Bargaining and Reputation with Ultimatums

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Abstract

This paper studies pre-trial bargaining between two players who want to divide a unit pie. Each player is either (i) justified to demand a fixed share, never accepting any offer below that, or (ii) unjustified to demand any share, nonetheless wanting as a big share of the pie as possible. A justified player receives evidence justifying his demand according to a Poisson process and lets the court settle the conflict in his favor as soon as he receives evidence. At any instant, an unjustified player can either concede to the opponent or send an ultimatum to let the court settle the conflict; he wins if the opponent backs down and loses if the opponent agrees to a trial. We study the equilibrium of the game when the opportunity to send an ultimatum is available to neither player (Abreu and Gul, 2000), to one player, or to both players.

Several interesting results arise. First, when evidence for justified players arrives sufficiently slowly, the rate at which an unjustified player sends an ultimatum may be non-monotonic in time: at first both players send an ultimatum at a positive rate, then one player sends at a positive rate and the other player does not send, and at last both players do not send and resort to a war of attrition. Second, when evidence arrives sufficiently quickly and players are sufficiently unlikely to be justified, players cannot build up their reputations and inefficient delay in bargaining occurs. Third, a justified player strictly prefers the presence of the court, but an unjustified player strictly prefers not to have the court, because it destroys his or her possibility of pretending to be justified.

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1 Introduction

This paper extends the two-player bargaining model of Abreu and Gul (2000) by allowing players to send an ultimatum to the opponent to end the bargaining process.

Namely, we consider the following setting. Two players divide a unit pie. Each player is either (i) justified to demand a fixed share of the pie (corresponding to a behavioral type in Abreu and Gul (2000)), or (ii) unjustified to demand any share but nonetheless wanting as a big share of the pie as possible (corresponding to a rational type in Abreu and Gul (2000)). Each player announces his or her demand sequentially at the beginning of the game. After announcing a demand, each player can hold on to the announced demand or give in to the opposing player's demand at any time. In the baseline bargaining model without ultimatums (Abreu and Gul, 2000), the game ends only when one player gives in to the other player. In our extensions, opportunities to challenge the opponent and end the bargaining process arise periodically. Think of these opportunities as opportunities to resolve the conflict in an arbitration court. A justified player always uses the opportunity to challenge the opposing side, but an unjustified player may use the opportunity strategically. The opponent, upon being challenged, must respond by seeing the challenge or giving in the challenger's demand. A justified challenger always wins in the court against an unjustified defendant in the court, but an unjustified challenger loses in the court against a justified defendant as well as against an unjustified defendant.

In the baseline bargaining model without ultimatums, the equilibrium bargaining dynamics and reputation dynamics are quite simple. There is a unique sequential equilibrium. After players announce their demands, at most one player concedes with a positive probability at time 0. Afterwards, both players concede at a constant rate, and their reputations – opponent's beliefs about a player being justified – increase at a constant rate until both players' reputations reach 1 at the same time at which point no unjustified player is left in the game and justified players continue to hold on to their demands. As the probabilities of justification tend to zero, the limit payoffs depend on the impatience factors only, as in Rubinstein (1982). A more patient unjustified player receives a higher payoff.

Because of the additional possibility to send ultimatums, the equilibrium bargaining dynamics as well as the equilibrium reputation dynamics are much richer than those in the baseline bargaining model without ultimatums. Consider first the setting in which only player 1 has the opportunity to send ultimatums. With sufficiently small initial reputations, the bargainers can experience three bargaining phases with different bargaining and reputation dynamics. In equilibrium, reputations always increase. In the first strategy phase, an unjustified player 1 mixes between challenging and not challenging when a challenge opportunity arrives and an unjustified player 2 mixes between responding and giving in to a challenge. In the second phase, an unjustified player 1 challenges whenever an opportunity presents and an unjustified player 2 never responds to a challenge but concedes at a higher rate than in the baseline model. In the third phase, because player 2's reputation is sufficiently high, an unjustified player 1 never challenges and simply concedes at a constant rate as in the baseline model. Both players' reputations reach 1 at the same time. The players either go through all three phases in equilibrium or player 1 transitions from sometimes challenging to not challenging at all immediately without going through the always-challenging phase. For sufficiently low frequency of challenge arrivals, the result that the limit payoffs depend on the impatience factors continues to hold. For sufficiently high frequency of challenge arrivals though, the limit payoffs depend on player 1's frequency of the challenge arrival and player 2's impatience factor: the higher the frequency of the arrival of challenges, the lower the limit payoff of player 1 is. In other words, an unjustified player 1 does not prefer to have the challenge opportunity, as it limits his commitment power of continuing to hold on to his demand. The challenge opportunity – the possibility to go to the court – helps to separate the justified from the unjustified.

Finally, we consider the bargaining problem with two-sided ultimatums: it is possible for both sides to take the conflict to the court. When players' reputations are relatively low and the challenge opportunities arrive sufficiently frequently, namely, when the rate of challenge arrival is greater than the equilibrium Abreu-Gul concession rate in the baseline model, reputations decrease in equilibrium and players cannot build up their reputations at all! Players mix between conceding and not conceding if no challenge opportunity arises, mix between challenging and not challenging if a challenge opportunity arises, and mix between seeing and not seeing a challenge when they are being challenged. The equilibrium may not be unique, however, as players can freely concede at time 0. When the players' reputations are sufficiently high or the challenge opportunities arrive sufficiently infrequently, equilibrium exists uniquely, reputations build up and players experience in general three phases: both players mix between challenging and not challenging, one player challenges and the other player does not, and neither player challenges. The limit payoffs again depend on the challenge arrival rates instead of discount rates when challenge arrival rates exceed the discount rates.

The paper contributes to the growing literature of reputational bargaining. In contrast to Fanning (2016) which studies reputational bargaining with exogenous deadlines, this paper can be viewed as studying reputational bargaining with endogenously chosen deadlines. Compared to Fanning (2018a) with a mediator neither player needs to obey, we have an arbitrator both players need to obey. In addition, the insights generated are also related to bargaining with outside options (Atakan and Ekmekci, 2013; Chang, 2016; Hwang and Li, 2017; Fanning, 2018b). This setup also has real-world implications to bargaining situations involving final-order arbitration. This model sheds light on the negotiation stage between two parties when they have a chance to have a court or any mediator to make an arbitration.

2 Baseline Model: Bargaining without Ultimatums

We start with the baseline model in which there is no challenge opportunity for either player. Abreu and Gul (2000) set up and solved this model. In this section, we review their setup and solutions to prepare for our subsequent extensions.

Two players – a male player 1 and a female player 2 – divide a unit pie. Each player $i = 1,2$ is either (i) justified to demand a fixed share, never accepting any offer below that share, or (ii) unjustified to demand a share, nonetheless wanting as a big share of the pie as possible. The prior probability of player i being justified is z_i. The two players announce their demands sequentially at the beginning of the game. A player's demand can be any a_i from the pre-determined set A_i, $i = 1,2$. Player i's distribution of demands conditional on being justified is known and characterized by probability distributions $\pi_i(\cdot)$. After the demands are announced, each player can either hold on to his or her demand, or give in to the other player's demand at any time. The game ends only when one player gives in to the other player. Time is continuous. Each player i discounts with rate r_i.

A bargaining game without an ultimatum is described by the two players' initial prior probabilities of being justified, conditional distributions of justified types, and discount rates: $B_0(\pi_1(\cdot), z_1, \pi_2(\cdot), z_2 | r_1, r_2)$, where the subscript 0 denotes that neither side can challenge the other side throughout the game. We denote a bargaining game without an ultimatum and with single justified types simply by $B_0(a_1, z_1, a_2, z_2)$.

Unjustified players' strategies in the game $B_0(\pi_1(\cdot), z_1, \pi_2(\cdot), z_2)$ are described as follows. An unjustified player 1 demands each $a_1 \in A_1$ with probability $\sigma_1(a_1)$, and upon seeing player 1's demand a_1, an unjustified player 2 either accepts player 1's demand with probability $\sigma_2(0|a_1)$ or demands $a_2 \in A_2$ with probability $\sigma_2(a_2|a_1)$. Whenever $a_1 + a_2 > 1$, the game does not end after players announce their demands. Observing each other's demand, each unjustified player i chooses the probability $Q_i(t|a_1, a_2)$ that he or she concedes to player j by time t (inclusive). $Q_1(0|a_1, a_2)$ may be strictly positive and represents the probability that player 1 may concede immediately to player 2's counter-offer a_2. Without loss of generality, let $Q_2(0|a_1, a_2) = 0$ for all $a_2 > 1 - a_1$, because immediate concession at time zero and immediate acceptance of player 1's offer are equivalent. Let $\Sigma_1 = (\sigma_1(\cdot), Q_1(\cdot|\cdot,\cdot))$ and $\Sigma_2 = (\sigma_2(\cdot|\cdot), Q_2(\cdot|\cdot,\cdot))$ represent unjustified players' strategies. From now on, when no confusion arises, whenever we talk about a player's strategy, we mean an unjustified player's strategy.

The conditional probability of player 1 being justified immediately after he is observed demanding a_1 when an unjustified player 1 demands a_1 with probability σ_1 at time zero is

$$x(a_1, \sigma_1) = \frac{z_1 \pi_1(a_1)}{z_1 \pi_1(a_1) + (1 - z_1)\sigma_1}$$

Similarly, the conditional probability of player 2 being justified immediately after player 2 is observed demanding a_2 when an unjustified player demands a_2 with probability σ_2 at time zero is

$$y(a_2,\sigma_2) = \frac{z_2\pi_2(a_2)}{z_2\pi_2(a_2)+(1-z_2)\sigma_2}.$$

Suppose an unjustified player 1 chooses each a_1 with probability $\sigma_1(a_1)$, and an unjustified player 2 chooses each a_2 with probability $\sigma_2(a_2|a_1)$ after observing player 1's demand a_1 and concedes by each time t with probability $Q_2(t|a_1,a_2)$ after a_1 and a_2 are announced. An unjustified player 1's expected utility when he concedes at time t is[1]

$$\begin{aligned} u_1(t,\Sigma_2|a_1,a_2) &= \int_{\tau<t} e^{-r_1\tau}a_1 d[(1-y(a_2,\sigma_2(a_2|a_1)))Q_2(\tau|a_1,a_2)] \\ &\quad +[1-(1-y(a_2,\sigma_2(a_2|a_1)))Q_2(t|a_1,a_2)]e^{-r_1 t}(1-a_2) \\ &\quad +(1-y(a_2,\sigma_2(a_2|a_1)))[Q_2(t|a_1,a_2)-\lim_{\tau\uparrow t}Q_2(\tau|a_1,a_2)]e^{-r_1 t}\frac{a_1+1-a_2}{2}. \end{aligned} \tag{1}$$

The expected utility of an unjustified player 1 who never concedes is

$$u_1(\infty,\Sigma_2|a_1,a_2) = \int_{\tau\in[0,\infty)} e^{-r_1\tau}a_1 d[(1-y(a_2,\sigma_2(a_2|a_1)))Q_2(\tau|a_1,a_2)].$$

Note that if both players never concede, they both get a payoff of 0. An unjustified player 1's expected payoff conditional upon (a_1,a_2) being observed at time zero is

$$u_1(\Sigma_1,\Sigma_2|a_1,a_2) = \int_{\tau\in[0,\infty)} u_1(\tau,\Sigma_2|a_1,a_2)dQ_1(\tau|a_1,a_2).$$

Finally, an unjustified player 1's expected payoff from the strategy profile (Σ_1,Σ_2) is

$$\begin{aligned} u_1(\Sigma_1,\Sigma_2) &= \sum_{a_1\in A_1}\sigma_1(a_1)\Bigg\{a_1\Bigg[(1-z_2)\sigma_2(0|a_1)+z_2\sum_{a_2\le 1-a_1}\pi_2(a_2)\Bigg]+ \\ &\qquad \sum_{a_2>1-a_1}u_1(\Sigma_1,\Sigma_2|a_1,a_2)((1-z_2)\sigma_2(a_2|a_1)+z_2\pi_2(a_2))\Bigg\}. \end{aligned}$$

An unjustified player 2's payoffs can be similarly represented. Suppose player 1 chooses each a_1 with probability $\sigma_1(a_1)$, player 2 chooses each a_2 with probability $\sigma_2(a_2|a_1)$ after observing player 1' s demand a_1, and player 1 concedes according to $Q_1(\cdot|a_1,a_2)$ after a_1 and a_2 are observed. An unjustified player 2's expected payoff when she concedes at time t is

$$\begin{aligned} u_2(t,\Sigma_1|a_1,a_2) &= \int_{\tau<t} e^{-r_2\tau}a_2 d[(1-x(a_1,\sigma_1(a_1)))Q_1(\tau|a_1,a_2)] \\ &\quad +[1-(1-x(a_1,\sigma_1(a_1)))Q_1(t|a_1,a_2)]e^{-r_2 t}(1-a_1) \\ &\quad +(1-x(a_1,\sigma_1(a_1)))[Q_1(t|a_1,a_2)-\lim_{\tau\uparrow t}Q_1(\tau|a_1,a_2)]e^{-r_2 t}\frac{a_2+1-a_1}{2}. \end{aligned} \tag{2}$$

[1]We have assumed an equal split of the surplus in the event of simultaneous concessions. This tie-breaking assumption may be replaced by any rule without affecting the result, because in equilibrium simultaneous concessions arise with probability zero.

The expected payoff of an unjustified player 2 who never concedes is

$$u_2(\infty, \Sigma_1|a_1,a_2) = \int_{\tau\in[0,\infty)} e^{-r_2\tau} a_2 d[(1-x(a_1,\sigma_1(a_1)))Q_1(\tau|a_1,a_2)].$$

An unjustified player 2's expected payoff conditional upon (a_1,a_2) being observed at time zero is

$$u_2(\Sigma_1,\Sigma_2|a_1,a_2) = \int_{\tau\in[0,\infty)} u_2(\tau,\Sigma_1|a_1,a_2)dQ_2(\tau|a_1,a_2).$$

Finally, an unjustified player 2's expected utility from the strategy profile (Σ_1,Σ_2) is

$$\begin{aligned} u_2(\Sigma_1,\Sigma_2) &= \sum_{a_1\in A_1} [(1-z_1)\sigma_1(a_1)+z_1\pi_1(a_1)] \\ &\quad \times \left[(1-a_1)\sigma_2(0|a_1) + \sum_{a_2>1-a_1} u_2(\Sigma_1,\Sigma_2|a_1,a_2)\sigma_2(a_2|a_1)\right]. \end{aligned}$$

2.1 Single Justified Types of Both Players

First, suppose $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$. With a slight abuse of notation, we denote such a game by $B_0(a_1,z_1,a_2,z_2)$.

2.1.1 Strategies

Both players can either concede or stay at each instant. We solve for a fully mixed equilibrium in which both players mix between conceding and staying at each instant. An unjustified player $i=1,2$ is indifferent between conceding and staying if

$$\begin{aligned} 1-a_j &= e^{-r_i t}(1-a_j)(1-\lambda_j dt)+\lambda_j dt a_i \\ 1-a_j &= (1-\lambda_j dt + r_i dt)(1-a_j)+\lambda_j dt a_i \\ (1-a_j)(\lambda_j dt + r_i dt) &= \lambda_j(a_i+a_j-1) \\ \lambda_j &= r_i\frac{1-a_j}{a_i+a_j-1} \end{aligned}$$

where λ_j is player j's unconditional conceding rate. More elaborate arguments in Proposition 1 of Abreu and Gul (2000) show that the described conceding strategies must be the unique equilibrium strategies.

2.1.2 Reputation Dynamics

Now we solve for the equilibrium reputation dynamics. Let $\mu_i(t)$ represent the probability belief that player i is justified. By the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathscr{F}_t]$ where $\mathscr{F}_t$ represents the information set up to time t,

$$\begin{aligned} \mu_i(t) &= \lambda_i dt\cdot 0 + (1-\lambda_i dt)\mu_i(t+dt) \\ \mu_i(t+dt)-\mu_i(t) &= \lambda_i dt \mu_i(t+dt) \end{aligned}$$

Taking $dt \to 0$,

$$\mu_i'(t) = \lambda_i \mu_i(t).$$

Solving the differential equation,

$$\mu_i(t) = C_i e^{\lambda_i t}.$$

2.1.3 Equilibrium

Finally, two more conditions pin down the initial and terminal conditions of the reputation dynamics. First, at most one player concedes with a positive probability at time 0. Second, both players' reputations reach 1 at the same time. Therefore, if the prior that player i is justified is z_i, then it takes $T_i(z_i) \equiv -(\ln z_i)/\lambda_i$ of time for $\mu_i(t)$ to reach 1. The player who takes longer to reach reputation 1 will concede with a positive probability at time 0 so that time 0 reputation is increased and the time it takes to reach reputation 1 is shortened. In particular, the prior needs to be raised to $C_i = 1/\exp(\lambda_i T_j)$. To raise the time 0 reputation to C_i, player i concedes with probability

$$Q_i = 1 - \frac{1-C_i}{C_i}\frac{z_i}{1-z_i} = 1 - [\exp(\lambda_i T_i) - 1]\frac{z_i}{1-z_i} = \frac{1-\exp(\lambda_i T_i) z_i}{1-z_i}$$

so that

$$C_i = \frac{z_i}{z_i + (1-z_i)(1-Q_i)}.$$

In summary, the equilibrium strategy is as follows. At time 0, player i, wether commitment type or unjustified type, demands a_i. Immediately after learning demand, both players concede with probability Q_i. Then each player $i = 1,2$ concedes with a constant rate λ_i thereafter until time T_i is reached. At that moment, any unjustified player has exited the game. Player i's payoff in the equilibrium of the bargaining game $B_0(a_1, a_2, z_1, z_2)$ is

$$u_i(a_1, a_2, z_1, z_2) = (1-z_j)Q_j a_i + [1-(1-z_j)Q](1-a_j) = 1 - a_j + (1-z_j)Q_j(a_1 + a_2 - 1).$$

If player j does not concede with a positive probability at time 0, then player i gets his or her reservation payoff $1-a_j$. If $Q_j > 0$, then player i "wins" and gets a strictly higher payoff than $1-a_j$. The difference with $1-a_j$ is $[1-\exp(\lambda_j T_j) z_j](a_1 + a_2 - 1)$.

Take a numerical example. Suppose that there is a probability of $z_2 = 0.2$ that player 1 is a commitment type that demands a share of $a_1 = 0.8$ and there is a probability of $z_1 = 0.1$ that player 2 is a commitment type that demands a share of $a_2 = 0.6$; and both players discount with rate 0.05. In equilibrium, an unjustified player 1 concedes with probability 0.459 at time 0 so that his reputation rises to 0.316 at time 0 from prior 0.2, and an unjustified player 2 does not concede at time 0 so that her reputation stays at 0.1. At positive times, an unjustified player 1 concedes with rate $\lambda_1 = 0.025$ and an unjustified player 2 concedes with rate $\lambda_2 = 0.05$ until both players'

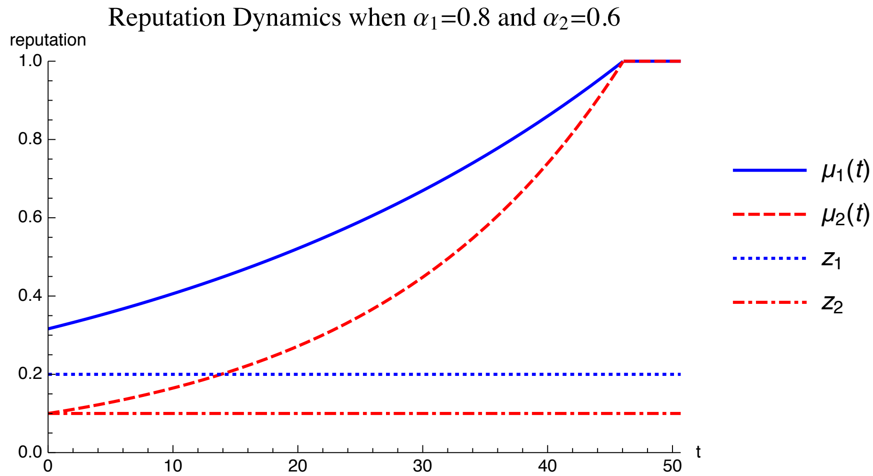


Figure 1: Equilibrium reputation dynamics in bargaining with no challenge opportunities and single justified types.

reputations reach 1 at the same time 46.0517. Both players' reputation dynamics on the path when neither concedes, $\mu_1(t) = 0.316\exp(0.025t)$ (in solid blue) and $\mu_2(t) = 0.1\exp(0.05t)$ (in dashed red), are illustrated in Figure 1.

2.2 Single Type of Player 1 and Multiple Types of Player 2

Suppose $|A_1| = 1$ and $|A_2| > 1$: a justified player 1 can only be one type and a justified player 2 can be multiple types. We denote such a game by $B_0(a_1, \pi_2(\cdot), x, z_2)$.

Denote by $B(a_1,x)$ the bargaining game in which a justified player 1's demand is a_1 and the probability player 1 is justified is x. If the game does not end at time 0, then player 2 has chosen some $a_2 > 1 - a_1$. Derived from the last section, after time 0, each player i concedes with rate

$$\lambda_i(a_i, a_j) = r_j \frac{1 - a_i}{a_1 + a_2 - 1}.$$

Determining player 2's equilibrium mimicking behavior suffices to determine full equilibrium strategy: define $\sigma_2(a_2)$ as the probability player 2 chooses a_2 and Q_2 the probability of conceding at time 0. Mimicking $a_2 \leq 1 - a_1$ is never optimal, so $\sigma_2(a_2) = 0$ for any $a_2 \leq 1 - a_1$.

We will show that there is a unique equilibrium of $B(a_1,x)$. If $x = 1$, then in equilibrium, $Q_2 = 1$. Assume $x < 1$ for the rest of the section. Define

$$T_1(a_1, a_2, x) = -\log x / \lambda_1(a_1, a_2)$$

and

$$T_2(a_1, a_2, y) = -\log y / \lambda_2(a_1, a_2).$$

They denote the times player i reaches reputation 1 if neither player concedes with positive probability at time 0.

Define player 2's reputation at time 0 when she plays a_2 with probability σ_2 as

$$y(a_2,\sigma_2) = \frac{z_2\pi_2(a_2)}{z_2\pi_2(a_2)+(1-z_2)\sigma_2}.$$

Given x, $\overline{\sigma}_2(a_1,a_2,x)$ is chosen so that the two players' reputations reach 1 at the same time: $\overline{\sigma}_2(a_1,a_2,x)$ is the unique solution to $T_1(a_1,a_2,x) = T_2(a_1,a_2,y(a_2,\sigma_2))$, which simplifies to

$$\overline{\sigma}_2(a_1,a_2,x) = \frac{z_2\pi_2(a_2)}{1-z_2}\frac{1-x^{\lambda_2(a_1,a_2)/\lambda_1(a_1,a_2)}}{x^{\lambda_2(a_1,a_2)/\lambda_1(a_1,a_2)}}.$$

Given y, $x^*(a_1,a_2,\sigma_2)$ is chosen so that the two players' reputations reach 1 at the same time: $x^*(a_1,a_2,\sigma_2)$ is the unique solution of x^* to $T_1(a_1,a_2,x^*) = T_2(a_1,a_2,y)$, which simplifies to

$$x^*(a_1,a_2,\sigma_2) = y(a_2,\sigma_2)^{\lambda_1(a_1,a_2)/\lambda_2(a_1,a_2)}.$$

To raise the time 0 reputation to

$$x^*(a_1,a_2,\sigma_2) = \frac{x}{x+(1-x)(1-Q_1(a_1,a_2,x,\sigma_2))}$$

when player 2 plays a_2 with probability σ_2, player 1 concedes at time 0 with probability

$$Q_1(a_1,a_2,x,\sigma_2) = 1-\frac{1-x^*(a_1,a_2,\sigma_2)}{x^*(a_1,a_2,\sigma_2)}\frac{x}{1-x}.$$

In equilibrium, both players' reputations must reach 1 at the same time, so $\sigma_2(a_2) \le \overline{\sigma}_2(a_1,a_2,x)$ (if player 1 does not concede with a positive probability at time 0, then player 2 mimics a_2 with probability $\overline{\sigma}_2(a_1,a_2,x)$, but if player 1 concedes with a positive probability at time 0, then player 2 mimics a_2 with a lower probability), and if player 2 mimics a_2 with probability $\sigma_2(a_2)$, then player 1 concedes at time 0 with probability $Q_1(a_1,a_2,x,\sigma_2(a_2))$.

Let $u_2(a_1,a_2,x,\sigma_2(a_2))$ denote player 2's utility if he mimics a_2 with probability $\sigma_2(a_2)$ in the game $B(a_1,x)$. Since player 2's payoff is a_2 when player 1 concedes at time 0 and is $1-a_1$ when player 1 does not concede at time 0,

$$u_2(a_1,a_2,x,\sigma_2) = [x+(1-x)(1-Q_1(a_1,a_2,x,\sigma_2)](1-a_1)+(1-x)Q_1(a_1,a_2,x,\sigma_2)a_2.$$

For any mimicking strategy $\sigma_2(\cdot)$, define

$$F_2(x,\sigma_2(\cdot)) = \min_{a_2:\sigma_2(a_2)>0} u_2(a_1,a_2,x,\sigma_2(a_2)).$$

For any equilibrium mimicking strategy $\sigma_2(\cdot)$, $\sigma_2(a_2)>0$ implies

$$u_2(a_1,a_2,x,\sigma_2(a_2)) \ge u_2(a_1,a_2',x,\sigma_2(a_2'))$$

for any $a_2' > 1-a_1$. Therefore, $\sigma_2(\cdot)$ is an equilibrium mimicking strategy for player 2 if and only if $\sigma_2(\cdot)$ solves

$$\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} F_2(x,\sigma_2(\cdot))$$

where

$$\Delta(a_1,x) = \{\sigma_2(\cdot)\in\Delta|\sigma_2(a_2)\leq\overline{\sigma}_2(a_1,a_2,x)\ \forall a_2>1-a_1 \text{ and } \sigma_2(a_2)=0 \text{ if } a_2\leq 1-a_1\}$$

and Δ is the set of probability distributions on $A_2\cup\{Q_2\}$. Player 1's equilibrium utility when player 2 plays $\sigma_2(\cdot)$ is

$$u_1(a_1,x) = (1-z_2)Q_2a_1 + \sum_{a_2\in A_2}[z_2\pi_2(a_2)+(1-z_2)\sigma_2(a_2)](1-a_2).$$

Two properties help us more simply determine the equilibrium conceding and mimicking strategies Q_2 and $\sigma_2(\cdot)$. First, whenever $\sum_{a_2>1-a_1}\overline{\sigma}_2(a_1,a_2,x)\leq 1$, $\sigma_2(a_2)=\overline{\sigma}_2(a_1,a_2,x)$ for all a_2 and $Q_2 = 1-\sum_{a_2>1-a_1}\overline{\sigma}_2(a_1,a_2,x)$. Second, if $\sigma_2(\cdot)$ is an equilibrium strategy, then $\sigma_2(a_2)>0$ implies $\sigma_2(a_2')>0$ for all $a_2'>a_2$. This is easily verified: if $\sigma_2(a_2')=0$, then $u_2(a_1,a_2',x,\sigma_2(a_2'))=(1-x)a_2'+x(1-a_1)$, while

$$\begin{aligned}
& u_2(a_1,a_2,x,\sigma_2(a_2)) \\
= \ & (1-x)Q_1(a_1,a_2,x,\sigma_2(a_2))a_2+[x+(1-x)(1-Q_1(a_1,a_2,x,\sigma_2(a_2)))](1-a_1) \\
\leq \ & (1-x)a_2+x(1-a_1)<(1-x)a_2'+x(1-a_1)=u_2(a_1,a_2',x,\sigma_2(a_2')).
\end{aligned}$$

The two properties together imply that we only need to check for $\sigma_2(\cdot)$ such that $\sigma_2(a_2')>0$ for all $a_2'\geq a_2$, for each $a_2\in A_2$. For example, when $A_2=\{0.8,0.6\}$, if $\overline{\sigma}_2(a_1,0.8,x)+\overline{\sigma}_2(a_1,0.6,x)>1$, equilibrium strategy $\sigma_2(\cdot)$ could only be (1) $0<\sigma_2(0.8)<1$, $0<\sigma_2(0.6)<1$, and $\sigma_2(0.8)+\sigma_2(0.6)=1$, or (2) $\sigma_2(0.8)=1$ and $\sigma_2(0.6)=0$, but could not be $\sigma_2(0.8)=0$ and $\sigma_2(0.6)=1$. We use this property to ease numerical calculation of the problem.

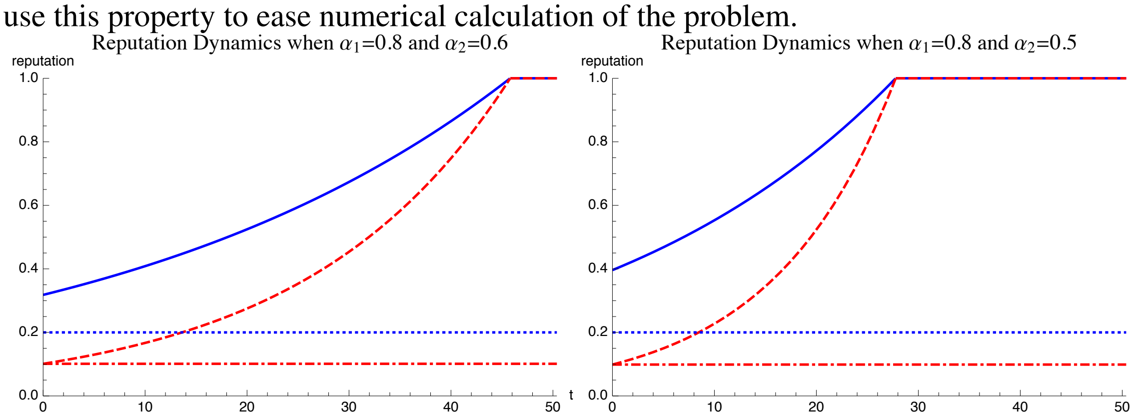


Figure 2: Equilibrium reputation dynamics in the bargaining game without ultimatums and with one type of player 1 and two types of player 2.

Take a numerical example. Suppose player 1 is justified with probability $z_1=0.2$ and player

2 is justified with probability $z_2 = 0.1$; a justified player 1 demands 0.8 for sure, and a justified player 2 demands either 0.7 or 0.5 with equal probability $1/2$; and two players discount with rate 0.05. In equilibrium, an unjustified player 2 demands 0.7 with probability 0.335, demands 0.5 with probability 0.665, and does not concede at time 0. Player 1 in response concedes at time 0 with probability 0.332 to player 2's demand of 0.7 and concedes at time 0 with probability 0.553 to player 2's demand of 0.5. When player 1 demands 0.8 and player 2 demands 0.7, player 1 concedes with rate 0.02 and player 2 concedes with rate 0.03; when player 1 demands 0.8 and player 2 demands 0.5, player 1 concedes with rate 0.0333 and player 2 concedes with rate 0.0833. Figure 2 illustrates the equilibrium reputation dynamics under the two scenarios of 0.8 versus 0.7 and 0.8 versus 0.5. Player 1's equilibrium utility is 0.4296 and player 2's equilibrium utility is 0.332799.

2.3 Multiple Justified Types of Both Players

Suppose $|A_1| > 1$ and $|A_2| > 1$. Suppose player 1 is unjustified with probability z_1 and conditional on being irrational, he demands $a_1 \in A_1$ with probability $\pi_1(a_1)$. Observing player 1's demand, player 2 chooses a demand $a_2 \in A_2$.

An unjustified player 1's strategy at the beginning of the game is $\sigma_1(\cdot)$ that specifies the probability $\sigma_1(a_1)$ he demands a_1. If player 1 plays $\sigma_1(\cdot)$, his posterior probability of irrationality conditional on choosing a_1 is

$$x(a_1, \sigma_1(a_1)) = \frac{z_1 \pi_1(a_1)}{z_1 \pi_1(a_1) + (1 - z_1)\sigma_1(a_1)}$$

and his expected utility is $u_1(a_1, x(a_1, \sigma_1(a_1)))$. His expected utility from choosing $\sigma_1(\cdot)$ is

$$\sum_{a_1 \in A_1} \sigma_1(a_1) u_1(a_1, x(a_1, \sigma_1(a_1))),$$

which equals

$$F_1(\sigma_1(\cdot)) = \min_{a_1 \in A_1 : \sigma_1(a_1) > 0} u_1(a_1, x(a_1, \sigma_1(a_1))).$$

It can be shown that, for different equilibrium strategies $(\sigma_1(\cdot), \sigma_2(\cdot))$ and $(\sigma_1'(\cdot), \sigma_2'(\cdot))$, the equilibrium outcomes are the same, so the equilibrium is unique.

Take a numerical example. Player 1 is justified with probability $z_1 = 0.2$ and player 2 is justified with probability $z_2 = 0.1$; a justified player 1 demands 0.8 or 0.7 with the same probability and a justified player 2 demands 0.6 or 0.5 with the same probability. In equilibrium, an unjustified player 1 demands 0.8 with probability 0.05545 and 0.6 with probability 0.94458. Facing player 1's demand of 0.8, player 2 concedes at time 0 with probability 0.333749, demands 0.6 with probability 0.166251, and demands 0.5 with probability 0.5; facing player 2's demand of either 0.6 or 0.5, player 1 does not concede. Facing player 1's demand of 0.7, player 2 concedes at time 0

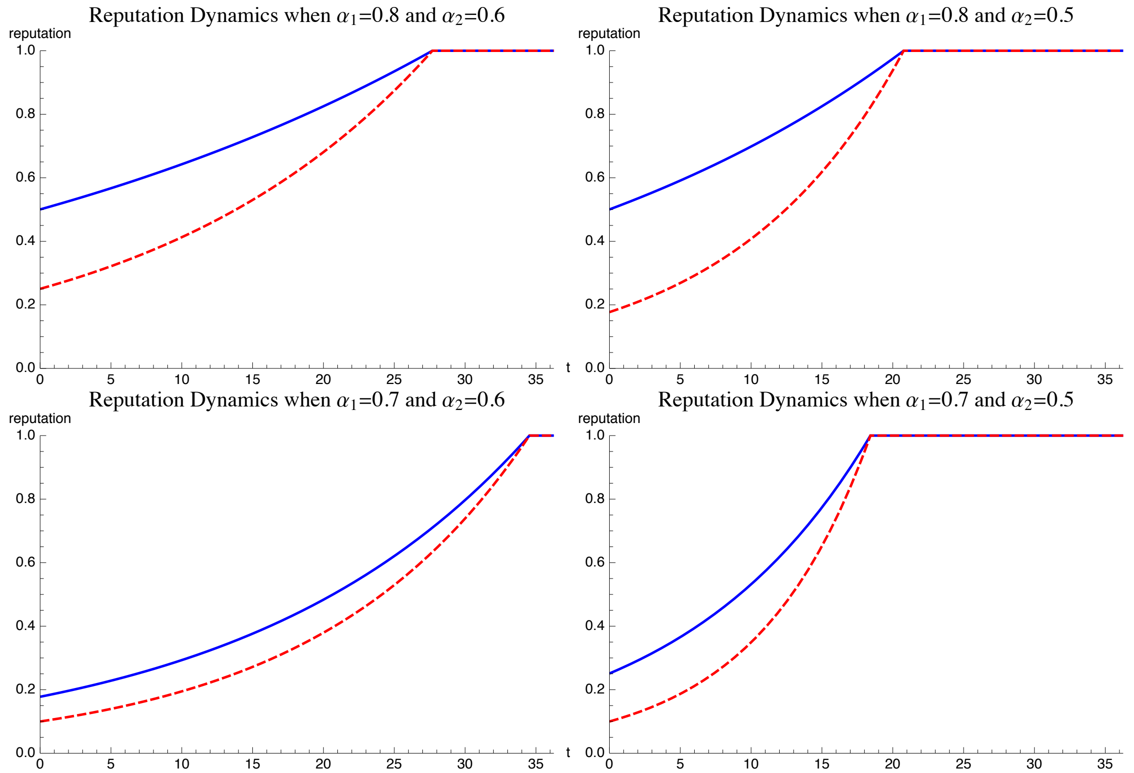


Figure 3: Equilibrium reputation dynamics in bargaining with no challenge opportunities: two justified types of player 1 and two justified types of player 2.

with probability 0.24202, demands 0.6 with probability 0.25798, and demands 0.5 with probability 0.5; facing player 2's demand of 0.6, player 1 concedes with probability 0.728 and facing player 2's demand of 0.5, player 1 concedes with probability 0.825. When player 1 demands 0.8 and player 2 demands 0.6, player 1 concedes with rate 0.025 and player 2 concedes with rate 0.05. When player 1 demands 0.8 and player 2 demands 0.5, player 1 concedes with rate 0.0333 and player 2 concedes with rate 0.0833. When player 1 demands 0.7 and player 2 demands 0.6, player 1 concedes with rate 0.05 and player 2 concedes with rate 0.0667. When player 1 demands 0.7 and player demands 0.5, player 1 concedes with rate 0.075 and player 2 concedes with rate 0.125. Figure 3 illustrates the reputation dynamics in the four separate bargaining games characterized by the two players' different demands.

3 Bargaining with Frictional One-Sided Ultimatums

A male player 1 and a female player 2 divide a unit pie. Each player $i = 1,2$ is either justified to demand a share of the pie never accepting any offer below that or unjustified to demand a share of the pie nonetheless wanting as a big share of the pie as possible. A justified player has favorable information supporting his or her demand, but an unjustified player has no evidence supporting his or her claim of the share. Players sequentially announce their demands at the beginning of the game. Each player i's demand can be any a_i from the pre-determined set A_i. The probability of player i being justified is z_i. The distributions are known and characterized by probability distributions $\pi_i(\cdot)$.

After both players announce their demands a_1 and a_2, if $a_1 + a_2 > 1$, then the game does not end. Time is continuous. Each player i discounts with rate r_i. At each instant $t \geq 0$, each player can decide to give in to the other player's demand or hold on to his or her demand. In addition, player 1 has a Poisson arrival of challenge opportunities with constant rate $g_1 > 0$. Player 1 can use the challenge opportunity, and if he challenges, player 2 can see or yield to player 1's challenge. If player 1 does not challenge when the opportunity arises, then the game continues and the current challenge opportunity disappears but the opportunity may arrive again in the future at the same rate. If player 1 challenges at time t, he incurs a cost c_1 right away and player 2 must respond to player 1's challenge. Player 2 may yield to the challenge right away and get $1 - a_1$, or may see the challenge by paying a cost c_2. The game ends either when one player gives in to the other player or when player 1 challenges player 2.

After player 2 sees the challenge, the shares of the pie are determined by the players' justified and unjustified types, as follows. If an unjustified player's opponents sees a challenge posed by an unjustified player, then the justified player loses and gets the payoff $1 - a_j$.[2]

In summary, a bargaining game $B_1(\pi_1(\cdot), z_1, \pi_2(\cdot), z_2 | r_1, r_2, c_1, c_2, g_1)$ is described by the two players' initial probabilities of being justified, conditional distributions of justified types, discount rates, costs of going to the court, as well as player 1's Poisson arrival rate of challenges. When both players have a single justified type, we simply denote the game by $B_1(a_1, z_1, a_2, z_2)$.

One application of the model is final-offer arbitration. Two parties announce their demands for a subject, like the wage of union workers, the division of a company after bankruptcy, or the salary of a baseball player (final-offer arbitrations are used frequently in firm-union bargaining, in bankruptcy cases, and in Major League Baseball). A justified player can have superior evidence supporting his or her claim, but needs time and effort to gather information about his or her claim and to appeal to the court. An unjustified player does not have proofs supporting his or her claim but nonetheless can credibly appeal to court. Whether or not a player could gather evidence and is justified is private information. While they gather evidence, they can negotiate with each other by repeatedly making offers to each other or choosing to let the case be settled by the court when possible.[3] A justified player can be done with collecting evidence at any moment, and as soon as he is done with collecting evidence and if the case has not been settled out of court, he submits his

[2]Finally, inconsequential to our results when justified players are non-strategic, assume that two justified players have the same chance of winning the case, so a justified player i's expected payoff is $(a_i + 1 - a_j)/2$.

[3]It is optimal for an unjustified player to continue to make the same demand, so the war-of-attrition structure of the bargaining game can be derived rather than assumed, just like in Abreu and Gul (2000). The addition to Abreu and Gul (2000) is an opportunity to appeal to the court or to any fair third-party arbitrator.

claim to the court. At that moment, the opposing player has to respond to the lawsuit, either by agreeing to the challenging player's demand out of court or by paying a cost to go on the court. In the court, an unjustified player loses to a justified player for sure and an unjustified challenger also loses to an unjustified defendant.

Players' strategies in the game are described as follows. An unjustified player 1 demands $a_1 \in A_1$ with probability $\sigma_1(a_1)$, and upon seeing player 1's demand, an unjustified player 2 either accepts player 1's demand with probability $\sigma_2(0|a_1)$ or demands $a_2 \in A_2$ with probability $\sigma_2(a_2|a_1)$. Whenever $a_1 + a_2 > 1$, the game does not end after players announce their demands. Observing each other's demand, each player i chooses the probability $Q_i(t|a_1,a_2)$ of conceding by time t (inclusive). $Q_1(0|a_1,a_2)$ may be strictly positive and represents the probability that player 1 may concede immediately to player 2's counter-offer a_2. Without loss of generality, let $Q_2(0|a_1,a_2) = 0$ for all $a_2 > 1-a_1$, because conceding at time zero and choosing immediate acceptance are equivalent for player 2. Furthermore, let $p_1(t|a_1,a_2)$ represent player 1's probability of challenging when a challenge opportunity arises at time t, and let $s_2(t|a_1,a_2)$ represent player 2's probability of seeing a challenge at time t. Let $\Sigma_1 = (\sigma_1(\cdot), Q_1(\cdot|\cdot,\cdot), p_1(\cdot|\cdot,\cdot))$ and $\Sigma_2 = (\sigma_2(\cdot), Q_2(\cdot|\cdot,\cdot), s_2(\cdot|\cdot,\cdot)))$ represent players' strategies, respectively.

The conditional probability of player 1 being justified immediately after he is observed demanding a_1 when an unjustified player 1 demands a_1 with probability σ_1 at time zero is

$$x(a_1,\sigma_1) = \frac{z_1\pi_1(a_1)}{z_1\pi_1(a_1) + (1-z_1)\sigma_1}$$

Similarly, the conditional probability of player 2 being justified immediately after player 2 is observed demanding a_2 when an unjustified player demands a_2 with probability σ_2 at time zero is

$$y(a_2,\sigma_2) = \frac{z_2\pi_2(a_2)}{z_2\pi_2(a_2) + (1-z_2)\sigma_2}.$$

Suppose player 1 chooses a_1 according to $\sigma_1(\cdot)$, and player 2 chooses a_2 according to $\sigma_2(\cdot|a_1)$ after observing player 1's demand a_1, concedes according to $Q_2(\cdot|a_1,a_2)$, and sees a challenge according to $s_2(\cdot|a_1,a_2)$. Player 1's expected utility when he challenges at time t is

$$\begin{aligned} v_1(t,\Sigma_2|a_1,a_2) \quad = \quad & -c_1 + [\mu_2(t,\Sigma_2|a_1,a_2) + (1-\mu_2(t,\Sigma_2|a_1,a_2))s_2(t|a_1,a_2)]\,(1-a_2) \\ & +(1-\mu_2(t,\Sigma_2|a_1,a_2))(1-s_2(t|a_1,a_2))a_1 \end{aligned}$$

where

$$\mu_2(t,\Sigma_2|a_1,a_2) = \frac{y(a_2,\sigma_2(a_2))}{1-(1-y(a_2,\sigma_2(a_2)))Q_2(t|a_1,a_2)}.$$

An justified player 1's expected utility when he challenges with probability $p_1(\tau)$ at each instant τ

before time t and concedes at time t is

$$
\begin{aligned}
& u_1(t, p_1(\cdot), \Sigma_2|a_1, a_2) \\
= & \int_{\tau<t} e^{-r_1\tau} v_1(\tau, \Sigma_2|a_1, a_2) d\left\{[1-(1-y(a_2, \sigma_2(a_2)))Q_2(\tau|a_1, a_2)]P_1(\tau, p_1(\cdot))\right\} \\
& + \int_{\tau<t} e^{-r_1\tau} a_1 d\left\{(1-P_1(\tau, p_1(\cdot)))[(1-y(a_2, \sigma_2(a_2)))Q_2(\tau|a_1, a_2)]\right\} \\
& +(1-P_1(t, p_1(\cdot)))(1-(1-y(a_2, \sigma_2(a_2)))Q_2(t|a_1, a_2))e^{-r_1 t}(1-a_2) \\
& +(1-P_1(\tau, p_1(\cdot)))(1-y(a_2, \sigma_2(a_2)))[Q_2(t|a_1, a_2) - \lim_{\tau\uparrow t} Q_2(\tau|a_1, a_2)]e^{-r_1 t}(1-a_2).
\end{aligned}
$$

where

$$
P_1(t, p_1(\cdot)) = 1 - \exp\left[-t\int_0^t g_1 p_1(\tau) d\tau\right].
$$

If player 1 challenges according to $p_1(\cdot)$ and never concedes, his expected payoff is

$$
\begin{aligned}
& u_1(\infty, p_1(\cdot), \Sigma_2|a_1, a_2) \\
= & \int_0^\infty e^{-r_1\tau} v_1(\tau, \Sigma_2|a_1, a_2)[1-(1-y(a_2, \sigma_2(a_2)))Q_2(\tau|a_1, a_2)] dP_1(\tau, p_1(\cdot)) \\
& + \int_0^\infty e^{-r_1\tau} a_1 (1-P_1(\tau, p_1(\cdot))) d[(1-y(a_2, \sigma_2(a_2)))Q_2(\tau|a_1, a_2)].
\end{aligned}
$$

We are assuming player 1 concedes when player 1 and player 2 concede simultaneously at time t, but this assumption does not affect the result because in equilibrium player 1 and player 2 simultaneously concede with probability zero. An unjustified player 1's expected utility given strategy profile (Σ_1, Σ_2) and observed demands a_1 and a_2 is

$$
u_1(\Sigma_1, \Sigma_2|a_1, a_2) = \int_0^\infty u_1(\tau, p_1(\cdot|a_1, a_2), \Sigma_2|a_1, a_2) dQ_1(\tau|a_1, a_2).
$$

Finally, an unjustified player 1's expected payoff from the strategy profile (Σ_1, Σ_2) is

$$
\begin{aligned}
u_1(\Sigma_1, \Sigma_2) = & \sum_{a_1\in A_1} \sigma_1(a_1)\left\{a_1\left[(1-z_2)\sigma_2(0|a_1) + z_2 \sum_{a_2\leq 1-a_1} \pi_2(a_2)\right] + \right. \\
& \left. \sum_{a_2>1-a_1} u_1(\Sigma_1, \Sigma_2|a_1, a_2)((1-z_2)\sigma_2(a_2|a-1) + z_2\pi_2\pi_2(a_2))\right\}.
\end{aligned}
$$

Let's now turn to an unjustified player 2's payoffs. Suppose player 1 chooses a_1 according to $\sigma_1(\cdot)$ and challenges according to $p_1(\cdot|a_1, a_2)$, and player 2 chooses a_2 according to $\sigma_2(\cdot|a_1)$. Player 2's expected utility from seeing a challenge at time t with probability s_2 is

$$
\begin{aligned}
w_2(t, s_2, \Sigma_1|a_1, a_2) = & (1-s_2)(1-a_1) \\
& + s_2[-c_2 + \mu_1(t, \Sigma_1|a_1, a_2)(1-a_1) + (1-\mu_1(t, \Sigma_1|a_1, a_2))a_2]
\end{aligned}
$$

where $\mu_1(t,\Sigma_1|a_1,a_2)$ is

$$\frac{x(a_1,\sigma_1(a_1))e^{-g_1t}}{x(a_1,\sigma_1(a_1))e^{-g_1t}+p_1(t|a_1,a_2)(1-x(a_1,\sigma_1(a_1)))(1-Q_1(t|a_1,a_2))(1-P_1(t,p_1(\cdot|a_1,a_2))}.$$

The expected payoff of an unjustified player 2 who sees a challenge with probability $s_2(\tau)$ at each time $\tau<t$ and concedes at time t is

$$\begin{aligned}
& u_2(t,s_2(\cdot),\Sigma_1|a_1,a_2) \\
= & \int_{\tau<t} e^{-r_2\tau} w_2(\tau,s_2(\tau),\Sigma_1|a_1,a_2) \\
& d\left[(1-x(a_1,\sigma_1(a_1)))(1-Q_1(\tau|a_1,a_2))P_1(\tau,p_1(\cdot|a_1,a_2))+x(a_1,\sigma_1(a_1))(1-e^{-g_1t})\right] \\
& +\int_{\tau<t} e^{-r_2\tau} a_2 d\left[(1-P_1(\tau,p_1(\cdot|a_1,a_2)))(1-x(a_1,\sigma_1(a_1))Q_1(\tau|a_1,a_2)\right] \\
& +(1-P_1(t,p_1(\cdot|a_1,a_2)))(1-(1-x(a_1,\sigma_1(a_1)))Q_1(t|a_1,a_2))e^{-r_2t}(1-a_1)+ \\
& +(1-P_1(t,p_1(\cdot|a_1,a_2)))(1-x(a_1,\sigma_1(a_1)))[Q_1(t|a_1,a_2)-\lim_{\tau\uparrow t}Q_1(\tau|a_1,a_2)]e^{-r_2t}a_2.
\end{aligned}$$

We are assuming player 1 concedes when players concede simultaneously but this assumption is innocuous because players concede at the same time with probability zero in equilibrium. The expected payoff of an unjustified player 2 who sees a challenges with probability $s_2(\tau)$ at each time $\tau<\infty$ and never concedes is

$$\begin{aligned}
& u_2(\infty,s_2(\cdot),\Sigma_1|a_1,a_2) \\
= & \int_{\tau\in[0,\infty)} e^{-r_2\tau} w_2(\tau,s_2(\tau),\Sigma_1|a_1,a_2) \\
& d\left[(1-x(a_1,\sigma_1(a_1)))(1-Q_1(\tau|a_1,a_2))P_1(\tau,p_1(\cdot|a_1,a_2))+x(a_1,\sigma_1(a_1))(1-e^{-g_1t})\right] \\
& +\int_{\tau\in[0,\infty)} e^{-r_2\tau} a_2 d\left[(1-P_1(\tau,p_1(\cdot|a_1,a_2)))(1-x(a_1,\sigma_1(a_1))Q_1(\tau|a_1,a_2)\right].
\end{aligned}$$

Her expected utility from the strategy profile (Σ_1,Σ_2) given a_1 and a_2 are observed is

$$u_2(\Sigma_1,\Sigma_2|a_1,a_2) \;=\; \int_0^\infty u_2(t,s_2(\cdot|a_1,a_2),\Sigma_1|a_1,a_2)dQ_2(\tau|a_1,a_2).$$

Finally, her expected utility from the strategy profile (Σ_1,Σ_2) is

$$\begin{aligned}
u_2(\Sigma_1,\Sigma_2) \;=\; & \sum_{a_1\in A_1}[(1-z_1)\sigma_1(a_1)+z_1\pi_1(a_1)] \\
& \times\left[(1-a_1)\sigma_2(0|a_1)+\sum_{a_2>1-a_1}u_2(\Sigma_1,\Sigma_2|a_1,a_2)\sigma_2(a_2|a_1)\right].
\end{aligned}$$

3.1 Single Justified Types of Both Players

To start, suppose each player can be of a single justified type: with probability z_1 player 1 is justified to demand $a_1>0.5$ and with probability z_2 player 2 is justified to demand $a_2>0.5$. Let $d\equiv a_i-(1-a_j)=a_1+a_2-1$ denote the conflicting difference between the two players.

$B\left(\{z_i, a_i, r_i, c_i\}_{i=1}^{2}, g_1, w\right)$ describes the one-sided challenge bargaining game with single justified types by players' prior justice probabilities z_1 and z_2, demands a_1 and a_2, discount rates r_1 and r_2, player 1's challenge arrival rate $g_1 > 0$, challenge costs c_1 and c_2, and a challenger's winning probability w.

3.1.1 Strategies

First, fixing players' beliefs about the opponent being justified, we solve for candidate equilibrium strategies of the game. Namely, given the opposing player's strategy and reputation at a moment, Lemma 1 characterizes player 1's equilibrium decision to challenge versus not to challenge when a challenge opportunity arrives and player 2's equilibrium decision to see a challenge versus to yield to a challenge, and Lemma 2 characterizes both players' equilibrium decisions to concede versus to stay.

Let $p_1(t)$ be an unjustified player 1's probability of challenging when a challenge opportunity arrives at time t, let $\lambda_i(t)$ be a player i's rate of conceding at time t so that an unjustified player's rate of conceding is $q_i(t) = \lambda_i(t)/[1-\mu_i(t)]$, and let $s_2(t)$ be an unjustified player 2's probability of seeing a challenge when she faces a challenge at time t. A player's optimal strategy depends on the opposing player's strategy as well as *reputation*, the probability of being justified. Let $\mu_i(t)$ denote player i's reputation when the game has not ended at time t.

Let $\underline{\mu}_1 \equiv 1 - \frac{c_2}{(1-w)d}$ and $\overline{\mu}_2 \equiv 1 - \frac{c_1}{d}$ represent key threshold reputations. An unjustified player 2 does not see a challenge from player 1 if player 1's reputation is sufficiently high: $\mu_1(t) > \underline{\mu}_1$ and an unjustified player 1 does not challenge player 2 if player 2's reputation is sufficiently high: $\mu_2(t) > \overline{\mu}_2$.

$\overline{\mu}_2 \leq \mu_2(t) \leq 1$	N	N
$0 < \mu_2(t) < \overline{\mu}_2$	S	A
	$0 < \mu_1(t) < \underline{\mu}_1$	$\underline{\mu}_1 \leq \mu_1(t) \leq 1$

Lemma 1. *The equilibrium strategies of challenging versus not challenging and seeing versus yielding to a challenge depend on players' reputations $\mu_1(t)$ and $\mu_2(t)$, as follows.*

(S). Suppose $\mu_2(t) \leq \overline{\mu}_2$ and $\mu_1(t) \leq \underline{\mu}_1$. An unjustified player 1 challenges with probability

$$p_1(t) = \frac{\mu_1(t)}{1-\mu_1(t)} \Big/ \frac{\underline{\mu}_1}{1-\underline{\mu}_1},$$

and if she faces a challenge, an unjustified player 2 sees the challenge with probability

$$s_2(t) = \frac{1}{1-w} \frac{\overline{\mu}_2 - \mu_2(t)}{1-\mu_2(t)}.$$

(A). Suppose $\mu_2(t) \leq \overline{\mu}_2$ and $\mu_1(t) \geq \underline{\mu}_1$. An unjustified player 1 always challenges and an unjustified player 2 always yields to a challenge.

(N). Suppose $\mu_2(t) \geq \overline{\mu}_2$. An unjustified player 1 does not challenge and an unjustified player 2 always yields to a challenge.

Lemma 2. *Equilibrium strategies of conceding versus not conceding at time t depend on players' reputations, as follows. Let $\lambda_i \equiv r_j(1-a_i)/d$. In any equilibrium,*

(S). Suppose $\mu_2(t) \leq \overline{\mu}_2$ and $\mu_1(t) \leq \underline{\mu}_1$. Player 1 concedes with constant rate λ_1 and player 2 concedes with constant rate λ_2.

(A). Suppose $\overline{\mu}_2 - \frac{\lambda_2}{g_1} < \mu_2(t) \leq \overline{\mu}_2$ and $\mu_1(t) > \underline{\mu}_1$. Player 1 concedes with constant rate λ_1 and player 2 concedes with rate

$$\lambda_2(t) = \lambda_2 - g_1\left[\overline{\mu}_2 - \mu_2(t)\right].$$

(N). Suppose $\mu_2(t) > \overline{\mu}_2$. Player 1 concedes with constant rate λ_1 and player 2 concedes with constant rate λ_2.

3.1.2 Reputation Dynamics

In the main text, assume $c_1 \geq wd$ and $c_2 < (1-w)d$ so that $\underline{\mu}_2 \leq 0$, $\overline{\mu}_2 < 1$, and $\underline{\mu}_1 < 1$. (When $w=0$, the assumption is equivalent to $c_1 \geq 0$ and $c_2 < d$.) There are potentially three distinct phases with different combinations of strategies and different reputation dynamics. They are (1) a sometimes-challenging sometimes-seeing (SS) phase in which an unjustified player 1 sometimes challenges and an unjustified player 2 sometimes sees a challenge, (2) an always-challenging never-seeing (AN) phase in which an unjustified player 1 always challenges and an unjustified player 2 never sees, and (3) a never-challenging never-seeing phase (NN) phase in which an unjustified player 1 never challenges and an unjustified player 2 never sees a challenge. Subsequently we describe the optimal strategies, reputation dynamics, and the duration of each of these three phases.[4],[5]

The following Bernoulli ordinary differential equation repeatedly appears in the characterization of reputation dynamics. We solve it here for subsequent convenient reference.

[4]When $c_1 > wd$, an always-challenging always-seeing phase in which player always challenges and player 2 always sees a challenge does not exist. When $c_1 < wd$, for sufficiently small initial probability of being justified, there is in addition an always-challenging always-seeing (AA) phase in which player 1 always challenges and player 2 always sees a challenge. There are multiple equilibria with addition of this phase; for sufficiently small initial probabilities, there might be an equilibrium in which player 1 always challenges and player 2 always sees a challenge without conceding at any time so that players' reputations stay constant. It is related to the inefficient bargaining outcome described in other papers such as Chang (2016).

[5]When $c_2 > (1-w)d$, the dynamics is rather trivial, as an justified player 2 would never see a challenge, so that an justified player would always challenge player 2 if player 2's reputation is relatively low and would never challenge player 2 if player 2's reputation is relatively high.

Lemma 3. *The solution to the following ordinary differential equation*

$$\mu'(t) = A\mu(t) + B\mu^2(t)$$

given $\mu(t^0) = \mu^0$ *is*

$$\mu(t;t^0,\mu^0,A,B) = 1 \Big/ \left[\left(\frac{1}{\mu^0} + \frac{B}{A} \right) \exp(-A(t-t^0)) - \frac{B}{A} \right]$$

if $A \neq 0$ *and*

$$\mu(t;t^0,\mu^0,0,B) = 1 \Big/ \left[-B(t-t^0) + \frac{1}{\mu^0} \right]$$

if $A = 0$.

If $\mu^0 > -A/B$, *then* $\mu'(t) > 0$ *for all* $t \geq t^0$, *and the time length it takes to reach reputation* $\mu > \mu^0$ *from* μ^0 *is*

$$t(\mu;\mu^0,A,B) = \frac{1}{A} \ln \left(\frac{\frac{1}{\mu^0} + \frac{B}{A}}{\frac{1}{\mu} + \frac{B}{A}} \right).$$

Note that, when there is no challenge opportunity, the ordinary differential equation $\mu_i'(t) = \lambda_i \mu_i(t)$ that determines the constant conceding rate in Abreu and Gul (2000) is $\mu(t;\cdot,\cdot,A,B)$ when $A = \lambda_i$ and $B = 0$. With the addition of a challenge opportunity by player 1, reputation building incorporates an additional non-zero square term $\mu_i^2(t)$, besides a possible change in the linear term $\mu_i(t)$.

Sometimes-Challenging Phase

When players' reputations are sufficiently low (but not too low for player 1 when the challenge arrival rate is high), an unjustified player 1 mixes between challenging and not challenging and an unjustified player 2 mixes between seeing a challenge and yielding to a challenge, and they concede at their respective constant Abreu-Gul rates. Players' reputations build up over time. Player 2's reputation follows Abreu-Gul's but player 1's is more complicated because of the presence of the challenge opportunity. The phase lasts until either player 1 reputation reaches the threshold.

Lemma 4 (S). *Suppose at time* t^0, $(1 - \frac{\lambda_1}{g_1})\underline{\mu}_1 < \mu_1(t^0) = \mu_1^0 < \underline{\mu}_1$ *and* $\mu_2(t^0) = \mu_2^0 < \overline{\mu}_2$. *For any time* t *between* t^0 *and* $t^0 + t^S(\mu_1^0, \mu_2^0)$, *an unjustified player 1 challenges with probability*

$$p_1^S(t) = \frac{\mu_1^S(t;t^0,\mu_1^0)}{1 - \mu_1^S(t;t^0,\mu_1^0)} \Big/ \frac{\underline{\mu}_1}{1 - \underline{\mu}_1},$$

and concedes with constant rate $\lambda_1^S(t) = \lambda_1$, *and an unjustified player 2 sees a challenge with probability*

$$s_2^S(t) = \frac{1}{1-w} \frac{\overline{\mu}_2 - \mu_2^S(t;t^0,\mu_2^0)}{1 - \mu_2^S(t;t^0,\mu_2^0)}$$

and concedes with constant rate $\lambda_2^S(t) = \lambda_2$. Player 1's reputation evolves according to

$$\mu_1^S(t;t^0,\mu_1^0) = \mu(t;t^0,\mu_1^0,\lambda_1 - g_1, g_1/\underline{\mu}_1),$$

player 2's reputation evolves according to

$$\mu_2^S(t;t^0,\mu_2^0) = \mu(t;t^0,\mu_2^0,\lambda_2,0),$$

the time it takes player 1's reputation to reach $\underline{\mu}_1$ from μ_1^0 is

$$t_1^S(\mu_1^0,\underline{\mu}_1) \equiv t(\underline{\mu}_1;\mu_1^0,\lambda_1 - g_1, g_1/\underline{\mu}_1),$$

and the time it takes player 2's reputation to reach $\overline{\mu}_2$ from μ_2^0 is

$$t_2^S(\mu_2^0,\overline{\mu}_2) \equiv t(\overline{\mu}_2;\mu_2^0,\lambda_2,0)$$

so that the duration of the phase counting from t^0 is $\min\left\{t_1^S(\mu_1^0,\underline{\mu}_1), t_2^S(\mu_2^0,\overline{\mu}_2)\right\}$.

Always-Challenging Phase

When player 1's reputation is sufficiently high and player 2's reputation is sufficiently low (but not too low when challenge arrival rate is high), an unjustified player 2 would never see a challenge from player 1 and an unjustified player 1 as a result always challenges. An unjustified layer 2 concedes at a rate lower than the Abreu-Gul rate to make an unjustified player 1 indifferent between challenging and conceding, and player 1 concedes at the constant Abreu-Gul rate. Player 1's reputation building follows Abreu-Gul's but player 2's reputation building is different. The phase lasts until player 1's reputation reaches 1 or player 2's reputation reaches the threshold reputation at which point player 1 no longer challenges player 2.

Lemma 5 (A). *Suppose at time t^0, $\mu_1(t^0) = \mu_1^0 \geq \underline{\mu}_1$ and $\overline{\mu}_2 - \frac{\lambda_2}{g_1} < \mu_2(t^0) = \mu_2^0 < \overline{\mu}_2$. For time t between t^0 and $t^0 + t^A(\mu_1^0,\mu_2^0)$, an unjustified player 1 always challenges and concedes with constant rate λ_1, and an unjustified player 2 never sees a challenge, concedes with rate*

$$\lambda_2^A(t) = \lambda_2 - g_1\left[\overline{\mu}_2 - \mu_2^A(t)\right]$$

where player 1's reputation evolves according to

$$\mu_1^A(t;t^0,\mu_1^0) = \mu(t;t^0,\mu_1^0,\lambda_1,0),$$

player 2's reputation evolves according to

$$\mu_2^A(t;t^0,\mu_2^0) = \mu(t;t^0,\mu_2^0,\lambda_2 - g_1\overline{\mu}_2, g_1),$$

the time it takes player 1 to reach reputation 1 *is*

$$t_1^A(\mu_1^0,1) \equiv t(1;\mu_1^0,\lambda_1,0)$$

the time it takes player 2 to reach reputation $\overline{\mu}_2$ *is*

$$t_2^A(\mu_2^0,\overline{\mu}_2) \equiv t(\overline{\mu}_2;\mu_2^0,\lambda_2 - g_1\overline{\mu}_2, g_1)$$

so that the duration of the phase counting from t^0 *is* $\min\left\{t_1^A(\mu_1^0,1), t_2^A(\mu_2^0,\overline{\mu}_2)\right\}$.

Never-Challenging Phase

Finally, when both players' reputations are sufficiently high, an unjustified player 1 never challenges and an unjustified player never sees a challenge, so the the challenge opportunity is essentially inconsequential in this phase. Players concede at constant Abreu-Gul rates, but their reputation building is different from Abreu-Gul's, because of the usefulness of the challenge opportunity for a justified player 1. The phase lasts until the reputation of one of the players reaches 1.

Lemma 6 **(N).** *Suppose at time* t^0, $\mu_2(t^0) = \mu_2^0 \geq \overline{\mu}_2$ *and* $\mu_1(t^0) = \mu_1^0 \geq \max\{\underline{\mu}_1, 1 - \frac{\lambda_1}{g_1}\}$. *For time t between* t^0 *and* $t^0 + t^N(\mu_1^0,\mu_2^0)$, *an unjustified player 1 never challenges and concedes with constant rate* λ_1, *and an unjustified player 2 never sees a challenge and concedes with constant rate* λ_2. *Player 1's reputation evolves according to*

$$\mu_1^N(t;t^0,\mu_1^0) = \mu(t;t^0,\mu_1^0,\lambda_1 - g_1, g_1),$$

player 2's reputation evolves according to

$$\mu_2^N(t;t^0,\mu_2^0) = \mu(t;t^0,\mu_2^0,\lambda_2,0),$$

the time it takes player 1 to reach reputation 1 is

$$t_1^N(\mu_1^0,1) \equiv t(1;\mu_1^0,\lambda_1 - g_1, g_1),$$

and the time it takes player 2 to reach reputation 1 is

$$t_2^N(\mu_2^0,1) \equiv t(1;\mu_2^0,\lambda_2,0)$$

so that the phase counting from t^0 *lasts for* $\min\left\{t_1^N(\mu_1^0,1), t_2^N(\mu_2^0,1)\right\}$.

3.1.3 Equilibrium

We need to close the model and find the equilibrium using the equilibrium property that both players' reputations reach 1 at the same time. If player i's reputation reaches 1 strictly before player j, then player j has a strict incentive to drop out of the game immediately after j infers i's reputation is 1 rather than to wait for a positive amount of time.

Initial Conceding Phase

Unjustified players may concede with strictly positive probabilities at time 0. Player 1's time 0 reputation after an unjustified player 1 concedes with probability Q_1 is

$$x = \frac{z_1}{z_1 + (1-z_1)(1-Q_1)} \geq z_1.$$

Player 2's time 0 reputation after an unjustified player 2 concedes with probability Q_2 is

$$y = \frac{z_2}{z_2 + (1-z_2)(1-Q_2)} \geq z_2.$$

By conceding with a positive probability, player i can increase his or her reputation at time 0 from the prior probability z_i of being justified to any arbitrary level between z_i and 1. In particular, in order to achieve reputation $x > z_1$ at time 0, player 1 needs to concede with a positive probability

$$Q_1 = 1 - \frac{z_1}{1-z_1} \Bigg/ \frac{x}{1-x},$$

and in order to achieve reputation $y > z_2$ at time 0, player 2 needs to concede with a positive probability

$$Q_2 = 1 - \frac{z_2}{1-z_2} \Bigg/ \frac{y}{1-y}.$$

Characterization of the Equilibrium

The central property to pin down the equilibrium is that two players' reputations simultaneously reach 1 on the equilibrium no-action path. If players' reputations do not reach 1 at the same time, then the player whose reputation has not reached 1 when the other's has reached will concede with a strictly positive probability at time 0 so that their reputations reach 1 at the same time. To characterize the equilibrium, we solve backwards.

Let's first suppose z_1 and z_2 are sufficiently small so that the game will start in the sometimes-challenging sometimes-seeing phase and all strategy phases potentially exist. The last phase in the game, the never-challenging never-seeing phase, lasts when player 2's reputation is between $\overline{\mu}_2$ and 1, so the length of the last phase is $t_2^N(\overline{\mu}_2, 1)$. At the beginning of the never-challenging never-seeing phase, player 2's reputation is $\overline{\mu}_2$ and player 1's reputation is $\mu_1^N = \mu_1^N(0; t_2^N(\overline{\mu}_2, 1), 1)$. It can be shown that $\mu_1^N > 1 - \frac{\lambda_1}{g_1}$ (to be completed). The penultimate phase, the always-challenging never-seeing phase may or may not exist. If $\mu_1^N \leq \underline{\mu}_1$, then the phase does not exist. If $\mu_1^N > \underline{\mu}_1$, then the phase exists. The phase lasts when player 1's reputation is between $\underline{\mu}_1$ and μ_1^N: the length of the phase is $t_1^A(\mu_1^N, 1)$. At the beginning of the phase, if the phase exists, player 2's reputation is $\mu_2^A = \mu_2^A(0; t_1^A(\mu_1^N, 1), \overline{\mu}_2)$. It can be shown that $\mu_2^A > \overline{\mu}_2 - \frac{\lambda_2}{g_1}$ (to be completed). Finally, at the end of the sometimes-challenging sometimes-seeing phase, if the AN phase exists, then the reputations are $\underline{\mu}_1$ and μ_2^A; if the always-challenging never-seeing phase does not exist, then the reputations are μ_1^N and $\overline{\mu}_2$. The time lengths it takes to reach these levels are $t_1^S(z_1, \underline{\mu}_1)$ and $t_2^S(z_2, \mu_2^A)$ when the always-challenging never-seeing phase exists and the time lengths it takes to reach these levels are $t_1^S(z_1, \mu_1^N)$ and $t_2^S(z_2, \overline{\mu}_2)$ if the always-challenging never-seeing phase does not exist.

If z_1 and z_2 are not sufficiently small, then not all three phases are present. The game starts in a latter phase and one of the players may concede with a positive probability so that both players' reputations reach 1 at the same time. The proof contains more detailed characterization of the equilibrium, including when z_1 and z_2 are not sufficiently small to have all three phases in equilibrium.

Figure 4 illustrates four different possible scenarios regarding the co-evolution of reputations on the equilibrium path.

Theorem 1. *There exists a unique equilibrium. For sufficiently small z_1 and z_2, the game goes through either all three strategy phases or two strategy strategy phases skipping the always-challenging never-seeing phase. At most one player concedes with a positive probability so that both players' reputations on the no-action path reach 1 at the same time.*

The equilibrium payoff player $i = 1, 2$ gets is

$$\begin{aligned} & u_i(a_1, a_2, z_1, z_2) \\ = \; & (1 - z_j)Q_j(a_1, a_2, z_1, z_2)a_i + [z_j + (1 - z_j)(1 - Q_j(a_1, a_2, z_1, z_2))](1 - a_j) \\ = \; & 1 - a_j + (1 - z_j)Q_j(a_1, a_2, z_1, z_2)(a_1 + a_2 - 1). \end{aligned}$$

Proposition 1. *The unique sequential equilibrium of the bargaining game is $(\widehat{F}_1(\cdot), \widehat{G}_1(\cdot), \widehat{F}_2(\cdot))$.*

Proof. Let $\overline{\sigma} = (F_1, G_1, F_2)$ define a sequential equilibrium. We will argue that $\overline{\sigma}$ must have the form specified (i.e., uniqueness) and that these strategies. □

Illustration of the Equilibrium

The solid lines in Figure 5 illustrates equilibrium reputation dynamics of a bargaining game with one-sided challenge in which both players have a single justified type. In the top panel, all three phases – sometimes-challenging, always-challenging, and never-challenging – are present. In the bottom panel, the sometimes-challenging phase is absent and only sometimes-challenging and never-challenging phases exist in equilibrium. The dashed lines in both graphs represent the equilibrium reputation dynamics of the same bargaining game but without available challenge opportunities for player 1. In both cases, the game ends more slowly because of the presence of the court. We need verify this comparative statics result: is it true that the addition of the challenge opportunity always slows down reputation building and the bargaining process?

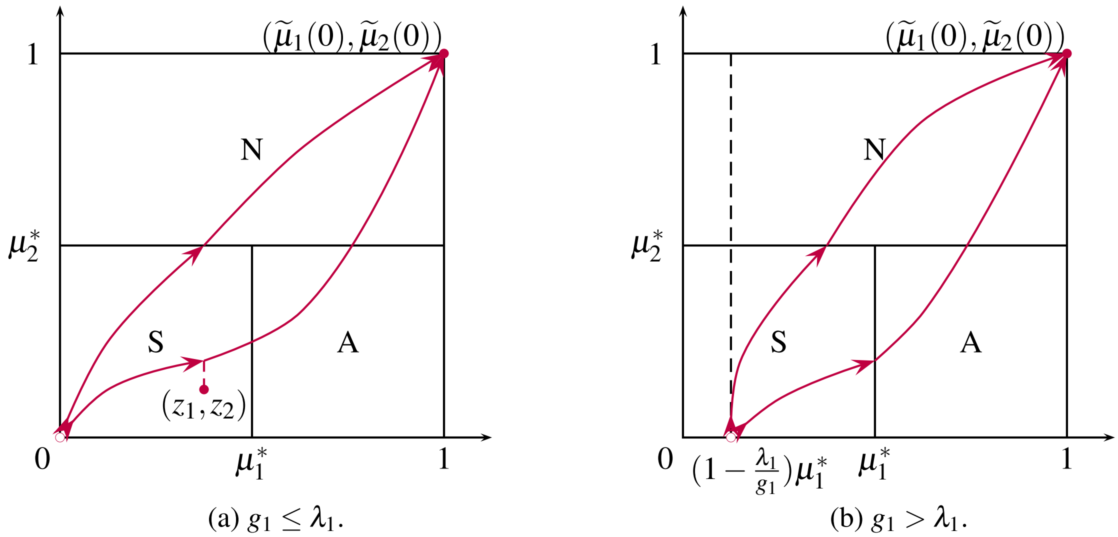


Figure 4: Co-evolution of reputations $(\widetilde{\mu}_1(-\tau), \widetilde{\mu}_2(-\tau))$ in bargaining with frictional one-sided challenges. Figure 4a illustrates two possible equilibrium reputation paths when $g_1 \leq \lambda_1$ and Figure 4b illustrates two possible equilibrium reputation paths when $g_1 > \lambda_1$.

3.2 One Type of Player 1 and Multiple Types of Player 2

Let's consider the intermediate case in which there is only one justified type of player 1 but there are several justified types of player 2: $|A_1| = 1$ and $|A_2| > 1$.

Denote by $B_1(a_1, x)$ the bargaining game with one-sided challenges in which player 1 is justified with probability x and a justified player 1's demand is always a_1. Given the equilibrium characterization solved in the previous subsections, determining player 2's equilibrium mimicking behavior suffices to characterize full equilibrium strategy. Define $\sigma_2(a_2)$ as player 2's probability of choosing $a_2 \in A_2$ and define $\sigma_2(0) \equiv Q_2$ as her probability of conceding at time 0. Player 2 chooses $\sigma_2(\cdot)$, a probability distribution over $A_2 \cup \{0\}$, to maximize

$$u_2(\sigma_2(\cdot); a_1, x) = \sigma_2(0)(1 - a_1) + \sum_{a_2 \in A_2} \sigma_2(a_2) u_2(a_1, a_2, x, \sigma_2(a_2))$$

subject to $\sum_{a_2 \in A_2 \cup \{0\}} \sigma_2(a_2) = 1$, where $u_2(a_1, a_2, x, \sigma_2(a_2))$ is player 2's expected payoff when player 2 chooses a_2 with probability $\sigma_2(a_2)$ and players play optimally in the subsequent bargaining game as described by the previous subsection.

If $x = 1$, then in equilibrium, $Q_2 = 1$. Assume $x < 1$ for the rest of the section. Define $T_i(a_1, a_2, x)$ as the time it takes for player i's reputation to increase from x to 1 on the equilibrium reputation path when each player i's demand is a_i, $i = 1, 2$. Define player 2's reputation at time 0 when she plays a_2 with probability σ_2 as

$$y(a_2, \sigma_2) = \frac{z_2 \pi_2(a_2)}{z_2 \pi_2(a_2) + (1 - z_2)\sigma_2}.$$

Because the more likely an unjustified player 2 announces a particular demand a_2, the more

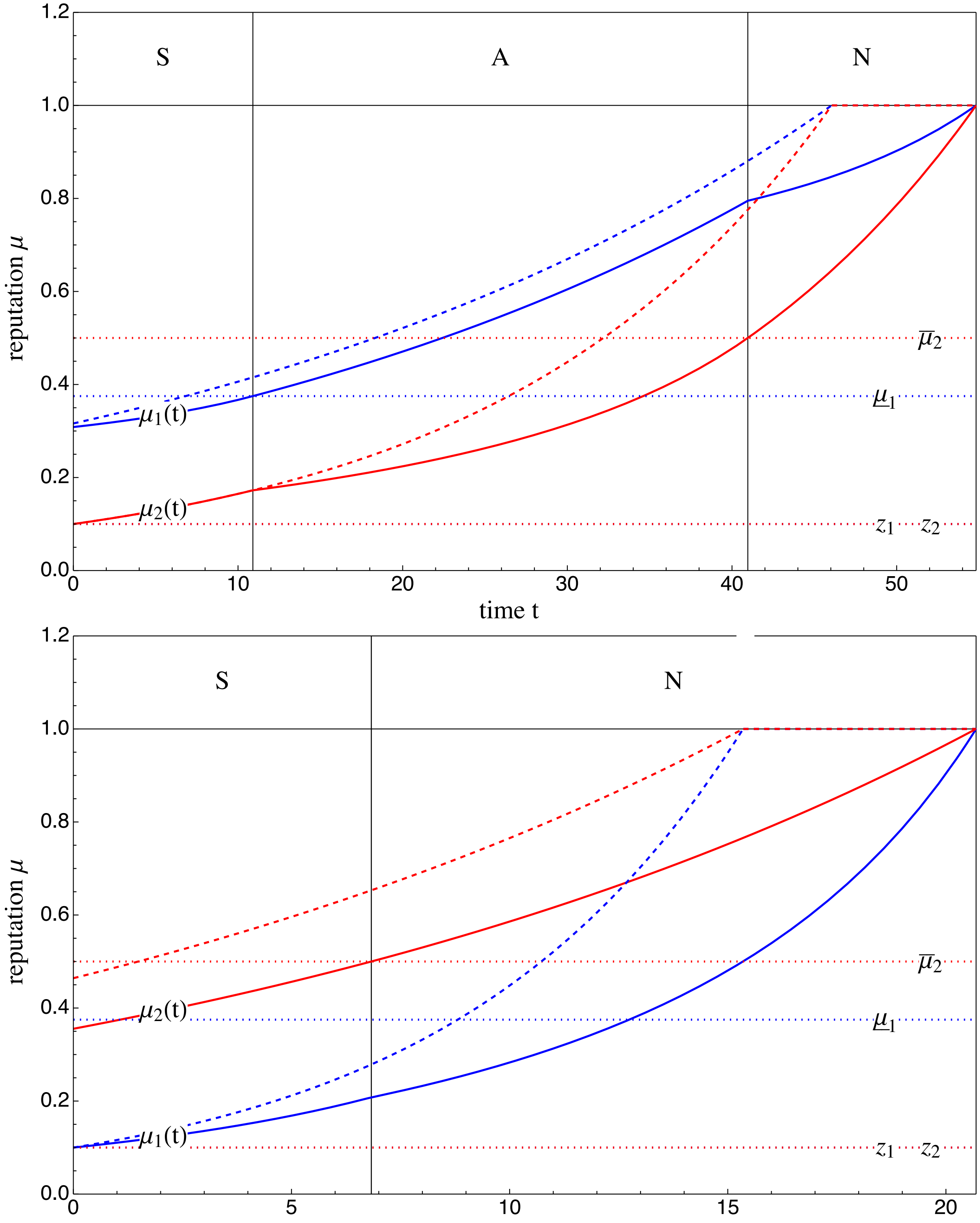


Figure 5: Solid lines represent equilibrium reputation dynamics of a bargaining game with one-sided challenge and single justified types. Dashed lines represent equilibrium reputation dynamics of the same bargaining game with no challenge opportunities.

likely she is believed to be unjustified, and the lower her payoff from demanding a_2 is. Let $\overline{\sigma}_2(a_1,a_2,x)$ be the maximum probability player 2 plays a_2 in equilibrium so that the expected payoff from demanding a_2 is higher than directly conceding to player 1's demand. For any $a_2 < 1-a_1$, $\overline{\sigma}_2(a_1,a_2,x) = 0$ because conceding to player 1's demand of a_1 and receiving $1-a_1$ is a strictly better strategy than demanding less than $1-a_1$. For any $a_2 \geq 1-a_1$, after choosing a_2, in any equilibrium, player 2 should not concede with a positive probability at time 0. First, if player 1's reputation can reach 1 without conceding with a positive probability at time 0 and player 2's reputation reaches 1 slower than player 1 when she demands a_2 with probability 1, $\overline{\sigma}_2(a_1,a_2,x)$ is the unique solution of σ_2 to $T_1(a_1,a_2,x) = T_2(a_1,a_2,y(a_2,\sigma_2))$ so that the two players' reputations reach 1 at the same time. Second, if player 1's reputation reaches 1 even slower than when player 2 demands a_2 with probability 1, $\overline{\sigma}_2(a_1,a_2,x) = 1$. The scenario happens whenever $T_1(a_1,a_2,x) > T_2(a_1,a_2,y(a_2,1))$. In particular, it happens whenever $x < \mu_1^*(1-\frac{\lambda_1}{g_1})$. In summary, in any equilibrium, $\sigma_2(a_2) \leq \overline{\sigma}_2(a_1,a_2,x)$, where $\overline{\sigma}_2(a_1,a_2,x) = 0$ if $a_2 \leq 1-a_1$, $\overline{\sigma}_2(a_1,a_2,x)$ is the unique solution of σ_2 in $T_1(a_1,a_2,x) = T_2(a_1,a_2,y(a_2,\sigma_2))$ if $a_2 > 1-a_1$ and $T_1(a_1,a_2,x) < T_2(a_1,a_2,y(a_2,1))$, and $\overline{\sigma}_2(a_1,a_2,x) = 1$ if $a_2 > 1-a_1$ and $T_1(a_1,a_2,x) \geq T_2(a_1,a_2,y(a_2,1))$.

When player 2 demands a_2 with probability $\sigma_2 \leq \overline{\sigma}_2(a_1,a_2,x)$, player 1 must raise his time 0 reputation to $x^*(a_1,a_2,\sigma_2)$ so that their reputations reach 1 at the same time:

$$T_1(a_1,a_2,x^*(a_1,a_2,\sigma_2)) = T_2(a_1,a_2,y(a_2,\sigma_2)).$$

In order to do so, player 1 concedes with probability

$$Q_1(a_1,a_2,x,\sigma_2) = 1 - \frac{x}{1-x}\frac{1-x^*(a_1,a_2,\sigma_2)}{x^*(a_1,a_2,\sigma_2)}$$

so that the reputation is raised to

$$x^*(a_1,a_2,\sigma_2) = \frac{x}{x+(1-x)(1-Q_1(a_1,a_2,x,\sigma_2))}.$$

When player 2 demands a_2 with probability σ_2 and an unjustified player 1 concedes with probability $Q_1(a_1,a_2,x,\sigma_2)$, player 2's expected payoff is

$$\begin{aligned} u_2(a_1,a_2,x,\sigma_2) &= (1-x)Q_1(a_1,a_2,x,\sigma_2)a_2 + [x+(1-x)(1-Q_1(a_1,a_2,x,\sigma_2))](1-a_1) \\ &= 1-a_1+(1-x)Q_1(a_1,a_2,x,\sigma_2)(a_1+a_2-1). \end{aligned}$$

Two additional properties restrict player 2's equilibrium strategies $\sigma_2(\cdot)$. First, for any a_2 and $a_2' > a_2$, if $\sigma_2(a_2) > 0$, then $\sigma_2(a_2') > 0$. We can prove this property by contradiction. Suppose $\sigma_2(a_2) > 0$ and $\sigma_2(a_2') = 0$. Because $\sigma_2(a_2') = 0$, $u_2(a_1,a_2',x,\sigma_2(a_2')) = 1-a_1+(1-x)(a_1+a_2'-1)$. Because $\sigma_2(a_2) > 0$, $u_2(a_1,a_2,x,\sigma_2(a_2)) = 1-a_1+(1-x)Q_1(a_1,a_2,x,\sigma_2(a_2))(a_1+a_2-1) \leq 1-a_1+(1-x)(a_1+a_2'-1) = u_2(a_1,a_2',x,\sigma_2(a_2'))$. Second, whenever $\sum_{a_2}\overline{\sigma}_2(a_1,a_2,x) \leq 1$,

$\sigma_2(a_2) = \overline{\sigma}_2(a_1,a_2,x)$ for all a_2, and $Q_2 = 1 - \sum_{a_2} \overline{\sigma}_2(a_1,a_2,x)$. The two properties together imply that we only need to check first if $\sum_{a_2} \overline{\sigma}_2(a_1,a_2,x) \leq 1$, and, if the first condition does not hold, then find the equilibrium strategy among $\sigma_2(\cdot)$ such that $\sigma_2(a_2') > 0$ for all $a_2' \geq a_2$, for each $a_2 \in A_2$.

For any mimicking strategy $\sigma_2(\cdot)$, define

$$F_2(x,\sigma_2(\cdot)) \equiv \min_{a_2:\sigma_2(a_2)>0} u_2(a_1,a_2,x,\sigma_2(a_2)).$$

$\sigma_2(\cdot)$ is an equilibrium strategy for player 2 if and only if $\sigma_2(\cdot)$ solves

$$\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} F_2(x,\sigma(\cdot))$$

where

$$\Delta(a_1,x) = \{\sigma(\cdot) \in \Delta | \sigma_2(a_2) \leq \overline{\sigma}_2(a_1,a_2,x)\ \forall a_2 \in A_2\}$$

and Δ is the set of probability distributions on $A_2 \cup \{0\}$. Player 1's equilibrium payoff when player 2 plays $\sigma_2(\cdot)$ is

$$u_1(a_1,a_2,x,\sigma_2(\cdot)) = (1-z_2)Q_2a_1 + \sum_{a_2\in A_2} [z_2\pi_2(a_2) + (1-z_2)\sigma_2(a_2)](1-a_2).$$

It remains to show that there is a unique equilibrium. Multiple equilibrium distributions over types being conceded to are in conflict with the requirement that types mimicked with positive probability must have equal payoffs that are not smaller than the payoffs of the types that are not mimicked. Suppose by contradiction there are two different equilibrium strategies for player 2: $\sigma_2(a_2) \neq \sigma_2'(a_2)$ for some a_2. If $\sigma_2(a_2) > 0$ and $\sigma_2'(a_2) > 0$, then $u_2(a_1,a_2,x,\sigma_2(a_2)) \neq u_2(a_1,a_2,x,\sigma_2'(a_2))$. But $u_2(a_1,a_2,x,\sigma_2(a_2)) = F_2(x,\sigma_2(\cdot))$ and $u_2(a_1,a_2,x,\sigma_2(a_2)) = F_2(x,\sigma_2'(\cdot))$. $F_2(x,\sigma_2(\cdot)) \neq F_2(x,\sigma_2'(\cdot))$ contradicts the fact that $\sigma_2(\cdot)$ and $\sigma_2'(\cdot)$ both solve $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} F_2(x,\sigma(\cdot))$. If $\sigma_2(a_2)$ or $\sigma_2'(a_2)$ is zero, then, by the first additional property of player 2's equilibrium strategy above, there is an $a_2' > a_2$ such that $\sigma_2(a_2') > 0$, $\sigma_2'(a_2') > 0$, and $\sigma_2(a_2') \neq \sigma_2'(a_2')$.

Player 1 receives $u_1(a_1,x)$ in the equilibrium of the bargaining game $B(a_1,x)$.

3.3 Multiple Justified Types of Both Players

Now we look at the case in which player 1 first chooses which type $a_1 \in A_1$ to mimic, and seeing this, player 2 responds with a type $a_2 \in A_2$ to mimic. Let $u_1(a_1,x)$ denote player 1's equilibrium payoff in the bargaining game $B_1(a_1,x)$. Player 1 chooses mimicking strategy $\sigma_1(\cdot)$ subject to $\sum_{a_1\in A_1} \sigma_1(a_1) \leq 1$ to maximize his payoff

$$u_1(a_1,\sigma_1(\cdot)) = \sum_{a_1\in A_1} u_1(a_1,x(a_1,\sigma_1(a_1)))$$

where

$$x(a_1,\sigma_1(a_1)) = \frac{z_1\pi_1(a_1)}{z_1\pi_1(a_1)+(1-z_1)\sigma_1(a_1)}.$$

It remains to show the equilibrium is unique.

Theorem 2. *For any bargaining game $B = \{(A_i,z_i,\pi_i,r_i,g_i,c_i)_{i=1}^2\}$ a sequential equilibrium exists. Furthermore, all equilibria yield the same distribution over outcomes.*

Proof of Theorem 2. Let $u_1(a_1,x)$ be the payoff of player 1 in the unique equilibrium of $B(a_1,x)$. We will show that $u_1(a_1,\cdot)$ is continuous and that there exists $\underline{x}$ such that for $x \leq \underline{x}$, $u_1(a_1,x) = u_1(a_1,\underline{x})$ and $\sigma_2(\max A_2) = 1$ where σ_2 is the equilibrium game in the game $B(a_1,x)$. We will also show that $u_1(a_1,x)$ is continuous and strictly increasing in x on $[\underline{x},1]$. □

3.4 The Limiting Case of Complete Rationality

Suppose that the commitment type set for each player sufficiently "finely" covers the interval $[0,1]$. We are interested in the limit equilibrium payoffs as $z_1,z_2 \to 0$.

First, suppose $g_1 > r_1$. We first start with the case in which player 1 chooses a type $a_1 > \frac{r_2}{r_2+g_1}$ with a probability that doesn't disappear as $z_1 \to 0$. If player 2 chooses $a_2 \approx 1$, then we have

$$\lambda_1 = \frac{(1-a_1)r_2}{a_1+a_2-1} \approx \frac{(1-a_1)r_2}{a_1} < g_1.$$

Hence, Player 2's payoff from this strategy approaches to approximately 1, and hence player 1's payoff approaches to approximately to 0.

If Player 1 chooses a type $a_1 < \frac{r_2}{r_2+g_1}$, then regardless of a_2, $\lambda_1 > g_1$, hence we need to solve for the equilibrium dynamics of the model in which $\lambda_1 > g_1$.

Player 1 wins if $\lambda_1 - g_1 > \lambda_2$, and player 2 wins otherwise. (Why?)

Plugging in λ_i the expression from Abreu and Gul, we have that Player 2 wins if

$$(1-a_1)r_2 - g_1 a_1 < (1-a_2)(r_1-g_1)$$

Because $g_1 > r_1$, the right hand side is negative, and left hand side is always positive, so Player 1 wins.[6]

Since this is true for every $a_1 < \frac{r_2}{r_2+g_1}$, player 1, by choosing a demand approximately equal to $\frac{r_2}{r_2+g_1}$ (more precisely, $\max\{a_1 \in A_1 | a_1 \leq \frac{r_2}{r_2+g_1}\}$) guarantees this payoff, and cannot do better, and Player 2 gets the rest of the surplus.

Second, suppose $g_1 < r_1$. In this case, if player 1 chooses $a_1 = \frac{r_2}{r_2+r_1}$, then $\lambda_1 > g_1$ for any

[6]It's important for this argument that z_1 goes to zero at a rate not smaller than that of z_2.

choice of a_2, so the winner is determined by comparison

$$(1-a_1)r_2 - g_1 a_1 < (1-a_2)(r_1 - g_1)$$

which for the choice of $a_1 = \frac{r_2}{r_2+r_1}$ makes player 1 the winner, and for any choice of a_1 lower, makes player 2 the winner by a choice that makes player 2 have a payoff larger than $\frac{r_1}{r_1+r_2}$, that leaves player 1 with a payoff smaller than $\frac{r_2}{r_1+r_2}$. Hence the solution is similar to Abreu and Gul in this case.

4 Bargaining with Frictional Two-Sided Ultimatums

There is a unit pie to be divided between a male player 1 and a female player 2. Each player $i = 1,2$ is either justified to demand a share of the pie and never accepts any offer below that, or unjustified to demand a share of the pie but nonetheless wants as a big share of the pie as possible. A justified player can be thought of as someone who could have evidence supporting his or her demand, and an unjustified player can be thought of as someone who does not have evidence supporting his or her claim of the share. To start, suppose each player can be of a single justified type: for $i = 1,2$, with probability z_i player i is justified to demand $a_i > 0.5$. Let $d \equiv a_i - (1-a_j) = a_1 + a_2 - 1$ denote the conflicting difference between the two players. Time is continuous.

At each instant t, each player can decide to give in to the other player's demand or hold on to his or her own demand. In addition, player $i = 1,2$ has Poisson arrival of challenge opportunities with constant rate $g_i > 0$. Player i can use the challenge opportunity, and if he or she challenges, opposing player $j \neq i$ can see or yield to player i's challenge. If player i does not challenge when the opportunity arises, then the game continues and the current challenge opportunity disappears but the opportunity may arrive again in the future at the same rate. If player i challenges at time t, he incurs a cost c_i right away and player j must respond to player i's challenge. Player j may yield to the challenge right away and get $1-a_i$, or may see the challenge by paying a cost c_j. To make challenging and seeing a challenge worthwhile for players, assume $c_i < d$ for $i = 1,2$. After opposing player j sees the challenge, the division of the pie is determined by the players' justified and unjustified types: an unjustified challenger always loses to a responder and a justified challenger always beats an unjustified responder.

$B(\{z_i, a_i, r_i, g_i, c_i\}_{i=1}^2)$ describes a bargaining game with two-sided challenges and single justified types by players' probabilities of being justified z_1 and z_2, demands a_1 and a_2, discount rates r_1 and r_2, players' challenge arrival rates g_1 and g_2, as well as challenge costs c_1 and c_2.

One application of the model is final-offer arbitration. Two parties announce their demands for a subject. A justified player has evidence supporting his or her claim, but needs time and effort to gather the evidence and appeal to the court. An unjustified player does not have evidence but nonetheless can pretend he or she does by appealing to the court when a chance presents. Whether or not a player could gather evidence and has gathered evidence is private information. While they gather evidence, they can negotiate with each other by repeatedly and frequently making offers to each other or choosing to let the case be settled by the court when chance presents. A justified player can be done with collecting evidence at any moment, and as soon as he is done with collecting evidence and if the case has not been settled out of the court by then, he or she appeals to the court. At that moment, the opposing player has to respond to the lawsuit either by agreeing to the challenging player's demand out of the court or by paying a cost to let the court settle the case. In the court, an unjustified player always loses the case.

We solve for the sequential equilibrium of the game. A player's profile consists of a player's strategy as well as a player's belief about the other player's type. A player's strategy is described by the probability $\chi_i(t)$ of challenging at each instant t if a challenge opportunity arises, the probability $\sigma_i(t)$ of seeing a challenging at time t, and the probability $L_i(t)$ of challenging by time t. A player i's belief about player j at time t is simply his or her probability of being justified at time t given that the game has not ended by time t. We call player i's belief about player j player j's reputation $\mu_j(t)$. Players' payoffs are denoted by u_i and u_j.

4.1 Strategies

When a challenge opportunity arrives, a player decides between challenging and not challenging and the opponent decides between seeing a challenge and yielding to a challenge. When a challenge opportunity does not arrive, a player decides between conceding and not conceding. When player j observes no action (no yielding or no challenging) by time t the belief that player i is justified is denoted by $\mu_i(t)$, which we call player i's *reputation*. Define an important threshold reputation $\mu_i^* \equiv 1 - \frac{c_j}{d}$ for $i = 1, 2$.

In a sequential equilibrium, at each instant, given the beliefs of the players at the moment, players are playing a Nash equilibrium. Using this property, we can derive the candidate equilibrium strategies of unjustified players. Unjustified players' equilibrium strategies of challenging are summarized in the following table. A represents always challenging, S represents sometimes challenging, and N represents never challenging. The first letter represents player 1's chance of challenging and the second letter represents player 2's chance of challenging.

$\mu_2^* \le \mu_2(t) \le 1$	NA	NN
$0 < \mu_2(t) < \mu_2^*$	SS	AN
	$0 < \mu_1(t) < \mu_1^*$	$\mu_1^* \le \mu_1(t) \le 1$

Unjustified players' equilibrium strategies of responding to a challenge turn out to uniquely pinned down by the chance of challenging: when an unjustified player always challenges, an unjustified opponent never sees a challenge; when an unjustified player sometimes challenges, an unjustified opponent sometimes sees a challenge; and when an unjustified player never challenges, an unjustified opponent never sees a challenge.

Lemma 7. *Equilibrium strategies of challenging versus not challenging and seeing versus yielding to a challenge depend on players' reputations, as follows. In any equilibrium, at time t, if a challenge opportunity arises to a player,*

1. *(SS) if $\mu_i(t) \leq \mu_i^*$ for each $i = 1,2$, an unjustified player i challenges with probability*
$$p_i^*(t) \equiv \frac{\mu_i(t)}{1-\mu_i(t)} \Big/ \frac{\mu_i^*}{1-\mu_i^*}$$
and sees a challenge with probability
$$s_i^*(t) \equiv \frac{\mu_i^* - \mu_i(t)}{1-\mu_i(t)}$$
so that each player i challenges with probability
$$\chi_i^*(t) \equiv \mu_i(t) + (1-\mu_i(t))p_i^*(t) = \frac{\mu_i(t)}{\mu_i^*}$$
and sees a challenge with probability
$$\sigma_i^*(t) \equiv \mu_i(t) + (1-\mu_i(t))s_i^*(t) = \mu_i^*;$$
2. *(AN or NA) if $\mu_i(t) > \mu_i^*$ and $\mu_j(t) \leq \mu_j^*$, an unjustified player i always challenges and never sees a challenge, and an unjustified player j never challenges and never sees a challenge.*
3. *(NN) if $\mu_i(t) > \mu_i^*$ for each $i = 1,2$ at time t, each unjustified player never challenges and never sees a challenge.*

Players' decisions to concede depend on their reputations as well as their decisions to challenge. When the opponent sometimes challenges or never challenges, a player simply concedes with a constant rate. When the opponent always challenges, a player needs to concede with a lower and varying rate to make an unjustified opponent indifferent between challenging and conceding. It is possible that the opponent's incentive to concede is so low that even if the player never concedes, the opponent would never concede; in this case, an unjustified player will concede with a positive probability to raise his or her reputation.

Lemma 8. *Equilibrium strategies of conceding versus not conceding depend on players' reputations, as follows. Let $\lambda_i \equiv r_j(1-a_i)/d$. In any equilibrium, at time t,*

1. *(SS) if $\mu_i(t) \leq \mu_i^*$ for $i = 1,2$, each player i concedes with constant rate λ_i.*
2. *(AN or NA)*

(a) *if $\mu_i(t) > \mu_i^*$ and $\mu_j(t) < \mu_j^* - \frac{\lambda_j}{g_i}$, player i does not concede and an unjustified player j concedes with a probability such that player j's reputation jumps to $\mu_j^* - \frac{\lambda_j}{g_i}$.*

(b) *if $\mu_i(t) > \mu_i^*$ and $\mu_j^* - \frac{\lambda_j}{g_i} \leq \mu_j(t) < \mu_j^*$, player i concedes with constant rate λ_i and player j concedes with rate $\lambda_j(t) = \lambda_j - g_i[\mu_j^* - \mu_j(t)]$.*

3. *(NN) if $\mu_i(t) > \mu_i^*$ for $i = 1, 2$, each player i concedes with constant rate λ_i.*

4.2 Reputation Dynamics

We have derived candidate equilibrium strategies fixing players' reputations at instances. But these reputations evolve over time and these evolutions are dictated by the equilibrium strategies. We now derive equilibrium reputation dynamics. In particular, we derive the evolutions of reputations along the path in which no action has appeared (i.e., no concession and no challenge); when an action has been taken, the game has ended and the reputation is fully revealed. Reputation dynamics differ in different phases when players play different strategies.

Lemma 9. *In any equilibrium, the reputation dynamics from time t^0 depend on players' reputations at time t^0, $\mu_i^0 \equiv \mu_i(t^0)$ and $\mu_j^0 \equiv \mu_j(t^0)$, as follows.*

1. *(SS) If $\mu_i^0 < \mu_i^*$ for each $i = 1, 2$, each player i's reputation evolves according to*

$$\mu_i^{SS}(t; t^0, \mu_i^0) \equiv \mu(t; t^0, \mu_i^0, \lambda_i - g_i, \frac{g_i}{\mu_i^*})$$

until either player i's reputation reaches μ_i^*.

2. *(AN or NA)*

(a) *If $\mu_i^0 > \mu_i^*$ and $\mu_j^* - \frac{\lambda_j}{g_i} \leq \mu_j^0 < \mu_j^*$, player i's reputation evolves according to*

$$\mu_i^{AN}(t; t^0, \mu_i^0) \equiv \mu(t; t^0, \mu_i^0, \lambda_i, 0)$$

and player j's reputation evolves according to

$$\mu_j^{AN}(t; t^0, \mu_j^0) = \mu(t; t^0, \mu_j^0, \lambda_j - g_i\mu_j^* - g_j, g_i + g_j)$$

until player i's reputation reaches 1 or player j's reputation reaches μ_j^*.

(b) *If $\mu_i^0 > \mu_i^*$ and $\mu_j^0 < \mu_j^* - \frac{\lambda_j}{g_i}$, player i's reputation evolves according to $\mu_i^{AN}(t; t^0, \mu_j^0)$ and player j's reputation jumps to $\mu_j^* - \frac{\lambda_j}{g_i}$ then evolves according to $\mu_j^{AN}(t; t^0, \mu_j^* - \frac{\lambda_j}{g_i})$ until player i's reputation reaches 1 or player j's reputation reaches μ_j^*.*

3. *(NN) If $\mu_i^0 \geq \mu_i^*$, each player i's reputation evolves according to*

$$\mu_i^{NN}(t; t^0, \mu_i^0) \equiv \mu(t; t^0, \mu_i^0, \lambda_i - g_i, g_i)$$

until either player's reputation reaches 1.

Notably, for a combination of sufficiently high challenge rates, patience factors, and demands, players' reputations may decrease and may never be built up in equilibrium.

4.3 Equilibrium

In this section, we fully characterize the sequential equilibrium and prove its generic uniqueness.

When $g_i > \lambda_i$ for each $i = 1,2$, for sufficiently small initial justice probabilities, there are multiple sequential equilibria, as who concedes at time 0 with what probability is not determined, but all equilibria yield the same expected payoff. In equilibrium, both players' reputations decrease but never reach zero (Figure 6a). Therefore, the game could potentially go on forever.

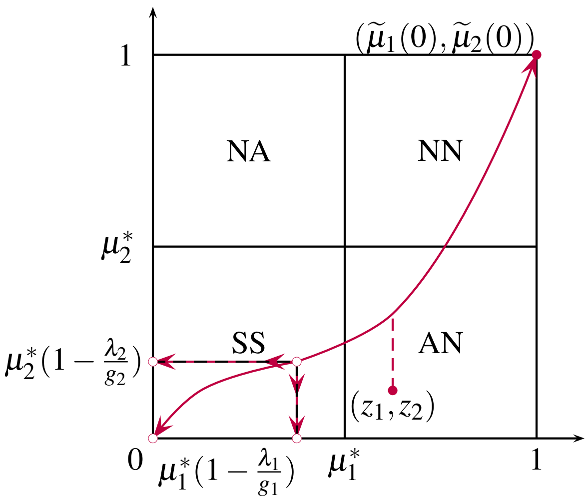
When $g_i \leq \lambda_i$ for either $i = 1,2$, there is a unique sequential equilibrium, and reputation always increases along the equilibrium no-action path and both players' reputations reach 1 at the same time (Figures 6b, 6c, 6d).

Theorem 3. *There exists a sequential equilibrium and each equilibrium yields the same expected payoff.*

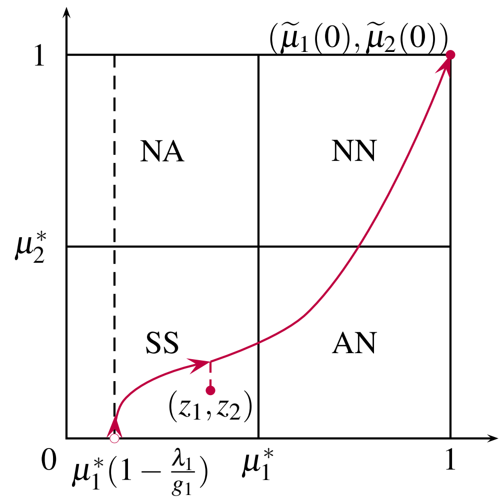
1. *If $g_i > \lambda_i$ for each $i = 1,2$, for sufficiently small z_1 and z_2 ($z_i < (1 - \frac{\lambda_i}{g_i})\mu_i^*$ for each $i = 1,2$ to be exact), at most one player concedes with a positive probability so that each player i's posterior time 0 reputation does not exceed $(1 - \frac{\lambda_i}{g_i})\mu_i^*$, at each instant when the game has not finished, both players' reputations decrease, and each unjustified player i challenges with probability $p_i^*(t)$, sees a challenge with probability $s_i^*(t)$, and concedes with rate $\lambda_i/(1 - \mu_i(t))$.*

2. *If $g_i \leq \lambda_i$ for either $i = 1,2$, for sufficiently small z_1 and z_2 ($z_i < \mu_i^*$ for each $i = 1,2$ to be exact), at most one player concedes with a positive probability at time 0, both players' reputations increase, and there are three strategy phases: (SS phase) both unjustified players challenge with a positive probability and see a challenge with a positive probability, (AN or NA phase) only one unjustified player always challenges whenever a challenge opportunity arises and the other unjustified player never challenges, and (NN phase) neither unjustified player challenges.*

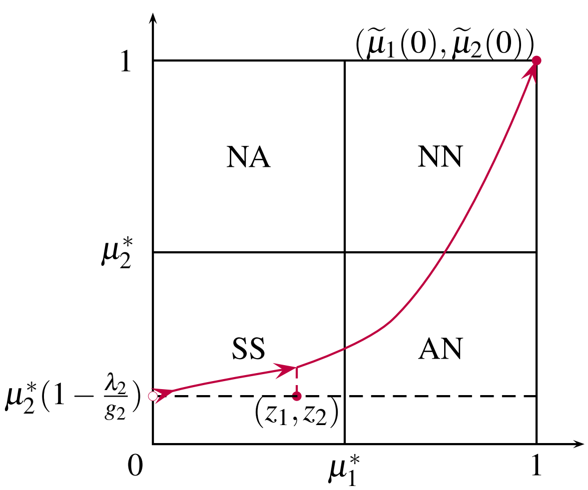
Figure 7 illustrates two cases of equilibrium reputation paths and strategy phases. In the top panel, $\lambda_i < g_i$ for each $i = 1,2$. Reputation never builds up for either player, and players mix between challenging, conceding, and waiting. The game could continue forever, with a diminishing probability. In the bottom panel, $\lambda_i \geq g_i$ for each $i = 1,2$. Reputation builds up for both players.



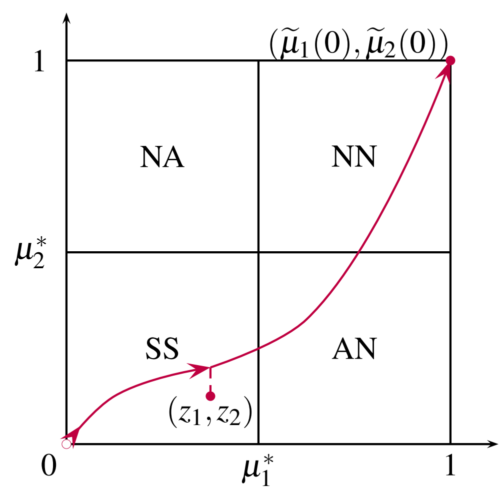
(a) $g_1 > \lambda_1$ and $g_2 > \lambda_2$.



(b) $g_1 > \lambda_1$ and $g_2 \leq \lambda_2$.



(c) $g_1 \leq \lambda_1$ and $g_2 > \lambda_2$.



(d) $g_1 \leq \lambda_1$ and $g_2 \leq \lambda_2$.

Figure 6: State space in bargaining with two-sided challenges.

The game goes through phases in equilibrium: two players both challenge with positive probabilities, only one player challenges with a positive probability, and no player challenges with a positive probability.

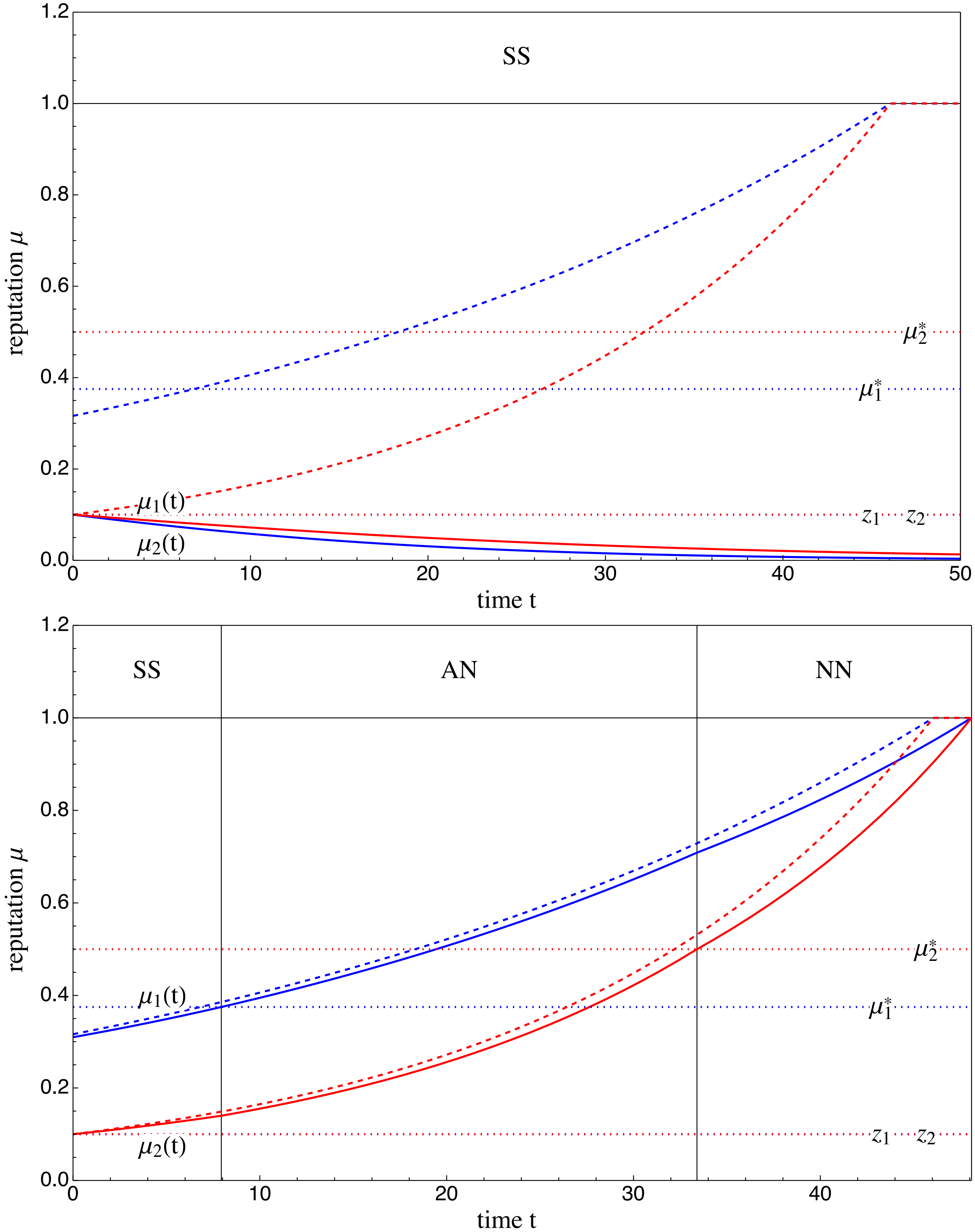


Figure 7: Equilibrium strategy phases and reputation dynamics of the bargaining game with two-sided challenge and single justified types.

References

Abreu, Dilip and Faruk Gul, "Bargaining and Reputation," *Econometrica*, 2000, *68* (1), 85–117.

Atakan, Alp E. and Mehmet Ekmekci, "Bargaining and Reputation in Search Markets," *Review of Economic Studies*, 2013.

Chang, Dongkyu, "Delay in Bargaining with Outisde Options," October 2016. Working Paper, Department of Economics and Finance, City University of Hong Kong.

Fanning, Jack, "Reputational Bargaining and Deadlines," *Econometrica*, 2016, *84* (3), 1131–1179.

_ , "Mediation in Reputational Bargaining," June 2018. Working Paper.

_ , "No Compromise: Uncertain Costs in Reputational Bargaining," *Journal of Economic Theory*, 2018, *175*, 518–555.

Hwang, Ilwoo and Fei Li, "Transparency of Outside Options in Bargaining," *Journal of Economics Theory*, 2017, *167*, 116–147.

Rubinstein, Ariel, "Perfect Equilibrium in a Bargaining Model," *Econometrica*, January 1982, *50* (1), 97–108.

Appendix

A Bargaining without Ultimatums

Proposition 2 (Proposition 1 in Abreu and Gul (2000)). *The unique sequential equilibrium of the bargaining game* $B(a_1, z_1, a_2, z_2)$ *is* $(\widehat{Q}_1(\cdot), \widehat{Q}_2(\cdot))$*, that is,* $\lambda_i = r_i(1-a_i)/(a_1+a_2-1)$*,* $T = \min\{(-\log z_1)/\lambda_1, (-\log z_2)/\lambda_2\}$*,* $C_i = z_i \exp(\lambda_i T)$ *and* $\widehat{Q}_i(t) = 1 - C_j \exp(-\lambda_i t)$*.*

Proof of Proposition 2 (Proof of Proposition 1 in Abreu and Gul (2000)). Let $\overline{\sigma} = (Q_1, Q_2)$ define a sequential equilibrium. We will argue that $\overline{\sigma}$ must have the form specified (i.e., uniqueness) and that these strategies do indeed define an equilibrium (i.e., existence). Let $u_i(s)$ denote the expected utility of an unjustified player i who concedes at time s. Define $T_i := \{t | u_i(t) = \max_s u_i(s)\}$ as the set of conceding times that attain the highest utility for player i; since $\overline{\sigma}$ is a sequential equilibrium, T_i is nonempty for $i = 1, 2$. Furthermore, define $\tau^i := \inf\{t \geq 0 | Q_i(t) = \lim_{t' \to \infty} Q_i(t')\}$ as the time of last concession for player i, where $\inf \emptyset := \infty$. Then we have the following results.

(a) $\tau_1 = \tau_2 \equiv \tau$. An unjustified player will not delay conceding once she knows that the opponent will never concede.

(b) If Q_1 jumps at $t \in \mathbb{R}$, then Q_2 does not jump at t. If Q_1 had a jump at t, then player 2 receives a strictly higher utility by conceding an instant after t than by conceding exactly at t.

(c) If Q_i is continuous at t, then $u_j(s)$ is continuous at $s = t$ for $j \neq i$. This follows immediately from the definitions of $u_1(s)$ and $u_2(s)$ in equations (1) and (2).

(d) There is no interval (t', t'') such that $0 \leq t' \leq t'' \leq \tau$ where both Q_1 and Q_2 are constant on the interval (t', t''). Assume the contrary and without loss of generality, let $t^* \leq \tau$ be the supremum of t'' for which (t', t'') satisfies the above properties. Fix $t \in (t', t^*)$ and note that for ε small there exists $\delta > 0$ such that $u_i(t) - \delta > u_i(s)$ for all $s \in (t^* - \varepsilon, t^*)$ for $i = 1, 2$; in words, conditional on the opponent not conceding in an interval, it is strictly better for a player to concede earlier within that interval. By (b) and (c), there exists i such that $u_i(s)$ is continuous at $s = t^*$, so for some $\eta > 0$, $u_i(s) < u_i(t)$ for all $s \in (t^*, t^* + \eta)$ for this player i; in words, by the continuity of the expected utility function at t^*, the expected utility of conceding a little bit after t^* is still lower than the expected utility of conceding at time t. Since Q_i is optimal, Q_i must be constant on the interval $(t', t^* + \eta)$. The optimality of Q_i implies Q_j is also constant on the interval $(t', t^* + \eta)$, because player j is strictly better off conceding before or after the interval than conceding during the interval. Hence, both functions are constant on the interval $(t', t^* + \eta)$. However, this contradicts the definition of t^*.

(e) If $t' < t'' < \tau$, then $Q_i(t'') > Q_i(t')$ for $i = 1,2$. If Q_i is constant on some interval, then the optimality of Q_j implies that Q_j is constant on the same interval, for $j \neq i$. However, (d) shows that Q_1 and Q_2 cannot be simultaneously constant.

(f) Q_i is continuous at $t > 0$. Assume the contrary: suppose Q_i has a jump at time t. Then Q_j is constant on interval $(t-\varepsilon, t)$ for $j \neq i$. This contradicts (e).

From (e) it follows that T_i is dense in $[0,\tau]$ for $i = 1,2$. From (c) and (f) it follows that $u_i(s)$ is continuous on $(0,\tau]$ and hence $u_i(s)$ is constant for all $s \in (0,\tau]$. Consequently, $T_i = (0,\tau]$. Hence, $u_i(t)$ is differentiable as a function of t and $du_i(t)/dt = 0$ for all $t \in (0,\tau)$. The expected utility is

$$u_i(t) = \int_{x=0}^{t} a_i e^{-r_i x} dQ_j(x) + (1-a_j)e^{-r_i t}(1-Q_j(t)). \tag{3}$$

The differentiability of Q_j follows from the differentiability of $u_i(t)$ on $(0,\tau)$. Differentiating equation (3) and applying Leibnitz's rule, we obtain

$$0 = a_i e^{-r_i t} f_j(t) - (1-a_j) r_i e^{-r_i t}(1-F_j(t)) - (1-a_j)e^{-r_i t} f_j(t)$$

where $q_j(t) = dQ_j(t)/dt$. This in turn implies $F_j(t) = 1 - C_j e^{-\lambda_j t}$ where C_j is yet to be determined. At $\tau_1 = \tau_2$, optimality for player i implies that $Q_i(\tau_i) = 1 - z_i$. At $t = 0$, if $Q_j(0) > 0$ then $Q_i(0) = 0$ by (b). Let T_i solve $1 - e^{-\lambda_i t} = 1 - z_i$. Then $\tau_1 = \tau_2 = T := \min\{T_1, T_2\}$ and c_i, c_j are determined by the requirement $1 - C_i e^{-\lambda_i T} = 1 - z_i$. So $Q_i = \widehat{Q}_i$ for $i = 1,2$. If j's strategy is $\widehat{Q}_j$, then $u_i(t)$ is constant on $(0,\tau]$ and $u_i(s) < u_i(T)$ for all $s > \tau$. Hence, for any mixed strategy on this support, and, in particular, $\widehat{Q}_i$ is optimal for player i. Hence $(\widehat{Q}_1, \widehat{Q}_2)$ is indeed an equilibrium.

□

B Bargaining with One-Sided Ultimatums

B.1 Equilibrium Strategies

Proof of Lemma 1. First, facing a challenge from player 1 with reputation $\mu_1(t)$ at time t, an unjustified player 2 decides between seeing a challenge and yielding to a challenge. When she yields to the challenge, her payoff is simply $1 - a_1$. When she sees a challenge and she knows that an unjustified player 1 challenges at time t with probability $p_1(t)$, her expected payoff is

$$-c_2 + 1 - a_1 + \frac{(1-\mu_1(t))p_1(t)}{\mu_1(t) + (1-\mu_1(t))p_1(t)}(1-w)d.$$

Economically, the payoff gain of seeing the challenge over yielding to the challenge is the expected payoff gain net the challenge cost:

$$\frac{(1-\mu_1(t))p_1(t)}{\mu_1(t) + (1-\mu_1(t))p_1(t)}(1-w)d - c_2,$$

which has the same sign as

$$\frac{(1-\mu_1(t))p_1(t)}{\mu_1(t)+(1-\mu_1(t))p_1(t)} - \frac{c_2}{(1-w)d}$$

and the same sign as

$$\left[1-\frac{c_2}{(1-w)d}\right] - \frac{\mu_1(t)}{\mu_1(t)+(1-\mu_1(t))p_1(t)} \equiv \underline{\mu}_1 - \frac{\mu_1(t)}{\mu_1(t)+(1-\mu_1(t))p_1(t)}.$$

Therefore, an unjustified player 2 strictly prefers to see the challenge / is indifferent between seeing and yielding to the challenge / strictly prefers to yield to the challenge if and only if

$$\frac{\mu_1(t)}{\mu_1(t)+(1-\mu_1(t))p_1(t)} < / = / > \underline{\mu}_1 \Leftrightarrow p_1(t) > / = / < \frac{\mu_1(t)}{1-\mu_1(t)} \Big/ \frac{\underline{\mu}_1}{1-\underline{\mu}_1}.$$

In particular, she always strictly prefers to yield to the challenge when

$$\frac{\mu_1(t)}{1-\mu_1(t)} \Big/ \frac{\underline{\mu}_1}{1-\underline{\mu}_1} > 1 \Leftrightarrow \mu_1(t) > \underline{\mu}_1.$$

In summary, an unjustified player 2 could prefer either seeing or yielding to a challenge if $\mu_1(t) \leq \underline{\mu}_1$ and strictly prefers to yield to a challenge if $\mu_1(t) > \underline{\mu}_1$.

Second, an unjustified player 1 decides between challenging and not challenging when a challenge opportunity arrives at time t. Given player 2's reputation $\mu_2(t)$ and an unjustified player 2's probability $s_2(t)$ of seeing a challenge, when a challenge opportunity arrives at time t, player 1's expected payoff of challenging is

$$-c_1+\mu_2(t)(1-a_2)+(1-\mu_2(t))[s_2(t)[1-a_2+wd]+(1-s_2(t))a_1]$$

If player 1 does not challenge, then his continuation payoff is simply $1-a_2$, the equilibrium payoff from conceding. The payoff gain of challenging over not challenging is

$$(1-\mu_2(t))[1-s_2(t)(1-w)]d-c_1$$

An unjustified player 1 strictly prefers to challenge / is indifferent between challenging and not challenging / strictly prefers not to challenge if and only if

$$(1-\mu_2(t))[1-s_2(t)(1-w)]d > / = / < c_1$$

which is rearranged to be

$$s_2(t) < / = / > \frac{1}{1-w}\left[1-\frac{c_1/d}{1-\mu_2(t)}\right] = \frac{1}{1-w}\left[1-\frac{1-\overline{\mu}_2}{1-\mu_2(t)}\right].$$

An unjustified player 1 is indifferent between challenging and not challenging only if $0 \leq s_2(t) \leq 1$. $s_2(t) \geq 0$ implies

$$\frac{1}{1-\mu_2(t)}\frac{c_1}{d} \leq 1 \Rightarrow \mu_2(t) \leq 1-\frac{c_1}{d} \equiv \overline{\mu}_2$$

and $s_2(t) \leq 1$ implies

$$1 - \frac{1}{1-\mu_2(t)} \frac{c_1}{a_1+a_2-1} \leq 1 - w \Rightarrow \mu_2(t) \geq 1 - \frac{c_1}{wd}.$$

By assumption $1 - \frac{c_1}{wd}$ is negative so the second condition is non-binding. Therefore, regardless of an unjustified player 2's strategy, an unjustified player 1 could prefer either challenging or not challenging when $\mu_2(t) \leq \overline{\mu}_2$ and strictly prefers not to challenge when $\mu_2(t) > \overline{\mu}_2$.

In summary, there are three different combinations of strategies under different reputations. First, when $\mu_2(t) \leq \overline{\mu}_2$ and $\mu_1(t) \leq \underline{\mu}_1$, player 1 mixes between challenging and not challenging and player 2 mixes between seeing and yielding to a challenge, with the appropriately defined probabilities $p_1(t)$ and $s_2(t)$. Second, when $\mu_2(t) \leq \overline{\mu}_2$ and $\mu_1(t) > \underline{\mu}_1$, an unjustified player 2 strictly prefers not to see a challenge and an unjustified player 1 strictly prefers to challenge. Third and finally, when $\mu_2(t) > \overline{\mu}_2$, an unjustified player 1 does not challenge, so when player 2 faces a challenge, it is from a justified player 1, so an unjustified player 2 does not see a challenge. □

Proof of Lemma 2. First, suppose $\mu_2(t) \leq \overline{\mu}_2$ and $\mu_1(t) \leq \underline{\mu}_1$. Since player 1 is indifferent between challenging and not, it must be that he gets a payoff of $1-a_2$ if he challenges. Given player 1's reputation $\mu_1(t)$, an unjustified player 2 who concedes at time t gets a payoff of $1-a_1$. An unjustified player 2 who concedes at time $t+dt$ gets a payoff of

$$(1-a_1)e^{-r_2 dt}(1-\lambda_1 dt - \chi_1(t)dt) + a_2\lambda_1 dt + (1-a_1)\chi_1(t)dt$$

where λ_1 is player 1's concession rate and $\chi_1(t)$ is player 1's challenging rate. The expression is simplified to, as $dt^2 \approx 0$,

$$(1-a_1)[(1-r_2 dt)(1-\lambda_1 dt - \chi_1(t)dt) + \chi_1(t)dt] + a_2\lambda_1 dt.$$

An unjustified player 2 is indifferent between conceding at time t and conceding at time $t+dt$ if

$$1-a_1 = (1-a_1)(1-r_2 dt - \lambda_1 dt) + a_2\lambda_1 dt$$

$$(1-a_1)(r_2+\lambda_1) = a_2\lambda_1$$

$$\lambda_1 = \frac{r_2(1-a_1)}{a_1+a_2-1}$$

the same concession rate as in Abreu and Gul (2000). An unjustified player 1 decides between conceding and staying at each instant. An unjustified player 1 who concedes gets a payoff of $1-a_2$. An unjustified player 1 who stays from t to $t+dt$ and has a reputation $\mu_1(t) < \underline{\mu}_1$ gets a payoff of

$$(1-a_2)e^{-r_1 dt}(1-\lambda_2 dt) + a_1\lambda_2 dt = (1-a_2)(1-r_1 dt - \lambda_2 dt) + a_1\lambda_2 dt.$$

where λ_2 is player 2's concession rate. An unjustified player 1 is indifferent between conceding and staying if

$$1 - a_2 = (1 - a_2)(1 - r_1 dt - \lambda_2 dt) + a_1 \lambda_2 dt$$

$$\lambda_2 = \frac{r_1(1 - a_2)}{a_1 + a_2 - 1}.$$

Second, suppose $\mu_2(t) \leq \overline{\mu}_2$ and $\mu_1(t) > \underline{\mu}_1$. An unjustified player 1 will get more than $1 - a_2$ because an unjustified player 1 always challenges and an unjustified player 2 always yields to a challenge. When an unjustified player 1 will challenge if a challenge opportunity arrives between t and $t + dt$, he is indifferent between conceding at time t and conceding at $t + dt$ and always optimally challenging between t and $t + dt$ if

$$\begin{aligned} 1 - a_2 &= (1 - a_2)e^{-r_1 dt}(1 - \lambda_2(t)dt - g_1 dt) + a_1 \lambda_2(t) dt \\ &\quad + g_1 dt \left[\mu_2(t)(1 - a_2 - c_1) + (1 - \mu_2(t))(a_1 - c_1)\right] \end{aligned}$$

which simplifies to

$$\begin{aligned} 1 - a_2 &= (1 - a_2)(1 - r_1 dt - \lambda_2(t)dt - g_1 dt) + a_1 \lambda_2(t) dt \\ &\quad + g_1 dt \left[a_1 - c_1 - \mu_2(t)(a_1 + a_2 - 1)\right]. \end{aligned}$$

Rearranged,

$$(1 - a_2)(r_1 + \lambda_2(t) + g_1) = a_1 \lambda_2(t) + g_1 \left[a_1 - c_1 - \mu_2(t)(a_1 + a_2 - 1)\right].$$

We have

$$\lambda_2(t) = \lambda_2 + g_1 \mu_2(t) + g_1 \left[\frac{c_1}{a_1 + a_2 - 1} - 1\right] = \lambda_2 + g_1 \left[\mu_2(t) - \overline{\mu}_2\right] \leq \lambda_2.$$

If $\lambda_2(t) < 0$, then at time t, an unjustified player 2 concedes with a strictly positive probability so that

$$\lambda_2 - g_1 \left[\overline{\mu}_2 - \frac{\mu_2(t)}{\mu_2(t) + (1 - \mu_2(t))(1 - Q_2)}\right] = 0$$

$$Q_2 = 1 - \frac{\mu_2(t)}{1 - \mu_2(t)} \Bigg/ \frac{\overline{\mu}_2 - \frac{\lambda_2}{g_1}}{1 - (\overline{\mu}_2 - \frac{\lambda_2}{g_1})}.$$

Third and finally, suppose $\mu_2(t) > \overline{\mu}_2$. Player 1 never challenges and an unjustified player 2 always yields to a challenge because any player 1 who challenges is justified. The challenge opportunity is essentially non-existent on equilibrium path. Players concede at the constant rates λ_1 and λ_2. $\square$

B.2 Reputation Dynamics

Proof of Lemma 3. Fix A, B, t^0 and $\mu(t^0) = \mu^0$. Rearrange $\mu'(t) = A\mu(t) + B\mu^2(t)$,

$$\frac{\mu'(t)}{\mu^2(t)} = A\frac{1}{\mu(t)} + B.$$

Let $\nu(t) = 1/\mu(t)$ and $-\nu'(t) = \mu'(t)/\mu^2(t)$,

$$\nu'(t) = -A\nu(t) - B$$

$$\nu(t) = C\exp(-At) - B/A$$

Given the initial condition $\mu(t^0) = \mu^0$,

$$1/\mu^0 = C\exp(-At^0) - B/A \Rightarrow C = \left[\frac{1}{\mu^0} + \frac{B}{A}\right]\exp(At^0).$$

Altogether,

$$\mu(t;t^0,\mu^0,A,B) = \frac{1}{\left(\frac{1}{\mu^0} + \frac{B}{A}\right)\exp(-A(t-t^0)) - \frac{B}{A}}.$$

$\mu'(t) > 0$ when $A + B\mu(t) > 0$, that is, $\mu(t) > -A/B$. Therefore, $\mu'(t) > 0$ for all $t \geq t^0$ if and only if $\mu'(t^0) > 0$, i.e. $\mu^0 = \mu(t^0) > -A/B$.

The time length t it takes for the reputation to increase from μ^0 to μ is determined by

$$\frac{1}{\mu} = \left(\frac{1}{\mu^0} + \frac{B}{A}\right)\exp(-At) - \frac{B}{A},$$

which is rearranged to

$$t(\mu;\mu^0,A,B) = \frac{1}{A}\left[\ln\left(\frac{1}{\mu^0} + \frac{B}{A}\right) - \ln\left(\frac{1}{\mu} + \frac{B}{A}\right)\right].$$

□

Proof of Lemma 4. At any time t, if $\mu_1(t) < \underline{\mu}_1$ and $\mu_2(t) < \overline{\mu}_2$, by Lemma 1(a), an unjustified player 1 challenges with probability

$$p_1(t) = \frac{\mu_1(t)}{1-\mu_1(t)} \bigg/ \frac{\underline{\mu}_1}{1-\underline{\mu}_1},$$

and at time t, given player 2's reputation $\mu_2(t) < \overline{\mu}_2$, an unjustified player 2 sees a challenge with probability

$$s_2(t) = \frac{1}{1-w}\frac{\overline{\mu}_2 - \mu_2(t)}{1-\mu_2(t)}.$$

Player 1's reputation dynamics is derived from the Martingale property $\mu_1(t) = \mathbb{E}[\mu_1(t+$

$dt)|\mathscr{F}]$.

$$\begin{aligned}\mu_1(t) &= \mu_1(t)g_1dt\cdot 1+(1-\mu_1(t))p_1(t)g_1dt\cdot 0+\lambda_1dt\cdot 0\\ &\quad+[1-\mu_1(t)g_1dt-(1-\mu_1(t))p_1(t)g_1dt-\lambda_1dt]\mu_1(t+dt)\end{aligned}$$

Rearrange,

$$\mu_1(t+dt)-\mu_1(t)=\left[\mu_1(t)g_1dt+\mu_1(t)g_1\frac{1-\underline{\mu}_1}{\underline{\mu}_1}dt+\lambda_1dt\right]\mu_1(t+dt)-\mu_1(t)g_1dt$$

Taking $dt\to 0$, we arrive at the reputation dynamics described by

$$\mu_1'(t)=(\lambda_1-g_1)\mu_1(t)+\frac{g_1}{\underline{\mu}_1}\mu_1^2(t).$$

By Lemma 3, player 1's reputation evolution is

$$\mu_1^S(t;t^0,\mu_1^0)=\mu(t;t^0,\mu_1^0,\lambda_1-g_1,g_1/\underline{\mu}_1).$$

Since player 2 concedes with a constant rate λ_2, player 2's reputation evolution in this sometimes-challenging phase is simply

$$\mu_2^S(t;t^0,\mu_2^0)=\mu(t;t^0,\mu_2^0,\lambda_2,0).$$

The sometimes-challenging sometimes-seeing phase ends as soon as $\mu_1(t)$ hits $\underline{\mu}_1$ or $\mu_2(t)$ hits $\overline{\mu}_2$. If the reputation building follows the equations above the whole time, it takes duration

$$t_1^S(\mu_1^0)\equiv t(\underline{\mu}_1;\mu_1^0,\lambda_1-g_1,g_1/\underline{\mu}_1)$$

for $\mu_1(t)$ to hit $\underline{\mu}_1$ and duration

$$t_2^S(\mu_2^0)\equiv t(\overline{\mu}_2;\mu_2^0,\lambda_2,0)$$

for $\mu_2(t)$ to hit $\overline{\mu}_2$, so the duration of the phase is

$$t^S(\mu_1^0,\mu_2^0)\equiv\min\{t_1^S(\mu_1^0),t_2^S(\mu_2^0)\}.$$

□

Proof of Lemma 5. By Lemma 1(c), when $\mu_1(t)>\underline{\mu}_1$, an unjustified player 2 never sees a challenge, and if $\mu_2(t)\le\underline{\mu}_1$ in addition, an unjustified player 1 always challenges whenever the opportunity arises. Given that an unjustified player 1 always challenges, player 1's reputation evolves according to

$$\mu_1(t)=\mu_1(t)g_1dt\cdot 1+(1-\mu_1(t))g_1dt\cdot 0+\lambda_1dt\cdot 0+(1-g_1dt-\lambda_1dt)\cdot\mu_1(t+dt).$$

Rearranged and simplified,

$$\frac{\mu_1(t+dt)-\mu_1(t)}{dt}=(g_1+\lambda_1)\mu_1(t+dt)-g_1\mu_1(t).$$

Taking $dt\to 0$,

$$\mu_1'(t)=\lambda_1\mu_1(t).$$

Therefore,

$$\mu_1^A(t)=\mu(t;t^0,\mu_1(t^0),\lambda_1,0).$$

Player 2's reputation evolves according to

$$\mu_2^A(t)=\lambda_2(\mu_2(t))dt\cdot 0+[1-\lambda_2^A(t)dt]\cdot\mu_2(t+dt).$$

Taking $dt\to 0$,

$$\mu_2'(t)=\lambda_2^A(t)\mu_2(t)=\left[\lambda_2-g_1\left[\overline{\mu}_2-\mu_2(t)\right]\right]\mu_2(t)$$

Rearrange,

$$\mu_2'(t)=\left(\lambda_2-g_1\overline{\mu}_2\right)\mu_2(t)+g_1\mu_2^2(t).$$

Therefore,

$$\mu_2^A(t)=\mu(t;t^0,\mu_2(t^0),\lambda_2-g_1\overline{\mu}_2,g_1).$$

Finally, the phase lasts until $\mu_2(t)$ reaches $\overline{\mu}_2$ or $\mu_1(t)$ reaches 1. □

Proof of Lemma 6. Player 1's reputation is determined as follows:

$$\mu_1(t)=\mu_1(t)g_1dt\cdot 1+\lambda_1dt\cdot 0+[1-\mu_1(t)g_1dt-\lambda_1dt]\cdot\mu_1(t+dt).$$

$$\mu_1(t+dt)-\mu_1(t)=-\mu_1(t)g_1dt+\mu_1(t)g_1dt\mu_1(t+dt)+\lambda_1dt\mu_1(t+dt)$$

$$\frac{\mu_1(t+dt)-\mu_1(t)}{dt}=g_1\mu_1(t+dt)\mu_1(t)+\lambda_1\mu_1(t+dt)-g_1\mu_1(t)$$

Taking $dt\to 0$,

$$\mu_1'(t)=(\lambda_1-g_1)\mu_1(t)+g_1\mu_1^2(t).$$

Player 2's reputation evolves according to

$$\mu_2(t)=(1-\lambda_2dt)\cdot\mu_2(t+dt)+\lambda_2dt\cdot 1$$

Taking $dt\to 0$,

$$\mu_2'(t)=\lambda_2\mu_2(t).$$

The phase lasts until $\mu_1(t)$ reaches 1 or $\mu_2(t)$ reaches 1. $\square$

B.3 Equilibrium

Proof 1 of Theorem 1. First, suppose $\lambda_1 \geq g_1$. Let $t_2^N(\mu_2^*, 1)$ denote the time it takes for player 2's reputation to increase from μ_2^* to 1 when an unjustified player 1 never challenges and an unjustified player 2 never responds to a challenge (i.e., in the N phase). Let $\mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1)$ denote player 1's reputation at time $-t_2^N(\mu_2^*, 1)$ so that player 1's reputation at time 0 is 1 when an unjustified player 1 never challenges and an unjustified player 2 never responds to a challenge (i.e., in the N phase). There are two sub-cases:

1. $\mu_1^* < \mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1) \leq 1$.
2. $0 < \mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1) \leq \mu_1^*$.

Second, suppose $\lambda_1 < g_1$. Before we consider sub-cases, we prove the following claim that provides a lower bound for $\mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1)$.

Claim 1. $\mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1) > (1 - \frac{\lambda_1}{g_1})\mu_1^*$.

Proof of Claim 1. If $g_1 \leq \lambda_1$, then $\mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1) > 0 \geq (1 - \frac{\lambda_1}{g_1})\mu_1^*$. If $g_1 > \lambda$, then

$$
\begin{aligned}
&\mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1) \\
=&\mu(-t_2^N(\mu_2^*, 1); 0, 1, \lambda_1 - g_1, g_1) && \text{by definition of } \mu_1^N(\cdot) \\
=&\mu(-t(1; \mu_2^*, \lambda_2, 0); 0, 1, \lambda_1 - g_1, g_1) && \text{by definition of } t_2^N(\cdot) \\
=&\mu(0; t(1, \mu_2^*, \lambda_2, 0), 1, \lambda_1 - g_1, g_1) && \text{by property of } t(\cdot) \\
=&\mu(0; \frac{1}{\lambda_2}\ln(\frac{1}{\mu_2^*}), 1, \lambda_1 - g_1, g_1) && \text{by definition of } t(\cdot) \\
=&1 \Big/ \left[\left(1 + \frac{g_1}{\lambda_1 - g_1}\right)\exp\left((\lambda_1 - g_1)\frac{1}{\lambda_2}\ln(\frac{1}{\mu_2^*})\right) - \frac{g_1}{\lambda_1 - g_1}\right] && \text{by definition of } \mu(\cdot) \\
=&\frac{\lambda_1 - g_1}{\lambda_1 \exp\left((\lambda_1 - g_1)\frac{1}{\lambda_2}\ln\left(\frac{1}{\mu_2^*}\right)\right) - g_1} && \text{by algebra} \\
=&\frac{g_1 - \lambda_1}{g_1 - \lambda_1 \exp\left((\lambda_1 - g_1)\frac{1}{\lambda_2}\ln\left(\frac{1}{\mu_2^*}\right)\right)} > \frac{g_1 - \lambda_1}{g_1} > \mu_1^* \frac{g_1 - \lambda_1}{g_1}
\end{aligned}
$$

where the first inequality follows from $\exp\left((\lambda_1 - g_1)\frac{1}{\lambda_2}\ln\left(\frac{1}{\mu_2^*}\right)\right) < 1$ because $\lambda_1 < g_1$ and $\mu_2^* < 1$, and the second inequality follows from $\mu_1^* < 1$. $\square$

1. $\mu_1^* < \mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1) \leq 1$.
2. $(1 - \frac{\lambda_1}{g_1})\mu_1^* < \mu_1^N(-t_2^N(\mu_2^*, 1); 0, 1) \leq \mu_1^*$.

Define

$$\widetilde{\mu}_i(t) \equiv \begin{cases} \mu_i^N(-t,1) & \text{if } t \le t^N \\ \mu_i^A(t^A - t, \mu_i^N) & \text{if } t^N < t \le t^N + t^A \\ \mu_i^S(t^A + t^N - t, \mu_i^A) & \text{if } t^N + t^A < t \end{cases}$$

where $t^N = t_2^N(\mu_2^*, 1)$, $\mu_2^N = \mu_2^*$, $\mu_1^N = \mu_1^N(-t^N, 1)$, $t^A = t_1^A(\mu_1^*, \max\{\mu_1^*, \mu_1^N\})$, $\mu_1^A = \min\{\mu_1^N, \mu_1^*\}$, and $\mu_2^A = \mu_2^A(-t^A, \mu_2^*)$. Define $t_i(z_i, 1)$ as the time it takes for player i to reach reputation 1 from reputation z_i. $t_i(z_i)$ is the unique solution to $\widetilde{\mu}_i(t) = z_i$ when $g_1 \le \lambda_1$, or $g_1 > \lambda_1$ and $z_1 \ge \mu_1^*(1 - \frac{\lambda_1}{g_1})$ (the solution exists and is unique because $\widetilde{\mu}_i(\cdot)$ has image $(0,1]$ and is strictly decreasing), and is $+\infty$ if $g_1 > \lambda_1$ and $z_1 < \mu_1^*(1 - \frac{\lambda_1}{g_1})$. Define

$$T(z_1, z_2) \equiv \min\{t_1(z_1), t_2(z_2)\}.$$

The equilibrium reputation of player $i = 1, 2$ at time t when player 1's initial reputation is z_1 and player 2's initial reputation is z_2 is

$$\mu_i^*(t, z_1, z_2) = \widetilde{\mu}_i(T(z_1, z_2) - t).$$

The probability that player i concedes is

$$Q_i(z_1, z_2) = 1 - \frac{1 - \mu_i^*(0, z_1, z_2)}{\mu_i^*(0, z_1, z_2)} \frac{z_i}{1 - z_i}.$$

□

Proof 2 of Theorem 1. Equilibrium strategies depend on the initial probabilities of being justified, z_1 and z_2. Equilibrium strategy phases are characterized as follows in 4 big cases and 19 sub-cases, ordered from large to small probabilities of being justified. Cases 3 and 4 characterize equilibrium strategy phases when z_1 and z_2 are sufficiently small.

1. Suppose $z_2 \ge \overline{\mu}_2$. Only the never-challenging never-seeing phase exists in equilibrium: player 1 never challenges, player 2 never sees a challenge, and both players concede at Abreu-Gul rates, until both players' reputations reach 1 simultaneously. The duration of the never-challenging never-seeing phase is $t^N = \min\{t_1^N(z_1, 1), t_2^N(z_2, 1)\}$. Who concedes with what probability at time 0 is determined as follows.

    (a) Suppose $z_1 > 1 - \frac{\lambda_1}{g_1}$ and $t_1^N(z_1, 1) < t_2^N(z_2, 1)$. Player 2 concedes with a positive probability at time 0 so that her reputation at time 0 jumps to $y > z_2$ where $t_1^N(z_1, 1) = t_2^N(y, 1)$.

    (b) Suppose $z_1 > 1 - \frac{\lambda_1}{g_1}$ and $t_1^N(z_1, 1) = t_2^N(z_2, 1)$. Neither player concedes with a positive probability at time 0.

    (c) Suppose $z_1 > 1 - \frac{\lambda_1}{g_1}$ and $t_1^N(z_1, 1) > t_2^N(z_2, 1)$. Player 1 concedes with a positive probability at time 0 so that his reputation at time 0 jumps to $x > z_1$ where $t_1^N(x, 1) = t_2^N(z_2, 1)$.

(d) Suppose $z_1 \leq 1 - \frac{\lambda_1}{g_1}$. Player 1 concedes with a positive probability at time 0 so that his reputation at time 0 jumps to $x > 1 - \frac{\lambda_1}{g_1}$ where $t_1^N(x,1) = t_2^N(z_2,1)$.

In the never-challenging never-seeing phase, it takes time $t_2^N(\overline{\mu}_2, 1)$ for player 2's reputation to reach 1 from $\overline{\mu}_2$. Time $t_2^N(\overline{\mu}_2, 1)$ before the end of the game (if the game lasts longer than that), player 1's reputation would be $\mu_1^N \equiv \mu_1^N(0; t_2^N(\overline{\mu}_2, 1), 1)$.

2. Suppose $z_2 < \overline{\mu}_2$ and $z_1 \geq \underline{\mu}_1$. The sometimes-challenging sometimes-seeing phase does not exist. The never-challenging never-seeing phase exists and an additional always-challenging never-seeing phase may exist.

   (a) Suppose $z_1 \geq \mu_1^N$ and $z_2 < \overline{\mu}_2$. The always-challenging never-seeing phase does not exist and only the never-challenging never-seeing phase exists. Player 2 concedes with a positive probability at time 0 so that player 2's reputation at time 0 jumps to $y > z_2$ where $t_2^N(y,1) = t_1^N(z_1,1)$; player 1 never challenges, player 2 never sees a challenge, and both players concede at constant Abreu-Gul rates until players' reputations reach 1 simultaneously.

   Otherwise, $\underline{\mu}_1 \leq z_1 < \mu_1^N$, so an additional always-challenging never-seeing phase exists. Namely, from time 0 to time t^A defined below, player 1 always challenges and player 2 never sees a challenge, and then from time $t^A + t^N$ on, player 1 never challenges and player 2 never sees a challenge, where $t^N = t_2^N(\overline{\mu}_2, 1)$. Who concedes with what probability at time 0 is determined as follows.

   (b) Suppose $\underline{\mu}_1 \leq z_1 < \mu_1^N$ and $z_2 \leq \overline{\mu}_2 - \frac{\lambda_2}{g_1}$. Player 2 concedes with a positive probability at time 0 so that player 2's reputation at time 0 jumps to $y > \overline{\mu}_2 - \frac{\lambda_2}{g_1}$ where $t_1^A(z_1, \mu_1^N) = t_2^A(y, \overline{\mu}_2)$.

   (c) Suppose $\underline{\mu}_1 \leq z_1 < \mu_1^N$, $\overline{\mu}_2 - \frac{\lambda_2}{g_1} < z_2 \leq \overline{\mu}_2$ and $t_1^A(z_1, \mu_1^N) < t_2^A(z_2, \overline{\mu}_2)$. Player 2 concedes with a positive probability at time 0 so that her reputation at time 0 jumps to $y > z_2$ where $t_1^A(z_1, \mu_1^N) = t_2^A(y, \overline{\mu}_2)$.

   (d) Suppose $\underline{\mu}_1 \leq z_1 < \mu_1^N$, $\overline{\mu}_2 - \frac{\lambda_2}{g_1} < z_2 \leq \overline{\mu}_2$ and $t_1^A(z_1, \mu_1^N) = t_2^A(z_2, \overline{\mu}_2)$. Neither player concedes with a positive probability.

   (e) Suppose $\underline{\mu}_1 \leq z_1 < \mu_1^N$, $\overline{\mu}_2 - \frac{\lambda_2}{g_1} < z_2 \leq \overline{\mu}_2$ and $t_1^A(z_1, \mu_1^N) > t_2^A(z_2, \overline{\mu}_2)$. Player 1 concedes with a positive probability at time 0 so that his reputation at time 0 jumps to $x > z_1$ where $t_1^A(x, \mu_1^N) = t_2^A(z_2, \overline{\mu}_2)$.

3. Suppose $z_2 < \overline{\mu}_2$, $z_1 < \underline{\mu}_1$, and $\mu_1^N \leq \underline{\mu}_1$. The always-challenging never-seeing phase does not exist and the sometimes-challenging sometimes-seeing phase may exist. If the

sometimes-challenging sometimes-seeing phase exists, then the reputations at the end of the sometimes-challenging sometimes-seeing phase are the same as those at the beginning of the never-challenging never-seeing phase: μ_1^N and $\overline{\mu}_2$. If $z_1 \geq \mu_1^N$, then the sometimes-challenging sometimes-seeing phase does not exist.

(a) Suppose $\mu_1^N \leq z_1 < \underline{\mu}_1$. Player concedes with a positive probability at time 0 so that player 2's reputation jumps to $y > \overline{\mu}_2$ where $t_1^N(z_1, 1) = t_2^N(y, 1)$. The game starts in the never-challenging never-seeing phase and does not have a sometimes-challenging sometimes-seeing phase, $t^S = 0$, and the never-challenging never-seeing phase lasts the entire game for $t^N = t_1^N(z_1, 1)$.

Otherwise, if $z_1 < \mu_1^N$ instead, then the sometimes-challenging sometimes-seeing phase exists. The length of the phase is determined as follows. The length of the subsequent never-challenging never-seeing phase is determined to be $t^N = t_2^N(\overline{\mu}_2, 1)$.

(b) Suppose $(1 - \frac{\lambda_1}{g_1})\underline{\mu}_1 < z_1 < \mu_1^N$ and $t_1^S(z_1, \mu_1^N) < t_2^S(z_2, \overline{\mu}_2)$. Player 2 concedes with a positive probability so that player 2's reputation jumps to $y > z_2$ where $t_1^S(z_1, \mu_1^N) = t_2^S(y, \overline{\mu}_2)$. The sometimes-challenging sometimes-seeing phase lasts for a length of $t^S = t_1^S(z_1, \mu_1^N)$.

(c) Suppose $(1 - \frac{\lambda_1}{g_1})\underline{\mu}_1 < z_1 < \mu_1^N$ and $t_1^S(z_1, \mu_1^N) = t_2^S(z_2, \overline{\mu}_2)$. Neither player concedes with a positive probability at time 0. The sometimes-challenging sometimes-seeing phase lasts for a length of $t^S = t_2^S(z_2, \overline{\mu}_2)$.

(d) Suppose $(1 - \frac{\lambda_1}{g_1})\underline{\mu}_1 < z_1 < \mu_1^N$ and $t_1^S(z_1, \mu_1^N) > t_2^S(z_2, \overline{\mu}_2)$. Player 1 concedes with a positive probability so that player 1's reputation jumps to $x > z_1$ where $t_1^S(x, \mu_1^N) = t_2^S(z_2, \overline{\mu}_2)$. The sometimes-challenging sometimes-seeing phase lasts for a length of $t^S = t_2^S(z_2, \overline{\mu}_2)$.

(e) Suppose $z_1 \leq (1 - \frac{\lambda_1}{g_1})\underline{\mu}_1 < \mu_1^N$. Player 1 concedes with a positive probability at time 0 so that player 1's reputation at time 0 jumps to $(1 - \frac{\lambda_1}{g_1})\underline{\mu}_1 < x < \mu_1^N$ where $t_1^S(x, \mu_1^N) = t_2^S(z_2, \overline{\mu}_2)$. The sometimes-challenging sometimes-seeing phase lasts for a length of $t^S = t_2^S(z_2, \overline{\mu}_2)$.

4. Suppose $z_2 < \overline{\mu}_2$ and $z_1 < \underline{\mu}_1 < \mu_1^N$. The always-challenging never-seeing phase exists, and the phase lasts for length of $t_1^A(\underline{\mu}_1, \mu_1^N)$ when player 1's reputation is between $\underline{\mu}_1$ and μ_1^N. If the phase lasts for the full length, player 2's reputation at the beginning of the phase is $\mu_2^A = \mu_2^A(0; t_1^A(\underline{\mu}_1, \mu_1^N), \overline{\mu}_2)$. If $z_2 \geq \mu_2^A$, then the sometimes-challenging sometimes-seeing phase does not exist and the game starts in the always-challenging never-seeing phase.

(a) Suppose $z_2 \geq \mu_2^A$. The sometimes-challenging sometimes-seeing phase does not exist and the game starts in the always-challenging never-seeing phase. Player 1 concedes with a positive probability at time 0 so that his reputation at time 0 jumps to $x > z_1$ where $t_1^A(x, \mu_1^N) = t_2^A(z_2, \overline{\mu}_2)$. The length of the phase is $t^A = t_2^A(z_2, \overline{\mu}_2)$.

Otherwise, if $z_2 < \mu_2^A$, then the sometimes-challenging sometimes-seeing phase exists. Both players' reputations at the end of the sometimes-challenging sometimes-seeing phase and equivalently at the beginning of the always-challenging never-seeing phase are $\underline{\mu}_1$ and μ_2^A. The length of the always-challenging never-seeing phase is $t^A = t_1^A(\underline{\mu}_1, \mu_1^N)$, the length of the never-challenging never-seeing phase is $t^N = t_2^N(\overline{\mu}_2, 1)$, and the length of the sometimes-challenging sometimes-seeing phase is determined as follows.

(b) Suppose $z_2 < \mu_2^A$, $z_1 > (1 - \frac{\lambda_1}{g_1})\underline{\mu}_1$ and $t_1^S(z_1, \underline{\mu}_1) < t_2^S(z_2, \mu_2^A)$. Player 2 concedes with a positive probability at time 0 so that her reputation at time 0 jumps to $y > z_2$ where $t_1^S(z_1, \underline{\mu}_1) = t_2^S(y, \mu_2^A)$. The length of the sometimes-challenging sometimes-seeing phase is $t^S = t_1^S(z_1, \underline{\mu}_1)$.

(c) Suppose $z_2 < \mu_2^A$, $z_1 > (1 - \frac{\lambda_1}{g_1})\underline{\mu}_1$ and $t_1^S(z_1, \underline{\mu}_1) = t_2^S(z_2, \mu_2^A)$. Neither player concedes with a positive probability at time 0. The length of the sometimes-challenging sometimes-seeing phase is $t^S = t_2^S(z_2, \mu_2^A)$.

(d) Suppose $z_2 < \mu_2^A$, $z_1 > (1 - \frac{\lambda_1}{g_1})\underline{\mu}_1$ and $t_1^S(z_1, \underline{\mu}_1) > t_2^S(z_2, \mu_2^A)$. Player 1 concedes at with a positive probability time 0 so that his reputation at time 0 jumps to $x > z_1$ where $t_1^S(x, \underline{\mu}_1) = t_2^S(z_2, \mu_2^A)$. The length of the sometimes-challenging sometimes-seeing phase is $t^S = t_2^S(z_2, \mu_2^A)$.

(e) Suppose $z_2 < \mu_2^A$ and $z_1 \leq (1 - \frac{\lambda_1}{g_1})\underline{\mu}_1$. Player 1 concedes with a positive probability at time 0 so that his reputation at time 0 jumps to $x > (1 - \frac{\lambda_1}{g_1})\underline{\mu}_1$ where $t_1^S(x, \underline{\mu}_1) = t_2^S(z_2, \mu_2^A)$. The length of the sometimes-challenging sometimes-seeing phase is $t^S = t_2^S(z_2, \mu_2^A)$.

□

C Bargaining with Two-Sided Ultimatums

C.1 Equilibrium Strategies

Proof of Lemma 7. An unjustified player i's expected payoff from challenging when an unjustified player j sees a challenge with probability $s_j(t)$ is

$$-c_i + \mu_j(t)(1 - a_j) + (1 - \mu_j(t))[s_j(t)(1 - a_j) + (1 - s_j(t))a_i]$$

$$= \quad -c_i + 1 - a_j + (1-\mu_j(t))(1-s_j(t))d.$$

An unjustified player i never challenges if challenging yields an even lower payoff than simply conceding and guaranteeing $1-a_j$:

$$-c_i + 1 - a_j + (1-\mu_j(t))(1-s_j(t))d < 1 - a_j$$

which simplifies to $\mu_j(t) > 1 - \frac{c_i}{d} = \mu_j^*$. An unjustified player i is indifferent between challenging and not challenging (and then conceding) if

$$-c_i + 1 - a_j + (1-\mu_j(t))(1-s_j(t))d = 1 - a_j,$$

that is, if an unjustified player sees a challenge with probability

$$s_j^*(t) = 1 - \frac{c_i}{d}\frac{1}{1-\mu_j(t)} = \frac{1-\mu_j(t)-\frac{c_i}{d}}{1-\mu_j(t)} = \frac{\mu_j^* - \mu_j(t)}{1-\mu_j(t)}.$$

Let's now consider a challenged player's optimal strategy. While his or her payoff from yielding to a challenge is simply $1-a_i$, an unjustified player j's expected payoff from seeing a challenge if an unjustified player i challenges with probability $p_i(t)$ is

$$-c_j + 1 - a_i + \frac{(1-\mu_i(t))p_i(t)}{\mu_i(t) + (1-\mu_i(t))p_i(t)}d.$$

The gain from seeing a challenge over yielding to a challenge is

$$\frac{(1-\mu_i(t))p_i(t)}{\mu_i(t) + (1-\mu_i(t))p_i(t)}d - c_j.$$

If $\mu_i(t) > 1 - \frac{c_j}{d} = \mu_i^*$, then player j never sees a challenge. If $\mu_i(t) \leq \mu_i^*$, then player j is indifferent between seeing and yielding to a challenge if an unjustified player i challenges with probability

$$p_i^*(t) = \frac{\mu_i(t)}{1-\mu_i(t)} \bigg/ \frac{\mu_i^*}{1-\mu_i^*}.$$

Consider three separate cases below.

1. First, if $\mu_i(t) \leq \mu_i^*$ for $i = 1, 2$, since there is no dominant strategy for an unjustified player, in any equilibrium, both mix between challenging and not challenging as well as mix between seeing a challenge and yielding to a challenge, with the appropriate probabilities $p_i^*(t)$ and $s_i^*(t)$ characterized above.

2. Second, if $\mu_i(t) > \mu_i^*$ and $\mu_j(t) \leq \mu_j^*$, an unjustified player j never challenges and never sees a challenge. As a result, an unjustified player i always challenges and never sees a challenge.

3. Third and finally, if $\mu_i(t) > \mu_i^*$ for $i = 1, 2$, each unjustified player never challenges, and since no unjustified player will challenge, a challenge must be from a justified player, so an unjustified player never sees a challenge.

$\square$

Proof of Lemma 8. We consider the three cases separately.

1. Suppose $\mu_i(t) < \mu_i^*$ for $i = 1,2$ at time t. An unjustified player i challenges with probability $p_i^*(t)$ and sees a challenge with probability $s_i^*(t)$. An unjustified player i's expected payoff from conceding at time $t + dt$ is

$$\begin{aligned}&(1-\lambda_j(t)dt - g_j\chi_j^*(t)dt - g_i dt)e^{-r_i t}(1-a_j)\\&\quad+\lambda_j(t)dta_i + g_j\chi_j^*(t)dt(1-a_j) + g_i dt(1-a_j)\end{aligned}$$

which is simplified to

$$1-a_j-(\lambda_j(t)dt + r_i dt)(1-a_j) + \lambda_j(t)dta_i = 1-a_j - r_i dt\cdot(1-a_j) + \lambda_j(t)dt\cdot d.$$

An unjustified player i would be indifferent between conceding at time t and conceding at time $t+dt$ if

$$1-a_j - r_i dt\cdot(1-a_j) + \lambda_j(t)dt\cdot d = 1-a_j,$$

that is,

$$\lambda_j(t) = \frac{r_i(1-a_j)}{d} = \lambda_j.$$

In any equilibrium, each player i challenges with probability $\mu_i(t)/\mu_i^*$, sees a challenge with probability μ_i^*, and concedes with constant rate λ_i.

2. Suppose $\mu_i(t) > \mu_i^*$ and $\mu_j(t) < \mu_j^*$. An unjustified player i always challenges and never sees a challenge, and an unjustified player j never challenges and never sees a challenge. An unjustified player i's expected payoff from conceding at time $t+dt$ is

$$\begin{aligned}&[1-r_i dt - \lambda_j(t)dt - g_i dt - g_j\mu_j(t)dt](1-a_j) + \lambda_j(t)dt\cdot a_i\\&+g_i dt\cdot[-c_i + \mu_j(t)(1-a_j) + (1-\mu_j(t))a_i] + g_j\mu_j(t)\cdot dt(1-a_j),\end{aligned}$$

which simplifies to

$$1-a_j - r_i dt\cdot(1-a_j) + \lambda_j(t)dt\cdot d + g_i dt\cdot[-c_i + (1-\mu_j(t))d].$$

An unjustified player i is indifferent between conceding at time $t+dt$ and at time t if

$$\lambda_j(t) = \lambda_j - g_i(\mu_j^* - \mu_j(t)).$$

If $\lambda_j(t) < 0$, that is, if $\mu_j(t) \le \mu_j^* - \frac{\lambda_j}{g_i}$, no matter how low frequency player j concedes, player i always prefers not conceding at time t, compared to conceding at time $t+dt$.

On the other hand, an unjustified player j never challenges and never sees a challenge. His or her time t expected payoff from conceding at time $t+dt$ is

$$[1-r_j dt-\lambda_i(t)dt-g_i dt](1-a_i)+\lambda_i(t)dt\cdot a_i+g_i dt\cdot(1-a_i),$$

which simplifies to

$$1-a_i-r_j dt(1-a_i)-\lambda_i(t)dt\cdot d.$$

An unjustified player j is indifferent between conceding at time t and at time $t+dt$ if

$$1-a_i-r_j dt(1-a_i)-\lambda_i(t)dt\cdot d=1-a_i\Rightarrow\lambda_i(t)=\lambda_i.$$

If player i concedes with a rate strictly lower than λ_i, then an unjustified player j strictly prefers to concede.

Therefore, there are two cases:

(a) If $\mu_i(t)>\mu_i^*$ and $\mu_j(t)<\mu_j^*-\frac{\lambda_j}{g_i}$, player i strictly prefers not to concede at time t and player j strictly prefers to concede at time t.

(b) If $\mu_i(t)>\mu_i^*$ and $\mu_j^*-\frac{\lambda_j}{g_i}\leq\mu_j(t)<\mu_j^*$, player i concedes with constant rate λ_i and player j concedes with positive rate $\lambda_j(t)=\lambda_j-g_i[\mu_j^*-\mu_j(t)]$.

3. Suppose $\mu_i(t)>\mu_1^*$. An unjustified player i never challenges and never sees a challenge and neither does an unjustified opponent, so his or her payoff from conceding at time $t+dt$ is

$$(1-r_i dt-\lambda_j(t)dt-g_j\mu_j(t)dt)\cdot(1-a_j)+\lambda_j(t)dt\cdot a_i+g_j\mu_j(t)dt\cdot(1-a_j)$$

which simplifies to

$$1-a_j-r_i dt\cdot(1-a_j)+\lambda_j(t)dt\cdot d.$$

An unjustified player i is indifferent between conceding at time $t+dt$ and at time t if

$$1-a_j-r_i dt\cdot(1-a_j)+\lambda_j(t)dt\cdot d=1-a_j\Rightarrow\lambda_j(t)=\frac{r_i(1-a_j)}{d}=\lambda_j.$$

□

C.2 Reputation Dynamics

We prove the three separate cases of Lemma 9 in three separate lemmas below and further characterize their reputation dynamics, especially whether the reputations are increasing or decreasing over time, in preparation for the proof of Theorem 3.

C.2.1 SS Phase

We start with the phase in which players' reputations are sufficiently low so that both players mix between challenging and not challenging, between seeing a challenge and yielding to a challenge, as well as between conceding and not conceding. In this SS strategy phase, depending on the initial reputations of both players, players' reputations may increase or decrease over time. When both players' reputations decrease, the phase may persist forever with a positive but vanishing probability.

Lemma 10 (SS Phase)**.** *Suppose $\mu_i^0 \equiv \mu_i(t^0) < \mu_i^*$ for $i = 1,2$. In any equilibrium, from time t^0, each player i's reputation evolves according to*

$$\mu_i^{SS}(t;t^0,\mu_i^0) \equiv \mu(t;t^0,\mu_i^0,\lambda_i - g_i,\frac{g_i}{\mu_i^*})$$

until either player i's reputation reaches μ_i^*. Namely,

1. *if $g_i \leq \lambda_i$ for each $i = 1,2$, each player i's reputation increases until either player i's reputation reaches μ_i^*.*
2. *Suppose $g_i > \lambda_i$ for either $i = 1,2$, $\mu_i^0 \equiv \mu_i(t^0) > \mu_i^*$ and $\mu_j^* - \frac{\lambda_j}{g_i} \leq \mu_j^0 \equiv \mu_j(t) < \mu_j^*$. In any equilibrium, from time t^0, player i's reputation evolves according to*

   $$\mu_i^{AN}(t;t^0,\mu_i^0) \equiv \mu(t;t^0,\mu_i^0,\lambda_i,0),$$

   and player j's reputation evolves according to

   $$\mu_j^{AN}(t;t^0,\mu_j^0) = \mu(t;t^0,\mu_j^0,\lambda_j - g_i\mu_j^* - g_j, g_i + g_j)$$

   until $\mu_i(t)$ reaches 1 or $\mu_j(t)$ reaches μ_j^*.

   (a) *if $\mu_i^0 \leq (1-\frac{\lambda_i}{g_i})\mu_i^*$, each player i's reputation decreases and never reaches zero.*

   (b) *if $\mu_i^0 \leq (1-\frac{\lambda_i}{g_i})\mu_i^*$ and $(1-\frac{\lambda_j}{g_j})\mu_i^* < \mu_i^0 < \mu_j^*$, player i's reputation decreases and player j's reputation increases until player j's reputation reaches μ_j^*.*

   (c) *if $(1-\frac{\lambda_i}{g_i})\mu_i^* < \mu_i^0 < \mu_i^*$, each player i's reputation increases until either player i's reputation reaches μ_i^*.*

Proof of Lemma 10. By Lemma 7-1, an unjustified player i challenges with probability $p_i^*(t)$ and by Lemma 8-1, player i concedes with constant rate λ_i. Player i's reputation at time t is

$$\mu_i(t) = [1-\lambda_i(t)dt - g_i dt(\mu_i(t) + (1-\mu_i(t))p_i^*(t))] \cdot \mu_i(t+dt) + g_i dt \cdot \mu_i(t).$$

Rearranged,

$$\mu_i(t) - \mu_i(t+dt) = -[\lambda_i(t) - g_i + g_i(\mu_i(t) + (1-\mu_i(t))p_i^*(t))]\mu_i(t)dt.$$

Taking $dt \to 0$,

$$\mu_i'(t) = (\lambda_i - g_i)\mu_i(t) + \frac{g_i}{\mu_i^*}\mu_i^2(t).$$

By Lemma 3,

$$\mu_i(t) = \mu(t; t^0, \mu_i^0, \lambda_i - g_i, \frac{g_i}{\mu_i^*}).$$

If $g_i \le \lambda_i$, $\mu_i^0 > (1 - \frac{\lambda_i}{g_i})\mu_i^*$, so player i's reputation always increases. If $g_i > \lambda_i$ and $\mu_i^0 < (1 - \frac{\lambda_i}{g_i})\mu_i^*$, then $\mu_i'(t^0) < 0$ but $\mu_i(t) > 0$ for any $t \ge t^0$, so player i's reputation decreases and never reaches zero. Different combinations of the two players' initial reputations at time t^0 yield the four sub-cases laid out in the statement of the lemma. □

C.2.2 AN or NA Phase

Lemma 11 (AN or NA Phase)**.** *Suppose $\mu_i^0 \equiv \mu_i(t^0) > \mu_i^*$ and $\mu_j^0 \equiv \mu_j(t) < \mu_j^*$. In any equilibrium, from time t^0 until player i's reputation reaches 1 or player j's reputation reaches μ_j^*, player i's reputation evolves according to*

$$\mu_i^{AN}(t; t^0, \mu_i^0) \equiv \mu(t; t^0, \mu_i^0, \lambda_i, 0),$$

and player j's reputation evolves according to

$$\mu_j^{AN}(t; t^0, \mu_j^0) = \mu(t; t^0, \mu_j^0, \lambda_j - g_i\mu_j^* - g_j, g_i + g_j)$$

if $\mu_j^0 \ge \mu_j^* - \frac{\lambda_j}{g_i}$, or jumps to $\mu_j^* - \frac{\lambda_j}{g_i}$ and evolves according to $\mu_j^{AN}(t; t^0, \mu_j^* - \frac{\lambda_j}{g_i})$ if $\mu_j^0 < \mu_j^* - \frac{\lambda_j}{g_i}$.

Proof of Lemma 11. By Lemma 7-2, an unjustified player i always challenges, and by Lemma 8-2, player i concedes with constant rate λ_i. Player i's reputation at time t is

$$\mu_i(t) = (1 - \lambda_i dt - g_i dt)\mu_i(t + dt) + g_i dt \mu_i(t) \cdot 1.$$

Rearrange,

$$\mu_i(t) - \mu_i(t + dt) = g_i dt \mu_i(t) - (\lambda_i dt + g_i dt)\mu_i(t + dt)$$

Take $dt \to 0$,

$$\mu_i'(t) = \lambda_i \mu_i(t).$$

Therefore,

$$\mu_i(t) = \mu(t; t^0, \mu_i^0, \lambda_i, 0).$$

By Lemma 7-2, an unjustified player j never challenges, and by Lemma 8-2, player j concedes with rate $\lambda_j(t) = \lambda_j - g_i[\mu_j^* - \mu_j(t)]$. Player j's reputation at time t is

$$\mu_j(t) = [1 - \lambda_j(t)dt - g_j\mu_j(t)dt]\mu_j(t + dt) + g_j\mu_j(t)dt.$$

Rearrange,

$$\mu_j(t + dt) - \mu_j(t) = \lambda_j(t)dt\mu_j(t + dt) + g_j dt \mu_j(t)\mu_j(t + dt) - g_j dt \mu_j(t)$$

$$\frac{\mu_j(t+dt)-\mu_j(t)}{dt}=[\lambda_j-g_i(\mu_j^*-\mu_j(t))]\mu_j(t+dt)+g_j\mu_j(t)\mu_j(t+dt)-g_j\mu_j(t).$$

Take $dt\to 0$,

$$\mu_j'(t)=(\lambda_j-g_i\mu_j^*-g_j)\mu_j(t)+(g_i+g_j)\mu_j^2(t).$$

By Lemma 3,

$$\mu_j(t)=\mu(t;t^0,\mu_j^0,\lambda_j-g_i\mu_j^*-g_j,g_i+g_j).$$

$\square$

C.2.3 NN Phase

Lemma 12 (NN Phase)**.** *Suppose $\mu_i^0\equiv\mu_i(t^0)\geq\mu_i^*$. In any equilibrium, from time t^0, each player i's reputation evolves according to*

$$\mu_i^{NN}(t;t^0,\mu_i^0)\equiv\mu(t;t^0,\mu_i^0,\lambda_i-g_i,g_i)$$

until one of the players' reputations reaches 1*. Namely,*

1. *if $\mu_i^*>1-\frac{\lambda_i}{g_i}$ for each $i=1,2$ (more primitively, $g_i>r_j(1-a_i)/c_j$), each player i's reputation increases until either player's reputation reaches 1;*
2. *if $\mu_i^*\leq 1-\frac{\lambda_i}{g_i}$ for either $i=1,2$, and*
   (a) *if $\mu_i^0\geq\mu_i^*>1-\frac{\lambda_i}{g_i}$ and $\mu_j^*\leq\mu_j^0\leq 1-\frac{\lambda_j}{g_j}$, player i's reputation increases and player j's reputation decreases until player i's reputation reaches* 1*;*
   (b) *if $\mu_i^*\leq\mu_i^0\leq 1-\frac{\lambda_i}{g_i}$ for $i=1,2$, each player i's reputation decreases and never reaches μ_i^0.*

Proof of Lemma 12. By Lemma 7-3, each unjustified player i never challenges and never sees a challenge, and by Lemma 8-3, each player i concedes with constant rate λ_i. Player i's reputation is

$$\mu_i(t)=[1-\lambda_i(t)dt-g_i\mu_i(t)dt]\cdot\mu_i(t+dt)+\lambda_i(t)dt\cdot 0+g_i\mu_i(t)dt\cdot 1$$

Simplify and take $dt\to 0$,

$$\mu_i'(t)=(\lambda_i-g_i)\mu_i(t)+g_i\mu_i^2(t).$$

Therefore, by Lemma 3,

$$\mu_i(t)=\mu(t;t^0,\mu_i^0,\lambda_i-g_i,g_i).$$

Note that if $\mu_i(t)\leq 1-\frac{\lambda_i}{g_i}$, then $\mu_i'(t)\leq 0$. If $\mu_i^0>1-\frac{\lambda_i}{g_i}$ for either player $i=1,2$, then that player i's reputations strictly increases until it reaches 1. If $\mu_i^*<\mu_i^0\leq 1-\frac{\lambda_i}{g_i}$ for either player $i=1,2$, then that player i's reputation decreases towards but never reaches μ_i^*. $\square$

C.3 Equilibrium

Proof of Theorem 3. We characterize the equilibrium and prove equilibrium uniqueness case by case. The key equilibrium property we rely on is that both players stop conceding at the same time. Formally, define $\tau_i \equiv \inf\{t | F_i(t) = \lim_{t' \to \infty} F_i(t')\}$ and $\inf \emptyset \equiv \infty$. $\tau_1 = \tau_2$.

First, suppose $g_i > \lambda_i$ for each $i = 1, 2$ and $z_i \leq 1 - \frac{\lambda_i}{g_i}$. Each unjustified player i challenges with probability $p_i^*(t)$, sees a challenge with probability $s_i^*(t)$, and concedes with rate $\lambda_i/(1-\mu_i(t))$ for any time $t < \infty$, and a player's reputation at time t is $\mu_i(t) = \mu_i^{SS}(t; 0, z_i)$. These strategies and reputations form an equilibrium because given these strategies, each player's reputation decreases and never reaches zero by Lemma 10-2(a), and given these reputations, by Lemmas 7-1 and 8-1, players always play the specified strategies as the game stays in the SS reputation and strategy phase. Note that the expected payoff for each player i is $1 - a_j$ at each instant on the equilibrium path. We now show this is a unique equilibrium with un-dominated strategies. If neither concedes with a positive probability at time 0, then the equilibrium specified is the only potential equilibrium. If one player concedes with a positive probability at time 0, we discuss two separate cases and argue there is no equilibrium under either case. First, suppose player i concedes with a positive probability Q_i so that player i's reputation at time 0 jumps to $\mu_i^0 > 1 - \frac{\lambda_i}{g_i}$. Then player i's reputation increases and player j's reputation decreases until player i's reputation reaches μ_i^*, by Lemma 10-2(b). Then player i's reputation increases and player j's reputation decreases until player i's reputation reaches 1, by Lemma 11. When player i's reputation reaches 1, there is no more possibility player j is unjustified, so an unjustified player j should concede right away.

Next, suppose $g_i \leq \lambda_i$ for either $i = 1, 2$. The reputations increase in equilibrium. Define the time it takes to reach a reputation from an initial reputation in different strategy phases:

$$t_i^{SS}(\mu_i^0, \mu_i) \equiv t(\mu_i; \mu_i^0, \lambda_i - g_i, g_i/\mu_i^*), i = 1, 2$$

$$t_1^{AN}(\mu_1^0, \mu_1) \equiv t(\mu_1; \lambda_1^0, \mu_1, 0)$$

$$t_2^{AN}(\mu_2^0, \mu_2) \equiv t(\mu_2; \mu_2^0, \lambda_1 - g_2\mu_1^* - g_1, g_1 + g_2)$$

$$t_1^{NA}(\mu_1^0, \mu_1) \equiv t(\mu_1; \mu_1^0, \lambda_2 - g_1\mu_2^* - g_2, g_1 + g_2)$$

$$t_2^{NA}(\mu_2^0, \mu_2) \equiv t(\mu_2; \lambda_2^0, \mu_2, 0)$$

$$t_i^{NN}(\mu_i^0, \mu_i) \equiv t(\mu_i; \mu_i^0, \lambda_i - g_i, g_i), i = 1, 2$$

In the following, we construct the reputation trajectory $\Gamma = \{(\mu_1(-t), \mu_2(-t)) | t \geq 0\}$. Let $t_1^{NN} \equiv t_1^{NN}(\mu_1^*, 1)$ and $t_2^{NN} \equiv t_2^{NN}(\mu_2^*, 1)$ denote the time it takes to reach reputation 1 from μ_i. Then the duration of the NN phase is

$$t^{NN} = \min\{t_1^{NN}, t_2^{NN}\}$$

and at the beginning of the NN phase, each player i' reputation is

$$\mu_i^{NN} \equiv \mu_i^{NN}(-t^{NN}; 0, 1).$$

If $t_2^{NN} > t^{NN} = t_1^{NN}$, $\mu_1^{NN} = \mu_1^*$ and $\mu_2^{NN} > \mu_2^*$. The game has an AN phase where $\mu_2^{AN} = \mu_2^*$,

$$t^{AN} \equiv t_2^{AN}(\mu_2^*, \mu_2^{NN}),$$

and $\mu_1^{AN} \equiv \mu_1^{AN}(-t^{AN}; 0, \mu_1^{NN})$. If $t_1^{NN} > t^{NN} = t_2^{NN}$, then $\mu_1^{NN} > \mu_1^*$ and $\mu_2^{NN} = \mu_2^*$. The game has an NA phase, and $\mu_1^{AN} = \mu_1^*$,

$$t^{NA} \equiv t_1^{NA}(\mu_1^*, \mu_1^{NN}),$$

and $\mu_2^{NA} \equiv \mu_2^{NA}(-t^{NA}, \mu_2^*)$. In general, let $t^{one} \equiv \max\left\{t^{AN}, t^{NA}\right\}$ denote the length of the phase in which one and only one player challenges. At the beginning of the phase, each player i's reputation is

$$\mu_i^{one} \equiv \mu_i^{one}(-t^{one}; 0, \mu_i^{NN})$$

where

$$\mu_i^{one}(-t; t^0, \mu_i^0) \equiv \begin{cases} \mu_i^{AN}(t; t^0, \mu_i^0) & t_1^{NN} > t_2^{NN} \\ \mu_i^{NA}(t; t^0, \mu_i^0) & t_2^{NN} > t_1^{NN} \end{cases}$$

In summary, each player i's reputation evolution, if the time line is reserved, where $\tau \geq 0$ represents the time until the end of the game, is

$$\widetilde{\mu}_i(-\tau) \equiv \begin{cases} \mu_i^{NN}(-\tau; 0, 1) & \tau \leq t^{NN} \\ \mu_i^{one}(t^{NN} - \tau, 0; \mu_i^{NN}) & t^{NN} < \tau \leq t^{NN} + t^{one} \\ \mu_i^{SS}(t^{one} + t^{NN} - \tau; 0, \mu_i^{one}) & \tau \geq t^{NN} + t^{one} \end{cases}$$

We can use $\{(\widetilde{\mu}_1(-\tau), \widetilde{\mu}_2(-\tau)) | \tau \geq 0\}$ to trace out the co-evolution of players' reputation on the equilibrium path. Using the co-evolution, we can characterize the unique equilibrium. Since $\widetilde{\mu}_i(-\tau)$ is strictly decreasing in τ asytomptically to 0 if $\lambda_i > g_i$, for at least one player i (whose primitives satisfy $\lambda_i > g_i$), there is a unique τ_i^* such that $\widetilde{\mu}_i(-\tau_i^*) = z_i > 0$. τ_i^* denotes the time it takes to have player i's reputation increase from z_i to 1 on the equilibrium path, while the opposing player j's reputation is lower than $\widetilde{\mu}_j(-\tau_i^*)$. If $\lambda_i > g_i$, then define $\tau_i^* \equiv \infty$.s The length of the game, T, equals $\min\{\tau_1^*, \tau_2^*\}$. Player i's reputation at time $t = 0$ is $\widetilde{\mu}_i(-T)$. If $z_i \neq \widetilde{\mu}_i(-T)$ (for at most one player), then player i concedes at time 0 with a positive probability

$$Q_i = 1 - \frac{z_i}{1-z_i} \Big/ \frac{\widetilde{\mu}_i(-T)}{1-\widetilde{\mu}_i(-T)}.$$

□

D Bargaining with Frictionless One-Sided Ultimatums

A male player 1 and a female player 2 divide a unit pie. Each player is either (i) justified to demand a share of the pie, never accepting any offer below that, or (ii) unjustified to demand a share of the pie but nonetheless wanting as a big share of the pie as possible. A justified player can find hard evidence supporting his or her demand, but an unjustified player has no hard evidence supporting his or her claim of the share. To start, suppose each player can be of a single justified type: with probability z_1 player 1 is justified to demand a_1 and with probability z_2 player 2 is justified to demand $a_2 > 1 - a_1$. Let $D \equiv a_1 + a_2 - 1$ denote the conflicting difference between the two players.

Time is continuous. At each instant t, each player can decide to give in to the other player's demand or hold on to his or her demand. In addition, player 1 has a challenge opportunity. A justified player 1 challenges when evidence arrives, and the evidence arrives according to a Poisson process with arrival rate $\gamma_1 > 0$. An unjustified player 1 can challenge at any time but he will time his challenge strategically. If player 1 does not challenge, then the game continues. If player 1 challenges at time t, he incurs a cost c_1 and player 2 must respond to player 1's challenge. Player 2 may either yield to the challenge right away and get $1-a_1$, or see the challenge by paying a cost c_2.

If player 2 sees the challenge, the shares of the pie are determined by the players' justified and unjustified types, as follows. If an unjustified player meets a justified player, then the justified player always wins, so an unjustified player i's payoff against a justified player j is $1-a_j$. If two unjustified players meet, then the challenging player 1 wins with probability $w < 1/2$: he gets a_1 with probability w and $1-a_2$ with probability $1-w$, so his expected payoff is $wa_1 + (1-w)(1-a_2) = 1-a_2+w(a_1+a_2-1) = 1-a_2+wD$. Player 2's expected payoff is $(1-w)(1-a_1)+wa_2 = 1-a_1+(1-w)(a_1+a_2-1) = 1-a_1+(1-w)D$. To make challenging worthwhile for player 1, assume $wD < c_1 < (1-w)D$; and to make seeing a challenge worthwhile for player 2, assume $wD < c_2 < (1-w)D$.[7]

The bargaining game $B\left(\{z_i, a_i, r_i, c_i\}_{i=1}^{2}, g_1, w\right)$ is described by players' prior justice probabilities z_1 and z_2, demands a_1 and a_2, discount rates r_1 and r_2, player 1's challenge arrival rate $g_1 > 0$, challenge costs c_1 and c_2, and a challenger's winning probability w.

[7]Inconsequential to our results because justified players are non-strategic, assume that two justified players have the same chance of winning the case, so a justified player i's expected payoff is $1-a_j+D/2$.

One application of the model is final-offer arbitration. Two parties announce their demands for a subject, like the wage of union workers, the division of a company after bankruptcy, or the salary of a baseball player (final-offer arbitrations are used frequently in firm-union bargaining, in bankruptcy cases, and in Major League Baseball). A justified player can have superior evidence supporting his or her claim, but needs time and effort to gather information about his or her claim and to appeal to the court. An unjustified player does not have good proofs supporting his or her claim but nonetheless can appeal to court. Whether or not a player could gather evidence and is justified is private information. While they gather evidence, they can negotiate with each other by repeatedly making offers to each other or choosing to let the case be settled by the court when possible.[8] A justified player can finish collecting evidence at any moment, and as soon as he is done with collecting evidence and if the case has not been settled out of court, he submits his claim to the court. At that moment, the opposing player has to respond to the lawsuit, either by agreeing to the challenging player's demand out of court or by paying a cost to go on the court. In the court, an unjustified player loses to a justified player for sure and an unjustified challenger loses to an unjustified defendant in expectation.

D.1 Formal Description of the Game

Let me formally describe the strategies and payoffs of the players when demands are fixed to be a_1 and a_2. Let $F_i(t)$ denote player i's probability of conceding by time t. Let $G_1(t)$ denote player 1's probability of challenging by time t. Let $q_2(t)$ denote player 2's probability of conceding to a challenge at time t. Player 1's strategy is described by $\Sigma_1 = (F_1, G_1)$, player 2's strategy is described by $\Sigma_2 = (F_2, q_2)$.

D.2 Single Justified Types

D.2.1 Strategies

We first consider the best response of player 2 when she faces a challenge and believes that a challenging player 1 is justified with probability ν_1. If she sees the challenge, her expected payoff is $\nu_1(1-a_1)+(1-\nu_1)(1-a_1+(1-w)D)-c_2 = 1-a_1+(1-\nu_1)(1-w)D-c_2$; if she concedes, her expected payoff is $1-a_1$. She is indifferent when $\nu_1 = 1-\frac{c_2}{(1-w)D}$.

We now consider the optimal challenging strategy of player 1 when he believes that player 2 is justified with probability μ_2 and an unjustified player 2 concedes to a challenge with probability q_2. The expected utility when he challenges is $1-a_2+(1-\mu_2)q_2D+(1-\mu_2)(1-q_2)wD-c_1$. The expected utility when he does not challenge (on equilibrium path) is $1-a_2$. He is indifferent if $\mu_2 = 1-\frac{c_1}{(q_2+(1-q_2)w)D}$.

[8] It is optimal for an unjustified player to continue to make the same demand, so the war-of-attrition structure of the bargaining game can be derived rather than assumed, just like in Abreu and Gul (2000). The addition compared to Abreu and Gul (2000) is an additional opportunity to appeal to the court or a third-party arbitrator.

In any equilibrium, an unjustified player 1 does not challenge player 2 if he believes that player 2 is justified with a probability more than $\mu_2^* = 1 - \frac{c_1}{D}$. When player 2's reputation is lower than μ_2^*, an unjustified player 1 challenges at rate χ_1 to make player 2 believe that a challenging player is justified with probability $v_1^* \equiv 1 - \frac{c_2}{(1-w)D}$:

$$\frac{\mu_1 g_1}{\mu_1 g_1 + (1-\mu_1)\chi_1} = v_1^* \Rightarrow \chi_1(\mu_1) = \frac{1-v_1^*}{v_1^*}\frac{\mu_1}{1-\mu_1}g_1.$$

On the other hand, to make player 1 indifferent between challenging and not challenging, player 2 concedes to a challenge with probability

$$q_2(\mu_2) = \frac{1}{1-w}\left[\frac{c_1}{D}\frac{1}{1-\mu_2} - w\right].$$

Players concede at the same rates as in the AG case, because the expected utility of conceding at time t is unchanged:

$$u_i(t) = \int_{s=0}^{t} e^{-r_i s} a_i dF_j(s) + (1-F_i(t))(1-a_j).$$

D.2.2 Reputation

Player 1's Reputation

Player 1's equilibrium reputation dynamics is different from the AG case. There are two strategy phases.

Challenging Phase. When $\mu_2 \leq \mu_2^*$, an unjustified player 1 challenges at a positive rate. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathscr{F}_t]$, we have

$$\begin{aligned}\mu_1(t) \quad &= \quad \mu_1(t)g_1 dt \cdot 1 + (1-\mu_1(t))\chi_1(t)dt \cdot 0 + \lambda_1 dt \cdot 0 + \\ &\quad\quad [1-\mu_1(t)g_1 dt - (1-\mu(t))\chi_1(t)dt - \lambda_1 dt]\mu_1(t+dt).\end{aligned}$$

Rearranging the equation and following the equilibrium property that $\mu_1 g_1 + (1-\mu_1)\chi_1(t) = \mu_1(t)\frac{v_1^*}{g_1}$, we get

$$\mu_1(t+dt) - \mu_1(t) = -\mu_1(t)g_1 dt + \mu_1(t)\frac{g_1}{v_1^*}dt\mu_1(t+dt) + \lambda_1 dt\mu_1(t+dt).$$

Dividing both sides by dt and taking $dt \to 0$, we have that player 1's reputation follows the following Bernoulli ODE:

$$\mu_1'(t) = (\lambda_1 - g_1)\mu_1(t) + \frac{g_1}{v_1^*}\mu_1^2(t).$$

Non-Challenging Phase. When $\mu_2 > \mu_2^*$, an unjustified player 1 does not challenge. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathscr{F}_t]$, we have

$$\mu_1(t) = \mu_1(t)g_1 dt \cdot 1 + \lambda_1 dt \cdot 0 + [1-\mu_1(t)g_1 dt - \lambda_1 dt]\mu_1(t+dt).$$

Rearrange,

$$\frac{\mu_1(t+dt)-\mu_1(t)}{dt} = -\mu_1(t)g_1 + \mu_1(t)\mu_1(t+dt)g_1 + \lambda_1\mu_1(t+dt).$$

Taking $dt \to 0$, we have that player 1's reputation follows the following Bernoulli ODE:

$$\mu_1'(t) = (\lambda_1 - g_1)\mu_1(t) + g_1\mu_1^2(t).$$

Player 2's Reputation

Player 2's reputation dynamics is the same as in the no-challenge benchmark model. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathscr{F}_t]$, we have

$$\mu_2(t) = \lambda_2 dt \cdot 0 + (1-\lambda_2)dt \cdot \mu_2(t+dt).$$

Rearranging, we get

$$\mu_2(t+dt) - \mu_2(t) = -\lambda_2 dt \mu_2(t+dt).$$

Dividing both sides by dt and taking $dt \to 0$, we get

$$\mu_2'(t) = -\lambda_2\mu_2(t).$$

D.2.3 Equilibrium

Proposition 3. *The unique sequential equilibrium of the bargaining game is* $(\widehat{\Sigma}_1, \widehat{\Sigma}_2)$.

Proof of Proposition 3. □