Online Appendix for "An Investment-and-Marriage Model with Differential Fecundity"
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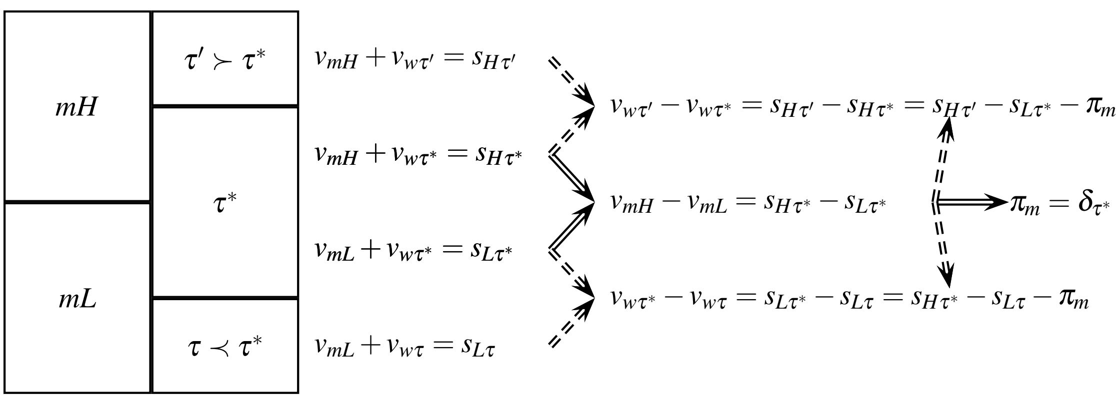
Appendix A. Omitted Proofs

A.1 Determination of Stable Marriage Payoff Differences

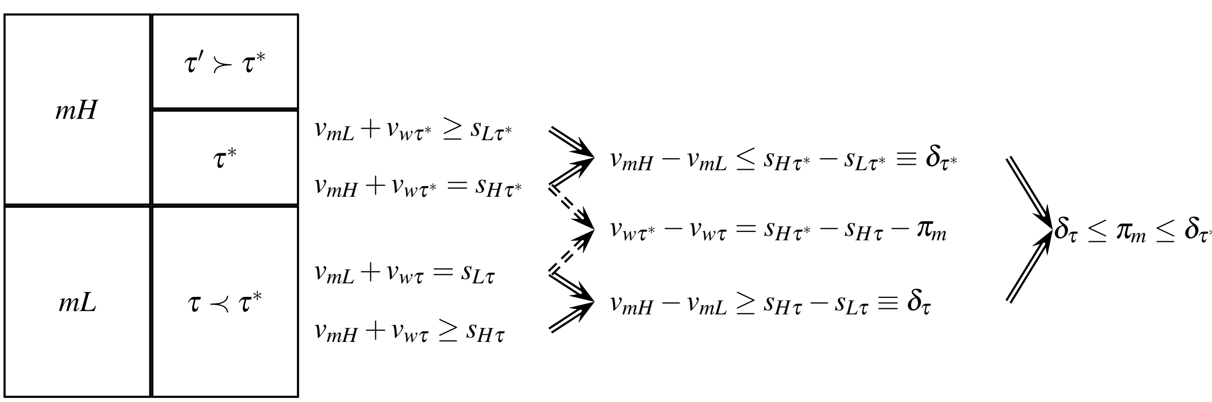
Figure A1 illustrates the two cases of the determination of stable marriage payoff difference between any two adjacently ranked types. In case (a), type-τ^* women marry both high-income men and low-income men with positive probabilities, so $v_{mH} + v_{w\tau^*} = s_{H\tau^*}$ and $v_{mL} + v_{w\tau^*} = s_{L\tau^*}$, which together imply $\pi_m = v_{mH} - v_{mL} = \delta_{\tau^*}$; the marriage payoff difference between any two adjacently ranked female types can be determined similarly, with the dashed arrows denoting the line of reasoning. In case (b), type $\tau \succeq \tau^*$ women almost surely marry high-income men, and type $\tau \prec \tau^*$ women almost surely marry low-income men. Since there is no type of women that marries both types of men with positive probabilities, π_m is indeterminate, and it can take any value between δ_τ and δ_{τ^*}, where τ is the female type ranked just below τ^*. This indeterminacy in π_m will dissipate in equilibrium, however, when marriage payoffs and investments are jointly determined. For women, almost all $\tau' \succeq \tau^*$ women marry high-income men and almost all $\tau \prec \tau^*$ women marry low-income men, so $v_{w\tau'} - v_{w\tau} = s_{H\tau'} - s_{L\tau} - \pi_m$.

Figure A1: Determination of Stable Marriage Payoff Difference

(a) The mass of high-income men is strictly between the mass of women strictly higher ranked than τ^* and the mass of women weakly higher ranked than type τ^* for some $\tau^* \in T$.



(b) The mass of high-income men equals the mass of women weakly higher ranked than τ^* for some type $\tau^* \in T$.



A.2 Proof of Theorem 1

Let $\theta_m(\pi_m)$, $\theta_{w1}(\pi_m)$, and $\theta_{w2}(\pi_m)$ denote the ability cutoffs characterizing optimal human capital investments when men's stable marriage premium is π_m (and women's stable marriage-payoff differences are pinned down by π_m). Let $G_m(\pi_m)$ and $G_w(\pi_m)$ denote the induced distributions of men's and women's marriage characteristics, respectively, when the investment strategies are the ones characterized by the ability cutoffs $\theta_m(\pi_m)$, $\theta_{w1}(\pi_m)$, and $\theta_{w2}(\pi_m)$. Let $\Pi_m(G_m, G_w)$ denote the set of men's stable marriage premiums (and associated stable marriage payoffs of women) in the marriage market (G_m, G_w). Construct the correspondence

$$D_{mH}(\pi_m) := \{G_{mH} \in [0,1] : \pi_m \in \Pi_m((G_{mH}, 1-G_{mH}), G_w(\pi_m))\}.$$

For any $\pi_m \in [\delta_l, \delta_H]$, each element in the set $D_{mH}(\pi_m)$ is a mass G_{mH} of high-income men such that π_m is men's stable marriage premium in the marriage market $((G_{mH}, 1-G_{mH}), G_w(\pi_m))$. Explicitly, (i) $D_{mH}(\pi_m) = [G_{w,\succ\tau_w^*}(\pi_m), G_{w,\succeq\tau_w^*}(\pi_m)]$ if $\pi_m = \delta_{\tau_w^*}$ for a certain type $\tau_w^* \in T_w$; and (ii) $D_{mH}(\pi_m) = G_{w,\succeq\tau_w^*}(\pi_m)$ if $\pi_m \in (\delta_{\tau_w'}, \delta_{\tau_w^*})$ for a pair of adjacently ranked types τ_w^* and $\tau_w' \prec \tau_w^*$.

I prove the claim that there exists an equilibrium in which men's stable marriage premium is π_m^* if and only if $G_{mH}(\pi_m^*) \in D_{mH}(\pi_m^*)$. First, the only if part. Suppose men's equilibrium marriage premium is π_m^*. The induced mass of high-income men is $G_{mH}(\pi_m^*)$, and the induced distribution of women's marriage characteristics is $G_w(\pi_m^*)$. Since $\pi_m^* \in \Pi_m((G_{mH}(\pi_m^*), 1-G_{mH}(\pi_m^*)), G_w(\pi_m^*))$, by definition of $D_{mH}(\pi_m^*)$, we have $G_{mH}(\pi_m^*) \in D_{mH}(\pi_m^*)$. Reversely, the if only part. If $G_{mH}(\pi_m^*) \in D_{mH}(\pi_m^*)$, then by definition of $D_{mH}(\pi_m^*)$, $\pi_m^* \in \Pi_m((G_{mH}(\pi_m^*), 1-G_{mH}(\pi_m^*)), G_w(\pi_m^*))$, so π_m^* is men's equilibrium marriage premium.

It follows from the claim above that an equilibrium exists if and only if the graph of function $G_{mH}(\cdot)$ and the graph of correspondence $D_{mH}(\cdot)$ intersect at least once. Equilibrium marriage-payoff differences and equilibrium investments are uniquely determined if and only if the graph of function $G_{mH}(\cdot)$ and the graph of correspondence $D_{mH}(\cdot)$ intersect once and only once. The existence of an equilibrium is guaranteed because $G_{mH}(\cdot)$ has a range $[0,1]$ and is continuous, and $D_{mH}(\cdot)$ has a range $[0,1]$ and is upperhemicontinuous.

It remains to prove equilibrium uniqueness. $G_{mH}(\pi_m) = \int_{\theta_m(\pi_m)}^1 \theta(2-\theta)dF_m(\theta)$ is strictly increasing in π_m because $\theta_m(\pi_m) = c_m/(z_{mH} - z_{mL} + \pi_m)$ is strictly decreasing in π_m. It suffices to show $D_{mH}(\pi_m)$ is weakly decreasing in the following sense: for any π_m and $\pi_m' > \pi_m$, $\max D_{mH}(\pi_m') \leq \min D_{mH}(\pi_m)$. For the remainder of the proof, we mechanically show that $D_{mH}(\pi_m)$ is decreasing. Depending on $\delta_h > \delta_L$, $\delta_h < \delta_L$, or $\delta_h = \delta_L$, $D_{mH}(\pi_m)$ is characterized differently. I discuss the three cases separately.

Case 1. Suppose $\delta_L > \delta_h$. Explicitly,

$$D_{mH}(\pi_m) = \begin{cases} [G_{w,\succeq h}(\pi_m), 1] & \text{if } \pi_m = \delta_l \\ G_{w,\succeq h}(\pi_m) & \text{if } \pi_m \in (\delta_l, \delta_h) \\ [G_{w,\succeq L}(\pi_m), G_{w,\succeq h}(\pi_m)] & \text{if } \pi_m = \delta_h \\ G_{w,\succeq L}(\pi_m) & \text{if } \pi_m \in (\delta_h, \delta_L) \\ [G_{wH}(\pi_m), G_{w,\succeq L}(\pi_m)] & \text{if } \pi_m = \delta_L \\ G_{wH}(\pi_m) & \text{if } \pi_m \in (\delta_L, \delta_H) \\ [0, G_{wH}(\pi_m)] & \text{if } \pi_m = \delta_H \end{cases}.$$

It remains to show that (i) $G_{w,\succeq h}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_h)$, (ii) $G_{w,\succeq L}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_h, \delta_L)$, and (iii) $G_{wH}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_L, \delta_H)$.

(i) To show $G_{w,\succeq h}(\pi_m) = 1 - \int_{\theta_{w2}(\pi_m)}^1 (1-\theta)^2 dF_w(\theta)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_h)$, it suffices to show $\theta_{w2}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_h)$. Men's stable marriage premium can be $\pi_m \in (\delta_l, \delta_h)$ only when $G_{mH} = G_{w,\succeq h}$. When $G_{mH} = G_{w,\succeq h}$, given men's stable marriage premium π_m, women's stable marriage-payoff differences are $v_{wL} - v_{wl} = s_{HL} - s_{Ll} - \pi_m$, $v_{wH} - v_{wL} = s_{HH} - s_{HL}$, and $v_{wh} - v_{wl} = s_{Hh} - s_{Ll} - \pi_m$, so

$$\begin{aligned} \theta_{w2}(\pi_m) &= \frac{c_w + (v_{wL} - v_{wl})}{z_{wH} - z_{wL} + (v_{wh} - v_{wl})} = \frac{c_w + (s_{HL} - s_{Ll} - \pi_m)}{z_{wH} - z_{wL} + (s_{Hh} - s_{Ll} - \pi_m)} \\ &= \frac{c_w + (s_{HL} - s_{Ll}) - \pi_m}{z_{wH} - z_{wL} + (s_{Hh} + s_{Ll}) - \pi_m}. \end{aligned}$$

Since $\theta_{w2}(\pi_m) < 1$, $\theta_{w2}'(\pi_m) < 0$ when $\pi_m \in (\delta_l, \delta_h)$.

(ii) To show $G_{w,\succeq L}(\pi_m) = 1 - \int_{\theta_{w2}(\pi_m)}^{1}(1-\theta)dF_w(\theta)$ is strictly decreasing when $\pi_m \in (\delta_h, \delta_L)$, it suffices to show $\theta_{w2}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_h, \delta_L)$. Men's stable marriage premium can be $\pi_m \in (\delta_h, \delta_L)$ only when $G_{mH} = G_{w,\succeq L}$. When $G_{mH} = G_{w,\succeq L}$, given men's stable marriage premium π_m, women's stable marriage-payoff differences are $v_{wL} - v_{wl} = s_{HL} - s_{Ll} - \pi_m$, $v_{wH} - v_{wL} = s_{HH} - s_{HL}$, and $v_{wh} - v_{wl} = s_{Lh} - s_{Ll}$, so

$$\theta_{w2}(\pi_m) = \frac{c_w + (s_{HL} - s_{Ll} - \pi_m)}{z_{wH} - z_{wL} + (s_{Lh} - s_{Ll})}.$$

Therefore, $\theta_{w2}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_h, \delta_L)$.

(iii) To show $G_{wH}(\pi_m) = \int_{\theta_{w1}(\pi_m)}^{1}\theta dF_w(\theta) + \int_{\theta_{w2}(\pi_m)}^{1}(1-\theta)\theta dF_w(\theta)$ is strictly decreasing when $\pi_m \in (\delta_L, \delta_H)$, it suffices to show $\theta_{w1}(\pi_m)$ and $\theta_{w2}(\pi_m)$ are strictly increasing when $\pi_m \in (\delta_L, \delta_H)$. Men's stable marriage premium is $\pi_m \in (\delta_L, \delta_H)$ only when $G_{mH} = G_{wH}(\pi_m)$. When $G_{mH} = G_{wH}$, given men's stable marriage premium π_m, women's stable marriage-payoff differences are $v_{wL} - v_{wl} = s_{LL} - s_{Ll}$, $v_{wH} - v_{wL} = s_{HL} - s_{Ll} - \pi_m$, and $v_{wh} - v_{wl} = s_{Lh} - s_{Ll}$, so

$$\theta_{w1}(\pi_m) = \frac{c_w}{z_{wH} - z_{wL} + s_{HH} - s_{LL} - \pi_m},$$

and

$$\theta_{w2}(\pi_m) = \frac{c_w + (s_{LL} - s_{Ll})}{z_{wH} - z_{wL} + (s_{Lh} - s_{Ll})}.$$

Therefore, both $\theta_{w1}(\pi_m)$ and $\theta_{w2}(\pi_m)$ are increasing when $\pi_m \in (\delta_L, \delta_H)$.

Case 2. Suppose $\delta_h \geq \delta_L$. Explicitly,

$$D_{mH}(\pi_m) = \begin{cases} [G_{w,\succeq L}(\pi_m), 1] & \text{if } \pi_m = \delta_l \\ G_{w,\succeq L}(\pi_m) & \text{if } \pi_m \in (\delta_l, \delta_L) \\ [G_{w,\succeq h}(\pi_m), G_{w,\succeq L}(\pi_m)] & \text{if } \pi_m = \delta_L \\ G_{w,\succeq h}(\pi_m) & \text{if } \pi_m \in (\delta_L, \delta_h) \\ [G_{wH}(\pi_m), G_{w,\succeq h}(\pi_m)] & \text{if } \pi_m = \delta_h \\ G_{wH}(\pi_m) & \text{if } \pi_m \in (\delta_h, \delta_H) \\ [0, G_{wH}(\pi_m)] & \text{if } \pi_m = \delta_H \end{cases}.$$

It suffices to show that (i) $G_{w,\succeq L}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_L)$, (ii) $G_{w,\succeq h}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_L, \delta_h)$, and (iii) $G_{wH}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_h, \delta_H)$.

(i) To show $G_{w,\succeq L}(\pi_m) = 1 - \int_{\theta_{w2}(\pi_m)}^{1}(1-\theta)^2 dF_w(\theta)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_L)$, it suffices to show $\theta_{w2}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_L)$. Men's stable marriage premium can be $\pi_m \in (\delta_l, \delta_L)$ only when $G_{mH} = G_{w,\succeq L}$. When $G_{mH} = G_{w,\succeq L}$, given men's stable marriage premium π_m, women's stable marriage-payoff differences are $v_{wL} - v_{wl} = s_{HL} - s_{Ll} - \pi_m$, $v_{wH} - v_{wL} = s_{HH} - s_{HL}$, and $v_{wh} - v_{wl} = s_{Hh} - s_{Hl} - \pi_m$, so

$$\theta_{w2}(\pi_m) = \frac{c_w + (s_{HL} - s_{Ll} - \pi_m)}{z_{wH} - z_{wL} + (s_{Hh} - s_{Hl} - \pi_m)}.$$

Since $\theta_{w2}(\pi_m) < 1$, $\theta'_{w2}(\pi_m) < 0$ when $\pi_m \in (\delta_l, \delta_L)$.

(ii) To show $G_{w,\succeq h}(\pi_m)$, it suffices to show both $\theta_{w1}(\pi_m)$ and $\theta_{w2}(\pi_m)$ are strictly increasing when $\pi_m \in (\delta_h, \delta_L)$. Men's stable marriage payoff can be $\pi_m \in (\delta_h, \delta_L)$ only when $G_{mH} = G_{w,\succeq h}$. When$G_{mH} = G_{w,\succeq h}$, given men's stable marriage premium π_m, women's stable marriage-payoff differences are $v_{wH} - v_{wL} = s_{HH} - s_{LL} - \pi_m$, $v_{wL} - v_{wl} = s_{LL} - s_{Ll}$, and $v_{wh} - v_{wl} = s_{Hh} - s_{Ll} - \pi_m$, so

$$\theta_{w1}(\pi_m) = \frac{c_w}{z_{wH} - z_{wL} + (s_{HH} - s_{LL} - \pi_m)}$$

and

$$\theta_{w2}(\pi_m) = \frac{c_w + (s_{LL} - s_{Ll})}{z_{wH} - z_{wL} + (s_{Hh} - s_{Ll} - \pi_m)}.$$

Therefore, both $\theta_{w1}(\pi_m)$ and $\theta_{w2}(\pi_m)$ are strictly increasing when $\pi_m \in (\delta_L, \delta_h)$.

(iii) To show $G_{wH}(\pi_m) = \int_{\theta_{w1}(\pi_m)}^{1} \theta dF_w(\theta)$ is strictly decreasing when $\pi_m \in (\delta_h, \delta_H)$, it suffices to show $\theta_{w1}(\pi_m)$ is strictly increasing when $\pi_m \in (\delta_h, \delta_L)$. Men's stable marriage premium can be $\pi_m \in (\delta_h, \delta_L)$ only when $G_{mH} = G_{wH}$. When $G_{mH} = G_{wH}$, given men's stable marriage premium π_m, women's stable marriage-payoff difference $v_{wH} - v_{wL} = s_{HH} - s_{LL} - \pi_m$, so

$$\theta_{w1}(\pi_m) = \frac{c_w}{z_{wH} - z_{wL} + s_{HH} - s_{LL} - \pi_m}.$$

Therefore, $\theta_{w1}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_h, \delta_L)$.

Case 3. Suppose $\delta_h = \delta_L$. Types are ranked as $H \succ L \sim h \succ l$. Let $\tau_2 := L \sim h$. Explicitly,

$$D_{mH}(\pi_m) = \begin{cases} [G_{w,\succeq\tau_2}(\pi_m), 1] & \text{if } \pi_m = \delta_l \\ G_{w,\succeq\tau_2}(\pi_m) & \text{if } \pi_m \in (\delta_l, \delta_{\tau_2}) \\ [G_{wH}(\pi_m), G_{w,\succeq\tau_2}(\pi_m)] & \text{if } \pi_m = \delta_{\tau_2} \\ G_{wH}(\pi_m) & \text{if } \pi_m \in (\delta_{\tau_2}, \delta_H) \\ [0, G_{wH}(\pi_m)] & \text{if } \pi_m = \delta_H \end{cases}.$$

It remains to show that (i) $G_{w,\succeq\tau_2}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_{\tau_2})$, and (ii) $G_{wH}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_{\tau_2}, \delta_H)$.

(i) To show $G_{w,\succeq\tau_2}(\pi_m) = 1 - \int_{\theta_{w2}(\pi_m)}^{1} (1-\theta)^2 dF_w(\theta)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_{\tau_2})$, it suffices to show $\theta_{w2}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_{\tau_2})$. Men's stable marriage premium can be $\pi_m \in (\delta_l, \delta_L)$ only when $G_{mH} = G_{w,\succeq\tau_2}$. When $G_{mH} = G_{w,\succeq\tau_2}$, given men's stable marriage premium π_m, women's stable marriage-payoff differences are $v_{wL} - v_{wl} = s_{HL} - s_{Ll} - \pi_m$, $v_{wH} - v_{wL} = s_{HH} - s_{HL}$, and $v_{wh} - v_{wl} = s_{Hh} - s_{Hl} - \pi_m$, so

$$\theta_{w2}(\pi_m) \quad = \quad \frac{c_w + s_{HL} - s_{Ll} - \pi_m}{z_{wH} - z_{wL} + s_{Hh} - s_{Hl} - \pi_m}.$$

Since $\theta_{w2}(\pi_m) < 1$, $\theta_{w2}(\pi_m)$ is strictly decreasing when $\pi_m \in (\delta_l, \delta_{\tau_2})$.

(ii) To show $G_{wH}(\pi_m) = \int_{\theta_{w1}(\pi_m)}^{1} \theta dF_w(\theta)$ is strictly decreasing when $\pi_m \in (\delta_{\tau_2}, \delta_H)$, it suffices to show $\theta_{w1}(\pi_m)$ is strictly increasing when $\pi_m \in (\delta_{\tau_2}, \delta_H)$. Men's stable marriage premium can be π_m only when $G_{mH} = G_{wH}$. When $G_{mH} = G_{wH}$, given men's stable marriage premium π_m, women's stable marriage-payoff difference $v_{wH} - v_{wL} = s_{HH} - s_{LL} - \pi_m$, so

$$\theta_{w1}(\pi_m) = \frac{c_w}{z_{wH} - z_{wL} + s_{HH} - s_{LL} - \pi_m}.$$

Therefore, $\theta_{w1}(\pi_m)$ is strictly increasing when $\pi_m \in (\delta_{\tau_2}, \delta_H)$. *QED*

A.3 Proof of Proposition 1

I first prove the college gender gap. Suppose by way of contradiction that weakly fewer women than men go to college in equilibrium: $1 - F_w(\theta_{w1}^*) \leq 1 - F_m(\theta_m^*)$. First, since $F_m = F_w$ by assumption, $F_w(\theta_{w1}^*) \geq F_m(\theta_m^*)$ implies $\theta_{w1}^* = c_w/(z_{wH} - z_{wL} + v_{wH}^* - v_{wL}^*) \geq \theta_m^* = c_m/(z_{mH} - z_{mL} + v_{mH}^* - v_{mL}^*)$. Since $z_{wH} - z_{wL} = z_{mH} - z_{mL}$ by assumption, $v_{wH}^* - v_{wL}^* \leq v_{mH}^* - v_{mL}^*$.

Second, $\theta_{w2}^* > \theta_{w1}^*$, so strictly fewer women than men make a career investment in equilibrium. Since weakly fewer women go to college by our premise and strictly fewer women make a career investment, strictly fewer women than men earn a high income, i.e., $G_{wH}^* + G_{wh}^* < G_{mH}^*$. As a result, there is a positive mass of type-L women marrying high-income men. By pairwise efficiency, $v_{wL}^* = s_{HL} - v_{mH}^*$. Since there is always a positive mass of (H,H) couples, by pairwise efficiency, $v^*_{wH} = s_{HH} - v^*_{mH}$. The two pairwise efficiency conditions together imply $v^*_{wH} - v^*_{wL} = s_{HH} - s_{HL}$. By $s_{HL} = s_{LH}$, $v^*_{wH} - v^*_{wL} = s_{HH} - s_{HL} = s_{HH} - s_{LH} = \delta_H$. Because a positive mass of type-H men marries type-L women in equilibrium, $v^*_{mH} = s_{HL} - v^*_{wL}$. Furthermore, by Pareto efficiency, $v^*_{mL} \geq s_{LL} - v^*_{wL}$. The two conditions together imply $v^*_{mH} - v^*_{mL} \leq s_{HL} - s_{LL}$. Since the surplus is strictly super-modular in incomes, $v^*_{wH} - v^*_{wL} = \delta_H > \delta_L = v^*_{mH} - v_{mL}$. The two conclusions, $v^*_{wH} - v^*_{wL} \leq v^*_{mH} - v^*_{mL}$ and $v^*_{wH} - v^*_{wL} > v^*_{mH} - v^*_{mL}$, contradict each other. Therefore, there must be strictly more women than men going to college.

I now prove the earnings gender gap. Consider the assumption $G_{mH}(\delta_l) > G_{wH}(\delta_l) + G_{wh}(\delta_l)$. It states that when men's stable marriage premium π_m is δ_l the lowest value possible, mass $G_{mH}(\delta_l)$ of high-income men is strictly greater than the mass $G_{wH}(\delta_l) + G_{wh}(\delta_l)$ of high-income women. That is, even when men have the smallest possible marriage premium $\pi_m = \delta_l = s_{Hl} - s_{Hl}$ and women have the largest possible marriage premium $\pi_w = s_{HH} - s_{HL}$, fewer women will end up with a high income than men. Therefore, the earnings gender gap always holds. *QED*