Reputational Bargaining in the Shadow of the Law

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Abstract

This paper analyzes reputational bargaining between two parties who can strategically send an ultimatum to resolve the conflict by law. Each party is either a *justified* player who is inflexible about demand and resolves the conflict by law whenever possible, or an *unjustified* player who is flexible about demand and *strategically* leverages an ultimatum. A strategic player is better off if the opponent concedes to an ultimatum, but is worse off if the opponent responds to it, making the ultimatum a risky choice. We study the equilibrium of the game when an ultimatum can be sent, and compare it with the equilibrium of the game when an ultimatum cannot be sent (i.e., Abreu and Gul (2000)). The ability to send an ultimatum can harm a strategic player by making it more difficult to build a reputation, but can also benefit the player who can pretend to be justified. When there are multiple types and the probability of being justified is small, the players share the surplus efficiently. In the limit of complete rationality, the outcome is the Rubinstein outcome if the ultimatum opportunity arrival rate for a justified player is smaller than the discount factor, and is the Rubinstein outcome with the discount factor replaced by the rate otherwise. When both players can send an ultimatum, players' reputations may decline, approaching 0.

Keywords: reputational bargaining, ultimatum, arbitration

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1 Introduction

The last resort of disputes between two parties is an arbitration or court trial: 15 of the 20 largest US credit card issues and 7 of the 8 largest cell phone companies include arbitration clauses in their contracts with consumers (Consumer Financial Protection Bureau, 2015). However, most disputes do not reach this terminal stage: 98% of federal criminal cases, 97% of state criminal cases, and 97% of civil lawsuits in the US have been resolved before trial in recent years (Gramlich, 2019). Although resolving the dispute through the final means is evidently rarely used, it is frequently leveraged as a strategic posture in the form of an ultimatum. Given the apparent prevalence of negotiation and settlement before trial, we study a model in which the parties have the option to use an ultimatum to resolve the conflict.

We build on the continuous-time war-of-attrition model of bargaining proposed and developed by Abreu and Gul (2000) (AG henceforth). Two players negotiate to divide a unit pie. Each player is either (i) *justified and inflexible* to demand a fixed share of the pie (corresponding to a behavioral type in AG), or (ii) *unjustified and flexible* to demand a share (corresponding to a rational type in AG). Players announce their demand sequentially at the beginning of the game. After announcing a demand, each player can hold on to the announced demand or give in to the opposing player's demand at any time. In the baseline AG model without ultimatums, the game ends only when one player gives in to the other player. In our model, opportunities to challenge the opponent and end the bargaining process arise periodically for a justified player, and an unjustified player may use the existence of such an opportunity strategically to bluff. The opponent, upon being challenged, must respond by seeing the challenge or giving in to the challenger's demand. If the opponent sees the challenge, a nonstrategic third party (e.g., a judge or an arbitrator) observes the players' types and executes a division of the surplus that rules in favor of a justified challenger and against an unjustified one.

The model captures the following settings of negotiation. Two parties announce their demands for the contract of union workers, the division of a dissolved company's remaining assets, or the salary of an athlete. A justified party insists on a demand that can be supported by evidence. However, finding the evidence requires time. An unjustified party does not have any evidence but nonetheless can threaten to take the case to the court. Whether or not a party could gather evidence is private information. A party can accept the opposing party's demand at any time. A justified party submits the case to the court once the needed evidence is collected (unless the case is closed already with the opposing party conceding). At that moment, the opposing party has to respond to the lawsuit, either by agreeing to the plaintiff's demand out of court or by paying a cost to go on the court. The court rules in favor of the plaintiff with evidence and against one without evidence.

We first study the setting in which one of the two players has the opportunity to send an ultimatum. In the unique sequential equilibrium, both players concede at the same rate as in the AG model. An unjustified player 1 challenges with a positive and increasing hazard rate as long as player 2's reputation is not high, and does not challenge at all after player 2's reputation increases past a threshold. In other words, there is a discontinuous drop in the challenge hazard rate. Consequently, there is a discontinuous drop in the hazard rate of negotiation termination. This hazard rate discontinuity is observed in the negotiation process in the Major League Baseball salary arbitrations from 2011 to 2020 we manually collected.

Without the challenge opportunity, players' reputations—probabilities of being justified—increase over time: Not conceding is evidence for a player being justified. With the introduction of the challenge opportunity, two additional opposite forces influence reputation building. On one hand, the challenge opportunity slows down reputation building, because "no news is bad news": Not challenging is evidence against player 1 being justified. On the other hand, the challenge opportunity can also increase player 1's payoff, because "no news is good news": Not challenging can be evidence for player 1 being justified, especially when an unjustified player challenges with a high rate. Although not having the challenge opportunity is a strong commitment device for reputation building, the challenge opportunity may prove to be beneficial when an unjustified player uses it more frequently than a justified player in equilibrium.

When the initial probabilities of being justified are small, the equilibrium outcome is efficient with one of the players yielding to the opponent's demand with a very high probability. The identity of the player who yields is determined by the discount factors, demands, and the rate at which the evidence arrives. The set of parameters for which player 1 is the player who yields expands with the arrival rate of the evidence for player 1. Finally we show that players' equilibrium payoffs in the limit do not depend on the details of the decision rule employed by the court.

We then analyze the scenario in which there are multiple justified demands a player can choose from. We show that there is a unique equilibrium outcome. Moreover, as the initial probability of being justified converges to zero, and as the set of justified demands gets larger and finer in the unit interval, the players' equilibrium payoffs converge to a unique payoff vector, and the outcome is efficient. Player 1's equilibrium payoff is his Rubinstein payoff if the arrival rate of ultimatums is smaller than his interest rate, and is equal to what would be his Rubinstein payoff if his interest rate was replaced by the arrival rate of the ultimatums if the latter is larger than his interest rate.

We then consider the setting in which both players have the opportunity to send an ultimatum. Perhaps interestingly, there may be equilibria in which both types lose their reputations over time, approaching complete rationality but never arriving there.

Our paper is most closely related to Abreu and Gul (2000). We model the negotiation phase prior to an ultimatum as the continuous-time war-of-attrition model developed in Abreu and Gul (2000). Our main difference is the additional possibility to send an ultimatum, which incorporates AG as a special case when the ultimatum cannot be sent, and leads to different qualitative predictions when the ultimatum can be sent.

The paper contributes to the growing subsequent literature of reputational bargaining. Compared with Fanning (2016) which studies reputational bargaining with exogenous deadlines, this paper can be viewed as studying reputational bargaining with endogenously chosen deadlines. Compared to Fanning (2019) with a mediator neither player needs to obey, we have an arbitrator both players need to obey if at least one player calls the arbitrator. In addition, the insights generated are also related to bargaining with outside options (Atakan and Ekmekci, 2014; Chang, 2016; Hwang and Li, 2017; Fanning, 2018). Most related are Chang (2016) and Hwang and Li (2017), where not taking an outside option opens up the possibility of declining reputations. Relatedly, Sandroni and Urgun (2017) and Sandroni and Urgun (2018) also study situations in which players can end the bargaining process. However, in these models, not ending the bargaining process is efficient, and the equilibrium dynamics are different from the war of attrition dynamics that result in our paper.

The rest of the paper proceeds as follows. Section 2 describes the model. Section 3 characterizes the equilibrium with one-sided challenges. Section 4 discusses multiple types and limiting payoffs. Section 5 discusses the results with two-sided challenges. Section 7 concludes. The appendix collects all omitted proofs.

2 Model

Player 1 ("he") and player 2 ("she") negotiate to divide a unit pie. Each player is either (i) *justified* to demand a share of the pie and never accepting any offer below that, or (ii) *unjustified* to demand a share of the pie but nonetheless wanting as a big share of the pie as possible. A justified player can justify their demand with verifiable evidence, but an unjustified player cannot.

We start by assuming that each player can be of a single justified type: with probability z_1 player 1 is justified to demand a_1, and with probability z_2 player 2 is justified to demand $a_2 > 1 - a_1$. Let $D \equiv a_1 + a_2 - 1$ denote the conflicting difference between the two players.

Time is continuous. At each instant t, each player can decide to give in to the other player's demand or hold on to their demand. In addition, player 1 has a challenge opportunity. A justified player 1 challenges with evidence, which arrives according to a Poisson process with rate $\gamma_1 \geq 0$. An unjustified player 1 can challenge at any time but he will time his challenge strategically. If player 1 does not challenge, then the game continues. If player 1 challenges at time t, he incurs a cost c_1 and player 2 must respond to player 1's challenge. Player 2 may either yield to the challenge and get $1 - a_1$, or see the challenge by paying a cost c_2.

If player 2 sees the challenge, the shares of the pie are determined as follows. If an unjustified player meets a justified player, then the justified player always wins, so an unjustified player i's payoff against a justified player j is $1-a_j$. If two unjustified players meet, then the challenging player 1 wins with probability $w_1 < 1/2$: he gets a_1 with probability w and $1-a_2$ with probability $1-w$, so his expected payoff is $1-a_2+w_1D$, and the defending player 2's expected payoff is $1-a_1+(1-w)D$. To make challenging worthwhile for player 1, assume $w_1D < c_1 < (1-w_1)D$; and to make seeing a challenge worthwhile for player 2, assume $w_1D < c_2 < (1-w_1)D$. Inconsequential to our results because justified players are nonstrategic, assume that two justified players have the same chance of winning the case so that a justified player i's expected payoff is $1-a_j+D/2$.

In summary, the bargaining game $\{z_i, a_i, r_i, \gamma_i, c_i, w_i\}_{i=1}^2$ with one-sided ultimatums is described by players' prior probabilities z_1 and z_2 of being justified, demands a_1 and a_2, discount rates r_1 and r_2, challenge arrival rates $\gamma_1 \geq 0$ and $\gamma_2 = 0$, challenge costs c_1 and c_2, and an unjustified challenger's winning probability w_i against an unjustified defender $j \neq i$.

Our model captures a range of situations in which one party has the opportunity to send an ultimatum.

Application 1 (Pretrial bargaining). *Plaintiffs demand what they believe they are entitled to based on a claim that needs to be proven in court, and defendants may disagree with and counter the claim. They engage in a series of negotiations—warning, arraignment, pretrial hearings, etc.—before reaching the trial.*

a. **Patent infringement**. *An inventor demands reparations from a firm for an alleged patent infringement. A patent owner needs to collect evidence to sue and beat the infringer, but a patent troll can take the firm to the court at anytime.*

b. **Alimony payment**. *A mother of a child demands overdue alimony payments from the father, but the father refuses to pay alleging that the mother frequently denied his child visitation rights. Both sides need to collect evidence defending their claims, and to receive the payment, the mother needs to sue the father.*

c. **Renter eviction**. *A landlord demands to evict a renter who allegedly violated the lease agreement (e.g., no-smoking policy or no-pet policy). The burden of proof falls on the landlord if the court provides an order.*

Application 2 (Major League Baseball salary arbitration). *Since 1974 in Major League Baseball, a player with between three and six years of service has been able to ask salary be determined by a final-offer arbitration. If the player and club have not agreed on a salary by a deadline in mid-January, they must report their final salary figures, and a hearing is scheduled to be in February. If no settlement can be reached by the hearing date, the case is brought before a panel of arbitrators.*

After hearing arguments from both sides, the panel selects the salary figure of either the player or the club—but not one in between—as the player's salary for the upcoming season.

Application 3 (Buyer-seller bargaining with a stochastically arriving outside option). *A buyer wants to buy a good from a seller. The buyer may have a purchasing opportunity of a similar product at a discounted price from another seller. Two sides can negotiate with each other while the buyer waits for the outside option to arrive. When the outside option arrives, the buyer sends an ultimatum to the seller, and the seller has to decide whether or not to strike a deal. The seller can verify with a cost the existence of the outside option. If the buyer presents proof, the seller sells to the buyer at the discounted price. If the buyer does not present proof, then the buyer reveals that he is bluffing and buys the good at the seller's requested price.*

Let us formally describe the strategies and payoffs of the (unjustified) players. Player 1's strategy is described by $\Sigma_1 = (F_1, G_1)$, and player 2's strategy is described by $\Sigma_2 = (F_2, q_2)$, where $F_i(t)$ denotes player i's probability of conceding by time t, $G_1(t)$ denotes player 1's probability of challenging by time t, and $q_2(t)$ denotes an player 2's probability of yielding to a challenge at time t.

Player 1's expected utility from conceding at time t is[1]

$$\begin{aligned} u_1(t, \Sigma_2) &= (1-z_2)\int_0^t a_1 e^{-r_1 s} dF_2(s) + (1-(1-z_2)F_2(t))e^{-r_1 t}(1-a_2) \\ &\quad + (1-z_2)\left[F_2(t) - \lim_{s\uparrow t} F_2(s)\right]\frac{a_1+1-a_2}{2}. \end{aligned} \tag{1}$$

Player 1's expected utility from challenging at time t is[2]

$$v_1(t,\Sigma_2) = (1-z_2)\int_0^t a_1 e^{-r_1 s} dF_2(s) + e^{-r_1 t}[1-a_2+(1-z_2)(1-F_2(t))((1-q_2(t))wD + q_2(t)D) - c_1].$$

Player 1's expected utility from strategy Σ_1 is $u_1(\Sigma_1,\Sigma_2) = \int_0^\infty u_1(s,\Sigma_2)dF_1(s) + \int_0^\infty v_1(s,\Sigma_2)dG_1(s)$.

Player 2's expected utility from conceding at time t and yielding to a challenge with probability $q_2(s)$ at time s is

$$\begin{aligned} u_2(t, q_2, \Sigma_1) &= (1-z_1)\int_0^t a_2 e^{-r_2 s} dF_1(s) + z_1\int_0^t (1-a_1)e^{-r_2 s}\gamma_1 e^{-\gamma_1 s} ds \\ &\quad + (1-z_1)\int_0^t [1-a_1+(1-q_2(s))((1-w)D-c_2)]e^{-r_2 s} d\gamma_1(s) \\ &\quad + e^{-r_2 t}(1-a_1)(1-(1-z_1)F_1(t)-(1-z_1)\gamma_1(t) - z_1(1-e^{-\gamma_1 t})) \end{aligned}$$

[1] We assume an equal split when two players concede at the same time. It is inconsequential to our results because simultaneous concession occurs with probability 0 in equilibrium.

[2] We assume that whenever concession and challenge occur simultaneously, the outcome is determined by the concession. It is an innocuous assumption because simultaneous concession and challenge occur with probability 0 in equilibrium.

$$+(1-z_1)\left[F_1(t)-\lim_{s\uparrow t}F_1(s)\right]\frac{a_2+1-a_1}{2}. \tag{2}$$

Player 2's expected utility from strategy Σ_2 is $u_2(\Sigma_2,\Sigma_1)=\int_0^\infty u_2(s,q_2,\Sigma_1)dF_2(s)$.

3 Equilibrium

In this section, we first solve for players' optimal strategies fixing their beliefs about the opponent being justified. We then characterize the reputation dynamics given the optimal strategies. To close the model, one player may concede with a positive probability at time 0. Finally, we summarize and analyze the equilibrium strategies and reputations.

3.1 Strategies

Player 2's optimal yielding strategy. We first consider the best response of player 2 when she faces a challenge and believes that the challenging player 1 is justified with probability v_1. If she yields to the challenge, her expected payoff is $1-a_1$. If she responds to the challenge, she pays a cost c_2 to realize a gain when player 1 is unjustified and loses in court, so her expected gain is $(1-v_1)(1-w)D-c_2$. She is indifferent between the two actions when

$$v_1=1-\frac{c_2}{(1-w)D}\equiv v_1^*.$$

Hence, she strictly prefers to respond to the challenge if the challenging player is justified with a probability strictly lower than v_1^*, and strictly prefers to yield to the challenge if the challenging player is justified with a probability strictly higher than v_1^*.

Player 1's optimal challenging strategy. We now consider the optimal challenging strategy of player 1 when he believes that player 2 is justified with probability μ_2 and an unjustified player 2 concedes to a challenge with probability q_2. The expected utility when he does not challenge is his continuation value, which is $1-a_2$ on the equilibrium path. The expected utility when he challenges is $1-a_2+(1-\mu_2)[q_2+(1-q_2)w]D-c_1$. He is indifferent if

$$\mu_2=1-\frac{c_1}{(q_2+(1-q_2)w)D}\equiv\mu_2.$$

Equilibrium challenging and yielding strategies. If player 2 is justified with a probability strictly higher than $\mu_2^*\equiv 1-\frac{c_1}{D}$, an unjustified player 1 strictly prefers not to challenge. If player 2 is justified with a probability strictly less than μ_2^*, an unjustified player 1 must challenge at rate χ_1 to make player 2 believe that the challenging player 1 is justified with probability $v_1^*\equiv 1-\frac{c_2}{(1-w)D}$:

$$\frac{\mu_1\gamma_1}{\mu_1\gamma_1+(1-\mu_1)\chi_1}=v_1^*\Rightarrow\chi_1(\mu_1)=\frac{1-v_1^*}{v_1^*}\frac{\mu_1}{1-\mu_1}\gamma_1.$$

If an unjustified player 1 challenges at a rate strictly higher/lower than the specified rate, then an unjustified player 2 is strictly better/worse off responding than yielding to the challenge. To make player 1 indifferent between challenging and not challenging, player 2 concedes to a challenge with probability

$$q_2(\mu_2) = \frac{1}{1-w}\left[\frac{c_1}{D}\frac{1}{1-\mu_2} - w\right].$$

Equilibrium conceding strategies. In equilibrium, players are indifferent between conceding and waiting to concede the next instant. Players concede at a rate to make their opponents indifferent between conceding and not conceding:

$$1-a_j = (1-\mu_j)\kappa_j dt\cdot a_i + e^{-r_i dt}\cdot(1-a_j)(1-(1-\mu_j)\kappa_j dt) \Rightarrow \kappa_i = \frac{r_j(1-a_i)/D}{1-\mu_i}.$$

Players concede at the same overall rate as in Abreu and Gul (2000), $\lambda_i = r_j(1-a_i)/D$.

Summary. Figure 1 shows a strategic player 1's equilibrium rate of concession, ultimatum, and negotiation termination overtime. Figure 2 shows the overall rate of ultimatum and negotiation termination over time.

3.2 Reputation

Both players' reputation dynamics will follow the Bernoulli differential equation, which is one of the few special cases of ordinary differential equations with exact solutions. We will take advantage of the exact characterizations of the reputations for our subsequent analyses.

Lemma 1. *The solution to the Bernoulli differential equation $\mu'(t) = A\mu(t) + B\mu^2(t)$ given $\mu(0) = \mu^0$ is*

$$\mu(t;\mu^0,A,B) = \begin{cases} 1\Big/\left[\left(\frac{1}{\mu^0}+\frac{B}{A}\right)\exp(-At) - \frac{B}{A}\right] & \text{if } A \neq 0 \\ 1\Big/\left[-Bt + \frac{1}{\mu^0}\right] & \text{if } A = 0 \end{cases}$$

If $\mu^0 > -A/B$, then $\mu'(t) > 0$ for all $t \geq t^0$, and the time length it takes to reach reputation $\mu > \mu^0$ from μ^0 is

$$t(\mu;\mu^0,A,B) = \frac{1}{A}\ln\left(\frac{\frac{1}{\mu^0}+\frac{B}{A}}{\frac{1}{\mu}+\frac{B}{A}}\right).$$

Player 2's Reputation. Player 2's reputation is affected by her rate of conceding. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathcal{F}_t]$, we have

$$\mu_2(t) = \lambda_2 dt\cdot 0 + (1-\lambda_2)dt\cdot\mu_2(t+dt).$$

Rearranging, we get

$$\mu_2(t+dt) - \mu_2(t) = -\lambda_2 dt\mu_2(t+dt).$$

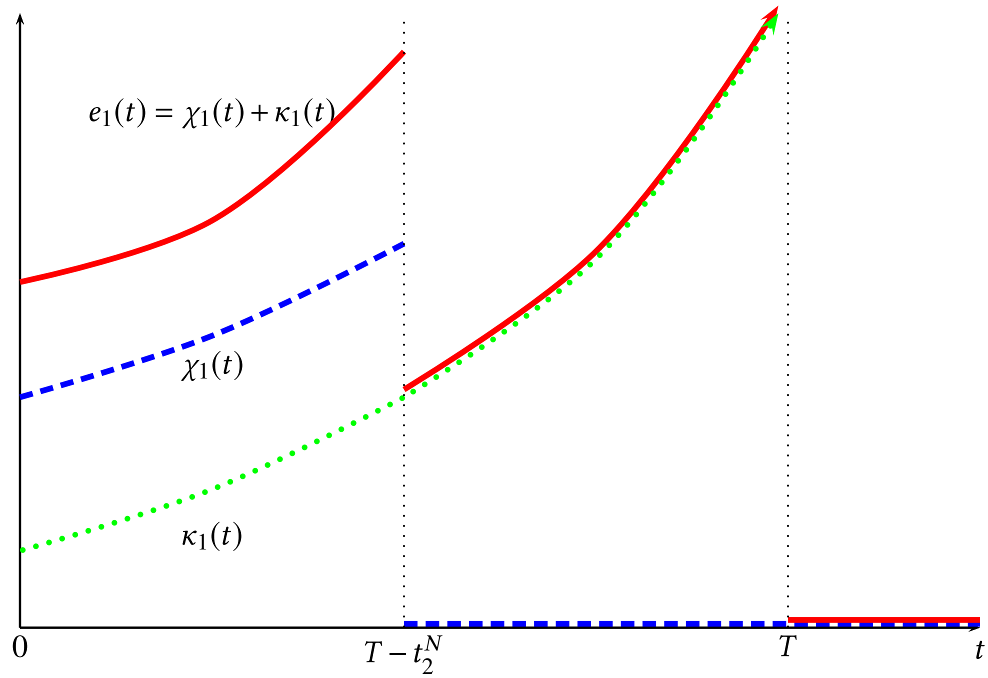


Figure 1: A strategic player 1's equilibrium hazard rates of ultimatum (blue dashed line), concession (green dotted line), and negotiation termination (red solid line).

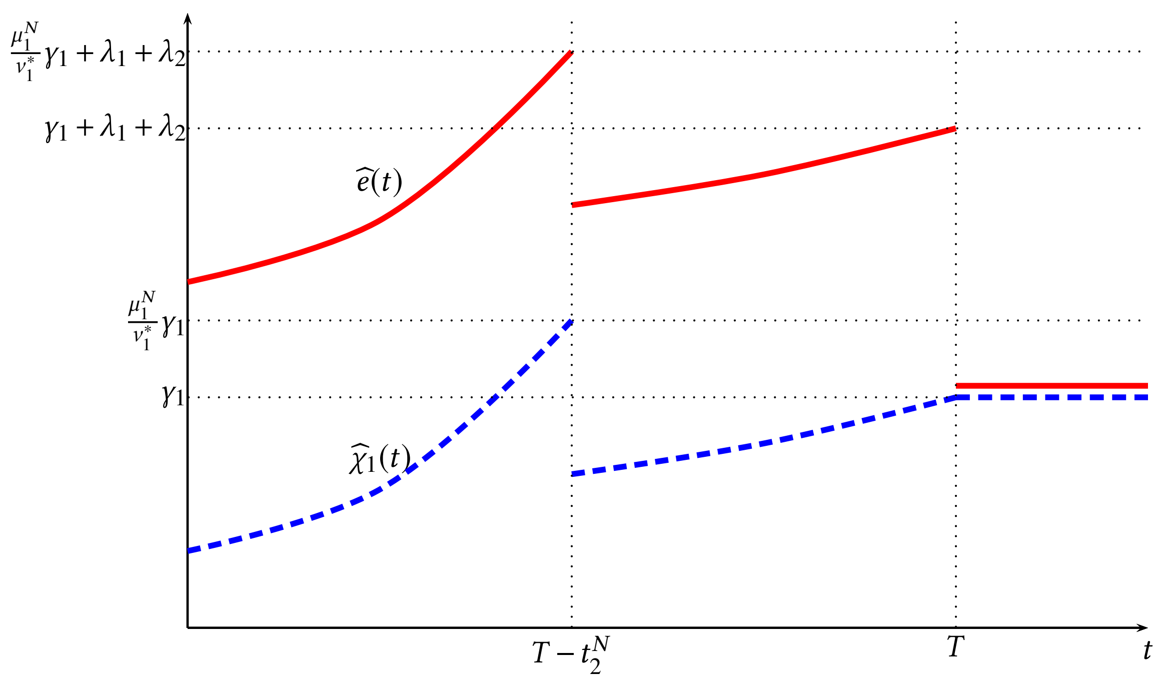


Figure 2: Overall hazard rates of ultimatum (blue dashed line) and negotiation termination (red solid line).

Dividing both sides by dt and taking $dt \to 0$, we get

$$\mu_2'(t) = \lambda_2 \mu_2(t). \tag{3}$$

Let $t(1; \mu_2^*, \lambda_2, 0)$ denote the time length it takes player 2 to reach reputation 1 from reputation μ_2^* when the reputation follows the dynamics above.

Player 1's Reputation in the No-Challenge Phase. When $\mu_2 > \mu_2^*$, player 1 does not challenge. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathcal{F}_t]$, we have

$$\mu_1(t) = \mu_1(t)\gamma_1 dt \cdot 1 + \lambda_1 dt \cdot 0 + [1 - \mu_1(t)\gamma_1 dt - \lambda_1 dt]\mu_1(t+dt).$$

Rearrange,

$$\frac{\mu_1(t+dt) - \mu_1(t)}{dt} = -\mu_1(t)\gamma_1 + \mu_1(t)\mu_1(t+dt)\gamma_1 + \lambda_1 \mu_1(t+dt).$$

Taking $dt \to 0$, we have that player 1's reputation follows the following Bernoulli ODE:

$$\mu_1'(t) = (\lambda_1 - \gamma_1)\mu_1(t) + \gamma_1 \mu_1^2(t), \tag{4}$$

which can be rearranged and decomposed as follows:

$$\frac{\mu_1'(t)}{\mu_1(t)} = \lambda_1 - (1 - \mu_1(t))\gamma_1.$$

Two effects influence player 1's reputation. First, "no news is good news": With player 1 conceding at rate λ_1, player 1's reputation increases exponentially at rate λ_1. Second, "no news is bad news": With only a justified player 1 challenging at rate γ_1 and an unjustified player not challenging at all, player 1's reputation decreases exponentially at rate $(1 - \mu_1(t))\gamma_1$. Because of the second effect, reputation building is slower with ultimatum than without ultimatum.

Player 1's Reputation in the Challenge Phase. When $\mu_2 \le \mu_2^*$, player 1 challenges at a positive rate. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathcal{F}_t]$, we have

$$\begin{aligned}\mu_1(t) \quad = \quad & \mu_1(t)\gamma_1 dt \cdot 1 + (1 - \mu_1(t))\chi_1(t) dt \cdot 0 + \lambda_1 dt \cdot 0 + \\ & [1 - \mu_1(t)\gamma_1 dt - (1 - \mu_1(t))\chi_1(t) dt - \lambda_1 dt]\mu_1(t+dt).\end{aligned}$$

Rearranging the equation and following the equilibrium property that $\mu_1\gamma_1 + (1-\mu_1)\chi_1(t) = \mu_1(t)\frac{\nu_1^*}{\gamma_1}$, we get $\mu_1(t+dt) - \mu_1(t) = -\mu_1(t)\gamma_1 dt + \mu_1(t)\frac{\gamma_1}{\nu_1^*} dt \mu_1(t+dt) + \lambda_1 dt \mu_1(t+dt)$. Dividing both sides by dt and taking $dt \to 0$, we have that player 1's reputation follows the following Bernoulli ODE:

$$\mu_1'(t) = (\lambda_1 - \gamma_1)\mu_1(t) + \frac{\gamma_1}{\nu_1^*}\mu_1^2(t), \tag{5}$$

which can be rearranged and decomposed as

$$\frac{\mu_1'(t)}{\mu_1(t)} = \lambda_1 - (1-\mu_1(t))\gamma_1 + (1-\mu_1(t))\chi_1(t).$$

The same two effects as those in the no-challenge phase are present, and there is a third effect because an unjustified player 1 challenges in this phase. First, the "no news is good news" effect is the same as in the no-challenge phase: Player 1 concedes at rate λ_1, so not conceding contributes an exponential growth rate λ_1 to reputation building. Second, the "no news is bad news" effect is also the same as that in the challenge phase: A justified player 1 challenges with rate γ_1, so not challenging yields an exponential decay at rate $(1-\mu_1(t))\gamma_1$. The new effect that aids reputation building is due to an unjustified player 1 challenging at rate $\chi_1(t)$, so the exponential growth rate increases by $(1-\mu_1(t))\chi_1(t)$.

Unlike the no-challenge phase in which a player's reputation builds slower than in AG, the reputation may build faster in the challenge phase than in AG. The reputation builds faster if an unjustified player challenges at a higher rate than a justified player, i.e., $\chi_1(t) > \gamma_1$.

3.3 Initial Concession

We have the key equilibrium property that players reach reputation 1 at the same time. Before the end of the game, when player 2's reputation is between μ_2^* and 1, player 1 does not challenge and his reputation evolves according to equation (4). (When players' initial reputations are low enough,) this challenging phase lasts as long as player 2's reputation evolves from μ_2^* to 1, that is, the phase lasts $t_2^N = t(1;\mu_2^*,\lambda_2,0)$, explicitly, $t_2^N = -\lambda_2 \ln(\mu_2^*)$. Player 1's reputation evolves from μ_1^N to 1 during this period, where μ_1^N is determined by $\mu(-t_2^N;1,\lambda_1-\gamma_1,\gamma_1)$. Explicitly,

$$\mu_1^N = \frac{\lambda_1-\gamma_1}{\lambda_1(\mu_2^*)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \gamma_1} \leq 1.$$

Before player 1's reputation reaches μ_1^N, player 1's reputation evolves according to equation (5).

One way to determine the player who needs to concede with a positive probability for an arbitrary pair of initial reputations is to trace out the parametric reputation coevolution curve $(\widetilde{\mu}_1(t),\widetilde{\mu}_2(t))$ in the belief plane. Because both reputations are characterized analytically, we can represent the graph of the coevolution curve as $\widetilde{\mu}_1(\mu_2)$ for $\mu_2 \in (0,1]$. The graph of the coevolution curve is characterized by

$$\widetilde{\mu}_1(\mu_2|\gamma_1) = \begin{cases} \frac{\lambda_1-\gamma_1}{\lambda_1(\mu_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}-\gamma_1} & \text{if } \mu_2^* < \mu_2 \leq 1 \\ \frac{\lambda_1-\gamma_1}{\lambda_1(\mu_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}+(\frac{\gamma_1}{\nu_1}-\gamma_1)(\frac{\mu_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}}-\frac{\gamma_1}{\nu_1^*}} & \text{if } 0 < \mu_2 \leq \mu_2^* \end{cases}.$$

In summary, equilibrium reputation evolves as follows.

Lemma 2. *Define*

$$\widetilde{\mu}_1(-t) = \begin{cases} \mu(-t; 1, \lambda_1 - \gamma_1, \frac{\gamma_1}{v_1^*}) & t < t_2^N \\ \mu(t_2^N - t; \mu_1^N, \lambda_1 - \gamma_1, \gamma_1) & t \geq t_2^N \end{cases},$$

and $\widetilde{\mu}_2(-t) = \mu(-t; 1, \lambda_2, 0)$. *Player i's reputation in equilibrium is* $\widehat{\mu}_i(T - t) = \widetilde{\mu}_i(-t)$, *where* $T = \min\{T_1, T_2\}$, *and* T_i *solves* $\widetilde{\mu}_i(-T_i) = z_i$.

3.4 Summary of Equilibrium

3.4.1 Equilibrium Strategies and Reputations

Proposition 1 shows that there is a unique equilibrium and that equilibrium strategies and reputations are differentiable in time.

Proposition 1. *For any bargaining game* $\{\pi_i, z_i, r_i, \gamma_i, c_i, w_i\}_{i=1}^2$ *with* $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, *and* $\gamma_2 = 0$, *there exists a unique sequential equilibrium* $(\widehat{\Sigma}_1, \widehat{\Sigma}_2, \widehat{\mu}_1, \widehat{\mu}_2)$ *in which*

$$\widehat{f}_i(t) = \exp\left[-\int_0^t \widehat{\kappa}_i(s)ds\right]\kappa_i(t), \text{ where } \widehat{\kappa}_i(s) = 1_{s<T} \cdot \frac{\lambda_i}{1 - \widehat{\mu}_i(s)};$$

$$\widehat{g}_1(t) = \exp\left[-\int_0^t \widehat{\chi}_1(s)ds\right]\widehat{\chi}_1(t), \text{ where } \widehat{\chi}_1(s) = 1_{s<T-t_2^N} \cdot \frac{1 - v_1^*}{v_1^*}\frac{\widehat{\mu}_1(s)}{1 - \widehat{\mu}_1(s)}\gamma_1;$$

$$\widehat{q}_2(t) = 1_{t<T-t_2^N} \cdot \frac{1}{1-w}\left[\frac{c_1}{D}\frac{1}{1 - \widehat{\mu}_2(t)} - w\right];$$

and $\widehat{\mu}_i$ *is defined as in Lemma 2.*

3.4.2 Equilibrium Hazard Rates

Figure 2 illustrates the empirical hazard rates—that is, the aggregate rates by justified and unjustified players—of sending an ultimatum and of terminating the game. The empirical hazard rate of ultimatum is

$$\widetilde{\chi}_1(t) = \begin{cases} \widehat{\mu}_1(t)\gamma_1 + (1 - \widehat{\mu}(t))\frac{\widehat{\mu}(t)}{\mu(t)}\frac{1-v_1^*}{v_1^*}\gamma_1 = \frac{\widehat{\mu}_1(t)}{v_1^*}\gamma_1 & \text{if } \widehat{\mu}_1(t) < \mu_1^N \\ \widehat{\mu}_1(t)\gamma_1 & \text{if } \mu_1^N \leq \widehat{\mu}_1(t) < 1 \\ \gamma_1 & \text{if } \widehat{\mu}_1(t) = 1 \end{cases}$$

Since the empirical rates of voluntary concession, λ_1 and λ_2, stay constant for the two players, the empirical rate of ending the bargaining process is

$$\widetilde{e}(t) = \begin{cases} \frac{\widehat{\mu}_1(t)}{v_1^*}\gamma_1 + \lambda_1 + \lambda_2 & \text{if } \widehat{\mu}_1(t) < \mu_1^N \\ \widehat{\mu}_1(t)\gamma_1 + \lambda_1 + \lambda_2 & \text{if } \mu_1^N \leq \widehat{\mu}_1(t) < 1 \\ \gamma_1 & \text{if } \widehat{\mu}_1(t) = 1 \end{cases}.$$

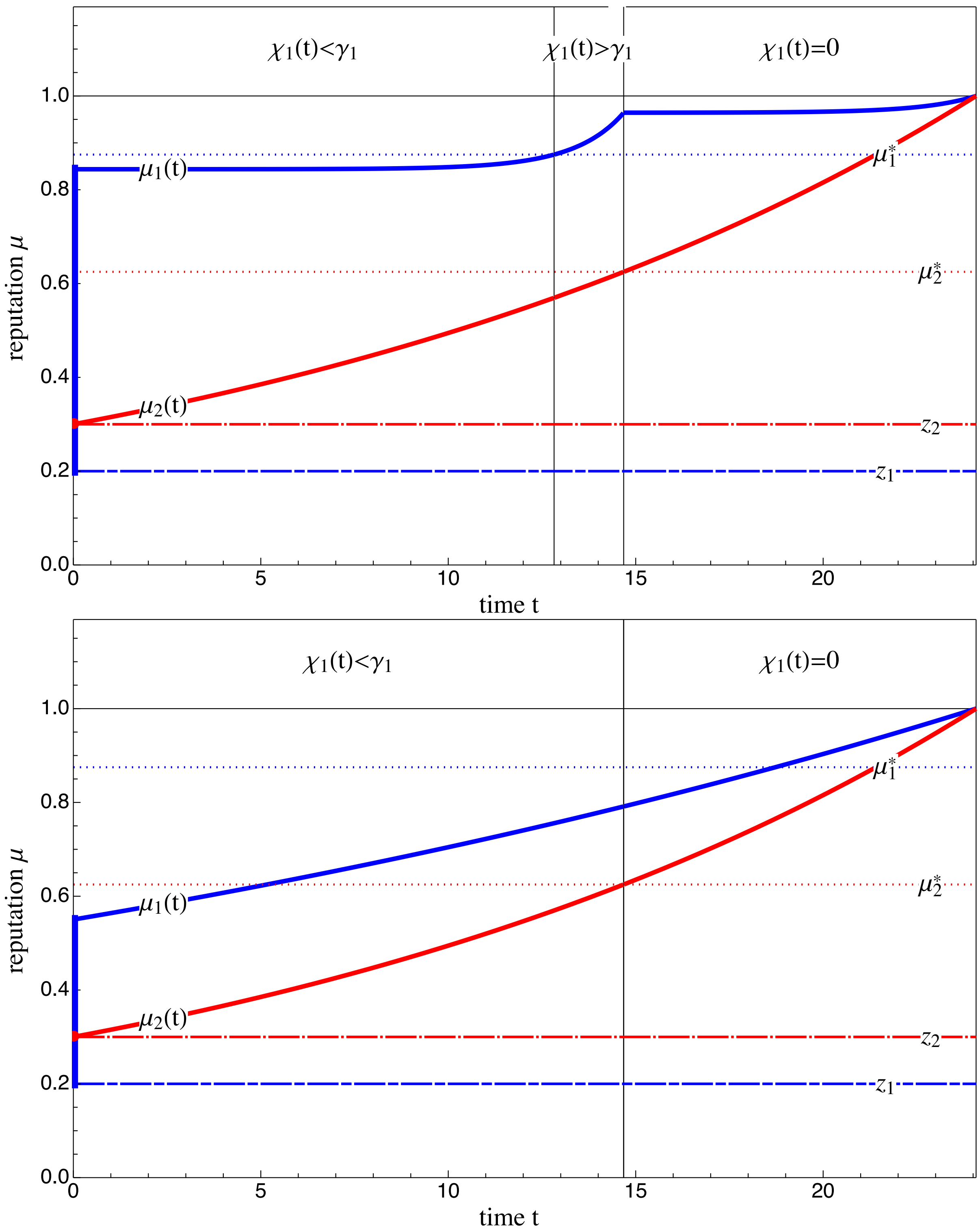


Figure 3: Equilibrium reputation dynamics. Solid lines represent equilibrium reputation dynamics of a bargaining game with one-sided challenge and single justified types. Dashed lines represent equilibrium reputation dynamics of the same bargaining game with no challenge opportunities. The first graph shows reputation building in which an unjustified player 1 challenges at a higher rate $\chi_1(t)$ than a justified player for a period. The second graph shows reputation building in which player 1 challenges at a rate lower than γ_1.

Figure 1 illustrates a strategic player 1's equilibrium rates of ultimatum, concession, and ending the bargaining process. For a strategic player 1, the rate of challenging is $\widehat{\chi}_1(t)$, and the rate of conceding is $\widehat{\kappa}_1(t) = \lambda_1/[(1-\widehat{\mu}_1(t)]$. Combined, the rate of ending the game is

$$\widehat{e}(t) = \begin{cases} \frac{1-v_1^*}{v_1^*}\frac{\widehat{\mu}_1(t)}{1-\widehat{\mu}_1(t)}\gamma_1 + \frac{\lambda_1}{1-\widehat{\mu}_1(t)} & \text{if } t < T - t_2^N \\ \frac{\lambda_1}{1-\widehat{\mu}_1(t)} & \text{if } T - t_2^N \leq t < T \\ 0 & \text{if } t \geq T \end{cases}$$

A testable implication of the model is that the empirical hazard rate of negotiation termination—which we may observe in many settings—should experience a discontinuous drop at some point. The first peak arises when player 2's reputation approaches the level beyond which player 1 has no incentive to challenge, and the second peak arises when player 2's reputation approaches 1 by which player 1 has no incentive to continue bargaining.

3.4.3 Who Benefits from the Challenge Opportunity?

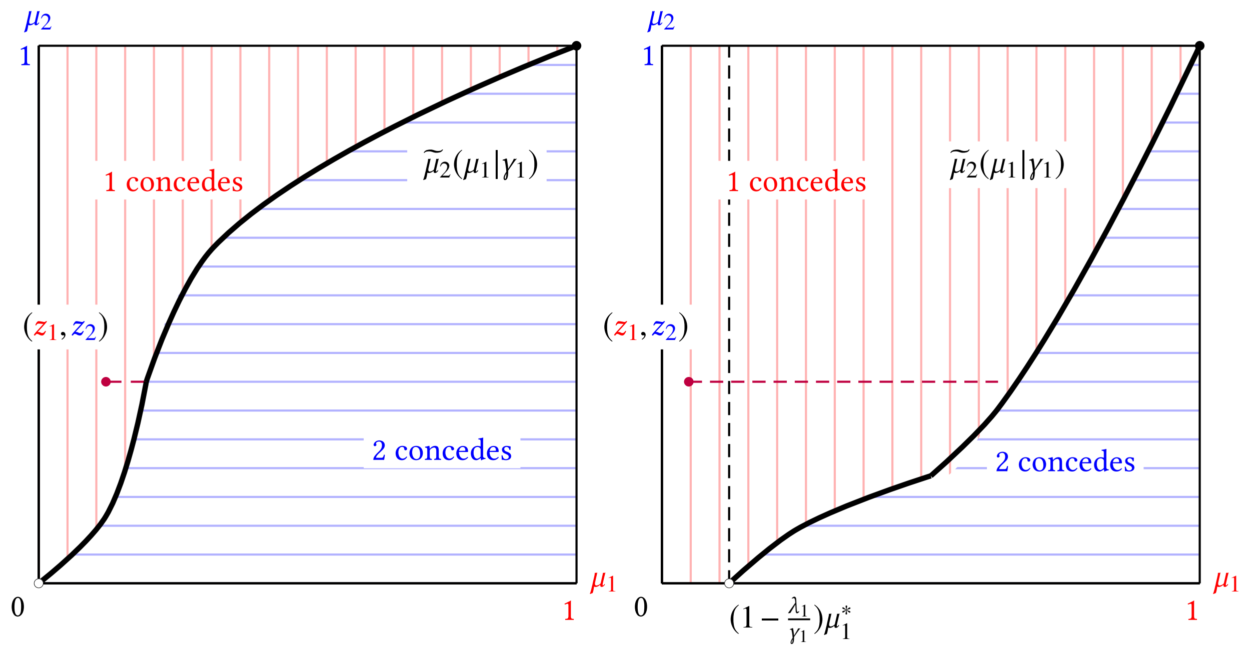
Because the players are conceding at exactly the Abreu-Gul rates, a strategic player challenging at a rate higher than the justified player's, γ_1, makes his reputation build faster. A necessary condition for some strategic player 1 to benefit from having an ultimatum is $\mu_1^N > v_1^*$: If $\mu_1^N \leq v_1^*$, a strategic player 1 never challenges at a higher rate than a justified player's, i.e., $\widehat{\chi}_1(t) \leq \gamma_1$. If we restrict that a strategic player cannot challenge at a rate higher than γ_1 (as in the case where every player receives a challenge opportunity at Poisson rate γ_1), then player 1 would never benefit from having the challenge opportunity. The sufficient condition for some strategic player 1 to benefit from having the challenge opportunity is that it takes a longer time to build a reputation from v_1^* to 1 in the current setting than in the Abreu-Gul setting. Figure 5 below illustrates who benefits from the challenge opportunity in the belief plane when $\mu_1^N \leq v_1^*$. The result is summarized in the proposition below.

Proposition 2. *Let $\widehat{\mu}_1(\mu_2|\gamma_1)$ denote the reputation curve in equilibrium and $t_1(\mu|\gamma_1)$ the time it takes for player 1's reputation to increase from μ to 1 in equilibrium when the challenge rate is γ_1. If (i) $\mu_1^N \geq v_1^*$ or (ii) $\mu_1^N < v_1^*$ and $t_1(v_1^*|\gamma_1) \leq t_1(v_1^*|0)$, then no player 1 benefits from the introduction of the challenge opportunity. If $\mu_1^N < v_1^*$ and $t_1(v_1^*|\gamma_1) > t_1(v_1^*|0)$, then player 1 strictly benefits from the introduction of the challenge opportunity if and only if $\underline{\mu}_1 < \mu_1 < \overline{\mu}_1$ and $\widehat{\mu}_1^{-1}(\underline{\mu}_1|\gamma_1) < \widehat{\mu}_1^{-1}(\mu_1|\gamma_1) < \widehat{\mu}_1^{-1}(\overline{\mu}_1|\gamma_1)$.*

4 Limiting Payoffs

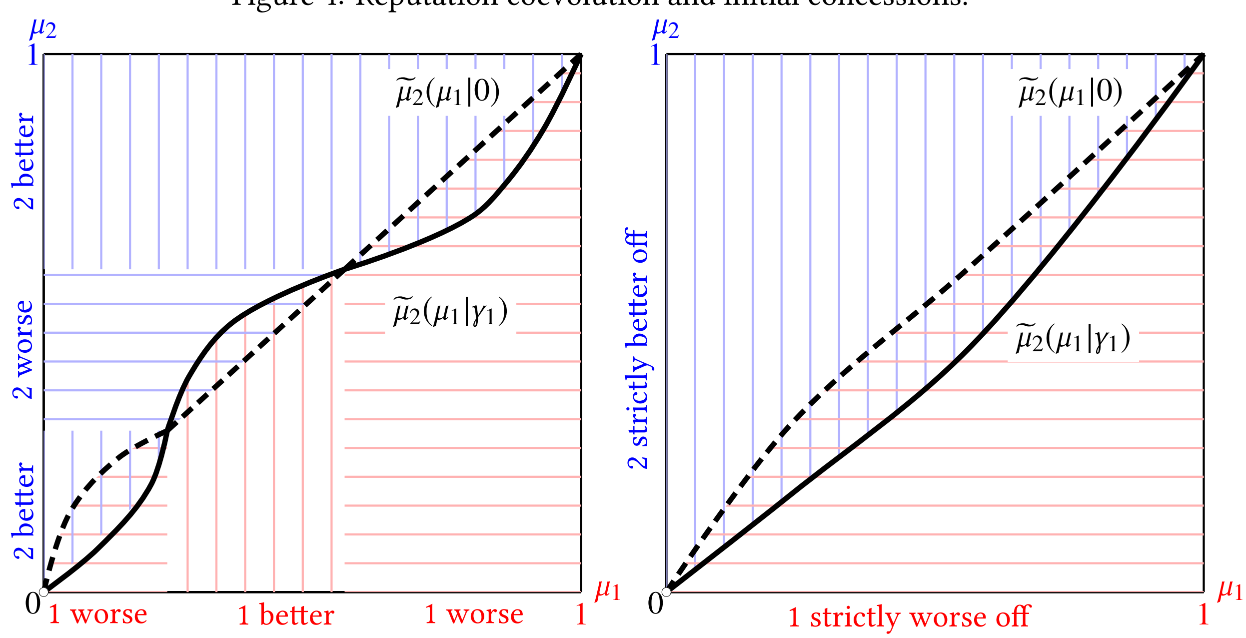
4.1 The Multiple-Type Case

Let's consider the intermediate case in which there is only one justified type of player 1 but there are several justified types of player 2: $|A_1| = 1$ and $|A_2| > 1$.



(a) Reputation coevolution when $\gamma_1 \leq \lambda_1$. (b) Reputation coevolution when $\gamma_1 > \lambda_1$.

Figure 4: Reputation coevolution and initial concessions.



(a) Player 1 may be better off with ultimatum. (b) Player 1 is never better off with ultimatum.

Figure 5: Who benefits from a challenge opportunity? The dark line is the reputation coevolution graph $\widetilde{\mu}_1(\mu_2|\gamma_1)$ with a challenge opportunity arriving at rate γ_1, and the dashed line is the reputation coevolution graph $\widetilde{\mu}_1(\mu_2|0)$ in AG without a challenge opportunity. With the introduction of a challenge opportunity, player 1 (2) is strictly worse off if the pair of initial reputations is in the region filled with red (blue) horizontal lines, and is strictly better off if the pair of initial reputations is in the region filled with red (blue) vertical lines.

Denote by $B_1(a_1, x)$ the bargaining game with one-sided challenges in which player 1 is justified with probability x and a justified player 1's demand is always a_1. Given the equilibrium characterization solved in the previous subsections, determining player 2's equilibrium mimicking behavior suffices to characterize full equilibrium strategy. Define $\sigma_2(a_2)$ as player 2's probability of choosing $a_2 \in A_2$ and define $\sigma_2(0) \equiv Q_2$ as her probability of conceding at time 0. Player 2 chooses $\sigma_2(\cdot)$, a probability distribution over $A_2 \cup \{0\}$, to maximize

$$u_2(\sigma_2(\cdot); a_1, x) = \sigma_2(0)(1 - a_1) + \sum_{a_2 \in A_2} \sigma_2(a_2) u_2(a_1, a_2, x, \sigma_2(a_2))$$

subject to $\sum_{a_2 \in A_2 \cup \{0\}} \sigma_2(a_2) = 1$, where $u_2(a_1, a_2, x, \sigma_2(a_2))$ is player 2's expected payoff when player 2 chooses a_2 with probability $\sigma_2(a_2)$ and players play optimally in the subsequent bargaining game as described by the previous subsection.

If $x = 1$, then in equilibrium, $Q_2 = 1$. Assume $x < 1$ for the rest of the section. Define $T_i(a_1, a_2, x)$ as the time it takes for player i's reputation to increase from x to 1 on the equilibrium reputation path when each player i's demand is a_i, $i = 1, 2$. Define player 2's reputation at time 0 when she plays a_2 with probability σ_2 as

$$y(a_2, \sigma_2) = \frac{z_2 \pi_2(a_2)}{z_2 \pi_2(a_2) + (1 - z_2)\sigma_2}.$$

Because the more likely an unjustified player 2 announces a particular demand a_2, the more likely she is believed to be unjustified, and the lower her payoff from demanding a_2 is. Let $\overline{\sigma}_2(a_1, a_2, x)$ be the maximum probability player 2 plays a_2 in equilibrium so that the expected payoff from demanding a_2 is higher than directly conceding to player 1's demand. For any $a_2 < 1 - a_1$, $\overline{\sigma}_2(a_1, a_2, x) = 0$ because conceding to player 1's demand of a_1 and receiving $1 - a_1$ is a strictly better strategy than demanding less than $1 - a_1$. For any $a_2 \geq 1 - a_1$, after choosing a_2, in any equilibrium, player 2 should not concede with a positive probability at time 0. First, if player 1's reputation can reach 1 without conceding with a positive probability at time 0 and player 2's reputation reaches 1 slower than player 1 when she demands a_2 with probability 1, $\overline{\sigma}_2(a_1, a_2, x)$ is the unique solution of σ_2 to $T_1(a_1, a_2, x) = T_2(a_1, a_2, y(a_2, \sigma_2))$ so that the two players' reputations reach 1 at the same time. Second, if player 1's reputation reaches 1 even slower than when player 2 demands a_2 with probability 1, $\overline{\sigma}_2(a_1, a_2, x) = 1$. The scenario happens whenever $T_1(a_1, a_2, x) > T_2(a_1, a_2, y(a_2, 1))$. In particular, it happens whenever $x < \mu_1^*(1 - \frac{\lambda_1}{\gamma_1})$. In summary, in any equilibrium, $\sigma_2(a_2) \leq \overline{\sigma}_2(a_1, a_2, x)$, where $\overline{\sigma}_2(a_1, a_2, x) = 0$ if $a_2 \leq 1 - a_1$, $\overline{\sigma}_2(a_1, a_2, x)$ is the unique solution of σ_2 in $T_1(a_1, a_2, x) = T_2(a_1, a_2, y(a_2, \sigma_2))$ if $a_2 > 1 - a_1$ and $T_1(a_1, a_2, x) < T_2(a_1, a_2, y(a_2, 1))$, and $\overline{\sigma}_2(a_1, a_2, x) = 1$ if $a_2 > 1 - a_1$ and $T_1(a_1, a_2, x) \geq T_2(a_1, a_2, y(a_2, 1))$.

When player 2 demands a_2 with probability $\sigma_2 \leq \overline{\sigma}_2(a_1, a_2, x)$, player 1 must raise his time 0 reputation to $x^*(a_1,a_2,\sigma_2)$ so that their reputations reach 1 at the same time:

$$T_1(a_1,a_2,x^*(a_1,a_2,\sigma_2)) = T_2(a_1,a_2,y(a_2,\sigma_2)).$$

In order to do so, player 1 concedes with probability

$$Q_1(a_1,a_2,x,\sigma_2) = 1 - \frac{x}{1-x}\frac{1-x^*(a_1,a_2,\sigma_2)}{x^*(a_1,a_2,\sigma_2)}$$

so that the reputation is raised to

$$x^*(a_1,a_2,\sigma_2) = \frac{x}{x+(1-x)(1-Q_1(a_1,a_2,x,\sigma_2))}.$$

When player 2 demands a_2 with probability σ_2 and an unjustified player 1 concedes with probability $Q_1(a_1,a_2,x,\sigma_2)$, player 2's expected payoff is

$$\begin{aligned} u_2(a_1,a_2,x,\sigma_2) &= (1-x)Q_1(a_1,a_2,x,\sigma_2)a_2 + [x+(1-x)(1-Q_1(a_1,a_2,x,\sigma_2))](1-a_1) \\ &= 1-a_1+(1-x)Q_1(a_1,a_2,x,\sigma_2)(a_1+a_2-1). \end{aligned}$$

Two additional properties restrict player 2's equilibrium strategies $\sigma_2(\cdot)$. First, for any a_2 and $a_2' > a_2$, if $\sigma_2(a_2) > 0$, then $\sigma_2(a_2') > 0$. We can prove this property by contradiction. Suppose $\sigma_2(a_2) > 0$ and $\sigma_2(a_2') = 0$. Because $\sigma_2(a_2') = 0$, $u_2(a_1,a_2',x,\sigma_2(a_2')) = 1-a_1+(1-x)(a_1+a_2'-1)$. Because $\sigma_2(a_2) > 0$, $u_2(a_1,a_2,x,\sigma_2(a_2)) = 1-a_1+(1-x)Q_1(a_1,a_2,x,\sigma_2(a_2))(a_1+a_2-1) \le 1-a_1+(1-x)(a_1+a_2'-1) = u_2(a_1,a_2',x,\sigma_2(a_2'))$. Second, whenever $\sum_{a_2}\overline{\sigma}_2(a_1,a_2,x) \le 1$, $\sigma_2(a_2) = \overline{\sigma}_2(a_1,a_2,x)$ for all a_2, and $Q_2 = 1-\sum_{a_2}\overline{\sigma}_2(a_1,a_2,x)$. The two properties together imply that we only need to check first if $\sum_{a_2}\overline{\sigma}_2(a_1,a_2,x) \le 1$, and, if the first condition does not hold, then find the equilibrium strategy among $\sigma_2(\cdot)$ such that $\sigma_2(a_2') > 0$ for all $a_2' \ge a_2$, for each $a_2 \in A_2$.

For any mimicking strategy $\sigma_2(\cdot)$, define

$$F_2(x,\sigma_2(\cdot)) \equiv \min_{a_2:\sigma_2(a_2)>0} u_2(a_1,a_2,x,\sigma_2(a_2)).$$

$\sigma_2(\cdot)$ is an equilibrium strategy for player 2 if and only if $\sigma_2(\cdot)$ solves

$$\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} F_2(x,\sigma(\cdot))$$

where

$$\Delta(a_1,x) = \{\sigma(\cdot)\in\Delta | \sigma_2(a_2) \le \overline{\sigma}_2(a_1,a_2,x)\ \forall a_2 \in A_2\}$$

and Δ is the set of probability distributions on $A_2 \cup \{0\}$. Player 1's equilibrium payoff when player 2 plays $\sigma_2(\cdot)$ is

$$u_1(a_1,a_2,x,\sigma_2(\cdot)) = (1-z_2)Q_2a_1 + \sum_{a_2\in A_2}[z_2\pi_2(a_2)+(1-z_2)\sigma_2(a_2)](1-a_2).$$

It remains to show that there is a unique equilibrium. Multiple equilibrium distributions over types being conceded to are in conflict with the requirement that types mimicked with positive probability must have equal payoffs that are not smaller than the payoffs of the types that are not mimicked. Suppose by contradiction there are two different equilibrium strategies for player 2: $\sigma_2(a_2) \neq \sigma_2'(a_2)$ for some a_2. If $\sigma_2(a_2) > 0$ and $\sigma_2'(a_2) > 0$, then $u_2(a_1, a_2, x, \sigma_2(a_2)) \neq u_2(a_1, a_2, x, \sigma_2'(a_2))$. But $u_2(a_1, a_2, x, \sigma_2(a_2)) = F_2(x, \sigma_2(\cdot))$ and $u_2(a_1, a_2, x, \sigma_2(a_2)) = F_2(x, \sigma_2'(\cdot))$. $F_2(x, \sigma_2(\cdot)) \neq F_2(x, \sigma_2'(\cdot))$ contradicts the fact that $\sigma_2(\cdot)$ and $\sigma_2'(\cdot)$ both solve $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} F_2(x, \sigma(\cdot))$. If $\sigma_2(a_2)$ or $\sigma_2'(a_2)$ is zero, then, by the first additional property of player 2's equilibrium strategy above, there is an $a_2' > a_2$ such that $\sigma_2(a_2') > 0$, $\sigma_2'(a_2') > 0$, and $\sigma_2(a_2') \neq \sigma_2'(a_2')$.

Player 1 receives $u_1(a_1, x)$ in the equilibrium of the bargaining game $B(a_1, x)$.

Proposition 3. *For any bargaining game $\{z_i, \pi_i, r_i, \gamma_i, c_i, w_i\}_{i=1}^2$ with $A_1 = \{a_1\}$, $|A_2| > 1$, and $\gamma_2 = 0$, there exists a unique sequential equilibrium $(\widehat{\Sigma}_1, \widehat{\Sigma}_2) = ((\widehat{F}_1, \widehat{\gamma}_1), (\widehat{\sigma}_2, \widehat{F}_2, \widehat{q}_2))$.*

Proposition 4. *Denote by $u_1^*(a_1, x)$ the payoff of player 1 in the unique sequential equilibrium of the bargaining game $\{\pi, z_i, r_i\}_{i=1}^2$ with $A_1 = \{a_1\}$ and $|A_2| \geq 1$. It is a continuous function of x. Moreover, there exists an $\underline{x}$ such that $u_1^*(a_1, x) = u_1^*(a_1, \underline{x})$ for any $x \leq \underline{x}$ and $u_1^*(a_1, x)$ is strictly increasing in x on the interval $(\underline{x}, 1)$.*

Now we look at the case in which player 1 first chooses which type $a_1 \in A_1$ to mimic, and seeing this, player 2 responds with a type $a_2 \in A_2$ to mimic. Let $u_1(a_1, x)$ denote player 1's equilibrium payoff in the bargaining game $B_1(a_1, x)$. Player 1 chooses mimicking strategy $\sigma_1(\cdot)$ subject to $\sum_{a_1\in A_1} \sigma_1(a_1) \leq 1$ to maximize his payoff

$$u_1(a_1, \sigma_1(\cdot)) = \sum_{a_1\in A_1} u_1(a_1, x(a_1, \sigma_1(a_1)))$$

where

$$x(a_1, \sigma_1(a_1)) = \frac{z_1\pi_1(a_1)}{z_1\pi_1(a_1) + (1 - z_1)\sigma_1(a_1)}.$$

It remains to show the equilibrium is unique.

Proposition 5. *For any bargaining game $\{\pi_i, z_i, r_i, \gamma_i, c_i, w_i\}_{i=1}^2$ with $\gamma_2 = 0$, there exists a sequential equilibrium $(\widehat{\Sigma}_1, \widehat{\Sigma}_2) = ((\widehat{\sigma}_1, \widehat{F}_1, \widehat{\gamma}_1), (\widehat{\sigma}_2, \widehat{F}_2, \widehat{q}_2))$. Furthermore, all equilibria yield the same distribution over outcomes.*

4.2 The Limiting Case of Complete Rationality

Suppose that the commitment type set for each player sufficiently "finely" covers the interval $[0, 1]$. We are interested in the limit equilibrium payoffs as $z_1, z_2 \to 0$.

First, suppose $\gamma_1 > r_1$. We first start with the case in which player 1 chooses a type $a_1 > \frac{r_2}{r_2+\gamma_1}$ with a probability that doesn't disappear as $z_1 \to 0$. If player 2 chooses $a_2 \approx 1$, then we have

$$\lambda_1 = \frac{(1-a_1)r_2}{a_1+a_2-1} \approx \frac{(1-a_1)r_2}{a_1} < \gamma_1.$$

Hence, Player 2's payoff from this strategy approaches to approximately 1, and hence player 1's payoff approaches to approximately to 0.

If Player 1 chooses a type $a_1 < \frac{r_2}{r_2+\gamma_1}$, then regardless of a_2, $\lambda_1 > \gamma_1$, hence we need to solve for the equilibrium dynamics of the model in which $\lambda_1 > \gamma_1$.

Player 1 wins if $\lambda_1 - \gamma_1 > \lambda_2$, and player 2 wins otherwise. (Why?)

Plugging in λ_i the expression from Abreu and Gul, we have that Player 2 wins if

$$(1-a_1)r_2 - \gamma_1 a_1 < (1-a_2)(r_1-\gamma_1)$$

Because $\gamma_1 > r_1$, the right hand side is negative, and left hand side is always positive, so Player 1 wins.[3]

Since this is true for every $a_1 < \frac{r_2}{r_2+\gamma_1}$, player 1, by choosing a demand approximately equal to $\frac{r_2}{r_2+\gamma_1}$ (more precisely, $\max\{a_1 \in A_1 | a_1 \leq \frac{r_2}{r_2+\gamma_1}\}$) guarantees this payoff, and cannot do better, and Player 2 gets the rest of the surplus.

Second, suppose $\gamma_1 < r_1$. In this case, if player 1 chooses $a_1 = \frac{r_2}{r_2+r_1}$, then $\lambda_1 > \gamma_1$ for any choice of a_2, so the winner is determined by comparison

$$(1-a_1)r_2 - \gamma_1 a_1 < (1-a_2)(r_1-\gamma_1)$$

which for the choice of $a_1 = \frac{r_2}{r_2+r_1}$ makes player 1 the winner, and for any choice of a_1 lower, makes player 2 the winner by a choice that makes player 2 have a payoff larger than $\frac{r_1}{r_1+r_2}$, that leaves player 1 with a payoff smaller than $\frac{r_2}{r_1+r_2}$. Hence the solution is similar to Abreu and Gul in this case.

Proposition 6. *Let $B_0^n = \{A_i, z_i^n, \pi_i, r_i\}_{i=1}^2$ be a sequence of continuous-time bargaining games. If $\lim z_1^n = \lim z_2^n = 0$, $\lim z_1^n/(z_1^n + z_2^n) \in (0,1)$ and v_i^n is the sequential equilibrium payoff for player i in the game B^n, then*

$$\liminf v_1^n \geq \max\left\{a \in A_1 \cup \{0\} \middle| a < \frac{r_2}{\max\{r_1,\gamma_1\}+r_2}\right\},$$

and

$$\liminf v_2^n \geq \max\left\{a \in A_2 \middle| a < \frac{\max\{r_1,\gamma_1\}}{\max\{r_1,\gamma_1\}+r_2}\right\}.$$

[3] It's important for this argument that z_1 goes to zero at a rate not smaller than that of z_2.

5 Two-Sided Ultimatums

A male player 1 and a female player 2 divide a unit pie. Each player is either (i) justified to demand a share of the pie and never accepting any offer below that, or (ii) unjustified to demand a share of the pie but nonetheless wanting as a big share of the pie as possible. A justified player can find hard evidence supporting his or her demand, but an unjustified player has no hard evidence supporting his or her claim of the share.

We initially assume that each player can be of a single justified type: with probability z_1 player 1 is justified to demand a_1 and with probability z_2 player 2 is justified to demand $a_2 > 1 - a_1$. Let $D \equiv a_1 + a_2 - 1$ denote the conflicting difference between the two players.

Time is continuous. At each instant t, each player can decide to give in to the other player's demand or hold on to his or her demand. In addition, player i has a challenge opportunity. A justified player i challenges when evidence arrives; the evidence arrives according to a Poisson process with arrival rate $\gamma_i > 0$. An unjustified player i can challenge at any time but he will time his challenge strategically. If the players neither challenge nor concede, then the game continues. If player i challenges at time t, he/she incurs a cost c_i and player $j \neq i$ must respond to the challenge. Player j may either yield to challenge and get $1 - a_j$, or see the challenge by paying a cost c_j.

The shares of the pie are determined by the players' justified and unjustified types in the court, as follows. An unjustified player i's payoff against a justified player j is $1 - a_j$. If two unjustified players meet, then the challenging player i wins with probability $w < 1/2$: he gets a_i with probability w and gets $1 - a_j$ with probability $1 - w$, so his expected payoff is $wa_i + (1 - w)(1 - a_j) = 1 - a_j + wD$, and the defending player j's expected payoff is $(1 - w)(1 - a_i) + wa_j = 1 - a_1 + (1 - w)D$. To make challenging and seeing a challenge worthwhile for player i, assume $wD < c_i < (1 - w)D$ for $i = 1, 2$.

In summary, the bargaining game $B\left(\{z_i, a_i, r_i, c_i, \gamma_i, w_i\}_{i=1}^{2}\right)$ with two-sided ultimatums is described by players' prior probabilities z_1 and z_2 of being justified, demands a_1 and a_2, discount rates r_1 and r_2, challenge arrival rates γ_1 and γ_2, challenge costs c_1 and c_2, and an unjustified challenger's winning probability w against an unjustified player.

5.1 The Single-Type Case

5.1.1 Formal Description of the Game

Let us formally describe the strategies and payoffs of the (unjustified) players when demands are fixed to be a_1 and a_2. Let $F_i(t)$ denote player i's probability of conceding by time t. Let $\gamma_i(t)$ denote player i's probability of challenging by time t. Let $q_i(t)$ denote player i's probability of conceding to a challenge at time t. Let $\Sigma_i = (F_i, \gamma_i, q_i)$ denote an unjustified player i's strategy.

Player i's expected utility of taking no action at any time $s < t$ while yielding to a challenge with probability $q_i(s)$ at time $s < t$ is

$$\begin{aligned} U_i(t^-, q_i, \Sigma_j) &= (1-z_j)\int_0^t a_i e^{-r_i s} dF_j(s) + z_j \int_0^t (1-a_i) e^{-r_i s} \gamma_j e^{-r_i s} ds \\ &\quad + (1-z_j)\int_0^t \left[1 - a_j + (1-q_i(s))((1-w)D - c_j)\right] e^{-r_j s} d\gamma_i(s). \end{aligned}$$

Player i's expected utility of conceding at time t is

$$\begin{aligned} u_i(t, q_i, \Sigma_j) &= U_i(t^-, q_i, \Sigma_j) + e^{-r_i t}(1-a_j)\left(1 - (1-z_j)F_j(t) - (1-z_j)G_j(t) - z_j(1-e^{-\gamma_j t})\right) \\ &\quad + (1-z_j)\left[F_j(t) - \lim_{s \uparrow t} F_j(s)\right] \frac{a_i + 1 - a_j}{2}. \end{aligned} \tag{6}$$

Player i's expected utility of challenging at time t is

$$\begin{aligned} v_i(t, q_i, \Sigma_j) &= U_i(t^-, q_i, \Sigma_j) + e^{-r_i t} \times \\ &\quad \left[1 - a_i + \Big(1 - (1-z_j)F_j(t) - (1-z_j)G_j(t) - z_j\Big)\Big(q_j(t) + (1-q_j(t))w\Big)D - c_i\right]. \end{aligned}$$

Player i's expected utility from strategy Σ_i is

$$u_i(\Sigma_i, \Sigma_j) = \int_0^\infty u_i(s, q_i, \Sigma_j) dF_i(s) + \int_0^\infty v_i(s, q_i, \Sigma_j) d\gamma_i(s).$$

5.1.2 Strategies

Player i's optimal yielding strategy. We consider the best response of an unjustified player i who faces a challenge and believes that the challenging player j is justified with probability ν_j. Responding to the challenge results in an expected utility of $1 - a_j + (1-\nu_j)(1-w)D - c_i$, and yielding to the challenge results in an expected utility of $1 - a_j$. An unjustified player i is indifferent between responding and yielding when player $j \neq i$ is believed to be justified with probability $\nu_j = 1 - \frac{c}{(1-w)D} \equiv \nu_j^*$, strictly prefers to respond when $\nu_j < \nu_j^*$, and strictly prefers to yield when $\nu_j > \nu_j^*$.

Player i's optimal challenging strategy. We consider the optimal challenging strategy of an unjustified player i who believes that player $j \neq i$ is justified with probability μ_j and an unjustified player j yields to a challenge with probability q_j. Challenging yields an expected utility of $1 - a_j + (1-\mu_j)[q_j + (1-q_j)w]D - c_i$, and not challenging yields an expected utility of $1 - a_j$ on any equilibrium path. An unjustified player i is indifferent between challenging and not challenging if $\mu_j = 1 - c_1/[(q_j + (1-q_j)w)D]$.

Candidate equilibrium challenging and yielding strategies. If player j is justified with a probability more than $\mu_j^* = 1 - \frac{c_i}{D}$, an unjustified player i strictly prefers not to challenge. If player j is justified with a probability less than μ_i^*, an unjustified player i must challenge at rate χ_j to make player i believe that a challenging player i is justified with probability $v_i^* \equiv 1 - \frac{c_j}{(1-w)D}$:

$$\frac{\mu_i \gamma_i}{\mu_i \gamma_i + (1-\mu_i)\chi_i} = v_i^* \Rightarrow \chi_i(\mu_i) = \frac{1-v_i^*}{v_i^*} \frac{\mu_i}{1-\mu_i} \gamma_i.$$

If an unjustified player i challenges at a rate higher than the specified rate, then an unjustified player j is strictly better off responding than yielding to the challenge. If an unjustified player i challenges at a rate lower than the specified rate, then an unjustified player 2 is strictly worse off responding than yielding to the challenge. On the other hand, to make player i indifferent between challenging and not challenging, player j yields to a challenge with probability

$$q_j(\mu_j) = \frac{1}{1-w}\left[\frac{c_i}{D}\frac{1}{1-\mu_j} - w\right].$$

Players' conceding strategies. In equilibrium, players concede at the same rates as in Abreu and Gul (2000). Players are indifferent between conceding and waiting to concede the next instant. An unjustified player concedes at a rate to make the opposing unjustified player indifferent between conceding and not conceding.

$$1 - a_j = \lambda_j dt \cdot a_i + e^{-r_i dt} \cdot (1-a_j)(1-\lambda_j dt),$$

$$\lambda_i = r_j(1-a_i)/D.$$

5.1.3 Reputation

Player i's reputation in i's challenge phase. When player i challenges, player i's reputation follows the following Bernoulli ODE:

$$\mu_i'(t) = (\lambda_i - \gamma_i)\mu_i(t) + \frac{\gamma_i}{v_i^*}\mu_i^2(t).$$

Player i's reputation in no-challenge phase. When player i does not challenge, player i's reputation follows the following Bernoulli ODE:

$$\mu_i'(t) = (\lambda_i - \gamma_i)\mu_i(t) + \gamma_i \mu_i^2(t).$$

5.1.4 Initial Concession Stage

We still have the key equilibrium property that players reach reputation 1 at the same time, if the reputations ever reach 1. Suppose the reputations reach 1 in finite time. We solve the equilibrium reputation dynamics backwards. Define $t_i^{NN} \equiv t(1; \mu_i^*, \lambda_i - \gamma_i, \gamma_i)$ as the time length it takes for player i's reputation evolve from μ_i^* to 1 following the reputation dynamics in the no-challenge phase, and define $t^{NN} \equiv \min\{t_1^{NN}, t_2^{NN}\}$, the shorter time length that it takes to evolve from μ_i^* to 1 in the non-challenging phase. Let I be the player (or one of the players) such that $t_I^N = t^{NN}$, and let $J \neq I$ denote the opposing player. Define $\mu_J^{NN} = \mu(-t^{NN}; 1, \lambda_J - \gamma_J, \gamma_J)$; note that if $t_I^N = t_J^N = t^{NN}$, then $\mu_J^{NN} = \mu_J^*$.

Time t^{NN} before the last concession, because player I's reputation drops below μ_I^*, player J challenges and player I does not challenge, player J's reputation evolves according to the dynamics in the challenging phase, and player I's reputation evolves according to the dynamics in the non-challenging phase. Define $t^N \equiv t(\mu_J^{NN}; \mu_J^*, \lambda_J - \gamma_J, \frac{\gamma_J}{v_J^*})$ as the time length it takes for player J's reputation to evolve from μ_J^* to μ_J^{NN}. Let $\mu_I^N \equiv \mu(-t^N; \mu_I^*, \lambda_I - \gamma_I, \gamma_I)$ denote player I's reputation when player J's reputation is μ_J^*.

Time $t^{NN} + t^N$ before the last concession, both players challenge, and their reputations evolve accordingly.

Lemma 3. *Define $t_i^{NN} \equiv t(1; \mu_i^*, \lambda_i - \gamma_i, \gamma_i)$. Let $t_I^{NN} \leq t_J^{NN}$. Define*

$$\mu_I(-t) \equiv \begin{cases} \mu(-t; 1, \lambda_I - \gamma_I, \gamma_I) & t \leq t^{NN} + t^N \\ \mu(-t; \mu_I^N, \lambda_I - \gamma_I, \frac{\gamma_I}{v_I^*}) & t > t^{NN} + t^N \end{cases}, \ \mu_J(-t) \equiv \begin{cases} \mu(-t; 1, \lambda_J - \gamma_J, \gamma_J) & t \leq t^{NN} \\ \mu(-t; \mu_J^{NN}, \lambda_J - \gamma_J, \frac{\gamma_J}{v_J^*}) & t > t^{NN} \end{cases}.$$

Player i's reputation in equilibrium is

$$\widehat{\mu}_i(T - t) = \mu_i(-t),$$

where $T = \min\{T_1, T_2\}$, and $T_i = \inf\{t; \mu_i(-t) = z_i\}$, $\inf \emptyset = \infty$.

5.1.5 Equilibrium

Proposition 7. *Define $(\widehat{F}_i, \widehat{\gamma}_i, \widehat{q}_i)$, where $\widehat{F}_i(t) = \frac{1 - C_i e^{-\lambda_i t}}{1 - z_i}$, $\widehat{\gamma}_i(t) = 1 - e^{-\int_0^t \widehat{\chi}_i(s) ds}$, $\widehat{\chi}_i(s) = 1_{s \leq \widehat{t}_i} \cdot \frac{v_i}{1 - v_i^*} \frac{\widehat{\mu}_i(s)}{1 - \widehat{\mu}_i(s)} \gamma_i$, $\widehat{q}_i(s) = 1_{s \leq \widehat{t}_j} \cdot \frac{1}{1 - w} \left[\frac{c_j}{D} \frac{1}{1 - \widehat{\mu}_j(s)} - w \right]$, $\widehat{\mu}_i(s)$ is as defined in lemma 3, and t_i solves $\widehat{\mu}_i(t_i) = \mu_i^*$. There exists a unique sequential equilibrium when $z_1 > 1 - \frac{\lambda_1}{\gamma_1}$ or $z_2 > 1 - \frac{\lambda_2}{\gamma_2}$. There exists a unique sequential equilibrium $(\widehat{F}_i, \widehat{\gamma}_i, \widehat{q}_i)$ in which both players do not concede with a positive probability at time 0, when $z_1 \leq 1 - \frac{\lambda_1}{\gamma_1}$ and $z_2 \leq 1 - \frac{\lambda_2}{\gamma_2}$.*

When $\gamma_i > \lambda_i$ for both $i = 1, 2$ and the initial reputations are sufficiently small, reputations may not build up.

Figure 6 illustrates two cases of equilibrium reputation paths and strategy phases. In the top panel, $\lambda_i < \gamma_i$ for each $i = 1, 2$. Reputation never builds up for either player, and players mix between challenging, conceding, and waiting. The game could continue forever, with a diminishing probability. In the bottom panel, $\lambda_i \geq \gamma_i$ for each $i = 1, 2$. Reputation builds up for both players. The game goes through phases in equilibrium: two players both challenge with positive probabilities, only one player challenges with a positive probability, and no player challenges with a positive probability.

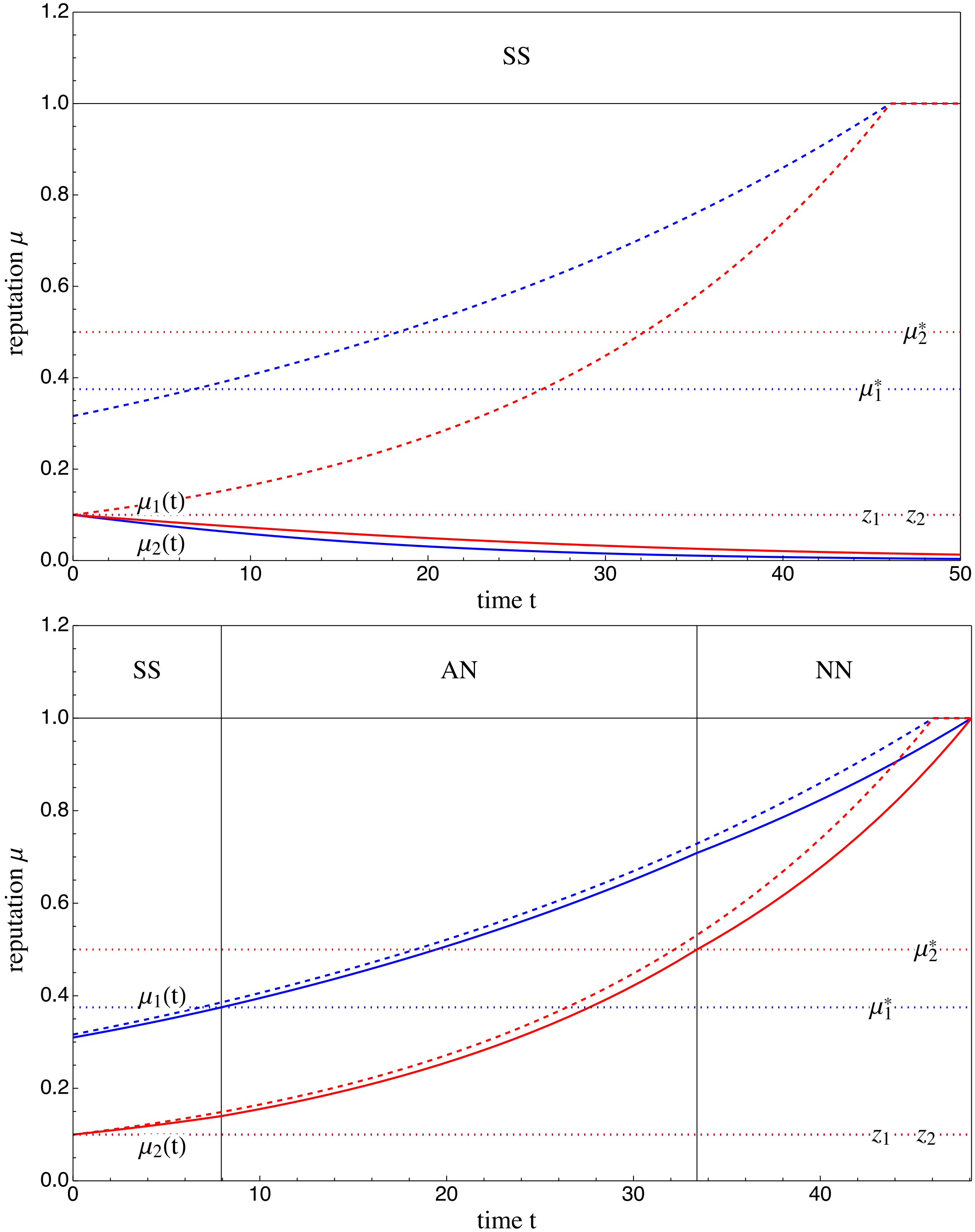


Figure 6: Equilibrium strategy phases and reputation dynamics of the bargaining game with two-sided challenge and single justified types.

6 Application

6.1 Theory: Bargaining with a Deadline

At the deadline $\overline{T}$, the case will be determined by the arbitration. In this case, if the equilibrium T^* is longer than $\overline{T}$, both players may need to concede at time 0 for the game to end at $\overline{T}$.

6.2 Application

We consider the process of negotiation preceding Major League Baseball salary arbitration. We consider this application because we can obtain from publicly available reports (i) the initial offers of the two parties (player and team), (ii) the time the challenge opportunity becomes credible (filing for arbitration and scheduling court date), (iii) the time the negotiation ends (signing the contract), and (iv) the outcome (the terms of the contract). In contrast, the initial proposals, the duration, and final outcome of the arbitration over economic disputes are often confidential.

From mid-January to mid-February each year, players with a defined amount of service time (i.e., number of years playing at the MLB level) will enter into the salary arbitration process with their teams where the player and the team will present their case to have the player's salary set by a neutral third party arbitration panel for the upcoming season by final-offer arbitration.

A team has the contractual rights to a player until that player has six years of service time and becomes a free agent. During the first three years of service a player will typically make around the major league minimum salary. Players with between three and six years of service time and high-caliber players with two years of service time become eligible for salary arbitration if they do not already have a contract with their team for the next season by mid-January.[4]

A player eligible for salary arbitration has to file by a prespecified date mid-January. Once the player files, the player and team will exchange salary offers by January 16. Because only the player can file a salary arbitration, the player is thought to be the side that has the challenge opportunity. At this point, the player and team can still have the opportunity to come to an agreement on a specific figure for the upcoming season prior to the hearing. If the player and team are unable to come to an agreement prior to the scheduled hearing, the player's salary will be determined by the arbitration panel. These hearings occur around mid-February.

During the salary arbitration hearing, both the player and the team will present their case to the arbitration panel. Following the hearing, the panel will choose between the player's and the team's salary offer. The information the two sides can use during the hearing to present their case

[4] A high-caliber second-year player—the so-called Super 2—is a player who has between two and three years of service time, and has at least 86 days of service time during the second year and ranks in the top 22 percent of players who fall into that classification. A Super 2 player will have three years as a pre-arbitration eligible player and four arbitration years while a player who doesn't earn Super 2 status will have three years of salary arbitration following their four pre-arbitration years (Sievert, 2018).

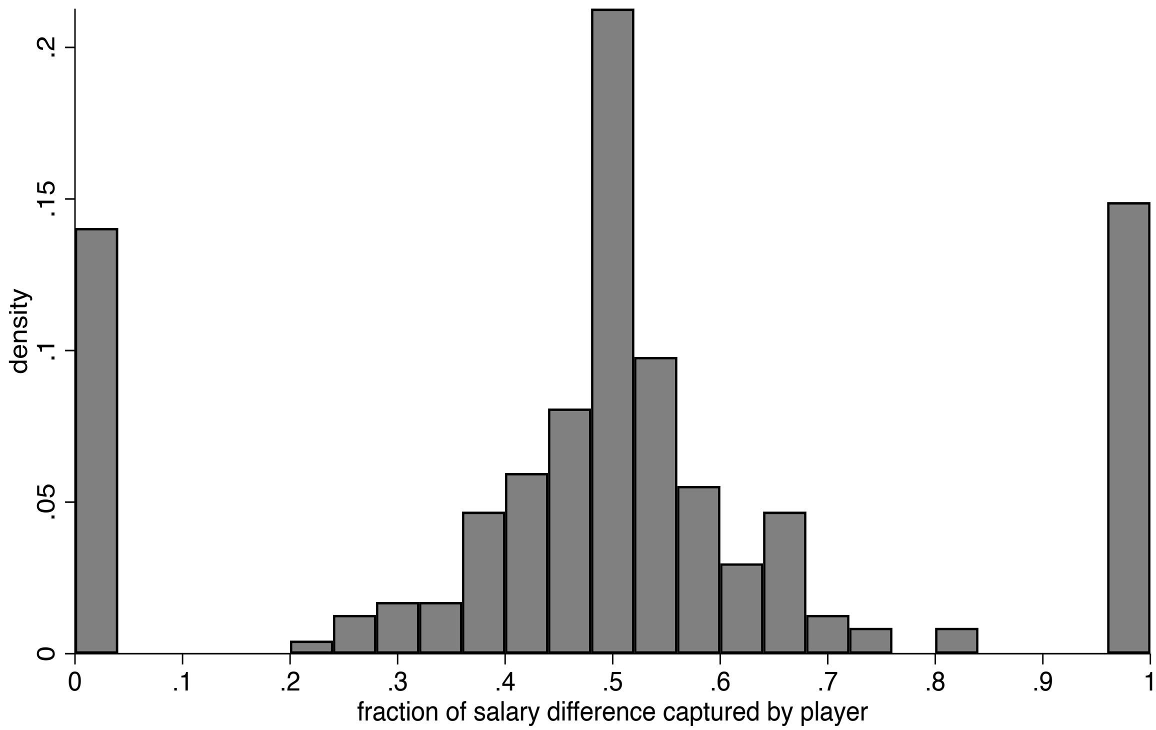


Figure 7: Distribution of fraction of salary difference captured by player.

includes the player's contribution to the team during the past season (e.g., the player's on-field performance and other qualities such as leadership and fan appeal), the length and consistency of the player's career contributions, the player's past compensation, the existence of any physical or mental defects, the team's recent performance (e.g., the team's record, improvement and attendance) and comparative baseball salaries.

The panel gives the most weight to each side's presentation of comparable baseball salaries. Here, the player and team can only compare the contracts of players whose service time does not exceed one annual service group above the player's service group. For example, a starting pitcher who enters the second year of salary arbitration would be compared other starting pitchers who are entering their second and third year of salary arbitration.

Information the panel cannot consider during the hearing includes the financial position of the team or player, testimonials or press comments regarding the team's or player's performance, prior contractual negotiations between the team and player, any costs associated with the salary arbitration process (i.e., attorney's fees), and salaries in other sports or occupations.

We collect all 292 cases in which the player has filed a salary arbitration from 2011 to 2018.[5] On average, these players have 3.6 years of service time, players' initial offers are 4.75 million USD, and teams' initial offers are 3.66 million USD, so their disputes are on average a little above

[5] Salary arbitration has been in effect since the 1970s.

a million dollars. On average, players' initial offers are 34.6% higher than teams' initial offers, and the final settled amounts are 16.7% higher than teams' initial offers. Overall 22.9% of the cases—13.8% in 2011-2016 and 64.3% in 2017 and 2018—were decided by the final arbitration. Of the 67 cases decided by arbitration, 33 are won by the player and 34 are won by the team.

Figure 7 shows the distribution of the outcome of the bargaining measured by the fraction of salary difference captured by the player. The outcome is fairly symmetrically distributed around .5, suggesting that the outcome does not systematically favor one side or another on aggregate and that negotiation is important.

The time it takes to reach an agreement ranges from 0 day to 39 days. Figure 8 illustrates the histogram and kernel density of days to reach an agreement. We can see a dip in the negotiation after two weeks from frequency and kernel density.

A unique prediction of our model is the existence of a discontinuity in empirical hazard rates in reaching an agreement. Figure 9 illustrates the empirical hazard rate of the end of the negotiation. We can see a dip in the negotiation after approximately two weeks, from frequency, kernel density, as well as hazard rate. We can think of the first two weeks as the time interval for the player to challenge the team. Besides the dip, the hazard rates of end of the game are increasing in time. Alternative specifications—(i) specifying business days rather than calendar days, (ii) varying the number of days in a time interval from 3 to 5, (iii) considering only the negotiations that did not end with arbitration, (iv) excluding years 2017 and 2018 with abnormally high rates of arbitration—show the dip in hazard rate around 10 days to two weeks.

Another observation that is consistent with the model: it is more likely to see a settled amount favoring the player in the first two weeks than in the latter two weeks. Figure 10 presents the result. For all negotiations that did not proceed to arbitration, we calculate the average fraction of the salary difference (i.e., pie) captured by the player by 4-day time interval. The average pie captured by the player increases initially and exceeds 50% in the second week, and drops below 50% as the days pass by. This is consistent with our model's equilibrium behavior: Initially, the player challenges with an increasing probability in the beginning of the game and the team quits in response to player's challenge until the team builds up its reputation enough that it is not worthy for the player to challenge the team.

Ideally, more detailed data need to be collected: (i) actual salary figures for extensions and (ii) the scheduled hearing dates even for the cases that did not go to hearing. We can also try to investigate when and why negotiation breaks in the cases decided by arbitration by comparing cases that avoided filing arbitration.

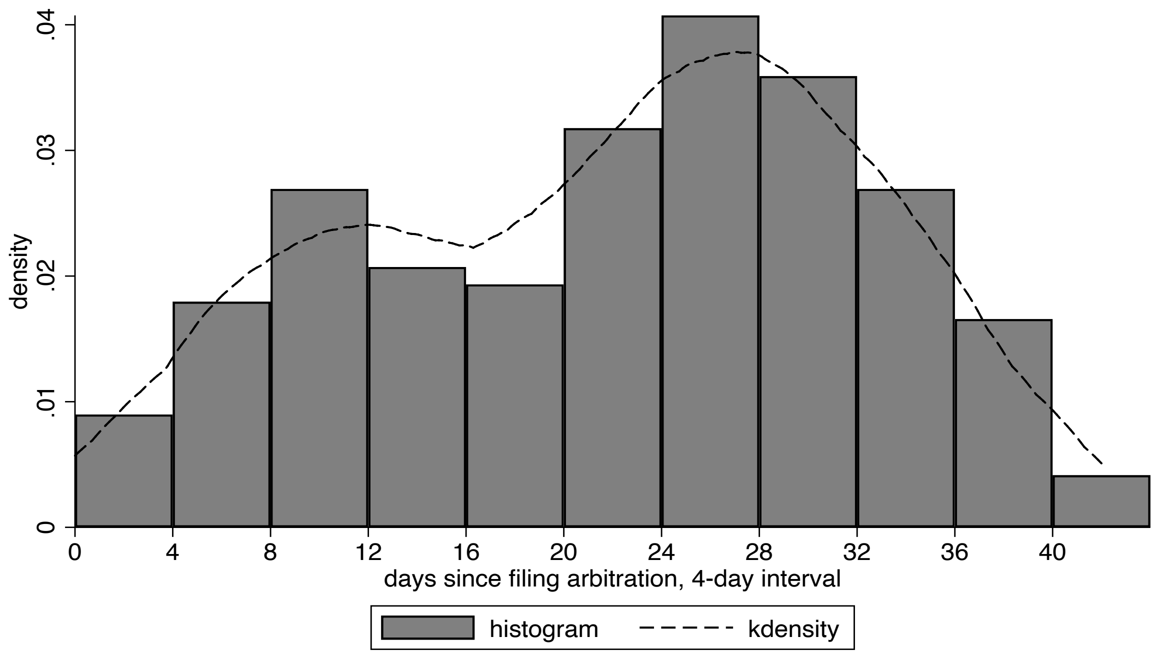


Figure 8: Distribution of days to reach an agreement.

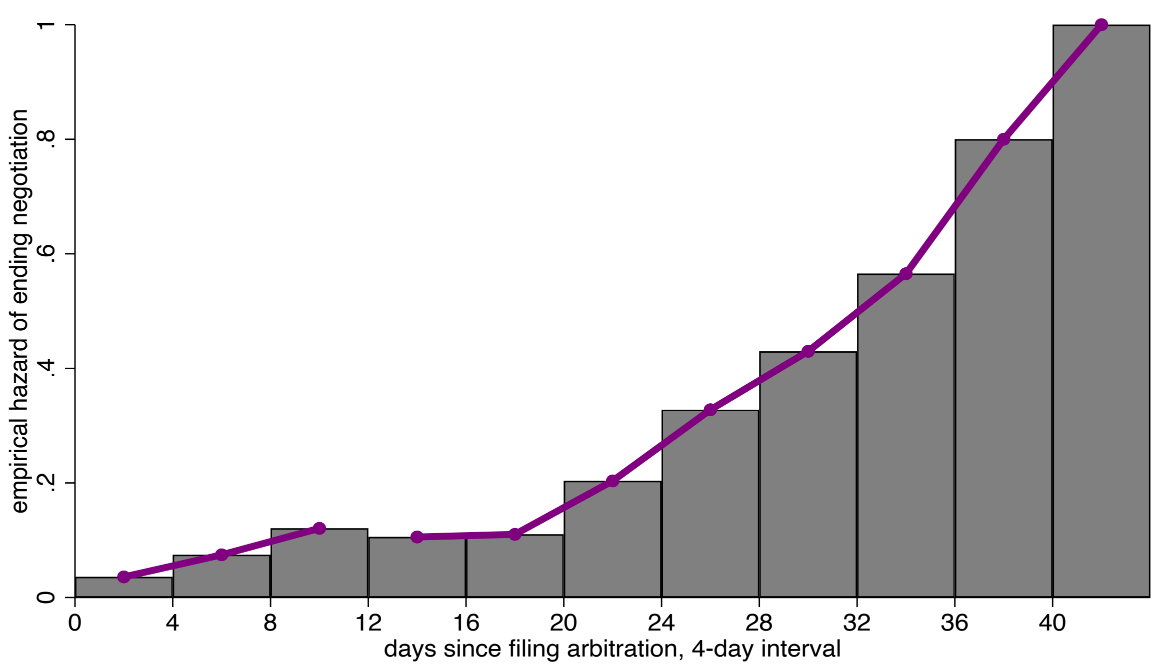


Figure 9: Hazard rate of days to reach an agreement.

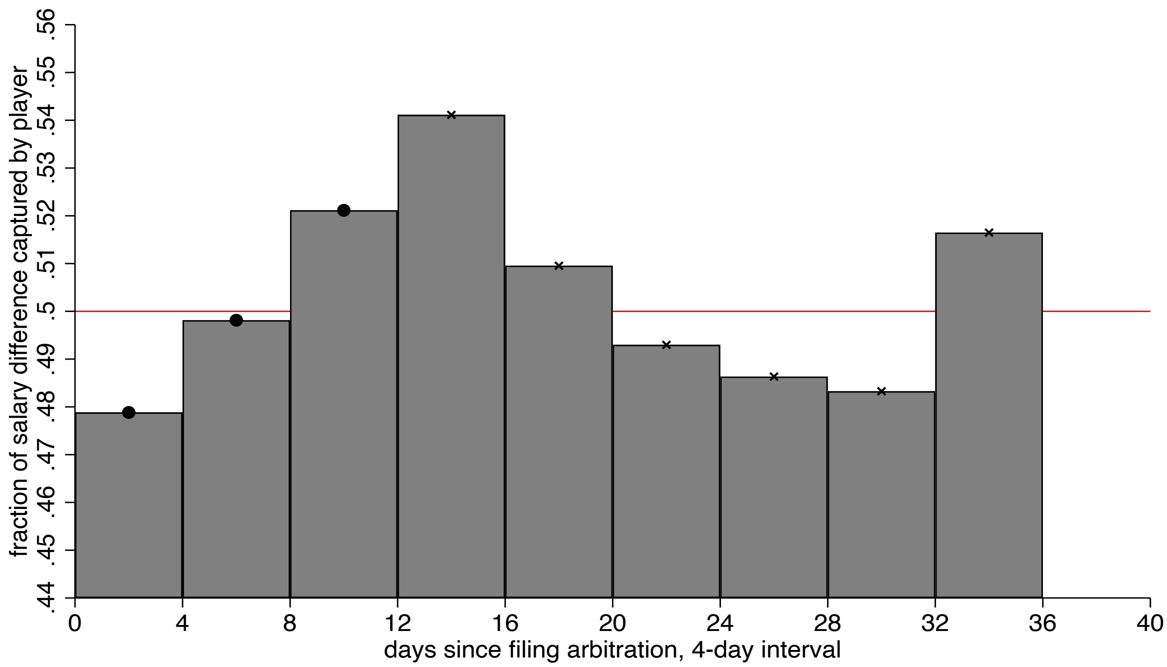


Figure 10: Average fraction of salary difference captured by player by days to reach an agreement.

7 Conclusion

We investigate the equilibrium strategies and reputation dynamics when players have chances to send an ultimatum to end the bargaining process. We find that having the extra ultimatum opportunity does not necessarily increase strategic players' payoffs, because the opportunity erodes a player's commitment power. For sufficiently frequent arrival of ultimatums, the challenge arrival rate replaces the discount rate in the determination of the limit payoff of the players. When both players have frequent opportunities to challenge, reputations do not build up in equilibrium.

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A Robustness Checks

Figure A1 shows the empirical hazard of ending with 3 days and 5 days pooled.

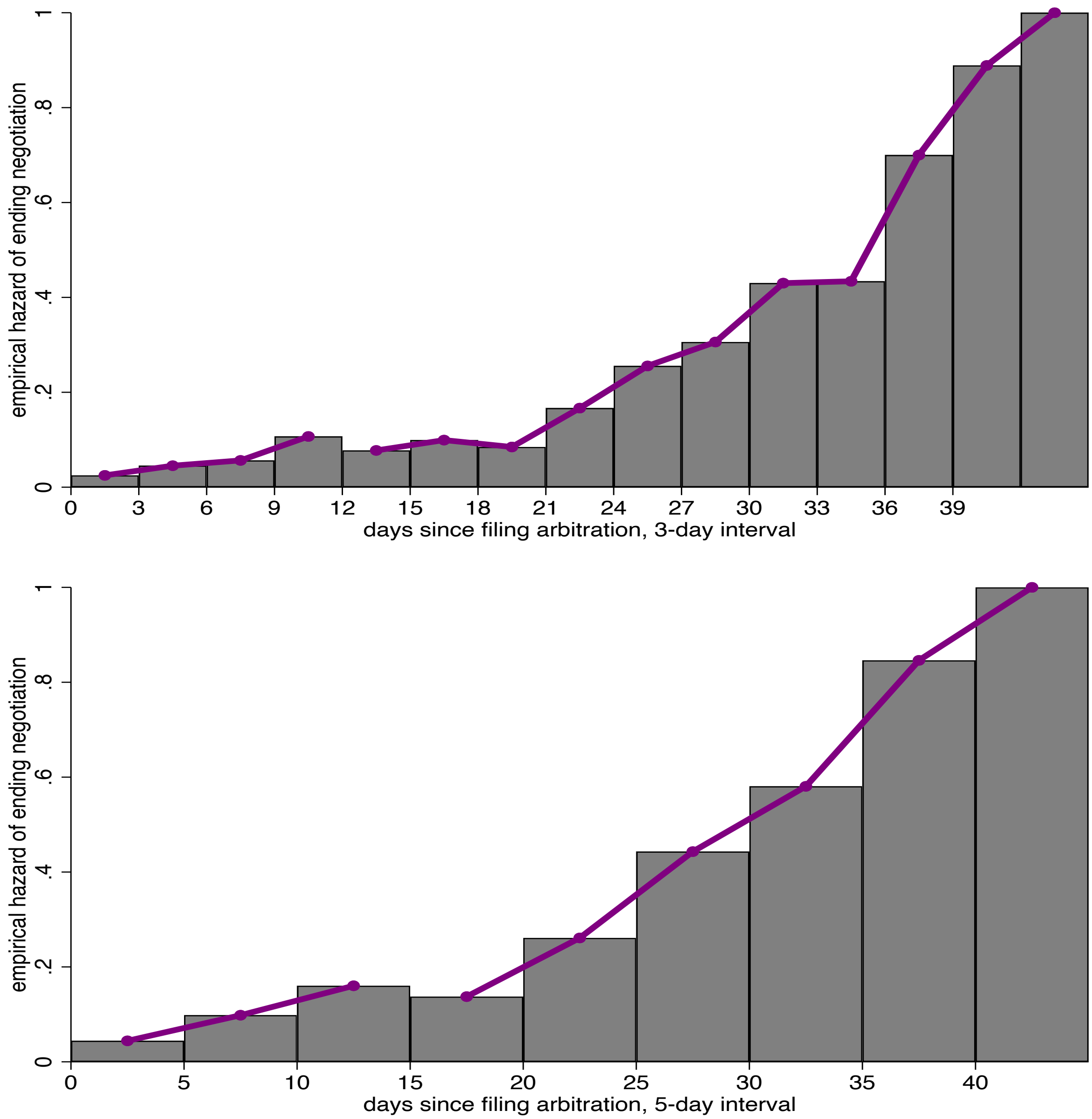

Figure A1: Empirical hazard of ending, 3 days pooled and 5 days pooled.

A Proofs for Section 3 (Equilibrium)

Proof. (**Proof of Proposition 1**)

Let $\Sigma = (\Sigma_1, \Sigma_2)$ define a sequential equilibrium. We will argue that Σ must have the form specified (i.e., uniqueness) and that these strategies do indeed define an equilibrium (i.e., existence). Let $u_i(t)$ denote the expected utility of an unjustified player i who concedes at time t. Define $T_i := \{t | u_i(t) = \max_s u_i(s)\}$ as the set of conceding times that attain the highest expected utility for player i; since Σ is a sequential equilibrium, T_i is nonempty for $i = 1, 2$. Furthermore, define $\tau_i = \inf\{t \geq 0 | F_i(t) = \lim_{t' \to \infty} F_i(t')\}$ as the time of last concession for player i, where $\inf \emptyset := \infty$. Then we have the following results.

(a) The last instant at which two unjustified players concede is the same: $\tau_1 = \tau_2$**.**
An unjustified player will not delay conceding once she knows that the opponent will never concede. Denote the last concession time by τ.

(b) If F_i **jumps at** $t \in \mathbb{R}$**, then** F_j **does not jump at** t **for** $j \neq i$**.**
If F_i had a jump at t, then player j receives a strictly higher utility by conceding an instant after t than by conceding exactly at t.

(c) If F_2 **is continuous at time** t**, then** $u_1(s)$ **is continuous at** $s = t$**. If** F_1 **and** γ_1 **are continuous at time** t**, then** $u_2(s)$ **is continuous at** $s = t$**.**
These claims follow immediately from the definition of $u_1(s)$ in equation (1) and the definition of $u_2(s)$ in equation (2), respectively.

(d) If γ_1 **is continuous, there is no interval** (t', t'') **such that** $0 \leq t' < t'' \leq \tau$ **where both** F_1 **and** F_2 **are constant on the interval** (t', t'')**.**
Assume the contrary and without loss of generality, let $t^* \leq \tau$ be the supremum of t'' for which (t', t'') satisfies the above properties. Fix $t \in (t', t^*)$ and note that for ϵ small enough there exists δ such that $u_i(t) - \delta > u_i(s)$ for all $s \in (t^* - \epsilon, t^*)$; in words, conditional on the opponent not conceding in an interval, it is strictly better for a player to concede earlier within that interval, and it is sufficiently significantly better by conceding early than by conceding close to the end of the time interval. By (b) and (c), there exists i such that $u_i(s)$ is continuous at $s = t^*$, so for some $\eta > 0$, $u_i(s) < u_i(t)$ for all $s \in (t^*, t^* + \eta)$; in words, because of the continuity of the expected utility function at time t^*, the expected utility of conceding a bit after time t^* is still lower than the expected utility of conceding at time t within the time interval. Since F_i is optimal, F_i must be constant on the interval $(t', t^* + \eta)$. The optimality of F_i implies F_j is also constant on the interval $(t', t^* + \eta)$, because player j is strictly better off conceding before or after the interval than conceding during the interval. Hence, both functions are constant on the interval $(t', t^* + \eta) \subseteq (t', \tau)$. However, this contradicts the definition of t^*.

(e) If $t' < t'' < \tau$, then $F_i(t'') > F_i(t')$ for $i = 1, 2$.

If F_i is constant on some interval, then the optimality of F_j implies that F_j is constant on the same interval, for $j \neq i$. However, (d) shows that F_1 and F_2 cannot be simultaneously constant.

(f) Cumulative concession probability F_i, $i = 1, 2$, is continuous at $t > 0$.

Assume the contrary: suppose F_i has a jump at time t. Then F_j is constant on interval $(t - \epsilon, t)$ for $j \neq i$. This contradicts (e).

(g) Cumulative ultimatum probability γ_1 is continuous at $t > 0$.

Suppose to the contrary that γ_1 jumps at time t, that is, an unjustified player 1 challenges with a positive probability. Given that an unjustified player 1 challenges with a positive probability and a justified player 1 challenges with probability 0, player 2 believes that a challenging player is unjustified with probability 1. Consequently, she is strictly better off responding to the challenge (obtaining a payoff of $1 - a_1 + (1 - w)D - c_2$, which is greater than $1 - a_1$ by the assumption that $(1 - w)D > c_2$) than yielding to the challenge (obtaining a payoff of $1 - a_1$). An unjustified player 1's payoff from challenging is less than $1 - a_1 + wD - c_1$, which is strictly less than his payoff from conceding, because $wD < c_1$.

(h) Player 1's continuation payoff at time $t > 0$ in any equilibrium is $1 - a_2$.

Suppose to the contrary that player 1's continuation payoff is strictly higher than $1 - a_2$ at time t. There exists $\epsilon > 0$ such that an unjustified player 1 must not have conceded at time $s \in (t - \epsilon, t)$, that is, $F_1(s)$ is constant on the interval $(t - \epsilon, t)$. The optimality of conceding implies that player 2 must also have not conceded on the interval $(t - \epsilon, t)$, that is, $F_2(s)$ is constant on the interval $(0, t)$. However, the fact that both F_1 and F_2 are constant on an open interval contradicts (d).

(i) Player 2's continuation payoff at time $t > 0$ is $1 - a_1$.

This result follows immediately from the fact that $F_2(t)$ is continuously strictly increasing.

From (e) it follows that T_i is dense in $[0, \tau]$ for $i = 1, 2$. From (c), (f), and (g), it follows that $u_i(s)$ is continuous on $(0, \tau]$ and hence $u_i(s)$ is constant for all $s \in (0, \tau]$. Consequently, $T_i = (0, \tau]$. Hence, $u_i(t)$ is differentiable as a function of t and $du_i(t)/dt = 0$ for all $t \in (0, \tau)$. The expected utility is

$$u_i(t) = (1 - z_j) \int_0^t a_i e^{-r_i s} dF_j(s) + (1 - a_j) e^{-r_i t} (1 - (1 - z_j) F_j(t)). \tag{7}$$

The differentiability of F_j follows from the differentiability of $u_i(t)$ on $(0, \tau)$. Differentiating equation (7) and applying Leibnitz's rule, we obtain

$$0 = a_i e^{-r_i t} (1 - z_j) f_j(t) - (1 - a_j) r_i e^{-r_i t} (1 - (1 - z_j) F_j(t)) - (1 - a_j) e^{-r_i t} (1 - z_j) f_j(t)$$

where $f_j(t) = dF_j(t)/dt$. This in turn implies $F_j(t) = \frac{1 - C_j e^{-\lambda_j t}}{1 - z_j}$, where C_j is yet to be determined. At $\tau_1 = \tau_2$, optimality for player i implies $F_i(\tau_i) = 1$. At $t = 0$, if $F_j(0) > 0$ then $F_i(0) = 0$ by (b).

From (c) and (f), it follows that $v_1(t)$ is continuous on $(0,\tau]$. Furthermore, $v_1(t)$ is strictly smaller than $1-a_1$ when $q_2(t)=1$ and $F_2(t) > 1-\frac{1}{1-z_2}\frac{c_2}{D}$. Therefore, after time t^*, a strategic player 1 does not challenge.

□

B Proofs for Section 4 (Limiting Payoffs)

Proof. (**Proof of Proposition 3**)

For any $a_1 \in A_1$ and $x \in (0,1]$, denote by $B_1(a_1,x)$ the bargaining game in which player 1 is persistent to demand a_1 with probability x and is strategic otherwise. We will show that there is a unique sequential equilibrium of the game $B_1(a_1,x)$.

If unjustified player 2 chooses some $a_2 \leq 1-a_1$, then the game ends at time zero. If player 2 chooses some $a_2 > 1-a_1$, then the game does not end at time zero, but from proposition 1 we know the unique sequential equilibrium for any $t>0$.

Denote by $\sigma_2(\cdot)$, a probability distribution over $A_2 \cup \{Q\}$, a mimicking strategy of unjustified player 2. Since mimicking $a_2 < 1-a_1$ is never optimal and mimicking $a_2 = 1-a-1$ is equivalent to conceding, we assume that in equilibrium $\sigma_2(a_2)=0$ for all $a_2 \leq 1-a_1$. If $x=1$, then in equilibrium $\sigma_2(Q)=1$, because unjustified player 2 will not delay conceding if she knows that player 1 is justified. For the remainder of the proof we assume $x<1$.

It remains to be shown that unjustified player 2's equilibrium behavior $\widehat{\sigma}_2(\cdot)$ and unjustified player 1's conceding behavior $\widehat{F}_1(\cdot|a_1,\cdot)$ at time zero are uniquely determined. Subsequently, we provide a series of definitions and use them to prove a series of claims that lead to equilibrium existence and uniqueness.

Denote by

$$y^*(a_2,\sigma_2) = \frac{z_2\pi_2(a_2)}{z_2\pi_2(a_2)+(1-z_2)\sigma_2}$$

player 2's initial reputation given that player 2 mimics a_2 with probability σ_2. Note that $y^*(a_2,\sigma_2)$ is continuous and strictly decreasing in σ_2.

Denote by $T_1(a_1,a_2,x)$ and $T_2(a_1,a_2,y)$ the time it takes for the reputation of player i to reach 1, given player i's initial reputation (x or y) and given that neither player concedes with a positive probability after initial demands a_1 and a_2 are announced. Explicitly,

$$T_1(a_1,a_2,x) \equiv \begin{cases} \infty & x \leq 1-\frac{\lambda_1}{\gamma_1} \\ t(\mu_1^N; x, \lambda_1-\gamma_1, \frac{\gamma_1}{v_1^*}) + t(1;\mu_1^N,\lambda_1-\gamma_1,\gamma_1) & 1-\frac{\lambda_1}{\gamma_1} < x < \mu_1^N \\ t(1;x,\lambda_1-\gamma_1,\gamma_1) & \mu_1^N \leq x \leq 1 \end{cases},$$

that is,

$$T_1(a_1,a_2,x) \equiv \begin{cases} \infty & \text{if } x \le 1-\frac{\lambda_1}{\gamma_1} \\ \frac{1}{\lambda_1-\gamma_1}\log\left[\frac{\frac{\lambda_1-\gamma_1}{x}+\frac{\gamma_1}{\nu_1^*}}{\frac{\lambda_1-\gamma_1}{\mu_1^N}+\frac{\gamma_1}{\nu_1^*}}\right]-\frac{1}{\lambda_2}\log\mu_2^* & \text{if } 1-\frac{\lambda_1}{\gamma_1}<x<\mu_1^N \\ \frac{1}{\lambda_1-\gamma_1}\log\left[\frac{\frac{\lambda_1-\gamma_1}{x}+\gamma_1}{\lambda_1}\right] & \text{if } \mu_1^N \le x \le 1 \end{cases},$$

and

$$T_2(a_1,a_2,y) \equiv -\frac{a_1+a_2-1}{r_1(1-a_2)}\log y.$$

Note that $T_1(a_1,a_2,x)$ is continuous and strictly decreasing in x on $(1-\frac{\lambda_1}{\gamma_1},1)$ and that $T_2(a_1,a_2,y)$ is continuous and strictly decreasing in y on $(0,1)$.

Denote by $\overline{\sigma}_2(a_1,a_2,x)$ the maximum probability with which player 2 can mimic a_2 in order for player 2's reputation to reach 1 weakly before player 1's reputation does, given that player 1's initial reputation at time zero is x and given that neither player concedes with a positive probability after initial demands a_1 and a_2 are announced. Formally, $\overline{\sigma}_2(a_1,a_2,x)$ is 1, if $x \le 1-\frac{\lambda_1}{\gamma_1}$ and $\overline{\sigma}_2(a_1,a_2,x)$ is the unique value of σ_2 that solves $T_1(a_1,a_2,x)=T_2(a_1,a_2,y^*(a_2,\sigma_2))$ if $x>1-\frac{\lambda_1}{\gamma_1}$. Explicitly, let $\Delta \equiv 1-\frac{\gamma_1}{\lambda_1}$,

$$\overline{\sigma}_2(a_1,a_2,x) \equiv \begin{cases} 1 & x \le 1-\frac{\lambda_1}{\gamma_1} \\ z_2\frac{\pi_2(a_2)}{1-\pi_2(a_2)}\left[\left(\frac{\Delta\frac{1}{x}+(1-\Delta)\frac{1}{\nu_1^*}}{\Delta\frac{1}{\mu_1^N}+(1-\Delta)\frac{1}{\nu_1^*}}\right)^{\frac{\lambda_2}{\lambda_1}\frac{1}{\Delta}}-1\right] & 1-\frac{\lambda_1}{\gamma_1}<x<\mu_1^N \\ z_2\frac{\pi_2(a_2)}{1-\pi_2(a_2)}\left[\left(\Delta\frac{1}{x}+(1-\Delta)\right)^{\frac{\lambda_2}{\lambda_1}\frac{1}{\Delta}}-1\right] & \mu_1^N \le x \le 1 \end{cases}$$

Note that in equilibrium $\sigma_2(a_2) \le \overline{\sigma}_2(a_1,a_2,\sigma_2)$ for all $a_2 > 1-a_1$. To see why this claim must hold, suppose player 2 mimics a_2 with a probability strictly higher than $\overline{\sigma}_2(a_1,a_2,\sigma_2)<1$. Then player 2 needs to concede with a strictly positive probability at time zero in order for players' reputations to reach 1 at the same time. However, we have specified that player 2 does not concede at time zero after announcing her demand.

Denote by

$$\Delta(a_1,x) \equiv \left\{\sigma_2(\cdot)\in\Delta \,\middle|\, \begin{array}{ll} \sigma_2(a_2)=0 & \forall a_2 \le 1-a_1 \\ \sigma_2(a_2) \le \overline{\sigma}_2(a_1,a_2,x) & \forall a_2 > 1-a_1 \end{array}\right\}$$

the set of candidate equilibrium strategies in the game $B_1(a_1,x)$, where Δ denotes the set of all probability distributions on $A_2 \cup \{Q\}$. Note that $\Delta(a_1,x)$ is nonempty, convex, and compact.

Denote by $x^*(a_1,a_2,\sigma_2)$ player 1's initial reputation in order for both players' reputations to reach 1 at the same time, given that a_2 is chosen with probability σ_2 and given that neither player concedes with a positive probability at time zero after initial demands a_1 and a_2 are announced. Formally, $x^*(a_1, a_2, \sigma_2)$ is the unique value of x that solves $T_1(a_1, a_2, x) = T_2(a_1, a_2, y^*(a_2, \sigma_2))$. Explicitly,

$$x^*(a_1, a_2, \sigma_2) \equiv \begin{cases} \dfrac{\lambda_1 - \gamma_1}{\left(\frac{\mu_2^*}{y}\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} \left[\frac{\lambda_1-\gamma_1}{\mu_1^N} + \frac{\gamma_1}{\nu_1^*}\right] - \frac{\gamma_1}{\nu_1^*}} & \text{if } y^*(a_2, \sigma_2) \le \mu_2^* \\ \dfrac{1 - \frac{\gamma_1}{\lambda_1}}{\left(\frac{1}{y}\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} - \frac{\gamma_1}{\lambda_1}} & \text{if } y^*(a_2, \sigma_2) > \mu_2^* \end{cases}.$$

Denote by $F_1^*(0|a_1, a_2, x, \sigma_2)$ the probability with which player 1 must concede at time zero so that the two players' reputations reach 1 at the same time, given player 1's initial reputation $x \le x^*(a_1, a_2, \sigma_2)$ and given that unjustified player 2 chooses a_2 with probability σ_2. In other words, if unjustified player 2 chooses a_2 with probability σ_2, then the probability that player 1 concedes at time zero is $F_1^*(0|a_1, a_2, x, \sigma_2)$. Formally, $F_1^*(0|a_1, a_2, x, \sigma_2)$ is the unique value of F_1 that solves

$$x^*(a_1, a_2, \sigma_2) = \frac{x}{x + (1-x)(1-F_1)}.$$

Explicitly,

$$F_1^*(0|a_1, a_2, x, \sigma_2) \equiv 1 - \frac{x}{1-x} \bigg/ \frac{x^*(a_1, a_2, \sigma_2)}{1 - x^*(a_1, a_2, \sigma_2)}.$$

Note that $F_1^*(0|a_1, a_2, x, \sigma_2)$ is continuous and strictly decreasing in σ_2.

Denote by $u_2^*(a_1, a_2, x, \sigma_2)$ player 2's utility of mimicking a_2 in the game $B_1(a_1, x)$ given that she mimics a_2 with probability σ_2 in equilibrium. Explicitly,

$$u_2^*(a_1, a_2, x, \sigma_2) \equiv 1 - a_1 + (1-x) F_1^*(0|a_1, a_2, x, \sigma_2)(a_1 + a_2 - 1).$$

To see why the equation holds, note that player 2's payoff is a_2 if player 1 concedes at time zero and is $1 - a_1$ if player 1 does not concede at time zero, since in equilibrium she is indifferent between conceding and not conceding at every time $t \ge 0$. Player 1 concedes with probability $(1-x)F_1^*(0|a_1, a_2, x, \sigma_2)$ in equilibrium. Furthermore, note that $u_2^*(a_1, a_2, x, \sigma_2)$ is continuous and strictly decreasing in σ_2.

For any $\sigma_2(\cdot) \in \Delta(a_1, x)$, define

$$\widehat{u}_2(x, \sigma_2(\cdot)) \equiv \begin{cases} \min\limits_{a_2 : \sigma_2(a_2) > 0} u_2^*(a_1, a_2, x, \sigma_2(a_2)) & \text{if } \sigma_2(Q) = 0 \\ 1 - a_1 & \text{if } \sigma_2(Q) \neq 0 \end{cases}.$$

Note that $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy if and only if $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot) \in \Delta(a_1, x)} \widehat{u}_2(x, \sigma_2(\cdot))$.

($\Rightarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy. Any equilibrium strategy $\sigma_2(\cdot)$ satisfies that for all $a_2 \in A_2 \cup \{Q\}$ such that $\sigma_2(a_2) > 0$, $u_2^*(a_1, a_2, x, \sigma_2(a_2))$ is the same. If $\sigma_2(Q) > 0$, then $u_2^*(a_1, a_2, x, \sigma_2(a_2)) = 1 - a_1$; if $\sigma_2(Q) = 0$, then $u_2^*(a_1, a_2, x, \sigma_2(a_2)) = \min_{a_2:\widehat{\sigma}_2(a_2)>0} u_2(a_1, a_2, x, \widehat{\sigma}_2(a_2))$. Hence, any equilibrium strategy $\sigma_2(\cdot)$ must generate an equilibrium utility of $\widehat{u}_2(x, \sigma_2(\cdot))$. Hence, $\widehat{\sigma}_2(\cdot)$ maximizes $\widehat{u}_2(x, \sigma_2(\cdot))$ among all candidate equilibrium strategies $\sigma_2(\cdot)$. ($\Leftarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x, \sigma_2(\cdot))$. By the strict monotonicity of $u_2^*(a_1, a_2, x, \cdot)$, for all $a_2 \in A_2$ such that $\widehat{\sigma}_2(a_2) > 0$, $u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)) = \widehat{u}_2(x, \widehat{\sigma}_2(\cdot))$. Coupled with the fact that $\widehat{\sigma}_2(\cdot)$ is the feasible strategy that maximizes $\widehat{u}_2(x, \sigma_2(\cdot))$, $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy.

Define Γ, a correspondence from $\Delta(a_1, x)$ to $\Delta(a_1, x)$, as follows:

$$\Gamma(\sigma_2(\cdot)) \equiv \{\widetilde{\sigma}_2(\cdot) \in \Delta(a_1, x) | \widetilde{\sigma}_2(a_2) > 0 \Rightarrow u_2^*(a_1, a_2, x, \sigma_2(a_2)) \geq u_2^*(a_1, a_2', x, \sigma_2(a_2'))\ \forall a_2' \in A_2\}.$$

Note that $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x, \sigma_2(\cdot))$ if and only if $\widehat{\sigma}_2(\cdot)$ is a fixed point of Γ. ($\Rightarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x, \sigma_2(\cdot))$. By the argument above, $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy. Therefore, $\widehat{\sigma}_2(a_2) > 0$ implies $u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)) \geq u_2^*(a_1, a_2', x, \widehat{\sigma}_2(a_2'))$ for any $a_2' \in A_2$. By the definition of Γ, $\widehat{\sigma}_2(\cdot) \in \Gamma(\widehat{\sigma}_2(\cdot))$. ($\Leftarrow$) Suppose $\widehat{\sigma}_2(\cdot) \in \Gamma(\widehat{\sigma}_2(\cdot))$. By the definition of Γ, $\widehat{\sigma}_2(a_2) > 0$ implies $u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)) \geq u_2^*(a_1, a_2', x, \widehat{\sigma}_2(a_2'))$ for any $a_2' \in A_2$. Assume by contradiction that $\widehat{\sigma}_2(\cdot)$ does not solve $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x, \sigma_2(\cdot))$ but $\widetilde{\sigma}_2(\cdot) \neq \widehat{\sigma}_2(\cdot)$ does. There must exist an $a_2 \in A_2$ such that $\widehat{\sigma}_2(a_2) > 0$ and $\widetilde{\sigma}_2(a_2) < \widehat{\sigma}_2(a_2)$ (otherwise, if $\widetilde{\sigma}_2(a_2) \geq \widehat{\sigma}_2(a_2)$ for all a_2 such that $\widehat{\sigma}_2(a_2) > 0$, then by the strict monotonicity of u_2^*, $u_2^*(a_1, a_2, x, \widetilde{\sigma}_2(a_2)) \leq u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2))$, and $\widehat{u}_2(x, \widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x, \widehat{\sigma}_2(\cdot)))$. However, that implies that there exists $a_2' \in A_2 \cup \{Q\}$ such that $\widetilde{\sigma}(a_2') > \widetilde{\sigma}(a_2')$. If $a_2' = Q$, then $\widehat{u}_2(x, \widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x, \widehat{\sigma}_2(\cdot))$. If $a_2' \in A_2$, then $\widehat{u}_2(x, \widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x, \widehat{\sigma}_2(\cdot))$.

Hence, from the two claims above, we have that $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy for player 2 in the game $B_0(a_1, x)$ if and only if $\widehat{\sigma}_2(\cdot)$ is a fixed point of Γ.

Equilibrium existence follows from the existence of a fixed point of Γ by Kakutani's fixed point theorem. By construction, $\Delta(a_1, x)$ is compact. By construction, Γ is convex-valued. Finally, Γ is upper-hemicontinuous because u_2^* is continuous in its last argument.

Equilibrium uniqueness follows from the strict monotonicity of u_2^*. Suppose there are two equilibrium strategies $\widehat{\sigma}_2(\cdot)$ and $\widetilde{\sigma}_2(\cdot)$; without loss of generality, suppose $\widehat{\sigma}_2(a_2) > \widetilde{\sigma}_2(a_2) > 0$ for some $a_2 > 1 - a_1$. The utilities of playing the two strategies are different: $\widehat{u}_2(x, \widehat{\sigma}_2(\cdot)) = u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)) < u_2^*(a_1, a_2, x, \widetilde{\sigma}_2(a_2)) = \widehat{u}_2(x, \widetilde{\sigma}_2(\cdot))$, where the strict inequality follows from the strict monotonicity of u_2^*. This contradicts the property that equilibrium strategies $\widehat{\sigma}_2(\cdot)$ and $\widetilde{\sigma}_2(\cdot)$ both maximize $\widehat{u}_2(x, \sigma_2(\cdot))$.

□

Proof. (**Proof of Proposition 6**)

Suppose that the commitment type set for each player sufficiently "finely" covers the interval

$[0,1]$. We are interested in the limit equilibrium payoffs as $z_1, z_2 \to 0$.

First, suppose $\gamma_1 > r_1$. We first start with the case in which player 1 chooses a type $a_1 > \frac{r_2}{r_2+\gamma_1}$ with a probability that doesn't disappear as $z_1 \to 0$. If player 2 chooses $a_2 \approx 1$, then we have

$$\lambda_1 = \frac{(1-a_1)r_2}{a_1+a_2-1} \approx \frac{(1-a_1)r_2}{a_1} < \gamma_1.$$

Hence, Player 2's payoff from this strategy approaches to approximately 1, and hence player 1's payoff approaches to approximately to 0.

If Player 1 chooses a type $a_1 < \frac{r_2}{r_2+\gamma_1}$, then regardless of a_2, $\lambda_1 > \gamma_1$, hence we need to solve for the equilibrium dynamics of the model in which $\lambda_1 > \gamma_1$.

Player 1 wins if $\lambda_1 - \gamma_1 > \lambda_2$, and player 2 wins otherwise. (Why?)

Plugging in λ_i the expression from Abreu and Gul, we have that Player 2 wins if

$$(1-a_1)r_2 - \gamma_1 a_1 < (1-a_2)(r_1-\gamma_1)$$

Because $\gamma_1 > r_1$, the right hand side is negative, and left hand side is always positive, so Player 1 wins. (It's important for this argument that z_1 goes to zero at a rate not smaller than that of z_2.)

Since this is true for every $a_1 < \frac{r_2}{r_2+\gamma_1}$, player 1, by choosing a demand approximately equal to $\frac{r_2}{r_2+\gamma_1}$ (more precisely, $\max\{a_1 \in A_1 | a_1 < \frac{r_2}{r_2+\gamma_1}\}$) guarantees this payoff, and cannot do better, and Player 2 gets the rest of the surplus.

Second, suppose $\gamma_1 < r_1$. In this case, if player 1 chooses $a_1 = \frac{r_2}{r_2+r_1}$, then $\lambda_1 > \gamma_1$ for any choice of a_2, so the winner is determined by comparison

$$(1-a_1)r_2 - \gamma_1 a_1 < (1-a_2)(r_1-\gamma_1)$$

which for the choice of $a_1 = \frac{r_2}{r_2+r_1}$ makes player 1 the winner, and for any choice of a_1 lower, makes player 2 the winner by a choice that makes player 2 have a payoff larger than $\frac{r_1}{r_1+r_2}$, that leaves player 1 with a payoff smaller than $\frac{r_2}{r_1+r_2}$. Hence the solution is similar to Abreu and Gul in this case. □

C Proofs for Section 5 (Two-Sided Ultimatums)

Proof. (**Proof of Proposition 7**)

Let $\Sigma = (\Sigma_1, \Sigma_2)$ define a sequential equilibrium. We will argue that Σ must have the form specified (i.e., uniqueness) and that these strategies do indeed define an equilibrium (i.e., existence). Let $u_i(t)$ denote the expected utility of an unjustified player i who concedes at time t. Define $T_i := \{t | u_i(t) = \max_s u_i(s)\}$ as the set of conceding times that attain the highest expected utility for player i; since Σ is a sequential equilibrium, T_i is nonempty for $i = 1, 2$. Furthermore, define $\tau_i = \inf\{t \geq 0 | F_i(t) = \lim_{t' \to \infty} F_i(t')\}$ as the time of last concession for player i, where $\inf \emptyset := \infty$.

Then we have the following results.

(a) The last instant at which two players concede is the same: $\tau_1 = \tau_2$. A player will not delay conceding once he/she knows that the opponent will never concede. Denote the instant of last concession by τ.

(b) If F_i jumps at $t \in \mathbb{R}$, then F_j does not jump at time t for $j \neq i$. If F_i had a jump at t, then player j receives a strictly higher utility by conceding an instant after time t than by conceding exactly at time t.

(c) If F_j is continuous at time t, then $u_i(s)$ is continuous at time $s = t$. If F_i and γ_i are continuous at time t, then $u_j(s)$ is continuous at time $s = t$. These two claims follow immediately from the definition of $u_i(s)$ in equation (6).

(d) If γ_1 and G_2 are continuous, there is no interval (t', t'') such that $0 \leq t' < t'' \leq \tau$ where both F_1 and F_2 are constant on the interval (t', t''). Assume the contrary and without loss of generality, let $t^* \leq \tau$ be the supremum of t'' for which (t', t'') satisfies the above property (i.e., both F_1 and F_2 are constant on the interval (t', t'')). Fix $t \in (t', t^*)$. For ϵ small enough there exists δ such that $u_i(t) - \delta > u_i(s)$ for all $s \in (t^* - \epsilon, t^*)$; in words, conditional on the opponent not conceding in an interval, it is strictly better to concede early than to concede close to the end of the time interval. By (b) and (c), there exists i such that $u_i(s)$ is continuous at $s = t^*$, so for some $\eta > 0$, $u_i(s) < u_i(t)$ for all $s \in (t^*, t^* + \eta)$; in words, because the expected utility function is continuous at t^*, the expected utility of conceding sufficiently immediately after time t^* is strictly lower than the expected utility of conceding at time t within the time interval. Since F_i is optimal, player i concedes with probability 0 at time t^*, so F_i must be constant on the interval $(t', t^* + \eta)$. The optimality of F_i implies F_j is also constant on the interval $(t', t^* + \eta)$, because player j is strictly better to concede before or after the interval than during the interval. Hence, both functions are constant on the interval $(t', t^* + \eta)$. However, this contradicts the definition of t^*.

(e) If $t' < t'' < \tau$, then $F_i(t'') > F_i(t')$ for $i = 1, 2$. If F_i is constant on an interval, then the optimality of F_j implies that F_j is constant on the same interval, for $j \neq i$. However, (d) shows that F_1 and F_2 cannot be simultaneously constant.

(f) Cumulative concession probability F_i, $i = 1, 2$, is continuous at time $t > 0$. Assume the contrary: suppose F_i has a jump at time t. Then F_j is constant on interval $(t - \epsilon, t)$ for $j \neq i$. This contradicts (e).

(g) Cumulative ultimatum probability γ_i, $i = 1, 2$, is continuous at time $t > 0$. Suppose to the contrary that γ_i jumps at time t, that is, player i challenges with a positive probability at an instant. player $j \neq i$ believes that a challenging player i is unjustified with probability 1, because an unjustified player i challenges with a positive probability and a justified player i challenges with probability 0. Player j is strictly better to respond to the challenge (obtaining a payoff of $1 - a_i + (1-w)D - c_j$, greater than $1 - a_i$ by the assumption that $(1-w)D > c_j$) than to yield to the challenge (obtaining a payoff of $1 - a_i$). Player i's payoff from challenging is less than $1 - a_i + wD - c_i$, which is strictly less than $1 - a_i$, the payoff from conceding, because $wD < c_i$.

(h) Player i's continuation payoff at time $t > 0$ in any equilibrium is $1 - a_j$. Suppose to the contrary that there is a player i such that player i's continuation payoff is strictly higher than $1 - a_j$ at time t. There exists an $\epsilon > 0$ such that player i must not have conceded at $s \in (t-\epsilon, t)$, that is, F_i is constant on the interval $(t-\epsilon, t)$. Player $j \neq i$ must also have not conceded on the interval $(t-\epsilon, t)$, that is, F_j is constant on the interval $(0, t)$. However, the fact that both F_1 and F_2 are constant on an open interval contradicts (d).

From (e) it follows that T_i is dense in $[0, \tau]$ for $i = 1, 2$. From (c), (f), and (g), it follows that $u_i(s)$ is continuous on $(0, \tau]$ and hence $u_i(s)$ is constant for all $s \in (0, \tau]$. Consequently, $T_i = (0, \tau]$. Hence, $u_i(t)$ is differentiable as a function of t and $du_i(t)/dt = 0$ for all $t \in (0, \tau)$. The expected utility is

$$u_i(t) = (1 - z_j) \int_0^t a_i e^{-r_i s} dF_j(s) + (1 - a_j) e^{-r_i t} (1 - (1 - z_j) F_j(t)). \tag{8}$$

The differentiability of F_j follows from the differentiability of $u_i(t)$ on $(0, \tau)$. Differentiating equation (8) and applying Leibnitz's rule, we obtain

$$0 = a_i e^{-r_i t}(1 - z_j) f_j(t) - (1 - a_j) r_i e^{-r_i t}(1 - (1 - z_j) F_j(t)) - (1 - a_j) e^{-r_i t}(1 - z_j) f_j(t),$$

where $f_j(t) = dF_j(t)/dt$. This in turn implies $F_j(t) = \frac{1 - C_j e^{-\lambda_j t}}{1 - z_j}$, where C_j is yet to be determined. Optimality for player i implies $F_i(\tau_i) = 1$. At $t = 0$, if $F_j(0) > 0$ then $F_i(0) = 0$ by (b). □