# Bargaining and Reputation with Ultimatums

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**Abstract**

This paper analyzes reputational bargaining between two players who can strategically send an ultimatum to end the bargaining process. Each player either (i) persists in a demand for a fixed share of a unit pie and sends an ultimatum according to a Poisson process, or (ii) wants as a big share of the pie as possible and can strategically send an ultimatum at any time. A strategic player wins if the opponent concedes to the ultimatum but expects to lose if the opponent responds to the ultimatum. We study the equilibrium of the game when the ultimatum may be sent by neither player (Abreu and Gul, 2000), by one player, or by both players. When at least one persistent player sends an ultimatum at a rate strictly lower than a cutoff rate, strategic players can build up their reputations and the rate at which a strategic player sends an ultimatum may be non-monotonic in time. When both persistent players send an ultimatum at a rate higher than the cutoff rate and both players are unlikely to be persistent, players cannot build up their reputations. The cutoff rate is precisely the equilibrium conceding rate in Abreu and Gul (2000).

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# 1 Introduction

This paper extends the two-player reputational bargaining model of Abreu and Gul (2000) by allowing players to send an ultimatum to the opponent to end the bargaining process.

Namely, we consider the following setting. Two players divide a unit pie. Each player is either (i) justified to demand a fixed share of the pie (corresponding to a behavioral type in Abreu and Gul (2000)), or (ii) unjustified to demand any share but nonetheless wanting as a big share of the pie as possible (corresponding to a rational type in Abreu and Gul (2000)). Each player announces his or her demand sequentially at the beginning of the game. After announcing a demand, each player can hold on to the announced demand or give in to the opposing player's demand at any time. In the baseline bargaining model without ultimatums (Abreu and Gul, 2000), the game ends only when one player gives in to the other player. In our extensions, opportunities to challenge the opponent and end the bargaining process arise periodically. Think of these opportunities as opportunities to resolve the conflict in an arbitration court. A justified player always uses the opportunity to challenge the opposing side, but an unjustified player may use the opportunity strategically. The opponent, upon being challenged, must respond by seeing the challenge or giving in the challenger's demand. A justified party wins in the court against an unjustified party in the court, and an unjustified challenger loses in expectation against an unjustified defendant.

In the baseline bargaining model without ultimatums, the equilibrium bargaining dynamics and reputation dynamics are quite simple. There is a unique sequential equilibrium. After players announce their demands, at most one player concedes with a positive probability at time 0. Afterwards, both players concede at a constant rate, and their reputations – opponent's beliefs about a player being justified – increase at a constant rate until both players' reputations reach 1 at the same time at which point no unjustified player is left in the game and justified players continue to hold on to their demands. As the probabilities of justification tend to zero, the limit payoffs depend on the impatience factors only, as in Rubinstein (1982). A more patient unjustified player receives a higher payoff.

Because of the additional possibility to send ultimatums, the equilibrium bargaining dynamics as well as the equilibrium reputation dynamics are much richer than those in the baseline bargaining model without ultimatums. Consider first the setting in which only player 1 has the opportunity to send ultimatums. The bargainers can experience two bargaining phases with different bargaining and reputation dynamics. A player challenges at an increasing rate in the initial strategy phase, and the opposing player mixes between seeing the challenge and yielding to the challenge. In the second phase, because player 2's reputation is sufficiently high, an unjustified player 1 never challenges and simply concedes at a constant rate as in the baseline model. Both players' reputations reach 1 at the same time. For sufficiently low frequency of challenge arrivals, the result that the limit payoffs depend on the impatience factors continues to hold. For sufficiently high frequency of challenge arrivals though, the limit payoffs depend on player 1's frequency of the challenge arrival and player 2's impatience factor: the higher the frequency of the arrival of challenges, the lower the limit payoff of player 1 is. In other words, an unjustified player 1 does not prefer to have the challenge opportunity, as it limits his commitment power of continuing to hold on to his demand. The challenge opportunity – the possibility to go to the court – helps to separate the justified from the unjustified.

Finally, we consider the bargaining problem with two-sided ultimatums: it is possible for both sides to take the conflict to the court. When players' reputations are relatively low and the challenge opportunities arrive sufficiently frequently, namely, when the rate of challenge arrival is greater than the equilibrium Abreu-Gul concession rate in the baseline model, reputations decrease in equilibrium and players cannot build up their reputations at all! Players mix between conceding and not conceding if no challenge opportunity arises, mix between challenging and not challenging if a challenge opportunity arises, and mix between seeing and not seeing a challenge when they are being challenged. The equilibrium may not be unique, however, as players can freely concede at time 0. When the players' reputations are sufficiently high or the challenge opportunities arrive sufficiently infrequently, equilibrium exists uniquely, reputations build up and players experience in general three phases: both players mix between challenging and not challenging, one player challenges and the other player does not, and neither player challenges. The limit payoffs again depend on the challenge arrival rates instead of discount rates when challenge arrival rates exceed the discount rates.

The paper contributes to the growing literature of reputational bargaining. In contrast to Fanning (2016) which studies reputational bargaining with exogenous deadlines, this paper can be viewed as studying reputational bargaining with endogenously chosen deadlines. Compared to Fanning (2018a) with a mediator neither player needs to obey, we have an arbitrator both players need to obey. In addition, the insights generated are also related to bargaining with outside options (Atakan and Ekmekci, 2013; Chang, 2016; Hwang and Li, 2017; Fanning, 2018b). Relatedly, Sandroni and Urgun (2018) and Sandroni and Urgun (2017) also study situations in which players can end the bargaining process.

This setup also has real-world implications to bargaining situations involving arbitration. This model sheds light on the negotiation stage between two parties when they have a chance to have a court or any mediator to make an arbitration. It provides an economic interpretation of the behavioral types.[1]

[1] To provide economic meaning for behavioral types is widely regarded as an important question: Abreu and Gul (2000) wrote that "it would be very interesting to extend this kind of analysis to specific institutional settings (for example, firm/union bargaining)"; Fanning (2016) wrote that "determining a plausible motivation for the commitment behavior of [behavioral] types is an important avenue for future research."

# 2 Baseline: Bargaining without Ultimatums

Player 1 ("he") and player 2 ("she") divide a unit pie. Each player either (i) persists to demand a share of the pie and never accepts any offer below that, or (ii) wants as a big share of the pie as possible. Players sequentially announce their demands $a_i \in A_i$. Time is continuous. At each instant, a player can either hold on to his or her demand, or give in to the other player's demand at any time. The game ends only when one player gives in to the other player. Each player $i$ discounts with rate $r_i$.

In summary, the bargaining game $\{z_i, \pi_i, r_i\}$ is described by players' prior probabilities $z_1$ and $z_2$ of being justified, conditional distributions $\pi_1$ and $\pi_2$, and discount rates $r_1$ and $r_2$.

## 2.1 The Single-Type Case

We initially assume that each player can be of a single justified type: with probability $z_1$ player 1 is justified to demand $a_1$, and with probability $z_2$ player 2 is justified to demand $a_2 > 1 - a_1$. Let $D \equiv a_1 + a_2 - 1$ denote the conflicting difference between the two players.

### 2.1.1 Formal Description of the Game

An unjustified player $i$ concedes with probability $F_i(t)$ by time $t$. An unjustified player $i$'s expected utility from conceding at time $t$, when an unjustified opponent $j \neq i$ concedes according to $F_j$, is

$$\begin{aligned} u_i(t, F_j) &= (1-z_j)\int_0^t a_i e^{-r_i s} dF_j(s) + (1-(1-z_j)F_j(t))e^{-r_i t}(1-a_j) \\ &\quad + (1-z_j)\left[F_j(t) - \lim_{s \uparrow t} F_j(s)\right]\frac{a_i + 1 - a_j}{2}. \end{aligned}$$

An unjustified player $i$'s expected utility from strategy $F_i$ is

$$u_i(F_i, F_j) = \int_0^\infty u_i(s, F_j)\, dF_i(s).$$

### 2.1.2 Strategy

A player can either concede or stay at each instant. We solve for a fully mixed equilibrium in which both players mix between conceding and staying at each instant. We will show that this is the unique sequential equilibrium of the bargaining game. An unjustified player $i = 1, 2$ is indifferent between conceding at time $t$ and conceding at time $t + dt$ if and only if

$$1 - a_j = e^{-r_i t}(1-a_j)(1-\lambda_j dt) + \lambda_j dt \cdot a_i.$$

Rearrange, we get that player $j \neq i$ has to challenge at a constant rate

$$\lambda_j = \frac{r_i(1-a_j)}{D}$$

to make player $i$ indifferent.

### 2.1.3 Reputation

Now we solve for the equilibrium reputation dynamics. Let $\mu_i(t)$ represent the probability that player $i$ is justified. By the Martingale property $\mu_i(t) = \mathbb{E}[\mu_i(t+dt)|\mathscr{F}_t]$ where $\mathscr{F}_t$ represents the information set up to time $t$,

$$\begin{aligned}\mu_i(t) &= \lambda_i dt \cdot 0 + (1-\lambda_i dt)\mu_i(t+dt)\\ \mu_i(t+dt) - \mu_i(t) &= \lambda_i dt \cdot \mu_i(t+dt)\end{aligned}$$

Taking $dt \to 0$,

$$\mu_i'(t) = \lambda_i \mu_i(t).$$

Solving the differential equation,

$$\mu_i(t) = C_i e^{\lambda_i t}.$$

### 2.1.4 Equilibrium

**Proposition 2.1.** *For any bargaining game* $\{z_i, \pi_i, r_i\}_{i=1}^2$ *with* $A_i = \{a_i\}$, *there exists a unique sequential equilibrium* $(\widehat{F}_1, \widehat{F}_2)$, *where* $\lambda_i = \frac{r_j(1-a_i)}{a_i+a_j-1}$, $\widehat{T} = \min\{-\frac{1}{\lambda_1}\log z_1, -\frac{1}{\lambda_2}\log z_2\}$, $C_i = z_i e^{-\lambda_i \widehat{T}}$, *and* $\widehat{F}_i(t) = \frac{1-C_i e^{-\lambda_i t}}{1-z_i}$.

**Proof of Proposition 2.1.** Let $\Sigma = (\Sigma_1, \Sigma_2)$ define a sequential equilibrium. We will argue that $\Sigma$ must have the form specified (i.e., uniqueness) and that these strategies do indeed define an equilibrium (i.e., existence). Let $u_i(t)$ denote the expected utility of an unjustified player $i$ who concedes at time $t$. Define $T_i := \{t | u_i(t) = \max_s u_i(s)\}$ as the set of conceding times that attain the highest expected utility for player $i$; since $\Sigma$ is a sequential equilibrium, $T_i$ is nonempty for $i = 1, 2$. Furthermore, define $\tau_i = \inf\{t \geq 0 | F_i(t) = \lim_{t' \to \infty} F_i(t')$ as the time of last concession for player $i$, where $\inf \emptyset := \infty$. Then we have the following results.

**(a) The last instant at which two unjustified players concede with a positive rate is the same: $\tau_1 = \tau_2$.**

An unjustified player will not delay conceding once he/she knows that the opponent is persistent and will never concede. Let $\tau$ denote $\tau_1 = \tau_2$.

**(b) If $F_i$ jumps at $t \in \mathbb{R}$, then $F_j$ does not jump at $t$.**

If $F_i$ had a jump at time $t$, then player $j$ receives a strictly higher utility by conceding an instant after time $t$ than by conceding exactly at time $t$.

**(c) If $F_i$ is continuous at time $t$, then $u_j(s)$ is continuous at $s = t$ for $j \neq i$.**
The continuity follows directly from the definition of $u_i(s)$ in equation (1).

**(d) There is no interval $(t', t'')$ such that $0 \leq t' \leq t'' \leq \tau$ where both $F_1$ and $F_2$ are constant on the interval $(t', t'')$.**
Assume the contrary. Without loss of generality, let $t^* \leq \tau$ be the supremum of $t''$ for which $F_1$ and $F_2$ are both constant on the interval $(t', t'')$. Fix $t \in (t', t^*)$. For $\varepsilon$ small there exists $\delta > 0$ such that $u_i(t) - \delta > u_i(s)$ for all $s \in (t^* - \varepsilon, t^*)$ for $i = 1, 2$; in words, conditional on the opponent not conceding in an interval, it is strictly better for a player to concede earlier within that interval. By (b) and (c), there exists $i$ such that $u_i(s)$ is continuous at $s = t^*$, so for some $\eta > 0$, $u_i(s) < u_i(t)$ for all $s \in (t^*, t^* + \eta)$ for this player $i$; in words, because the expected utility function is continuous at time $t$, the expected utility of conceding a little bit after $t^*$ is still lower than the expected utility of conceding at time $t$. Since $F_i$ is optimal, $F_i$ must be constant on the interval $(t', t^* + \eta)$. The optimality of $F_i$ implies $F_j$ is also constant on the same interval $(t', t^* + \eta)$, because player $j$ is strictly better off conceding right before or right after the interval than conceding at any time during the interval. Hence, both functions are constant on the interval $(t', t^* + \eta)$. However, this contradicts the definition of $t^*$.

**(e) Players concede at a strictly positive rate: if $t' < t'' < \tau$, then $F_i(t'') > F_i(t')$ for $i = 1, 2$.**

If $F_i$ is constant on some interval, then the optimality of $F_i$ implies that $F_j$ is also constant on the same interval, for $j \neq i$. However, (d) shows that $F_1$ and $F_2$ cannot be simultaneously constant.

**(f) Cumulative concession probability $F_i$ is continuous at any time $t > 0$.**
Assume the contrary: suppose $F_i$ has a jump at time $t$. Then by (b), $F_j$ is constant on interval $(t - \varepsilon, t)$ for $j \neq i$. This contradicts (e), however.

From (e) it follows that $T_i$ is dense on $[0, \tau_i]$ for $i = 1, 2$. From (c) and (f) it follows that $u_i(s)$ is continuous on $(0, \tau]$ and hence $u_i(s)$ is constant for all $s \in (0, \tau]$. Consequently, $T_i = (0, \tau]$. Hence, $u_i(t)$ is differentiable as a function of $t$ and $du_i(t)/dt = 0$ for all $t \in (0, \tau)$. The expected utility is

$$u_i(t) = (1 - z_i) \int_0^t a_i e^{-r_i s} dF_j(s) + (1 - a_j) e^{-r_i t} (1 - (1 - z_j) F_j(t)). \tag{1}$$

Differentiability of $F_j$ follows from differentiability of $u_i$ on $(0, \tau)$. Differentiating equation (1) and applying Leibnitz's rule, we obtain

$$0 = a_i e^{-r_i t} (1 - z_j) f_j(t) - (1 - a_j) r_i e^{-r_i t} (1 - (1 - z_j) F_j(t)) - (1 - a_j) e^{-r_i t} (1 - z_j) f_j(t),$$

where $f_j(t) = dF_j(t)/dt$. This in turn implies $F_j(t) = \frac{1-C_j e^{-\lambda_j t}}{1-z_j}$, where $C_j$ is yet to be determined. Optimality for player $i$ implies $F_i(\tau_i) = 1$. At $t = 0$, if $F_j(0) > 0$ then $F_i(0) = 0$ by (b). Let $\ell_i$ solve $1 - e^{-\lambda_i \ell_i} = 1 - z_i$. Then $\tau_1 = \tau_2 = \ell := \min\{\ell_1, \ell_2\}$, and $C_i$ and $C_j$ are determined by the boundary condition $1 - C_i e^{-\lambda_i \ell} = 1 - z_i$. If $j$'s strategy is $\widehat{F}_j$, then $u_i(t)$ is constant on $(0, \tau]$, and $u_i(s) < u_i(\ell)$ for all $s > \tau$. Hence, for any mixed strategy on this support, and in particular, $\widehat{F}_i$, is optimal for player $i$. Hence, $(\widehat{F}_1, \widehat{F}_2)$ is indeed an equilibrium. □

In the unique equilibrium derived above, an unjustified player $i$'s utility is

$$1 - a_j + (1 - z_j)\widehat{F}_j(0)D.$$

## 2.2 The Multiple-Type Case

**Proposition 2.2.** *For any bargaining game $\{\pi_i, z_i, r_i\}_{i=1}^2$ with $A_1 = \{a_1\}$ and $|A_2| \geq 1$, there exists a unique sequential equilibrium $(\widehat{F}_1, (\widehat{\sigma}_2, \widehat{F}_2))$.*

**Proof of Proposition 2.2.** For any $a_1 \in A_1$ and $x \in (0, 1]$, denote by $B_0(a_1, x)$ the bargaining game in which player 1 is justified to demand $a_1$ with probability $x$ and is unjustified otherwise. We will show that there is a unique sequential equilibrium of the game $B_0(a_1, x)$.

If player 2 chooses some $a_2 \leq 1 - a_1$, then the game ends at time zero. If player 2 chooses some $a_2 > 1 - a_1$, then the game does not end at time zero, but we know from proposition 2.1 that after time zero, player $i = 1, 2$ must concede at rate $\lambda_i = r_j(1 - a_i)/(a_i + a_j - 1)$, where $j \neq i$. That is, $\widehat{F}_i(t|a_1, a_2)$ for any $t > 0$ is uniquely determined by $\widehat{F}_i(0|a_1, a_2)$. Without loss of generality, assume $\widehat{F}_2(0|a_1, a_2) = 0$ for all $a_2$.

Denote by $\sigma_2(\cdot)$, a probability distribution over $A_2 \cup \{Q\}$, a mimicking strategy of unjustified player 2. Since mimicking $a_2 < 1 - a_1$ is never optimal and mimicking $a_2 = 1 - a_1$ is equivalent to conceding, we assume that in equilibrium $\sigma_2(a_2) = 0$ for all $a_2 \leq 1 - a_1$. If $x = 1$, then in equilibrium $\sigma_2(Q) = 1$, because unjustified player 2 will not delay conceding if she knows that player 1 is justified. For the remainder of the proof we assume $x < 1$.

It remains to be shown that unjustified player 2's equilibrium mimicking behavior $\widehat{\sigma}_2(\cdot)$ and unjustified player 1's conceding behavior $\widehat{F}_1(0|a_1, \cdot)$ at time zero are uniquely determined. Subsequently, we provide a series of definitions and use them to prove a series of claims that lead to equilibrium existence and uniqueness.

Denote by

$$y^*(a_2, \sigma_2) \equiv \frac{z_2 \pi_2(a_2)}{z_2 \pi_2(a_2) + (1 - z_2)\sigma_2}$$

player 2's initial reputation given that unjustified player 2 mimics $a_2$ with probability $\sigma_2$. Note that $y^*(a_2, \sigma_2)$ is continuous and strictly decreasing in $\sigma_2$.

Denote by $T_1(a_1, a_2, x)$ and $T_2(a_1, a_2, y)$ the time it takes for the reputation of player $i = 1, 2$ to reach 1, given his/her initial reputation ($x$ or $y$) at time zero and given that neither player concedes with a positive probability after initial demands $a_1$ and $a_2$ are announced. Explicitly,

$$T_1(a_1, a_2, x) \equiv -\frac{a_1 + a_2 - 1}{r_2(1 - a_1)} \log x$$

and

$$T_2(a_1, a_2, y) \equiv -\frac{a_1 + a_2 - 1}{r_1(1 - a_2)} \log y.$$

Note that $T_1(a_1, a_2, \cdot)$ is continuous and strictly decreasing in $x$, and $T_2(a_1, a_2, \cdot)$ is continuous and strictly decreasing in $y$.

Denote by $\overline{\sigma}_2(a_1, a_2, x)$ the probability with which player 2 must mimic $a_2$ for both players' reputations to reach 1 at the same time, given that neither player concedes with a positive probability after initial demands $a_1$ and $a_2$ are announced. Formally, $\overline{\sigma}_2(a_1, a_2, x)$ is the unique value of $\sigma_2$ that solves $T_1(a_1, a_2, x) = T_2(a_1, a_2, y^*(a_2, \sigma_2))$. Explicitly,

$$\overline{\sigma}_2(a_1, a_2, x) \equiv \frac{1 - x^{\frac{r_1(1-a_2)}{r_2(1-a_1)}}}{x^{\frac{r_1(1-a_2)}{r_2(1-a_1)}}} \frac{z_2 \pi_2(a_2)}{1 - z_2}.$$

Note that in equilibrium $\sigma_2(a_2) \leq \overline{\sigma}_2(a_1, a_2, x)$ for all $a_2 > 1 - a_1$. To see why this claim must hold, suppose unjustified player 2 mimics $a_2$ with a probability strictly higher than $\overline{\sigma}_2(a_1, a_2, x)$. Then player 2 needs to concede with a strictly positive probability at time zero in order for players' reputations to reach 1 at the same time. However, we have specified (without loss of generality) that player 2 does not concede at time zero after announcing her demand.

Denote by

$$\Delta(a_1, x) \equiv \left\{ \sigma_2(\cdot) \in \Delta \;\middle|\; \begin{array}{ll} \sigma_2(a_2) = 0 & \forall a_2 \leq 1 - a_1 \\ \sigma_2(a_2) \leq \overline{\sigma}_2(a_1, a_2, x) & \forall a_2 > 1 - a_1 \end{array} \right\}$$

the set of candidate equilibrium strategies, where $\Delta$ is the set of all probability distributions on $A_2 \cup \{Q\}$.

Denote by $x^*(a_1, a_2, \sigma_2)$ player 1's initial reputation in order for both players' reputations to reach 1 at the same time, given that $a_2$ is chosen with probability $\sigma_2$ and given that neither player concedes with a positive probability at time zero after initial demands $a_1$ and $a_2$ are announced. Formally, $x^*(a_1, a_2, \sigma_2)$ is the unique value of $x$ that solves $T_1(a_1, a_2, x) = T_2(a_1, a_2, y^*(a_2, \sigma_2))$. Explicitly,

$$x^*(a_1, a_2, \sigma_2) \equiv y^*(a_2, \sigma_2)^{\frac{r_2(1-a_1)}{r_1(1-a_2)}}.$$

Note that $x^*(a_1,a_2,\sigma_2) \in [x,1]$, and note that $x^*(a_1,a_2,\sigma_2)$ is continuous and strictly decreasing in $\sigma_2$.

Denote by $F_1^*(0|a_1,a_2,x,\sigma_2)$ the probability with which unjustified player 1 must concede at time 0, so that the two players' reputations reach 1 at the same time, given player 1's initial reputation $x \le x^*(a_1,a_2,\sigma_2)$ and given that unjustified player 2 chooses $a_2$ with probability $\sigma_2$. In other words, in equilibrium, if unjustified player 2 chooses $a_2$ with probability $\sigma_2$, then the probability that unjustified player 1 concedes at time zero is $F_1^*(0|a_1,a_2,x,\sigma_2)$. Formally, $F_1^*(0|a_1,a_2,x,\sigma_2)$ is the unique value of $F_1$ that solves

$$x^*(a_1,a_2,\sigma_2) = \frac{x}{x+(1-x)(1-F_1)}.$$

Explicitly,

$$F_1^*(0|a_1,a_2,x,\sigma_2) \equiv 1 - \frac{x}{1-x}\bigg/\frac{x^*(a_1,a_2,\sigma_2)}{1-x^*(a_1,a_2,\sigma_2)}.$$

Note that $F_1^*(0|a_1,a_2,x,\sigma_2)$ is continuous and strictly decreasing in $\sigma_2$.

Denote by $u_2^*(a_1,a_2,x,\sigma_2)$ the utility of unjustified player 2 if she mimics $a_2$ in the game $B_0(a_1,x)$ given that equilibrium specifies that she mimics $a_2$ with probability $\sigma_2$. Explicitly,

$$u_2^*(a_1,a_2,x,\sigma_2) \equiv 1-a_1+(1-x)F_1^*(0|a_1,a_2,x,\sigma_2)(a_1+a_2-1).$$

To see why the equation holds, note that player 2's payoff is $a_2$ if player 1 concedes at time zero and is $1-a_1$ if player 1 does not concede at time zero, since in equilibrium she is indifferent between conceding and not at every $t \ge 0$. Player 1 concedes with probability $(1-x)F_1^*(0|a_1,a_2,x,\sigma_2)$ in equilibrium. Furthermore, note that $u_2^*(a_1,a_2,x,\sigma_2)$ is continuous and strictly decreasing in $\sigma_2$ on $[0,\overline{\sigma}_2(a_1,a_2,x)]$. The properties follow from the fact that $F_1^*(0|a_1,a_2,x,\sigma_2)$ is continuous and strictly decreasing in $\sigma_2$.

For any $\sigma_2(\cdot) \in \Delta(a_1,x)$, define

$$\widehat{u}_2(x,\sigma_2(\cdot)) \equiv \begin{cases} \min_{a_2:\sigma_2(a_2)>0} u_2^*(a_1,a_2,x,\sigma_2(a_2)) & \text{if } \sigma_2(Q)=0 \\ 1-a_1 & \text{if } \sigma_2(Q)\neq 0 \end{cases}.$$

Note that $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy if and only if $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$. ($\Rightarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy. Any equilibrium strategy $\sigma_2(\cdot)$ satisfies that for all $a_2 \in A_2 \cup \{Q\}$ such that $\sigma_2(a_2)>0$, $u_2^*(a_1,a_2,x,\sigma_2(a_2))$ is the same. If $\sigma_2(Q)>0$, then $u_2^*(a_1,a_2,x,\sigma_2(a_2)) = 1-a_1$; if $\sigma_2(Q)=0$, then $u_2^*(a_1,a_2,x,\sigma_2(a_2)) = \min_{a_2:\widehat{\sigma}_2(a_2)>0} u_2(a_1,a_2,x,\widehat{\sigma}_2(a_2))$. Hence, any equilibrium strategy $\sigma_2(\cdot)$ must generate an equilibrium utility of $\widehat{u}_2(x,\sigma_2(\cdot))$. Hence, $\widehat{\sigma}_2(\cdot)$ maximizes $\widehat{u}_2(x,\sigma_2(\cdot))$ among all candidate equilibrium strategies $\sigma_2(\cdot)$. ($\Leftarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ solves

$\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$. By the strict monotonicity of $u_2^*(a_1,a_2,x,\cdot)$, for all $a_2 \in A_2$ such that $\widehat{\sigma}_2(a_2) > 0$, $u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2)) = \widehat{u}_2(x,\widehat{\sigma}_2(\cdot))$. Coupled with the fact that $\widehat{\sigma}_2(\cdot)$ is the feasible strategy that maximizes $\widehat{u}_2(x,\sigma_2(\cdot))$, $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy.

Define $\Gamma$, a correspondence from $\Delta(a_1,x)$ to $\Delta(a_1,x)$, as follows:

$$\Gamma(\sigma_2(\cdot)) \equiv \{\widetilde{\sigma}_2(\cdot) \in \Delta(a_1,x) | \widetilde{\sigma}_2(a_2) > 0 \Rightarrow u_2^*(a_1,a_2,x,\sigma_2(a_2)) \geq u_2^*(a_1,a_2',x,\sigma_2(a_2')) \, \forall a_2' \in A_2\}.$$

Note that $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$ if and only if $\widehat{\sigma}_2(\cdot)$ is a fixed point of $\Gamma$. ($\Rightarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$. By the argument above, $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy. Therefore, $\widehat{\sigma}_2(a_2) > 0$ implies $u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2)) \geq u_2^*(a_1,a_2',x,\widehat{\sigma}_2(a_2'))$ for any $a_2' \in A_2$. By the definition of $\Gamma$, $\widehat{\sigma}_2(\cdot) \in \Gamma(\widehat{\sigma}_2(\cdot))$. ($\Leftarrow$) Suppose $\widehat{\sigma}_2(\cdot) \in \Gamma(\widehat{\sigma}_2(\cdot))$. By the definition of $\Gamma$, $\widehat{\sigma}_2(a_2) > 0$ implies $u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2)) \geq u_2^*(a_1,a_2',x,\widehat{\sigma}_2(a_2'))$ for any $a_2' \in A_2$. Assume by contradiction that $\widehat{\sigma}_2(\cdot)$ does not solve $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$ but $\widetilde{\sigma}_2(\cdot) \neq \widehat{\sigma}_2(\cdot)$ does. There must exist an $a_2 \in A_2$ such that $\widehat{\sigma}_2(a_2) > 0$ and $\widetilde{\sigma}_2(a_2) < \widehat{\sigma}_2(a_2)$ (otherwise, if $\widetilde{\sigma}_2(a_2) \geq \widehat{\sigma}_2(a_2)$ for all $a_2$ such that $\widehat{\sigma}_2(a_2) > 0$, then by the strict monotonicity of $u_2^*$, $u_2^*(a_1,a_2,x,\widetilde{\sigma}_2(a_2)) \leq u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2))$, and $\widehat{u}_2(x,\widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x,\widehat{\sigma}_2(\cdot))$). However, that implies that there exists $a_2' \in A_2 \cup \{Q\}$ such that $\widetilde{\sigma}(a_2') > \widetilde{\sigma}(a_2')$. If $a_2' = Q$, then $\widehat{u}_2(x,\widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x,\widehat{\sigma}_2(\cdot))$. If $a_2' \in A_2$, then $\widehat{u}_2(x,\widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x,\widehat{\sigma}_2(\cdot))$.

Hence, from the two claims above, we have that $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy for player 2 in the game $B_0(a_1,x)$ if and only if $\widehat{\sigma}_2(\cdot)$ is a fixed point of $\Gamma$.

Equilibrium existence follows from the existence of a fixed point of $\Gamma$ by Kakutani's fixed point theorem. By construction, $\Delta(a_1,x)$ is compact. By construction, $\Gamma$ is convex-valued. Finally, $\Gamma$ is upper-hemicontinuous because $u_2^*$ is continuous in its last argument.

Equilibrium uniqueness follows from the strict monotonicity of $u_2^*$. Suppose there are two equilibrium strategies $\widehat{\sigma}_2(\cdot)$ and $\widetilde{\sigma}_2(\cdot)$; without loss of generality, suppose $\widehat{\sigma}_2(a_2) > \widetilde{\sigma}_2(a_2) > 0$ for some $a_2 > 1 - a_1$. The utilities of playing the two strategies are different: $\widehat{u}_2(x,\widehat{\sigma}_2(\cdot)) = u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2)) < u_2^*(a_1,a_2,x,\widetilde{\sigma}_2(a_2)) = \widehat{u}_2(x,\widetilde{\sigma}_2(\cdot))$, where the strict inequality follows from the strict monotonicity of $u_2^*$. This contradicts the property that equilibrium strategies $\widehat{\sigma}_2(\cdot)$ and $\widetilde{\sigma}_2(\cdot)$ both maximize $\widehat{u}_2(x,\sigma_2(\cdot))$.

$\square$

**Proposition 2.3.** *Denote by $u_1^*(a_1,x)$ the payoff of player 1 in the unique sequential equilibrium of the bargaining game $\{\pi,z_i,r_i\}_{i=1}^2$ with $A_1 = \{a_1\}$ and $|A_2| \geq 1$. It is a continuous function of $x$. Moreover, there exists an $\underline{x}$ such that $u_1^*(a_1,x) = u_1^*(a_1,\underline{x})$ for any $x \leq \underline{x}$ and $u_1^*(a_1,x)$ is strictly increasing in $x$ on the interval $(\underline{x},1)$.*

**Proof of Proposition 2.3.** Player 1's equilibrium payoff can be written as

$$\begin{aligned} u_1^*(a_1,x) &= \left[\sum_{a_2\le 1-a_1} z_2\pi_2(a_2)+(1-z_2)\sigma_2(Q)\right]a_1+\sum_{a_2>1-a_1}\left[(z_2\pi_2(a_2)+(1-z_2)\sigma_2(a_2))(1-a_2)\right] \\ &= a_1-\sum_{a_2>1-a_1}\left[(z_2\pi_2(a_2)+(1-z_2)\sigma_2(a_2))(a_1+a_2-1)\right] \end{aligned}$$

We show that $u_1^*(a_1,\cdot)$ is a continuous function of $x$. Note that $\widehat{u}_2(\cdot,\sigma_2(\cdot))$ is continuous and $\widehat{u}_2(\cdot,\cdot)$ is upper semi-continuous. Hence a straightforward extension of the Theorem of the Maximum yields that $\arg\max_{\sigma_2(\cdot)}\widehat{u}_2(x,\sigma_2(\cdot))$ is a continuous function of $x$. This implies $u_1^*(a_1,\cdot)$ is a continuous function of $x$.

To prove the monotonicity of $u_1^*(a_1,\cdot)$ in $x$, we divide the proof into three steps. First, we show that it is strictly increasing on $[x_*,1)$. Second, we show that it is constant on $(0,\underline{x}]$. Third, we show that it is strictly increasing on $[\underline{x},x_*]$.

**Step 1**

Suppose $\sum_{a_2>1-a_1}\overline{\sigma}_2(a_1,a_2,x)\le 1$. Then $\sigma_2(a_2)<\overline{\sigma}_2(a_1,a_2,x)$ for some $a_2>1-a_1$ implies $\sigma_2(Q)>0$, so the equilibrium payoff of player 2 in $B_0(a_1,x)$ must be $1-a_1$. But $\sigma_2(a_2)<\overline{\sigma}_2(a_1,a_2,x)$ implies $F_1^*(0|a_1,a_2,x,\sigma_2(a_2))>0$. Hence, $u_2^*(a_1,a_2,x,\sigma_2(a_2))>1-a_1$. Therefore player 2 can get a higher payoff than her equilibrium payoff by mimicking $a_2$, a contradiction. Hence we have the following property: when $\sum_{a_2>1-a_1}\overline{\sigma}_2(a_1,a_2,x)\le 1$, $\sigma_2(a_2)=\overline{\sigma}_2(a_1,a_2,x)$ for all $a_2>1-a_1$ and $\sigma_2(Q)=1-\sum_{a_2>1-a_1}\overline{\sigma}_2(a_1,a_2,x)$. Since $\overline{\sigma}_2(a_1,a_2,\cdot)$ is strictly decreasing in $x$, there is a unique value of $\widehat{x}$ that solves $\sum_{a_2>1-a_1}\overline{\sigma}_2(a_1,a_2,\widehat{x})=1$; let $x_*$ denote the value. For any $x\in[x_*,1)$ the $\sigma_2(a_2)$ in the expression of $u_1^*(a_1,x)$ can be replaced by $\overline{\sigma}_2(a_1,a_2,x)$:

$$u_1^*(a_1,x)=a_1-\sum_{a_2>1-a_1}\left[(z_2\pi_2(a_2)+(1-z_2)\overline{\sigma}_2(a_1,a_2,x))(a_1+a_2-1)\right].$$

Hence, $u_1^*(a_1,\cdot)$ is strictly increasing in $x$ on the interval $[x_*,1)$.

**Step 2**

Next, we show that for some $\underline{x}\le x_*$, $u_1(a_1,x)=u_1(a_1,\underline{x})$ and $\sigma_2(\max A_2)=1$ for all $x\le\underline{x}$. Let $\overline{a}_2\equiv\max A_2$ and $\underline{x}\equiv\sup\{x|\widehat{\sigma}_2(\overline{a}_2)=1\}$.

First we show that $\underline{x}$ is well-defined. Note that as $x$ approaches 0, $F_1^*(0|a_1,a_2,x,1)$ approaches 1 and since $F_1(a_1,a_2,x,\cdot)$ is strictly decreasing in $\sigma_2$, $u_2^*(a_1,\overline{a}_2,x,\sigma_2(\overline{a}_2))$ approaches $\overline{a}_2$. But for any $a_2$, $u_2(a_1,a_2,\cdot,\cdot)\le a_2$, so mimicking $a_2<\overline{a}_2$ cannot be optimal when $x$ is sufficiently small. Since $\sigma_2(\cdot)$ is continuous in $x$, $\sigma_2(\overline{a}_2)=1$ for $x=\underline{x}$. It remains to be shown that $u_1^*(a_1,x)=u_1^*(a_1,\underline{x})$ for all $x<\underline{x}$.

From the optimality of player 2's equilibrium strategy $\sigma_2(\cdot)$ it follows that $\sigma_2(\widehat{a}_2)>0$

implies $u_2^*(a_1,\widehat{a}_2,x,\sigma_2(\widehat{a}_2)) \geq u_2(a_1,a_2,x,\sigma_2(a_2))$ for all $a_2 \in A_2$:

$1-a_1+(1-x)(a_1+\widehat{a}_2-1)F_1^*(0|a_1,\widehat{a}_2,x,\sigma_2(\widehat{a}_2)) \geq 1-a_1+(1-x)(a_1+a_2-1)F_1^*(0|a_1,a_2,x,\sigma_2(a_2))$.

Therefore, for any $x \leq x_*$, $\sigma_2(\widehat{a}_2) > 0$ implies

$$F_1^*(0|a_1,\widehat{a}_2,x,\sigma_2(\widehat{a}_2)) \geq F_1^*(0|a_1,a_2,x,\sigma_2(a_2)) \cdot \frac{a_1+a_2-1}{a_1+\widehat{a}_2-1} \quad \forall a_2.$$

In particular, the inequality holds for $x = \underline{x}$, $\widehat{a}_2 = \overline{a}_2$, and $\sigma_2(\cdot)$ such that $\sigma_2(\overline{a}_2) = 1$ and $\sigma_2(a_2) = 0$ for any $a_2 \neq \overline{a}_2$:

$$F_1^*(0|a_1,\overline{a}_2,\underline{x},1) \geq F_1^*(0|a_1,a_2,\underline{x},0) \cdot \frac{a_1+a_2-1}{a_1+\overline{a}_2-1} \quad \forall a_2 \neq \overline{a}_2.$$

Since $F_1^*(0|a_1,a_2,\cdot,1)$ is strictly decreasing in $x$, for any $x < \underline{x}$,

$$F_1^*(0|a_1,\overline{a}_2,x,1) > F_1^*(0|a_1,\overline{a}_2,\underline{x},1)$$

Moreover, since a strategic player 1 will concede immediately if he knows player 2 is persistent,

$$F_1^*(0|a_1,a_2,\underline{x},0) = 1 \geq F_1^*(0|a_1,a_2,x,\sigma_2(a_2)).$$

For any $x \leq \underline{x}$,

$$F_1^*(0|a_1,\overline{a}_2,x,\sigma_2(\overline{a}_2)) > F_1^*(0|a_1,a_2,x,\sigma_2(a_2)) \cdot \frac{a_1+a_2-1}{a_1+\overline{a}_2-1} \quad \forall a_2 \neq \overline{a}_2.$$

Therefore, for any $x < \underline{x}$,

$$u_2^*(a_1,\overline{a}_2,x,\sigma_2(\overline{a}_2)) > u_2^*(a_1,a_2,x,\sigma_2(a_2)) \quad \forall a_2 \neq \overline{a}_2.$$

Hence, the optimality of $\sigma_2(\cdot)$ implies $\sigma_2(a_2) = 0$ for all $a_2 \neq \overline{a}_2$ and $\sigma_2(\overline{a}_2) = 1$ whenever $x \leq \underline{x}$, and $u_1^*(a_1,x) = u_1^*(a_1,\underline{x})$ for all $x \leq \underline{x}$.

**Step 3**

Finally, we prove that $u_1^*(a_1,\cdot)$ is strictly increasing on $[\underline{x},x_*]$.

We first prove that if $\sigma_2(\cdot)$ is an equilibrium strategy, then $\sigma_2(a_2) > 0$ implies $\sigma_2(\widehat{a}_2) > 0$ for all $\widehat{a}_2 > a_2$. Suppose by contradiction $\sigma_2(\widehat{a}_2) = 0$. Then $u_2^*(a_1,\widehat{a}_2,x,0) = (1-x)\widehat{a}_2 + x(1-a_1) > (1-x)a_2 + x(1-a_1) = u_2^*(a_1,a_2,x,0) \geq u_2^*(a_1,a_2,x,\sigma_2(a_2))$. It is strictly better to play $\widehat{a}_2$ than to play $a_2$.

There exists an $\widetilde{a}_2 > 1-a_1$ such that $\sigma_2(a_2) = 0$ for any $a_2 < \widetilde{a}_2$, $\sigma_2(a_2) > 0$ for any $a_2 \geq \widetilde{a}_2$, and $\sum_{a_2 \geq \widetilde{a}_2} \sigma_2(a_2) = 1$, and for any $\widehat{a}_2 \geq \widetilde{a}_2$,

$$F_1^*(0|a_1,\widehat{a}_2,x,\sigma_2(\widehat{a}_2)) \quad = \quad F_1^*(0|a_1,a_2,x,\sigma_2(a_2)) \cdot \frac{a_1+a_2-1}{a_1+\widehat{a}_2-1} \quad \forall a_2 \geq \widetilde{a}_2,$$

$$\geq \quad \frac{a_1+a_2-1}{a_1+\widehat{a}_2-1} \quad \forall a_2 < \widetilde{a}_2.$$

We have $\sigma_2(\cdot|a_1,x)$ stochastically dominates $\sigma_2(\cdot|a_1,x')$ for any $x<x'$. Hence $u_1^*(a_1,x) > u_1^*(a_1,x')$. This conclusion holds for any $x,x' \in [\underline{x}, x_*]$.

$\square$

**Proposition 2.4.** *For any bargaining game* $\{\pi_i, z_i, r_i\}_{i=1}^2$*, there exists a sequential equilibrium* $((\widehat{\sigma}_1, \widehat{F}_1), (\widehat{\sigma}_2, \widehat{F}_2))$*. Furthermore, all equilibria yield the same distribution over outcomes.*

**Proof of Proposition 2.4.** Denote by

$$x^*(a_1,\sigma_1) \equiv \frac{z_1\pi_1(a_1)}{z_1\pi_1(a_1)+(1-z_1)\sigma_1(a_1)}$$

the initial reputation of player 1 given that player 1 demands $a_1$ with probability $\sigma_1$. Following the proof of proposition 2.2, we can characterize player 1's equilibrium strategy $\widehat{\sigma}_1(\cdot)$, a probability distribution over $A_1$, as the solution to $\max_{\sigma_1(\cdot)} \widehat{u}_1(\sigma_1(\cdot))$, where

$$\widehat{u}_1(\sigma_1(\cdot)) \equiv \min_{a_1:\sigma_1(a_1)>0} u_1^*(a_1, x^*(a_1, \sigma_1(a_1))),$$

The continuity of $u_1^*(a_1,\cdot)$ ensures that $\widehat{\sigma}_1(\cdot)$ exists.

It remains to be shown that different equilibrium strategies of player 1 lead to the same distribution of equilibrium outcomes.

Let $\overline{u}_1 \equiv \max_{\sigma_1(\cdot)} \widehat{u}_1(\sigma_1(\cdot))$ be the maximized value above. Hence, $\overline{u}_1$ is the utility that player 1 attains in any equilibrium. Clearly, $\overline{u}_1 \geq u_1^*(a_1, \underline{x}(a_1))$ for all $a_1$. Let $\sigma_1$ and $\widehat{\sigma}_1$ denote two equilibrium strategies for player 1. If $\overline{u}_1 > u_1^*(a_1,\underline{x})$, then $\sigma_1(a_1) = \widehat{\sigma}_1(a_1)$. To see this note that either $u_1^*(a_1,1) > \overline{u}_1$, in which case there is a unique $\sigma_1^*$ such that $u_1^*(a_1, x^*(a_1,\sigma_1^*)) = \overline{u}_1$ and hence $\sigma_1(a_1) = \widehat{\sigma}_1(a_1) = \sigma_1^*$, or $u_1(a_1,1) \leq \overline{u}_1$, in which case $\sigma_1(a_1) = \widehat{\sigma}(a_1) = 0$ by the strict monotonicity of $u_1(a_1,\cdot)$. Let $A_1^* \equiv \{a_1 \in A_1 | u_1^*(a_1, \underline{x}(a_1)) = \overline{u}_1\}$. We have just noted that $\sigma_1(a_1) = \widehat{\mu}_1(a_1)$ for all $a_1 \in A_1 \backslash A_1^*$, so $\sum_{a_1\in A_1^*} \sigma_1(a_1) = \sum_{a_1 \in A_1^*} \widehat{\sigma}_1(a_1)$.

We will conclude the proof that $\widetilde{\sigma}_1(\cdot)$ and $\widehat{\sigma}_1(\cdot)$ lead to the same random outcome $\widetilde{\theta}$ by first verifying that the probability that player 1 chooses some $a_2 \in A_1^*$ and agreement is reached at time zero is the same with either $\widetilde{\sigma}_1$ or $\widehat{\sigma}_1$. (This will imply that the random outcome, conditional on agreement at time zero, is the same with either $\widetilde{\sigma}_1$ or $\widehat{\sigma}_1$.) Finally, we show that for each $a_1 \in A_1^*$, the probability that a strategic player 1 will mimic $a_1$ and not concede is the same with either $\widetilde{\sigma}_1$ or $\widehat{\sigma}_1$.

Let $A(\sigma_1(\cdot))$ be the probability that player 1 mimics some $a_1 \in A_1^*$ and agreement is reached at time zero given the equilibrium strategy $\sigma_1(\cdot)$. Since $a_1 \in A_1^*$ implies $\sigma_2(\overline{a}_2|a_1) = 1$, it follows that $a_1 \geq 1-\overline{a}_2$; otherwise player 1 would achieve a higher utility by mimicking $\max A_1 > 1-\overline{a}_2$. For any equilibrium $\sigma_1(\cdot)$,

$$\begin{aligned}
A(\sigma_1(\cdot)) &= \sum_{a_1\in A_1^*} F_1^*(0|a_1,\overline{a}_2,x(a_1,\sigma_1(a_1)),1)\cdot(1-x(a_1,\sigma_1(a_1)))\times(z_1\pi_1(a_1)+(1-z_1)\cdot\sigma_1(a_1)) \\
&= \sum_{a_1\in A_1^*} \frac{x^*(a_1,\overline{a}_2,1)-x(a_1,\sigma_1(a_1))}{x^*(a_1,\overline{a}_2,1)}(z_1\pi_1(a_1)+(1-z_1)\sigma_1(a_1)) \\
&= \sum_{a_1\in A_1^*}(z_1\pi_1(a_1)+(1-z_1)\pi_1(a_1)) - \sum_{a_1\in A_1^*}\frac{z_1\pi_1(a_1)}{x^*(a_1,\overline{a}_2,1)}.
\end{aligned}$$

But since $\sum_{a_1\in A_1^*}\widetilde{\sigma}_1(a_1) = \sum_{a_1\in A_1^*}\widehat{\sigma}_1(a_1)$ we have $A(\widetilde{\sigma}_1(\cdot)) = A(\widehat{\sigma}_1(\cdot))$.

Finally, for any $a_1 \in A_1^*$, the probability that a strategic player 1 will mimic $a_1$ and not concede at time zero is

$$\begin{aligned}
& \sigma_1(a_1)(1-F_1^*(0|a_1,\overline{a}_2,x^*(a_1,\sigma_1(a_1),1))) \\
= \ & \sigma_1(a_1)\frac{x^*(a_1,\sigma_1(a_1))(1-x^*(a_1,\overline{a}_2,1))}{(1-x^*(a_1,\sigma_1(a_1))x^*(a_1,\overline{a}_2,1)} \\
= \ & \frac{\pi_1(a_1)z_1(1-x^*(a_1,\overline{a}_2,1))}{(1-z_1)x^*(a_1,\overline{a}_2,1)}.
\end{aligned}$$

Hence, this term is independent of $\sigma_1(\cdot)$ and therefore the same for both $\widetilde{\sigma}_1(\cdot)$ and $\widehat{\sigma}_1(\cdot)$.

□

**Proposition 2.5.** *Let $B_0^n = \{(z_i^n,\pi_i^n,r_i^n)_{i=1}^2\}$ be a sequence of bargaining games and $v_i^n$ the corresponding sequence of sequential equilibrium payoffs for player i.*

1. *If all parameters other than $r_1^n$ and $r_2^n$ are constant along the sequence $B_0^n$, then, for $i \neq j$,*

$$\lim(r_i^n/r_j^n) = 0 \Rightarrow \liminf v_i^n \geq (1-z_j)\max A_i$$

$$\lim(r_i^n/r_j^n) = \infty \Rightarrow \limsup v_i^n \leq 1-(1-z_j)\max A_j$$

2. *If all parameters other than $z_i^n$ for some $i = 1,2$ are constant along the sequence $B_0^n$, then for $i \neq j$,*

$$\lim z_i^n = 1 \Rightarrow \liminf v_i^n \geq (1-z_j)\max A_i$$

$$\lim z_i^n = 0 \Rightarrow \lim v_j^n = \max A_j$$

**Proof of Proposition 2.5.** Pick a subsequence of $B_0^n$ such that $\sigma_1^{n_k}(\cdot)$, $F_1^{n_k}(0|\cdot,\cdot)$, $v_1^{n_k}$, $\sigma_2^{n_k}(\cdot|\cdot)$, and $v_2^{n_k}$ converge to their respective limits $\sigma_1(\cdot)$, $F_1(0|\cdot,\cdot)$, $v_1$, $\sigma_2(\cdot|\cdot)$, and $v_2$. Without loss of generality assume that the subsequence is the sequence itself.

Suppose $\lim(r_1^n/r_2^n) = 0$ (i.e., player 1 is infinitely more patient than player 2). It follows from the definition of

$$\overline{\sigma}_2(a_1,a_2,x_n) \equiv \left[ x_n^{-\frac{r_1^n(1-a_2)}{r_2^n(1-a_1)}} - 1 \right] \frac{z_2\pi_2(a_2)}{1-z_2}$$

that $\lim \overline{\sigma}_2(a_1,a_2,x_n) = 0$ and hence $\sigma_2(Q|a_1) = 1$, where

$$x_n = \frac{z_1\pi_1(a_1)}{z_1\pi_1(a_1)+(1-z_1)\sigma_1^n(a_1)}.$$

whenever $\lim(r_1^n/r_2^n) = 0$. Let $x = \lim x_n$. Consequently, the payoff associated with mimicking each $a_1$ is at least $(1-z_2)a_1$ for player 1; hence his equilibrium payoff is at least $(1-z_2)\max A_1$ in the limit.

Suppose $\lim(r_1^n/r_2^n) = \infty$ (i.e., player 1 is infinitely more impatient than player 2). Let $\underline{\sigma}_1(a_1,a_2,\sigma_2)$ be the unique value of $\sigma_1$ that solves $T_1(a_1,a_2,x^*(a_1,\sigma_1)) = T_2(a_1,a_2,y^*(a_2,\sigma_2))$. Then for any $a_1$ such that $\sigma_1(a_1) > 0$ and any $a_2 > 1-a_1$, $\lim \underline{\sigma}_1(a_1,a_2,x_n,\sigma_2(a_2|a_1)) = 0$. Hence, no matter what player 2 demands, player 1 concedes if he is rational. It must be that after any $a_1$, player 2 demands $\max A_2$ for sure. Therefore, $\liminf v_2^n \geq (1-z_1)\max A_2$ and $\limsup v_1^n \leq 1-(1-z_2)\max A_2$ as desired.

The proof of part (2) is similar and omitted.

□

## 2.3 The Limiting Case of Complete Rationality

**Proposition 2.6.** *Let $B_0^n = \{z_i^n, A_i, \pi_i, r_i\}_{i=1}^2$ be a sequence of continuous-time bargaining games. If $\lim z_1^n = \lim z_2^n = 0$, $\lim z_1^n/(z_1^n+z_2^n) \in (0,1)$, and $v_i^n$ is the sequential equilibrium payoff for player i in the game $B^n$, then*

$$\liminf v_i^n \geq \max\{a \in A_i \cup \{0\} | a < r_j/(r_i+r_j)\} \text{ for } i = 1,2.$$

**Proof of Proposition 2.6.** Without loss of generality, we will assume that the sequence $(\sigma_1^n(\cdot), \sigma_2^n(\cdot|\cdot))$ converges to some $(\sigma_1(\cdot), \sigma_2(\cdot|\cdot))$.

Assume $\sigma_1(a_1) > 0$ and

$$a_1 > \frac{r_2}{r_1+r_2} \quad \text{and} \quad a_2 < \frac{r_1}{r_1+r_2}.$$

The key observation is the following: if $a_1$ and $a_2$ are demanded at time zero, then player 1 must concede to player 2 with unconditional probability $\sigma_1(a_1)$.

To see this, $\underline{\sigma}_1$, the conditional probability that player 1 does not concede to player 2 must solve

$$T_1(a_1,a_2,x^*(a_1,\underline{\sigma}_1)) = T_2(a_1,a_2,y^*(a_2,\sigma_2(a_2|a_1))).$$

Explicitly,

$$\frac{\log x^*(a_1,\underline{\sigma}_1)}{r_2(1-a_1)} = \frac{\log y^*(a_2,\sigma_2(a_2|a_1))}{r_1(1-a_2)}$$

$$\frac{r_1/(1-a_1)}{r_2/(1-a_2)} = \frac{\log\left[1+\frac{(1-z_2)\sigma_2(a_2|a_1)}{z_2\pi_2(a_2)}\right]}{\log\left[1+\frac{(1-z_1)\underline{\sigma}_1}{z_1\pi_1(a_1)}\right]}.$$

Since $z_1$ is converging to 0, $\sigma_2(a_2|a_1)$ is converging to zero. Since $z_1$ and $z_2$ are converging to zero at the same rate, $\underline{\sigma}_1$ must also be converging to zero.

Thus, if player 1's demand $a_1 > r_2/(r_1+r_2)$, by choosing any $a_2 < r_1/(r_1+r_2)$ player 2 can guarantee that her opponent concedes immediately if rational. If player 1's demand $a_1 \le r_2/(r_1+r_2)$, player 2 can guarantee a payoff of $r_1/(r_1+r_2)$ by accepting player 1's demand. Hence player 2 can guarantee a payoff of

$$\max\left\{a_2 \in A_2 \,\middle|\, a_2 < \frac{r_1}{r_1+r_2}\right\}.$$

A similar argument establishes that player 1 can guarantee a payoff of

$$\max\left\{a_1 \in A_1 \,\middle|\, a_1 < \frac{r_2}{r_1+r_2}\right\}.$$

In equilibrium, player 2 mimics $a_2$ with a probability less than

$$\begin{aligned}
\overline{\sigma}_2(a_1,a_2,x^*(a_1,\sigma_1)) &= \left[x^*(a_1,\sigma_1)^{-\frac{r_1(1-a_2)}{r_2(1-a_1)}}-1\right]\frac{z_2\pi_2(a_2)}{1-z_2} \\
&= \left[\left(\frac{z_1\pi_1(a_1)}{z_1\pi_1(a_1)+(1-z_1)\sigma_1}\right)^{-\frac{r_1(1-a_2)}{r_2(1-a_1)}}-1\right]\frac{z_2\pi_2(a_2)}{1-z_2} \\
&= \left[\left(1+\frac{(1-z_1)\sigma_1}{z_1\pi_1(a_1)}\right)^{\frac{r_1(1-a_2)}{r_2(1-a_1)}}-1\right]\frac{z_2\pi_2(a_2)}{1-z_2}
\end{aligned}$$

which converges to zero when $a_1 < \frac{r_2}{r_1+r_2}$ and $a_2 > \frac{r_1}{r_1+r_2}$. Therefore, player 1 can guarantee a payoff of

$$\max\left\{a_1 \in A_1 \middle| a_1 < \frac{r_2}{r_1+r_2}\right\}.$$

□

Call $(A_1,A_2)$ generic if $r_1/(1-a_1) \neq r_2/(1-a_2)$ for all $(a_1,a_2) \in A_1 \times A_2$.

**Proposition 2.7.** *Let $B_0^n = \{z_i^n, \pi_i, r_i\}^2$ be a sequence of continuous-bargaining games such that $\lim z_1^n = \lim z_2^n = 0$ and $\lim z_1^n/(z_1^n + z_2^n) \in (0,1)$. For generic $(A_1,A_2)$ there exists a compromise outcome $(a_1^c, a_2^c)$, where $a_1^c + a_2^c = 1$, such that the equilibrium payoff of player $i$ converges to $a_i^c$, $i = 1,2$.*

**Proof of Proposition 2.7.** Consider the following artificial constant-sum game: $i$ chooses $a_i \in A_i$; $i$ wins iff

$$\frac{r_i}{1-a_i} < \frac{r_j}{1-a_j}.$$

Note that by the genericity assumption there are no ties. We will consistently assume $j \neq i$. The payoff to $i$ is $a_i$ for winning and is $1-a_j$ for losing. Denote by

$$\widetilde{a}_j(a_i) \equiv \min\{\{a_j \in A_j | a_j > 1 - a_i\} \cup \{\max A_j\}\}$$

player $j$'s lowest demand that is in conflict with player $i$'s demand $a_i$. Denote by

$$\widehat{a}_i \equiv \max\{\{a_i \in A_i | a_i + \widetilde{a}_j(a_i) > 1 \text{ and } a_i \text{ beats } \widetilde{a}_j(a_i)\} \cup \{0\}\}$$

player $i$'s highest demand that beats player $j$'s demand. Denote by

$$\widehat{a}_i^+ \equiv \min\{\{a_i \in A_i | a_i > \widehat{a}_i\} \cup \max A_i\}$$

player $i$'s lowest demand strictly higher than $\widehat{a}_i$.

We will demonstrate that our artificial constant-sum game has a pure strategy equilibrium $(\widehat{a}_1, \widehat{a}_2)$. If $i$ is the winner in this equilibrium, we set $a_i^c = \widehat{a}_i$ and $a_j^c = 1 - \widehat{a}_i$. Furthermore, in this equilibrium $a_i^* + a_j^* > 1$, a fact that simplifies the final step of the proof.

The argument is as follows. By assumption there exists $(a_1,a_2) \in A_1 \times A_2$ such that $a_1 + a_2 > 1$. Furthermore, we may assume that this pair satisfies $a_i + \widetilde{a}_j(a_i) > 1$, $i = 1,2$. Suppose $a_1$ beats $a_2$. Clearly by definition $\widehat{a}_1 > 0$. Thus $\widehat{a}_i > 0$ for some $i = 1,2$.

Suppose $\widehat{a}_1 > 0$ and $(\widehat{a}_1, \widetilde{a}_2(\widehat{a}_1))$ is not an equilibrium. Then it must be the case that $\widehat{a}_1^+ > \widehat{a}_1$ beats $\widetilde{a}_2(\widetilde{a}_1)$.

TBA. □

## 3 Bargaining with One-Sided Ultimatums

Player 1 ("he") and player 2 ("she") divide a unit pie. Each player is either (i) justified to demand a share of the pie and never accepting any offer below that, or (ii) unjustified to demand a share of the pie but nonetheless wanting as a big share of the pie as possible. A justified player can justify his or her demand, but an unjustified player cannot.

We initially assume that each player can be of a single justified type: with probability $z_1$ player 1 is justified to demand $a_1$ and with probability $z_2$ player 2 is justified to demand $a_2 > 1 - a_1$. Let $D \equiv a_1 + a_2 - 1$ denote the conflicting difference between the two players.

Time is continuous. At each instant $t$, each player can decide to give in to the other player's demand or hold on to his or her demand. In addition, player 1 has a challenge opportunity. Justified player 1 challenges when evidence arrives; the evidence arrives according to a Poisson process with arrival rate $\gamma_1 > 0$. Unjustified player 1 can challenge at any time but he will time his challenge strategically. If player 1 does not challenge, then the game continues. If player 1 challenges at time $t$, he incurs a cost $c_1$ and player 2 must respond to player 1's challenge. Player 2 may either yield to the challenge and get $1 - a_1$, or see the challenge by paying a cost $c_2$.

If player 2 sees the challenge, the shares of the pie are determined by the players' justified and unjustified types, as follows. If an unjustified player meets a justified player, then the justified player always wins, so an unjustified player $i$'s payoff against a justified player $j$ is $1 - a_j$. If two unjustified players meet, then the challenging player 1 wins with probability $w < 1/2$: he gets $a_1$ with probability $w$ and $1 - a_2$ with probability $1 - w$, so his expected payoff is $wa_1 + (1 - w)(1 - a_2) = 1 - a_2 + wD$, and the defending player 2's expected payoff is $(1-w)(1-a_1) + wa_2 = 1 - a_1 + (1-w)D$. To make challenging worthwhile for player 1, assume $wD < c_1 < (1-w)D$; and to make seeing a challenge worthwhile for player 2, assume $wD < c_2 < (1-w)D$.[2]

In summary, the bargaining game $\{z_i, a_i, r_i, \gamma_i, c_i, w_i\}_{i=1}^{2}$ with one-sided ultimatums is described by players' prior probabilities $z_1$ and $z_2$ of being justified, demands $a_1$ and $a_2$, discount rates $r_1$ and $r_2$, player 1's challenge arrival rate $\gamma_1 > 0$, challenge costs $c_1$ and $c_2$, and an unjustified challenger's winning probability $w$ against an unjustified defender.

**Application 1: Arbitration.** One application of the model is final-offer arbitration. Two parties announce their demands for a subject, such as the wage of union workers, the division of a company after bankruptcy, or the salary of a baseball player (final-offer arbitrations are used frequently in firm-union bargaining, in bankruptcy cases, and in Major League Baseball). A justified player can have superior evidence supporting his or her claim, but needs time and effort to gather information about his or her claim and to appeal to the court. An unjustified player does not have

[2] Inconsequential to our results because justified players are non-strategic, assume that two justified players have the same chance of winning the case, so a justified player $i$'s expected payoff is $1 - a_j + D/2$.

good proofs supporting his or her claim but nonetheless can appeal to court. Whether or not a player could gather evidence and is justified is private information. While they gather evidence, they can negotiate with each other by repeatedly making offers to each other or choosing to let the case be settled by the court when possible.[3] A justified player can finish collecting evidence at any moment, and as soon as he is done with collecting evidence and if the case has not been settled out of court, he submits his claim to the court. At that moment, the opposing player has to respond to the lawsuit, either by agreeing to the challenging player's demand out of court or by paying a cost to go on the court. In the court, an unjustified player loses to a justified player for sure and an unjustified challenger loses to an unjustified defendant in expectation.

**Application 2: Outside Option.** Another application of the model is bargaining in the presence of a verifiable outside option. Two parties announce their reservation values for a good. A justified player has an outside option that always guarantees him/her a payoff that equals the claim. An unjustified player does not have an outside option. Two players can negotiate with each other while waiting for the outside option to arrive. When a justified player's outside option arrives, he/she sends an ultimatum to the opponent. At the moment, the opponent has to decide whether or not to strike a deal by asking the other side to present evidence that outside option has already arrived. If the challenger presents the outside option, the challenged has to agree to the challenger's claim. If the challenger does not present the outside option, then the challenged wins.

**Application 3: Ultimatum.** Players make repeated offers to each other. A player who makes an offer can claim that this is the final offer he/she will make. If the offer is rejected, a rational player immediately reveals rationality.

## 3.1 The Single-Type Case

### 3.1.1 Formal Description of the Game

Let us formally describe the strategies and payoffs of the (unjustified) players when demands are fixed to be $a_1$ and $a_2$. Let $F_i(t)$ denote player $i$'s probability of conceding by time $t$. Let $G_1(t)$ denote player 1's probability of challenging by time $t$. Let $q_2(t)$ denote player 2's probability of conceding to a challenge at time $t$. Player 1's strategy is described by $\Sigma_1 = (F_1, G_1)$, and player 2's strategy is described by $\Sigma_2 = (F_2, q_2)$.

Player 1's expected utility from conceding at time $t$ is[4]

$$u_1(t, \Sigma_2) \quad = \quad (1-z_2)\int_0^t a_1 e^{-r_1 s} dF_2(s) + (1-(1-z_2)F_2(t))e^{-r_1 t}(1-a_2)$$

[3] It is optimal for an unjustified player to continue to make the same demand, so the war-of-attrition structure of the bargaining game can be derived rather than assumed, just like in Abreu and Gul (2000). The addition compared to Abreu and Gul (2000) is an additional opportunity to appeal to a third-party arbitrator.

[4] We assume an equal split when two players concede at the same time. It is inconsequential to our results because simultaneous concession occurs with probability 0 in equilibrium.

$$+(1-z_2)\left[F_2(t)-\lim_{s\uparrow t}F_2(s)\right]\frac{a_1+1-a_2}{2}. \tag{2}$$

Player 1's expected utility from challenging at time $t$ is[5]

$$\begin{aligned} v_1(t,\Sigma_2) &= (1-z_2)\int_{s=0}^{t} a_1 e^{-r_1 s} dF_2(s) \\ &\quad + e^{-r_1 t}[1-a_2+(1-z_2)(1-F_2(t))((1-q_2(t))wD+q_2(t)D)-c_1]. \end{aligned}$$

Player 1's expected utility from strategy $\Sigma_1$ is

$$u_1(\Sigma_1,\Sigma_2)=\int_0^\infty u_1(s,\Sigma_2)dF_1(s)+\int_0^\infty v_1(s,\Sigma_2)dG_1(s).$$

Player 2's expected utility from conceding at time $t$ and yielding to a challenge with probability $q_2(s)$ at time $s$ is

$$\begin{aligned} u_2(t,q_2,\Sigma_1) &= (1-z_1)\int_0^t a_2 e^{-r_2 s} dF_1(s)+z_1\int_0^t (1-a_1)e^{-r_2 s}\gamma_1 e^{-\gamma_1 s}ds \\ &\quad +(1-z_1)\int_0^t [1-a_1+(1-q_2(s))((1-w)D-c_2)]e^{-r_2 s}dG_1(s) \\ &\quad +e^{-r_2 t}(1-a_1)(1-(1-z_1)F_1(t)-(1-z_1)G_1(t)-z_1(1-e^{-\gamma_1 t})) \\ &\quad +(1-z_1)\left[F_1(t)-\lim_{s\uparrow t}F_1(s)\right]\frac{a_2+1-a_1}{2}. \end{aligned} \tag{3}$$

Player 2's expected utility from strategy $\Sigma_2$ is

$$u_2(\Sigma_2,\Sigma_1)=\int_0^\infty u_2(s,q_2,\Sigma_1)dF_2(s).$$

### 3.1.2 Strategies

**Player 2's optimal yielding strategy.** We first consider the best response of player 2 when she faces a challenge and believes that the challenging player 1 is justified with probability $v_1$. If she responds to the challenge, her expected payoff is $v_1(1-a_1)+(1-v_1)(1-a_1+(1-w)D)-c_2 = 1-a_1+(1-v_1)(1-w)D-c_2$; if she yields to the challenge, her expected payoff is $1-a_1$. She is indifferent between the two actions when $v_1 = 1-\frac{c_2}{(1-w)D} \equiv v_1^*$. She strictly prefers to respond to the challenge if the challenging player is justified with a probability strictly lower than $v_1^*$, and strictly prefers to yield to the challenge if the challenging player is justified with a probability strictly higher than $v_1^*$.

**Player 1's optimal challenging strategy.** We now consider the optimal challenging strategy of player 1 when he believes that player 2 is justified with probability $\mu_2$ and an unjustified player 2 concedes to a challenge with probability $q_2$. The expected utility when he challenges is $1-$

[5]We assume that whenever concession and challenge occur simultaneously, the outcome is determined by the concession. It is an innocuous assumption because simultaneous concession and challenge occur with probability 0 in equilibrium.

$a_2 + (1-\mu_2)q_2D + (1-\mu_2)(1-q_2)wD - c_1$. The expected utility when he does not challenge is his continuation value, which is $1-a_2$ on the equilibrium path. He is indifferent if $\mu_2 = 1 - \frac{c_1}{(q_2+(1-q_2)w)D}$.

**Candidate equilibrium challenging and yielding strategies.** If player 2 is justified with a probability strictly higher than $\mu_2^* \equiv 1 - \frac{c_1}{D}$, an unjustified player 1 strictly prefers not to challenge. If player 2 is justified with a probability strictly less than $\mu_2^*$, an unjustified player 1 must challenge at rate $\chi_1$ to make player 2 believe that the challenging player 1 is justified with probability $v_1^* \equiv 1 - \frac{c_2}{(1-w)D}$:

$$\frac{\mu_1\gamma_1}{\mu_1\gamma_1 + (1-\mu_1)\chi_1} = v_1^* \Rightarrow \chi_1(\mu_1) = \frac{1-v_1^*}{v_1^*}\frac{\mu_1}{1-\mu_1}\gamma_1.$$

If an unjustified player 1 challenges at a rate strictly higher/lower than the specified rate, then an unjustified player 2 is strictly better/worse off responding than yielding to the challenge. To make player 1 indifferent between challenging and not challenging, player 2 concedes to a challenge with probability

$$q_2(\mu_2) = \frac{1}{1-w}\left[\frac{c_1}{D}\frac{1}{1-\mu_2} - w\right].$$

**Players' conceding strategies.** In equilibrium, players concede at the same rates as in Abreu and Gul (2000). Players are indifferent between conceding and waiting to concede the next instant. Players concede at a rate to make their opponents indifferent between conceding and not conceding.

$$1 - a_j = \lambda_j dt \cdot a_i + e^{-r_i dt} \cdot (1-a_j)(1-\lambda_j dt),$$

$$\lambda_i = r_j(1-a_i)/D.$$

### 3.1.3 Reputation

**Player 1's Reputation**

Player 1's equilibrium reputation dynamics is different from the AG case. There are two strategy phases.

**Player 1's Reputation in Challenging Phase.** When $\mu_2 \leq \mu_2^*$, player 1 challenges at a positive rate. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathscr{F}_t]$, we have

$$\begin{aligned}\mu_1(t) \quad = \quad & \mu_1(t)\gamma_1 dt \cdot 1 + (1-\mu_1(t))\chi_1(t)dt \cdot 0 + \lambda_1 dt \cdot 0 + \\ & [1-\mu_1(t)\gamma_1 dt - (1-\mu_1(t))\chi_1(t)dt - \lambda_1 dt]\mu_1(t+dt).\end{aligned}$$

Rearranging the equation and following the equilibrium property that $\mu_1\gamma_1 + (1-\mu_1)\chi_1(t) = \mu_1(t)\frac{v_1^*}{\gamma_1}$, we get

$$\mu_1(t+dt) - \mu_1(t) = -\mu_1(t)\gamma_1 dt + \mu_1(t)\frac{\gamma_1}{v_1^*} dt \mu_1(t+dt) + \lambda_1 dt \mu_1(t+dt).$$

Dividing both sides by $dt$ and taking $dt \to 0$, we have that player 1's reputation follows the following Bernoulli ODE:

$$\mu_1'(t) = (\lambda_1 - \gamma_1)\mu_1(t) + \frac{\gamma_1}{v_1^*}\mu_1^2(t). \tag{4}$$

**Player 1's Reputation in Non-Challenging Phase.** When $\mu_2 > \mu_2^*$, player 1 does not challenge. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathscr{F}_t]$, we have

$$\mu_1(t) = \mu_1(t)\gamma_1 dt \cdot 1 + \lambda_1 dt \cdot 0 + [1 - \mu_1(t)\gamma_1 dt - \lambda_1 dt]\mu_1(t+dt). \tag{5}$$

Rearrange,

$$\frac{\mu_1(t+dt) - \mu_1(t)}{dt} = -\mu_1(t)\gamma_1 + \mu_1(t)\mu_1(t+dt)\gamma_1 + \lambda_1\mu_1(t+dt).$$

Taking $dt \to 0$, we have that player 1's reputation follows the following Bernoulli ODE:

$$\mu_1'(t) = (\lambda_1 - \gamma_1)\mu_1(t) + \gamma_1\mu_1^2(t).$$

**Player 2's Reputation**

Player 2's reputation dynamics is the same as in the no-challenge benchmark model. Following the Martingale property $\mu_i(t) = E[\mu_i(t+dt)|\mathscr{F}_t]$, we have

$$\mu_2(t) = \lambda_2 dt \cdot 0 + (1-\lambda_2)dt \cdot \mu_2(t+dt).$$

Rearranging, we get

$$\mu_2(t+dt) - \mu_2(t) = -\lambda_2 dt \mu_2(t+dt).$$

Dividing both sides by $dt$ and taking $dt \to 0$, we get

$$\mu_2'(t) = \lambda_2\mu_2(t).$$

Let $t(1;\mu_2^*,\lambda_2,0)$ denote the time length it takes player 2 to reach reputation 1 from reputation $\mu_2^*$ when the reputation follows the dynamics above.

#### 3.1.4 Initial Concession Stage

**Lemma 1.** *The solution to the Bernoulli ordinary differential equation*

$$\mu'(t) = A\mu(t) + B\mu^2(t)$$

*given* $\mu(0) = \mu^0$ *is*

$$\mu(t;\mu^0,A,B) = 1 \bigg/ \left[\left(\frac{1}{\mu^0} + \frac{B}{A}\right)\exp(-At) - \frac{B}{A}\right]$$

*if* $A \neq 0$ *and*

$$\mu(t;\mu^0,A,B) = 1 \bigg/ \left[-Bt + \frac{1}{\mu^0}\right]$$

*if* $A = 0$*. If* $\mu^0 > -A/B$*, then* $\mu'(t) > 0$ *for all* $t \geq t^0$*, and the time length it takes to reach reputation* $\mu > \mu^0$ *from* $\mu^0$ *is*

$$t(\mu;\mu^0,A,B) = \frac{1}{A}\ln\left(\frac{\frac{1}{\mu^0} + \frac{B}{A}}{\frac{1}{\mu} + \frac{B}{A}}\right).$$

We have the key equilibrium property that players reach reputation 1 at the same time. Before the end of the game, when player 2's reputation is between $\mu_2^*$ and 1, player 1 does not challenge and his reputation evolves according to equation (5). (When players' initial reputations are low enough,) this challenging phase lasts as long as player 2's reputation evolves from $\mu_2^*$ to 1, that is, $t_2^N = t(1;\mu_2^*,\lambda_2,0)$, explicitly, $t_2^N = -\lambda_2 \ln(\mu_2^*)$. Player 1's reputation evolves from $\mu_1^N$ to 1 during this period, where $\mu_1^N$ is determined by $\mu(-t_2^N;1,\lambda_1-\gamma_1,\gamma_1)$. Explicitly,

$$\mu_1^N = \frac{\lambda_1 - \gamma_1}{\lambda_1(\mu_2^*)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \gamma_1} \leq 1.$$

Before player 1's reputation reaches $\mu_1^N$, player 1's reputation evolves according to equation (4). In summary, equilibrium reputation evolves as follows.

**Lemma 2.** *Define*

$$\mu_2(-t) = \mu(-t;1,\lambda_2,0)$$

*and*

$$\mu_1(-t) = \begin{cases} \mu(-t;1,\lambda_1-\gamma_1,\frac{\gamma_1}{v_1^*}) & t < t_2^N \\ \mu(t_2^N - t;\mu_1^N,\lambda_1-\gamma_1,\gamma_1) & t \geq t_2^N \end{cases}.$$

*Player i's reputation in equilibrium is*

$$\widehat{\mu}_i(T-t) = \mu_i(-t),$$

*where* $T = \min\{T_1,T_2\}$*, and* $T_i$ *solves* $\mu_i(-T_i) = z_i$*.*

#### 3.1.5 Equilibrium

We summarize the equilibrium strategies in the proposition below. We show rigorously that there is indeed a unique equilibrium and that the utilities and strategies are differentiable in time.

**Proposition 3.1.** *For any bargaining game $\{\pi_i, z_i, r_i, \gamma_i, c_i, w_i\}_{i=1}^2$ with $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, and $\gamma_2 = 0$, there exists a unique sequential equilibrium $(\widehat{\Sigma}_1, \widehat{\Sigma}_2) = ((\widehat{F}_1, \widehat{G}_1), (\widehat{F}_2, \widehat{q}_2))$, where $\widehat{F}_i(t) = \frac{1-C_i e^{-\lambda_i t}}{1-z_i}$, $\widehat{G}_1(t) = 1 - e^{-\int_0^t \chi_1(s)ds}$, $\widehat{\chi}_1(s) = 1_{s<T-t_2^N} \cdot \frac{1-v^*}{v^*} \frac{\widehat{\mu}_1(s)}{1-\widehat{\mu}_1(s)} \gamma_1$, $\widehat{q}_2(s) = 1_{s<T-t_2^N} \cdot \frac{1}{1-w} \left[ \frac{c_1}{D} \frac{1}{1-\mu_2(s)} - w \right]$, and $\widehat{\mu}_i$ is defined as in lemma 2.*

**Proof of Proposition 3.1.** Let $\Sigma = (\Sigma_1, \Sigma_2)$ define a sequential equilibrium. We will argue that $\Sigma$ must have the form specified (i.e., uniqueness) and that these strategies do indeed define an equilibrium (i.e., existence). Let $u_i(t)$ denote the expected utility of an unjustified player $i$ who concedes at time $t$. Define $T_i := \{t | u_i(t) = \max_s u_i(s)\}$ as the set of conceding times that attain the highest expected utility for player $i$; since $\Sigma$ is a sequential equilibrium, $T_i$ is nonempty for $i = 1, 2$. Furthermore, define $\tau_i = \inf\{t \geq 0 | F_i(t) = \lim_{t' \to \infty} F_i(t')\}$ as the time of last concession for player $i$, where $\inf \emptyset := \infty$. Then we have the following results.

**(a) The last instant at which two unjustified players concede is the same: $\tau_1 = \tau_2$.**
An unjustified player will not delay conceding once she knows that the opponent will never concede. Denote the last concession time by $\tau$.

**(b) If $F_i$ jumps at $t \in \mathbb{R}$, then $F_j$ does not jump at $t$ for $j \neq i$.**
If $F_i$ had a jump at $t$, then player $j$ receives a strictly higher utility by conceding an instant after $t$ than by conceding exactly at $t$.

**(c) If $F_2$ is continuous at time $t$, then $u_1(s)$ is continuous at $s = t$. If $F_1$ and $G_1$ are continuous at time $t$, then $u_2(s)$ is continuous at $s = t$.**
These claims follow immediately from the definition of $u_1(s)$ in equation (2) and the definition of $u_2(s)$ in equation (3), respectively.

**(d) If $G_1$ is continuous, there is no interval $(t', t'')$ such that $0 \leq t' < t'' \leq \tau$ where both $F_1$ and $F_2$ are constant on the interval $(t', t'')$.**
Assume the contrary and without loss of generality, let $t^* \leq \tau$ be the supremum of $t''$ for which $(t', t'')$ satisfies the above properties. Fix $t \in (t', t^*)$ and note that for $\varepsilon$ small enough there exists $\delta$ such that $u_i(t) - \delta > u_i(s)$ for all $s \in (t^* - \varepsilon, t^*)$; in words, conditional on the opponent not conceding in an interval, it is strictly better for a player to concede earlier within that interval, and it is sufficiently significantly better by conceding early than by conceding close to the end of the time interval. By (b) and (c), there exists $i$ such that $u_i(s)$ is continuous at $s = t^*$, so for some $\eta > 0$, $u_i(s) < u_i(t)$ for all $s \in (t^*, t^* + \eta)$; in words, because of the continuity of the expected utility function at time $t^*$, the expected utility of conceding a bit after time $t^*$ is still lower than the expected utility of conceding at time $t$ within the time interval. Since $F_i$ is

optimal, $F_i$ must be constant on the interval $(t', t^* + \eta)$. The optimality of $F_i$ implies $F_j$ is also constant on the interval $(t', t^* + \eta)$, because player $j$ is strictly better off conceding before or after the interval than conceding during the interval. Hence, both functions are constant on the interval $(t', t^* + \eta) \subseteq (t', \tau)$. However, this contradicts the definition of $t^*$.

**(e) If $t' < t'' < \tau$, then $F_i(t'') > F_i(t')$ for $i = 1, 2$.**

If $F_i$ is constant on some interval, then the optimality of $F_j$ implies that $F_j$ is constant on the same interval, for $j \neq i$. However, (d) shows that $F_1$ and $F_2$ cannot be simultaneously constant.

**(f) Cumulative concession probability $F_i$, $i = 1, 2$, is continuous at $t > 0$.**

Assume the contrary: suppose $F_i$ has a jump at time $t$. Then $F_j$ is constant on interval $(t - \varepsilon, t)$ for $j \neq i$. This contradicts (e).

**(g) Cumulative ultimatum probability $G_1$ is continuous at $t > 0$.**

Suppose to the contrary that $G_1$ jumps at time $t$, that is, an unjustified player 1 challenges with a positive probability. Given that an unjustified player 1 challenges with a positive probability and a justified player 1 challenges with probability 0, player 2 believes that a challenging player is unjustified with probability 1. Consequently, she is strictly better off responding to the challenge (obtaining a payoff of $1 - a_1 + (1 - w)D - c_2$, which is greater than $1 - a_1$ by the assumption that $(1 - w)D > c_2$) than yielding to the challenge (obtaining a payoff of $1 - a_1$). An unjustified player 1's payoff from challenging is less than $1 - a_1 + wD - c_1$, which is strictly less than his payoff from conceding, because $wD < c_1$.

**(h) Player 1's continuation payoff at time $t > 0$ in any equilibrium is $1 - a_2$.**

Suppose to the contrary that player 1's continuation payoff is strictly higher than $1 - a_2$ at time $t$. There exists $\varepsilon > 0$ such that an unjustified player 1 must not have conceded at time $s \in (t - \varepsilon, t)$, that is, $F_1(s)$ is constant on the interval $(t - \varepsilon, t)$. The optimality of conceding implies that player 2 must also have not conceded on the interval $(t - \varepsilon, t)$, that is, $F_2(s)$ is constant on the interval $(0, t)$. However, the fact that both $F_1$ and $F_2$ are constant on an open interval contradicts (d).

**(i) Player 2's continuation payoff at time $t > 0$ is $1 - a_1$.**

This result follows immediately from the fact that $F_2(t)$ is continuously strictly increasing.

From (e) it follows that $T_i$ is dense in $[0, \tau]$ for $i = 1, 2$. From (c), (f), and (g), it follows that $u_i(s)$ is continuous on $(0, \tau]$ and hence $u_i(s)$ is constant for all $s \in (0, \tau]$. Consequently, $T_i = (0, \tau]$. Hence, $u_i(t)$ is differentiable as a function of $t$ and $du_i(t)/dt = 0$ for all $t \in (0, \tau)$. The expected utility is

$$u_i(t) = (1 - z_j) \int_0^t a_i e^{-r_i s} dF_j(s) + (1 - a_j) e^{-r_i t} (1 - (1 - z_j) F_j(t)). \tag{6}$$

The differentiability of $F_j$ follows from the differentiability of $u_i(t)$ on $(0,\tau)$. Differentiating equation (6) and applying Leibnitz's rule, we obtain

$$0 = a_i e^{-r_i t}(1-z_j) f_j(t) - (1-a_j) r_i e^{-r_i t}(1-(1-z_j)F_j(t)) - (1-a_j)e^{-r_i t}(1-z_j) f_j(t)$$

where $f_j(t) = dF_j(t)/dt$. This in turn implies $F_j(t) = \frac{1-C_j e^{-\lambda_j t}}{1-z_j}$, where $C_j$ is yet to be determined. At $\tau_1 = \tau_2$, optimality for player $i$ implies $F_i(\tau_i) = 1$. At $t = 0$, if $F_j(0) > 0$ then $F_i(0) = 0$ by (b).

From (c) and (f), it follows that $v_1(t)$ is continuous on $(0,\tau]$. Furthermore, $v_1(t)$ is strictly smaller than $1-a_1$ when $q_2(t) = 1$ and $F_2(t) > 1 - \frac{1}{1-z_2}\frac{c_2}{D}$. Therefore, after time $t^*$, a strategic player 1 does not challenge.

□

## 3.2 The Multiple-Type Case

**Proposition 3.2.** *For any bargaining game $\{z_i, \pi_i, r_i, \gamma_i, c_i, w_i\}_{i=1}^2$ with $A_1 = \{a_1\}$, $|A_2| > 1$, and $\gamma_2 = 0$, there exists a unique sequential equilibrium $(\widehat{\Sigma}_1, \widehat{\Sigma}_2) = ((\widehat{F}_1, \widehat{G}_1), (\widehat{\sigma}_2, \widehat{F}_2, \widehat{q}_2))$.*

**Proof of Proposition 3.2.** For any $a_1 \in A_1$ and $x \in (0,1]$, denote by $B_1(a_1, x)$ the bargaining game in which player 1 is persistent to demand $a_1$ with probability $x$ and is strategic otherwise. We will show that there is a unique sequential equilibrium of the game $B_1(a_1, x)$.

If unjustified player 2 chooses some $a_2 \le 1 - a_1$, then the game ends at time zero. If player 2 chooses some $a_2 > 1 - a_1$, then the game does not end at time zero, but from proposition 3.1 we know the unique sequential equilibrium for any $t > 0$.

Denote by $\sigma_2(\cdot)$, a probability distribution over $A_2 \cup \{Q\}$, a mimicking strategy of unjustified player 2. Since mimicking $a_2 < 1 - a_1$ is never optimal and mimicking $a_2 = 1 - a - 1$ is equivalent to conceding, we assume that in equilibrium $\sigma_2(a_2) = 0$ for all $a_2 \le 1 - a_1$. If $x = 1$, then in equilibrium $\sigma_2(Q) = 1$, because unjustified player 2 will not delay conceding if she knows that player 1 is justified. For the remainder of the proof we assume $x < 1$.

It remains to be shown that unjustified player 2's equilibrium behavior $\widehat{\sigma}_2(\cdot)$ and unjustified player 1's conceding behavior $\widehat{F}_1(\cdot|a_1, \cdot)$ at time zero are uniquely determined. Subsequently, we provide a series of definitions and use them to prove a series of claims that lead to equilibrium existence and uniqueness.

Denote by

$$y^*(a_2, \sigma_2) = \frac{z_2 \pi_2(a_2)}{z_2 \pi_2(a_2) + (1-z_2)\sigma_2}$$

player 2's initial reputation given that player 2 mimics $a_2$ with probability $\sigma_2$. Note that

$y^*(a_2,\sigma_2)$ is continuous and strictly decreasing in $\sigma_2$.

Denote by $T_1(a_1,a_2,x)$ and $T_2(a_1,a_2,y)$ the time it takes for the reputation of player $i$ to reach 1, given player $i$'s initial reputation ($x$ or $y$) and given that neither player concedes with a positive probability after initial demands $a_1$ and $a_2$ are announced. Explicitly,

$$T_1(a_1,a_2,x) \equiv \begin{cases} \infty & x \le 1-\frac{\lambda_1}{\gamma_1} \\ t(\mu_1^N; x, \lambda_1-\gamma_1, \frac{\gamma_1}{v_1^*}) + t(1;\mu_1^N, \lambda_1-\gamma_1,\gamma_1) & 1-\frac{\lambda_1}{\gamma_1} < x < \mu_1^N \\ t(1;x,\lambda_1-\gamma_1,\gamma_1) & \mu_1^N \le x \le 1 \end{cases},$$

that is,

$$T_1(a_1,a_2,x) \equiv \begin{cases} \infty & \text{if } x \le 1-\frac{\lambda_1}{\gamma_1} \\ \frac{1}{\lambda_1-\gamma_1}\log\left[\frac{\frac{\lambda_1-\gamma_1}{x}+\frac{\gamma_1}{v_1^*}}{\frac{\lambda_1-\gamma_1}{\mu_1^N}+\frac{\gamma_1}{v_1^*}}\right] - \frac{1}{\lambda_2}\log\mu_2^* & \text{if } 1-\frac{\lambda_1}{\gamma_1} < x < \mu_1^N \\ \frac{1}{\lambda_1-\gamma_1}\log\left[\frac{\frac{\lambda_1-\gamma_1}{x}+\gamma_1}{\lambda_1}\right] & \text{if } \mu_1^N \le x \le 1 \end{cases},$$

and

$$T_2(a_1,a_2,y) \equiv -\frac{a_1+a_2-1}{r_1(1-a_2)}\log y.$$

Note that $T_1(a_1,a_2,x)$ is continuous and strictly decreasing in $x$ on $(1-\frac{\lambda_1}{\gamma_1},1)$ and that $T_2(a_1,a_2,y)$ is continuous and strictly decreasing in $y$ on $(0,1)$.

Denote by $\overline{\sigma}_2(a_1,a_2,x)$ the maximum probability with which player 2 can mimic $a_2$ in order for player 2's reputation to reach 1 weakly before player 1's reputation does, given that player 1's initial reputation at time zero is $x$ and given that neither player concedes with a positive probability after initial demands $a_1$ and $a_2$ are announced. Formally, $\overline{\sigma}_2(a_1,a_2,x)$ is 1, if $x \le 1-\frac{\lambda_1}{\gamma_1}$ and $\overline{\sigma}_2(a_1,a_2,x)$ is the unique value of $\sigma_2$ that solves $T_1(a_1,a_2,x) = T_2(a_1,a_2,y^*(a_2,\sigma_2))$ if $x > 1-\frac{\lambda_1}{\gamma_1}$. Explicitly, let $\Delta \equiv 1-\frac{\gamma_1}{\lambda_1}$,

$$\overline{\sigma}_2(a_1,a_2,x) \equiv \begin{cases} 1 & x \le 1-\frac{\lambda_1}{\gamma_1} \\ z_2\frac{\pi_2(a_2)}{1-\pi_2(a_2)}\left[\left(\frac{\Delta\frac{1}{x}+(1-\Delta)\frac{1}{v_1^*}}{\Delta\frac{1}{\mu_1^N}+(1-\Delta)\frac{1}{v_1^*}}\right)^{\frac{\lambda_2}{\lambda_1}\frac{1}{\Delta}}-1\right] & 1-\frac{\lambda_1}{\gamma_1} < x < \mu_1^N \\ z_2\frac{\pi_2(a_2)}{1-\pi_2(a_2)}\left[\left(\Delta\frac{1}{x}+(1-\Delta)\right)^{\frac{\lambda_2}{\lambda_1}\frac{1}{\Delta}}-1\right] & \mu_1^N \le x \le 1 \end{cases}$$

Note that in equilibrium $\sigma_2(a_2) \le \overline{\sigma}_2(a_1,a_2,\sigma_2)$ for all $a_2 > 1-a_1$. To see why this claim must hold, suppose player 2 mimics $a_2$ with a probability strictly higher than $\overline{\sigma}_2(a_1,a_2,\sigma_2) < 1$. Then player 2 needs to concede with a strictly positive probability at time zero in order for players' reputations to reach 1 at the same time. However, we have specified that player 2 does not concede at time zero after announcing her demand.

Denote by

$$\Delta(a_1,x) \equiv \left\{ \sigma_2(\cdot) \in \Delta \;\middle|\; \begin{array}{ll} \sigma_2(a_2) = 0 & \forall a_2 \le 1-a_1 \\ \sigma_2(a_2) \le \overline{\sigma}_2(a_1,a_2,x) & \forall a_2 > 1-a_1 \end{array} \right\}$$

the set of candidate equilibrium strategies in the game $B_1(a_1,x)$, where $\Delta$ denotes the set of all probability distributions on $A_2 \cup \{Q\}$. Note that $\Delta(a_1,x)$ is nonempty, convex, and compact.

Denote by $x^*(a_1,a_2,\sigma_2)$ player 1's initial reputation in order for both players' reputations to reach 1 at the same time, given that $a_2$ is chosen with probability $\sigma_2$ and given that neither player concedes with a positive probability at time zero after initial demands $a_1$ and $a_2$ are announced. Formally, $x^*(a_1,a_2,\sigma_2)$ is the unique value of $x$ that solves $T_1(a_1,a_2,x) = T_2(a_1,a_2,y^*(a_2,\sigma_2))$. Explicitly,

$$x^*(a_1,a_2,\sigma_2) \equiv \begin{cases} \dfrac{\lambda_1-\gamma_1}{\left(\frac{\mu_2^*}{y}\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\left[\frac{\lambda_1-\gamma_1}{\mu_1^N}+\frac{\gamma_1}{\nu_1^*}\right]-\frac{\gamma_1}{\nu_1^*}} & \text{if } y^*(a_2,\sigma_2) \le \mu_2^* \\ \dfrac{1-\frac{\gamma_1}{\lambda_1}}{\left(\frac{1}{y}\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}-\frac{\gamma_1}{\lambda_1}} & \text{if } y^*(a_2,\sigma_2) > \mu_2^* \end{cases}.$$

Denote by $F_1^*(0|a_1,a_2,x,\sigma_2)$ the probability with which player 1 must concede at time zero so that the two players' reputations reach 1 at the same time, given player 1's initial reputation $x \le x^*(a_1,a_2,\sigma_2)$ and given that unjustified player 2 chooses $a_2$ with probability $\sigma_2$. In other words, if unjustified player 2 chooses $a_2$ with probability $\sigma_2$, then the probability that player 1 concedes at time zero is $F_1^*(0|a_1,a_2,x,\sigma_2)$. Formally, $F_1^*(0|a_1,a_2,x,\sigma_2)$ is the unique value of $F_1$ that solves

$$x^*(a_1,a_2,\sigma_2) = \frac{x}{x+(1-x)(1-F_1)}.$$

Explicitly,

$$F_1^*(0|a_1,a_2,x,\sigma_2) \equiv 1 - \frac{x}{1-x}\bigg/\frac{x^*(a_1,a_2,\sigma_2)}{1-x^*(a_1,a_2,\sigma_2)}.$$

Note that $F_1^*(0|a_1,a_2,x,\sigma_2)$ is continuous and strictly decreasing in $\sigma_2$.

Denote by $u_2^*(a_1,a_2,x,\sigma_2)$ player 2's utility of mimicking $a_2$ in the game $B_1(a_1,x)$ given that she mimics $a_2$ with probability $\sigma_2$ in equilibrium. Explicitly,

$$u_2^*(a_1,a_2,x,\sigma_2) \equiv 1-a_1+(1-x)F_1^*(0|a_1,a_2,x,\sigma_2)(a_1+a_2-1).$$

To see why the equation holds, note that player 2's payoff is $a_2$ if player 1 concedes at time zero and is $1-a_1$ if player 1 does not concede at time zero, since in equilibrium she is indifferent between conceding and not conceding at every time $t \geq 0$. Player 1 concedes with probability $(1-x)F_1^*(0|a_1,a_2,x,\sigma_2)$ in equilibrium. Furthermore, note that $u_2^*(a_1,a_2,x,\sigma_2)$ is continuous and strictly decreasing in $\sigma_2$.

For any $\sigma_2(\cdot) \in \Delta(a_1,x)$, define

$$\widehat{u}_2(x,\sigma_2(\cdot)) \equiv \begin{cases} \min\limits_{a_2:\sigma_2(a_2)>0} u_2^*(a_1,a_2,x,\sigma_2(a_2)) & \text{if } \sigma_2(Q)=0 \\ 1-a_1 & \text{if } \sigma_2(Q) \neq 0 \end{cases}.$$

Note that $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy if and only if $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$. ($\Rightarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy. Any equilibrium strategy $\sigma_2(\cdot)$ satisfies that for all $a_2 \in A_2 \cup \{Q\}$ such that $\sigma_2(a_2) > 0$, $u_2^*(a_1,a_2,x,\sigma_2(a_2))$ is the same. If $\sigma_2(Q) > 0$, then $u_2^*(a_1,a_2,x,\sigma_2(a_2)) = 1-a_1$; if $\sigma_2(Q) = 0$, then $u_2^*(a_1,a_2,x,\sigma_2(a_2)) = \min_{a_2:\widehat{\sigma}_2(a_2)>0} u_2(a_1,a_2,x,\widehat{\sigma}_2(a_2))$. Hence, any equilibrium strategy $\sigma_2(\cdot)$ must generate an equilibrium utility of $\widehat{u}_2(x,\sigma_2(\cdot))$. Hence, $\widehat{\sigma}_2(\cdot)$ maximizes $\widehat{u}_2(x,\sigma_2(\cdot))$ among all candidate equilibrium strategies $\sigma_2(\cdot)$. ($\Leftarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$. By the strict monotonicity of $u_2^*(a_1,a_2,x,\cdot)$, for all $a_2 \in A_2$ such that $\widehat{\sigma}_2(a_2) > 0$, $u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2)) = \widehat{u}_2(x,\widehat{\sigma}_2(\cdot))$. Coupled with the fact that $\widehat{\sigma}_2(\cdot)$ is the feasible strategy that maximizes $\widehat{u}_2(x,\sigma_2(\cdot))$, $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy.

Define $\Gamma$, a correspondence from $\Delta(a_1,x)$ to $\Delta(a_1,x)$, as follows:

$$\Gamma(\sigma_2(\cdot)) \equiv \{\widetilde{\sigma}_2(\cdot) \in \Delta(a_1,x) | \widetilde{\sigma}_2(a_2) > 0 \Rightarrow u_2^*(a_1,a_2,x,\sigma_2(a_2)) \geq u_2^*(a_1,a_2',x,\sigma_2(a_2'))\ \forall a_2' \in A_2\}.$$

Note that $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$ if and only if $\widehat{\sigma}_2(\cdot)$ is a fixed point of $\Gamma$. ($\Rightarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$. By the argument above, $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy. Therefore, $\widehat{\sigma}_2(a_2) > 0$ implies $u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2)) \geq u_2^*(a_1,a_2',x,\widehat{\sigma}_2(a_2'))$ for any $a_2' \in A_2$. By the definition of $\Gamma$, $\widehat{\sigma}_2(\cdot) \in \Gamma(\widehat{\sigma}_2(\cdot))$. ($\Leftarrow$) Suppose $\widehat{\sigma}_2(\cdot) \in \Gamma(\widehat{\sigma}_2(\cdot))$. By the definition of $\Gamma$, $\widehat{\sigma}_2(a_2) > 0$ implies $u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2)) \geq u_2^*(a_1,a_2',x,\widehat{\sigma}_2(a_2'))$ for any $a_2' \in A_2$. Assume by contradiction that $\widehat{\sigma}_2(\cdot)$ does not solve $\max_{\sigma_2(\cdot)\in\Delta(a_1,x)} \widehat{u}_2(x,\sigma_2(\cdot))$ but $\widetilde{\sigma}_2(\cdot) \neq \widehat{\sigma}_2(\cdot)$ does. There must exist an $a_2 \in A_2$ such that $\widehat{\sigma}_2(a_2) > 0$ and $\widetilde{\sigma}_2(a_2) < \widehat{\sigma}_2(a_2)$ (otherwise, if $\widetilde{\sigma}_2(a_2) \geq \widehat{\sigma}_2(a_2)$ for all $a_2$ such that $\widehat{\sigma}_2(a_2) > 0$, then by the strict monotonicity of $u_2^*$, $u_2^*(a_1,a_2,x,\widetilde{\sigma}_2(a_2)) \leq u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2))$, and $\widehat{u}_2(x,\widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x,\widehat{\sigma}_2(\cdot))$). However, that implies that there exists $a_2' \in A_2 \cup \{Q\}$ such that $\widetilde{\sigma}(a_2') > \widetilde{\sigma}(a_2')$. If $a_2' = Q$, then $\widehat{u}_2(x,\widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x,\widehat{\sigma}_2(\cdot))$. If $a_2' \in A_2$, then $\widehat{u}_2(x,\widetilde{\sigma}_2(\cdot)) \leq \widehat{u}_2(x,\widehat{\sigma}_2(\cdot))$.

Hence, from the two claims above, we have that $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy for player 2 in the game $B_0(a_1,x)$ if and only if $\widehat{\sigma}_2(\cdot)$ is a fixed point of $\Gamma$.

Equilibrium existence follows from the existence of a fixed point of $\Gamma$ by Kakutani's fixed point theorem. By construction, $\Delta(a_1,x)$ is compact. By construction, $\Gamma$ is convex-valued. Finally, $\Gamma$ is upper-hemicontinuous because $u_2^*$ is continuous in its last argument.

Equilibrium uniqueness follows from the strict monotonicity of $u_2^*$. Suppose there are two equilibrium strategies $\widehat{\sigma}_2(\cdot)$ and $\widetilde{\sigma}_2(\cdot)$; without loss of generality, suppose $\widehat{\sigma}_2(a_2) > \widetilde{\sigma}_2(a_2) > 0$ for some $a_2 > 1-a_1$. The utilities of playing the two strategies are different: $\widehat{u}_2(x,\widehat{\sigma}_2(\cdot)) = u_2^*(a_1,a_2,x,\widehat{\sigma}_2(a_2)) < u_2^*(a_1,a_2,x,\widetilde{\sigma}_2(a_2)) = \widehat{u}_2(x,\widetilde{\sigma}_2(\cdot))$, where the strict inequality follows from the strict monotonicity of $u_2^*$. This contradicts the property that equilibrium strategies $\widehat{\sigma}_2(\cdot)$ and $\widetilde{\sigma}_2(\cdot)$ both maximize $\widehat{u}_2(x,\sigma_2(\cdot))$.

□

**Proposition 3.3.** *Denote by $u_1^*(a_1,x)$ the payoff of player 1 in the unique sequential equilibrium of the bargaining game $\{\pi, z_i, r_i\}_{i=1}^2$ with $A_1 = \{a_1\}$ and $|A_2| \geq 1$. It is a continuous function of $x$. Moreover, there exists an $\underline{x}$ such that $u_1^*(a_1,x) = u_1^*(a_1,\underline{x})$ for any $x \leq \underline{x}$ and $u_1^*(a_1,x)$ is strictly increasing in $x$ on the interval $(\underline{x},1)$.*

**Proof of Proposition 3.3.** TBA. □

**Proposition 3.4.** *For any bargaining game $\{\pi_i, z_i, r_i, \gamma_i, c_i, w_i\}_{i=1}^2$ with $\gamma_2 = 0$, there exists a sequential equilibrium $(\widehat{\Sigma}_1,\widehat{\Sigma}_2) = ((\widehat{\sigma}_1,\widehat{F}_1,\widehat{G}_1),(\widehat{\sigma}_2,\widehat{F}_2,\widehat{q}_2))$. Furthermore, all equilibria yield the same distribution over outcomes.*

**Proof of Proposition 3.4.** TBA. □

### 3.3 The Limiting Case of Complete Rationality

**Proposition 3.5.** *Let $B_0^n = \{A_i, z_i^n, \pi_i, r_i\}_{i=1}^2$ be a sequence of continuous-time bargaining games. If $\lim z_1^n = \lim z_2^n = 0$, $\lim z_1^n/(z_1^n + z_2^n) \in (0,1)$ and $v_i^n$ is the sequential equilibrium payoff for player $i$ in the game $B^n$, then*

$$\liminf v_1^n \geq \max\left\{ a \in A_1 \cup \{0\} \,\middle|\, a < \frac{r_2}{\max\{r_1,\gamma_1\} + r_2} \right\},$$

*and*

$$\liminf v_2^n \geq \max\left\{ a \in A_2 \,\middle|\, a < \frac{\max\{r_1,\gamma_1\}}{\max\{r_1,\gamma_1\} + r_2} \right\}.$$

**Proof of Proposition 3.5.** Suppose that the commitment type set for each player sufficiently "finely" covers the interval $[0,1]$. We are interested in the limit equilibrium payoffs as $z_1, z_2 \to 0$.

First, suppose $g_1 > r_1$. We first start with the case in which player 1 chooses a type $a_1 > \frac{r_2}{r_2+g_1}$ with a probability that doesn't disappear as $z_1 \to 0$. If player 2 chooses $a_2 \approx 1$, then we have

$$\lambda_1 = \frac{(1-a_1)r_2}{a_1+a_2-1} \approx \frac{(1-a_1)r_2}{a_1} < g_1.$$

Hence, Player 2's payoff from this strategy approaches to approximately 1, and hence player 1's payoff approaches to approximately to 0.

If Player 1 chooses a type $a_1 < \frac{r_2}{r_2+g_1}$, then regardless of $a_2$, $\lambda_1 > g_1$, hence we need to solve for the equilibrium dynamics of the model in which $\lambda_1 > g_1$.

Player 1 wins if $\lambda_1 - g_1 > \lambda_2$, and player 2 wins otherwise. (Why?)

Plugging in $\lambda_i$ the expression from Abreu and Gul, we have that Player 2 wins if

$$(1-a_1)r_2 - g_1 a_1 < (1-a_2)(r_1-g_1)$$

Because $g_1 > r_1$, the right hand side is negative, and left hand side is always positive, so Player 1 wins. (It's important for this argument that $z_1$ goes to zero at a rate not smaller than that of $z_2$.)

Since this is true for every $a_1 < \frac{r_2}{r_2+g_1}$, player 1, by choosing a demand approximately equal to $\frac{r_2}{r_2+g_1}$ (more precisely, $\max\{a_1 \in A_1 | a_1 < \frac{r_2}{r_2+g_1}\}$) guarantees this payoff, and cannot do better, and Player 2 gets the rest of the surplus.

Second, suppose $g_1 < r_1$. In this case, if player 1 chooses $a_1 = \frac{r_2}{r_2+r_1}$, then $\lambda_1 > g_1$ for any choice of $a_2$, so the winner is determined by comparison

$$(1-a_1)r_2 - g_1 a_1 < (1-a_2)(r_1-g_1)$$

which for the choice of $a_1 = \frac{r_2}{r_2+r_1}$ makes player 1 the winner, and for any choice of $a_1$ lower, makes player 2 the winner by a choice that makes player 2 have a payoff larger than $\frac{r_1}{r_1+r_2}$, that leaves player 1 with a payoff smaller than $\frac{r_2}{r_1+r_2}$. Hence the solution is similar to Abreu and Gul in this case. □

# 4 Bargaining with Two-Sided Ultimatums

A male player 1 and a female player 2 divide a unit pie. Each player is either (i) justified to demand a share of the pie and never accepting any offer below that, or (ii) unjustified to demand a share of the pie but nonetheless wanting as a big share of the pie as possible. A justified player can find hard evidence supporting his or her demand, but an unjustified player has no hard evidence supporting his or her claim of the share.

We initially assume that each player can be of a single justified type: with probability $z_1$ player 1 is justified to demand $a_1$ and with probability $z_2$ player 2 is justified to demand $a_2 > 1-a_1$. Let $D \equiv a_1+a_2-1$ denote the conflicting difference between the two players.

Time is continuous. At each instant $t$, each player can decide to give in to the other player's demand or hold on to his or her demand. In addition, player $i$ has a challenge opportunity. A justified player $i$ challenges when evidence arrives; the evidence arrives according to a Poisson process with arrival rate $\gamma_i > 0$. An unjustified player $i$ can challenge at any time but he will time his challenge strategically. If the players neither challenge nor concede, then the game continues. If player $i$ challenges at time $t$, he/she incurs a cost $c_i$ and player $j \neq i$ must respond to the challenge. Player $j$ may either yield to challenge and get $1-a_j$, or see the challenge by paying a cost $c_j$.

The shares of the pie are determined by the players' justified and unjustified types in the court, as follows. An unjustified player $i$'s payoff against a justified player $j$ is $1-a_j$. If two unjustified players meet, then the challenging player $i$ wins with probability $w < 1/2$: he gets $a_i$ with probability $w$ and gets $1-a_j$ with probability $1-w$, so his expected payoff is $wa_i+(1-w)(1-a_j) = 1-a_j+wD$, and the defending player $j$'s expected payoff is $(1-w)(1-a_i)+wa_j = 1-a_1+(1-w)D$. To make challenging and seeing a challenge worthwhile for player $i$, assume $wD < c_i < (1-w)D$ for $i = 1,2$.

In summary, the bargaining game $B(\{z_i,a_i,r_i,c_i,\gamma_i\}_{i=1}^{2},w)$ with two-sided ultimatums is described by players' prior probabilities $z_1$ and $z_2$ of being justified, demands $a_1$ and $a_2$, discount rates $r_1$ and $r_2$, challenge arrival rates $\gamma_1$ and $\gamma_2$, challenge costs $c_1$ and $c_2$, and an unjustified challenger's winning probability $w$ against an unjustified player.

## 4.1 The Single-Type Case

### 4.1.1 Formal Description of the Game

Let us formally describe the strategies and payoffs of the (unjustified) players when demands are fixed to be $a_1$ and $a_2$. Let $F_i(t)$ denote player $i$'s probability of conceding by time $t$. Let $G_i(t)$ denote player $i$'s probability of challenging by time $t$. Let $q_i(t)$ denote player $i$'s probability of conceding to a challenge at time $t$. Let $\Sigma_i = (F_i,G_i,q_i)$ denote an unjustified player $i$'s strategy.

Player $i$'s expected utility of taking no action at any time $s<t$ while yielding to a challenge with probability $q_i(s)$ at time $s<t$ is

$$\begin{aligned} U_i(t^-,q_i,\Sigma_j) &= (1-z_j)\int_0^t a_i e^{-r_i s} dF_j(s) + z_j \int_0^t (1-a_i)e^{-r_i s}\gamma_j e^{-r_i s} ds \\ &\quad +(1-z_j)\int_0^t \left[1-a_j+(1-q_i(s))((1-w)D-c_j)\right] e^{-r_j s} dG_i(s). \end{aligned}$$

Player $i$'s expected utility of conceding at time $t$ is

$$\begin{aligned} u_i(t,q_i,\Sigma_j) &= U_i(t^-,q_i,\Sigma_j)+e^{-r_it}(1-a_j)\left(1-(1-z_j)F_j(t)-(1-z_j)G_j(t)-z_j(1-e^{-\gamma_jt})\right) \\ &\quad +(1-z_j)\left[F_j(t)-\lim_{s\uparrow t}F_j(s)\right]\frac{a_i+1-a_j}{2}. \end{aligned} \tag{7}$$

Player $i$'s expected utility of challenging at time $t$ is

$$\begin{aligned} v_i(t,q_i,\Sigma_j) &= U_i(t^-,q_i,\Sigma_j)+e^{-r_it}\times \\ &\quad \left[1-a_i+\Big(1-(1-z_j)F_j(t)-(1-z_j)G_j(t)-z_j\Big)\Big(q_j(t)+(1-q_j(t))w\Big)D-c_i\right]. \end{aligned}$$

Player $i$'s expected utility from strategy $\Sigma_i$ is

$$u_i(\Sigma_i,\Sigma_j)=\int_0^\infty u_i(s,q_i,\Sigma_j)dF_i(s)+\int_0^\infty v_i(s,q_i,\Sigma_j)dG_i(s).$$

#### 4.1.2 Strategies

**Player $i$'s optimal yielding strategy.** We consider the best response of an unjustified player $i$ who faces a challenge and believes that the challenging player $j$ is justified with probability $\nu_j$. Responding to the challenge results in an expected utility of $1-a_j+(1-\nu_j)(1-w)D-c_i$, and yielding to the challenge results in an expected utility of $1-a_j$. An unjustified player $i$ is indifferent between responding and yielding when player $j\neq i$ is believed to be justified with probability $\nu_j=1-\frac{c}{(1-w)D}\equiv\nu_j^*$, strictly prefers to respond when $\nu_j<\nu_j^*$, and strictly prefers to yield when $\nu_j>\nu_j^*$.

**Player $i$'s optimal challenging strategy.** We consider the optimal challenging strategy of an unjustified player $i$ who believes that player $j\neq i$ is justified with probability $\mu_j$ and an unjustified player $j$ yields to a challenge with probability $q_j$. Challenging yields an expected utility of $1-a_j+(1-\mu_j)[q_j+(1-q_j)w]D-c_i$, and not challenging yields an expected utility of $1-a_j$ on any equilibrium path. An unjustified player $i$ is indifferent between challenging and not challenging if $\mu_j=1-c_1/[(q_j+(1-q_j)w)D]$.

**Candidate equilibrium challenging and yielding strategies.** If player $j$ is justified with a probability more than $\mu_j^*=1-\frac{c_i}{D}$, an unjustified player $i$ strictly prefers not to challenge. If player $j$ is justified with a probability less than $\mu_i^*$, an unjustified player $i$ must challenge at rate $\chi_j$ to make player $i$ believe that a challenging player $i$ is justified with probability $\nu_i^*\equiv1-\frac{c_j}{(1-w)D}$:

$$\frac{\mu_i\gamma_i}{\mu_i\gamma_i+(1-\mu_i)\chi_i}=\nu_i^*\Rightarrow\chi_i(\mu_i)=\frac{1-\nu_i^*}{\nu_i^*}\frac{\mu_i}{1-\mu_i}\gamma_i.$$

If an unjustified player $i$ challenges at a rate higher than the specified rate, then an unjustified player $j$ is strictly better off responding than yielding to the challenge. If an unjustified player $i$ challenges at a rate lower than the specified rate, then an unjustified player 2 is strictly worse off responding than yielding to the challenge. On the other hand, to make player $i$ indifferent between challenging and not challenging, player $j$ yields to a challenge with probability

$$q_j(\mu_j) = \frac{1}{1-w}\left[\frac{c_i}{D}\frac{1}{1-\mu_j} - w\right].$$

**Players' conceding strategies.** In equilibrium, players concede at the same rates as in Abreu and Gul (2000). Players are indifferent between conceding and waiting to concede the next instant. An unjustified player concedes at a rate to make the opposing unjustified player indifferent between conceding and not conceding.

$$1 - a_j = \lambda_j dt \cdot a_i + e^{-r_i dt} \cdot (1-a_j)(1-\lambda_j dt),$$

$$\lambda_i = r_j(1-a_i)/D.$$

### 4.1.3 Reputation

**Player $i$'s reputation in challenging phase.** When $\mu_j \le \mu_j^*$, an unjustified player $i$ challenges at a positive rate. Following the Martingale property, we have

$$\begin{aligned}\mu_i(t) \quad &= \quad \mu_i(t)\gamma_i dt \cdot 1 + (1-\mu_i(t))\chi_i(t)dt \cdot 0 + \lambda_i dt \cdot 0 \\ &\quad\quad [1 - \mu_i(t)\gamma_i dt - (1-\mu_i(t))\chi_i(t)dt - \lambda_i dt]\mu_i(t+dt).\end{aligned}$$

Rearranging the equation and following the equilibrium property that

$$\mu_i\gamma_i + (1-\mu_i)\chi_i(t) = \mu_i(t)\frac{v_i^*}{\gamma_i},$$

we get

$$\mu_i(t+dt) - \mu_i(t) = -\mu_i(t)\gamma_i dt + \mu_i(t)\frac{\gamma_i}{v_i^*}dt \cdot \mu_i(t+dt) + \lambda_i dt \cdot \mu_i(t+dt).$$

Dividing both sides by $dt$ and taking $dt \to 0$, we have that player $i$'s reputation follows the following Bernoulli ODE:

$$\mu_i'(t) = (\lambda_i - \gamma_i)\mu_i(t) + \frac{\gamma_i}{v_i^*}\mu_i^2(t).$$

**Player $i$'s reputation in non-challenging phase.** When $\mu_j > \mu_j^*$, an unjustified player $i$ does not challenge. Following the Martingale property, we have

$$\mu_i(t) = \mu_i(t)\gamma_i dt \cdot 1 + \lambda_i dt \cdot 0 + [1 - \mu_i(t)\gamma_i dt - \lambda_i dt]\mu_i(t+dt).$$

Taking $dt \to 0$, we have that player $i$'s reputation follows the following Bernoulli ODE:

$$\mu_i'(t) = (\lambda_i - \gamma_i)\mu_i(t) + \gamma_i\mu_i^2(t).$$

### 4.1.4 Initial Concession Stage

We still have the key equilibrium property that players reach reputation 1 at the same time (possibly at time infinity). Suppose the reputations reach 1 in finite time. We solve the equilibrium reputation dynamics backwards. Define $t_i^{NN} \equiv t(1; \mu_i^*, \lambda_i - \gamma_i, \gamma_i)$ as the time length it takes for player $i$'s reputation evolve from $\mu_i^*$ to 1 following the reputation dynamics in the no-challenge phase, and define $t^{NN} \equiv \min\{t_1^{NN}, t_2^{NN}\}$, the shorter time length that it takes to evolve from $\mu_i^*$ to 1 in the non-challenging phase. Let $I$ be the player (or one of the players) such that $t_I^N = t^{NN}$, and let $J \neq I$ denote the opposing player. Define $\mu_J^{NN} = \mu(-t^{NN}; 1, \lambda_J - \gamma_J, \gamma_J)$; note that if $t_I^N = t_J^N = t^{NN}$, then $\mu_J^{NN} = \mu_J^*$.

Time $t^{NN}$ before the last concession, because player $I$'s reputation drops below $\mu_I^*$, player $J$ challenges and player $I$ does not challenge, player $J$'s reputation evolves according to the dynamics in the challenging phase, and player $I$'s reputation evolves according to the dynamics in the non-challenging phase. Define $t^N \equiv t(\mu_J^{NN}; \mu_J^*, \lambda_J - \gamma_J, \frac{\gamma_J}{v_J^*})$ as the time length it takes for player $J$'s reputation to evolve from $\mu_J^*$ to $\mu_J^{NN}$. Let $\mu_I^N \equiv \mu(-t^N; \mu_I^*, \lambda_I - \gamma_I, \gamma_I)$ denote player $I$'s reputation when player $J$'s reputation is $\mu_J^*$.

Time $t^{NN} + t^N$ before the last concession, both players challenge, and their reputations evolve accordingly.

**Lemma 3.** *Define $t_i^{NN} \equiv t(1; \mu_i^*, \lambda_i - \gamma_i, \gamma_i)$. Let $t_I^{NN} \leq t_J^{NN}$. Define*

$$\mu_I(-t) \equiv \begin{cases} \mu(-t; 1, \lambda_I - \gamma_I, \gamma_I) & t \leq t^{NN} + t^N \\ \mu(-t; \mu_I^N, \lambda_I - \gamma_I, \frac{\gamma_I}{v_I^*}) & t > t^{NN} + t^N \end{cases}, \quad \mu_J(-t) \equiv \begin{cases} \mu(-t; 1, \lambda_J - \gamma_J, \gamma_J) & t \leq t^{NN} \\ \mu(-t; \mu_J^{NN}, \lambda_J - \gamma_J, \frac{\gamma_J}{v_J^*}) & t > t^{NN} \end{cases}.$$

*Player i's reputation in equilibrium is*

$$\widehat{\mu}_i(T - t) = \mu_i(-t),$$

*where $T = \min\{T_1, T_2\}$, and $T_i = \inf\{t; \mu_i(-t) = z_i\}$, $\inf \emptyset = \infty$.*

### 4.1.5 Equilibrium

**Proposition 4.1.** *Define $(\widehat{F}_i, \widehat{G}_i, \widehat{q}_i)$, where $\widehat{F}_i(t) = \frac{1 - C_i e^{-\lambda_i t}}{1 - z_i}$, $\widehat{G}_i(t) = 1 - e^{-\int_0^t \widehat{\chi}_i(s) ds}$, $\widehat{\chi}_i(s) = 1_{s \leq \widehat{t}_i} \cdot \frac{v_i}{1 - v_i^*} \frac{\widehat{\mu}_i(s)}{1 - \widehat{\mu}_i(s)} \gamma_i$, $\widehat{q}_i(s) = 1_{s \leq \widehat{t}_j} \cdot \frac{1}{1 - w} \left[ \frac{c_j}{D} \frac{1}{1 - \widehat{\mu}_j(s)} - w \right]$, $\widehat{\mu}_i(s)$ is as defined in lemma 3, and $t_i$ solves $\widehat{\mu}_i(t_i) = \mu_i^*$. There exists a unique sequential equilibrium when $z_1 > 1 - \frac{\lambda_1}{\gamma_1}$ or $z_2 > 1 - \frac{\lambda_2}{\gamma_2}$. There exists a unique sequential equilibrium $(\widehat{F}_i, \widehat{G}_i, \widehat{q}_i)$ in which both players do not concede with a positive probability at time 0, when $z_1 \leq 1 - \frac{\lambda_1}{\gamma_1}$ and $z_2 \leq 1 - \frac{\lambda_2}{\gamma_2}$.*

**Proof of Proposition 4.1.** Let $\Sigma = (\Sigma_1, \Sigma_2)$ define a sequential equilibrium. We will argue that $\Sigma$ must have the form specified (i.e., uniqueness) and that these strategies do indeed define an equilibrium (i.e., existence). Let $u_i(t)$ denote the expected utility of an unjustified player $i$ who concedes at time $t$. Define $T_i := \{t | u_i(t) = \max_s u_i(s)\}$ as the set of conceding times that attain the highest expected utility for player $i$; since $\Sigma$ is a sequential equilibrium, $T_i$ is nonempty for $i = 1, 2$. Furthermore, define $\tau_i = \inf\{t \geq 0 | F_i(t) = \lim_{t' \to \infty} F_i(t')\}$ as the time of last concession for player $i$, where $\inf \emptyset := \infty$. Then we have the following results.

**(a) The last instant at which two players concede is the same: $\tau_1 = \tau_2$.** A player will not delay conceding once he/she knows that the opponent will never concede. Denote the instant of last concession by $\tau$.

**(b) If $F_i$ jumps at $t \in \mathbb{R}$, then $F_j$ does not jump at time $t$ for $j \neq i$.** If $F_i$ had a jump at $t$, then player $j$ receives a strictly higher utility by conceding an instant after time $t$ than by conceding exactly at time $t$.

**(c) If $F_j$ is continuous at time $t$, then $u_i(s)$ is continuous at time $s = t$. If $F_i$ and $G_i$ are continuous at time $t$, then $u_j(s)$ is continuous at time $s = t$.** These two claims follow immediately from the definition of $u_i(s)$ in equation (7).

**(d) If $G_1$ and $G_2$ are continuous, there is no interval $(t', t'')$ such that $0 \leq t' < t'' \leq \tau$ where both $F_1$ and $F_2$ are constant on the interval $(t', t'')$.** Assume the contrary and without loss of generality, let $t^* \leq \tau$ be the supremum of $t''$ for which $(t', t'')$ satisfies the above property (i.e., both $F_1$ and $F_2$ are constant on the interval $(t', t'')$). Fix $t \in (t', t^*)$. For $\varepsilon$ small enough there exists $\delta$ such that $u_i(t) - \delta > u_i(s)$ for all $s \in (t^* - \varepsilon, t^*)$; in words, conditional on the opponent not conceding in an interval, it is strictly better to concede early than to concede close to the end of the time interval. By (b) and (c), there exists $i$ such that $u_i(s)$ is continuous at $s = t^*$, so for some $\eta > 0$, $u_i(s) < u_i(t)$ for all $s \in (t^*, t^* + \eta)$; in words, because the expected utility function is continuous at $t^*$, the expected utility of conceding sufficiently immediately after time $t^*$ is strictly lower than the expected utility of conceding at time $t$ within the time interval. Since $F_i$ is optimal, player $i$ concedes with probability 0 at time $t^*$, so $F_i$ must be constant on the interval $(t', t^* + \eta)$. The optimality of $F_i$ implies $F_j$ is also constant on the interval $(t', t^* + \eta)$, because player $j$ is strictly better to concede before or after the interval than during the interval. Hence, both functions are constant on the interval $(t', t^* + \eta)$. However, this contradicts the definition of $t^*$.

**(e) If $t' < t'' < \tau$, then $F_i(t'') > F_i(t')$ for $i = 1, 2$.** If $F_i$ is constant on an interval, then the optimality of $F_j$ implies that $F_j$ is constant on the same interval, for $j \neq i$. However, (d) shows that $F_1$ and $F_2$ cannot be simultaneously constant.

**(f) Cumulative concession probability $F_i$, $i = 1,2$, is continuous at time $t > 0$.** Assume the contrary: suppose $F_i$ has a jump at time $t$. Then $F_j$ is constant on interval $(t-\varepsilon, t)$ for $j \neq i$. This contradicts (e).

**(g) Cumulative ultimatum probability $G_i$, $i = 1,2$, is continuous at time $t > 0$.** Suppose to the contrary that $G_i$ jumps at time $t$, that is, player $i$ challenges with a positive probability at an instant. player $j \neq i$ believes that a challenging player $i$ is unjustified with probability 1, because an unjustified player $i$ challenges with a positive probability and a justified player $i$ challenges with probability 0. Player $j$ is strictly better to respond to the challenge (obtaining a payoff of $1-a_i+(1-w)D-c_j$, greater than $1-a_i$ by the assumption that $(1-w)D > c_j$) than to yield to the challenge (obtaining a payoff of $1-a_i$). Player $i$'s payoff from challenging is less than $1-a_i+wD-c_i$, which is strictly less than $1-a_i$, the payoff from conceding, because $wD < c_i$.

**(h) Player $i$'s continuation payoff at time $t > 0$ in any equilibrium is $1-a_j$.** Suppose to the contrary that there is a player $i$ such that player $i$'s continuation payoff is strictly higher than $1-a_j$ at time $t$. There exists an $\varepsilon > 0$ such that player $i$ must not have conceded at $s \in (t-\varepsilon, t)$, that is, $F_i$ is constant on the interval $(t-\varepsilon, t)$. Player $j \neq i$ must also have not conceded on the interval $(t-\varepsilon, t)$, that is, $F_j$ is constant on the interval $(0,t)$. However, the fact that both $F_1$ and $F_2$ are constant on an open interval contradicts (d).

From (e) it follows that $T_i$ is dense in $[0,\tau]$ for $i = 1,2$. From (c), (f), and (g), it follows that $u_i(s)$ is continuous on $(0,\tau]$ and hence $u_i(s)$ is constant for all $s \in (0,\tau]$. Consequently, $T_i = (0,\tau]$. Hence, $u_i(t)$ is differentiable as a function of $t$ and $du_i(t)/dt = 0$ for all $t \in (0,\tau)$. The expected utility is

$$u_i(t) = (1-z_j)\int_0^t a_i e^{-r_i s} dF_j(s) + (1-a_j)e^{-r_i t}(1-(1-z_j)F_j(t)). \tag{8}$$

The differentiability of $F_j$ follows from the differentiability of $u_i(t)$ on $(0,\tau)$. Differentiating equation (8) and applying Leibnitz's rule, we obtain

$$0 = a_i e^{-r_i t}(1-z_j)f_j(t) - (1-a_j)r_i e^{-r_i t}(1-(1-z_j)F_j(t)) - (1-a_j)e^{-r_i t}(1-z_j)f_j(t),$$

where $f_j(t) = dF_j(t)/dt$. This in turn implies $F_j(t) = \frac{1-C_j e^{-\lambda_j t}}{1-z_j}$, where $C_j$ is yet to be determined. Optimality for player $i$ implies $F_i(\tau_i) = 1$. At $t = 0$, if $F_j(0) > 0$ then $F_i(0) = 0$ by (b). □

## 4.2 The Multiple-Type Case

## 4.3 The Limiting Case of Complete Rationality

## 4.4 The Discrete Model and Convergence

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