Generating Tractable Cubic Cost Functions

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Abstract

Classes in microeconomics typically use cubic cost functions, because they can exhibit marginal costs that fall as output increases to some efficient level, and then rise thereafter. Cubic cost functions embody economies of scale, making it easy to illustrate that concept with quadratic average cost curves. However, designing problems with cubic cost functions is harder than it looks, because well behaved functions must meet several mathematical and economic restrictions. Yet as instructors develop more online assignments and exam questions, they face the need to produce varied problems that support the same learning objectives. This article explains the restrictions needed to generate well behaved cubic cost functions. It proceeds to show how to generate random parameters for well-behaved, cubic cost functions for problems that meet common student learning objectives. An associated workbook contains the algorithms described here.

Motivation

How a manager maximizes profit is central to courses in microeconomics and managerial economics. On the output side, this entails understanding unit costs. Classic textbook presentations portray a cubic cost function that exhibits marginal costs falling as output increases to some efficient level, and then rising after that. Cubic cost functions embody economies of scale, making it easy to illustrate that concept with quadratic average cost curves.

Many instructors (certainly these two!) have discovered that generating cubic cost functions is harder than it looks. A number of standard assumptions used in microeconomic theory constrain the set of parameters that can generate valid cubic cost functions. Davis (2014) presents a set of three similar looking cubic cost functions and explains why only one of them is economically sound. Because finding valid cubic cost functions is tricky, many textbooks offer problems based on quadratic cost functions with linear marginal cost curves—in spite of the conceptual appeal of cubic cost functions (Baye and Prince 2017; Bernheim and Whinston 2014; McGuigan, Moyer, and Harris 2014). If instructors do employ cubic cost functions, they typically identify one or two well behaved examples and simply tweak the parameters each time they need a new assignment or exam question. Not an ideal system, but it works—so long as one or two well behaved functions is sufficient.

As communications technology has enabled test takers to share information faster, the need to generate multiple formulations of the same problem has grown. In 2020, cloning cost functions from one or two exemplars ceased to be sufficient. The global shift to online teaching prompted by the COVID-19 pandemic meant that virtually all university economics instructors faced a learning environment familiar to those who have long taught online: Students frequently communicate with peers when completing assignments and taking exams. Instructors can respond by varying the problems that different students see. One common approach is for the instructor to build a database of problems that are organized by question type, so that individual problems of the same type may be drawn at random for an online student quiz or examination. Quantitative microeconomic problems that are built from linear or quadratic functions can be varied with little difficulty while remaining theoretically consistent. But building tractable problems from cubic cost functions is more complicated, because the parameters must satisfy several criteria.

To illustrate the problem, consider the three total cost (TC) functions in Figure 1. Each shows TC as a function of the quantity (Q) of products made. The TC functions arise from these parameterizations of the cubic function:

A: $TC(Q) = 3600 + 10Q + 0.5Q^2 + 0.5Q^3$

B: $TC(Q) = 3600 + 118Q - 15Q^2 + 0.5Q^3$

C: $TC(Q) = 3600 + 177Q - 15Q^2 + 0.5Q^3$

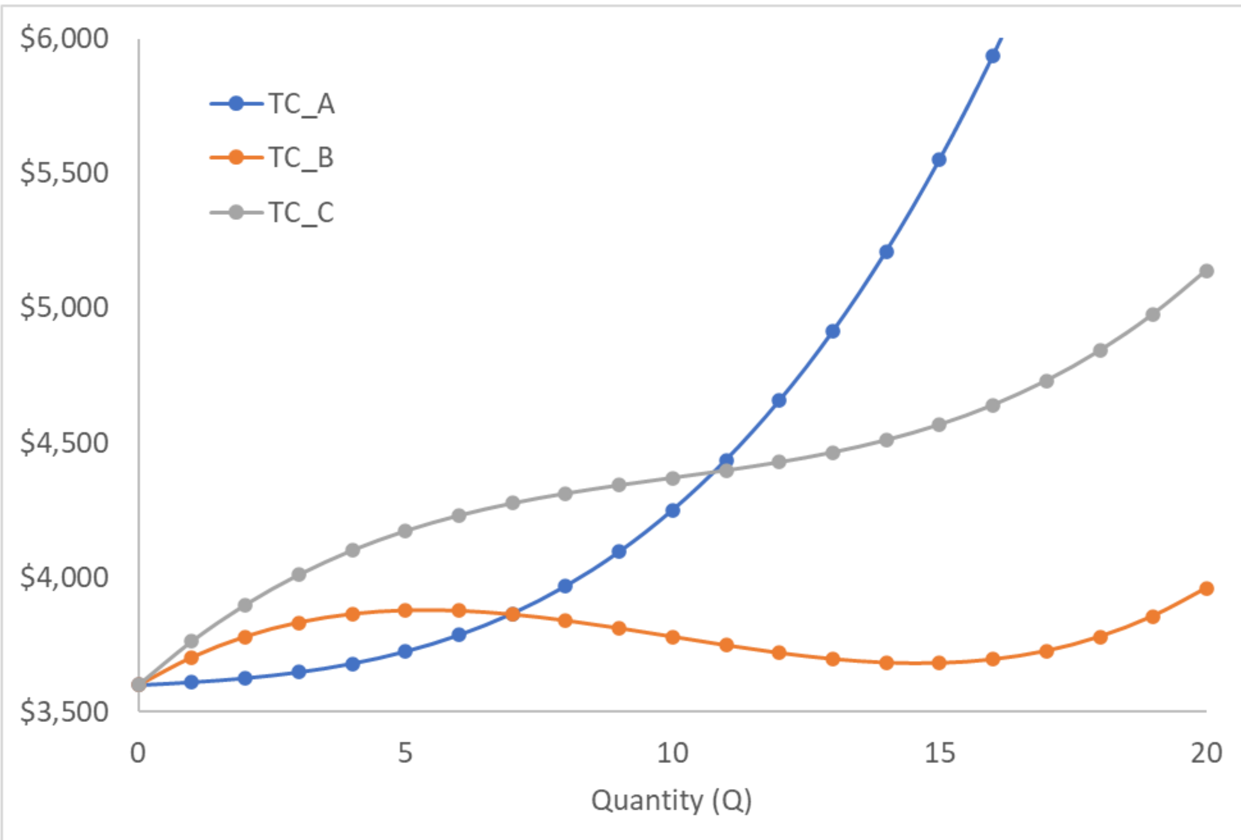

Figure 1: Total Cost Functions A, B, and C

All three functions have identical parameters for the fixed cost and cubic terms. All have positive costs. But only one suits the purpose of a cost function. TC_A has the most obvious flaw: marginal costs that only increase, making it unable to illustrate the intuition behind economies of scale. Function TC_B has costs that decline between 5 and 15 units of Quantity, which is economically nonsensical. TC_C is the only one of the three that has both increasing costs and an inflection point where marginal costs switch from declining to increasing.

The problem of how to generate parameters for tractable cubic cost functions is not new. Davis (2014) offers guidelines for quadratic functions and then presents a somewhat arbitrary set of parameters for cubic cost functions that incorporates output price (for reasons never made clear). In a more mathematically grounded article, Erfle (2014) derives signs and restrictions that must govern cubic cost functions. His article is accompanied by a helpful, downloadable spreadsheet where the user can modify cost function parameters and generate updated graphs of the total cost, marginal cost, average variable cost, and average total cost functions.

What has changed since those articles appeared is not the economic and mathematical principles shaping well-behaved cubic cost functions, but rather the need to generate multiple problems without difficulty. The objective of this article and the associated workbook is to provide an automated means of generating random cubic cost functions that conform with standard microeconomic cost theory and that students can readily solve.

Learning objectives and instructor goals

Central learning objectives for cost analysis in microeconomics and managerial economics courses are to understand how a firm maximizes profit from the output side, via cost functions. (A complementary approach focuses on the input side, via production functions that have a dual, mirror relationship to associated cost functions (Debertin, 2012).)

Cost functions measure the cost of production as a function of the quantity of products produced (Q). The central lessons of cost function analysis are that 1) the firm's profit is maximized when the marginal revenue (MR) earned from the last unit sold equals its marginal cost (MC) in the region where MC is rising, 2) in the short run, the firm's MR must at least cover its average variable cost (AVC), and 3) in the long run, the firm's MR must also cover its average total cost (ATC). An important secondary lesson is that firms may achieve economies of scale when the long-run ATC declines with the scale of output.

For instructors of firm-level cost analysis, the primary learning objective is to convey an understanding of the four ideas above. A second, intimately related, learning objective is to do so with numerical examples that fully illustrate the concepts. In this instance, "fully illustrate" means using cost functions that can generate sets of results that are consistent with standard microeconomic theory and supporting assumptions. As noted above, cubic cost functions do this best. There is no simpler way to illustrate economies of scale than with quadratic ATC and MC functions that are derived from cubic cost functions. This article aims to facilitate the teaching of these core ideas by identifying the properties of "well behaved" cubic cost functions and applying those properties to generate valid problems. By facilitating the task of generating problems, the algorithm presented here can produce many variations on the same basic problem. It also enables the instructor to identify versions with relatively simple parameters that facilitate clear solutions.

Mathematical properties of a "well-behaved" cubic cost function

Economists use the term "well behaved" as code that means a function meets several criteria for it to make sense economically. For cost functions, those criteria include:

1. All costs are positive, both variable costs and fixed costs, so $VC(Q) > 0$ and $FC > 0,\ \forall Q$.
2. Total cost is increasing in output quantity: $\dfrac{dTC}{dQ} > 0$, so $MC(Q) > 0, \forall Q$.

3. Production process exhibits economies of scale, so average total cost declines to a minimum at quantity $Q_{\min}^{ATC}$, after which it increases.

From criteria (2) and (3) follow two consequences:

4. Quadratic MC that is convex to the origin, so $\frac{d^2TC}{dQ^2} = \frac{dMC}{dQ} > 0$, which ensures that ATC declines to a minimum and rises thereafter;
5. The minimum MC (which is the inflection point on the cubic TC function) must lie in the region where both Q > 0 and TC > 0, which ensures that total cost is increasing in Q.

Combining these criteria with the mathematical properties of cubic polynomials, we can identify limits on the parameters of the cubic cost function. This cubic formulation follows standard economic ordering of terms, starting with the constant, as opposed to standard mathematical form, which starts with the cubic term.

As in the example above, let the total cost (TC) function be:

$$TC(Q) = \alpha + \beta Q + \gamma Q^2 + \delta Q^3 \tag{1}$$

The constant term, α, represents fixed cost (FC):

$$FC = \alpha \tag{2}$$

VC are represented by the remaining terms in the TC equation, and AVC is VC/Q:

$$VC(Q) = \beta Q + \gamma Q^2 + \delta Q^3 \tag{3}$$

$$AVC = \beta + \gamma Q + \delta Q^2 \tag{4}$$

MC is the first derivative of TC:

$$\frac{dTC}{dQ} = MC(Q) = \beta + 2\gamma Q + 3\delta Q^2 \tag{5}$$

Overlaying the economic criteria for a "well-behaved" cubic cost function with the definitions above, we can deduce several parameter restrictions:

1. $\alpha > 0$, from Criterion #1 that all costs are positive and $FC = \alpha$ (Eq. (2)).
2. $\delta > 0$, from Criterion #2 that TC is increasing in Q and δ is the coefficient on the cubic term, the largest in this polynomial.
3. $\gamma < 0$. This follows from Criterion #5 (and indirectly from Criteria #3 and #4) that the minimum of the MC curve must lie in the positive orthant. Differentiating Eq. (5) and setting the 2nd order condition for the minimum MC = 0 yields $Q_{\min}^{MC} = \frac{-\gamma}{3\delta}$. Since Q>0 and $\delta > 0$, therefore $\gamma < 0$.
4. $\beta > 0$. This too follows also from Criteria #4 and #5 that the MC minimum lies in the positive orthant. To guarantee positive roots for the MC quadratic (meaning no intersection with the Q axis), its discriminant must be negative: $\gamma^2 - 3\beta\delta < 0$ (Paul 2017), so $\gamma^2 < 3\beta\delta$. Since γ^2 and δ are positive, β must be also.
5. $\gamma^2 < 3\beta\delta$, as noted above, this discriminant condition satisfies Criterion #5. (Paul 2017).

Generating well-behaved cubic cost functions

The workbook accompanying this paper implements the restrictions above by randomly generating values for α, β, γ, and δ that meet the four sign restrictions above. Then a conditional function checks whether the β parameter meets the fifth restriction, restated to require $\beta > \frac{\gamma^2}{3\delta}$.

Although the five conditions above are all that are necessary to generate cubic cost functions that meet economic criteria, two other conditions are desirable. The first is that the functions should yield problem solutions in whole or simple rational numbers that students can readily interpret. Certain key relationships can guide parameter relationships to generate clean solutions. Three relationships describe the Q levels that minimize the MC, AVC, and ATC curves:

a. $\frac{dAVC}{dQ} = \gamma + 2\delta Q$, so at the AVC minimum: $Q_{\min}^{AVC} = \frac{-\gamma}{2\delta}$

b. From Eq (5) the MC-minimizing Q, $Q_{\min}^{MC} = \frac{-\gamma}{3\delta}$

c. $\frac{dATC}{dQ} = -\alpha Q^{-2} + \gamma + 2\delta Q$, so at the ATC minimum,

$Q_{\min}^{ATC} = \frac{-\gamma}{2\delta} + \frac{2\alpha\delta}{(Q_{\min}^{ATC})^2} = Q_{\min}^{AVC} + \frac{2\alpha\delta}{(Q_{\min}^{ATC})^2}$, showing how FC and Q cause the ATC-minimizing Q always to exceed the AVC-minimizing Q.

Taking advantage of the fact that $Q_{\min}^{MC}$ occurs at 2/3 of $Q_{\min}^{AVC}$ (Erfle, 2014), we can ensure whole-number MC and AVC minima by generating values for $\gamma = -6\delta k$, where k is a randomly drawn, scaling parameter.

The second desirable condition for economic problems is to identify a domain that can generate valid β parameters that do not get excessively large. One parsimonious way to do so is first to generate values for the γ and δ parameters, and then to draw a random β from the k-scaled interval $\left[\frac{\gamma^2}{3\delta}, \frac{k\gamma^2}{3\delta}\right]$, which simplifies to $[12\delta k^2, 12\delta k^3]$ if $\gamma = -6\delta k$. The instructor can choose the interval for random draws on k to suit the scale of interest.

We can generate whole-number ATC values by appropriate choice of the α parameter. The expression for Q_{min}^{ATC} can be rearranged as $\alpha = \gamma(Q_{\min}^{ATC})^2 + 2\delta(Q_{\min}^{ATC})^3$. Employing a reverse engineering approach to find α, we can first select a valid random value for Q_{min}^{ATC}, and then generate the α parameter that will yield this Q_{min}^{ATC}. Recognizing that Q_{min}^{ATC} must exceed Q_{min}^{AVC} we draw a random integer for Q_{min}^{ATC} from the interval $[(Q_{min}^{AVC}+1),(2Q_{min}^{AVC})]$. Then plug it into the equation at the beginning of this paragraph.

The workbook accompanying this paper facilitates generating sets of random parameters that meet the criteria for valid cost functions. It allows the instructor to adjust domains for random parameter draws but offers default domains. For the cubic (δ) parameter, the default domain is [0.3, 1.0] in increments of 0.1, in recognition of the strong exponential effect from values greater than one. For the scaling (k) parameter, the default domain is integers in [1,20], with the minimum value required to satisfy the formula for drawing random values of β. After Q_{min}^{ATC} is drawn, the fixed cost (α) parameter is calculated from the equation above.

In order to meet the learning objectives identified early in this note, instructors typically wish for students to answer the questions: 1) What is the profit maximizing level of Q? 2) Should the firm stay in business in the short run? 3) Should the firm stay in business in the long run?, and 4) How can you tell? To accompany randomly generated cost function parameters, the associated workbook includes a table of derived TC, MC, AVC, and ATC values. It graphs the functions and calculates the Q values that minimize the derived MC and AVC functions. These tools are intended to aid the instructor in constructing workable problems that answer the questions above.

To conclude, this teaching resource note summarizes the restrictions needed to generate cubic cost functions that are both economically valid and readily solved by students. It further introduces an associated workbook that generates well-behaved, cubic cost functions with supporting information to assist instructors in building cost analysis problems for courses related to managerial economics and intermediate microeconomics.

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