design. Finally, analogues are drawn between posted price mechanisms and reserve price auctions with their revenues directly compared, and it is probably the first paper to consider simultaneous existence of posted prices, auctions, and Buy-It-Now options to investigate their welfare effects in dynamic settings.

The paper is organized as follows. Section 2 introduces the basic dynamic setup and defines the steady state solution concept. Section 3 solves a generalized monopolist's problem when buyers have sequential outside options and examines the effects of changes in the outside options. Section 4 characterizes the stationary equilibrium and Section 5 presents comparative static results and their market implications. Section 6 introduces the posted price mechanism with its revenue compared to auction's and Section 7 discusses the welfare effects when posted prices are used. Section 8 concludes.

2 The Setup

In this section, we describe the dynamic setup and define the equilibrium. There are countably infinite periods: $t = 0, 1, \cdots$. In the beginning of each period, there are measure 1 continuum of unit-supply sellers and integer measure n continuum of unit-demand buyers in a homogeneous good market. We refer to n as the **buyer-seller ratio**. All the agents are risk-neutral, expected utility maximizers with quasilinear preferences, and have common discount factor δ for the next period. Each seller lives for one period and has normalized value zero for the good. Each buyer possibly lives forever and has persistent private value v independently and identically drawn from value distribution F with positive density f on support $[0, 1]$. Let H_t denote period t's active buyer value distribution composed of old and newborn buyers, except for that in period 0 when all the buyers are newborn ($H_0 = F$).

The market proceeds as follows. Buyers and sellers are randomly matched. Sellers each choose a sales mechanism, and buyers participate in the mechanisms. The number of buyers each seller is matched with is n.

Seller s chooses a direct anonymous mechanism (DAM) $M_{s,t}$ in which the buyers cannot be distinguished by their ages or identities but only by reported values. It consists of probability assignment and cost functions $\{P_i(\cdot), C_i(\cdot)\}_i$ which, for any vector of value reports

$\mathbf{z} = (z_1, \cdots, z_n)$ specifies each buyer's probability of winning and payment. Therefore, A buyer i's expected probability of winning and expected payment by reporting z_i are

$$\begin{aligned}\overline{P}_i(z_i) &= \int_{\mathbf{z}_{-i}} P_i(z_i, \mathbf{z}_{-i})\, dH_{-i}(\mathbf{z}_{-i}),\\ \overline{C}_i(z_i) &= \int_{\mathbf{z}_{-i}} C_i(z_i, \mathbf{z}_{-i})\, dH_{-i}(\mathbf{z}_{-i})\end{aligned}$$

where $H_{-i}(\mathbf{z}_{-i}) = \prod_{j=1}^{n} H_j(z_j)$.

A buyer i of value v, after knowing what mechanism $M_{s,t}$ she participates in, plays a feasible strategy $\sigma_{i,t}(v, M_{s,t}) \in [0,1]$. If buyer i obtains the good from the seller she is matched with, she is a **winner** ($i \in W_t$) and exits the market; otherwise, she is a **loser** ($i \in L_t$) and survives with probability s to period $t+1$, and we refer to s as the **survival rate**. Newborn buyers enter the market in the beginning of period $t+1$ to replace the winners and exiting losers to to keep a constant measure n of buyers.

Let $l_{i,t}(v|\sigma_{i,t}(v, M_{s,t}), \sigma_{-i,t}(\cdot, M_{s,t}), M_{s,t})$ denote value v buyer i's expected probability of losing when she plays $\sigma_{i,t}(v, M_{s,t})$ and her opponents play $\sigma_{-i,t}(\cdot, M_{s,t})$ in the mechanism $M_{s,t}$ she participates in, and $l_t\left(\{\sigma_i(\cdot,\cdot)\}_i, \{M_{s,t}\}_s\right)$ is the proportion of losers in period t given the agents' behaviors. We subsequently denote them $l_{i,t}(v)$ and l_t, but keep in mind that they depend on behavior of the active agents. Period $t+1$ active buyers are composed of newborn buyers and surviving losers from period t, so the expected active value distribution in period $t+1$ is

$$\mathbb{E}\left[H_{t+1}(v)\,|H_t\right] = (1 - sl_t)\,F(v) + sH_t(v|L_t) \tag{1}$$

where $H_t(v|L_t)$ is the value conditional distribution for losing buyers.

Before defining the equilibrium, we shall comment on some aspects of the model. Wolinsky (1988) employed similar frictional matching technology reflecting information and search frictions. The stochastic matching however adds no further insights but computational complexities, so we restrict our attention to the uniform matching with deterministic buyer-seller ratio. Furthermore, a seller can hold multiple goods in the same period, and as long as he does not own a positive measure of the goods in the market, his actions cannot alter the subsequent market compositions.

Two tunnels of learning arise in finite markets but not in infinite and anonymous ones. In a market with finite number of buyers, there is significant probability that a buyer's opponents may become her opponents again in subsequent periods, so when the transaction price is not announced in an auction, a bidder can infer from her bid about the possible transaction price and values of her opponents. In a market with finite number of items for sale (sellers), a buyer knowing whether the item from the mechanism he has participated in is sold or not helps her to learn about nontrivial portion of the market. Information asymmetry arises in sellers facing markets of asymmetric bidders knowing different amounts of information from history of actions and outcomes despite of symmetric newborn distributions. Milgrom and Weber (2000) studies the setting of n buyers, and one seller with $k \leq n$ goods and shows these learning effects make the expected price path a martingale. Said (2012) shows that a simultaneous ascending auction can alleviate these effects resulted from stochastic arrival of the agents.

Continuum of agents and restrictions to the direct anonymous mechanisms guarantee information symmetry among all the participating agents. Numerical results show that the learning effect is relatively small and vanishes rather quickly even in medium-sized markets[1]. Furthermore, Bodoh-Creed (2012) shows that the equilibrium with continuum of agents is approximated by that of finite markets. In this particular setting, the sellers have symmetric beliefs about buyers' values. Anonymous mechanisms further restrict sellers to discriminate by the buyers' ages that potentially reveal buyers' losing histories and thus values. Learning by bidding is also not plausible as the online market is so enormous that no buyer aims at beating any particular opponent.

The assumption that masses of agents stay constant is nonrestrictive as well. Suppose that the market size grows each period so that masses of sellers and buyers both increase by $g \geq 0$. This market growth effect is actually equivalent to decreasing the buyer's survival rate by $1/(1+g)$. Instead of sl_t, there is only $sl_t/(1+g)$ of surviving losers in period $t+1$. The active value distribution is then

$$\mathbb{E}\left[H_{t+1}(v)\,|H_t\right] = \left(1 - \frac{s}{1+g}l_t\right)F(v) + \frac{s}{1+g}H_t(v|L_t)\,.$$

When $s' = s/(1+g)$, the buyer value evolution coincides with (1). Although each buyer survives with probability s, the market growth rate essentially makes the survival rate $s' = s/(1+g)$.

For the remainder of the section, we define the equilibrium concept we want to solve, the stationary symmetric sequential equilibrium (SSSE) where buyers and sellers of the same value play the same utility maximizing strategies in each period when the expected active buyer value distribution remains stationary, and the belief of the buyers and sellers updates according to the equilibrium behavior.

Agents' strategies and beliefs are defined as follows. Since all the sellers are identical and

[1] The beneficial probability of learning from bidding and outcome is of order $1-(1-s/k)^{n-1}$ where k is the total number of sellers. It vanishes rather quickly in market of small size with relatively low survival rate s. For example, for $s=1/2$, $k=10$ and $n=5$, the probability of a buyer encountering a previous opponent in the next period is 20%, conditional on her own survival.

the buyers symmetric, we restrict our attention to symmetric behavior where all the sellers chooses the same mechanism M_t and each value v buyer behaves according to strategy $\sigma_t(v, M_t)$; that is, agents are only distinguished by their types but not by indices. In period t, each agent has belief $\mu_t(\{H_{t'}, M_{t'}, \sigma_{t'}\}_{t'=t}^{\infty})$ on the active buyer value distribution, the seller mechanism choice and the buyer strategies for the current and every subsequent period.

The payoffs of the agents are specified as follows. Each seller gets expected revenue $r(M_t|\mu_t)$ by choosing mechanism M_t under belief μ_t. Each value v buyer receives total discounted expected utility $u(v, \sigma_t(v, M_t), \sigma_{-i,t}(\cdot, M_t)|M_t, \mu_t)$ when her $n-1$ opponents play $\sigma_{-i,t}(\cdot, M_t)$.

Subsequent active buyer value distributions $\mathbb{E}[H_{t+1}(v)|H_t, M_t, \sigma_t]$ can be inferred from the buyer and seller behavior and pre-specified market and matching dynamics. If the active buyer value distribution is stationary, then the buyer composition will be expected to be the same across periods, so the stationary value distribution is $H_*(v) \equiv \mathbb{E}[H_t(v)|\mu_*] = \mathbb{E}[H_{t+1}(v)|\mu_*]$.

In particular, given the equilibrium mechanism M_* and strategy $\sigma_*(\cdot,\cdot)$, the stationary value PDF is the weighted average of newborn PDF f and the previous period PDF h_* in stationary equilibrium. The stationary value PDF and CDF are respectively

$$\begin{aligned} h_*(v) &= (1 - sl_*) f(v) + sl_*(v) h_*(v) && (2) \\ H_*(v) &= (1 - sl_*) F(v) + s\int_0^v l_*(z) dH_*(z) && (3) \end{aligned}$$

where $l_*(\cdot)$ and l_* depend on the equilibrium seller and buyer behaviors and matching dynamics.

Now we can define the equilibrium where 1) all the agents play symmetric stationary strategies, 2) the active value distribution is stationary, 3) agents' beliefs about the equilibrium behavior and active value distribution are correctly updated, and 4) every agent maximizes the expected payoff given beliefs.

Definition 1. $(M_*, \sigma_*, H_*, \mu_*)$ constitutes a **stationary symmetric sequential equilibrium (SSSE)** where every seller runs the same mechanism M_* and every value v buyer plays according to strategy $\sigma_*(v, M)$ in mechanism $M \in \mathcal{M}$ and it satisfies the following conditions.

1. **(Stationarity)** Stationary value distribution is a martingale with respect to the equilibrium mechanism M_* and strategy σ_*:

$$H_*(v) = \mathbb{E}[H_{t+1}(v)|H_t = H_*, M_*, \sigma_*],$$

where the stationary value PDF and CDF are determined by (2) and (3), respectively.

2. **(Consistency)** Every agent has the correct belief μ_* about the stationary value distribution, equilibrium mechanism and strategy:

$$\mu_* \left(H_*, M_*, \sigma_*\right) = 1.$$

3. **(Sequential Rationality of Seller)** M_* maximizes seller's expected profit with respect to belief μ_*: $\forall M \in \mathcal{M}$, $\pi \left(M_* | \mu_*\right) \geq \pi \left(M | \mu_*\right)$.

4. **(Sequential Rationality of Buyer)** $\sigma_* (\cdot, M)$ is a symmetric Bayes-Nash equilibrium in every mechanism M: $\forall v$, $\forall M \in \mathcal{M}$, $\forall \sigma (\cdot, M) \in \Sigma_M$,

$$u \left(v | M, \mu_*\right) \geq u \left(v, \tilde{\sigma} \left(v, M\right), \sigma_{-i,*} \left(\cdot, M\right) | M, \mu_*\right),$$

where $u \left(v | M, \mu_*\right) \equiv u \left(v, \sigma_* \left(v, M\right), \sigma_{-i,*} \left(\cdot, M\right) | M, \mu_*\right)$ denotes value v buyer's equilibrium expected payoff in M.

Note in particular that the symmetry assumption does not need to be assumed but arises as an equilibrium outcome. Since all the optimal incentive-compatible mechanisms yield the same revenues and the same expected buyer payoffs, all the sellers will choose the same mechanism in equilibrium and all the buyers will have the same expected equilibrium payoffs.

Characterizing the equilibrium requires 1) characterizing the stationary value distribution, 2) solving the monopolist profit-maximizing problem with value-dependent outside options, and 3) solving the buyers' equilibrium strategies when they have sequential opportunities. In essence, in the dynamic setting, the existence of sequential markets affects both seller mechanism choice and buyer behavior, and the sequential market conditions depend on buyer composition H_*.

In Section 3, we solve a general version of the monopolist's problem and specifies the buyer's equilibrium strategy in the optimal mechanism and the agents' expected equilibrium payoffs. The SSSE is characterized and proven to uniquely exist in Section 4 and comparative statics results on the survival rate, the buyer-seller ratio and the discount factor are presented in Section 5.

3 Seller's Problem Generalized

In this section, we study revenue maximization of a monopolist who faces symmetric buyers in the presence of a sequential market where the identical good is expected to be sold by other sellers. We show that the optimal mechanism is implementable by a standard reserve price auction, but the reserve price changes with respect to the sequential market condition. In particular, the reserve price is always lower than the monopoly reserve price in Myerson (1981), an extreme setting of the current setup which incorporates the monopolist problem in the SSSE as a special case. We generalize the marginal revenue curve defined by Bulow and Roberts (1989) to this sequentially competitive setting to determine optimal reserve prices. The general revenue maximizing mechanism with ex-ante *a*symmetric buyers and related results regarding incentive-compatibility, buyer indifference and revenue-equivalence of different mechanisms are derived and presented in Appendix A.

Suppose that there are n buyers and one seller which we call a monopolist. In addition to the assumptions defined above, the value distribution is assumed to have decreasing inverse hazard rate.

Assumption 1. $d\eta\left(F\left(v\right)\right)/dv \leq 0$ *where* $\eta\left(F\left(v\right)\right) \equiv \left(1-F\left(v\right)\right)/f\left(v\right)$.

In particular, the setting so far is the same as Myerson (1981), which we refer to as the **pure monopoly** setting. We introduce a sequential market where each buyer is expected to receive a price offer drawn from distribution Λ with support $[\underline{x}, \overline{x}]$ and density λ. Since Λ incorporates all the information about the subsequent period, we simply refer to the **sequential market** by the distribution Λ. Not to be interrupted by technical details, we assume that the buyers are symmetric and the sequential market distribution is continuous and differentiable.

The presence of a sequential market incorporates a wide range of settings. Myerson (1981) solves the monopolist's problem when there is no price offer (lower than 1) in the sequential market ($\Lambda\left(x\right) = 0 \forall x \leq 1$), or the buyers discount the sequential market infinitely ($\delta = 0$). In the SSSE, a buyer's continuation payoff is her total discounted expected utility from all mechanisms she participates in the future periods. Any outcome she faces can be characterized by the expected payment with the probability of such payment. The set of outcomes can be summarized by a probability distribution, thus a sequential market. In particular, the sequential opportunities each buyer faces in the stationary equilibrium are captured by an equilibrium sequential market. In addition, the sequential market can be viewed as signals buyers receive to update their valuations of the good. The dynamic mechanism design literature takes this particular stance and yield analogous results as this current section (Esö and Szentes (2007)).

Although all the buyers face the same sequential market, they may receive different realized price offers and may make different decisions based on their own valuations even if the realized price offers are the same. A buyer purchases the good if and only if her value exceeds the realized price offer she receives, so a value v buyer's expected payoff in the sequential market Λ is

$$\underline{u}_{\Lambda}(v) = \int_{\underline{x}}^{v} (v - x)\, d\Lambda(x) = \int_{\underline{x}}^{v} \Lambda(x)\, dx.$$

It increases whenever the buyers receive lower price offers with higher probability, so we say that Λ is more buyer-friendly than $\tilde{\Lambda}$ if Λ is first-order stochastically dominated by $\tilde{\Lambda}$.

Definition 2. A sequential market is **more buyer-friendly** than another if and only if it is first order stochastically dominated by the other: $\Lambda \succeq_{\mathrm{B}} \tilde{\Lambda}$ if and only if $\Lambda(x) \geq \tilde{\Lambda}(x)\, \forall x$.

A value v buyer will not pay more than the expected utility if she waits, so her willingness-to-pay (WTP) to the monopolist is

$$w_{\Lambda}(v) = v - \delta \underline{u}_{\Lambda}(v),$$

and it is differentiable, weakly increasing and weakly concave[2]. The monopolist's problem with a sequential market corresponds to a monopolist's problem with buyers' values transformed according to an increasing, concave function. Conversely, for any concave value transformation $\tilde{v}(\cdot)$, there is a sequential market Λ that induces a WTP function $w_{\Lambda}(\cdot) = \tilde{v}(\cdot)$.

In essence, the monopolist maximizes expected revenue with respect to n buyers who have WTP $w(v)$ drawn from the **WTP distribution** $\tilde{F}_{\Lambda}(\tilde{v}) = F\left(w_{\Lambda}^{-1}(\tilde{v})\right)$ and $\tilde{f}_{\Lambda}(\tilde{v}) = f\left(w_{\Lambda}^{-1}(\tilde{v})\right) / w'\left(w_{\Lambda}^{-1}(\tilde{v})\right)$, so by Myerson (1981), the optimal mechanism is to run a standard auction with reserve price determined by a transformed virtual utility curve. The virtual utility in this setting is defined as the **competitive auction marginal revenue (MR) in the presence of sequential market Λ**

$$\mathrm{MR}_{\Lambda}^{\mathrm{A}}(v) = w_{\Lambda}(v) - \eta\left(\tilde{F}_{\Lambda}(w_{\Lambda}(v))\right) = w_{\Lambda}(v) - \eta(F(v))\, w'_{\Lambda}(v). \tag{4}$$

In section 7, we will define the competitive posted price marginal revenue and compare those with the auction's when the sequential market changes to derive results regarding changes in revenues of different mechanisms and provide further economic insights, especially parallels between auctions and posted prices. Since WTP is increasing and concave, and

[2] $w'_{\Lambda}(v) = 1 - \delta \underline{u}'_{\Lambda}(v) = 1 - \delta\Lambda(v) \geq 0$ and $w''_{\Lambda}(v) = -\delta \underline{u}''_{\Lambda}(v) = -\delta\lambda(v) \leq 0$.

inverse hazard rate is decreasing, the competitive auction marginal revenue is continuously increasing, so the optimal reserve type exists and is unique.

Proposition 1 (The Optimal Mechanism)**.** *Let $A(\rho)$ represent a standard auction that implements reserve price $w_\Lambda(\rho)$ in the presence of sequential market Λ. The revenue-maximizing mechanism in the presence of the sequential market Λ is $A^*_\Lambda \equiv A(\rho^*_\Lambda)$ where $MR^A_\Lambda(\rho^*_\Lambda) = 0$.*

Not all the buyers who have values above the reserve price will participate in the auction, but the reserve price screens for participation of buyers with willingness-to-pay above it, who have values above ρ which we call the **the reserve type**. Without the sequential market, a buyer's value is her WTP, so the pure monopoly reserve price and reserve type coincide. When there is a nontrivial sequential market, the optimal competitive reserve price is lower than the pure monopoly reserve price. The importance of determining the reserve type is its relation with the probability of sale, $1 - F^n(\rho)$: the bigger the optimal reserve type, the lower the probability of sale and the higher the expected sale efficiency to transform the good from seller to buyers.

Proposition 2. *The optimal competitive reserve price in the presence of a sequential market is lower than the optimal pure monopoly reserve price.*

Remark 1. The optimal competitive reserve price is equal to the pure monopoly reserve price when there is no price offer (weakly) lower than the pure monopoly reserve price in the sequential market, because the competitive auction marginal revenue curve is unchanged for values below the pure monopoly reserve price.

The following numerical example illustrates that we can not compare optimal reserve type across different markets as a more buyer-friendly sequential market does not guarantee a lower optimal reserve price or reserve type.

Example 1. Suppose there are two buyers ($n = 2$) with values drawn uniformly from $[0, 1]$ ($F(x) = x$). In the first market, they may receive a uniform price offer from 0 to 1 ($\Lambda_1(x) = x$), and in the second market, they may receive a lower price offer with lower probability ($\Lambda_2(x) = 2x^2 \forall x \in [0, 0.5]\,; x \forall x \in [0.5, 1]$). Therefore, the first market is more buyer-friendly. However, the optimal reserve type and price are higher in the first market ($\rho^*_{\Lambda_1} \approx 0.4227 > \rho^*_{\Lambda_2} \approx 0.4221$ and $w_{\Lambda_1}\left(\rho^*_{\Lambda_1}\right) \approx 0.333 > w_{\Lambda_2}\left(\rho^*_{\Lambda_2}\right) \approx 0.303$).

Finally, we can calculate the buyers' equilibrium strategies and payoffs in the standard auctions in the presence of a sequential market. The first price and second price auctions (as well as Dutch and English, and all-pay auctions) with the same reserve prices generate the same revenues, from the generalized revenue equivalence and buyer indifference result

(Corollary 1), a corollary of the characterization of incentive compatible mechanisms (Proposition 11), and both are presented in Appendix A. In the second price auction, buyers bid their WTPs rather than their values (which are equivalent in the setting without sequential market). In the first price auction, buyers continue to shade their bids, but according to their WTPs as well.

Proposition 3 (Equilibrium in Standard Auctions)**.** *Consider a standard reserve price auction $A(\rho)$ with n buyers in the presence of Λ. A value $v \geq \rho$ buyer in the second price auction bids her willingness-to-pay $w_\Lambda(v)$. In the first price auction, the equilibrium bidding strategy of buyer $v \geq \rho$ is*

$$\sigma(v, A(\rho)) = w_\Lambda(v) - \int_\rho^v F^{n-1}(z)\, dw_\Lambda(z) / F^{n-1}(v)\,. \tag{5}$$

A value v buyer's total discounted payoff is

$$u(v|A(\rho)) = \int_\rho^v F^{n-1}(z)\, dw_\Lambda(z) + \delta \underline{u}_\Lambda(v)\,, \tag{6}$$

and the seller's expected revenue is

$$r(A(\rho)) = \int_\rho^1 MR_\Lambda^A(v)\, dF^n(v)\,. \tag{7}$$

These equilibrium characterizations, along with the characterization of the optimal auction, directly apply to the dynamic setting.

4 Stationary Symmetric Sequential Equilibrium

In this section, we characterize the stationary symmetric sequential equilibrium and prove its existence and uniqueness under mild technical conditions. Based on the results from the previous section, every seller runs the same reserve price auction. Higher value buyers are more likely to win the auctions and to exit the market, resulting in lower value buyers staying longer and a stationary value distribution that is first order stochastically dominated by the newborn value distribution. Such an equilibrium exists and is unique under mild conditions, and convergence of equilibrium from initial period is also discussed.

The equilibrium is characterized by the optimal reserve type and the stationary distribution. Since the buyers' sequential options in the SSSE can be summarized by a sequential market Λ_* which we will characterize, Proposition 1 shows the optimal mechanism is a standard auction with reserve type ρ_* determined by the equilibrium auction MR,

$$\mathrm{MR}_*^{\mathrm{A}}(\rho_*) \equiv w_*(\rho_*) - \frac{1 - H_*(\rho_*)}{h_*(\rho_*)} w_*'(\rho_*) = 0,$$

where $w_*(v)$ is value v buyer's equilibrium WTP. However, since the buyers' realized payments are all higher than ρ_*, by Remark 1, and the equilibrium reserve type is the same as the equilibrium reserve price, and is determined by

$$\rho_* - \frac{1 - H_*(\rho_*)}{h_*(\rho_*)} = 0. \tag{8}$$

The equilibrium reserve price is always positive, in contrast to the result obtained by McAfee (1993) that it converges to zero. As results from the next section demonstrate, the equilibrium reserve price varies with different parameters of the model, indicating level of market competitiveness. These competitive forces alleviate the frictions in matching by reducing the equilibrium reserve price to be close to the efficient level. The sellers are able to set nontrivial prices even if there is no excess demand which resonate the natural unemployment resulted from frictional search and matching models (Diamond and Maskin, 1979). When $n = 1$, this setting is a dynamic bargaining setting with equal measure of buyers and sellers, the model is similar to Rubinstein and Wolinsky (1985).

As a result a value v buyer wins the auction if and only if her value is above ρ_* and no opponent has value above v, so her probability of winning is $\overline{P}_*(v) = 1 - \mathbf{1}_{v>\rho_*} H_*^{n-1}(v)$. Conversely, a value v buyer's losing probability $l_*(v)$ is 1 if $v \leq \rho_*$ and $1 - H_*^{n-1}(v)$ if $v > \rho_*$. Equilibrium loser proportion is then

$$l_* = \int_0^1 l_*(v)\, dH_*(v) = 1 - \int_{\rho_*}^1 H_*^{n-1}(v)\, dH_*(v) = 1 - \frac{1}{n}\left(1 - H_*^n(\rho_*)\right).$$

In order words, measure $1 - H_*^n(\rho_*)$ of the sellers sell their goods.

The stationary value distributions can be determined by 2 and 3 by substituting for $l(v) = 1 - \overline{P}(v)$, a value v buyer's expected probability of losing in a direct anonymous mechanism. Value v buyer's expected utility of reporting z is $\overline{P}(z)\, v - \overline{C}(z)$ in the mechanism and is $\overline{P}_*(v)\, v - \overline{C}_*(v)$ if she does not win. The stationary value distributions are determined as follows.

$$h_*(v) = \left[1 - s\int_0^1 \left(1 - \overline{P}_*(z)\right) dH_*(z)\right] f(v) + s\left(1 - \overline{P}_*(v)\right) dh_*(v) \tag{9}$$

$$H_*(v) = \left[1 - s\int_0^1 \left(1 - \overline{P}_*(z)\right) dH_*(z)\right] F(v) + s\int_0^v \left(1 - \overline{P}_*(z)\right) dH_*(z) \tag{10}$$

Given the equilibrium losing probability and proportion of losers, the stationary value distributions characterized by (2) and (3) are

$$h_*(v) = \left[1 - s\left(1 - \frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right)\right] f(v) / \left(1 - s\left(1 - \mathbf{1}_{v>\rho_*} H_*^{n-1}(v)\right)\right), \quad (11)$$

$$H_*(v) = \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right] F(v) - \mathbf{1}_{v>\rho_*}\frac{s}{1-s}\frac{1}{n}\left(H_*^n(v) - H_*^n(\rho_*)\right). \quad (12)$$

In particular, the stationary distribution at the equilibrium reserve type is determined by

$$\frac{h_*(\rho_*)}{f(\rho_*)} = \frac{H_*(\rho_*)}{F(\rho_*)} = \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right]. \quad (13)$$

As a consequence of the auction's allocative efficiency that the buyer with the highest value above the reserve type is allocated the good, higher value buyers exit the market faster, resulting in a stationary value distribution that is first order stochastically dominated by the newborn distribution[3].

In summary, $(\mathrm{A}(\rho_*), \sigma_*, H_*, \mu_*)$ constitutes a SSSE. Each seller runs reserve price auction $\mathrm{A}(\rho_*)$ with ρ_* determined by the system of equations, (8) and (13) which also pin down $H_*(\rho_*)$. The stationary value distribution H_* is characterized by (8) and (12) given $H_*(\rho_*)$. Each buyer reports truthfully in each incentive-compatible direct anonymous mechanism (bids $w_*(v)$ in second price auction by Proposition (3)) and the equilibrium belief is the same across periods, $\mu_*(\mathrm{A}(\rho_*), \sigma_*, H_*) = 1$.

Such an equilibrium exists when losing buyers exit the market sufficiently fast. When enough higher value newborns enter the market to ensure the active value distribution with decreasing inverse hazard rate, the reserve price auction remain optimal. The exact sufficient assumptions are as follows.

Assumption 2. *The newborn value distribution is convex:* $F''(v) \geq 0$.

Assumption 3. *The survival rate is sufficiently small:* s *and* δ *satisfy* $\delta s^2 - 2s + 1 \geq 0$. *Equivalently,* $s \leq \left(1 - \sqrt{1-\delta}\right)/\delta$ *for all* δ.

These are reasonable assumptions. Although convex distribution assumption is more restrictive than that of monotone hazard rate, it includes some standard distributions such

[3]Rearrange (12),

$$\frac{H_*(v)}{F(v)} = \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right] / \left[1 + \mathbf{1}_{v>\rho_*}\frac{s}{1-s}\frac{1}{n}\frac{H_*^n(v) - H_*^n(\rho_*)}{H_*(v)}\right].$$

Since $\left[H_*^n(v) - H_*^n(\rho_*)\right]/H_*(v)$ achieves its maximum $1 - H_*^n(\rho_*)$ at $v = 1$, RHS is bigger than 1.

as uniform and exponential distributions. Although in this paper we do not have a buyer entry stage, the equilibrium distribution resulted from a reasonable entry model should be one where higher value buyers are more likely to enter. The assumption on the survival rate and discount factor is not restrictive. First of all, as long as $s \leq 1/2$, the equilibrium exists regardless of δ. And for higher discount factor, the range of survival rate that supports a unique equilibrium becomes larger. For example, for $\delta = 0.95$, $s \leq 0.876$ satisfies the condition. Furthermore, note that these assumptions are by means necessary conditions for existence or uniqueness but rather loose sufficient conditions. Finding the necessary conditions for equilibrium existence is not key to the arguments of the paper. Furthermore, they are conditions that not only guarantee equilibrium existence but also uniqueness.

Proposition 4. *When Assumptions 2 and 3 hold, there exists a unique SSSE.*

This equilibrium is reached from periods of plays where the reserve price monotonically decreases. Starting with the initial period in which all the buyers are newborns, the sellers post auctions with a reserve price higher than the equilibrium one, because buyers value distribution FOSDs the stationary distribution and the buyers face worse sequential market than the buyers in the steady state. However, as the time progresses, lower value buyers congest the market and expect more buyer-friendly environment in the future periods, so the sellers post lower reserve price in response. The sequence of reserve price monotonically converges to the equilibrium reserve price. The monotonicity of adjusted inverse hazard rate is guaranteed because the sequential market improves and the active buyer distribution decreases as higher value buyers win more often and exit faster.

The equilibrium sequential market is

$$\Lambda_* (x) = \mathbf{1}_{v>\rho_*} \frac{H_*^{n-1}(x)}{1 - \delta s \left(1 - H_*^{n-1}(x)\right)} \quad \forall x > \rho_*. \tag{14}$$

It indicates market competitiveness or buyer-friendliness of the market as we have mentioned. In the next section, we partially use the change in the equilibrium sequential market to explore effects of the changes in different parameters of the model - the survival rate, the buyer-seller ratio, and the discount factor, on the equilibrium reserve price, stationary value distribution, buyer utility, seller revenue, and sale efficiency.

5 Comparative Statics

In this section, we investigate how the equilibrium is affected by changes in the market environment manifested in three parameters of the model - the buyer survival rate, the buyer-seller ratio, and the discount factor. Although there are only three parameters (apart from the newborn value distribution) in the model, each of them represents different characteristics of the market which we shall elaborate on.

We consider the change in the equilibrium from three perspectives: the sellers', the buyers', and the social planner's. It should not be a surprise that the sellers and the buyers always exhibit conflict of interests, but it is not directly obvious that the social planner's welfare aligns with the sellers'. In general, the changes that benefit the buyers and harm the sellers include prolonging buyer expected age, decreasing competition, and making them more patient. However, from the social efficiency point of view, benefiting the buyer worsens the allocative efficiency as the probability of sale decreases.

A seller's expected revenue from her optimally chosen auction $\mathrm{A}\left(\rho_*\right)$ is

$$r_* = \int_{\rho_*}^{1} \left[w_*\left(v\right) - \eta\left(H_*\left(v\right)\right) w_*'\left(v\right)\right] dH_*^n\left(v\right). \tag{15}$$

We need to characterize the changes in the optimal auction, the stationary buyer value distribution, and the buyer's WTP. In addition, a value v buyer's total discounted expected utility is

$$\underline{u}_*\left(v\right) = \mathbf{1}_{v \geq \rho_*} \int_{\rho_*}^{v} \frac{H_*^{n-1}\left(z\right)}{1 - s\delta\left(1 - H_*^{n-1}\left(z\right)\right)} dz. \tag{16}$$

There are different types of efficiencies associated with a sale mechanism. When the good is transferred from the seller who values the good at zero to any of the buyers, we say the transaction is **sale efficient**. When the sale occurs, and it is transferred to the buyer with the highest value, then we say the sale is **allocative efficient**. A zero reserve price second price auction is both sale efficient and allocative efficient as the good always ends up in the hand of the buyer with the highest valuation. A positive reserve price auction is not always sale efficient but is allocative efficient as the buyers' competition results in the allocative efficiency. However, an optimal posted price is neither sale nor allocative efficient. The significant posted price results in more ex-post sale inefficient allocation than the optimal auction because the optimal reserve price is lower than the optimal posted price, and it also results in allocative inefficiency as the good does not necessarily end in the hand of the buyer with the highest value.

The total social welfare is the discounted sum of the buyers and the sellers across all periods, but with a little careful consideration reveals that the equilibrium probability of sale is the key indicator of social efficiency. Since quasi-efficient auctions, the mechanisms that assign the good to the highest valued agent if not withheld by the seller, are run, and buyers and sellers divide the profit, transaction occurring is always preferred to otherwise as the sellers holding onto the good being the most socially inefficient allocation. The social efficiency monotonically increases in sale probability.

The platform designer's profit closely ties in with sale and allocative efficiency. If his profit is proportional to the total surplus he generates, given that a mechanism is allocative efficient as it is the case in an auction, higher sale efficiency results in higher total welfare. The idea that platform designer's maximum profit is proportional to the total welfare is supported by Oi (1971) and Armstrong (1999). If instead the designer charges a participation fee for new agents who arrive in the market (shown to be a revenue maximizing fee structure in Bodoh-Creed (2012)), he wishes that more new agents arrive, which is realized when equilibrium probability of sale increases.

Therefore, in order to consider the changes in the three perspectives, we solve the comparative statics of these variables - the seller's chosen equilibrium reserve price, the seller's utility; the stationary buyer value distribution, the buyer's utility; and the probability of sale. We summarize these results in the next three Propositions. All of the changes have definite signs and monotonic effects over the range as long as an additional convexity assumption on the newborn value distribution is imposed in addition to the existing one. It further restricts Assumption 2 of convex value distribution that guarantees the existence of equilibria, but not by much: the uniform distribution and any distribution with CDF $F(v) = v^k, k \geq 1$ still satisfies the assumption.

Assumption 4. *The newborn value distribution satisfies that* $vf(v)/F(v)$ *is increasing for* $v < \rho^*_{mon}$.

First, let us look at the change in the survival rate. These variations in the survival rate may reflect elasticities of demand across different types of goods: people are more likely to keep searching for a cellphone until they get one, but people may quit searching for a novel if they cannot find it at a satisfactory price. Furthermore, it can also represent search friction in the market; higher survival rate means that the current buyers can easily find a perfect substitute in the existing market. Lastly, recall that the market size growth rate is inversely related to the survival rate, an increase in survival rate also corresponds to a slower market expansion or faster market contraction.

Suppose that the survival rate increases so that the good becomes more inelastically demanded, the consumer's search friction decreases, or the consumer base remains relatively more stable. When the sellers survive to the next period with higher probability, the equilibrium stationary value distribution will be depressed because lower type buyers crowd the market, preventing newcomers to enter. The equilibrium reserve price the sellers choose goes down as a result. The probability of sale, positively related to the equilibrium reserve price, decreases. The buyers' expected utilities increase as they can search longer and the seller's revenue decreases.

Proposition 5 (Survival Rate)**.** *When the survival rate increases, the equilibrium reserve price decreases, the equilibrium probability of sale decreases, the stationary value distribution shifts down first-order stochastically, and the buyer's utility increases, and the seller's revenue decreases.*

The proof relies on the two equations that pin down the equilibrium reserve price and the stationary value distribution at the reserve price, (8) and (13). Algebraic rearrangements of the differentiation by the Implicit Function Theorem with the continuity of the solution guaranteed specifies the change in the reserve price, and with Assumption 4, the sign is definite. The ensuing changes follow from the change in reserve price. The proofs of the following two Propositions follow the same method, with the last one particularly easy as the discount factor does not affect the equilibrium conditions.

Decrease in the buyer-seller ratio, reduction in relative number of buyers to sellers, indicates that relative demand decreases or relative supply increases. Similar to the increase in survival rate, if the buyer-seller ratio decreases, the buyers are better off and the sellers are worse off. What is not obvious is that, although the reserve price decreases, the probability of sale decreases as well - the demand plummets more than what the supply side can optimally adjust for.

Proposition 6 (Buyer-Seller Ratio)**.** *When the buyer-seller ratio increases, the equilibrium reserve price decreases, the equilibrium probability of sale decreases, the stationary value distribution shifts down first-order stochastically, the buyer's utility increases, and the seller's revenue decreases.*

Finally, let us examine the effects of change in discount factor. If the discount factor increases, the buyers become more patient, or the interval between the time periods decreases and buyers have more sequential opportunities in the imminent future. The stationary value distribution and the reserve price are not affected by the change in discount factor, as the sellers do not change their optimal mechanism, as equilibrium reserve price only depends on the survival rate and the buyer-seller ratio.

In particular, we need to distinguish the shrinking time interval interpretation of the market expansion and the market growth rate as mentioned in Section 2. When the interval shrinks, the market is being replicated more times, increasing measures of the old buyers, the new buyers and the sellers. However, increasing the market growth rate only increases the measure of newborn buyers and sellers, and in particular decreasing the relative proportion of the old buyers whose opportunities are worsened rather than improved.

Proposition 7 (Discount Factor)**.** *When the discount factor increases, the equilibrium reserve price, the equilibrium probability of sale, the stationary value distribution are not affected, the buyer's utility increases and the seller's revenue decreases.*

Overall, when the survival rate increases and/or the buyer-seller ratio decreases, the buyers' expected utilities increase as their WTPs are depressed. The revenue of the sellers decreases, as they face a buyer composition that is first-order stochastically worse (of course, as a result of the precedent sellers' responses to buyer-friendly market environment). However, social sale efficiency decreases in response, resulting in lower probability of transactions. When buyers become more patient, the market composition does not change, but the seller's revenue will be hit hard, as the buyers are more willing to wait for future chances.

6 Posted Prices and Their Market Effects

Despite of being sub-optimal, the posted prices are robust revenue preservers to sequential price competitions. We define the posted prices marginal revenue and utilize it to explain why posted prices are an increasingly attractive for sellers when the market becomes more buyer friendly.

Let's consider the widely used posted price mechanism and its relation with the reserve price auction. A posted price $\mathrm{P}(\phi)$ is the mechanism where a seller posts price $w_\Lambda(\phi)$ and the buyers who are willing to buy at the price enter the lottery to be picked as a winner with equal chances. The posted price mechanism is an incentive compatible direct-revelation mechanism: only the buyers with willingness-to-pay $w_\Lambda(\phi)$, or values above ϕ enter the lottery. The lottery seems unrealistic, but it is equivalent to the following natural selling process when the buyers arrive stochastically. The seller posts price $w_\Lambda(\phi)$ and the n buyers uniformly randomly arrive within a time interval. The first buyer with willingness-to-pay greater than $w_\Lambda(\phi)$ takes the item and pays the posted price.

From the mechanism, all the buyers with value v above the **posted type** ϕ have the same expected utility,

$$u_\Lambda(v|\mathrm{P}(\phi)) = \frac{1}{n}\frac{1-F^n(\phi)}{1-F(\phi)}(v - w_\Lambda(\phi)),$$

and the seller's expected revenue is

$$r(\mathrm{P}(\phi)) = (1 - F^n(\phi))\, w_\Lambda(\phi).$$

Define the **posted price marginal revenue in the presence of sequential market** Λ

to be

$$\mathrm{MR}^{\mathrm{P}}_{\Lambda}(v) = w_{\Lambda}(v) - \eta\left(F^{n}(v)\right) w'_{\Lambda}(v)\,. \tag{17}$$

then the revenue-maximizing posted price mechanism in the presence of sequential market Λ is $\mathrm{P}^{*} = \mathrm{P}(\phi^{*})$ with $\mathrm{MR}^{\mathrm{P}}_{\Lambda}(\phi^{*}) = 0$.

The posted price and the reserve price auction have a lot of similarities in terms of the marginal revenue curve. Both reserve type and posted type are determined by equating the marginal revenue to zero, and the types calculated specify the set of willing buyers. Together, we call the reserve price auction and the posted prices critical type mechanisms.

Remark 2. A critical type mechanism (CTM) $M(\tau)$ is either a reserve price auction $\mathrm{A}(\rho)$ or a posted price $\mathrm{P}(\phi)$. Its revenue is

$$r(M(\tau)) = \int_{\tau}^{1} \mathrm{MR}^{M}_{\Lambda}(v)\, dF^{n}(v)$$

and the optimal CTM is $M(\tau^{*})$ where $\mathrm{MR}^{M}_{\Lambda}(\tau^{*}) = 0$. The probability of sale is $1 - F^{n}(\tau)$.

$\mathrm{MR}^{\mathrm{A}}(v) \geq \mathrm{MR}^{\mathrm{P}}(v)$ for all v because $\eta\left(F^{n}(v)\right) > \eta\left(F(v)\right)$[4]. Therefore, the auction MR curve always dominates the posted price MR curve and the optimal posted type ϕ^{*}_{Λ} is always greater than the optimal reserve type ρ^{*}_{Λ}. The difference in revenues between the auction and the posted price of the same critical type is positive, and we call it the auction premium over posted price, $\Delta(\tau) \equiv r(\mathrm{A}(\tau)) - r(\mathrm{P}(\tau))$.

Whenever the sequential market becomes more buyer-friendly, the monopolist's revenues from the optimal auction and the optimal posted price decreases. Kultti (1999), realizing the disadvantages of an auction facing sequential competition, shows that in the limit when the market is infinitely competitive, an auction and a posted price are equivalent, generating the same revenue.

Proposition 8. *When the sequential market becomes more buyer-friendly or buyers become more patient, the auction premium over posted price decreases for any critical type.*

Wang (1993) shows in a stochastic setting that auction is more advantageous than posted price when the MR is steeper, Proposition (8) shows that more buyer friendly sequential market flattens the competitive auction marginal revenue more so that the gain in revenue is reduced. In addition to listing a critical type (the price posted or the reserve price in the auction), an auction involves additional procedures such as sorting the bids, announcing the winner and the payment calculated from the bids. The proof shows that the competitive

[4] $\eta\left(F^{n}(v)\right) = \eta\left(F(v)\right)\left(1 + F(v) + \cdots + F^{n-1}(v)\right) / \left(nF^{n-1}(v)\right)$ is greater than $\eta\left(F(v)\right)$ and decreasing when $F(\cdot)$ has decreasing inverse hazard rate.

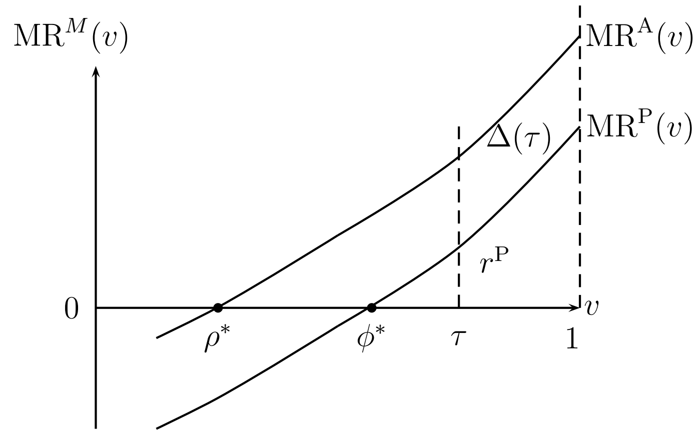

Figure 1: Critical type mechanisms.

marginal revenues of auction and posted price are closer to each other when the market becomes more buyer friendly or the buyers become more patient.

Proof of Proposition 8. The difference between the competitive marginal revenues is

$$\begin{aligned}\mathrm{MR}^{\mathrm{A}}_{\Lambda}(v) - \mathrm{MR}^{\mathrm{P}}_{\Lambda}(v) &= \left[w_{\Lambda}(v) - \eta\left(F^n(v)\right) w'_{\Lambda}(v)\right] - \left[w_{\Lambda}(v) - \eta\left(F^n(v)\right) w'_{\Lambda}(v)\right] \\ &= \left[\eta\left(F^n(v)\right) - \eta\left(F(v)\right)\right]\left(1 - \delta\Lambda(v)\right)\end{aligned}$$

The term in the square bracket is positive, so when $\tilde{\Lambda}(v) \geq \Lambda(v)$,

$$\mathrm{MR}^{\mathrm{A}}_{\tilde{\Lambda}}(v) - \mathrm{MR}^{\mathrm{P}}_{\tilde{\Lambda}}(v) \leq \mathrm{MR}^{\mathrm{A}}_{\Lambda}(v) - \mathrm{MR}^{\mathrm{P}}_{\Lambda}(v).$$

Also, when the discount factor becomes larger, the difference becomes strictly smaller as well as $\Lambda(v) > 0$. Take any critical type τ and the difference between the revenues in the presence of Λ is

$$r\left(\mathrm{A}(\tau)\right) - r\left(\mathrm{P}(\tau)\right) = \int_{\tau}^{1} \left[\mathrm{MR}^{\mathrm{A}}_{\Lambda}(v) - \mathrm{MR}^{\mathrm{P}}_{\Lambda}(v)\right] dF^n(v).$$

□

This result offers a possible explanation to the rapid demise of small auctions on eBay (Einav et al., 2016). The paper argues that auctions are widely used in the early 2000s because the sellers are auction-fervent, but as time progresses with more entrance of general sellers, they do not have an inclination for posted prices. Another effect as the general population enters is increase in market liquidity so that buyers have more opportunities to purchase. When there are more better purchasing opportunities for the buyers in sequential

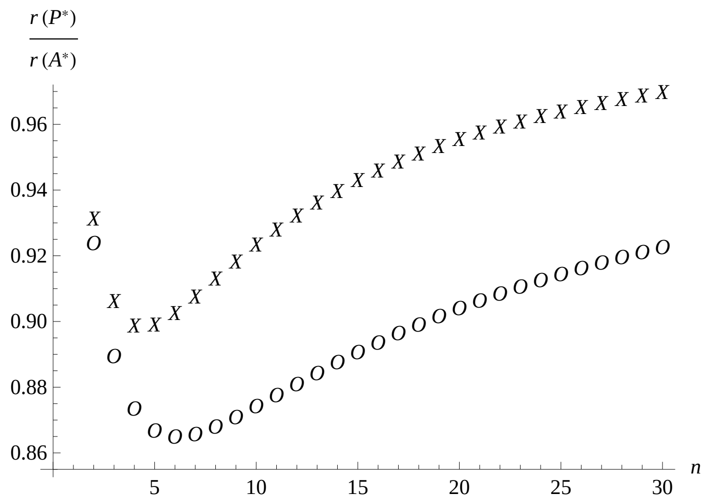

Figure 2: The ratio of the optimal auction and the optimal price revenues.

periods, the revenue gain from running an auction, which declines as shown, may not cover the extra cost associated with displaying and storage.

It is also worth mentioning the change in optimal auction premium with respect to the number of buyers as it is not a monotonic relationship. When there is only one buyer, the seller essentially faces a bargaining problem: a buyer either accepts or rejects the price offer the seller proposes, so the optimal auction and the optimal posted price yield the same revenue. When there are several buyers, the auction yields the benefit that competition between buyers drives up the transaction price. However, when the number of buyers approaches infinity, the two revenues are equal again. There is probability 1 that a buyer has any arbitrarily high value, so full revenue of is guaranteed by posting a high price. What the sequential market limits is the maximum revenue any mechanism obtains, which is the willingness-to-pay of the highest value buyer, $w_\Lambda(1)$.

Figure 2 shows the ratio of the two revenues with respect to different number of buyers, when the buyers' values are drawn from uniform distribution $(r(\mathrm{P}^*_\Lambda)/r(\mathrm{A}^*_\Lambda),\ F(v) = v)$. 'X' shows the ratio when it is a pure monopoly setting, and 'O's shows the ratio when the sequential market is $\Lambda(x) = x$, i.e., each buyer expects to receive a price offer drawn from the uniform distribution. As the plot illustrates, the percentage of expected revenue the optimal posted price attains with respect to the optimal auction increases when there is a sequential market. For example, when there are five buyers, the seller can get 86% of the optimal revenue from a posted price as a monopoly but almost 90% of the optimal revenue when a sequential market is present. The absolute revenue difference exhibits a similar pattern - as the market gets more buyer friendly, the optimal auction premium is likely to decrease.

If the seller is impatient and buyers arrive stochastically over time, posted price brings immediacy to transaction in the way that as soon as there is one willing buyer the transaction is realized and the payment is received, whereas the auction involves waiting longer. Therefore, when the buyer arrival rate is low, an impatient seller is more inclined to post a price instead of running the reserve price auction. Board and Skrzypacz (2014), for example, shows with continuos stochastic arrival of buyers, the optimal mechanism is to dynamically adjust prices with a reserve price auction in the end, so a posted price or an auction approximates the optimal revenue a seller. Skreta (2006) shows that posted price is optimal when the sellers cannot commit in a sequential setting.

We investigate market effects of such auction cost on equilibrium mechanism choice and market efficiency in the next section.

7 A Market with Heterogeneous Auction Costs

We have shown in the previous section that all the homogenous sellers run reserve price auctions in equilibrium. In this section, we investigate how seller's fixed cost of setting an auction alters his mechanism choice and what effects the switch to alternative mechanisms brings to the market.

Each seller chooses a critical type mechanism: $\mathcal{M} = \{\mathrm{A}(\cdot), \mathrm{P}(\cdot)\}$, but has heterogenous fixed cost associated with running an auction. The cost c is iid drawn from the cost distribution $G(\cdot)$ on support $[-1, 1]$ with positive PDF g. The expected profit $\pi(\mathrm{A}(\cdot))$ is the expected revenue $r(\mathrm{A}(\cdot))$ minus the cost c. The matching and market processes remain the same as in the previous setting. We consider both settings in which the sellers live for one period and two periods.

A SSSE $(M_*(\cdot), \sigma_*(\cdot,\cdot), H_*(\cdot), \mu_*(\cdot))$ in this setting only differs from Definition 1 by that each cost c seller chooses a possibly different mechanism $M_*(c)$. In equilibrium, there is a cutoff cost c_* such that any seller with cost $c > c_*$ uses posts price mechanism $\mathrm{P}(\phi_*)$ and any other with cost $c < c_*$ runs reserve price auction $\mathrm{A}(\rho_*)$. Cost c_* seller is indifferent between the two mechanisms because he yields the same profit,

$$\int_{\rho_*}^{1} \mathrm{MR}_*^{\mathrm{A}}(v)\, dH_*^n(v) - c_* = \int_{\phi_*}^{1} \mathrm{MR}_*^{\mathrm{P}}(v)\, dH_*^n(v).$$

The buyers with the highest values above exit slower than in the previous setting because all the participating buyers have the same probability of winning in a posted price. Furthermore, a posted price makes the sale probability smaller, resulting in more lower value buyers crowding the market, bringing down the stationary value distribution. In particular, such an equilibrium exists and is unique under the same assumptions as in the previous section.

The derivations and characterizations of the SSSE are included in the Appendix.

We are particularly interested in the market effects of the auction costs and the posted price mechanisms as a consequence of the auction costs. The introduction of posted prices mechanisms reduces allocative efficiency possibly because sellers value immediacy of the sale to some buyer who arrives the earliest, not necessarily of the highest value. However, even though sale efficiency depreciates with the posted price mechanism compared to the auction mechanisms, the downward change in stationary value distribution brought by the posted prices actually improves sale efficiency of the equilibrium auction.

Proposition 9. *Suppose Assumption 4 holds. When there is auction cost, allocative inefficiency increases but change in sale efficiency is ambiguous.*

Although analytically it is ambiguous, under most circumstances, the sale efficiency decreases as the decrease in sale efficiency brought about by the switch to posted price dominates the gain in social efficiency in auctions. In the extreme case when every seller posts price, sale efficiency depreciates the most.

Note that the high cost sellers are the ones using inefficient, suboptimal posted price mechanisms. The sellers are gaining less from the posted price, because the sellers who use auctions get strictly higher profit $(r(\mathrm{A}(\rho_*)) - c > r(\mathrm{P}(\phi_*)))$. The highest value buyers' expected utilities are depressed too. If the sellers and buyers make decisions whether to enter the market based on ex-ante expected utilities, expected prevalence of posted prices mechanisms may deter their entrance, and possibly causes the market to collapse if there is a sufficiently competitive rival who may offer market of auctions. As a result, incentivizing the sellers to use auctions may be an attractive option. Facilitating buyer bids by allowing bidding bots, reducing fixed cost of auctions for sellers and buyers, and deterring posted prices may be options worth trying.

The theoretical predictions qualitatively match most qualitative results of empirical evidence. On the market level, the relative proportion of auctions to posted prices has rapidly decreased from over 95% in 2003 to only 25% in 2011 (Einav et al., 2016, Figure 1). However, share of revenues from auctions has not dropped as much, meaning that auctions generate relatively higher average revenues.

Empirical evidence suggests that the speed of convergence to SSSE is rather fast - three weeks in Ockenfels and Roth (2004). A deck of cards with each showing a most wanted terrorist was issued by the US military to its solders on April 11, 2003, but was sold on eBay and retail immediately (on day 2 - April 12, and on day 3 - April 13, respectively). The deck, for its novelty, origin, and rarity, sold for nearly \$70 in the first week, and mostly by auctions. However, the US Playing Cards Company released the identical copies of the cards for \$5.95 and got publicly known, the prices gradually dropped to the competitive prices in about three weeks and gradually higher proportion of decks were sold through posted prices.

8 Conclusion

The paper shows that reserve price auction remains a desirable choice for a seller facing uncertain demand and uncertain future competition. The reserve price, determined by a generalized marginal revenue curve, screens for buyers' willingnesses-to-pay but also indicates market competitiveness. The optimal posted price mechanism is shown to exhibit similar critical type determination procedure.

Although the optimal mechanism remains reserve price auction, sequential outside options reduce the desirability of an auction as it curtails some of the most important features of an auction. The advantage of running auction comes from the uniqueness of the item and the surplus extracted from the possible high valuations of the buyers, but the sequential opportunities make the current item dispensable, especially to buyers with high values for the item. A mechanism particularly immune to such turns in market conditions is posted price mechanism that does not have the transaction price dependent on unique willing buyers. Variants include dynamically adjusting prices and Buy-It-Now featured on eBay. However, posted prices are allocative inefficient and impose market externalities that affect sale efficiency and affect active buyer composition that lowers profit for future sellers.

We make some concluding remarks focusing on the current work's shortcomings and possible amendments for future works. An issue is the seller's commitment in case he does not sell, which is embodied by the assumption that we have maintained throughout the majority of the paper, sellers live only for one period and thus choose a static selling mechanism. When transaction costs of posting to intermediaries are high, when the items for sale have high depreciation rate or face intense competition from substitutable goods, a seller is better off using one-shot mechanisms. Reputation and rating systems also punish sellers for selling low quality goods for repeatedly many times. Furthermore, many goods that have substitutable competitions and upgrades have steep price drops in short period which essentially prevents a seller to choose a dynamically optimal pricing rule or repeated auctions. Zhang (2017) shows that even if sellers can adjust their mechanisms periodically, it is going to be some predictable combination of auctions and posted prices. A dynamic model taking consideration of possible resale is of important interest.

Although we suggest several possibilities to the rise of auction cost - risk aversion, discount factor, fixed costs and idiosyncratic taste, the simple one-dimensional cost imposed is seemingly ad-hoc and does not tackle the more fundamental question *why* auction is less desirable to a rational agent who should supposedly only care about revenue. Works in dynamic mechanism design and revenue management can offer some insights.

A SSSE is proven to exist uniquely in the settings but we have been ignorant of the speed of convergence to such a stationary equilibrium behavior from an initial value distribution. If the speed of convergence to the SSSE is low (as it might be, suggested by numerical simulation of similar trading markets of Cho and Matsui (2013)), then it is important to investigate the adjustment on the equilibrium path to see how sellers switch from auctions to posted prices for example. Furthermore, we are agnostic about whether it is guaranteed that any change in the market environment results in global convergence to the new SSSE.

Furthermore, there is a line of empirical research that needs to be done, especially to the two latter dynamic markets. Equilibrium characterizations are complicated enough already that definitive comparative statics results were not obtainable because of complexity and non-monotonicity resulted from sellers' possible switch to alternative sub-efficient mechanisms. Numerical simulations and empirical works can be done to investigate different inefficiencies and externalities, to quantify the effects such as changes in revenues and probability of sale, and to test validity of assumptions made in the paper.

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Appendix

A The Monopolist's Problem with Asymmetric Buyers

In this section, we solve the revenue-maximizing mechanism for n buyers with possibly asymmetric value distributions. Each buyer i has independent private value drawn distribution F_i on support $[\underline{v}_i, \overline{v}_i]$ such that $(1 - F_i(v))/f_i(v)$ is weakly decreasing. The sequential market is Λ, which is fixed throughout the section, so we do not include subscript. Let $\mathbf{v} = (v_1, \cdots, v_n)$ be the vector of values and $F(\mathbf{v}) = \prod_i F_i(v_i)$ be the value distribution.

A DSM $(P(\cdot), C(\cdot)) \equiv (P_i(\cdot), C_i(\cdot))_{i=1}^n$ consists of a collection of probability assignment functions and cost functions. The probability assignment functions take reports of the buyers to assign each buyer a probability of obtaining the item, with the properties that $0 \leq P_i(z_1, \cdots, z_n) \leq 1$ and $\sum_{i=1}^n P_i(z_1, \cdots, z_n) \leq 1$ for each buyer i. The cost function $C_i(z_1, \cdots, z_n)$ specifies the transfer from buyer i to the monopolist given all buyers' reports. Define the **expected probability assignment** and **expected cost function** as

$$\begin{aligned}
\overline{P}_i(z_i) &\equiv \int_{\prod_{j\neq i} z_j \in \prod_{j\neq i}\left[\underline{v}_j, \overline{v}_j\right]} P_i(z_1, \cdots, z_n) \prod_{j\neq i} dF_j(z_j), \\
\overline{C}_i(z_i) &\equiv \int_{\prod_{j\neq i} z_j \in \prod_{j\neq i}\left[\underline{v}_j, \overline{v}_j\right]} P_i(z_1, \cdots, z_n) \prod_{j\neq i} dF_j(z_j).
\end{aligned}$$

We want to restrict our attention to **incentive-compatible (IC)** mechanisms. Buyer i's expected utility of reporting z_i in the mechanism is the expected utility when he gets the object from the mechanism plus the expected utility if she does not get and waits until the next period,

$$u(z_i|v_i) = \overline{P}_i(z_i) v_i - \overline{C}_i(z_i) + \left(1 - \overline{P}_i(z_i)\right) \underline{u}(v_i),$$

Proposition 10. *$(P_i(\cdot), C_i(\cdot))_{i=1}^n$ is incentive-compatible if and only if for every buyer i,*

1. *$\overline{P}_i(v_i)$ is non-decreasing in v_i, and*
2. *$\overline{C}_i(v_i) = \overline{C}_i(\underline{v}_i) - \overline{P}_i(\underline{v}_i) w(\underline{v}_i) + \overline{P}_i(v_i) w(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i(x) dw(x)$ for all $v_i \in [\underline{v}_i, \overline{v}_i]$.*

Proof of Proposition 10. First I show that if the mechanism is IC, then both conditions 1 and 2 hold. Define benefit of lying,

$$\psi_i(z|v_i) = u_i(z_i|v_i) - u_i(v_i|v_i).$$

IC implies that for any v_i, there is no benefit in lying, so $\psi_i(z_i|v_i) \leq 0$ for all z_i, v_i, so

$$
\begin{aligned}
0 &\geq \psi_i(z_i|v_i) + \psi_i(v_i|z_i) \\
&= \left[\overline{P}_i(z_i)\, w(v_i) - \overline{C}_i(z_i) + (v_i - w(v_i)) - \left(\overline{P}_i(v_i)\, w(v_i) - \overline{C}_i(v_i) + (v_i - w(v_i))\right)\right] - \\
&\quad \left[\overline{P}_i(v_i)\, w(z_i) - \overline{C}_i(v_i) + (z_i - w(z_i)) - \left(\overline{P}_i(z_i)\, w(z_i) - \overline{C}_i(z_i) + (z_i - w(z_i))\right)\right] \\
&= \left[\overline{P}_i(z_i)\, w(v_i) - \overline{C}_i(z_i) - \left(\overline{P}_i(v_i)\, w(v_i) - \overline{C}_i(v_i)\right)\right] - \\
&\quad \left[\overline{P}_i(v_i)\, w(z_i) - \overline{C}_i(v_i) - \left(\overline{P}_i(z_i)\, w(z_i) - \overline{C}_i(z_i)\right)\right] \\
&= \left(\overline{P}_i(z_i) - \overline{P}_i(v_i)\right)(w(v_i) - w(z_i))
\end{aligned}
$$

Since $w(v_i)$ is non-decreasing in v_i, $\overline{P}_i(v_i)$ is non-decreasing in v_i. Furthermore, IC implies that $u_i(z_i|v_i)$ is maximized at $z_i = v_i$, then by envelope theorem,

$$
\left.\frac{\partial u_i(z_i|v_i)}{\partial z_i}\right|_{z_i = v_i} = \overline{P}'_i(v_i)\, w(v_i) - \overline{C}'_i(v_i) = 0.
$$

By fundamental theorem of calculus,

$$
\overline{C}_i(v_i) = \overline{C}_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} \overline{P}'_i(v_i)\, w(v_i)\, dv_i
$$

Integration by parts yields condition 2 as desired. The converse is shown directly by definition of incentive-compatibility:

$$
\begin{aligned}
\psi_i(z_i|v_i) &= \left[\overline{P}_i(z_i) w(v_i) - \overline{C}_i(z_i) + (v_i - w(v_i))\right] \\
&\quad - \left[\overline{P}_i(v_i)\, w(v_i) - \overline{C}_i(v_i) + (v_i - w(v_i))\right] \\
&= \left[\overline{P}_i(z_i) - \overline{P}_i(v_i)\right] w(v_i) + \overline{C}_i(v_i) - \overline{C}_i(z_i)
\end{aligned}
$$

Plugging in condition 2,

$$
\begin{aligned}
\psi_i(z_i|v_i) &= \left[\overline{P}_i(z_i) - \overline{P}_i(v_i)\right] w(v_i) + \overline{P}_i(v_i)\, w(v_i) - \int_{z_i}^{v_i} \overline{P}_i(x)\, dw(x) - \overline{P}_i(z_i)\, w(z_i) \\
&= \overline{P}_i(z_i)\,[w(v_i) - w(z_i)] - \int_{z_i}^{v_i} \overline{P}_i(x)\, dw(x) = \int_{z_i}^{v_i} \left(\overline{P}_i(z_i) - \overline{P}_i(x)\right) dw(x)
\end{aligned}
$$

Then by condition 1, the expression is non-positive. □

Because the expected probability assignment function pins down the expected cost function, if the expected cost function of the buyer of the lowest value is the same across two bidders and the expected probability assignment is the same, then the expected revenue is the same for the seller, and all buyers are indifferent between the incentive-compatible mechanisms the seller runs.

Corollary 1 (Buyer Indifference and Revenue Equivalence)**.** *If two incentive-compatible mechanisms have 1) the same expected probability assignment functions, and 2) the same expected costs for the buyer of the lowest value, then all the agents are indifferent between the two mechanisms, as they yield the same expected revenue, and the same expected payoffs for the buyers of the same value.*

Proof to Corollary 1. Denote the two mechanisms $M^I = \left(\overline{P}_i^I(\cdot), \overline{C}_i^I(\cdot)\right)_{i=1}^n$ and $M^{II} = \left(\overline{P}_i^{II}(\cdot), \overline{C}_i^{II}(\cdot)\right)_{i=1}^n$, respectively. In any IC mechanism $M = \left(\overline{P}_i(\cdot), \overline{C}_i(\cdot)\right)_{i=1}^n$, the expected cost function

$$\overline{C}_i(v_i) = \overline{C}_i(\underline{v}_i) - \overline{P}_i(\underline{v}_i)\,\tilde{v}(\underline{v}_i) + \overline{P}_i(v_i)\,w(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i(x)\,dw(x)$$

Because $\overline{P}_i^I(\cdot) = \overline{P}_i^{II}(\cdot)$ for all i by condition 1 and $\overline{C}_i^I(\underline{v}_i) = \overline{C}_i^{II}(\underline{v}_i)$ by condition 2, $\overline{C}_i^I(v_i) = \overline{C}_i^{II}(v_i)$ for all v_i for all i. Expected utility of v_i in mechanism M is

$$u_i^M(v_i|v_i) \;=\; \overline{P}_i(v_i)\,w(v_i) - \overline{C}_i(v_i) + v_i - w(v_i)$$

$u_i^{M^I}(v_i|v_i) = u_i^{M^{II}}(v_i|v_i)$ for all v_i for i. Revenue is determined by

$$r(M) = \sum_{i=1}^n \int_{\underline{v}}^{\overline{v}} \overline{C}^M(v_i)\,dF_i(v_i)$$

Then as shown that $\overline{C}_i^I(v_i) = \overline{C}_i^{II}(v_i)\ \forall v_i\ \forall i$, $r\left(M^I\right) = r\left(M^{II}\right)$. □

The seller maximizes expected revenue subject to the incentive compatibility and individual rationality constraints.

Definition 3. A revenue-maximizing IC, IR DSM is $M = (P_i(\cdot), C_i(\cdot))_{i=1}^n$ that maximizes

$$r \;=\; \sum_{i=1}^n \int_{\underline{v}_i}^{\overline{v}_i} \overline{C}_i(v_i)\,dF_i(v_i)$$

subject to $\forall i$: $\tilde{u}_\Lambda(v_i|v_i) \geq \tilde{u}_\Lambda(z_i|v_i)\ \forall z_i \neq v_i$ and $\tilde{u}_\Lambda(v_i|v_i) \geq 0\ \forall v_i$.

Proposition 11. *The revenue-maximizing IC, IR DSM $(P_i^*(\cdot), C_i^*(\cdot))_{i=1}^n$ is that $P_i^*(\mathbf{v}) = 1$*

if $\tilde{v}(v_i) - \frac{1-F_i(v_i)}{f_i(v_i)} w'(v_i) > \max\left\{\tilde{v}(v) - \frac{1-F_j(v_j)}{F_j(v_j)} w'(v_i), 0\right\}$ *and* $=0$ *otherwise, and*

$$\overline{C}_i^*(v_i) = \overline{C}_i^*(\underline{v}_i) - \overline{P}_i^*(\underline{v}_i)\tilde{v}(\underline{v}_i) + \overline{P}_i^*(v_i)\tilde{v}(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i^*(z)\, d\tilde{v}(z)$$

The several key steps to the proof are i) characterizing the individual rationality constraint, ii) exchanging integrals, and iii) expanding expected probability assignment functions to obtain a weighted average of probability assignment functions.

Proof of Proposition 11. The constraints are

1. IC1: $\overline{P}_i(v_i) \geq \overline{P}_i(z_i)\ \forall v_i \geq z_i$,
2. IC2: $\overline{C}_i(v_i) = \overline{C}_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} \overline{P}_i'(v_i) w(v_i)\, dv_i\ \forall v_i$, and
3. IR: $u_i(v_i|v_i) \geq \underline{u}_i(v_i)$.

In particular, The IR implies that $\overline{C}_i(\underline{v}_i) - \overline{P}_i(\underline{v}_i) w(\underline{v}_i) \leq 0$.

$$r = \sum_{i=1}^{n} \int_{\underline{v}_i}^{\overline{v}_i} \left[\overline{P}_i(v_i) w(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i(z)\, dw(z)\right] dF_i(v_i) + \sum_{i=1}^{n} \left[\overline{C}_i(\underline{v}_i) - \overline{P}_i(\underline{v}_i) w(\underline{v}_i)\right]$$

where

$$\begin{aligned}
& \int_{\underline{v}_i}^{\overline{v}_i} \left[\overline{P}_i(v_i) w(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i(x)\, dw(x)\right] dF_i(v_i) \\
= \; & \int_{\underline{v}_i}^{\overline{v}_i} \overline{P}_i(v_i) w(v_i) f_i(v_i)\, dv_i - \int_{\underline{v}_i}^{\overline{v}_i} \int_{x}^{\overline{v}_i} \overline{P}_i(x) w'(x) f_i(v_i)\, dv_i dx \\
= \; & \int_{\underline{v}_i}^{\overline{v}_i} \overline{P}_i(v_i) w(v_i) f_i(v_i)\, dv_i - \int_{\underline{v}_i}^{\overline{v}_i} \overline{P}_i(x) w'(x) (1 - F_i(x))\, dx \\
= \; & \int_{\underline{v}_i}^{\overline{v}_i} \overline{P}_i(v_i) \left[w(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} w'(v_i)\right] f_i(v_i)\, dv_i \\
= \; & \int_{\underline{v}_1}^{\overline{v}_1} \cdots \int_{\underline{v}_n}^{\overline{v}_n} P_i(\mathbf{v}) \left[w(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} w'(v_i)\right] dF(\mathbf{v})
\end{aligned}$$

Therefore,

$$r \;\leq\; \int_{\underline{v}_1}^{\overline{v}_1} \cdots \int_{\underline{v}_n}^{\overline{v}_n} \sum_{i=1}^{n} P_i(\mathbf{v}) \left[w(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} w'(v_i)\right] dF(\mathbf{v}),$$

and any mechanism that implements the specified probability assignment function will achieve the upper bound. □

B Proofs

Proof of Proposition 1. It follows as a corollary of Proposition 11 when all the value distributions are identically F. When the WTP function $w(\cdot)$ is not continuously differentiable, the optimal reserve type ρ^* satisfies that $\mathrm{MR}^{\mathrm{A}}\left(\rho_+^*\right) \geq 0$ and $\mathrm{MR}^{\mathrm{A}}\left(\rho_-^*\right) \leq 0$. The existence and uniqueness of ρ^* are guaranteed by the monotonicity of the auction marginal revenue curve. □

Proof of Proposition 2. The optimal reserve type is determined by

$$w_\Lambda\left(\rho_\Lambda^*\right)/w_\Lambda'\left(\rho_\Lambda^*\right) = \eta\left(F\left(\rho_\Lambda^*\right)\right)$$

The LHS is increasing and the RHS is decreasing. If we show that LHS is greater than ρ_Λ^*, then $\rho_\Lambda^* < \rho_{\mathrm{mon}}^*$. $w_\Lambda(v)/w_\Lambda'(v) \geq v$ if

$$\begin{aligned} w_\Lambda(v) - v w_\Lambda'(v) &= v - \delta \int_{\underline{x}}^{v} \Lambda(x)\, dx - v\left(1 - \delta\Lambda(v)\right) \\ &= v\delta\Lambda(v) - \delta\int_{\underline{x}}^{v} \Lambda(x)\, dx \geq 0 \end{aligned}$$

as $\Lambda(x) \leq \Lambda(v)$ for all $x \leq v$. □

Proof of Proposition 3. The derivation modifies Jehle and Reny (2011). Value v buyer's utility bidding according to the strictly increasing bidding function $\sigma(z)$ is

$$u\left(v, \sigma(z), \sigma_{-i}(\cdot)\,|\mathrm{A}(\rho)\right) = F^{n-1}(z)\left(v - \sigma(z)\right) + l(z)\,\delta\underline{u}_\Lambda(v).$$

where $l(z) = 1 - \mathbf{1}_{v \geq \rho} F^{n-1}(z)$ is the expected probability of losing by reporting z. Rearrange,

$$u\left(v, \sigma(z), \sigma_{-i}(\cdot)\,|\mathrm{A}(\rho)\right) = F^{n-1}(z)\left(w_\Lambda(v) - \sigma(z)\right) + \delta\underline{u}_\Lambda(v) \tag{18}$$

Differentiate (18) with respect to z and the buyer optimality condition requires that it is zero at $z = v$. Coupled with the boundary condition that $\sigma(\rho) = w_\Lambda(\rho)$, the equilibrium bidding function is obtained. The candidate bidding function is indeed the equilibrium as the utility is maximized (rather than minimized) at $z = v$. The equilibrium payoffs are obtained by replugging in the equilibrium bidding function, and revenue is obtained by the MR curve by Riley and Samuelson (1981). □

Proof of Proposition 4. We show first that there exists a unique solution of ρ_* to (8) and (13), and it is the only candidate equilibrium. It is then shown to be the unique optimal reserve type that a seller chooses in the equilibrium. Coupled with the stationary distributions determined by (11) and (12), it constitutes an equilibrium.

First we show the existence and uniqueness of ρ_* and $H_*(\rho_*)$ for the system of equations, (8) and (13). If every seller chooses the same reserve type ρ auction (not necessarily optimal), the stationary value distribution resulted from it is characterized by (12) (substitute ρ for ρ_*). In particular, $x \equiv H_*(\rho)$ is determined by (13),

$$x/\left[1+\frac{s}{1-s}\frac{1}{n}(1-x^n)\right] = F(\rho).$$

LHS is monotonically strictly increasing in x, ranging from 0 (when $x=0$) to 1 (when $x=1$). Since RHS is a fixed number between 0 and 1 for any ρ, there is solution $x(\rho) = H_*(\rho)$ for each ρ, and is strictly increasing in ρ.

In order to show that (8) has a solution, it is sufficient to show that LHS is increasing in ρ, and it holds when $1/\eta(H_*(\rho))$ is increasing,

$$\frac{h_*(\rho)}{1-H_*(\rho)} = \frac{H_*(\rho)}{1-H_*(\rho)}\frac{f(\rho)}{F(\rho)} = f(\rho)\left[\frac{1}{1-H_*(\rho)} + \frac{s}{1-s}\frac{1}{n}\frac{1-H_*{}^n(\rho)}{1-H_*(\rho)}\right]$$

where both equalities follow from (13). f is increasing by Assumption (2) and the second term is increasing in ρ because $H_*(\rho)$ is increasing. Overall, LHS of (8) continuously and monotonically increases from $-1/f(0)$ to 1 as ρ increases from 0 to 1. Therefore, the solution to the system of equations, ρ_*, is unique.

In fact, it is the only candidate equilibrium. Any ρ can be the symmetric equilibrium reserve type but the stationary distribution from such choice of ρ is pinned down by solution $H_*(\rho) = x(\rho)$. Only the pairs $(\rho, x(\rho))$ satisfy stationarity condition and only one pair is candidate equilibrium by the seller's optimality condition that determines the reserve price. Finally, it is sufficient to show that under $H_*(\rho_*)$, ρ_* is indeed the unique optimal reserve price.

ρ_* is optimal if $\mathrm{MR}_*^{\mathrm{A}}(v)$ is increasing, so it is sufficient to show that $h_*(v)/[(1-H_*(v))w_*'(v)]$ is increasing.

$$\frac{d\log\left[h_*(v)/\left[(1-H_*(v))\,w_*'(v)\right]\right]}{dv} = \frac{h_*'(v)}{h_*(v)} + \frac{h_*(v)}{1-H_*(v)} - \frac{w_*''(v)}{w_*'(v)}.$$

By (2),

$$(1-sl_*(v))\,h_*(v) = (1-sl_*)\,f(v) \Rightarrow (1-sl_*(v))\,h_*'(v) - sl_*'(v)\,h_*(v) = (1-sl_*)\,f'(v)$$

so

$$\frac{h_*'(v)}{h_*(v)}=\frac{f'(v)}{f(v)}+\frac{sl_*'(v)}{1-sl_*(v)}$$

By differentiating (20),

$$w_*''(v)=w_*'(v)\,\frac{\delta sl_*'(v)}{1-\delta sl_*(v)}.$$

Pulling together, we get

$$\begin{aligned}
& \frac{f'(v)}{f(v)}+\frac{sl_*'(v)}{1-sl_*(v)}+\frac{h_*(v)}{1-H_*(v)}-\frac{\delta sl_*'(v)}{1-\delta sl_*(v)} \\
= \;& \frac{f'(v)}{f(v)}+\frac{h_*(v)}{1-H_*(v)}+\frac{(1-\delta)\,sl_*'(v)}{(1-sl_*(v))\,(1-\delta sl_*(v))} \\
= \;& \frac{f'(v)}{f(v)}+h_*(v)\left[\frac{1}{1-H_*(v)}-\frac{(1-\delta)\,s}{(1-sl_*(v))\,(1-\delta sl_*(v))}\,(n-1)\,H_*^{n-2}(v)\right]
\end{aligned}$$

Note that $\frac{1}{1-H_*(v)}\geq\sum_{j=0}^{n-2}H_*^j(v)\geq(n-1)\,H_*^{n-2}(v)$, the term in the bracket is greater than 0 if

$$\frac{(1-\delta)\,s}{(1-sl_*(v))\,(1-\delta sl_*(v))}\leq 1$$

and the LHS is bounded by $\frac{(1-\delta)s}{(1-s)(1-\delta s)}$ when $l_*(v)=1$. The numerator is smaller than the (positive) denominator when Assumption (3) holds. Coupled with Assumption (2) that guarantees f' to be positive, the auction marginal revenue curve is strictly increasing, so ρ_* is the only solution to $\mathrm{MR}_*^{\mathrm{A}}(\rho)=0$, thus the only equilibrium. □

Calculations of Sequential Market, WTP, Equilibrium Buyer Utility and Seller Revenue Because the equilibrium behavior and buyer composition are stationary, value $v>\rho_*$ buyer's total discounted expected payoff is the same as her continuation payoff, which is the same as the equilibrium expected payoff in $\mathrm{A}(\rho_*)$, which by (6) is,

$$\underline{u}_*(v)=u(v|\mathrm{A}(\rho_*))=\int_{\rho_*}^{v}H_*^{n-1}(z)\,w_*'(z)\,dz+s\delta\underline{u}_*(v)\,. \tag{19}$$

Her WTP is her value net her present value of expected continuation payoff, depressed by her survival rate and discount factor,

$$w_*(v)=v-s\delta\underline{u}_*(v)=v-\frac{s\delta}{1-s\delta}\int_{\rho_*}^{v}H_*^{n-1}(z)\,w_*'(z)\,dz \tag{20}$$

where (20) is from plugging in (19). Differentiate both sides of (20) and rearrange, we get that

$$w_*'(v) = \frac{1-s\delta}{1-s\delta\left(1-H_*^{n-1}(v)\right)} = \frac{1-s\delta}{1-s\delta l_*(v)}.$$

In equilibrium, value v buyer's equilibrium WTP is

$$w_*(v) = v - \mathbf{1}_{v\geq\rho_*}\int_{\rho_*}^{v}\frac{s\delta H_*^{n-1}(z)}{1-s\delta+s\delta H_*^{n-1}(z)}dz. \tag{21}$$

Proof of Proposition 5. The equilibrium ρ_* and $H_*(\rho_*)$ simultaneously satisfy

$$\begin{aligned} H_*(\rho_*) &\equiv \left[1+\frac{s}{1-s}\frac{1}{n}\left(1-H_*^n(\rho_*)\right)\right]F(\rho_*) \quad (22)\\ \rho_*\cdot h_*(\rho_*)+H_*(\rho_*) &\equiv 1 \quad (23)\end{aligned}$$

Plugging in the equality $H_*(\rho_*)/h_*(\rho_*) = F(\rho_*)/f(\rho_*)$ to (23),

$$H_*(\rho_*) = F(\rho_*)/\left(F(\rho_*)+\rho_* f(\rho_*)\right) \tag{24}$$

and plug that into (22),

$$\frac{1}{F(\rho_*)+\rho_* f(\rho_*)} = 1+\frac{s}{1-s}\frac{1}{n}\left(1-\left(\frac{F(\rho_*)}{F(\rho_*)+\rho_* f(\rho_*)}\right)^n\right) \tag{25}$$

This holds for all s, n at $\rho_*(s,n)$, the equilibrium reserve price at the respective environment. By Implicit Function Theorem on (24),

$$\frac{dH_*(\rho_*)}{ds} = \left(\frac{1}{1+\frac{\rho_* f(\rho_*)}{F(\rho_*)}}\right)'\frac{d\rho_*}{ds} \tag{26}$$

has the opposite sign as $d\rho_*/ds$, because the sign of the first term is negative by Assumption 4. By Implicit Function Theorem again, differentiate (25) with respect to s,

$$\left(\frac{1}{F(\rho_*)+\rho_* f(\rho_*)}\right)'\frac{d\rho_*}{ds} = \left(\frac{s}{1-s}\right)'\frac{1}{n}\left(1-H_*^n(\rho_*)\right) - \frac{s}{1-s}H_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{ds}$$

Plug (26) in and rearrange,

$$\left[\left(\frac{1}{F(\rho_*)+\rho_* f(\rho_*)}\right)' + \frac{s}{1-s}H_*^{n-1}(\rho_*)\left(\frac{1}{1+\frac{\rho_* f(\rho_*)}{F(\rho_*)}}\right)'\right]\frac{d\rho_*}{ds}$$

$$= \left(\frac{s}{1-s}\right)' \frac{1}{n}\left(1 - H_*^n(\rho_*)\right)$$

Since $\left(\frac{s}{1-s}\right)' > 0$, the sign of $\frac{d\rho_*}{ds}$ is the same as

$$\left(\frac{1}{F(\rho_*) + \rho_* f(\rho_*)}\right)' + \frac{s}{1-s} H_*^{n-1}(\rho_*) \left(\frac{1}{1 + \frac{\rho_* f(\rho_*)}{F(\rho_*)}}\right)'.$$

The first term is negative because $f'(v) \geq 0$ and the second term is negative by Assumption 4. Therefore, $d\rho_*/ds < 0$: the sign is strict because $\rho_* f(\rho_*)$ is strictly increasing.

The change in probability of sale with respect to higher survival rate has opposite sign as that of $dH_*(\rho_*)/ds$ which has opposite sign as $d\rho_*/ds$, so it is negative.

By (12), for all $v < \rho_*$,

$$H_*(v) = F(v) \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right].$$

Since the term in the bracket equals $1/\left(F(\rho_*) + \rho_* f(\rho_*)\right)$, it increases as s increases:

$$\frac{d\left[1/\left(F(\rho_*) + \rho_* f(\rho_*)\right)\right]}{ds} = \left(\frac{1}{F(\rho_*) + \rho_* f(\rho_*)}\right)' \frac{d\rho_*}{ds} > 0,$$

as both multiplicands are negative. For all $v \geq \rho_*$,

$$H_*(v) = F(v) \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right] - \frac{s}{1-s}\frac{1}{n}\left(H_*^n(v) - H_*^n(\rho_*)\right)$$

Rearrange,

$$H_*(v) + \frac{s}{1-s}\frac{1}{n} H_*^n(v) = F(v) + F(v)\frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right) + \frac{s}{1-s}\frac{1}{n} H_*^n(\rho_*) \tag{27}$$

Differentiate with respect to s and let $x \equiv H_*(v)$, the LHS is

$$\frac{dx}{ds} + \frac{s}{1-s} x^{n-1} \frac{dx}{ds} + \left(\frac{s}{1-s}\right)' \frac{1}{n} H_*^n(v),$$

and the RHS is

$$F(v)\left(\frac{s}{1-s}\right)'\frac{1}{n}\left(1 - H_*^n(\rho_*)\right) + \left(\frac{s}{1-s}\right)'\frac{1}{n} H_*^n(\rho_*) + (1 - F(v))\frac{s}{1-s}\frac{1}{n} n H_*^{n-1}(\rho_*) \frac{dH_*(\rho_*)}{ds}$$

Rearrange, $\left(1+\frac{s}{1-s}x^{n-1}\right)\frac{dx}{ds}$ equals

$$\left[F(v)\frac{1}{n}\left(1-H_*^n(\rho_*)\right)-\frac{1}{n}\left(H_*^n(v)-H_*^n(\rho_*)\right)\right]\left(\frac{s}{1-s}\right)'$$
$$+\left(1-F(v)\right)\frac{s}{1-s}\frac{1}{n}nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{ds}$$

To show that $dx/ds \geq 0$, it suffices to show that the first term in the bracket is positive, because $\frac{dH_*(\rho_*)}{ds} \geq 0$ is already shown. By (12), the term equals

$$\left[H_*(v)-F(v)\right]/\left(\frac{s}{1-s}\right)\geq 0$$

as the newborn distribution always first order stochastically dominates the stationary value distribution in equilibrium.

The buyer's utility then increases as the sequential market becomes more buyer-friendly. For (16), the integrand increases as $H_*(v)$ increases for all $v>\rho_*$, s increases, and ρ_* decreases. On the other hand, the seller's revenue decreases because the willingnesses to pay of the buyers all decrease, and the buyer stationary value distribution first stochastically increases, resulting in less equilibrium probability of sale. □

Proof of Proposition 6. The comparative statics results still derive from the equilibrium conditions of (22) and (23). Differentiate (25) with respect to n, the LHS is

$$\left(\frac{1}{F(\rho_*)+\rho_* f(\rho_*)}\right)'\frac{d\rho_*}{dn}$$

and the RHS becomes

$$\begin{aligned}
&\frac{s}{1-s}\left(-\frac{1}{n^2}\right)\left(1-H_*^n(\rho_*)\right)+\frac{s}{1-s}\left(-nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{dn}-H_*^n(\rho_*)\log H_*(\rho_*)\right)\\
=\quad&-\frac{s}{1-s}nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{dn}-\frac{s}{1-s}\frac{1}{n^2}\left[1-H_*^n(\rho_*)+H_*^n(\rho_*)\log H_*^n(\rho_*)\right]
\end{aligned}$$

Equating the two sides and rearrange,

$$\begin{aligned}
&-\left[\left(\frac{1}{F(\rho_*)+\rho_* f(\rho_*)}\right)'+\frac{s}{1-s}nH_*^{n-1}(\rho_*)\left(\frac{1}{1+\frac{\rho_* f(\rho_*)}{F(\rho_*)}}\right)'\right]\frac{d\rho_*}{dn}\\
=\quad&\frac{s}{1-s}\frac{1}{n^2}\left[1-H_*^n(\rho_*)+H_*^n(\rho_*)\log H_*^n(\rho_*)\right]
\end{aligned}$$

The two terms in the bracket in the LHS are both negative as shown in the previous proof (first by Assumption 2 and second by Assumption 4). Therefore $\frac{d\rho_*}{dn}$ has the same sign as that of RHS. Since $H_*^n(\rho_*) < 1$,

$$H_*^n(\rho_*)(1 - \log H_*^n(\rho_*)) - 1 = (1 + x)/\exp(x) - 1 < 0$$

for $x = -\log H_*^n(\rho_*) > 0$ as $\exp(x) > 1 + x$ by Taylor expansion. Differentiation of (22) yields the same result as with s, so it is (26) with n replacing s, so $dH_*(\rho_*)/dn < 0$. Furthermore,

$$\begin{aligned} d(1 - H_*^n(\rho_*))/dn &= -H_*^n(\rho_*)\log H_*(\rho_*) - nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{dn} \\ &= -H_*^{n-1}(\rho_*)\left(H_*(\rho_*)\log H_*(\rho_*) + n\frac{dH_*(\rho_*)}{dn}\right) > 0. \end{aligned} \tag{28}$$

In summary thus far, $d\rho_*/dn > 0$, $dH_*(\rho_*)/dn < 0$, and $d(1 - H_*^n(\rho_*))/dn > 0$.

Next, we show that $dH_*(v)/dn < 0$ for all v. First, $\left[1 + \frac{s}{1-s}\frac{1}{n}(1 - H_*^n(\rho_*))\right]$ is decreasing because its change equals

$$\left(\frac{1}{F(\rho_*) + \rho_* f(\rho_*)}\right)' \frac{d\rho_*}{dn}$$

which is negative, as the first term is negative and the second term positive. Therefore, $H_*(v)$ decreases for all $v < \rho_*$ as n increases. Next, differentiate (27) with respect to n, the LHS equals ($x \equiv H_*(v)$),

$$\left[1 + \frac{s}{1-s}H_*^{n-1}(v)\right]\frac{dx}{dn} - \frac{s}{1-s}\frac{1}{n^2}\left(H_*^n(v) - H_*^n(v)\log H_*^n(v)\right).$$

The RHS equals the derivative of

$$\frac{s}{1-s}\frac{1}{n} - (1 - F(v))\left[1 - \frac{s}{1-s}\frac{1}{n}(1 - H_*^n(\rho_*))\right]$$

which is

$$\frac{s}{1-s}\left[-\frac{1}{n^2} + (1 - F(v))\, d\left(\frac{1}{n}(1 - H_*^n(\rho_*))\right)/dn\right]$$

Rearrange the terms, then equals

$$\begin{aligned} \left[1 + \frac{s}{1-s}H_*^{n-1}(v)\right]\frac{dx}{dn} &= -\frac{s}{1-s}\frac{1}{n^2}\left[1 - H_*^n(v) + H_*^n(v)\log H_*^n(v)\right] \\ &\quad + \frac{s}{1-s}(1 - F(v))\, d\left(\frac{1}{n}(1 - H_*^n(\rho_*))\right)/dn < 0 \end{aligned}$$

where the first term being negative follows from (28) and the second term being negative follows from $\left[1+\frac{s}{1-s}\frac{1}{n}\left(1-H_*^n\left(\rho_*\right)\right)\right]$ decreasing in n.

Finally, the buyer's utility decreases and the seller's revenue increases because the WTP increases for all buyers of different values. $\square$

Proof of Proposition 7. Change in δ does not affect the equilibrium reserve price, so the equilibrium probability of sale and stationary value distribution are not affected either. The buyer's utility increases because the sequential market becomes more buyer-friendly: directly by (16), increase in δ increases the integral. The seller's revenue decreases as each buyer's WTP decreases and the buyer composition does not change. The reduction in WTP is by (21),

$$w_*\left(v\right)=v-\mathbf{1}_{v\geq\rho_*}\int_{\rho_*}^{v}\frac{H_*^{n-1}\left(z\right)}{\left(\frac{1}{\delta s}-1\right)+H_*^{n-1}\left(z\right)}dz.$$

$\square$

In the equilibrium, for a cost c seller facing n symmetric buyers whose WTP is determined by $w_*\left(\cdot\right)$, the profit-maximizing auction is $\mathrm{A}\left(\rho_*\right)$ where ρ_* is determined by

$$\rho_*-\frac{1-H_*\left(\rho_*\right)}{h_*\left(\rho_*\right)}=0$$

and the expected profit from it is $\pi\left(\mathrm{A}\left(\rho_*\right)\right)=r\left(\mathrm{A}\left(\rho_*\right)\right)-c$, where

$$r\left(\mathrm{A}\left(\rho_*\right)\right)=\int_{\rho_*}^{1}\left[w_*\left(v\right)-\frac{1-H_*\left(v\right)}{h_*\left(v\right)}w_*'\left(v\right)\right]dH_*^n\left(v\right).$$

The profit-maximizing posted price is $\mathrm{P}\left(\phi_*\right)$ where ϕ_* is determined by

$$w_*\left(\phi_*\right)-\frac{1-H_*^n\left(\phi_*\right)}{\left(H_*^n\left(\phi_*\right)\right)'}w_*'\left(\phi_*\right)\quad=\quad 0$$

and the expected profit and revenue from it is

$$\pi\left(\mathrm{P}\left(\phi_*\right)\right)=r\left(\mathrm{P}\left(\phi_*\right)\right)=\left(1-H_*^n\left(\phi_*\right)\right)w_*\left(\phi_*\right).$$

Since a seller can only run an auction or post a price, he essentially makes a choice between $\mathrm{A}\left(\rho_*\right)$ and $\mathrm{P}\left(\phi_*\right)$, and he will choose the auction if and only if the auction generates higher expected profit, or equivalent, the equilibrium auction premium is greater than the cost,

$$\pi\left(\mathrm{A}\left(\rho_*\right)\right)\geq\pi\left(\mathrm{P}\left(\phi_*\right)\right)\Leftrightarrow\Delta_*=r\left(\mathrm{A}\left(\rho_*\right)\right)-r\left(\mathrm{P}\left(\phi_*\right)\right)\geq c,$$

and posts the optimal price otherwise. The cost $c_* \equiv \Delta_*$ seller is indifferent between the two mechanisms, and we call c_* the equilibrium cutoff cost such that measure $p_* = G(c_*)$ sellers with cost lower than c_* runs $\mathrm{A}(\rho_*)$ and measure $1 - p_*$ of the sellers with cost higher than c_* chooses $\mathrm{P}(\phi_*)$.

The total discounted payoff is the expected utilities from participation in auctions and posted prices,

$$\underline{u}_*(v) = \mathbf{1}_{v \geq \rho_*} G(c_*)\, u(v|\mathrm{A}(\rho_*)) + \mathbf{1}_{v \geq \phi_*}(1 - G(c_*))\, u(v|\mathrm{P}(\phi_*))$$

where the payoffs in the auction and the posted price are respectively,

$$\begin{aligned} u_*(v|\mathrm{A}(\rho_*)) &= \mathbf{1}_{v \geq \rho_*}\left[\int_{\rho_*}^{v} H_*^{n-1}(z)\, dw(z) + \delta s \underline{u}(v)\right] \\ u_*(v|\mathrm{P}(\phi_*)) &= \mathbf{1}_{v \geq \phi_*}\left[\frac{1}{n}\frac{1 - H_*^n(\phi_*)}{1 - H_*(\phi_*)}(v - w(\phi_*)) + \left(1 - \frac{1}{n}\frac{1 - H_*^n(\phi_*)}{1 - H_*(\phi_*)}\right)\delta s \underline{u}(v)\right] \\ &= \mathbf{1}_{v \geq \phi_*}\left[\frac{1}{n}\frac{1 - H_*^n(\phi_*)}{1 - H_*(\phi_*)}(w(v) - w(\rho)) + \delta s \underline{u}(v)\right] \end{aligned}$$

For value $v \in (\rho_*, \phi_*)$, the derivation is similar to the previous section and only differs by the extra $G(c_*)$ term.

$$u_*(v|\mathrm{A}(\rho_*)) = \frac{1}{1 - G(c_*)\delta s}\int_{\rho_*}^{v} H_*^{n-1}(z)\, dw(z)$$

and

$$w_*'(v) = \frac{1 - s\delta G(c_*)}{1 - s\delta G(c_*) + s\delta G(c_*) H_*^{n-1}(v)}.$$

For $v \geq \phi_*$, the calculation is more convoluted,

$$(1 - \delta s)\,\underline{u}(v) = \int_{\rho_*}^{v} H_*^{n-1}(z)\, dw_*(z) + \frac{1}{n}\frac{1 - H_*^n(\phi_*)}{1 - H_*(\phi_*)}(w_*(v) - w_*(\phi_*)).$$

In summary, an equilibrium $(M_*(\cdot), \sigma_*(\cdot,\cdot), H_*(\cdot), \mu_*(\cdot))$ is characterized by c_*, the cutoff cost, ϕ_*, the optimal posted type, ρ_*, the optimal reserve type, and the stationary value distribution H_*, where

1. Mass p_* of sellers with cost $c \leq c_*$ run the same optimal reserve price auction $\mathrm{A}(\rho_*)$ and the other mass $1 - p_*$ of sellers with cost $c > c_*$ uses the same optimal posted price mechanism $\mathrm{P}(\phi_*)$, where ρ_* and ϕ_* are determined by $\mathrm{MR}_*^{\mathrm{A}}(\rho_*) = \mathrm{MR}_*^{\mathrm{P}}(\phi_*) = 0$.

2. Value v buyer's equilibrium WTP is

$$w_*(v) = v - \frac{\delta s \left[\mathbf{1}_{v\geq\rho_*} p_* u(v|\mathrm{A}(\rho_*)) + \mathbf{1}_{v\geq\phi_*}(1-p_*) u(v|\mathrm{P}(\phi_*))\right]}{1-\delta s\left[1-\mathbf{1}_{v\geq\rho_*} p_* H_*^{n-1}(v) - \mathbf{1}_{v\geq\phi_*}(1-p_*)\frac{1}{n}\frac{1-H_*^n(\phi_*)}{1-H_*(\phi_*)}\right]}$$

3. Stationary value distributions h_* and H_* are characterized by (2) and (3) with the expected losing probability function $l_*(\cdot)$,

$$l_*(v) \;=\; \begin{cases} 1 & v \leq \rho_* \\ p_*\left(1-H_*^{n-1}(v)\right) & v \in (\rho_*, \phi_*] \\ p_*\left(1-H_*^{n-1}(v)\right) + (1-p_*)\left(1-\frac{1}{n}\frac{1-H_*^n(\phi_*)}{1-H_*(\phi_*)}\right) & v \in (\phi_*, 1] \end{cases}$$

4. The equilibrium belief μ_* is

$$\mu_*\left(H_*(\cdot), (p\circ\mathrm{A}(\rho_*), (1-p)\circ\mathrm{P}(\phi_*)), \sigma_*(\cdot,\cdot)\right) = 1.$$

Proof of Proposition 9. Similar to the previous model, the equilibrium is

$$\begin{aligned} H_*(\rho_*) &= \left\{1+\frac{s}{1-s}\frac{1}{n}\left[(1-H_*^n(\rho_*))G(c_*) + (1-H_*^n(\phi_*))(1-G(c_*))\right]\right\}F(\rho_*) \\ &= \left[1+\frac{s}{1-s}\frac{1}{n}(1-H_*^n(\rho_*)) + \frac{s}{1-s}\frac{1}{n}(H_*^n(\rho_*)-H_*^n(\phi_*))(1-G(c_*))\right]F(\rho_*) \\ &\equiv \left[1+\frac{s}{1-s}\frac{1}{n}(1-H_*^n(\rho_*)) - \epsilon(\rho_*)\right]F(\rho_*) \end{aligned}$$

where $\epsilon(\rho_*)$ is strictly positive. As before, substitute in the equilibrium condition

$$H_*(\rho_*) = F(\rho_*)/(F(\rho_*)+\rho_* f(\rho_*)),$$

$$\frac{1}{F(\rho_*)+\rho_* f(\rho_*)} = 1+\frac{s}{1-s}\frac{1}{n}\left(1-\left(\frac{F(\rho_*)}{F(\rho_*)+\rho_* f(\rho_*)}\right)^n\right) - \epsilon(\rho_*). \tag{29}$$

Compared with (25), the first two terms of RHS are increasing, but with the last term, the curve shifts down and intersects the LHS at a bigger ρ_* than before. However, bigger ρ_* means smaller $H_*(\rho_*)$ and bigger $1-H_*^n(\rho_*)$. Therefore, sale probability and efficiency increases for the auctions, but it increases when there are sellers switching to posted prices which have sale efficiency $1-H_*^n(\phi_*)$, so the total effect is ambiguous. However, more posted price mechanisms bring more allocative inefficiency. □