Reputational Bargaining with Ultimatum Opportunities

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Abstract

Two parties negotiate in the presence of external resolution opportunities (court, arbitration, mediation, or war). The outcome of external resolution depends on the justifiability of their claims. A justified party issues an ultimatum for resolution whenever possible; a strategic unjustified party bluffs with ultimatum to establish a reputation for being justified. Even when external resolution disfavors the unjustified, its availability can benefit them in equilibrium. If prior reputations vanish, agreement is immediate and efficient; and if the set of justifiable demands is rich, our solution incorporates ultimatum in the Rubinstein division of Abreu and Gul (2000) in a simple way.

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1 Introduction

Many negotiations share the common features that (i) involved parties can seek external resolution (e.g., court, arbitration, mediation, or war) as last resort if internal resolution fails and (ii) they hold private information about the potential outcome of external resolution. For example, two parties involved in a patent infringement dispute can seek intellectual property court if settlement fails, and a court or arbitrator can determine if the plaintiff is a legitimate victim or a patent troll. A country can choose to invade another country if peaceful negotiation fails, and the outcome of the invasion depends on the countries' private military strength and devotion to the dispute. The buyer of a firm or product may invite an auditor or mediator to verify the seller's claims and negotiate afterwards.[1] In these settings, negotiation and external resolution is often time-consuming and costly.[2]

However, many disputes are resolved before external resolution is invoked, so threatening external resolution is frequently leveraged as a strategic posture in the form of an ultimatum for internal resolution.[3] The ability of making such a threat may differ across situations and locations. For example, several large jurisdictions (e.g., California, Illinois, and Texas) have rules that explicitly bar attorneys from threatening disciplinary or criminal action to gain the upper hand in settlement talks. Some states (e.g., New York) only prohibit threatening criminal action, some states (e.g., Michigan) haven't promulgated any rules covering this subject at all. We study the strategic and welfare effects of the presence of external resolution opportunities on bargaining. What happens to the negotiation process, and what are the determinants of negotiation outcome?

Our model incorporates external resolution opportunities into the continuous-time war-of-attrition bargaining model of Abreu and Gul (2000) (AG henceforth), which allows for only internal resolution. Players 1 ("he") and 2 ("she") negotiate to divide a unit pie. Privately, each player is either justified or unjustified in their demand. A justified player demands a fixed share of the pie and never gives in to an offer smaller than their demand (corresponding to the behavioral type in AG), and an unjustified player can demand any share and give in to any demand (corresponding to the rational type in AG).

Players announce their demands sequentially at the beginning of the game. Afterward, each player can continue the negotiation by holding on to the announced demand, or end the negotiation by either giving in to the opposing demand (internal resolution) or challenging the opponent before external resolution with an ultimatum for internal resolution. The challenge opportunities arrive frictionally for justified players, and we consider both the case in which these opportunities arrive frictionlessly and the case in which they arrive frictionally for unjustified players. Upon being challenged, the opponent must respond either by giving in to the challenger's demand (internal resolution), or seeing the challenge (external resolution). The external resolution mechanism specifies a division of the pie that more likely favors a justified party

[1] Arbitration is also used widely. For example, 15 of the 20 largest US credit card issuers and seven of the eight largest cell phone companies include arbitration clauses in their contracts with consumers (Consumer Financial Protection Bureau, 2015); Major League Baseball (MLB) and the National Hockey League (NHL) have used arbitration to resolve salary conflicts since the 1970s and 1990s, respectively.

[2] Appendix B.1 provides five applications within our consideration.

[3] For example, 98% of criminal cases and 97% of civil lawsuits have been resolved before trial, and 80% of financial arbitration cases and 95% of NHL salary arbitration cases are settled before their scheduled hearings (Gramlich, 2019; Financial Industry Regulatory Authority, 2020; National Hockey League Players' Association, 2020).

and more likely disfavors an unjustified party, and executes a compromise division if both players are unjustified.[4]

The key assumption we make about the external resolution mechanism is that its outcome depends on the types of the players. In court, the outcome can be enforced by a court that observes players' types in patent conflicts. In war, the outcome depends on countries' strengths. If an auditor or mediator that reveals information is invoked, the outcome is the equilibrium payoff in the continuation game after players' claims are verified (as in Fanning (2021a)).

In the model in which neither player has external resolution opportunities—the AG model—the equilibrium bargaining and reputation dynamics are unique. After players announce their demands, at most one player concedes with a positive probability at time zero. Afterward, both players concede at overall constant rates, and their reputations—the opponent's beliefs about a player's being justified—increase exponentially at the respective constant concession rates until both reputations reach one at the same time, at which point no unjustified player is left in the game and justified players continue to hold on to their demands.

We start our analysis with the case in which only one player—player 1—has challenge opportunities.[5] This case is a building block for the case in which both players have challenge opportunities, and most of the new economic forces from challenge opportunities on behavior, reputation, and outcome are present and transparent in this case. We start with the setting in which each player has a single justified demand. In the unique equilibrium, as in the AG equilibrium, at most one player concedes with a positive probability at time zero, both players' overall concession rates are the same constant rates as in AG, and both players' reputations increase to 1 at the same time. In addition, an unjustified player 1 challenges with a positive and increasing hazard rate as long as player 2's reputation is not too high, and does not challenge at all after player 2's reputation increases past a threshold (Theorem 1). Hence, there is a challenge phase followed by a no-challenge phase.

The *interdependence* of players' reputation-building processes differs from AG's: The rate of change of player 1's reputation depends on player 2's reputation at each instance. This interdependence renders AG's solution method inapplicable to our model. To solve the model, we introduce reputation coevolution diagrams, which are inherently backward looking. This uses our new technique to characterize the initial concession for any pair of initial reputations, and more generally, the whole equilibrium play. Reputations evolve according to phase-dependent Bernoulli differential equations, which include exponential growth in the ultimatum-free model of AG as a special case.

We find that the overall hazard rate of resolution is *discontinuous* and *piecewise monotonic* in time, which differs from existing reputational bargaining models (e.g., AG and Fanning (2016)). Because an unjustified player 1 does not challenge after player 2's reputation passes a threshold, there is a discontinuous drop in the equilibrium hazard rate of ultimatum usage by an unjustified player, and consequently a

[4] We assume that justified players follow commitment behavior while unjustified players are strategic. We relax this assumption in the extensions.

[5] For example, in MLB and the NHL, essentially, only players can elect to have salary arbitration hearings; in civil lawsuits, usually only one side has the incentive to sue the other side; in price negotiations, typically either buyer or seller—but not both—waits for outside options; and in international conflicts, one side may consider aggression.

discontinuous drop in the overall hazard rate of dispute resolution.

Two forces in our model determine the speed and dynamics of reputation building. The first is reputation building by not conceding (not invoking internal resolution, as in AG): Persisting longer in the negotiation increases a player's reputation. The second, which is new in our model, is reputation gain or loss by not challenging (not invoking external resolution). On one hand, the presence of challenge opportunities can hurt player 1 by slowing reputation building, when an unjustified player 1 is expected to challenge at a lower rate than a justified player 1. This is because not challenging is evidence against his being justified (bad news). On the other hand, the presence of challenge opportunities can benefit player 1 by speeding up reputation building and resulting in player 2 conceding with a higher probability at the beginning of the game, when an unjustified player 1 is expected to challenge at a higher rate than a justified player 1 (good news). What is the net equilibrium impact of challenge opportunities? Player 1's equilibrium payoff may be higher or lower with the presence of challenge opportunities.

Players' payoffs depend on the details of external resolution in natural ways (Proposition 1), but when initial reputations approach zero, the equilibrium outcome does not depend on these details. In this limit case of rationality, the equilibrium outcome is efficient with one of the players yielding to the opponent's demand at time zero with a probability approaching one (Proposition 2). The identity of the loser—the player who concedes with probability one at time zero—and the division of the pie are determined by the discount rates, demands, and ultimatum opportunity arrival rate via a simple formula. The set of parameters for which player 1 loses expands with the ultimatum opportunity arrival rate; hence, ultimatum opportunities always hurt player 1 in the limit case of rationality. In the context of war, this result suggests that being able to start a war as external resolution may hurt a country's ability to receive concessions from its rival.

Moreover, in the rich demand space, equilibrium outcome is unique (Theorem 2), and the presence of ultimatum opportunities affects players' bargaining power in a remarkably simple way. As initial reputations approach zero, and as the set of justified demands gets larger and finer, the players' equilibrium payoffs converge to a unique vector and the outcome is efficient (Proposition 3). Player 1's equilibrium payoff is the AG payoff if the ultimatum opportunity arrival rate is smaller than his discount rate, and is equal to what his AG payoff would be if his discount rate were replaced by the ultimatum opportunity arrival rate if otherwise. In the former case, players tend to compromise; in the latter case, player 2 chooses the greediest demand.

In summary, our model captures bargaining situations with privately held information and external resolution whose outcome depends on the information. The reputation coevolution diagram that collapses the time dimension provides a convenient and general way of characterizing equilibrium behavior and reputation. Notably, even when the external resolution does not favor unjustified players, they may nonetheless benefit from its availability: Although the potential arrival of ultimatum opportunities slows down their reputation building, bluffing with ultimatum when they have built a sufficiently high reputation may deter the opposing party from responding. Hence, in this setting, the commitment power of not accessing and seeking resolution is not necessarily beneficial. In the limit in which the private information vanishes, immediate agreement and efficiency ensue, and the determination of winner and payoff division incorporates ultimatum opportunity arrival rate in a parsimonious and intuitive way.

To demonstrate the robustness of our findings to alternative specifications in ultimatum arrival and commitment behavior, we consider the setting in which ultimatum opportunities arrive equally frictionally for both justified and unjustified players. In this setting, there may be an additional strategy phase in which an unjustified player's equilibrium ultimatum usage rate is capped by the frictional ultimatum opportunity arrival rate, which complicates equilibrium characterization and uniqueness proof. Nonetheless, the frictional arrival of ultimatum opportunities does not alter our key qualitative results (e.g., discontinuous ultimatum and resolution rates, the potential benefits of player 1 by the introduction of a frictional ultimatum, and the payoffs in the limit case of rationality). We also demonstrate that *public* arrival of ultimatum opportunities does not alter players' equilibrium payoffs. Moreover, we relax our assumption that the justified players act like the commitment types in reputation literature.

We then consider additional extensions: (i) external resolution is costless, random, compromising, or noisy, and (ii) both players have the opportunity to challenge. The first set of extensions clarifies the roles of bluffing opportunities and external resolution mechanism in the determination of bargaining process and bargaining outcome, and showcases the generality of our model and solution method to alternative settings of conflicts. In the extension to two-sided challenge opportunities, if at least one player's exogenous ultimatum opportunity arrival rate is lower than the AG equilibrium concession rate, there exists a unique equilibrium outcome that is similar to the one in the setting with one-sided ultimatum opportunities (Theorem 4). Otherwise, inefficient delays arise in equilibrium even in the limit case of rationality due to an overabundant availability of access to the court. This result suggests that more convenient access to the external resolution may be counterproductive and socially inefficient for resolution.

The rest of the paper proceeds as follows. Section 2 describes the basic model with one-sided ultimatum opportunities. Section 3 characterizes its equilibrium. Section 4 discusses the determinants of bargaining outcome, including the case with multiple justifiable demands and the limit payoffs in the case of rationality and rich type spaces. Section 5 discusses the extension with frictional bluffing opportunities to clarify public versus private arrival of ultimatum opportunities and the role of commitment. Section 6 discusses additional extensions. Section 7 discusses related literature, and Section 8 concludes. Appendix A collects omitted proofs.

2 Model

Players 1 ("he") and 2 ("she") decide on how to split a unit surplus. Each player is either (i) *justified and committed* in demanding a fixed share of the pie, or (ii) *unjustified and strategic* in demanding any fixed share.

We start by assuming that each player can be of a single justified type: With probability z_1 player 1 is justified in demanding $a_1 \in (0,1)$, and with probability z_2 player 2 is justified in demanding $a_2 > 1 - a_1$. Let $D := a_i - (1 - a_j)$ denote the amount of disagreement between the two players.

Time is continuous and the horizon is infinite. At time zero, player 1 announces his demand first, and upon observing player 1's announcement, player 2 accepts it or announces her demand.[6] At each instant,

[6] Because for now there is only a single justifiable type, the initial demand announcement stage is redundant. When we allow

each player can either concede to their opponent or not concede. We assume that each justified player never concedes. When an unjustified player i concedes to player j, player i gets a payoff of $1 - a_j$ and player j gets a payoff of a_j. In addition, we start by assuming a one-sided challenge model: Player 1 has an opportunity to challenge player 2 with an ultimatum. The game ends upon a concession, and moves to the challenge phase if a player challenges.

Challenge. It costs $c_1 D$ for player 1 to challenge. A justified player 1's challenge opportunities arrive according to a Poisson process with rate $\gamma_1 \in [0, \infty)$, and he challenges whenever such an opportunity arrives. An unjustified player 1 can challenge at any time, so he can time his challenge strategically and *bluff* with ultimatum. In Section 5, we consider the model in which an unjustified player's challenge opportunities also arrive according to a Poisson process with rate γ_1 and he can decide whether or not to challenge.

Response to a challenge. Player 2 can respond to a challenge either by yielding to the challenge or by seeing it. A justified player 2 always sees a challenge, and an unjustified player 2 chooses between the two actions. If player 2 yields, she gets $1 - a_1$, and player 1 gets a_1. It costs $k_2 D$ for player 2 to see a challenge, and in this case the division of the pie is determined by external resolution.

External resolution. We start with the extreme case in which the external resolution favors a justified player against an unjustified player. An interpretation maybe is that external resolution simply reveals the types publicly. If an unjustified player meets a justified player, the unjustified player i receives $1 - a_j$. If two unjustified players meet, the challenging player 1 is favored and gets a_1 with probability w_1, or is disfavored and gets $1 - a_2$ with probability $1 - w_1$. Therefore, his expected share is $1 - a_2 + w_1 D$ and the defending player 2's expected share is $1 - a_1 + (1 - w_1)D$. The players' payoffs are linear in the share of the surplus they receive, so we could equivalently interpret that the third party decides on a deterministic compromise division that gives each player their respective expected share. We do not specify the outcome for justified players, since this does not play any role in the strategic decisions of the unjustified players. Table 1 summarizes the outcome of the external resolution considered in the benchmark model. In the

	Unjustified defender	Justified defender
Unjustified challenger	$1 - a_2 + w_1 D, 1 - a_1 + (1 - w_1)D$	$1 - a_2, \cdot$
Justified challenger	$\cdot, 1 - a_1$	$\cdot, \cdot$

Table 1: Outcome in the external resolution

Note. The $\cdot$ indicates that the payoff is irrelevant for the strategic consideration of an unjustified player.

benchmark model, we assume that if player 2 is expected to see a challenge, then player 1 prefers conceding to challenging: $w_1 < c_1 < 1$; and that player 2 prefers seeing a challenge from an unjustified player 1 to yielding to it: $0 < k_2 < 1 - w_1$. If $w_1 = 0$—i.e., the external resolution never favors an unjustified plaintiff—we are simply assuming that c_1 and k_2 are strictly between 0 and 1. In Section 6.1, we explore alternative external resolution mechanisms.

We note that when the external resolution mechanism is an auditor or mediator that publicly reveals

for multiple types in Section B.4, we add the demand announcement stage.

players' types, the perfect association between the commitment behavior and being justified is natural. This is because when an auditor or mediator is called upon to reveal the commitment behavior of players, there is a perfect association of committed and justified and that of strategic and unjustified in the following sense. When the auditor reveals a party to be committed and the other party to be rational, in the continuation game the rational party concedes, and when both parties are rational, then the continuation payoffs are efficient and captured by the division share w_1 (as in Fanning (2021a)).

In summary, a bargaining game $B = (a_1, a_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$ with ultimatum opportunities for one player and single demand types for both players is described by players' justified demands a_1 and a_2, prior probabilities z_1 and z_2 of being justified, discount rates r_1 and r_2, challenge opportunity arrival rate γ_1 for a justified player 1, challenge cost c_1 and seeing cost k_2 as proportions of the conflicting difference, and an unjustified player 1's winning probability w_1 against an unjustified opponent. Appendix B.1 provides several applications that can be thought of as negotiation with one-sided and/or two-sided resolution opportunities.

2.1 Discussion of modeling choices and extensions

We assume that challenge opportunities arrive according to a Poisson process, implying a constant arrival rate. This assumption eases some of the calculation and exposition of our results. Our analyses do not rely on the arrival process to be Poisson. Assuming a stationary arrival process helps tease out the sources of nonmonotonicity and discontinuity of dispute resolution.

For relative expositional ease of equilibrium characterization, we start with the "asymmetric" case in which the unjustified player can challenge at any time and the justified player challenges only when the opportunity arrives. We extend in Section 5.1 to the "symmetric" case in which challenge opportunities arrive equally frictionally for justified and unjustified player 1, and extend the equilibrium characterization and demonstrate the generality of the key results established in the "asymmetric" benchmark model. In the "symmetric" case, we also demonstrate equilibrium equivalence of public and private frictional arrival of ultimatum opportunities in Section 5.2.

Also for relative expositional ease, we start with the perfect association of commitment behavior (i.e., always challenging and always seeing a challenge) with justified players, who get a favorable outcome in the external resolution. We also relax this perfect association by allowing unjustified players to exhibit commitment behavior and justified players to exhibit strategic behavior in Section 5.3. The equilibrium in the extended model is a generalization of the equilibrium in the benchmark model.

In the current specification of external resolution, challenge is dominated by concession if an unjustified opponent always sees the challenge: $w_1 < c_1$. In Section 6.1, we explore alternative external resolution specifications to showcase the versatility of our solution method. We first consider the setting in which challenge dominates concession (e.g., if external resolution is random and/or if the challenge is costless) and then the setting in which challenge neither dominates nor is dominated by concession (e.g., external resolution is noisy): $w_1 > c_1$. Moreover, our results continue to hold if player 1 pays the court cost only when player 2 sees the challenge.

We focus on the model with one-sided ultimatum opportunities, as it has ample applications (e.g., patent infringement, debt collection, country aggression) and captures most of the economic channels in consideration. Section 6.2 studies the model with two-sided ultimatum opportunities (which has a different set of applications, e.g., division of financial assets in a dissolved firm), highlighting the similarities (Section 6.2.1) and differences (Section 6.2.2) with the one-sided model.

We model the negotiation process directly as a concession game in the style of war of attrition with the addition of ultimatum opportunities. We could alternatively model the negotiations in a *continuous-discrete-time* model in which a player can change his demand at any positive integer time, but can concede to an outstanding demand (or challenge in our case) at any time $t \in [0, \infty)$. This formulation was introduced by Abreu and Pearce (2007) in a setting of repeated games with contracts and adopted by Abreu, Pearce, and Stacchetti (2015) in a bargaining context. In that formulation, without ultimatum opportunities, whenever a player makes a demand different from a commitment (justified) type she reveals her rationality, and there is a unique equilibrium continuation payoff vector, which coincides with the payoff vector from concession. With ultimatum opportunities, however, when player 2 reveals rationality, there are multiple equilibria with different continuation payoffs. For example, there is an equilibrium in which player 2 chooses a fixed demand, players concede to each other at constant hazard rates, player 1 challenges at a constant rate, and player 1's reputation stays constant. However, when player 1 reveals rationality, there is a unique equilibrium continuation payoff vector, which coincides with the payoff vector from concession. In particular, all of the equilibria we identify in our model have an analogous equilibrium in the continuous-discrete-time bargaining model that yields identical behavior.

2.2 Formal description of strategies and payoffs

Since only unjustified players can choose their strategies, we drop the qualifier "unjustified" or "strategic" whenever no confusion can arise. An unjustified player 1's strategy is described by $\Sigma_1 = (F_1, G_1)$, where F_1 and G_1, the probabilities of conceding and challenging by time (including) t, respectively, are right-continuous and increasing functions with $F_1(t) + G_1(t) \leqslant 1$ for every $t \geqslant 0$. A strategic player 2's strategy is described by $\Sigma_2 = (F_2, q_2)$, where F_2, the probability of conceding by time t, is a right-continuous and increasing function with $F_2(t) \leqslant 1$ for every $t \geqslant 0$, and $q_2(t) \in [0, 1]$, her probability of yielding to a challenge at time t, is a measurable function. Each strategy profile induces a distribution over action profiles, which we refer to as *equilibrium play*.

A strategic player 1's (time-zero) expected utility from conceding at time t is[7]

$$\begin{aligned} U_1(t, \Sigma_2) \quad = \quad & (1 - z_2) \int_0^t a_1 e^{-r_1 s} dF_2(s) + \Big[1 - (1 - z_2) F_2(t)\Big] e^{-r_1 t} (1 - a_2) \\ & + (1 - z_2) \Big[F_2(t) - F_2(t^-)\Big] \frac{a_1 + 1 - a_2}{2}, \end{aligned} \tag{1}$$

where $F_2(t^-) := \lim_{s \uparrow t} F_2(s)$. His expected utility from challenging at time t is[8]

$$V_1(t, \Sigma_2) \quad = \quad (1 - z_2) \int_0^t a_1 e^{-r_1 s} dF_2(s) + \Big[1 - (1 - z_2) F_2(t)\Big] e^{-r_1 t} (1 - a_2 - c_1 D) +$$

[7] We assume an equal split when two players concede at the same time. It is inconsequential for our results, because simultaneous concession occurs with probability 0 in equilibrium.

[8] We assume that whenever concession and challenge occur simultaneously, the outcome is determined by the concession. This is an innocuous assumption, because simultaneous concession and challenge occur with probability 0 in equilibrium.

$$(1 - z_2)[1 - F_2(t)]e^{-r_1 t}[(1 - q_2(t))w_1 + q_2(t)]D.$$

His expected utility from strategy Σ_1 is

$$u_1(\Sigma_1, \Sigma_2) = \int_0^\infty U_1(s, \Sigma_2)dF_1(s) + \int_0^\infty V_1(s, \Sigma_2)dG_1(s).$$

A strategic player 2's expected utility from conceding at time t and yielding according to $q_2(\cdot)$ when facing a challenge is

$$\begin{aligned} U_2(t, q_2(\cdot), \Sigma_1) \quad = \quad & (1 - z_1)\int_0^t a_2 e^{-r_2 s}dF_1(s) + z_1 \int_0^t \left[1 - a_1 - (1 - q_2(s))k_2 D\right]e^{-r_2 s}\gamma_1 e^{-\gamma_1 s}ds \\ & +(1 - z_1)\int_0^t \left\{1 - a_1 + \left[1 - q_2(s)\right]\left[1 - w_1 - k_2\right]D\right\}e^{-r_2 s}dG_1(s) \\ & +e^{-r_2 t}(1 - a_1)\left[1 - (1 - z_1)F_1(t) - (1 - z_1)G_1(t^-) - z_1\left(1 - e^{-\gamma_1 t}\right)\right] \\ & +e^{-r_2 t}(1 - z_1)\left[F_1(t) - F_1(t^-)\right]\frac{a_2 + 1 - a_1}{2}, \end{aligned} \tag{2}$$

where $F_1(t^-) := \lim_{s\uparrow t} F_1(s)$. Her expected utility from strategy Σ_2 is

$$u_2(\Sigma_2, \Sigma_1) = \int_0^\infty U_2(s, q_2, \Sigma_1)dF_2(s).$$

We study this game's Bayesian Nash equilibria. Because the game is dynamic, it is natural to define public beliefs about players' types, i.e., *reputations*, throughout the game. We define the reputation process $\mu_i(t)$ in the natural way, as the posterior belief that player i is justified conditional on the game not ending by time t. Bayes' rule gives us this process explicitly as

$$\mu_1(t) = \frac{z_1\left[1 - \int_0^t \gamma_1 e^{-\gamma_1 s}ds\right]}{z_1\left[1 - \int_0^t \gamma_1 e^{-\gamma_1 s}ds\right] + (1 - z_1)\left[1 - F_1(t^-) - G_1(t^-)\right]},$$

and

$$\mu_2(t) = \frac{z_2}{z_2 + (1 - z_2)\left[1 - F_2(t^-)\right]}.$$

Finally, let $\nu_1(t)$ be player 2's posterior belief that player 1 is justified conditional on player 1 challenging at time t. Namely, $\nu_1(t) = 0$ at any $t \geqslant 0$ where G_1 has an atom, and at any $t \geqslant 0$ where G_1 is differentiable,

$$\nu_1(t) = \frac{\mu_1\gamma_1}{\mu_1\gamma_1 + [1 - \mu_1(t)]\beta_1(t)}, \tag{3}$$

where

$$\beta_1(t) = \frac{G_1'(t)}{1 - F_1(t^-) - G_1(t^-)}$$

is an unjustified player 1's hazard rate of challenging.[9]

[9]The function G_1 is differentiable almost everywhere, because it is right-continuous and monotone. Moreover, the posterior

3 Equilibrium

In this section, we solve and characterize equilibrium strategies and reputations. The bargaining game entails a unique equilibrium play, which satisfies the following four properties.

Theorem 1. *Consider $B = (a_1, a_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$, a bargaining game with one-sided ultimatum opportunities and single demand types. There exists an equilibrium. There exist finite times T and $T_1 \in [0, T)$ such that every equilibrium strategy profile $(\widehat{F}_1, \widehat{G}_1, \widehat{F}_2, \widehat{q}_2)$ satisfies the following properties.*

1. *$\widehat{F}_1$ and $\widehat{F}_2$ are strictly increasing in $(0, T)$ and constant for $t \geqslant T$;*
2. *$\widehat{F}_1$ and $\widehat{F}_2$ are atomless in $(0, T]$ and at most one of the two has an atom at $t = 0$;*
3. *(a) $\widehat{F}_1(T) + \widehat{G}_1(T_1) = 1$;*
   (b) $\widehat{F}_2(T) = 1$;
4. *(a) $\widehat{G}_1$ is atomless in $[0, T]$, strictly increasing in $[0, T_1]$, and constant for $t \geqslant T_1$;*
   (b) For almost every $t \in [0, T]$, $\widehat{q}_2(t) \in (0, 1)$ if $t \in [0, T_1]$ and $\widehat{q}_2(t) = 1$ if $t \in (T_1, T]$.

Moreover, $\widehat{F}_1$, $\widehat{F}_2$, and $\widehat{G}_1$ are unique, and $\widehat{q}_2$ is unique almost everywhere for $t \leqslant T$.

Property 1 states that there is a finite time $T > 0$ such that players concede to each other with a strictly positive probability in every subinterval of $(0, T]$, and never concede after time T. Property 2 states that distributions of concession are atomless except at time zero, and there can be an atom in at most one of these distributions. Property 3a states that an unjustified player 1 has either conceded before time T or challenged before time T_1, and Property 3b states that an unjustified player 2 has conceded before time T. Properties 1, 2, and 3b coincide with the three properties in AG, and Property 3a modifies AG to characterize equilibrium challenge usage.

Property 4 extends AG's equilibrium characterization when there are ultimatum opportunities. There are difficulties, however, due to players' larger strategy spaces: In addition to the timing of concession, player 1 chooses the timing of challenge and player 2 chooses how to respond to a potential challenge at each instant. A priori, players' incentives to concede may change—for better or for worse—due to the arrival or anticipated arrival of challenge opportunities at each instant. We first show that in every equilibrium, player 2 does not benefit from challenges, i.e., at each instant she weakly prefers conceding to seeing a challenge. Second, we show that $\widehat{G}_1$ is atomless. These findings allow us to show that players' concession distributions are strictly increasing and atomless in an interval $(0, T_1)$. This implies that player 2's reputation is increasing, which allows us to show the novelty of our characterization.

Property 4a asserts that player 1 challenges his opponent with an atomless distribution until some time $T_1 < T$, and never challenges afterward. Property 4b asserts that player 2 responds to a challenge by both seeing the challenge and yielding to it with positive probabilities until time T_1, and yields to it afterward. Because this is a new property, let us provide an intuition for why this property must hold. Property 1

beliefs are well defined at the jump points of G_1, and hence, they are well defined almost everywhere in both the G_1 measure and Lebesgue measure.

implies that at any time $t \in (0,T)$, player i's continuation payoff at time t is equal to $1-a_j$. If $\widehat{G}_1$ is constant in some interval, after observing a challenge in that time interval, player 2's posterior belief that player 1 is justified is one, and player 2 optimally yields to any challenge. However, if player 2's reputation is smaller than $\mu_2^* := 1-c_1$, then challenging gives player 1 a payoff that strictly exceeds $1-a_2$, which yields a contradiction. Similarly, if $\widehat{G}_1$ had an atom at some time t, then after observing a challenge at time t, player 1's reputation would be 0, and player 2 would optimally see the challenge. However, then player 1 would receive a payoff strictly lower than $1-a_2$, leading again to a contradiction. Furthermore, as we will argue in the next section, player 2's reputation increases over time, and at some time $T_1 < T$ reaches μ_2^*. After this time, player 1 never challenges. Finally, for time $t < T_1$, player 1 is indifferent between conceding and challenging, and player 2's reputation is smaller than μ_2^*. Therefore, $\widehat{q}_2(t) \in (0,1)$ for time $t < T_1$.

We now use the four properties to derive the closed-form solutions of equilibrium strategies $\widehat{F}$ and $\widehat{G}$. In the next subsection, we first derive the equilibrium concession rates at time $t > 0$, player 1's challenge rate, and player 2's challenge response. We then derive reputation evolution based on these rates and construct a reputation coevolution diagram, which allows us to compute the probabilities of concession at time $t = 0$.

3.1 Equilibrium strategies and reputations

3.1.1 Challenge and response to challenge

Property 2 implies that player 1 is indifferent between challenging and conceding at any time $t \in (0, T_1)$. At any such time t, $\mu_2(t)$ denotes player 2's reputation and $q_2(t)$ denotes the probability that player 2 yields if a challenge comes at time t. Compared to conceding, the benefit of challenging comes from winning against an unjustified opponent who yields or sees, $\left[1-\mu_2(t)\right]\left[q_2(t) + (1-q_2(t))w_1\right]D$, and the cost of challenging is $c_1 D$. Hence, we obtain that

$$q_2(\mu_2) := \frac{c_1 - w_1(1-\mu_2)}{1-\mu_2 - w_1(1-\mu_2)}. \tag{4}$$

The yielding probability is interior if $1 - c_1/w_1 < \mu_2 < \mu_2^* := 1 - c_1$. The lower bound is negative given the assumption that $c_1 > w_1$, and when μ_2 exceeds the upper bound, player 2 strictly prefers yielding to a challenge. At any time $t \leqslant T_1$, player 2 is indifferent between seeing and yielding to a challenge when player 1's reputation conditional on challenging player 2 is

$$v_1^* := 1 - \frac{k_2}{1-w_1} \Longleftarrow (1-v_1^*)(1-w_1)D - k_2 D = 0. \tag{5}$$

This implies, by Bayes' rule and Equation (3), that player 1's overall challenge rate seen as a function of player 1's reputation is

$$\chi_1(\mu_1) := \frac{\mu_1}{v_1^*}\gamma_1 \Longleftarrow \frac{\mu_1 \gamma_1}{\chi_1} = v_1^*. \tag{6}$$

Equivalently, an unjustified player 1's rate of bluffing with ultimatum is

$$\beta_1(\mu_1) := \frac{1-v_1^*}{v_1^*}\frac{\mu_1}{1-\mu_1}\gamma_1 \Longleftarrow \frac{\mu_1\gamma_1}{\mu_1\gamma_1 + (1-\mu_1)\beta_1} = v_1^*. \tag{7}$$

To summarize, Equation (4) holds almost everywhere for $t \leqslant T$, because actions after time T are off equilibrium path for an unjustified player 2, and Equation (6) holds almost everywhere for $t \leqslant T_1$ and $\beta_1(t) = 0$ almost everywhere for $t \in (T_1, T]$.

3.1.2 Concessions

Property 1 says that player 1 concedes with a positive probability in every subinterval of $(0, T)$, so player 1's continuation payoff at every time t is equal to $1 - a_2$, and he is indifferent between conceding at any time in $(0, T)$. Hence, player 2 concedes at the constant rate λ_2 in the interval $(0, T)$ that sustains this indifference:

$$1 - a_2 = a_1\lambda_2 dt + e^{-r_1 dt}(1-a_2)(1-\lambda_2 dt) \Longrightarrow \lambda_2 = \frac{r_1(1-a_2)}{a_1+a_2-1},$$

as in AG; an unjustified player 2 concedes at rate $\kappa_2 = \lambda_2/(1-\mu_2)$. An immediate implication is that player 2's reputation conditional on negotiation continuing at time $t < T$, $\mu_2(t)$, is an increasing function.

Property 1 says that player 2 concedes with a positive probability in every subinterval of $(0, T)$, as is the case for player 1. However, from player 2's perspective, in any time interval, player 1 may concede or challenge. As Property 4b indicates, because player 2 sees the challenge with an interior probability, her continuation payoff when she is challenged is equal to her payoff from conceding to player 1. Hence, the indifference condition for player 2 in yielding across all times $t \in (0, T)$ implies that the overall hazard rate of player 1 conceding to player 2 is $\lambda_1 = r_2(1-a_1)/D$, as in AG. To summarize, each player i, $i = 1, 2$, concedes at the overall rate of

$$\lambda_i := \frac{r_j(1-a_i)}{a_1+a_2-1} = \frac{r_j(1-a_i)}{D}. \tag{8}$$

3.1.3 Reputation evolution

We now characterize the evolution of the players' reputations. To do so, we use the concession rates and the challenge rate of player 1 found in the previous section. We start with player 2's reputation building for $t \in (0, T]$. Player 1's reputation dynamics depend on both his concession rate and challenge rate. We start with the *no-challenge phase*, $t \in (T_1, T]$, and then characterize the *challenge phase*, $t \in (0, T_1]$.

Note that Property 3 implies that $\mu_i(T) = 1$ for $i = 1, 2$. Using this property and the reputation dynamics we derive, we characterize the *reputation coevolution curve*. This curve shows the locus of the reputation vectors at times $t > 0$. The curve will determine the identity of the player who yields with a positive probability at time 0 and the magnitude of that atom. This will complete the characterization of the unique equilibrium.

We use the Martingale property $\mu_i(t) = \mathbb{E}_t\mu_i(t+dt)$ to characterize players' reputation evolution in different phases, which can be succinctly summarized in the following lemma.

Lemma 1. *Player 1's reputation evolution can be characterized as*

$$\dot{\mu}_1(t) := \frac{\mu_1'(t)}{\mu_1(t)} = \lambda_1 - \gamma_1 + \chi_1(t) \tag{9}$$

$$= \begin{cases} \lambda_1 - [1 - \frac{\mu_1(t)}{v_1^*}]\gamma_1 & \text{if } 0 < t \leqslant T_1 \\ \lambda_1 - [1 - \mu_1(t)]\gamma_1 & \text{if } T_1 < t \leqslant T \end{cases} \tag{10}$$

and player 2's reputation evolution is

$$\dot{\mu}_2(t) := \frac{\mu_2'(t)}{\mu_2(t)} = \lambda_2. \tag{11}$$

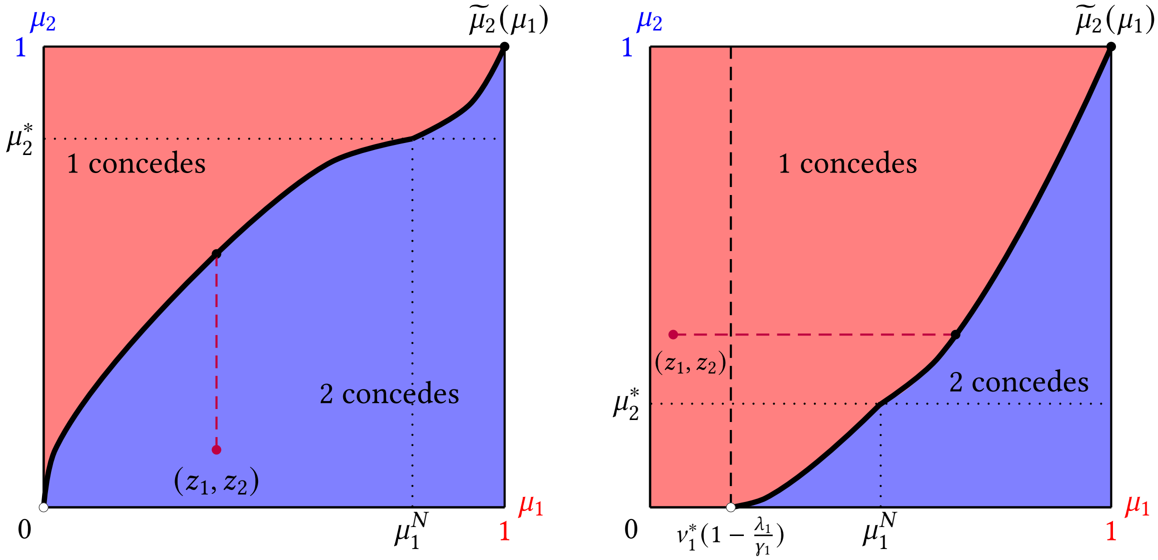
Bad-news and good-news effects. Two forces shape the evolution of player 1's reputation. First, no concession is "good news": With player 1 conceding at rate λ_1, his reputation conditional on not having conceded increases exponentially at rate λ_1. The second force, which is new, comes from the equilibrium challenges. Observe that when $\gamma_1 = \beta_1(t) = 0$, Equation (9) boils down to the exponential growth reputation dynamics in AG.

This second force can *decelerate* or *accelerate* reputation building. In the no-challenge phase, no challenge is bad news: With a justified player 1 challenging and an unjustified player 1 not challenging at all, player 1's reputation declines at rate $[1 - \mu_1(t)]\gamma_1$. In the challenge phase, however, the unjustified player 1 also challenges at a positive rate. Hence, the "bad-news" effect of no challenge is less severe in this phase compared to the no-challenge phase. This is captured by the third term in Equation (9). In fact, when $\beta_1(t) > \gamma_1$, player 1's reputation building accelerates with no challenge, and no challenge becomes "good news." Player 1's reputation builds faster when $\mu_1(t) > v_1^*$, equivalently, $\beta_1(t) > \gamma_1$ and $\chi_1(t) > \gamma_1$, while player 2's reputation is not too high, $\mu_2(t) < \mu_2^*$. This effect provides a benefit from the presence of ultimatum opportunities for an unjustified player 1 who has an intermediate range of reputations. We characterize the range of initial reputations for ultimatum opportunities to be beneficial for an unjustified player 1 in Section 4.1; this range may not exist in equilibrium. The decomposition of the bad-news and good-news effects clarifies the potential benefit of external resolution opportunities for an unjustified player 1.

3.1.4 Reputation coevolution diagram and initial concession

Both players' reputation dynamics in each phase follow the Bernoulli differential equation, which is one of the few cases of ordinary differential equations with closed-form solutions and includes the exponential growth of AG as the special case when ultimatum opportunities are absent. Hence, it is feasible to combine the reputation-building dynamics at different phases of the game to find the evolution of both players' reputations in equilibrium. To do so, we "run" the Bernoulli differential equations that describe players' reputation dynamics backward, starting from time T.

Recall that the finiteness of T in Property 3 of Theorem 1 implies that $\mu_1(T) = \mu_2(T) = 1$. Moreover, $\mu_2(T_1) = \mu_2^*$. Hence, $T - T_1$ can be found using player 2's reputation dynamics given by Equation (11). Then we can use player 1's reputation dynamics in the no-challenge phase, Equation (10), to find $\mu_1(T_1)$. Then we let T_1^* be the time it takes for player 1 to build a reputation from z_1 to $\mu_1(T_1)$ using the dynamics in



(a) $\gamma_1 \leqslant \lambda_1$

(b) $\gamma_1 > \lambda_1$

Figure 1: Reputation coevolution and initial concession in games with one-sided ultimatum opportunities
Note. The solid line in each panel depicts the reputation coevolution curve $\widetilde{\mu}_2(\mu_1)$. Player 1 concedes with a positive probability at time 0 when (z_1, z_2) is strictly to the left of the curve, player 2 concedes with a positive probability at time 0 when (z_1, z_2) is strictly to the right of the curve, and neither player concedes with a positive probability at time 0 when (z_1, z_2) is on the curve. The probability of initial concession ensures that the posterior reputation vector after initial concession lies on the curve. The reputations coevolve to $(1, 1)$ according to the curve. When player 2's reputation reaches μ_2^*, player 1 stops challenging, and player 1's reputation μ_1^N at the time is derived from the reputation coevolution curve.

Equation (10), and T_2^* the time it takes for player 2 to build a reputation from z_2 to μ_2^* using the dynamics in Equation (11). Finally, we let $T_2 := \min\{T_1^*, T_2^*\}$, and conclude that if $T_i^* > T_2$, then player i concedes at time 0 with a strictly positive probability.

Alternatively, we can trace out a parametric *reputation coevolution curve* $(\mu_1(t), \mu_2(t))$ in the belief plane, which represents the locus of players' reputations for any initial reputations at any time $t > 0$. Because both reputations are characterized analytically, we can represent the graph of the coevolution curve as $\widetilde{\mu}_1(\mu_2)$ for $\mu_2 \in (0, 1]$, or equivalently, its inverse $\widetilde{\mu}_2(\mu_1)$ for $\mu_1 \in \big(\max\{0, \phi_1^* v_1^*\}, 1\big]$, where $\phi_1^* := 1 - \lambda_1/\gamma_1$. The coevolution curve is characterized by

$$\widetilde{\mu}_1(\mu_2) = \begin{cases} \dfrac{\lambda_1 - \gamma_1}{\lambda_1 (\mu_2)^{\frac{\gamma_1 - \lambda_1}{\lambda_2}} - \gamma_1} & \text{if } \mu_2^* < \mu_2 \leqslant 1, \\ \dfrac{\lambda_1 - \gamma_1}{\lambda_1 (\mu_2)^{\frac{\gamma_1 - \lambda_1}{\lambda_2}} + \left(\frac{\gamma_1}{v_1^*} - \gamma_1\right)\left(\frac{\mu_2}{\mu_2^*}\right)^{\frac{\gamma_1 - \lambda_1}{\lambda_2}} - \frac{\gamma_1}{v_1^*}} & \text{if } 0 < \mu_2 \leqslant \mu_2^*, \end{cases}$$

when $\gamma_1 \neq \lambda_1$. When $\gamma_1 = \lambda_1$, this curve is obtained directly from $\mu_1(t)$ and $\mu_2(t)$ or by applying L'Hospital's rule to the above formula, and is explicitly given in the appendix. We can obtain the reputation $\mu_1^N = \widetilde{\mu}_1(\mu_2^*)$ of player 1 when player 2's reputation is μ_2^*.

Figure 1 provides examples of the reputation coevolution curve under two cases. When $\gamma_1 \leqslant \lambda_1$, the curve tends toward $(0,0)$ (Figure 1a), and when $\gamma_1 > \lambda_1$, since player 1's reputation is decreasing for reputation lower than $\phi_1^* \nu_1^*$ in the challenge phase, the curve tends toward $(\phi_1^* \nu_1^*, 0)$ (Figure 1b). When (z_1, z_2) is on the coevolution curve, their reputations situate on the equilibrium path to $(1,1)$, so neither player concedes at time 0 with a strictly positive probability. When (z_1, z_2) is to the left of the curve, that is, $\widetilde{\mu}_2(z_1) < z_2$, or equivalently, $\widetilde{\mu}_1(z_2) > z_1$, player 1 will be the player who concedes with a positive probability at time 0. He must concede with a probability Q_1 such that the pair of his posterior reputation and player 2's initial reputation z_2 exactly falls on the curve:

$$\frac{z_1}{z_1 + (1-z_1)(1-Q_1)} = \widetilde{\mu}_1(z_2) \Longrightarrow Q_1 = 1 - \frac{z_1}{1-z_1} \Bigg/ \frac{\widetilde{\mu}_1(z_2)}{1-\widetilde{\mu}_1(z_2)}. \tag{12}$$

When (z_1, z_2) is to the right of the reputation coevolution curve, player 2 will be the one who concedes with a positive probability at time 0, which raises her reputation if she does not concede at time 0 to lie on the coevolution curve.

This completes our equilibrium characterization. We summarize the resulting equilibrium strategies and beliefs explicitly in Appendix B.

3.2 Equilibrium rates of challenge and resolution

While distributions of challenging and dispute resolution depend on model primitives such as prior reputations and ultimatum opportunity arrival rates, some qualitative features of equilibrium hazard rates do not depend on the fine details of the model. For an unjustified player 1, the equilibrium hazard rate of bluffing is increasing as t approaches T_1, and the rate of conceding is increasing to infinity as t approaches T.[10] Building on these rates, we can derive the overall hazard rates—that is, the aggregate rates by justified and unjustified players—of challenge and resolution.

Namely, an unjustified player 1's equilibrium hazard rate of ultimatum usage increases between time 0 and time T_1 and drops to zero afterward, and an unjustified player 1's equilibrium concession rate increases between time 0 and time T. The overall hazard rate of ultimatum usage increases between time 0 and time T_1, drops from $\frac{\mu_1^N}{\nu_1^*}\gamma_1$—which might be above or below γ_1— to a rate below γ_1, and increases to γ_1 between time T_1 and time T. The overall hazard rate of dispute resolution adds the concession rate $\lambda_1 + \lambda_2$ to the challenge rate before time T, and hence exhibits discontinuities at both times T_1 and T.

A testable prediction of the model is that the hazard rate of resolution in negotiations, which we can observe in many settings, experiences (local) peaks and subsequent discontinuities in three instances: (i) the onset of negotiation, (ii) the moment when an unjustified player stops challenging, and (iii) the moment players stop conceding. The first peak arises when the agreement is reached at the onset of the negotiation, the second peak arises when player 2's reputation approaches the level beyond which player 1 has no incentive to challenge, and the last peak arises when both players' reputations approach 1, beyond which neither player has an incentive to continue the negotiation. We predict some resolution in the middle of the negotiation, in addition to agreements at the onset of the game (predicted by Abreu and Gul (2000) and Fanning (2016)) and before the deadline (predicted by Fanning (2016); Simsek and Yildiz (2016); and

[10] These rates are unique almost everywhere with respect to the F_2 measure and Lebesgue measure.

Vasserman and Yildiz (2019)).[11]

3.3 Multiple demand types

In this section, we consider the case in which there are multiple justifiable demands for both players. Player 1 announces a demand $a_1 \in A_1$ first, and upon observing player 1's announcement, player 2 either accepts the demand, or rejects the demand and announces her own demand $a_2 \in A_2$. Assume A_1 and A_2 are finite; assume that player i's maximal demand is incompatible with all demands of player j: $\max A_i + \min A_j > 1$. The prior conditional probability distribution π_i of demands by a justified player i, in which $\pi_i(a_i)$ specifies the conditional probability of demanding a_i by a justified player, is commonly known. The game then proceeds as in the previous case with one-sided ultimatum opportunities and single demand types for both players. Hence, a game with one-sided ultimatum opportunities and multiple demands is described by the bargaining game $B = (\pi_1, \pi_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$. In addition to choosing their subsequent challenge, concession, and response to challenges, unjustified players choose initial demands to mimic.

Let $\sigma_1 \in \Delta(A_1)$ denote an unjustified player 1's mimicking strategy at the beginning of the game, and $\sigma_2(\cdot|a_1)$ an unjustified player 2's mimicking strategy upon observing player 1's announced demand a_1, where the argument can be either any $a_2 \in A_2$ or $\{0\}$, which indicates the acceptance of player 1's demand a_1.

Theorem 2. *For any bargaining game with ultimatum opportunities for one player and multiple demand types for both players, all equilibria yield the same distribution over outcomes.*

The proof is similar to the proof in AG. The key property—that players' payoffs are monotonic in z_i—is preserved in the current setting, as we will show in the comparative statics exercises. In the proof, we will first consider the intermediate case in which there is only one justified type of player 1 but there are several justified types of player 2. In this case, a unique equilibrium exists. Then we look at the general case in which player 1 first chooses which type $a_1 \in A_1$ to mimic, and seeing this, player 2 responds with a type $a_2 \in A_2$ to mimic. In this case, we show that the distribution of equilibrium outcomes is unique.

Note that the equilibrium outcome does depend on the order of the move. If player 2 announces the demand before player 1, then the distribution of equilibrium outcomes is still unique but potentially different from that when player 1 announces first. However, as we will show, these orders will be irrelevant in the limit case of rationality and rich demand space.

4 Determinants of the bargaining outcome

4.1 Comparative statics

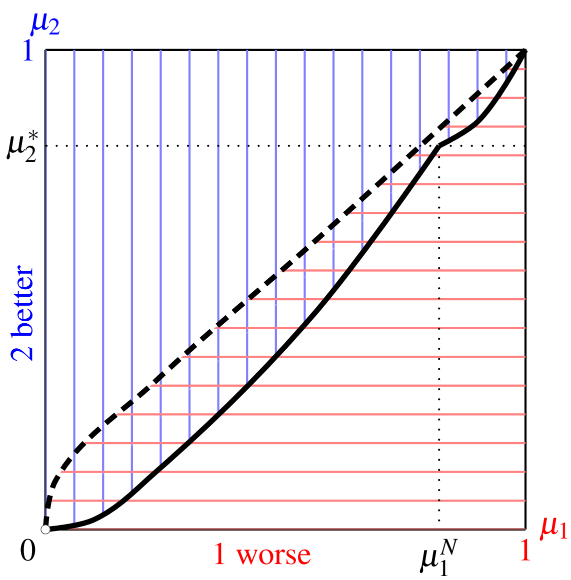
4.1.1 Effects of ultimatum opportunities

The introduction of ultimatum opportunities can have mixed effects. Reputation building may be faster when an unjustified player bluffs at a rate $\widehat{\beta}_1(t)$ higher than γ_1 (when the "no ultimatum is good news" effect dominates the "no ultimatum is bad news" effect). A necessary condition for player 1 to benefit

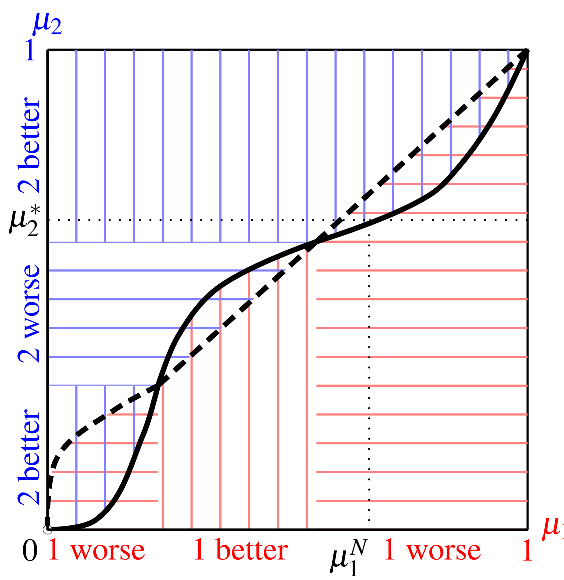
[11] We do not explicitly add a deadline to the model, but if we do, the discontinuity in the hazard rates of challenge and resolution in the middle of the negotiation remains, and there will be a mass of deals near the deadline.

from having ultimatum opportunities is $\mu_1^N > v_1^*$; otherwise, if $\mu_1^N \leqslant v_1^*$, in equilibrium, an unjustified player 1 never challenges at a higher rate than a justified player, and no ultimatum is always bad news on net. However, this condition $\mu_1^N > v_1^*$ is not sufficient for player 1 to benefit from the introduction of the ultimatum opportunity. The sufficient condition for player 1 to benefit from having the challenge opportunity is that it takes a longer time to build a reputation in the current setting than in AG.

Figure 2 illustrates who benefits from the challenge opportunity in the belief plane when $\mu_1^N \leqslant v_1^*$. There may be an intermediate range of initial reputations of player 1 in which he benefits from the introduction of the ultimatum opportunity. We can show that this is always a connected interval bounded away from 0 and 1 when it exists.



(a) 1 is never better off



(b) 1 may be better off

Figure 2: Who benefits from ultimatum opportunities?

Note. The solid line represents the reputation coevolution curve with $\gamma_1 > 0$, and the dashed line is the reputation coevolution curve with $\gamma_1 = 0$, as in AG. With the introduction of a challenge opportunity, player 1 (resp., 2) is strictly worse off if the pair of initial reputations is in the region filled with red (resp., blue) horizontal lines, and is strictly better off if the pair of initial reputations is in the region filled with red (resp., blue) vertical lines.

As a result of the indeterminacy in the benefit in the introduction of the ultimatum opportunity, the effects of a local increase in the ultimatum opportunity arrival rate on strategic players' payoffs are also ambiguous.[12]

However, as the ultimatum opportunities arrive very frequently (i.e., as $\gamma_1 \to \infty$), $\phi_1^* v_1^* \to 1$, and for any given prior, players' payoffs converge to $(1 - a_2, (1 - z_1)a_2 + z_1(1 - a_1))$. In other words, frequent ultimatum opportunities for player 1 cancel out player 1's reputation effects, resulting in player 2 winning the game.

[12]Players' payoffs are also nonmonotonic in demands a_1 and a_2, a result that is similar to Abreu and Gul (2000) and used by Sanktjohanser (2020).

4.1.2 Effects of external resolution

Proposition 1. *Start with a bargaining game $B = (a_1, a_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$ with one-sided ultimatum opportunities and single demand types. When either c_1 increases, k_2 decreases, or w_1 decreases, u_1 strictly decreases if and only if $\widetilde{\mu}_1(z_2) \leqslant z_1 < \mu_1^N$, and u_2 strictly increases if and only if $\widetilde{\mu}_2(z_1) \leqslant z_2 < \mu_2^*$.*

It may not be straightforward to see the unambiguous effects of changes in c_1, k_2, and w_1. For example, when c_1 increases, there are two opposite effects. On one hand, an unjustified player 1 is less likely to challenge because it is more costly. On the other hand, because an unjustified player 1 is less likely to challenge, when facing a challenge player 2 is less likely to face an unjustified player 1, and hence is more likely to yield, which may increase the value of a challenge and hence player 1's payoff. However, in equilibrium, this second effect is moot, because in equilibrium the value of a challenge is taken away by player 2's adjustment of her strategy to render player 1 indifferent between challenging and not challenging. While similar logic applies to changes in k_2 and w_1, obtaining the unambiguous results requires accounting for various shifts in both the speed of reputation building and the threshold beliefs that divide the challenge and no-challenge phases. These details of the court will not affect players' payoffs in the limit case of rationality—i.e., when priors z_1 and z_2 approach 0.

4.1.3 Effects of prior reputations and discount rates

When z_i increases or r_i decreases, u_i strictly increases if and only if $z_i \geqslant \widetilde{\mu}_i(z_j)$, and u_j decreases if and only if $z_i \leqslant \widetilde{\mu}_i(z_j)$. It is unambiguous and relatively straightforward that an unjustified player's payoff strictly decreases when their initial reputation declines or they become more impatient. The first is due to the strict monotonicity of the reputation coevolution curve, and the second is because of the monotonic shift of the curve with respect to a player's concession rate.

4.2 Limit case of rationality

We now investigate the *limit case of rationality*, the case in which the prior probability that each player is justified is small. This case captures situations in which being justified is a rare event and ultimatum is prominently used for strategic posturing.

4.2.1 Single type space

We start with the case in which each player i may have a single justifiable demand a_i. Generically (when $\lambda_1 \neq \gamma_1 + \lambda_2$, to be precise), players divide the surplus efficiently, with one player immediately conceding at time zero in equilibrium.

Proposition 2. *Let $\{B^n\}_n$ be a sequence of games in which for each $n \in \mathbb{N}$, $B^n = (a_1,\ a_2,\ z_1^n,\ z_2^n,\ r_1,\ r_2,\ \gamma_1,\ c_1,\ k_2,\ w_1)$ is a bargaining game with one-sided ultimatum opportunities and single demand types. If $\lim_{n\to\infty} z_1^n = \lim_{n\to\infty} z_2^n = 0$, and u_i^n is the equilibrium payoff for player i in the game B^n, then*

$$\left(\lim_{n\to\infty} u_1^n, \lim_{n\to\infty} u_2^n\right) = \begin{cases} (1-a_2, a_2) & \text{if } \lambda_1 < \gamma_1, \text{ or} \\ & \text{if } \gamma_1 \leqslant \lambda_1 < \gamma_1 + \lambda_2 \text{ and } \lim_{n\to\infty} z_1^n/z_2^n \in (0,\infty), \\ (a_1, 1-a_1) & \text{if } \lambda_1 > \gamma_1 + \lambda_2 \text{ and } \lim_{n\to\infty} z_1^n/z_2^n \in (0,\infty). \end{cases}$$

If $\lambda_1 < \gamma_1$, the reputation coevolution curve approaches the x-axis at the belief $\phi_1^* v_1^*$ (Figure 1b). Hence, for small z_1 and z_2, player 1 concedes at time 0 with a large probability such that conditional on no concession, player 1's reputation jumps above $\phi_1^* v_1^*$; we can verify this from Equation (12).

If $\lambda_1 \geqslant \gamma_1$, the reputation coevolution curve approaches the x-axis at the belief 0. In this case, when the prior probability of being justified goes to zero on the same order for the two players, agreement is efficient, is on the terms of player 1 if $\lambda_1 - \gamma_1 > \lambda_2$, and is on the terms of player 2 if $\lambda_1 - \gamma_1 < \lambda_2$. To see this, note that the derivative of the reputation coevolution curve, $\widetilde{\mu}_2'(\mu_1)$, as μ_2 goes to 0, tends to ∞ if $\lambda_1 - \gamma_1 > \lambda_2$ and tends to 0 if $\lambda_1 - \gamma_1 < \lambda_2$. Hence, as z_1 and z_2 go to 0 on the same order, player 2 in the former case and player 1 in the latter case concede at time 0 with a probability that approaches 1.

Note that the limit payoffs are independent of the details of the arbitration, the cost c_1 of challenging, the cost k_2 of seeing the challenge, and the probability w_1 of winning the challenge. The discount rates r_1 and r_2 and the ultimatum opportunity arrival rate γ_1 do not affect efficiency, although they determine who is the winner (the player who is conceded to immediately) and the loser (the player who concedes immediately) in the game. In particular, the higher the ultimatum opportunity arrival rate γ_1, the more likely player 1 the loser. Hence, unlike the general case in which the ultimatum opportunity may benefit or harm an unjustified player 1, in the limit case of rationality, the ultimatum opportunity is always detrimental to an unjustified player 1.

The intuition for this "independence from the details of external resolution" finding can be gained from the reputation dynamics. When z_1 and z_2 are small, negotiation may last for a long time—i.e., T is long. Moreover, reputation building for player 1 spends most of its time when $\mu_1(t)$ is small. Hence, player 1's reputation increases approximately exponentially, and at the rate $\lambda_1 - \gamma_1$. In other words, it is as if the bad-news effect of not challenging slows the rate of reputation building exactly by γ_1. In light of our discussion in Section 3.1.3, this result shows that the good-news effect of challenging disappears and the bad-news effect persists for player 1 in the limit case of rationality.

Finally, the player who builds reputation with the higher rate is the "winner," i.e., their opponent concedes at time 0 with a positive probability. Because reputations grow exponentially (approximately for player 1), the initial concession probability converges to 1 as z_1 and z_2 approach 0 on the same order. This final part of our analysis is similar to the analysis of Abreu and Gul (2000) and Kambe (1999).

4.2.2 Rich type space

We investigate the limit case of rationality when the set of available demand types for each player is sufficiently rich. The purpose of the analysis is to investigate which types stand out as the ones that are mimicked most often.

For $K \in \mathbb{Z}_{>0}$, let $A^K := \{2/K, 3/K, ..., (K-1)/K\}$ be a set of demands. Each element of A^K corresponds to a commitment type whose demand coincides with that element. Suppose that $\pi_i \in \Delta(A^K)$ with full support, i.e., the prior distribution of player i's type conditional on player i being justified has full support on A^K. Finally, let z_i^n be the probability that player i is a justified type. Hence, $z_i^n \pi_i(k/K)$ is the probability that player i is a justified type who demands k/K, for $k = 2, .., K-1$.

In what follows, we fix K and analyze the equilibrium sequence of a sequence of bargaining games in which the probabilities of each player being justified go to zero on the same order for the two players.

Proposition 3. *Let $\{B^n\}_n$ be a sequence of games in which for each $n \in \mathbb{N}$, $B^n = (\pi_1, \pi_2, z_1^n, z_2^n, r_1, r_2, \gamma_1, c_1, k_2, w_1)$ is a bargaining game with one-sided ultimatum opportunities and rich type spaces. If $\lim_{n\to\infty} z_1^n = \lim_{n\to\infty} z_2^n = 0$, $\lim_{n\to\infty} z_1^n/z_2^n \in (0,\infty)$, and u_i^n is the equilibrium payoff for player i in the n^{th} game of the sequence, then*

$$\liminf u_1^n > \frac{r_2}{\max\{r_1,\gamma_1\}+r_2} - 1/K,$$

$$\liminf u_2^n > \frac{\max\{r_1,\gamma_1\}}{\max\{r_1,\gamma_1\}+r_2} - 1/K.$$

Remark 1. *Proposition 3 implies that* $\limsup u_i^n \leqslant 1 - \liminf u_{-i}^n$*, because the size of the pie is 1. Therefore, as K grows without bound, player 1's limit equilibrium payoff converges to $\frac{r_2}{\max\{r_1,\gamma_1\}+r_2}$, and player 2's limit equilibrium payoff converges to $\frac{\max\{r_1,\gamma_1\}}{\max\{r_1,\gamma_1\}+r_2}$.*

Proposition 3 illustrates how the bargaining power depends on the arrival of ultimatum opportunities in a remarkably simple way. The specific outcome of external resolution does not affect players' payoffs. Moreover, ultimatums have no impact if their arrival rate is smaller than the discount rate, while their arrival rate takes the role of the discount rate otherwise. Finally, when ultimatum opportunities are arbitrarily frequent—i.e., as $\gamma_1 \to \infty$—player 2 guarantees herself the highest justifiable demand.

Proposition 2 shows that the limit equilibrium outcome when each side has a single type is (generically) efficient, i.e., agreement is immediate. Moreover, player 1 wins if $\lambda_1 - \gamma_1 > \lambda_2$, and player 2 wins if $\lambda_1 - \gamma_1 < \lambda_2$. Writing this comparison in terms of the primitives of the model, we have that player 1 wins if

$$r_2(1-a_1) > r_1(1-a_2) + \gamma_1(a_1+a_2-1),$$

and player 2 wins if the strict inequality sign is flipped. Note that in AG, the comparison is between $r_2(1-a_1)$ and $r_1(1-a_2)$—two terms that resemble the marginal costs of waiting that involve only demands and discount rates—to determine the winner. The comparison in our model is complicated by an additional term involving the ultimatum opportunity arrival rate γ_1 and the amount of disagreement D. The addition of the ultimatum opportunity cannot simply be thought of as a discount rate. Player i's problem is to maximize a_i subject to being the winner.

In the case of $\gamma_1 \leqslant r_1$, which includes $\gamma_1 = 0$ in AG as a special case, player 1 can guarantee being the winner by choosing the demand $\max\left\{a_1 \in A^K \middle| a_1 \leqslant \frac{r_2}{r_1+r_2}\right\}$. The result holds because the inequality above can be rearranged as

$$r_2(1-a_1) > (\gamma_1 - r_1)(a_1+a_2-1) + r_1 a_1 \iff r_2 - (r_1+r_2)a_1 > (\gamma_1 - r_1)(a_1+a_2-1).$$

Given the negative term on the right-hand side of the inequality, player 1's Rubinstein-like demand guarantees his being the winner. Analogously, player 2 is the winner if

$$r_1 - (r_1+r_2)a_2 > (-\gamma_1 - r_2)(a_1+a_2-1),$$

and she can guarantee being the winner by demanding $\max\left\{a_2 \in A^K \middle| a_2 \leqslant \frac{r_1}{r_1+r_2}\right\}$.

However, when $r_1 < \gamma_1$, player 1 can no longer guarantee $\max\left\{a_1 \in A^K \middle| a_1 \leqslant \frac{r_2}{r_1+r_2}\right\}$. Rearranging the inequality, we have that player 1 wins if

$$r_2(1-a_1) > (r_1-\gamma_1)(1-a_2)+\gamma_1 a_1 \Longleftrightarrow r_2-(r_2+\gamma_1)a_1 > (r_1-\gamma_1)(1-a_2).$$

Given that the right-hand side of the inequality is negative, but can be close to 0, player 1 can guarantee winning by choosing any demand $a_1 \leqslant \frac{r_2}{\gamma_1+r_2}$.

Conversely, player 2 can guarantee the payoff $\frac{\gamma_1}{\gamma_1+r_2} - 1/K$ by choosing the demand $1-1/K$ (the inequality is flipped whenever a_1 is at least $\frac{r_2}{\gamma_1+r_2}+1/K$). Observe that player 2 guarantees this high payoff by choosing the greediest demand, which increases the disagreement D between the two players, that lowers concession rates λ_i, amplifying the disadvantage to player 1. This is in contrast to the existing results in the literature, in which players tend to make compromise demands to get their Rubinstein-like payoffs.

Note that none of the arguments above depends on the order of moves, so the limit payoffs in a rich type space are independent of the order of players' moves.

5 Frictional bluffing opportunities

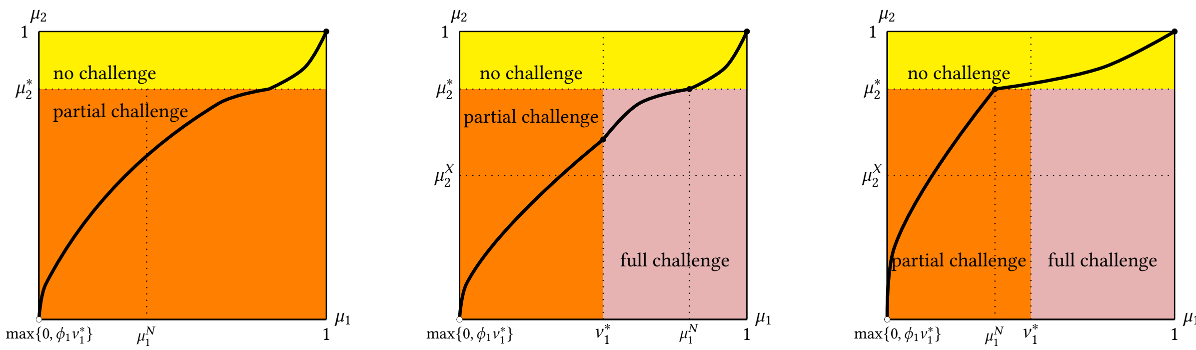
In this section, we consider the alternative situation in which both justified and unjustified players face equally frictional arrival of ultimatum opportunities. Because the ultimatum usage rate is capped by ultimatum opportunity arrival rate, an additional strategy phase with a capped rate of ultimatum usage of player 1 and lower concession of player 2 may arise in equilibrium. Theorem 1 extends and the detailed proof is in Online Appendix C.1. We show that the key qualitative results (e.g., discontinuous ultimatum and resolution rates, the potential benefits of player 1 by the introduction of a frictional ultimatum, and the payoffs in the limit case of rationality) do not change in this setting. We also demonstrate the irrelevance of public versus private arrival of ultimatum opportunities in equilibrium behavior and outcome, and separate justified demand and commitment behavior.

5.1 Frictional arrival of ultimatum opportunities

Suppose an unjustified player 1's ultimatum opportunities arrive according to the same process as a justified player 1's, that is, a Poisson process with rate γ_1, and he can choose whether or not to bluff. Basic properties of the equilibrium will be sustained, that is, there is a positive concession rate over the full support of interval $(0, T]$, and consequently, player i' continuation payoff at time t in that interval is $1-a_j$.

When player 1's reputation exceeds ν_1^*, regardless of his strategy—even when he challenges with probability one—his reputation conditional on challenging exceeds ν_1^*. Hence, when $\mu_1(t) > \nu_1^*$, an unjustified player 2 does not see a challenge. And when $\mu_2(t) < \mu_2^*$, an unjustified player 1 challenges with probability one when an opportunity arrives.

With frictionless arrival of bluffing opportunities, there are two strategy phases: challenge and no-challenge phases that indicate whether an unjustified player 1 challenges with a positive rate (Figure 3a). With frictional arrival of bluffing opportunities, a new "full-challenge" phase may arise between the (partial) challenge and no-challenge phases. This phase arises in equilibrium if and only if $\mu_1^N > \nu_1^*$. In this



(a) Frictionless bluffing (b) Frictional bluffing, $\mu_1^N > v_1^*$ (c) Frictional bluffing, $\mu_1^N \leqslant v_1^*$

Figure 3: Strategy phases in games with one-sided ultimatum opportunities and frictionless or frictional bluffing opportunities

phase, player 1's reputation builds from v_1^* to μ_1^N. Because player 1 receives a strictly higher payoff from challenging than from conceding, player 2's equilibrium concession rate adjusts so that an unjustified player 1 is indifferent between conceding and persisting; the overall concession rate is lowered to

$$\lambda_2^*(t) := \lambda_2 - \gamma_1[\mu_2^* - \mu_2(t)]. \tag{13}$$

In the (partial) challenge phase—when $\mu_1(t) < v_1^*$ and $\mu_2(t) < \mu_2^*$—an unjustified player 1 challenges with probability $\beta_1(t)/\gamma_1 \in (0,1)$ when an opportunity arises, resulting in an effective bluffing rate $\beta_1(t)$, and an unjustified player 2 mixes between yielding to and seeing a challenge with the same probability as specified in the frictionless bluffing opportunity model.

Figure 3 illustrates the two possibilities: The full-challenge phase may (Figure 3b) or may not (Figure 3c) exist. It is worth mentioning that equilibrium existence and uniqueness requires additional arguments than in the benchmark case. The additional complication arises from the lower concession rate of player 2 in the full-challenge phase. If player 2's reputation falls below $\mu_2^X := \mu_2^* - \lambda_2/\gamma_1$ in the full-challenge phase, then her reputation may decline. Additional arguments are needed to show that player 2's reputation exceeds μ_2^X in the full-challenge phase, which ensures an ever-increasing reputation for player 2.

Claim 1. *Suppose player 1 has a Poisson arrival of ultimatum opportunities at rate γ_1. In the unique equilibrium, the overall challenge and concession rates are*

$$(\chi_1(t), \lambda_1(t), \lambda_2(t)) = \begin{cases} (\mu_1(t)\gamma_1, \lambda_1, \lambda_2) & \text{if } \mu_2(t) \geqslant \mu_2^*, \\ (\gamma_1, \lambda_1, \lambda_2^*(t)) & \text{if } \mu_2(t) < \mu_2^* \text{ and } \mu_1(t) \geqslant v_1^*, \\ (\frac{\mu_1(t)}{v_1^*}\gamma_1, \lambda_1, \lambda_2) & \text{if } \mu_2(t) < \mu_2^* \text{ and } \mu_1(t) < v_1^*, \end{cases}$$

and players' reputation dynamics are

$$\dot{\mu}_1(t) = \lambda_1 - \gamma_1 + \chi_1(t) \text{ and } \dot{\mu}_2(t) = \lambda_2(t).$$

Because bluffing rate $\beta_1(t)$ is restricted to be less than γ_1 and a justified player 1's challenge rate is specified to be γ_1, the overall challenge rate $\chi_1(t)$ is less than γ_1. Hence, player 1's reputation building is slower than λ_1, while player 2's reputation building is also slower than λ_2 in the full challenge phase; see Equation (13). This lower $\lambda_2(t)$ may sometimes lead player 2 to build a reputation slowly and lead player 1 to benefit from the introduction of ultimatum opportunities.

Regarding the rates of challenge and resolution, there is an additional discontinuity when the dynamics transition from the partial-challenge phase to the full-challenge phase; in the full-challenge phase, player 2 concedes at a lower rate. Because both players' reputations may build more slowly, the introduction of frictional ultimatum opportunities continues to have an ambiguous effect on an unjustified player 1's payoff in general.

The frictional arrival of bluffing opportunities does not alter the results about the payoffs in the limit case of rationality. This is because, at near zero reputations, the play will be in the partial challenge phase for a long period of time; hence, the impact of the additional phase vanishes.

5.2 Outcome equivalence of public and private arrival of ultimatum opportunities

Suppose the arrival of ultimatum opportunity is public. Player 1 decides whether to use it when it publicly, and if he decides not to use it, he essentially reveals his rationality and gets $1 - a_2$. In this case, player 1's challenge probability given $\mu_1(t)$ is the same as specified above. However, the consequence of no challenge changes. No challenge automatically reveals player 1's rationality and benefits player 2. As a consequence, when $\mu_1(t) < \nu_1^*$, player 1 concedes at a lower rate $\lambda_1^{\text{public}}(t) = \lambda_1 - \gamma_1[1 - \mu_1(t)/\nu_1^*]$, and when $\mu_2(t) > \mu_2^*$, player 1 also concedes at a lower rate $\lambda_1^{\text{public}}(t) = \lambda_1 - \gamma_1[1 - \mu_1(t)]$.

Somewhat surprisingly, whether the ultimatum opportunities are public or private does not affect the outcome of the game. Although the optimal concession behavior changes, the overall concession rates will stay the same. In addition to active/voluntary concession by player 1, there is also passive/involuntary concession by player 1 when the ultimatum opportunity arrives and is publicly known. In addition, the reputation coevolution stays the same. Recall that when the arrival of ultimatum opportunity is private, player 1's reputation is $\dot{\mu}_1^{\text{private}}(t) = \lambda_1^{\text{private}}(t) - [\gamma_1 - \chi_1(t)]$. In contrast, when the arrival is public, its arrival ends the game and its nonarrival does not affect players' reputations. Player 1's reputation evolution is simply $\dot{\mu}_1^{\text{public}}(t) = \lambda_1^{\text{public}}(t)$. When the arrival is public, player 1 still challenges with the overall rate $\chi_1(t)$, but without challenging, player 1's rationality is revealed and player 1 is essentially conceding involuntarily, the overall rate of involuntary concession by player 1 is $\ell_1(t) := \gamma_1 - \chi_1(t)$. Because player 1 may be forced to concede due to the public arrival of ultimatum opportunity, anticipating such passive concession, player 1 will actively concede at a lower rate, and the active concession rate is reduced by exactly the passive concession rate: $\lambda_1^{\text{public}}(t) = \lambda_1^{\text{private}}(t) - \ell_1(t)$. Coupled with the fact that $\ell_1(t)$ and $\chi_1(t)$ sum to γ_1, player 1's reputation evolution with the public and private arrival of ultimatum opportunities is the same.

$$\dot{\mu}_1^{\text{public}}(t) = \lambda_1^{\text{public}}(t) = \lambda_1^{\text{private}}(t) - \ell_1(t) = \lambda_1^{\text{private}}(t) - [\gamma_1 - \chi_1(t)] = \dot{\mu}_1^{\text{private}}(t).$$

In addition, player 2's concession rates ensure player 1's indifference in concessions over time, so her concession rates and consequently her reputation evolution also are the same across the two settings.

5.3 Separation of demand justifiability and commitment behavior

To separate the association between justifiability and commitment behavior, we extend the model to allow a committed player to be unjustified.[13] More precisely, suppose each player i is justified and committed with probability $z_i \psi_i$, is unjustified and committed with probability $z_i(1-\psi_i)$, and is unjustified and strategic with probability $1-z_i$. In the benchmark model, $\psi_i = 1$: a committed player is justified with probability one. In this extension, z_i is the probability of commitment, and we track players' reputation $\mu_i(t)$ of commitment to characterize equilibrium behavior. When a committed player is justified with a sufficiently high probability, the equilibrium structure remains similar to our benchmark case.

A strategic player 2 is indifferent between seeing and yielding to a challenge if player 1's conditional reputation v_1^{**} of commitment satisfies

$$v_1^{**} = \frac{1}{\psi_1}\left(1 - \frac{k_2}{1-w_1}\right) \Longleftarrow [v_1^{**}(1-\psi_1) + 1 - v_1^{**}](1-w_1) = k_2.$$

Note that when $\psi_1 = 1$, $v_1^{**} = v_1^* < 1$, but when $\psi_1 < 1$, it is possible that $v_1^{**} \geqslant 1$.

First, suppose $v_1^{**} < 1$. In this case, the equilibrium characterization is quite similar to the case when $\psi_i = 1$, summarized in Claim 1. All $*$ functions in the claim are replaced by the $**$ functions defined subsequently. The reputation coevolution diagram is qualitatively the same as Figures 3b and 3c. A strategic player does not have an incentive to challenge if the maximal expected payoff is less than concession:

$$\mu_2(t) > \mu_2^{**} = \frac{1-c_1}{1-(1-\psi_2)w_1} \Longleftarrow 1 - \mu_2(t) + \mu_2(t)(1-\psi_2) < c_1.$$

Since $c_1 > (1-\psi_2)w_1$, $\mu_2^{**} < 1$.

Partial-challenge phase. When $\mu_2(t) < \mu_2^{**}$ and $\mu_1(t) < v_1^{**}$, a strategic player 1 challenges with a probability less than one so that player 2 indifferent between seeing and yielding to a challenge:

$$\beta_1^{**}(t)/\gamma_1 = \frac{\mu_1(t)}{1-\mu_1(t)} \Bigg/ \frac{v_1^{**}}{1-v_1^{**}},$$

and a strategic player 2 yields with probability

$$q_2^{**}(\mu_2) = \frac{c_1 - w_1(1-\mu_2\psi_1)}{1-\mu_2-w_1(1-\mu_2)},$$

which coincides with $q_2(\mu_2)$ in Equation (4) when $\psi_1 = 1$. In this phase, player i concedes at AG rate λ_i.

Full-challenge phase. When $\mu_2(t) < \mu_2^{**}$ and $\mu_1(t) \geqslant v_1^{**}$, player 1 challenges with probability one, a strategic player 2 does not see a challenge, but she concedes at a lower rate than λ_2 so that a strategic

[13] Allowing a strategic player to be justified at the same time will completely separate the association between justifiability and commitment behavior. However, it will require tracking multiple state variables of reputation, which is beyond the machinery developed in this paper, but warrants further investigation.

player 1 is indifferent between conceding and not:

$$\lambda_2^{**}(t) = \lambda_2 - \gamma_1[\mu_2^{**} - \mu_2(t)][1 - (1 - \psi_2)w_1].$$

No-challenge phase. When $\mu_2(t) \geqslant \mu_2^{**}$, a strategic player 1 does not challenge; a strategic player 2 sees a challenge if and only if $\mu_1(t) \leqslant \nu_1^{**}$. Each player i concedes at rate λ_i.

Second, suppose $\nu_1^{**} \geqslant 1$. In this case, regardless of a strategic player 1's choice of challenge, player 2 strictly prefers seeing a challenge. Then given player 2's optimal response to a challenge, because $c_1 > w_1$, a strategic player 1 strictly prefers not to challenge, and he is indifferent in concession time when player 2 concedes at rate λ_2. He can adjust his concession rate so that player 2 is indifferent in concession time:

$$\lambda_1^{**}(t) = \lambda_1 - \gamma_1[\mu_1(t)(1 - \psi_1)(1 - w_1) - k_2],$$

which is positive when

$$\mu_1(t) < \left(\frac{\lambda_1}{\gamma_1} + k_2\right)\frac{1}{1 - \psi_1}\frac{1}{1 - w_1} =: \mu_1^{**}.$$

Equilibrium. If $\mu_1^{**} \geqslant 1$, there is a unique equilibrium in which after time zero, player 1 concedes at rate $\lambda_1^{**}(t)$ and player 2 concedes at rate $\lambda_2^{**}(t)$, and their reputation of commitment reaches one at the same time. If $\mu_1^{**} < 1$, there is a unique equilibrium in which a strategic player 1 does not challenge and concedes right away and a strategic player 2 does not concede and waits for a challenge.

6 Extensions

6.1 Alternative resolution mechanisms

We examine alternative specifications of the external resolution outcome. For this examination, we stick with the benchmark model for all other components, including frictionless arrival of challenges for unjustified players. These alternative specifications show the generality of our analysis.

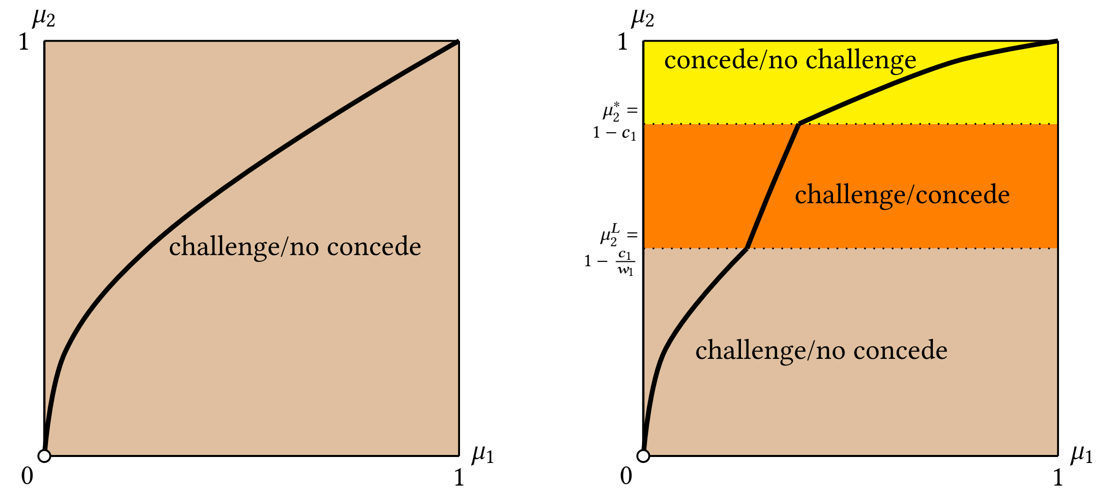
6.1.1 Costless resolution

Consider the case in which external resolution is costless: $c_1 = k_2 = 0$. Because $k_2 = 0$, an unjustified player 2 strictly prefers seeing a challenge. Given player 2's strategy of always seeing, an unjustified player 1 challenges if $\mu_2 < 1 - c_1/w_1$, and does not challenge if $\mu_2 \geqslant 1 - c_1/w_1$. When $c_1 = 0$, an unjustified player 1 strictly prefers challenging to conceding.[14]

There is a unique equilibrium in which player 1 mixes between challenge time without ever conceding. An unjustified player 1 challenges at rate $\beta_1(t)$ so that player 2 is indifferent between conceding now and conceding a moment later, as the cost of waiting balances the benefit:

$$\beta_1(t) = \frac{\lambda_1}{(1 - \mu_1(t))(1 - w_1)}.$$

[14]If $c_1 \geqslant w_1$, then an unjustified player 1 never challenges, and the current setting boils down to a setting without profitable ultimatum opportunities for unjustified players (note that it still differs from AG, because justified players challenge).



(a) Costless or random external resolution
$c_1 = 0$ or win against justified

(b) Noisy external resolution
$0 < c_1 < w_1$

Figure 4: Strategy phases in games with one-sided ultimatum opportunities and alternative external resolution mechanisms

Note. In games with costless or random external resolution ($c_1 = 0$ or an unjustified player can win against a justified player), an unjustified player strictly prefers challenging to conceding, and challenges at a positive rate throughout the game. In games with costly noisy external resolution, there are three phases: an unjustified player 1 (i) challenges at positive rate without concession while 2 concedes and always sees a challenge, (ii) mixes between challenging and conceding while 2 mixes between seeing and yielding to a challenge, and (iii) does not challenge while 2 does not see a challenge.

Player 1's overall challenge rate is

$$\chi_1(t) = \mu_1(t)\gamma_1 + \frac{\lambda_1}{1 - w_1}.$$

Player 2 concedes at overall rate

$$\lambda_2(t) = \frac{\lambda_2 + r_1(1 - \mu_2(t))w_1}{1 - w_1},$$

higher than λ_2, for player 1 to sustain the indifference in challenge across time.

Player 2's reputation evolution is $\dot{\mu}_2(t) = \lambda_2(t)$. Player 1's reputation evolution is

$$\dot{\mu}_1(t) = \frac{\lambda_1}{1 - w_1} - (1 - \mu_1(t))\gamma_1,$$

which is positive if $\mu_1 > 1 - \lambda_1/[\gamma_1(1 - w_1)]$. It is possible that the reputation coevolution curve tending toward the x-intercept $1 - \lambda_1/[\gamma_1(1 - w_1)]$ if it is positive. If it is nonpositive, the curve tends toward the origin instead; Figure 4a illustrates one such reputation coevolution curve. At time zero, either player 1 challenges with a positive probability (if the initial reputations are strictly to the left of the curve), or player 2 concedes with a positive probability (strictly to the right of the curve), or neither (on the curve).

6.1.2 Random resolution

Suppose external resolution is independent of demand justifiability: Player i gets a_i and $1 - a_j$ with equal probability.[15] In this case, for sufficiently small cost k_2, it is a dominant strategy for an unjustified player 2 to see the challenge, and an unjustified player 1 strictly prefers challenging to conceding for sufficiently small cost c_1. Hence, the game is in a similar strategy phase as illustrated by Figure 4a.

An unjustified player 1 never concedes, since he always benefits from challenging. An unjustified player 2 does not yield to a challenge, but mixes between conceding and waiting at each instant. She is indifferent between conceding and waiting when player 1 challenges at the overall rate

$$\chi_1(t) = \frac{r_2(1-a_1)}{D/2 - k_2 D} = \frac{\lambda_1}{1/2 - k_2} =: \chi_{1k},$$

a constant rate. Note that when $k_2 = 0$, $\chi_{1k} = 2\lambda_1$.

An unjustified player 1 is indifferent between challenging at time t and challenging at time $t+dt$, when player 2 chooses a concession rate

$$\lambda_2(t) = \frac{r_1[a_1 + (1-a_2) - 2c_1 D]}{D - 2c_1 D} =: \lambda_{2k},$$

which is again a constant rate. Note that when $c_1 = 0$, $\lambda_{2k} = \lambda_2 + r_1 a_1/D > 2\lambda_2$.

If $\mu_1(t)\gamma_1 > \chi_{1k}$, then the overall challenge rate must be higher than χ_{1k}. Hence, for $\mu_1(t) > \chi_{1k}/\gamma_1$, the indifference cannot be sustained, and an unjustified player 2 strictly prefers conceding at time $t+dt$ to conceding at time t. Because player 2 does not have an incentive to concede, an unjustified player 1 does not have an incentive to challenge over time. In this case, an unjustified player 1 challenges at time zero.

Therefore, when $\gamma_1 > \chi_{1k}$, an unjustified player 1 challenges right away; and when $\gamma_1 \leqslant \chi_{1k}$, player 1 challenges at rate χ_{1k} and player 2 concedes at rate λ_{2k}.

6.1.3 Equal-split resolution

Suppose that equal-split is the external resolution regardless of players' claims or the justifiability of their claims. For example, there is a court that rules randomly or a war that has an equal chance of both sides winning and claiming the entire pie. Suppose $1/2 - c_1 > 1 - a_2$ so that player 1 strictly prefers challenging to conceding, and suppose $1/2 - k_2 > 1 - a_1$ so that player 2 always sees a challenge. At time $t > 0$, player 1 challenges at rate $\chi_1(t)$ and player 2 concedes at rate $\lambda_2(t)$, where $\chi_1(t) = r_2(1-a_1)/(a_1 - k_2 - 1/2)$ and $\lambda_2(t) = r_1(1/2 - c_1)/(a_1 + c_1 - 1/2)$.

6.1.4 Noisy resolution

We have assumed that $w_1 < c_1$ in the benchmark model. Now assume $w_1 > c_1$.[16] In this case, an unjustified player 1 strictly prefers challenging to conceding when player 2 has a sufficiently low reputation, so there is a strategy phase in which player 1 challenges without concession. For a sufficiently high reputation of player 2, challenging is weakly dominated by concession, and the two phases of challenge (with concession)

[15] For example, Lee and Liu (2013) consider bargaining with random settlement in a repeated game setting with a long-run player and a sequence of short-run players.

[16] We maintain the assumption that $k_2 < 1 - w_1$, though it can be relaxed too.

and no challenge exist as in the benchmark model. Figure 4b illustrates the three strategy phases, which combine the strategy phase illustrated in the first three alternative specifications and the two strategy phases in the benchmark model.

Concretely, an unjustified player 1 strictly prefers challenging to conceding at time t if $\mu_2(t) < 1 - c_1/w_1 =: \mu_2^L$.[17] Because player 1 does not concede, for an unjustified player 2 to have an incentive to wait, she must be yielding to a challenge with probability zero, $q_2(t) = 0$, and receives a strictly higher payoff than $1 - a_1$; otherwise, yielding to a challenge results in the same payoff as conceding, which can be done without time delay. Hence, $q_2(t) = 0$, and she is indifferent between conceding at time t and conceding at time $t + dt$ when an unjustified player 1 bluffs at rate

$$\beta_1(t) = \frac{\lambda_1 + k_2\mu_1(t)\gamma_1}{[1-\mu_1(t)](1-w_1-k_2)} = \frac{\lambda_1}{1-w_1}\frac{1}{1-\mu_1(t)}\frac{1}{v_1^*} + \frac{\mu_1(t)}{1-\mu_1(t)}\frac{1-v_1^*}{v_1^*}\gamma_1.$$

Player 2 concedes at overall rate

$$\lambda_2(t) = \frac{\lambda_2 + (1-\mu_2(t))w_1 - c_1}{1-w_1+c_1}.$$

Player 1's reputation evolution is characterized by

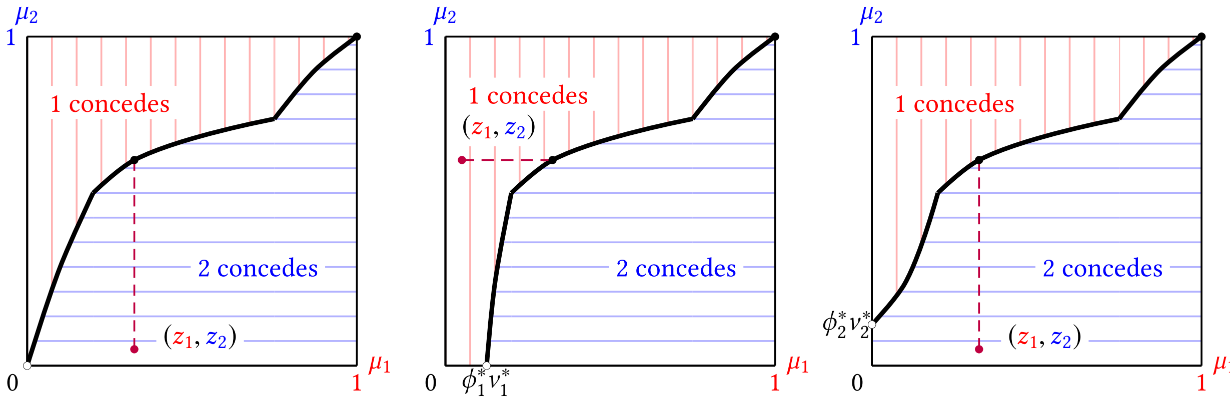
$$\dot{\mu}_1(t) = \chi_1(t) - \gamma_1 = \frac{\lambda_1}{1-w_1}\frac{1}{v_1^*} + \frac{\mu_1(t)}{v_1^*}\gamma_1 - \gamma_1.$$

6.2 Two-sided ultimatum opportunities

Now consider the setting in which both players can challenge. Each player $i \in \{1, 2\}$ has a single demand type a_i such that the amount of disagreement is $D = a_1 + a_2 - 1 > 0$. Specifically, a justified player i challenges according to a Poisson process with arrival rate $\gamma_i \in [0, \infty)$, and to illustrate the main points, an unjustified player i can time a challenge strategically ($\rho_i = \infty$). At each instant t, each unjustified player can (i) give in to the other player's demand, (ii) hold on to their demand, or if an opportunity arrives, (iii) challenge. If the players neither challenge nor concede, then the game continues. Player i who challenges at time t incurs a cost c_iD and player $j \neq i$ must respond to the challenge, by either yielding to the challenge and getting $1 - a_j$, or seeing the challenge by paying a cost k_jD. When player j sees the challenge, the shares of the pie are determined as follows. An unjustified player i's payoff against a justified player j is $1-a_j$. If two unjustified players meet, then the challenging player i wins with probability $w_i < 1/2$: Player i gets a_i with probability w_i and $1-a_j$ with probability $1-w_i$, so the challenging player i's expected payoff is $1 - a_j + w_iD$, and the defending player j's expected payoff is $1 - a_i + (1 - w_i)D$. To make challenging and seeing a challenge worthwhile for player i, assume $w_i < c_i < 1$ and $0 < k_i < 1 - w_i$ for $i = 1, 2$.

In summary, $B = (\{a_i, z_i, r_i, \gamma_i, c_i, k_i, w_i\}_{i=1}^2)$, *a bargaining game with two-sided ultimatum opportunities and single demand types*, is described by demands a_1 and a_2, players' prior probabilities z_1 and z_2 of being justified, discount rates r_1 and r_2, challenge opportunity arrival rates γ_1 and γ_2, bluffing opportunity arrival rates ρ_1 and ρ_2, challenge costs c_1D and c_2D, seeing costs k_1D and k_2D, and unjustified challengers'

[17] The bound is derived from the inequality $[1-\mu_2(t)](1-a_2+w_1D) + \mu_2(t)(1-a_2) - c_1D > 1-a_2$.



(a) $\gamma_1 \leqslant \lambda_1$ and $\gamma_2 \leqslant \lambda_2$. (b) $\gamma_1 > \lambda_1$ and $\gamma_2 \leqslant \lambda_2$. (c) $\gamma_1 \leqslant \lambda_1$ and $\gamma_2 > \lambda_2$.

Figure 5: Reputation coevolution curves and initial concessions in games with two-sided ultimatum opportunities and single demand types when $\gamma_i \leqslant \lambda_i$ for some $i = 1, 2$.

Note. There is a unique equilibrium outcome, and the game ends in finite time. The reputation coevolution curve divides the plane into two regions that differ in the player who concedes with a positive probability at time 0. The curve tends to $(0, 0)$ when $\gamma_1 \leqslant \lambda_1$ and $\gamma_2 \leqslant \lambda_2$, to $(\phi_1^* \nu_1^*, 0)$ when $\gamma_1 > \lambda_1$ and $\gamma_2 \leqslant \lambda_2$, and to $(0, \phi_2^* \nu_2^*)$ when $\gamma_1 \leqslant \lambda_1$ and $\gamma_2 > \lambda_2$.

winning probabilities w_1 and w_2 against unjustified defendants.

Formally, let $\Sigma_i = (F_i, G_i, p_i, q_i)$ denote an unjustified player i's strategy, where $F_i(t)$ is player i's probability of conceding by time t or $p_i(t)$ is player i's probability of challenging when a bluffing opportunity arrives with probability $\beta_i < \infty$, probability of challenging, $G_i(t)$ is player i's probability of challenging by time t, and $q_i(t)$ is player i's probability of conceding to a challenge at time t. Restrict F_i and G_i to be right-continuous and increasing functions with $F_i(t) + G_i(t) \leqslant 1$ for every $t \geqslant 0$, and $q_i(t) \in [0, 1]$ to be a measurable function. We again study the Bayesian Nash equilibrium of this game. The belief process is naturally defined, with $\mu_i(t)$, $\nu_i(t)$, and $\chi_i(t)$ analogously defined as in the game with one-sided ultimatum opportunities.

6.2.1 Slow ultimatum opportunity arrival for at least one player

There is a unique equilibrium outcome under the assumption that $\gamma_i \leqslant \lambda_i := r_j(1 - a_i)/D$ for some $i = 1, 2$. This assumption is automatically satisfied in the the setting with one-sided ultimatum, opportunities which is essentially a setting with two-sided ultimatum opportunities but $\gamma_2 = 0 < \lambda_2$. This condition guarantees that the reputations always increase in equilibrium and the game ends in finite time. The four properties in Theorem 1 are modified to incorporate the possibility of player 2 challenging, as follows. We include Theorem 4 in the online appendix.

Equilibrium strategies are analogous to those in the setting with one-sided ultimatum opportunities: After at most one player concedes initially, each player i concedes at the overall AG concession rate λ_i, each player i challenges at an increasing overall rate $\chi_i(t) = \mu_i(t)\gamma_i/\nu_i^*$ up to time T_i to guarantee a challenger i a reputation $\nu_i^* := 1 - k_j/(1 - w_i)$, the level that renders an unjustified opponent j indifferent between seeing and yielding to a challenge.

Again, the reputation coevolution diagram can be used to determine the player and magnitude of the initial concession. Figure 5 illustrates the three possible reputation coevolution curves when $\gamma_i \leqslant \lambda_i$ for some $i = 1, 2$. When $\gamma_i \leqslant \lambda_i$ for both players (Figure 5a), the reputation coevolution curve tends to $(0, 0)$. When $\gamma_i > \lambda_i$ for some $i = 1, 2$ (Figures 5b and 5c), the reputation coevolution curve tends to the intercept $\phi_i^* v_i^*$, where $\phi_i^* := 1 - \lambda_i/\gamma_i$.

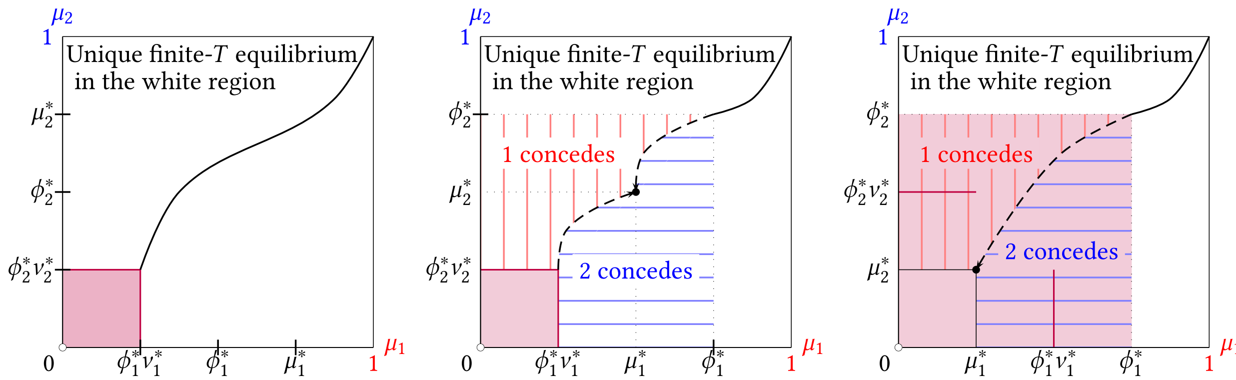
The implications in this setting with two-sided ultimatum opportunities and slow arrival for at least one side are mostly analogous to those in the setting with one-sided ultimatum opportunities. Namely, the hazard rates are discontinuous and piecewise monotonic, with the possibility of having two discontinuities at the finite times when each player ends challenging (modifying the one discontinuity at the finite time when player 1 ends challenging). Ultimatum opportunities may benefit or hurt players, but definitely hurt them in the limit case of rationality, i.e., the case with vanishing probabilities of being justified (preserving the qualitative results of Proposition 2). More precisely, in the limit case of rationality, the outcome is efficient if $\lambda_1 - \gamma_1 \neq \lambda_2 - \gamma_2$, and the winner is player i if $\lambda_i - \gamma_i > \lambda_j - \gamma_j$ (modifying Proposition 2). The comparative statics results in Proposition 1 are generalized for both $i = 1, 2$, with player i's payoff (weakly) hurt by decreasing initial reputation z_i, increasing discount rate r_i, increasing challenging cost c_i, increasing challenge response cost k_i, and decreasing challenge winning probability w_i.

6.2.2 Fast ultimatum opportunity arrival for both players

One main difference from the setting with one-sided ultimatum opportunities is that when $\gamma_i > \lambda_i$ for both $i = 1, 2$ and both players' initial reputations are sufficiently small, there are equilibria in which reputations do not reach 1 and/or do not build up at all, and possibly equilibria with varying initial concession possibilities. Consequently, inefficient infinite delay (i.e., $T = \infty$) may arise. The inefficient infinite delays manifest in two classes of equilibria. In the first class, players concede at AG rates, but their reputations cannot build up because of the fast arrival of ultimatum opportunities for justified types, and consequently they challenge at decreasing rates. Players' reputations approach zero but never reach it. This type of equilibria, with ever declining reputations, exists when both players' initial reputations are sufficiently small. In this case, one of the players may concede with a strictly positive—but sufficiently small—probability at time zero, and still both players experience subsequent declining reputations. This creates the indeterminacy of the initial concessions and the existence of a continuum of equilibria with different initial concession probabilities by different players.[18] In the second class of equilibria, the players concede at AG rates and reputations may decrease or increase toward an absorbing belief $\mu_i^* := 1 - c_i$, the reputation level that renders the opponent indifferent between challenging and not challenging. Upon the reputation reaching this absorbing level, the challenge rates balance the exit of unjustified and justified types for each player such that their reputations, conditional on the game not ending, stay constant at μ_1^* and μ_2^*, respectively. This second class of equilibria may or may not exist, depending on the parameters of the model.

Figure 6 illustrates the regions of initial reputations with these two classes of equilibria with possibly infinite delays. The first class of equilibria always exists when $\gamma_i > \lambda_i$ for both $i = 1, 2$ for a range of initial reputations (the purple areas in the graphs, with the boundary highlighted if such equilibria may exist on it). The second type of equilibria (indicated by the player of initial concession in the graphs)

[18] There may also be equilibria in which one player's reputation stays constant and the other's reputation declines to zero but never reaches it. If the reputations before or after initial concessions lie on the purple lines in Figure 6, such equilibria arise.



(a) $\mu_i^* > \phi_i^*$, $i = 1, 2$. (b) $\mu_i^* \in (\phi_i^* v_i^*, \phi_i^*)$, $i = 1, 2$. (c) $\mu_i^* < \phi_i^* v_i^*$, $i = 1, 2$.

Figure 6: Demonstration of the range of initial reputations with infinite-delay equilibria in bargaining games with two-sided ultimatum opportunities and single demand types when $\gamma_i > \lambda_i$ for both $i = 1, 2$. (a) Type-1 equilibria in which players concede at AG rates for $t > 0$ exist if $z_i \leqslant \phi_i^* v_i^*$, and there is no type-2 equilibrium, one in which both players' reputations eventually converge to (μ_1^*, μ_2^*). (b) Type-1 equilibria exist if $z_i \leqslant \phi_i^* v_i^*$ for both i, and type-2 equilibria exist if $z_i \in (\phi_i^* v_i^*, \phi_i^*)$ for at least one i and $z_i < \phi_i^*$ for both i; (c) Type-1 equilibria exist if $z_i < \phi_i^* v_i^*$ for both i, and type-2 equilibria exist if $z_i \in (\phi_i^* v_i^*, \phi_i^*)$ for at least one i and $z_i < \phi_i^*$ for both i. In the regions not covered, a unique finite-T equilibrium exists.

may not exist (Figure 6a), may exist as the unique equilibrium in a range of initial reputations (Figure 6b), and may coexist with the first class of equilibria for a range of parameters (Figure 6c). Appendix C.3.3 provides a comprehensive description of equilibrium reputations and strategies in this setting.[19] Multiplicity of equilibria arises in previous reputational bargaining models (e.g., Atakan and Ekmekci (2014) and Sanktjohanser (2020)), but to the best of our knowledge, multiplicity due to inefficient infinite delays and reputations not building up is a new feature in the literature.

When $\lambda_i < \gamma_i$ for both $i = 1, 2$, payoffs in the limit case of rationality are indeterminate due to the multiplicity of equilibria. As in the other cases, efficient equilibria with no delay can be sustained in the limit. However, different from the other cases, the most inefficient equilibrium in which player i's payoff is $1 - a_j$ for both $i = 1, 2$ can also be sustained. Thus, the fast arrival of challenge opportunities may be detrimental for efficiency.

The analysis of limit results under multiple demand types is feasible, but will inevitably lead to a multiplicity of outcomes. This multiplicity also carries into the limit case of rationality with a rich type space. Performing a more predictive analysis requires additional criteria to select from multiple equilibria.

7 Relation to literature

Our paper builds on the seminal work of Abreu and Gul (2000), which introduces the two-sided reputational bargaining model.[20] They show the convergence of the equilibrium outcomes of discrete-time bargaining

[19] Note that the three demonstrations do not encapsulate all possible scenarios of the model. For example, the game in which $\mu_1^* > \phi_1^*$ but $\mu_2^* < \phi_2^*$ is not captured. However, in the cases not covered in the demonstrations, no new type of equilibria arises, and the characterization of equilibria falls into one of the three categories described.

[20] Myerson (1991) introduces one-sided reputational bargaining. Subsequent contributions to reputational bargaining include Kambe (1999); Abreu and Pearce (2007); Wolitzky (2011, 2012); Atakan and Ekmekci (2013); Abreu, Pearce, and Stacchetti (2015); and Sanktjohanser (2020). See Fanning and Wolitzky (2020) for a comprehensive survey.

games with incomplete information to the unique equilibrium of a continuous-time war-of-attrition model. We build on their war-of-attrition model by adding the opportunity for players to challenge and seek external resolution. When the exogenous arrival rate of ultimatum opportunities to the justified type is zero, our model is equivalent to AG's model. When this arrival rate is strictly positive, a new possibility of negotiations being resolved arises. Compared with AG, our model requires new techniques and leads to new predictions. Specifically, (i) the addition of ultimatum opportunities results in richer yet tractable strategic behavior and reputation dynamics, solved by new methods and aided by the introduction of reputation coevolution diagrams; (ii) even though external resolution disfavors unjustified players, its availability may benefit them in equilibrium through reputation building; and (iii) the payoffs in the limit case of rationality and rich type spaces depart from AG's payoffs in a simple way when the ultimatum opportunity arrival rate exceeds the discount rate.

Our analysis has two main technical differences from AG's. First, in our model players have a larger strategy space due to the additional challenge opportunities. A priori, players may have more or less incentive for waiting to concede due to anticipation of challenges. However, we show that in equilibrium, a player's payoff when being challenged is equal to the payoff from conceding. Moreover, equilibrium distribution of challenges is continuously strictly increasing up to a finite time, and halts afterward. These findings show that the equilibrium structure of our model is a tractable enrichment of AG's.

Second, in AG's model, players' equilibrium behavior does not depend on their opponent's reputation, whereas in our model it inevitably does, as we note above. AG develops a "forward-looking" method that first calculates the time it takes for each player's reputation to reach 1 in the absence of an initial concession to determine the winning player, and then characterizes the initial concession probability to ensure that players' reputations reach 1 at the same time. This method no longer applies to our model, because of the interdependence of the evolution of players' reputations. Instead, we develop a "backward-looking" method that characterizes players' reputations jointly on a diagram. The reputation coevolution curve, which depicts players' reputations as functions of each other's reputation, characterizes the locus of players' reputations in any equilibrium of all games with all possible initial reputations, after the start of the game. This locus divides the reputation plane into two regions that identify the winning player and the initial concession of the losing player.[21]

Three important features differ from previous literature of reputational bargaining: (i) each player's disagreement payoff depends on the opponent's type, (ii) the distribution of deadlines is endogenous, and (iii) players' outside options are endogenously evolving. The dependence of players' payoffs on players' and opponents' types, has not been studied in reputational bargaining. See Pei (2020) for reputation effects under interdependent values.

The ultimatum in our model can be seen as invoking an immediate deadline. Fanning (2016) studies reputational bargaining with exogenous deadlines, and obtain a monotonic hazard rate of dispute resolution when the deadline distribution is tightly compressed in a time interval. In our model, we assume

[21] We also generalize the locus to regions in the setting with two-sided ultimatum opportunities to represent all equilibrium reputations after initial concessions. Kreps and Wilson (1982) and Fudenberg and Kreps (1987) have a similar representation of the state space by two players' reputations, but they do not use the reputation coevolution curve to derive the probability of initial concession or pin down additional strategy dynamics.

that the arrival rate of ultimatum opportunities to the justified type is constant, yet we obtain a piecewise monotonic rate of dispute resolution in the middle of the negotiation due to the endogeneity of the ultimatum usage rates by strategic players. In addition, we obtain a discontinuity in the hazard rate of resolution due to the endogeneity of the payoffs when an ultimatum is issued. Relatedly, Fanning (2021a,b) studies a reputational bargaining model in which a mediator makes nonbinding recommendations at the beginning of negotiation. In our model, our third party resembles an arbitrator who makes binding resolution when consulted during the negotiation.

Another interpretation of the ultimatum is an endogenously evolving outside option. A player can use an ultimatum to have a third party cast a division of the surplus. Compte and Jehiel (2002) study exogenous outside options that generate a value strictly higher than concession, and show that these high-value outside options cancel out reputation effects. Atakan and Ekmekci (2014) study reputational bargaining in a market setting with many buyers and sellers. In their model, the market serves as the endogenous outside option, and they show that even in the limit case of rationality inefficiency may arise. We obtain a similar inefficiency result when both players can challenge frequently and when the probability of being justified is small. Whereas in Atakan and Ekmekci (2014) the cause of the inefficiency is that the players exercise their outside option when their opponent has built a reputation for being a commitment type, in our model the cause of the inefficiency is the inability of the players to build a reputation. In addition, the models of Özyurt (2014, 2015) share the similarity whereby the value of the outside option depends on the players' evolving reputations, but the motivations and the modeling choices of the papers are different otherwise. There is a further related literature on the exogenous arrival of outside options in bargaining with one-sided incomplete information. In Hwang and Li (2017) and Hwang (2018), not taking an outside option opens up the possibility of nonincreasing reputations and equilibrium multiplicity. Lee and Liu (2013) study the role of incomplete information and outside options in bargaining, but between a long-run player and a sequence of short-run players.

The paper is also related to conflict bargaining models in international relations (Fearon, 1994; Sandroni and Urgun, 2017, 2018). The literature studies situations in which players can end the bargaining process by confronting each other. However, in these models, not ending the bargaining process is more efficient, and the equilibrium dynamics are different from the war-of-attrition dynamics in our paper. Fearon (1994) shows the importance of audience costs (i.e., waiting costs) in bargaining outcome; our limit result shows that the bargaining outcome depends on waiting costs (interest rates) in a simple way, but does not depend on the court costs.

8 Conclusion

We study negotiation when two parties have private information about the justifiability of their demand and have chances to issue an ultimatum to end the bargaining process by verifying the demand justifiability. In our stationary setting, equilibrium hazard rates of ultimatum and conflict resolution are discontinuous and piecewise monotonic in time. The presence of ultimatum opportunity affects reputation building in two opposite directions: The opportunity erodes a player's commitment power, but if used appropriately, the ultimatum can be an effective strategic posture. However, in the limit case of rationality, the ultimatum opportunity is detrimental. For sufficiently fast arrival of ultimatum opportunities, the opportunity arrival rate replaces the discount rate in the determination of the limit payoff of the players.

There are further questions worth exploring. For example, we can model continuous-discrete-time games and study other equilibria in which players' continuation payoffs after revealing rationality do not coincide with their concession payoffs. Another direction would be to include deadlines, and finally, nonstationary arrival rates of ultimatums opportunities or more complex demands such as nonstationary justified demands.

A Omitted proofs

Proof of Theorem 1. Let $\widehat{\Sigma} = (\widehat{\Sigma}_1, \widehat{\Sigma}_2) = ((\widehat{F}_1(\cdot), \widehat{G}_1(\cdot)), (\widehat{F}_2(\cdot), \widehat{q}_2(\cdot)))$ denote an equilibrium strategy profile. We argue that $\widehat{\Sigma}$ must have the form specified in the theorem (hence proving the uniqueness of equilibrium outcome) and that these strategies indeed define an equilibrium (hence proving the existence of equilibrium strategies). Let $u_i(t)$ denote the expected utility of an unjustified player i who concedes at time t. Define $\mathcal{T}_i := \{t | u_i(t) = \max_s u_i(s)\}$ as the set of conceding times that attain the highest expected utility for player i given opponent j's strategy $\widehat{\Sigma}_j$. Because $\widehat{\Sigma}$ is an equilibrium, $\mathcal{T}_i$ is nonempty for $i = 1, 2$. Furthermore, define $\tau_i := \inf\{t \geqslant 0 | \widehat{F}_i(t) = \lim_{s\to\infty} \widehat{F}_i(s)\}$ as the time of last concession for player i, with $\inf \emptyset := \infty$. Finally, the support of player 1's challenge distribution is $[0, \infty)$ due to the justified type's challenge behavior. Hence, in any equilibrium, $\widehat{q}_2(t)$ maximizes player 2's expected payoff at time t when she faces a challenge when player 1's reputation is $\nu_1(t)$ upon challenging, for almost every $t \leqslant \tau_2$ in both the $\widehat{G}_1$ measure and the Lebesgue measure. In the remainder of the proof, we will drop the "almost everywhere" qualifier. We have the following results.

(*a*) **Player 1's challenging strategy $\widehat{G}_1$ is continuous for $t \geqslant 0$.** To show that $\widehat{G}_1$ does not have any atoms, suppose to the contrary that $\widehat{G}_1$ jumps at time t so that an unjustified player 1 challenges with a positive probability at time t; that is, $\widehat{G}_1(t) > 0$ for $t = 0$, or $\widehat{G}_1(t) - \widehat{G}_1(t^-) > 0$ for $t > 0$. Given that an unjustified player 1 challenges with a positive probability and a justified player 1 challenges with probability 0, player 2 facing a challenge believes that a challenging player 1 is unjustified with probability 1: $\nu_1(t) = 0$. Consequently, she is strictly better off responding to the challenge and obtaining a payoff of $1 - a_1 + (1 - w_1)D - k_2 D$ than yielding to the challenge and obtaining a payoff of $1 - a_1$, because $k_2 < 1 - w_1$ by assumption. But if player 2 responds to a challenge with probability 1, an unjustified player 1's payoff from challenging is less than $1 - a_1 + w_1 D - c_1 D$ (an unjustified player 1's expected payoff when the player 2 who responds to a challenge is unjustified with probability 1), which is strictly less than his payoff from conceding, because $c_1 > w_1$ by assumption, so an unjustified player 1 has a profitable deviation to conceding at t from challenging with a positive probability at t, a contradiction.

(*b*) **Player 2's yielding probability $\widehat{q}_2(t)$ is positive for almost all $t \leqslant \tau_2$.** Suppose to the contrary that $\widehat{q}_2(t) = 0$ on a set A of positive Lebesgue measure. Then $\int_A d\widehat{G}_1(t)dt = 0$. Then $\nu_1(t) = 1$ for almost every $t \in A$. Then $\widehat{q}_2(t) = 1$ for $t \in A$ is a profitable deviation, a contradiction.

(c) **Player 2's payoff when being challenged at time t is $1 - a_1$ for almost all $t \leqslant \tau_2$.** Whenever an unjustified player 2 yields to a challenge with a positive probability at time t in equilibrium, her payoff when being challenged at time t is equal to $1 - a_1$. By (b), player 2 yields to a challenge with a positive probability for almost all $t \leqslant \tau_2$, so her payoff when being challenged at time t is $1 - a_1$.

(d) **The last instant at which two unjustified players concede is the same: $\tau_1 = \tau_2$.** An unjustified player will not delay conceding upon learning that the opponent will never concede. Note that even if an unjustified player 1 might challenge with a positive probability but never concedes, an unjustified player 2's payoff from being challenged is $1 - a_1$ (by (c)), so she does not benefit from waiting for a challenge. Denote the last concession time by τ.

(e) **If $\widehat{F}_i$ jumps at t, then $\widehat{F}_j$ does not jump at t for $j \neq i$.** If $\widehat{F}_i$ has a jump at t, then player j receives a strictly higher utility by conceding an instant after t than by conceding exactly at t; note that whether or not player 1 challenges at t does not affect the result, by (c).

(f) **If $\widehat{F}_2$ is continuous at time t, then $u_1(s)$ is continuous at $s = t$. If $\widehat{F}_1$ and $\widehat{G}_1$ are continuous at time t, then $u_2(s)$ is continuous at $s = t$.** These claims follow immediately from the definition of $u_1(s)$ in Equation (1) and the definition of $u_2(s)$ in Equation (2), respectively.

(g) **There is no interval $(t', t'') \subseteq [0, \tau]$ such that both $\widehat{F}_1$ and $\widehat{F}_2$ are constant on the interval (t', t'').** Assume the contrary and without loss of generality, let $t^* \leqslant \tau$ be the supremum of t'' for which (t', t'') satisfies the above properties. Fix $t \in (t', t^*)$ and note that for ε small enough there exists $\delta > 0$ such that $u_i(t) - \delta > u_i(s)$ for all $s \in (t^* - \varepsilon, t^*)$. In words, conditional on the opponent not conceding in an interval, it is strictly better for a player to concede earlier within that interval, and it is sufficiently significantly better by conceding early than by conceding close to the end of the time interval. By (e) and (f), there exists i such that $u_i(s)$ is continuous at $s = t^*$, so for some $\eta > 0$, $u_i(s) < u_i(t)$ for all $s \in (t^*, t^* + \eta)$ (observe that this relies on player 2 not benefiting from waiting for a challenge from player 1, by (c)). In words, because of the continuity of the expected utility function at time t^*, the expected utility of conceding a bit after time t^* is still lower than the expected utility of conceding at time t within the time interval. Since $\widehat{F}_i$ is optimal, $\widehat{F}_i$ must be constant on the interval $(t', t^* + \eta)$. The optimality of $\widehat{F}_i$ implies that $\widehat{F}_j$ is also constant on the interval $(t', t^* + \eta)$, because player j is strictly better off conceding before or after the interval than conceding during the interval. Hence, both functions are constant on the interval $(t', t^* + \eta) \subseteq (t', \tau)$. However, this contradicts the definition of t^*.

(h) **If $t' < t'' < \tau$, then $\widehat{F}_i(t'') > \widehat{F}_i(t')$ for $i = 1, 2$.** If $\widehat{F}_i$ is constant on some interval, then the optimality of $\widehat{F}_j$ implies that $\widehat{F}_j$ is constant on the same interval, for $j \neq i$ (again, by (c)). However, (g) shows that $\widehat{F}_1$ and $\widehat{F}_2$ cannot be constant simultaneously.

(i) **$\widehat{F}_i$ is continuous for $t > 0$.** Assume the contrary: Suppose $\widehat{F}_i$ has a jump at time t. Then $\widehat{F}_j$ is constant on interval $(t - \varepsilon, t)$ for $j \neq i$. This contradicts (h).

1. Strictly increasing $\widehat{F}_1$ and $\widehat{F}_2$ for $t < T$ follows from *(h)*, and constant $\widehat{F}_1$ and $\widehat{F}_2$ for $t \geqslant T$ follows from *(d)*.

2. No atom for $\widehat{F}_i$ follows from *(i)*. At most one atom for $\widehat{F}_1$ and $\widehat{F}_2$ at $t = 0$ follows from *(e)*.

3. (a) $\widehat{G}_1$ has no atom follows from *(a)*, and *(b)* implies that $\widehat{G}_1$ is strictly increasing; if $\widehat{G}_1$ is constant, then $\widehat{q}_2(t) = 1$, which contradicts *(b)*.

   (b) $\widehat{q}_2(t) \in (0, 1)$ for $t \in [0, T_1]$ follows from *(b)*. From *(f)* and *(i)*, it follows that $v_1(t)$ is continuous on $(0, \tau]$. Furthermore, $v_1(t)$ is strictly smaller than $1 - a_1$ when $\mu_2(t) > \mu_2^*$ (i.e., $\widehat{F}_2(t) > 1 - \frac{k_2}{1-z_2}$). Therefore, after $\mu_2(t) > \mu_2^*$, an unjustified player 1 does not challenge. Since player 2's reputation strictly increases over time, there is a finite time T_1 such that player 1 challenges from time 0 to T_1 and does not challenge from T_1 onward. Hence, $\widehat{q}_2(t) = 0$ for $t \geqslant T_1$.

4. It follows from *(h)* that $\mathcal{T}_i$ is dense in $[0, \tau]$ for $i = 1, 2$. From *(d)*, *(f)*, and *(i)*, it follows that $u_i(s)$ is continuous on $(0, \tau]$, and hence $u_i(s)$ is constant for all $s \in (0, \tau]$. Consequently, $\mathcal{T}_i = (0, \tau]$. Hence, $u_i(t)$ is differentiable as a function of t and $du_i(t)/dt = 0$ for all $t \in (0, \tau)$.

   In particular, player 1's expected utility from conceding at time t is

$$u_1(t) = (1 - z_2) \int_0^t a_1 e^{-r_1 s} d\widehat{F}_2(s) + (1 - a_2) e^{-r_1 t} [1 - (1 - z_2) \widehat{F}_2(t)]. \tag{14}$$

   The differentiability of $\widehat{F}_2$ follows from the differentiability of $u_1(t)$ on $(0, \tau)$. Differentiating Equation (14) and applying Leibnitz's rule, we obtain

$$0 = a_1 e^{-r_1 t}(1 - z_2)\widehat{f}_2(t) - (1 - a_2) r_1 e^{-r_1 t}(1 - (1 - z_2)\widehat{F}_2(t)) - (1 - a_2) e^{-r_1 t}(1 - z_2)\widehat{f}_2(t),$$

   where $\widehat{f}_2(t) = d\widehat{F}_2(t)/dt$. This in turn implies $\widehat{F}_2(t) = \frac{1 - C_2 e^{-\lambda_2 t}}{1 - z_2}$, where constant C_2 is yet to be determined. This characterization implies that τ_2 is finite. At $\tau_1 = \tau_2$, optimality for player i implies $\widehat{F}_1(\tau_1) + \widehat{G}_1(\tau_1) = 1$ and $\widehat{F}_2(\tau_2) = 1$.

This completes the proof that the structure of equilibrium strategies is unique. We now proceed to show the uniqueness of equilibrium strategies. We derive the reputation coevolution diagram using the reputation dynamics in Section 3.1.3. The reputation coevolution curve is strictly increasing, and $\widetilde{\mu}_1(\mu_2)$ is well defined for $\mu_2 \in (0, 1]$. Hence, the unique equilibrium entails $F_1(0) = 0$ and $\widehat{F}_2(0) > 0$ if $z_1 < \widetilde{\mu}_1(z_2)$; $\widehat{F}_1(0) > 0$ and $\widehat{F}_2(0) = 0$ if $z_1 > \widetilde{\mu}_1(z_2)$; and $\widehat{F}_1(0) = 0$ and $\widehat{F}_2(0) = 0$ if $z_1 = \widetilde{\mu}_1(z_2)$. Moreover, $F_1(0)$ is uniquely determined by Equation (12), and $\widehat{F}_2(0)$ is uniquely determined analogously. This completes the uniqueness of equilibrium strategies. □

Proof of Proposition 2. We now consider a sequence of games in which all parameters of the game are fixed but the initial probabilities of commitment types, $\{z_1^n, z_2^n\}_n$, satisfy that $\lim \frac{z_1^n}{z_2^n} \in (0, \infty)$ and

$\lim z_1^n = \lim z_2^n = 0$. Recall the reputation coevolution curve for $\mu_2 < \mu_2^F$,

$$\widetilde{\mu}_1(\mu_2|\gamma_1) = \frac{\lambda_1 - \gamma_1}{\lambda_1(\mu_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} + (\frac{\gamma_1}{v_1^*} - \gamma_1)(\frac{\mu_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{v_1^*}}.$$

(i) If $\lambda_1 < \gamma_1$, then

$$\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1(\mu_2|\gamma_1) = v_1^*(\gamma_1 - \lambda_1)/\gamma_1 = [1 - k_2/(1-w)](1 - \lambda_1/\gamma_1) > 0.$$

Therefore, in this case, along the equilibrium sequence of the sequence of games with vanishing probability of commitment types, player 1 concedes at time 0 with a probability converging to 1 (since otherwise after time 0, the reputations would not land on the reputation coevolution diagram). Hence, we obtain efficiency in this case, where players agree on player 2's terms right away, i.e., player 2 is the "winner."
(ii) If $\lambda_1 = \gamma_1$, the expression of $\widetilde{\mu}_1(\mu_2|\gamma_1 \neq \lambda_1)$ becomes

$$\widetilde{\mu}_1(\mu_2|\gamma_1) = \begin{cases} \frac{1}{-\frac{\gamma_1}{\lambda_2}\log(\mu_2)+1} & \text{if } \mu_2^* < \mu_2 < 1, \\ \frac{1}{-\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\log\left(\frac{\mu_2}{\mu_2^*}\right)+\mu_1^N} & \text{if } 0 < \mu_2 \leqslant \mu_2^*, \end{cases}$$

where in this case $\mu_1^N = 1/\left[-\frac{\gamma_1}{\lambda_2}\log(\mu_2^*) + 1\right]$. Hence,

$$\begin{aligned}
\lim_{\mu_2 \to 0} \widetilde{\mu}_1'(\mu_2|\gamma_1 \neq \lambda_1) &= \lim_{\mu_2 \to 0} \frac{\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\frac{1}{\mu_2}}{\left[-\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\log\left(\frac{\mu_2}{\mu_2^*}\right) + \mu_1^N\right]^2} \\
&= \lim_{\mu_2 \to 0} \frac{-\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\frac{1}{\mu_2^2}}{-2\left[-\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\log\left(\frac{\mu_2}{\mu_2^*}\right) + \mu_1^N\right]\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\frac{1}{\mu_2}} = \lim_{\mu_2 \to 0} \frac{\frac{1}{\mu_2}}{2\left[-\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\log\left(\frac{\mu_2}{\mu_2^*}\right) + \mu_1^N\right]} \\
&= \lim_{\mu_2 \to 0} \frac{-\frac{1}{\mu_2^2}}{-2\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\frac{1}{\mu_2}} = \lim_{\mu_2 \to 0} \frac{1}{2\frac{\gamma_1}{v_1^*}\frac{1}{\lambda_2}\mu_2} = \infty,
\end{aligned}$$

where L'Hospital's rule is applied once on each line. Hence, player 2 will be the "winner."
(iii) If $\lambda_1 > \gamma_1$, then

$$\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1(\mu_2|\gamma_1) = 0.$$

If $\lambda_1 > \gamma_1 + \lambda_2$, then

$$\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1'(\mu_2|\gamma_1) = 0,$$

if $\lambda_1 = \gamma_1 + \lambda_2$, then

$$\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1'(\mu_2|\gamma_1) > 0,$$

and if $\lambda_1 < \gamma_1 + \lambda_2$, then

$$\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1'(\mu_2|\gamma_1) = \infty.$$

The limits of $\widetilde{\mu}_1'(\mu_2|\gamma_1)$ above can be derived from the expression of $\widetilde{\mu}_1(\mu_2|\gamma_1)$ for $\mu_2 \leqslant \mu_2^*$, which can be rearranged as

$$\widetilde{\mu}_1(\mu_2|\gamma_1) = \frac{(\lambda_1 - \gamma_1)(\mu_2)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}{\lambda_1 + \gamma_1 \frac{1-\nu_1^*}{\nu_1^*}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*}(\mu_2)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}.$$

The derivative is

$$\widetilde{\mu}_1'(\mu_2|\gamma_1) = (\mu_2)^{\frac{\lambda_1-\gamma_1-\lambda_2}{\lambda_2}} \frac{\left[\lambda_1 + \gamma_1 \frac{1-\nu_1^*}{\nu_1^*}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right](\lambda_1 - \gamma_1)}{\left[\lambda_1 + \gamma_1 \frac{1-\nu_1^*}{\nu_1^*}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*}(\mu_2)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right]^2},$$

which in the limit is

$$\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1'(\mu_2|\gamma_1) = \lim_{\mu_2 \to 0^+} (\mu_2)^{\frac{\lambda_1-\gamma_1-\lambda_2}{\lambda_2}} \frac{\lambda_1 - \gamma_1}{\lambda_1 + \gamma_1 \frac{1-\nu_1^*}{\nu_1^*}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}.$$

The "winner" is player 1 (resp., player 2) if $\lambda_1 >$ (resp., $<$) $\gamma_1 + \lambda_2$, so there is efficiency. □

Proof of Proposition 3. Our result does not depend on the initial order of moves of the players in their demand choice. We will perform the analysis for the case in which player 1 first picks a demand, and then player 2, observing this, chooses her demand, and then the war of attrition starts. Let $\sigma_1^n(i)$ be the equilibrium probability that player 1 chooses type i/K in the n^{th} game, and let $\sigma_2^n(j|i)$ be the equilibrium probability that player 2 chooses type j/K after observing that player 1 chooses i/K in the n^{th} game. Let $\left(\sigma_1, \{\sigma_2(\cdot|i)\}_{i\in\{2,\dots,K-1\}}\right)$ be the limits of these strategies (along a convergent subsequence).

The first case is $\gamma_1 \leqslant r_1$. In this case, if player 1 chooses

$$a_1 = \max\left\{a \in A^K \,\middle|\, a \leqslant \frac{r_2}{r_1 + r_2}\right\},$$

then for any incompatible demand of player 2, $\lambda_1 = \frac{r_2(1-a_1)}{a_1+a_2-1}$ is decreasing in a_2, so it is minimized at $a_2 = (K-1)/K$. In that case, $\lambda_1 > \gamma_1$. Hence, when player 2 makes an incompatible demand, either $\sigma_2(\cdot|a_1) = 0$ or $\sigma_1(a_1) = 0$, and player 1 is the winner, or the winner is determined by the comparison of $\lambda_1 - \gamma_1$ versus λ_2.

$$\begin{aligned} \lambda_1 - \gamma_1 > \lambda_2 \quad &\Longleftrightarrow \quad r_2(1-a_1) - \gamma_1(a_1 + a_2 - 1) > r_1(1-a_2) \\ &\Longleftrightarrow \quad r_2(1-a_1) - \gamma_1 a_1 > (1-a_2)(r_1 - \gamma_1). \end{aligned} \tag{15}$$

It is then routine to verify that if $a_1 = \max\left\{a \in A^K \,\middle|\, a \leqslant \frac{r_2}{r_1+r_2}\right\}$, and if $a_2 > 1 - a_1$, player 1 is the winner.

Turning to player 2 in this case, for any $a_1 > \frac{r_2}{r_1+r_2}$ such that $\sigma_1(a_1) > 0$, player 2 is the winner if she demands $\max\left\{a \in A^K \,\middle|\, a \leqslant \frac{r_1}{r_1+r_2}\right\}$. This is again routine to verify. This completes the proof for $r_1 \geqslant \gamma_1$.

The second case is $\gamma_1 > r_1$. In this case, if player 1 chooses

$$\max\left\{a \in A^K \,\middle|\, a \leqslant \frac{r_2}{\gamma_1 + r_2}\right\},$$

then for any incompatible demand of player 2, $\lambda_1 > \gamma_1$. This is because λ_1 is decreasing in player 2's demand, a_2, and when $a_2 < 1$ and when player 1's demand is not more than $\frac{r_2}{\gamma_1+r_2}$, $\lambda_1 > \gamma_1$. Moreover, the right-hand side of Equation (15), $(1-a_2)(r_1-\gamma_1) < 0$, and the left-hand side, $r_2(1-a_1)-\gamma_1 a_1 \geqslant 0$. Hence, whenever player 2 chooses an incompatible demand a_2 with $\sigma_2(a_2|a_1) > 0$, player 1 is the winner. Hence, player 1 secures the payoff of $\frac{r_2}{\gamma_1+r_2} - 1/K$.

Turning to player 2 in this case, consider the strategy for player 2 of always choosing $a_2 = (K-1)/K$. When player 1's demand, a_1, is less than $\frac{r_2}{r_2+\gamma_1} + 1/K$, player 2's payoff is at least $1 - a_1$, and our claim is true. If $a_1 \geqslant \frac{r_2}{r_2+\gamma_1} + 1/K$, and if $\sigma_1(a_1) > 0$, then

$$\lambda_1 = \frac{(1-a_1)r_2}{a_1 + a_2 - 1} = \frac{(1-a_1)r_2}{a_1 - 1/K} < \gamma_1,$$

which implies that player 2 is the winner. Hence, player 2 secures the payoff of $\frac{\gamma_1}{\gamma_1+r_2} - 1/K$. □

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Online appendices (not for publication)

B Additional omitted details

We present detailed proofs and derivations regarding (i) equilibrium strategies and reputations with one-sided ultimatum opportunities, (ii) comparative statics, (iii) equilibrium existence and uniqueness with multiple demand types, and (iv) equilibrium strategies and reputations with two-sided ultimatum opportunities.

B.1 Applications

We provide a brief description of several applications that can be thought of as negotiations with one-sided and/or two-sided ultimatum opportunities.

(1) Negotiation in the shadow of the law. A plaintiff claims to be entitled to a demand, which must be proven in court, and a defendant may disagree with the claim. Before trial, they engage in negotiations—warning, arraignment, pretrial hearings, and so on. **(a) Patent infringement**. An inventor demands reparations from a firm for an alleged patent infringement. A justified patent owner collects evidence to sue and beat the infringer, but an unjustified patent troll can take the firm to court anytime. **(b) Child support**. A mother of a child demands overdue alimony payments from the father, but the father refuses to pay, alleging that the mother frequently denied his visitation rights. Both sides must collect evidence to defend their claims; to receive the payment, the mother must sue the father. **(c) Renter eviction**. A landlord demands to evict a renter who allegedly violated the terms of the lease agreement (e.g., no smoking or no pets). The burden of proof falls on the landlord, who must share a proportion of the gain with their attorney.

(2) Negotiation in the presence of arbitration. Since 1974 in MLB, a player with between 3 and 6 years of service has been able to ask that his salary be determined by a final-offer arbitration. If the player and club have not agreed on a salary by a deadline in mid-January, they must report their final salary figures and a hearing is scheduled to be held in February. If no settlement can be reached by the hearing date, the case is brought before a panel of arbitrators. After hearing arguments from both sides, the panel selects the salary figure of either the player or the club—but not any price in between—as the player's salary for the upcoming season. The NHL has used a similar arbitration procedure since 1994.

(3) Evidence procurement for auditing. Two parties independently claim their valuations of a firm for sale (in merger and acquisition or bankruptcy cases). They either settle on one party's claim or invite an independent third-party auditor to come up with an estimate, which is expected to be between the two valuations. In bankruptcy cases the seller usually has the right to invite an auditor, and it takes time for the accounting department to submit the necessary files for audit (e.g., some investors still have not received any payment in 2020 from the 2008 Lehman bankruptcy). The auditing and attorneys' fee can be costly; for example, lawyers can claim up to 40% of the winning proceeds.

(4) Negotiation with the threat of war. Two countries are involved in a border dispute. They can peacefully negotiate or settle the conflict by war. Their preparedness for a war is privately known. A country can issue an ultimatum before initiating a war, and the rival country can back down or escalate the situation. If an armed conflict ensues, the stronger side prevails (Fearon, 1994).

(5) Negotiation with the chance to match competing offers. A buyer wants to buy a good from a seller. The buyer may have a purchasing opportunity for a similar product at a discounted price from another seller. Two sides can negotiate with each other while the buyer waits for the outside option to arrive. When the outside option arrives, the buyer issues an ultimatum to the seller, and the seller must decide whether to strike a deal. The seller can verify, for a cost, the existence of the outside option (e.g., by spending time and effort to verify the existence of the claimed outside option). If the buyer presents proof, the seller sells to the buyer at the discounted price. If the buyer does not present proof, the buyer reveals that he is bluffing and buys the good at the seller's requested price.

B.2 One-sided ultimatum opportunities and single demand types

B.2.1 Bernoulli differential equations

Lemma 2. *The solution to the Bernoulli differential equation $\mu'(t) = A\mu(t) + B\mu^2(t)$ given $\mu(0) = \mu^0$ is*

$$\mu(t;\mu^0,A,B) = \begin{cases} 1\Big/\left[\left(\frac{1}{\mu^0}+\frac{B}{A}\right)\exp(-At) - \frac{B}{A}\right] & \text{if } A \neq 0, \\ 1\Big/\left[-Bt + \frac{1}{\mu^0}\right] & \text{if } A = 0. \end{cases}$$

If $\mu^0 > -A/B$, then $\mu'(t) > 0$ for $t \geqslant t^0$, and the time length it takes to reach reputation μ from μ^0 is

$$t(\mu;\mu^0,A,B) = \frac{1}{A}\ln\left(\frac{\frac{1}{\mu^0}+\frac{B}{A}}{\frac{1}{\mu}+\frac{B}{A}}\right).$$

B.2.2 Equilibrium strategies, reputations, and payoffs

Theorem 3. *Consider a bargaining game $B = (a_1, a_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$ with one-sided ultimatum opportunities and single demand types. Equilibrium strategies and reputations $(\widehat{F}_1, \widehat{G}_1, \widehat{F}_2, \widehat{q}_2, \widehat{\mu}_1, \widehat{\mu}_2)$ satisfy*

$$\widehat{f}_i(t) = \exp\left[-\int_0^t \widehat{\kappa}_i(s)ds\right]\widehat{\kappa}_i(t), \text{ where } \widehat{\kappa}_i(s) = 1_{s<T}\frac{\lambda_i}{1-\widehat{\mu}_i(s)};$$

$$\widehat{g}_1(t) = \exp\left[-\int_0^t \widehat{\chi}_1(s)ds\right]\widehat{\chi}_1(t), \text{ where } \widehat{\chi}_1(s) = 1_{s<T-t_2^N}\frac{1-v_1^*}{v_1^*}\frac{\widehat{\mu}_1(s)}{1-\widehat{\mu}_1(s)}\gamma_1;$$

$$\widehat{q}_2(t) = 1_{t<T-t_2^N}\frac{1}{1-w}\left[\frac{c_1}{1-\widehat{\mu}_2(t)} - w\right];$$

$$\widehat{\mu}_i(T-t) = \check{\mu}_i(-t),$$

where

$$\check{\mu}_1(-t) = \begin{cases} \mu(-t;1,\lambda_1-\gamma_1,\frac{\gamma_1}{v_1^*}) & \text{if } t < T - T_1, \\ \mu(t_2^N - t;\mu_1^N,\lambda_1-\gamma_1,\gamma_1) & \text{if } t \geqslant T - T_1, \end{cases}$$

$$\check{\mu}_2(-t) = \mu(-t;1,\lambda_2,0),$$

T_i solves $\breve{\mu}_i(-T_i) = z_i$, and $T = \min\{T_1, T_2\}$. Player i's equilibrium payoff is

$$\widehat{u}_i = 1 - a_j + 1_{z_i \geqslant \widetilde{\mu}_i(z_j)} \left[1 - \frac{z_j}{1 - z_j} \Big/ \frac{\widetilde{\mu}_j(z_i)}{1 - \widetilde{\mu}_j(z_i)} \right] D.$$

B.2.3 Reputation coevolution curves

When player 2's reputation is μ_2^*, player 1's reputation is

$$\mu_1^N := \frac{\lambda_1 - \gamma_1}{\lambda_1 (\mu_2^*)^{\frac{\gamma_1 - \lambda_1}{\lambda_2}} - \gamma_1}.$$

The reputation coevolution curve can be represented by

$$\widetilde{\mu}_1(\mu_2) = \begin{cases} \frac{1}{-\frac{\gamma_1}{\lambda_2} \log(\mu_2) + 1} & \text{if } \mu_2^* < \mu_2 \leqslant 1, \\ \frac{1}{-\frac{\gamma_1}{\nu_1^*} \frac{1}{\lambda_2} \log\left(\frac{\mu_2}{\mu_2^*}\right) + \frac{1}{1 - \frac{\gamma_1}{\lambda_2} \log(\mu_2^*)}} & \text{if } 0 < \mu_2 \leqslant \mu_2^*, \end{cases}$$

when $\gamma_1 = \lambda_1$.Equivalently, the curve is represented by the inverse

$$\widetilde{\mu}_2(\mu_1) = \begin{cases} \left[\left(1 - \frac{\gamma_1}{\lambda_1}\right) \frac{1}{\mu_1} + \frac{\gamma_1}{\lambda_1} \right]^{\frac{\lambda_2}{\gamma_1 - \lambda_1}} & \text{if } \mu_1^N < \mu_1 \leqslant 1, \\ \left[\frac{\frac{\lambda_1 - \gamma_1}{\lambda_1} \frac{1}{\mu_1} + \frac{\gamma_1}{\lambda_1} \frac{1}{\nu_1^*}}{1 + \frac{1 - \nu_1^*}{\nu_1^*} \frac{\gamma_1}{\lambda_1} (\mu_2^*)^{\frac{\lambda_1 - \gamma_1}{\lambda_2}}} \right]^{\frac{\lambda_2}{\gamma_1 - \lambda_1}} & \text{if } \max\left\{0, \left(1 - \frac{\lambda_1}{\gamma_1}\right) \nu_1^*\right\} < \mu_1 \leqslant \mu_1^N. \end{cases}$$

and

$$\widetilde{\mu}_2(\mu_1) = \begin{cases} \exp\left[\frac{\lambda_2}{\gamma_1} \left(1 - \frac{1}{\mu_1}\right) \right] & \text{if } \mu_1^N < \mu_1 \leqslant 1, \\ \mu_2^* \exp\left[\frac{\frac{1}{1 - \frac{\gamma_1}{\lambda_2} \log(\mu_2^*)} - \frac{1}{\mu_1}}{\frac{\gamma_1}{\nu_1^*} \frac{1}{\lambda_2}} \right] & \text{if } \max\left\{0, 1 - \frac{\lambda_1}{\gamma_1}\right\} < \mu_1 \leqslant \mu_1^N. \end{cases}$$

B.3 Comparative statics

B.3.1 Proof of Proposition 1

(i) Effects of z_i. Player 1's payoff can be rearranged as

$$u_1 = 1 - a_2 + 1_{z_1 \geqslant \widetilde{\mu}_1(z_2)} \left[1 - \frac{z_2}{1 - z_2} / \frac{\widetilde{\mu}_2(z_1)}{1 - \widetilde{\mu}_2(z_1)} \right] D.$$

Player 2's payoff can be rearranged as

$$u_2 = 1 - a_1 + 1_{z_2 \leqslant \widetilde{\mu}_2(z_1)} \left[1 - \frac{z_1}{1 - z_1} / \frac{\widetilde{\mu}_1(z_2)}{1 - \widetilde{\mu}_1(z_2)} \right] D.$$

Note that z_i only influences the term that involves the indicator function and that D does not depend on z_i. In particular, the indicator function $1_{z_i \geqslant \widetilde{\mu}_i(z_j)}$ is increasing in z_i and decreasing in z_j, and the term enclosed in the square brackets is also increasing in z_i and decreasing in z_j. Therefore, when z_i increases, player i's payoff strictly increases and player j's payoff strictly decreases only when the condition of the indicator function is satisfied. □

(ii) Effects of r_i. (i) Consider $\partial u_1/\partial r_1$ first. The only term affected by r_1 is

$$-1_{z_1 > \widetilde{\mu}_1(z_2)} \frac{z_1}{1-z_1} / \frac{\widetilde{\mu}_1(z_2)}{1-\widetilde{\mu}_1(z_2)},$$

whose derivative has the same sign as that of $-1_{z_1 > \widetilde{\mu}_1(z_2)} / \widetilde{\mu}_2(z_1)$. The derivative of $-1/\widetilde{\mu}_2(z_1)$ is $\frac{1}{\widetilde{\mu}_2^2(z_1)} \frac{\partial \widetilde{\mu}_2(z_1)}{\partial r_1}$. Therefore, the sign of $\partial u_1/\partial r_1$ is the same as that of $\frac{\partial \widetilde{\mu}_2(z_1)}{\partial r_1}$, whenever $z_1 \geqslant \widetilde{\mu}_1(z_2)$. In the expression of $\widetilde{\mu}_2$, r_1 only enters through the expression of λ_2, which is strictly increasing in r_1. Therefore, the sign of the expression is the same as $\frac{\partial \widetilde{\mu}_2(z_1)}{\partial \lambda_2}$. Mathematically,

$$\frac{\partial u_1}{\partial r_1} = 1_{z_1 > \widetilde{\mu}_1(z_2)} \frac{z_1}{1-z_1} \frac{1}{\widetilde{\mu}_2^2(z_1)} \frac{\partial \widetilde{\mu}_2(z_1)}{\partial \lambda_2} \frac{\partial \lambda_2}{\partial r_1}.$$

For $z_1 \geqslant \mu_1^N$,

$$\widetilde{\mu}_2(z_1) = \left[(1 - \frac{\gamma_1}{\lambda_1}) \frac{1}{z_1} + \frac{\gamma_1}{\lambda_1} \right]^{\frac{\lambda_2}{\gamma_1 - \lambda_1}}.$$

Its derivative with respect to λ_2 is

$$\frac{\partial \widetilde{\mu}_2(z_1)}{\partial \lambda_2} = \widetilde{\mu}_2(z_1) \log \left[\widetilde{\mu}_2^{\frac{1}{\lambda_2}}(z_1) \right] = \widetilde{\mu}_2(z_1) \frac{1}{\lambda_2} \log \left[\widetilde{\mu}_2(z_1) \right] \leqslant 0,$$

as $\widetilde{\mu}_2(z_1) \leqslant 1$. For $z_1 < \mu_1^N$,

$$\widetilde{\mu}_2(z_1) = \left[\frac{\frac{\lambda_1 - \gamma_1}{\lambda_1} \frac{1}{\mu_1} + \frac{\gamma_1}{\lambda_1} \frac{1}{\nu_1^*}}{1 + \frac{1-\nu_1^*}{\nu_1^*} \frac{\gamma_1}{\lambda_1} (\mu_2^*)^{\frac{\lambda_1 - \gamma_1}{\lambda_2}}} \right]^{\frac{\lambda_2}{\gamma_1 - \lambda_1}}.$$

Its partial derivative with respect to λ_2 is

$$\begin{aligned} \frac{\partial \widetilde{\mu}_2(z_1)}{\partial \lambda_2} &= \widetilde{\mu}_2(z_1) \frac{1}{\lambda_2} \underline{\log[\widetilde{\mu}_2(z_1)]} \frac{1}{\gamma_1 - \lambda_1} [\widetilde{\mu}_2(z_1)]^{\frac{\lambda_2}{\gamma_1 - \lambda_1} [\frac{\lambda_2}{\gamma_1 - \lambda_1} - 1]} \underline{(-1)} \frac{\frac{\lambda_1 - \gamma_1}{\lambda_1} \frac{1}{\mu_1} + \frac{\gamma_1}{\lambda_1} \frac{1}{\nu_1^*}}{\left[1 + \frac{1-\nu_1^*}{\nu_1^*} \frac{\gamma_1}{\lambda_1} (\mu_2^*)^{\frac{\lambda_1 - \gamma_1}{\lambda_2}} \right]^2} \cdot \\ & \quad (\mu_2^*)^{\frac{\lambda_1 - \gamma_1}{\lambda_2}} (\lambda_1 - \gamma_1) \underline{\log(\mu_2^*)} \underline{(-\frac{1}{\lambda_2^2})}. \end{aligned}$$

The expression above is negative because the four underlined terms are negative, the terms $\frac{1}{\gamma_1 - \lambda_1}$ and $\lambda_1 - \gamma_1$ multiply to -1, and the other terms are positive.

(ii) Consider $\partial u_2/\partial r_1$ next. We have

$$\frac{\partial u_2}{\partial r_1} = 1_{z_2>\widetilde{\mu}_2(z_1)}\frac{z_1}{1-z_1}\frac{1}{\widetilde{\mu}_1^2(z_2)}(1-a_2)\frac{\partial\widetilde{\mu}_1(z_2)}{\partial\lambda_2}.$$

It remains to show that $\partial\widetilde{\mu}_1(z_2)/\partial\lambda_2 > 0$. It suffices to show that $\partial\log\widetilde{\mu}_1(z_2)/\partial\lambda_2 > 0$. For $z_2 \geqslant \mu_2^*$,

$$\log\widetilde{\mu}_1(z_2) = \log(\lambda_1-\gamma_1) - \log\left[\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \gamma_1\right],$$

and

$$\begin{aligned}\frac{\partial\log\widetilde{\mu}_1(z_2)}{\partial\lambda_2} &= -\frac{\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}(\gamma_1-\lambda_1)[\log(z_2)](-1/\lambda_2^2)}{\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}-\gamma_1}\\ &= \frac{1}{\lambda_1}(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}[-\log(z_2)]\frac{\lambda_1-\gamma_1}{\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}-\gamma_1}\\ &= \frac{1}{\lambda_1}(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\left[\log\left(\frac{1}{z_2}\right)\right]\widetilde{\mu}_1(z_2) > 0.\end{aligned}$$

For $z_2 < \mu_2^*$,

$$\log\widetilde{\mu}_1(z_2) = \log(\lambda_1-\gamma_1) - \log\left[\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} + (\frac{\gamma_1}{v_1^*}-\gamma_1)(\frac{z_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{v_1^*}\right],$$

and

$$\begin{aligned}\frac{\partial\log\widetilde{\mu}_1(z_2)}{\partial\lambda_2} &= -(\gamma_1-\lambda_1)\frac{\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}[\log(z_2)](-\frac{1}{\lambda_2^2}) + (\frac{\gamma_1}{v_1^*}-\gamma_1)(\frac{z_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\left[\log(\frac{z_2}{\mu_2^*})\right](-\frac{1}{\lambda_2^2})}{\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} + (\frac{\gamma_1}{v_1^*}-\gamma_1)(\frac{z_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{v_1^*}}\\ &= -\widetilde{\mu}_1(z_2)\frac{1}{\lambda_2^2}\left\{\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}[\log(z_2)] + (\frac{\gamma_1}{v_1^*}-\gamma_1)(\frac{z_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\left[\log\left(\frac{z_2}{\mu_2^*}\right)\right]\right\} > 0,\end{aligned}$$

where the strict inequality follows from $\log(z_2) < 0$ and $z_2 \leqslant \mu_2^*$.

(iii) Consider $\partial u_1/\partial r_2$ next. We have

$$\frac{\partial u_1}{\partial r_2} = 1_{z_1\geqslant\widetilde{\mu}_1(z_2)}\frac{z_2}{1-z_2}\frac{1}{\widetilde{\mu}_2^2(z_1)}(1-a_1)\frac{\partial\widetilde{\mu}_2(z_1)}{\partial\lambda_1}.$$

It remains to show the sign of $\partial\widetilde{\mu}_2(z_1)/\partial\lambda_1$, which is equivalent to showing the sign of $\partial\log\widetilde{\mu}_2(z_1)/\partial\lambda_1$. For $z_1 \geqslant \mu_1^N$,

$$\log[\widetilde{\mu}_2(z_1)] = \frac{\lambda_2}{\gamma_1-\lambda_1}\log\left[\left(1-\frac{\gamma_1}{\lambda_1}\right)\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\right].$$

Hence,

$$\frac{\partial \log[\widetilde{\mu}_2(z_1)]}{\partial \lambda_1} = \frac{\lambda_2}{(\gamma_1-\lambda_1)^2}\log\left[\left(1-\frac{\gamma_1}{\lambda_1}\right)\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\right]+\frac{\frac{\gamma_1}{\lambda_1^2}\left(\frac{1}{z_1}-1\right)}{\left(1-\frac{\gamma_1}{\lambda_1}\right)\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}}\frac{\lambda_2}{\gamma_1-\lambda_1}.$$

Let $z \equiv [\widetilde{\mu}_2(z_1)]^{\frac{\gamma_1-\lambda_1}{\lambda_2}} = \left(1-\frac{\gamma_1}{\lambda_1}\right)\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}$. Since $z_1 = \widetilde{\mu}_1(\widetilde{\mu}_2(z_1)) = (\lambda_1-\gamma_1)/(\lambda_1 z-\gamma_1)$, $1/z_1-1 = \frac{\lambda_1}{\lambda_1-\gamma_1}(z-1)$. The expression above is simplified to

$$\frac{\partial \log[\widetilde{\mu}_2(z_1)]}{\partial \lambda_1} = \frac{\lambda_2}{(\gamma_1-\lambda_1)^2}\log z - \frac{z-1}{z}\frac{\gamma_1}{\lambda_1^2}\frac{\lambda_1\lambda_2}{(\gamma_1-\lambda_1)^2},$$

which has the same sign as

$$\Delta(z) \equiv \log z + \left(\frac{1}{z}-1\right)\frac{\gamma_1}{\lambda_1}.$$

The first derivative of $\Delta(z)$ above is

$$\Delta'(z) = \frac{1}{z} - \frac{1}{z^2}\frac{\gamma_1}{\lambda_1} = \frac{1}{z}\left(1-\frac{1}{z}\frac{\gamma_1}{\lambda_1}\right),$$

which reaches its extreme at $z^* = \gamma_1/\lambda_1$. The second derivative of $\Delta(z)$ is

$$\Delta''(z) = -\frac{1}{z^2} - (-2)\frac{1}{z^3}\frac{\gamma_1}{\lambda_1} = \frac{1}{z^2}\left(2\frac{1}{z}\frac{\gamma_1}{\lambda_1}-1\right),$$

which is $1/(z^*)^2$, positive at z^*. Therefore, the minimum is reached at the point, and $\Delta(z)$ is decreasing for $z < z^*$ and increasing for $z > z^*$. On one hand, when $\gamma_1 > \lambda_1$, $z \leqslant 1$, the minimum is achieved at $z^* = \gamma_1/\lambda_1 > 1$. As $\Delta(z)$ is decreasing for $z \leqslant 1$, the minimum of $\Delta(z)$ is achieved when $z \leqslant 1$ and $\gamma_1 > \lambda_1$ is achieved at $z^{**} = 1$, which is

$$\Delta(1) = \log 1 + \left(\frac{1}{1}-1\right)\frac{\gamma_1}{\lambda_1} = 0.$$

When $\gamma_1 < \lambda_1$, $z \geqslant 1$, the minimum is achieved at $z^* = \gamma_1/\lambda_1 < 1$. As $\Delta(z)$ is increasing for $z \geqslant 1$, the minimum of $\Delta(z)$ is achieved when $z \geqslant 1$ and $\gamma_1 < \lambda_1$ is achieved at $z^{**} = 1$, which, as calculated above, is $\Delta(1) = 0$. Finally, when $\gamma_1 = \lambda_1$, the minimum is achieved at $z^* = 1$, and the minimum is 0. Therefore, overall, regardless of the parameter, $\Delta(z) \geqslant 0$. Because $\widetilde{\mu}_2(z_1) \neq 1$ and consequently $z \neq 1$, the inequality holds strictly: $\Delta(z) > 0$ for $z \neq 1$. Therefore, $\partial u_1/\partial r_2 > 0$ for $z_1 \geqslant \mu_1^N$.

For $z_1 < \mu_1^N$,

$$\log\widetilde{\mu}_2(z_1) = \frac{\lambda_2}{\gamma_1-\lambda_1}\log\left[\frac{\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\frac{1}{\nu_1^*}}{1+\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}\left(\mu_2^*\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}\right] = \frac{\lambda_2}{\gamma_1-\lambda_1}\log\left[\frac{\lambda_1\frac{1}{z_1}+\gamma_1\frac{1}{\nu_1^*}-\gamma_1\frac{1}{z_1}}{\lambda_1+\frac{1-\nu_1^*}{\nu_1^*}\gamma_1\left(\mu_2^*\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}\right].$$

Denote $x \equiv \left[\widetilde{\mu}_2\left(z_1\right)\right]^{\frac{\gamma_1-\lambda_1}{\lambda_2}} = \frac{\lambda_1 \frac{1}{z_1}+\gamma_1 \frac{1}{v_1^*}-\gamma_1 \frac{1}{z_1}}{\lambda_1+\frac{1-v_1^*}{v_1^*}\gamma_1\left(\mu_2^*\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}$ and $\mu \equiv \left(\mu_2^*\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}$. Then,

$$
\begin{aligned}
& \frac{\partial \log \widetilde{\mu}_2\left(z_1\right)}{\partial \lambda_1} \\
= \quad & \frac{\lambda_2}{\left(\gamma_1-\lambda_1\right)^2} \log x+\frac{\lambda_2}{\gamma_1-\lambda_1} \frac{1}{x} \cdot \\
& \left\{-\frac{\lambda_1 \frac{1}{z_1}+\gamma_1 \frac{1}{v_1^*}-\gamma_1 \frac{1}{z_1}}{\left[\lambda_1+\frac{1-v_1^*}{v_1^*}\gamma_1\left(\mu_2^*\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right]^2}\left[1+\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu} \frac{1}{\lambda_2} \log \left(\mu_2^*\right)\right]+\frac{\frac{1}{z_1}}{\lambda_1+\frac{1-v_1^*}{v_1^*}\gamma_1\left(\mu_2^*\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}\right\},
\end{aligned}
$$

which, because $\frac{\lambda_2}{\left(\gamma_1-\lambda_1\right)^2} \frac{1}{\lambda_1+\frac{1-v_1^*}{v_1^*}\gamma_1\left(\mu_2^*\right)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}>0$, has the same sign as

$$
\begin{aligned}
& \left(\lambda_1+\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu}\right) \log x+\left(\gamma_1-\lambda_1\right) \frac{1}{x}\left[-x-\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu} \frac{1}{\lambda_2} \log \left(\mu_2^*\right) x+\frac{1}{z_1}\right] \\
= & \left(\lambda_1+\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu}\right) \log x+\left(\gamma_1-\lambda_1\right) \frac{1}{x}\left(\frac{1}{z_1}-x\right)-\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu} \log \mu
\end{aligned}
$$

Since $1 / z_1=\frac{\lambda_1 x+\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{x}{\mu}-\frac{\gamma_1}{v_1^*}}{\lambda_1-\gamma_1}=\frac{\frac{\gamma_1}{v_1^*} \frac{1}{x}-\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu}-\lambda_1}{\gamma_1-\lambda_1} x$,

$$
\frac{1}{z_1}-x=\frac{\frac{\gamma_1}{v_1^*} \frac{1}{x}-\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu}-\gamma_1}{\gamma_1-\lambda_1} x .
$$

Plugging this into the previous expression, we have

$$
\left(\lambda_1+\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu}\right) \log x+\frac{\gamma_1}{v_1^*} \frac{1}{x}-\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu}-\gamma_1-\frac{1-v_1^*}{v_1^*}\gamma_1 \frac{1}{\mu} \log \mu
$$

which has the same sign as

$$
\Delta(x)=\left(\frac{\lambda_1}{\gamma_1}+\frac{1-v_1^*}{v_1^*} \frac{1}{\mu}\right) \log x+\frac{1}{v_1^*} \frac{1}{x}-\frac{1-v_1^*}{v_1^*} \frac{1}{\mu}-\frac{1-v_1^*}{v_1^*} \frac{1}{\mu} \log \mu-1 .
$$

Its first derivative is

$$
\Delta^{\prime}(x)=\left(\frac{\lambda_1}{\gamma_1}+\frac{1-v_1^*}{v_1^*} \frac{1}{\mu}\right) \frac{1}{x}-\frac{1}{v_1^*} \frac{1}{x^2} .
$$

Its second derivative is

$$
\Delta^{\prime \prime}(x)=-\left(\frac{\lambda_1}{\gamma_1}+\frac{1-v_1^*}{v_1^*} \frac{1}{\mu}\right) \frac{1}{x^2}+2 \frac{1}{v_1^*} \frac{1}{x^3}=-\frac{1}{x}\left[\left(\frac{\lambda_1}{\gamma_1}+\frac{1-v_1^*}{v_1^*} \frac{1}{\mu}\right) \frac{1}{x}-\frac{1}{v_1^*} \frac{1}{x^2}\right]+\frac{1}{v_1^*} \frac{1}{x^3} .
$$

Therefore, when $\Delta'(x^*) = 0$, $\Delta''(x^*) > 0$. The minimum is reached at

$$x^* = \frac{\frac{1}{v_1^*}}{\frac{\lambda_1}{\gamma_1} + \frac{1-v_1^*}{v_1^*}\frac{1}{\mu}} = \frac{\frac{1}{v_1^*}\frac{\gamma_1}{\lambda_1}}{1 + \frac{1-v_1^*}{v_1^*}\frac{\gamma_1}{\lambda_1}\frac{1}{\mu}},$$

and $\Delta'(x) < 0$ for $x < x^*$ and $\Delta'(x) > 0$ for $x > x^*$. Since

$$(x^*)^{\frac{\lambda_2}{\gamma_1-\lambda_1}} = \left(\frac{\frac{1}{v_1^*}\frac{\gamma_1}{\lambda_1}}{1 + \frac{1-v_1^*}{v_1^*}\frac{\gamma_1}{\lambda_1}\mu}\right)^{\frac{\lambda_2}{\gamma_1-\lambda_1}} > \left(\frac{\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{z_1} + \frac{1}{v_1^*}\frac{\gamma_1}{\lambda_1}}{1 + \frac{1-v_1^*}{v_1^*}\frac{\gamma_1}{\lambda_1}\mu}\right)^{\frac{\lambda_2}{\gamma_1-\lambda_1}} = \widetilde{\mu}_2\left(z_1\right),$$

for $z_1 \leqslant \mu_1^N$, it holds for $z_1 = \mu_1^N$, it holds that $(x^*)^{\frac{\lambda_2}{\gamma_1-\lambda_1}} > \mu_1^N$. When $\gamma_1 > \lambda_1$, $x < \mu < x^*$, so the minimum is attained at μ; when $\gamma_1 < \lambda_1$, $x > \mu > x^*$, so the minimum value is also attained at μ.

$$\Delta(\mu) = \left(\frac{\lambda_1}{\gamma_1} + \frac{1-v_1^*}{v_1^*}\frac{1}{\mu}\right)\log\mu + \frac{1}{v_1^*}\frac{1}{\mu} - \frac{1-v_1^*}{v_1^*}\frac{1}{\mu} - \frac{1-v_1^*}{v_1^*}\frac{1}{\mu}\log\mu - 1 = \frac{\lambda_1}{\gamma_1}\log\mu + \frac{1}{\mu} - 1.$$

Denote this function of μ by ψ:

$$\psi\left(\mu\right) \equiv \frac{\lambda_1}{\gamma_1}\log\mu + \frac{1}{\mu} - 1.$$

Its first derivative is

$$\psi'\left(\mu\right) = \frac{\lambda_1}{\gamma_1}\frac{1}{\mu} - \frac{1}{\mu^2}$$

and its second derivative is

$$\psi''\left(\mu\right) = -\frac{\lambda_1}{\gamma_1}\frac{1}{\mu^2} + \frac{2}{\mu^3} = -\left(\frac{\lambda_1}{\gamma_1}\frac{1}{\mu} - \frac{1}{\mu^2}\right)\frac{1}{\mu} + \frac{1}{\mu^3}.$$

When $\psi'(\mu^*) = 0$, $\psi''(\mu^*) > 0$, so the minimum of $\psi(\mu)$ is achieved at $\mu^* = \gamma_1/\lambda_1$, and $\psi'(\mu) < 0$ for $\mu < \mu^*$ and $\psi'(\mu) > 0$ for $\mu > \mu^*$. When $\gamma_1 > \lambda_1$, $\mu \leqslant 1 < \mu^*$, so the minimum of $\psi(\mu)$ is achieved at $\mu^{**} = 1$; when $\gamma_1 < \lambda_1$, $\mu \geqslant 1 > \mu^*$, so the minimum of $\psi(\mu)$ is also achieved at $\mu^{**} = 1$; and when $\gamma_1 = \lambda_1$, $\mu^* = 1$, so the minimum of $\psi(\mu)$ is also achieved at $\mu^{**} = 1$. The minimum value is $\psi(\mu^{**}) = 0$. Therefore, the sign of $\partial\log\widetilde{\mu}_2\left(z_1\right)/\partial\lambda_1$, and consequently the sign of $\partial u_1/\partial r_2$, is positive for any $z_1 < \mu_1^N$.

(ii) Consider $\partial u_2/\partial r_2$. We have

$$\frac{\partial u_2}{\partial r_2} = 1_{z_2 > \widetilde{\mu}_2(z_1)}\frac{z_1}{1-z_1}\frac{1}{\widetilde{\mu}_1^2(z_2)}(1-a_1)\frac{\partial\widetilde{\mu}_1(z_2)}{\partial\lambda_1}.$$

It remains to show the sign of $\partial\widetilde{\mu}_1(z_2)/\partial\lambda_1$, which is equivalent to showing the sign of $\partial\log\widetilde{\mu}_1(z_2)/\partial\lambda_1$. When $\mu_2^* \leqslant z_2 \leqslant 1$,

$$\widetilde{\mu}_1(z_2) = \frac{\lambda_1 - \gamma_1}{\lambda_1\left(z_2\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \gamma_1}.$$

Hence

$$\frac{\partial\widetilde{\mu}_1(z_2)}{\partial\lambda_1} = -\frac{(\lambda_1-\gamma_1)\left[(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\log\left(z_2^{\frac{1}{\lambda_2}}\right)\right]}{\left[\lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \gamma_1\right]^2} + \frac{1}{\lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \gamma_1}$$
$$= \frac{\lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \gamma_1 + (\gamma_1-\lambda_1)\left[(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\log\left(z_2^{\frac{1}{\lambda_2}}\right)\right]}{\left[\lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \gamma_1\right]^2},$$

which has the same sign as

$$\lambda_1 - \gamma_1/\left(z_2^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\right) + \gamma_1 - \lambda_1 - \lambda_1\log\left(z_2^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\right).$$

Let $z \equiv z_2^{\frac{\gamma_1-\lambda_1}{\lambda_2}}$. The expression becomes

$$\Delta(z) \equiv \gamma_1 - \gamma_1/z - \lambda_1 \log z.$$

Its first derivative is

$$\Delta'(z) = \gamma_1/z^2 - \lambda_1/z = \frac{1}{z^2}\left(\gamma_1 - \lambda_1 z\right).$$

Its second derivative is

$$\Delta''(z) = -2\gamma_1/z^3 + \lambda_1/z^2 = \frac{1}{z^3}\left(\lambda_1 z - 2\gamma_1\right).$$

It reaches the extreme at $z^* = \gamma_1$, and at the extreme, $\Delta''(z^*) = -\gamma_1/(z^*)^3 < 0$. Therefore, the maximum is reached at $z^* = \gamma_1/\lambda_1$ for any given γ_1, and $\Delta(z)$ is increasing in z for $z < z^*$ and decreasing in z for $z > z^*$. When $\gamma_1 > \lambda_1$, $z \leqslant 1$, so the maximum is reached at $z^{**} = 1 < \gamma_1/\lambda_1$. If $\gamma_1 \leqslant \lambda_1$, $z \geqslant 1$, so the maximum is reached at $z^{**} = 1 > \gamma_1/\lambda_1$. Altogether, $\Delta(z) \leqslant 0$, and when $z_2 \neq 1$, $\Delta(z) < 0$, thus $\partial u_2/\partial r_2 < 0$ for $z_2 \geqslant \mu_2^*$. For $z_2 < \mu_2^*$, For $z_2 < \mu_2^*$,

$$\widetilde{\mu}_1(z_2) = \frac{\lambda_1 - \gamma_1}{\lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} + \left(\frac{\gamma_1}{\nu_1^*} - \gamma_1\right)\left(\frac{z_2}{\mu_2^*}\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*}}.$$

Then,

$$\frac{\partial\widetilde{\mu}_1(z_2)}{\partial\lambda_1} = \frac{1}{\left[\lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} + \left(\frac{\gamma_1}{\nu_1^*} - \gamma_1\right)\left(\frac{z_2}{\mu_2^*}\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*}\right]^2} \times$$
$$\left\{\lambda_1 (z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} + \left(\frac{\gamma_1}{\nu_1^*} - \gamma_1\right)\left(\frac{z_2}{\mu_2^*}\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*} + \right.$$

$$
(\gamma_1-\lambda_1)\left[(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}-\lambda_1(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\log\left(z_2^{\frac{1}{\lambda_2}}\right)-\left(\frac{\gamma_1}{\nu_1^*}-\gamma_1\right)\left(\frac{z_2}{\mu_2^*}\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}\frac{1}{\lambda_2}\log\left(\frac{z_2}{\mu_2^*}\right)\right]\Bigg\}.
$$

Let $z \equiv z_2^{\frac{\gamma_1-\lambda_1}{\lambda_2}}$ and $\mu \equiv \left(\mu_2^*\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}$. The expression has the same sign as

$$
\lambda_1 z+\left(\frac{\gamma_1}{\nu_1^*}-\gamma_1\right)\frac{z}{\mu}-\frac{\gamma_1}{\nu_1^*}+(\gamma_1-\lambda_1)z-\lambda_1 z\log z-\left(\frac{\gamma_1}{\nu_1^*}-\gamma_1\right)\frac{z}{\mu}\log\left(\frac{z}{\mu}\right),
$$

which, because $z>0$, has the same sign as

$$
\lambda_1+\frac{1-\nu_1^*}{\nu_1^*}\frac{1}{\mu}\gamma_1-\frac{\gamma_1}{\nu_1^*}\frac{1}{z}+\gamma_1-\lambda_1-\lambda_1\log z-\frac{1-\nu_1^*}{\nu_1^*}\frac{1}{\mu}\gamma_1\log\left(\frac{z}{\mu}\right),
$$

and, because $\gamma_1>0$, has the same sign as

$$
1+\frac{1-\nu_1^*}{\nu_1^*}\frac{1}{\mu}-\frac{1}{\nu_1^*}\frac{1}{z}-\frac{\lambda_1}{\gamma_1}\log z-\frac{1-\nu_1^*}{\nu_1^*}\frac{1}{\mu}\log\left(\frac{z}{\mu}\right)\equiv\Delta(z).
$$

Its first derivative is

$$
\Delta'(z)=\frac{1}{z}\left[\frac{1}{\nu_1^*}\frac{1}{z}-\left(\frac{\lambda_1}{\gamma_1}+\frac{1-\nu_1^*}{\nu_1^*}\frac{1}{\mu}\right)\right].
$$

It reaches the extreme at

$$
z^*=\frac{\frac{1}{\nu_1^*}}{\frac{\gamma_1}{\lambda_1}+\frac{1-\nu_1^*}{\nu_1^*}\frac{1}{\mu}}.
$$

Its second derivative is

$$
\Delta''(z)=-\frac{1}{z^2}\left[2\frac{1}{\nu_1^*}\frac{1}{z}-\left(\frac{\lambda_1}{\gamma_1}+\frac{1-\nu_1^*}{\nu_1^*}\frac{1}{\mu}\right)\right].
$$

At z^*, $\Delta''(z^*)=-\frac{1}{(z^*)^2}\left(\frac{\lambda_1}{\gamma_1}+\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}\frac{1}{\mu}\right)<0$, so the maximum value is reached at z^*.

$$
(z^*)^{\frac{\lambda_2}{\gamma_1-\lambda_1}}=\left(\frac{\frac{\gamma_1}{\lambda_1}\frac{1}{\nu_1^*}}{1+\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}\frac{1}{\mu}}\right)^{\frac{\lambda_2}{\gamma_1-\lambda_1}}>\left(\frac{\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\frac{1}{\nu_1^*}}{1+\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}\frac{1}{\mu}}\right)^{\frac{\lambda_2}{\gamma_1-\lambda_1}}
$$

for any $z_1 \leqslant \mu_1^N$; in particular, the inequality holds for $z_1=\mu_1^N$, in which case the right-hand side is μ. Hence, $z^*>\mu$. However, when $\gamma_1>\lambda_1$, $z<\mu$. Therefore, $\Delta(z)$ reaches maximum value at μ, and has value

$$
\Delta(\mu)=1-\frac{1}{\mu}-\frac{\lambda_1}{\gamma_1}\log\mu.
$$

Let's denote by $\psi(\mu)$ the function that treats μ as a variable. Its first derivative is

$$
\psi'(\mu)=\frac{1}{\mu^2}-\frac{\lambda_1}{\gamma_1}\frac{1}{\mu}=\frac{1}{\mu^2}\left(1-\frac{\lambda_1}{\gamma_1}\mu\right).
$$

The extreme value of $\psi(\mu)$ is reached at $\mu^* = \gamma_1/\lambda_1$, and it is strictly decreasing when $\mu > \mu^*$ and strictly increasing when $\mu < \mu^*$, so the maximum value is reached at μ^*. The maximum of $\psi(\mu)$ is

$$\psi(\mu^*) = 1 - \frac{\lambda_1}{\gamma_1} - \frac{\lambda_1}{\gamma_1}\log\left(\frac{\gamma_1}{\lambda_1}\right).$$

When $\gamma_1 > \lambda_1$, $\mu \leqslant 1 < \mu^*$, so when $\gamma_1 > \lambda_1$, $\psi(\mu)$ for $\mu \leqslant 1$ reaches its maximum value $\psi(1) = 0$ at 1. When $\gamma_1 < \lambda_1$, $\mu \geqslant 1 > \mu^*$, so when $\gamma_1 < \lambda_1$, $\psi(\mu)$ for $\mu \geqslant 1$ also reaches its maximum value $\psi(1) = 0$ at 1. Finally, when $\gamma_1 = \lambda_1$, $\psi(\mu) < \psi(\mu^*) = \Delta(1) = 0$.

Altogether, $\Delta(z) \leqslant \Delta(\mu) = \psi(\mu) < \psi(1) = 0$. Hence, $\partial\widetilde{\mu}_1(z_2)/\partial\lambda_1 < 0$ for $z_2 < \mu_2^*$. □

(iii) Effects of c_1**.** Consider $\partial u_1/\partial c_1$ first. The cost c_1 only affects u_1 through μ_2^* in $\widetilde{\mu}_2(z_1)$ when z_1 is sufficiently small (to be precise, when $z_1 \leqslant \mu_1^N$). Formally,

$$\frac{\partial u_1}{\partial c_1} = 1_{z_1 \geqslant \widetilde{\mu}_1(z_2)} \frac{z_2}{1-z_2} \frac{1}{\widetilde{\mu}_2^2(z_1)} \frac{\partial \widetilde{\mu}_2(z_1)}{\partial \mu_2^*} \frac{\partial \mu_2^*}{\partial c_1}.$$

Because $\mu_2^* = 1 - c_1$, $\partial\mu_2^*/\partial k = -1$, and as a result, $\partial u_1/\partial c_1$ has the opposite sign as $\partial\widetilde{\mu}_2(z_1)/\partial\mu_2^*$, which is zero when $z_1 > \mu_1^N$, and is the following expression when $z_1 \leqslant \mu_1^N$:

$$\frac{\lambda_2}{\gamma_1-\lambda_1}\left[\frac{\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{\mu_1}+\frac{\gamma_1}{\lambda_1}\frac{1}{\nu_1^*}}{1+\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}\right]^{\frac{\lambda_2}{\gamma_1-\lambda_1}-1} \frac{\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{\mu_1}+\frac{\gamma_1}{\lambda_1}\frac{1}{\nu_1^*}}{\left[1+\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right]^2} \frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}-1}\frac{\gamma_1=\lambda_1}{\lambda_2}.$$

The expression above is positive, because it can be simplified to

$$\widetilde{\mu}_2(z_1)\frac{\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}{1+\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}},$$

which is a product of all positive terms. Overall, the expression of $\partial u_1/\partial c_1$ becomes

$$\frac{\partial u_1}{\partial c_1} = -1_{z_1 \geqslant \widetilde{\mu}_1(z_2)} \frac{1}{1-z_2} \frac{z_2}{\widetilde{\mu}_2(z_1)} \frac{\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}{1+\frac{1-\nu_1^*}{\nu_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}.$$

Therefore, when $z_1 \leqslant \mu_1^N$, $\partial u_1/\partial c_1 < 0$.

Consider $\partial u_2/\partial c_1$ next. The cost coefficient c_1 only affects u_2 through μ_2^* in $\widetilde{\mu}_1(z_2)$ when z_2 is sufficiently small, when $z_2 \leqslant \mu_2^*$, to be precise. Formally,

$$\frac{\partial u_2}{\partial c_1} = 1_{z_2 \geqslant \widetilde{\mu}_2(z_1)} \frac{z_1}{1-z_1} \frac{1}{\widetilde{\mu}_1^2(z_2)} \frac{\partial \widetilde{\mu}_1(z_2)}{\partial \mu_2^*} \frac{\partial \mu_2^*}{\partial k_2}.$$

In the terms of the product, $\partial\mu_2^*/\partial k_2 = -1$ and when $z_2 \leqslant \mu_2^*$,

$$
\begin{aligned}
& \frac{\partial\widetilde{\mu}_1(z_2)}{\partial\mu_2^*} \\
= & -\frac{\lambda_1-\gamma_1}{\left[\lambda_1(\mu_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}+(\frac{\gamma_1}{v_1^*}-\gamma_1)(\frac{\mu_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}}-\frac{\gamma_1}{v_1^*}\right]^2}(\frac{\gamma_1}{v_1^*}-\gamma_1)(\mu_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}-1}\frac{\lambda_1-\gamma_1}{\lambda_2} \\
= & -\widetilde{\mu}_1^2(z_2)\frac{1}{\lambda_2}\left(\frac{\gamma_1}{v_1^*}-\gamma_1\right)(z_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}-1}.
\end{aligned}
$$

Overall, with some rearrangements,

$$
\frac{\partial u_2}{\partial c_1} = 1_{\widetilde{\mu}_2(z_1)\leqslant z_2\leqslant\mu_2^*}\frac{z_1}{1-z_1}\frac{1-v_1^*}{v_1^*}\frac{\gamma_1}{\mu_2^*\lambda_2}\left(\frac{z_2}{\mu_2^*}\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}}.
$$

Therefore, $\partial u_2/\partial c_1$ is positive when $\widetilde{\mu}_2(z_1) \leqslant z_2 \leqslant \mu_2^*$. □

(iv) Effects of k_2. Consider $\partial u_1/\partial k_2$ first.

$$
\begin{aligned}
\frac{\partial u_1}{\partial k_2} &= 1_{z_1\geqslant\widetilde{\mu}_1(z_2)}1_{z_1\leqslant\mu_1^N}\frac{z_2}{1-z_2}\frac{1}{\widetilde{\mu}_2^2(z_1)}\frac{\partial\widetilde{\mu}_2(z_1)}{\partial v_1^*}\frac{\partial v_1^*}{\partial k_2} \\
&= -1_{\widetilde{\mu}_1(z_2)\leqslant z_1\leqslant\mu_1^N}\frac{z_2}{1-z_2}\frac{1}{\widetilde{\mu}_2^2(z_1)}\frac{1}{1-w_1}\frac{\partial\widetilde{\mu}_2(z_1)}{\partial v_1^*}.
\end{aligned}
$$

The sign of $\partial\widetilde{\mu}_2(z_1)/\partial v_1^*$ is the same as the sign of $\partial\log\widetilde{\mu}_2(z_1)/\partial v_1^*$, so we consider the latter. When $z_1 \leqslant \mu_1^N$,

$$
\log\widetilde{\mu}_2(z_1) = \frac{\lambda_2}{\gamma_1-\lambda_1}\left\{\log\left[\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\frac{1}{v_1^*}\right]-\log\left[1+\frac{1}{v_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}-\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right]\right\}.
$$

Hence,

$$
\begin{aligned}
\frac{\partial\log\widetilde{\mu}_2(z_1)}{\partial v_1^*} &= \frac{\lambda_2}{\gamma_1-\lambda_1}\Bigg\{\left[\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\frac{1}{v_1^*}\right]^{-1}\frac{\gamma_1}{\lambda_1}\frac{1}{(v_1^*)^2}(-1) \\
&\quad -\left[1+\frac{1}{v_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}-\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right]^{-1}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\left(\frac{1}{v_1^*}\right)^2(-1)\Bigg\} \\
&= \frac{\lambda_2}{\lambda_1-\gamma_1}\frac{\gamma_1}{\lambda_1}\frac{1}{(v_1^*)^2}\left[\frac{1}{\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\frac{1}{v_1^*}}-\frac{(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}{1+\frac{1}{v_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}-\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}\right],
\end{aligned}
$$

which has the same sign as

$$
\begin{aligned}
& \frac{1}{\lambda_1-\gamma_1}\left[1+\frac{1}{v_1^*}\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}-\frac{\gamma_1}{\lambda_1}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}-(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\left(\frac{\lambda_1-\gamma_1}{\lambda_1}\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\frac{1}{v_1^*}\right)\right] \\
= & \frac{1}{\lambda_1-\gamma_1}\left\{1-\left[\left(1-\frac{\gamma_1}{\lambda_1}\right)\frac{1}{z_1}+\frac{\gamma_1}{\lambda_1}\right](\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\lambda_1-\gamma_1} - \left(\frac{1}{\lambda_1}\frac{1}{z_1} + \frac{1}{\lambda_1-\gamma_1}\frac{\gamma_1}{\lambda_1}\right)(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} \\
&< \frac{1}{\lambda_1-\gamma_1} - \left(\frac{1}{\lambda_1}\frac{1}{\mu_1^N} + \frac{1}{\lambda_1-\gamma_1}\frac{\gamma_1}{\lambda_1}\right)(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} \\
&= \frac{1}{\lambda_1-\gamma_1} - \left\{\frac{1}{\lambda_1-\gamma_1}\left[(\mu_2^*)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{\lambda_1}\right] + \frac{1}{\lambda_1-\gamma_1}\frac{\gamma_1}{\lambda_1}\right\}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} = 0,
\end{aligned}
$$

where the strict inequality follows from $z_1 < \mu_1^N$. Therefore, $\partial\widetilde{\mu}_2(z_1)/\partial v_1^*$ is negative, and consequently, $\partial u_1/\partial k_2$ is positive.

Consider $\partial u_2/\partial k_2$ next.

$$
\frac{\partial u_2}{\partial k_2} = 1_{z_2>\widetilde{\mu}_2(z_1)}\frac{z_1}{1-z_1}\frac{1}{\widetilde{\mu}_1^2(z_2)}\frac{\partial\widetilde{\mu}_1(z_2)}{\partial v_1^*}\frac{\partial v_1^*}{\partial k_2} = -1_{\widetilde{\mu}_2(z_1)<z_2<\mu_2^*}\frac{z_1}{1-z_1}\frac{1}{\widetilde{\mu}_1^2(z_2)}\frac{1}{1-w_1}\frac{\partial\widetilde{\mu}_1(z_2)}{\partial v_1^*}.
$$

For $z_2 < \mu_2^*$,

$$
\widetilde{\mu}_1(z_2) = \frac{\lambda_1-\gamma_1}{\lambda_1(\mu_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} + (\frac{\gamma_1}{v_1^*}-\gamma_1)(\frac{z_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{v_1^*}}.
$$

Then,

$$
\frac{\widetilde{\mu}_1(z_2)}{\partial v_1^*} = \widetilde{\mu}_1^2(z_2)\frac{\gamma_1}{(v_1^*)^2}\frac{\left(\frac{z_2}{\mu_2^*}\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - 1}{\lambda_1-\gamma_1},
$$

Altogether,

$$
\frac{\partial u_2}{\partial k_2} = -1_{\widetilde{\mu}_2(z_1)<z_2<\mu_2^*}\frac{z_1}{1-z_1}\frac{1}{1-w_1}\frac{\gamma_1}{(v_1^*)^2}\frac{\left(\frac{z_2}{\mu_2^*}\right)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - 1}{\lambda_1-\gamma_1},
$$

and it is negative if $\widetilde{\mu}_2(z_1) < z_2 < \mu_2^*$. □

(v) Effects of w_1. The sign of the change due to w_1 is exactly the same as that due to k_2, because both w_1 and k_2 affect players' payoffs through v_1^*, and the changes in v_1^* due to w_1 and k_2 are both negative. □

B.4 Equilibrium existence and uniqueness with one-sided ultimatum opportunities and multiple demand types

Before proving Theorem 2, we prove a lemma that shows the uniqueness of equilibrium when player 1 has a single demand type and player 2 has multiple demand types.

Lemma 3. *For any game $(a_1, \pi_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$ with ultimatum opportunities for player 1, a single demand for player 1, and multiple demands for player 2, there exists a unique equilibrium.*

Proof of Lemma 3. Denote by $\sigma_2(\cdot)$, a probability distribution over $A_2 \cup \{Q\}$, a mimicking strategy of an unjustified player 2. Since mimicking $a_2 < 1 - a_1$ is never optimal and mimicking $a_2 = 1 - a_1$ is equivalent to conceding, we assume that in equilibrium $\sigma_2(a_2) = 0$ for all $a_2 \leqslant 1 - a_1$. If $x = 1$, then in equilibrium $\sigma_2(Q) = 1$, because unjustified player 2 will not delay conceding if she knows that player 1 is justified. For the remainder of the proof we assume $x < 1$.

Define $T_i(a_1, a_2, x)$ as the time it takes for player i's reputation to increase from x to 1 on the equilibrium reputation path when each player i's demand is a_i. Explicitly,

$$T_1(a_1, a_2, x) := \begin{cases} \infty & x \leqslant \left(1 - \frac{\lambda_1}{\gamma_1}\right) v_1^*, \\ t(\mu_1^N; x, \lambda_1 - \gamma_1, \frac{\gamma_1}{v_1^*}) + t(1; \mu_1^N, \lambda_1 - \gamma_1, \gamma_1) & \left(1 - \frac{\lambda_1}{\gamma_1}\right) v_1^* < x < \mu_1^N, \\ t(1; x, \lambda_1 - \gamma_1, \gamma_1) & \mu_1^N \leqslant x \leqslant 1, \end{cases}$$

that is,

$$T_1(a_1, a_2, x) := \begin{cases} \infty & \text{if } x \leqslant \left(1 - \frac{\lambda_1}{\gamma_1}\right) v_1^*, \\ \frac{1}{\lambda_1 - \gamma_1} \log \left[\frac{\frac{\lambda_1 - \gamma_1}{x} + \frac{\gamma_1}{v_1^*}}{\frac{\lambda_1 - \gamma_1}{\mu_1^N} + \frac{\gamma_1}{v_1^*}} \right] - \frac{1}{\lambda_2} \log \mu_2^* & \text{if } \left(1 - \frac{\lambda_1}{\gamma_1}\right) v_1^* < x < \mu_1^N, \\ \frac{1}{\lambda_1 - \gamma_1} \log \left[\frac{\frac{\lambda_1 - \gamma_1}{x} + \gamma_1}{\lambda_1} \right] & \text{if } \mu_1^N \leqslant x \leqslant 1, \end{cases}$$

and

$$T_2(a_1, a_2, y) := -\frac{a_1 + a_2 - 1}{r_1(1 - a_2)} \log y.$$

Note that $T_1(a_1, a_2, x)$ is continuous and strictly decreasing in x on $(1 - \frac{\lambda_1}{\gamma_1}, 1)$ and that $T_2(a_1, a_2, y)$ is continuous and strictly decreasing in y on $(0, 1)$.

It remains to be shown that an unjustified player 2's equilibrium behavior $\sigma_2(\cdot)$ and an unjustified player 1's conceding behavior $Q_1(a_1, a_2, x, \sigma_2)$ at time zero are uniquely determined. Subsequently, we provide a series of definitions and use them to prove a series of claims that lead to equilibrium existence and uniqueness. Define player 2's reputation at time 0 when she plays a_2 with probability σ_2 as

$$y^*(a_2, \sigma_2) = \frac{z_2 \pi_2(a_2)}{z_2 \pi_2(a_2) + (1 - z_2)\sigma_2}.$$

Note that the more likely an unjustified player 2 announces a particular demand a_2, the more likely she is believed to be unjustified, and the lower her payoff from demanding a_2 is.

Let $\overline{\sigma}_2(a_1, a_2, x)$ be the maximum probability player 2 plays a_2 in equilibrium so that the expected payoff from demanding a_2 is higher than directly conceding to player 1's demand. For any $a_2 < 1 - a_1$, $\overline{\sigma}_2(a_1, a_2, x) = 0$ because conceding to player 1's demand a_1—which results in a payoff of $1 - a_1$—is a strictly better strategy than demanding strictly less than $1 - a_1$ and a weakly better strategy than demanding $1 - a_1$. For any $a_2 > 1 - a_1$, after choosing a_2, in any equilibrium, player 2 should not concede with a positive probability at time 0. First, if player 1's reputation can reach 1 without conceding with a positive probability at time 0 and player 2's reputation reaches 1 slower than player 1 when she demands a_2 with probability 1, $\overline{\sigma}_2(a_1, a_2, x)$ is the unique solution of σ_2 to $T_1(a_1, a_2, x) = T_2(a_1, a_2, y^*(a_2, \sigma_2))$ so that the two players' reputations reach 1 at the same time. Explicitly, when we let $\psi := 1 - \frac{\gamma_1}{\lambda_1}$, and we have

$$\overline{\sigma}_2(a_1, a_2, x) := \begin{cases} 1 & x \leqslant \left(1 - \frac{\lambda_1}{\gamma_1}\right) \nu_1^*, \\ z_2 \frac{\pi_2(a_2)}{1-\pi_2(a_2)} \left[\left(\frac{\psi \frac{1}{x} + (1-\psi) \frac{1}{\nu_1^*}}{\psi \frac{1}{\mu_1^N} + (1-\psi) \frac{1}{\nu_1^*}} \right)^{\frac{\lambda_2}{\lambda_1} \frac{1}{\psi}} - 1 \right] & \left(1 - \frac{\lambda_1}{\gamma_1}\right) \nu_1^* < x < \mu_1^N, \\ z_2 \frac{\pi_2(a_2)}{1-\pi_2(a_2)} \left[\left(\psi \frac{1}{x} + (1-\psi)\right)^{\frac{\lambda_2}{\lambda_1} \frac{1}{\psi}} - 1 \right] & \mu_1^N \leqslant x < 1. \end{cases}$$

Note that in equilibrium $\sigma_2(a_2) \leqslant \overline{\sigma}_2(a_1, a_2, \sigma_2)$ for all $a_2 > 1 - a_1$. To see why this claim must hold, suppose player 2 mimics a_2 with a probability strictly higher than $\overline{\sigma}_2(a_1, a_2, \sigma_2) < 1$. Then player 2 must concede with a strictly positive probability at time zero in order for players' reputations to reach 1 at the same time. However, we have specified that player 2 does not concede at time zero after announcing her demand. Second, if player 1's reputation reaches 1 even slower than when player 2 demands a_2 with probability 1, $\overline{\sigma}_2(a_1, a_2, x) = 1$. The scenario happens whenever $T_1(a_1, a_2, x) > T_2(a_1, a_2, y^*(a_2, 1))$. In particular, it happens whenever $x < \mu_1^*(1 - \frac{\lambda_1}{\gamma_1})$. In summary, in any equilibrium, $\sigma_2(a_2) \leqslant \overline{\sigma}_2(a_1, a_2, x)$, where $\overline{\sigma}_2(a_1, a_2, x) = 0$ if $a_2 \leqslant 1 - a_1$; $\overline{\sigma}_2(a_1, a_2, x)$ is the unique solution of σ_2 in $T_1(a_1, a_2, x) = T_2(a_1, a_2, y^*(a_2, \sigma_2))$ if $a_2 > 1 - a_1$ and $T_1(a_1, a_2, x) < T_2(a_1, a_2, y^*(a_2, 1))$; and $\overline{\sigma}_2(a_1, a_2, x) = 1$ if $a_2 > 1 - a_1$ and $T_1(a_1, a_2, x) \geqslant T_2(a_1, a_2, y^*(a_2, 1))$.

When player 2 demands a_2 with probability $\sigma_2 \leqslant \overline{\sigma}_2(a_1, a_2, x)$, player 1 must raise his time 0 reputation to $x^*(a_1, a_2, \sigma_2)$ so that their reputations reach 1 at the same time:

$$T_1(a_1, a_2, x^*(a_1, a_2, \sigma_2)) = T_2(a_1, a_2, y^*(a_2, \sigma_2)).$$

In order to do so, an unjustified player 1 concedes with probability

$$Q_1(a_1, a_2, x, \sigma_2) = 1 - \frac{x}{1-x} \frac{1 - x^*(a_1, a_2, \sigma_2)}{x^*(a_1, a_2, \sigma_2)},$$

so that player 1's reputation is raised to

$$x^*(a_1, a_2, \sigma_2) = \frac{x}{x + (1-x)[1 - Q_1(a_1, a_2, x, \sigma_2)]}.$$

Explicitly,

$$x^*(a_1, a_2, \sigma_2) := \begin{cases} \dfrac{\lambda_1 - \gamma_1}{\left(\frac{\mu_2^*}{y}\right)^{\frac{\lambda_1 - \gamma_1}{\lambda_2}} \left(\frac{\lambda_1 - \gamma_1}{\mu_1^N} + \frac{\gamma_1}{\nu_1^*}\right) - \frac{\gamma_1}{\nu_1^*}} & \text{if } y^*(a_2, \sigma_2) \leqslant \mu_2^* \\ \dfrac{1 - \frac{\gamma_1}{\lambda_1}}{\left(\frac{1}{y}\right)^{\frac{\lambda_1 - \gamma_1}{\lambda_2}} - \frac{\gamma_1}{\lambda_1}} & \text{if } y^*(a_2, \sigma_2) > \mu_2^* \end{cases}.$$

When player 2 demands a_2 with probability σ_2 and an unjustified player 1 concedes with probability

$Q_1(a_1, a_2, x, \sigma_2)$, an unjustified player 2's expected payoff is

$$u_2^*(a_1, a_2, x, \sigma_2) = 1 - a_1 + (1-x)Q_1(a_1, a_2, x, \sigma_2)(a_1 + a_2 - 1).$$

Two additional properties restrict player 2's equilibrium strategy $\sigma_2(\cdot)$. First, for any a_2 and $a_2' > a_2$, if $\sigma_2(a_2) > 0$, then $\sigma_2(a_2') > 0$. We can prove this property by contradiction. Suppose $\sigma_2(a_2) > 0$ and $\sigma_2(a_2') = 0$. Because $\sigma_2(a_2') = 0$, $u_2^*(a_1, a_2', x, \sigma_2(a_2')) = 1 - a_1 + (1-x)(a_1 + a_2' - 1)$. Because $\sigma_2(a_2) > 0$, $u_2^*(a_1, a_2, x, \sigma_2(a_2)) = 1 - a_1 + (1-x)Q_1(a_1, a_2, x, \sigma_2(a_2))(a_1 + a_2 - 1) \leqslant 1 - a_1 + (1 - x)(a_1 + a_2' - 1) = u_2^*(a_1, a_2', x, \sigma_2(a_2'))$. Second, whenever $\sum_{a_2} \overline{\sigma}_2(a_1, a_2, x) \leqslant 1$, $\sigma_2(a_2) = \overline{\sigma}_2(a_1, a_2, x)$ for all a_2, and $Q_2 = 1 - \sum_{a_2} \overline{\sigma}_2(a_1, a_2, x)$. The two properties together imply that we only need to check first if $\sum_{a_2} \overline{\sigma}_2(a_1, a_2, x) \leqslant 1$, and, if the first condition does not hold, then we find the equilibrium strategy among the set of strategies $\sigma_2(\cdot)$ such that $\sigma_2(a_2') > 0$ for all $a_2' \geqslant a_2$, for each $a_2 \in A_2$.

Denote by

$$\Delta_2(a_1, x) := \left\{ \sigma_2(\cdot) \in \Delta \;\middle|\; \begin{array}{ll} \sigma_2(a_2) = 0 & \forall a_2 \leqslant 1 - a_1 \\ \sigma_2(a_2) \leqslant \overline{\sigma}_2(a_1, a_2, x) & \forall a_2 > 1 - a_1 \end{array} \right\}$$

the set of candidate equilibrium mimicking strategies of player 2 in the game $B_1(a_1, x)$, where Δ denotes the set of all probability distributions on $A_2 \cup \{Q\}$. Note that the set $\Delta_2(a_1, x)$ is nonempty, convex, and compact. For any candidate equilibrium mimicking strategy $\sigma_2(\cdot) \in \Delta_2(a_1, x)$, define

$$\widehat{u}_2(x, \sigma_2(\cdot)) := \min_{a_2 : \sigma_2(a_2) > 0} u_2^*(a_1, a_2, x, \sigma_2(a_2)).$$

Explicitly,

$$\widehat{u}_2(x, \sigma_2(\cdot)) := \begin{cases} \min\limits_{a_2 : \sigma_2(a_2) > 0} u_2^*(a_1, a_2, x, \sigma_2(a_2)) & \text{if } \sigma_2(Q) = 0 \\ 1 - a_1 & \text{if } \sigma_2(Q) \neq 0 \end{cases}.$$

Note that $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy if and only if $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot) \in \Delta_2(a_1, x)} \widehat{u}_2(x, \sigma_2(\cdot))$. ($\Rightarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy. Any equilibrium strategy $\sigma_2(\cdot)$ satisfies that for all $a_2 \in A_2 \cup \{Q\}$ such that $\sigma_2(a_2) > 0$, $u_2^*(a_1, a_2, x, \sigma_2(a_2))$ is the same. If $\sigma_2(Q) > 0$, then

$$u_2^*(a_1, a_2, x, \sigma_2(a_2)) = 1 - a_1;$$

if $\sigma_2(Q) = 0$, then

$$u_2^*(a_1, a_2, x, \sigma_2(a_2)) = \min_{a_2 : \widehat{\sigma}_2(a_2) > 0} u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)).$$

Hence, any equilibrium strategy $\widehat{\sigma}_2(\cdot)$ must generate an equilibrium utility of $\widehat{u}_2(x, \sigma_2(\cdot))$. Hence, $\widehat{\sigma}_2(\cdot)$ maximizes $\widehat{u}_2(x, \sigma_2(\cdot))$ among all candidate equilibrium strategies $\sigma_2(\cdot)$. ($\Leftarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot) \in \Delta_2(a_1, x)} \widehat{u}_2(x, \sigma_2(\cdot))$. By the strict monotonicity of $u_2^*(a_1, a_2, x, \cdot)$, for all $a_2 \in A_2$ such that $\widehat{\sigma}_2(a_2) > 0$, $u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)) = \widehat{u}_2(x, \widehat{\sigma}_2(\cdot))$. Coupled with the fact that $\widehat{\sigma}_2(\cdot)$ is the feasible strategy that maximizes $\widehat{u}_2(x, \sigma_2(\cdot))$, $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy.

Define $\Gamma(\sigma_2(\cdot))$, a correspondence from $\Delta_2(a_1, x)$ to $\Delta_2(a_1, x)$, as follows:

$$\{\widetilde{\sigma}_2(\cdot) \in \Delta_2(a_1, x) | \widetilde{\sigma}_2(a_2) > 0 \Rightarrow u_2^*(a_1, a_2, x, \sigma_2(a_2)) \geqslant u_2^*(a_1, a_2', x, \sigma_2(a_2')) \ \forall a_2' \in A_2\}.$$

Note that $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta_2(a_1,x)} \widehat{u}_2(x, \sigma_2(\cdot))$ if and only if $\widehat{\sigma}_2(\cdot)$ is a fixed point of Γ. ($\Rightarrow$) Suppose $\widehat{\sigma}_2(\cdot)$ solves $\max_{\sigma_2(\cdot)\in\Delta_2(a_1,x)} \widehat{u}_2(x, \sigma_2(\cdot))$. By the argument above, $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy. Therefore, $\widehat{\sigma}_2(a_2) > 0$ implies $u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)) \geqslant u_2^*(a_1, a_2', x, \widehat{\sigma}_2(a_2'))$ for any $a_2' \in A_2$. By the definition of Γ, $\widehat{\sigma}_2(\cdot) \in \Gamma(\widehat{\sigma}_2(\cdot))$. ($\Leftarrow$) Suppose $\widehat{\sigma}_2(\cdot) \in \Gamma(\widehat{\sigma}_2(\cdot))$. By the definition of Γ, $\widehat{\sigma}_2(a_2) > 0$ implies $u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)) \geqslant u_2^*(a_1, a_2', x, \widehat{\sigma}_2(a_2'))$ for any $a_2' \in A_2$. Assume by contradiction that $\widehat{\sigma}_2(\cdot)$ does not solve $\max_{\sigma_2(\cdot)\in\Delta_2(a_1,x)} \widehat{u}_2(x, \sigma_2(\cdot))$ but $\widetilde{\sigma}_2(\cdot) \neq \widehat{\sigma}_2(\cdot)$ does. There must exist an $a_2 \in A_2$ such that $\widehat{\sigma}_2(a_2) > 0$ and $\widetilde{\sigma}_2(a_2) < \widehat{\sigma}_2(a_2)$ (otherwise, if $\widetilde{\sigma}_2(a_2) \geqslant \widehat{\sigma}_2(a_2)$ for all a_2 such that $\widehat{\sigma}_2(a_2) > 0$, then by the strict monotonicity of u_2^*, $u_2^*(a_1, a_2, x, \widetilde{\sigma}_2(a_2)) \leqslant u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2))$, and $\widehat{u}_2(x, \widetilde{\sigma}_2(\cdot)) \leqslant \widehat{u}_2(x, \widehat{\sigma}_2(\cdot))$). However, that implies that there exists $a_2' \in A_2 \cup \{Q\}$ such that $\widetilde{\sigma}(a_2') > \widetilde{\sigma}(a_2')$. If $a_2' = Q$, then $\widehat{u}_2(x, \widetilde{\sigma}_2(\cdot)) \leqslant \widehat{u}_2(x, \widehat{\sigma}_2(\cdot))$. If $a_2' \in A_2$, then $\widehat{u}_2(x, \widetilde{\sigma}_2(\cdot)) \leqslant \widehat{u}_2(x, \widehat{\sigma}_2(\cdot))$.

Hence, from the two claims above, we have that $\widehat{\sigma}_2(\cdot)$ is an equilibrium strategy for player 2 in the game $B_0(a_1, x)$ if and only if $\widehat{\sigma}_2(\cdot)$ is a fixed point of Γ. Equilibrium existence follows from the existence of a fixed point of Γ by Kakutani's fixed point theorem. By construction, $\Delta_2(a_1, x)$ is compact. By construction, Γ is convex-valued. Finally, Γ is upper-hemicontinuous because u_2^* is continuous in its last argument.

It remains to show the existence of a unique equilibrium. Equilibrium uniqueness follows from the strict monotonicity of u_2^* in x. Suppose there are two equilibrium strategies $\widehat{\sigma}_2(\cdot)$ and $\widetilde{\sigma}_2(\cdot)$; without loss of generality, suppose $\widehat{\sigma}_2(a_2) > \widetilde{\sigma}_2(a_2) > 0$ for some $a_2 > 1 - a_1$. The utilities of playing the two strategies are different:

$$\widehat{u}_2(x, \widehat{\sigma}_2(\cdot)) = u_2^*(a_1, a_2, x, \widehat{\sigma}_2(a_2)) < u_2^*(a_1, a_2, x, \widetilde{\sigma}_2(a_2)) = \widehat{u}_2(x, \widetilde{\sigma}_2(\cdot)),$$

where the strict inequality follows from the strict monotonicity of u_2^*. This contradicts the property that equilibrium strategies $\widehat{\sigma}_2(\cdot)$ and $\widetilde{\sigma}_2(\cdot)$ both maximize $\widehat{u}_2(x, \sigma_2(\cdot))$. Multiple equilibrium distributions over types being conceded to are in conflict with the requirement that types mimicked with a positive probability must have equal payoffs that are not smaller than the payoffs of the types that are not mimicked. Suppose by contradiction there are two different equilibrium strategies for player 2: $\sigma_2(a_2) \neq \sigma_2'(a_2)$ for some a_2. If $\sigma_2(a_2) > 0$ and $\sigma_2'(a_2) > 0$, then $u_2(a_1, a_2, x, \sigma_2(a_2)) \neq u_2(a_1, a_2, x, \sigma_2'(a_2))$. But $u_2(a_1, a_2, x, \sigma_2(a_2)) = \widehat{u}_2(x, \sigma_2(\cdot))$ and $u_2(a_1, a_2, x, \sigma_2(a_2)) = \widehat{u}_2(x, \sigma_2'(\cdot))$, but $\widehat{u}_2(x, \sigma_2(\cdot)) \neq \widehat{u}_2(x, \sigma_2'(\cdot))$ contradicts the fact that $\sigma_2(\cdot)$ and $\sigma_2'(\cdot)$ both solve $\max_{\sigma_2(\cdot)\in\Delta_2(a_1,x)} \widehat{u}_2(x, \sigma(\cdot))$. If $\sigma_2(a_2)$ or $\sigma_2'(a_2)$ is zero, then by the first additional property of player 2's equilibrium strategy above, there is an $a_2' > a_2$ such that $\sigma_2(a_2') > 0$, $\sigma_2'(a_2') > 0$, and $\sigma_2(a_2') \neq \sigma_2'(a_2')$, so that the contradiction arises again. Player 1 receives $u_1(a_1, x)$ in the equilibrium of the bargaining game $B(a_1, x)$. □

Proof of Theorem 2. Denote by $u_1(a_1, x)$ the payoff of player 1 in the unique equilibrium of the bargaining game $B_1(a_1, x)$ with $A_1 = \{a_1\}$ and $|A_2| \geqslant 1$. Note that it is a continuous function of x. Moreover, there exists an $\underline{x}$ such that $u_1^*(a_1, x) = u_1^*(a_1, \underline{x})$ for any $x \leqslant \underline{x}$ and $u_1^*(a_1, x)$ is strictly increasing in x on the interval $\big(\underline{x}, 1\big)$.

We characterize the equilibrium distribution σ_1 as the solution to

$$\max_{\sigma_1} \widehat{u}(\sigma_1),$$

where $\widehat{u}(\sigma_1) = \min_{a_1 \text{ s.t. } \sigma_1(a_1)>0} u_1(a_1, x(\sigma_1(a_1)))$, and

$$x(\sigma_1(a_1)) = \frac{z_1\pi_1(a_1)}{z_1\pi_1(a_1) + (1-z_1)\sigma_1(a_1)}.$$

The continuity of $u_1(a_1, x)$ in x ensures that an equilibrium exists; see the fixed-point argument establishing the existence of an equilibrium strategy σ_2 in $B(a_1, x)$ above.

Let $\overline{u}_1$ be the maximized value above; $\overline{u}_1$ is the utility that player 1 attains in any equilibrium. Clearly, $\overline{u}_1 \geqslant u_1(a_1, \underline{x})$ for all a_1. Let σ_1 and $\widehat{\sigma}_1$ be two equilibrium strategies for player 1.

Claim: If $\overline{u}_1 > u_1(a_1, \underline{x})$, then $\widetilde{\sigma}_1(a_1) = \widehat{\sigma}_1(a_1)$. Proof: To see this note that either $u_1(a_1, 1) > \overline{u}_1$ or $u_1(a_1, 1) \leqslant \overline{u}_1$. If $u_1(a_1, 1) > \overline{u}_1$, then there is a unique σ_1 such that $\sigma_1(a_1, x(\sigma_1)) = \overline{u}_1$, and hence $\widetilde{\sigma}_1(a_1) = \widehat{\sigma}_1(a_1) = \sigma_1$. If $u_1(a_1, 1) \leqslant \overline{u}_1$, then by the strict monotonicity of $u_1(a_1, x)$ in x for $x > \underline{x}$ and monotonicity of $u_1(a_1, x)$ in x for $x \leqslant \underline{x}$, $u_1(a_1, x) < u_1(a_1, 1) \leqslant \overline{u}_1$ for any $x < 1$. Hence, $\widetilde{\sigma}_1(a_1) = \widehat{\sigma}_1(a_1) = 0$.

Define $D_1 = \{a_1 \in A_1 | u_1(a_1, \underline{x}) = \overline{u}_1\}$. Recall that $\underline{x}$ depends on a_1. We have already noted that $\widetilde{\sigma}_1(a_1) = \widehat{\sigma}_1(a_1)$ for $a_1 \in A_1 \backslash D_1$. Hence, $\sum_{a_1 \in D_1} \widetilde{\sigma}_1(a_1) = \sum_{a_1 \in D_1} \widehat{\sigma}_1(a_1)$.

We will conclude the proof that $\widetilde{\sigma}_1$ and $\widehat{\sigma}_1$ lead to the same random outcome $\widetilde{\theta}$ by first verifying that the probability that player 1 chooses $a_1 \in D_1$ and agreement is reached at time 0 is the same with either $\widetilde{\sigma}_1$ or $\widehat{\sigma}_1$. This will imply that the random outcome, conditional on agreement at time 0, is the same with either $\widetilde{\sigma}_1$ or $\widehat{\sigma}_1$. Finally, we show that for each $a_1 \in D_1$, the probability that an unjustified player 1 will mimic a_1 and not concede is the same with either $\widetilde{\sigma}_1$ or $\widehat{\sigma}_1$.

Let $A(\sigma_1)$ denote the probability that player 1 mimics some $a_1 \in D_1$ and agreement is reached at time 0 given the equilibrium strategy σ_1. Since $a_1 \in D_1$ implies $\sigma_2(\overline{a}_2 | a_1) = 1$, it follows that $a_1 \geqslant 1 - \overline{a}_2$; otherwise, player 1 would achieve a higher utility by mimicking $\max C_1 > 1 - \overline{a}_2$. Hence,

$$\begin{aligned}
A(\sigma_1) &= \sum_{a_1 \in D_1} q_1(a_1, \overline{a}_2, x(\sigma_1(a_1)), 1)\,[1 - x(\sigma_1(a_1))]\,[z_1\pi_1(a_1) + (1-z_1)\sigma_1(a_1)] \\
&= \sum_{a_1 \in D_1} \frac{K(a_1, \overline{a}_2, 1) - x(\sigma_1(a_1))}{K(a_1, \overline{a}_2, 1)}\,[z_1\pi_1(a_1) + (1-z_1)\sigma_1(a_1)] \\
&= \sum_{a_1 \in D_1} [z_1\pi_1(a_1) + (1-z_1)\sigma_1(a_1)] - \sum_{a_1 \in D_1} \frac{x(\sigma_1(a_1))}{K(a_1, \overline{a}_2, 1)}\,[z_1\pi_1(a_1) + (1-z_1)\sigma_1(a_1)] \\
&= \sum_{a_1 \in D_1} (1-z_1)\sigma_1(a_1) + \sum_{a_1 \in D_1} z_1\pi_1(a_1) - \sum_{a_1 \in D_1} \frac{z_1\pi_1(a_1)}{K(a_1, \overline{a}_2, 1)}.
\end{aligned}$$

Since $\sum_{a_1 \in D_1} \widetilde{\sigma}_1(a_1) = \sum_{a_1 \in D_1} \widehat{\sigma}_1(a_1)$, we have $A(\widetilde{\sigma}_1) = A(\widehat{\sigma}_1)$. For any $a_1 \in D_1$, the probability that an unjustified player 1 will mimic a_1 and not concede at time 0 is

$$\sigma_1(a_1)\,[1 - q_1(a_1, \overline{a}_2, x(\sigma_1(a_1)), 1)]$$

$$
\begin{aligned}
&=\sigma_1(a_1)\frac{x(\sigma_1(a_1))}{1-x(\sigma_1(a_1))}\frac{1-K(a_1,\overline{a}_2,1)}{K(a_1,\overline{a}_2,1)}\\
&=\pi_1(a_1)\frac{z_1}{1-z_1}\frac{1-K(a_1,\overline{a}_2,1)}{K(a_1,\overline{a}_2,1)},
\end{aligned}
$$

which is independent of σ_1. Hence, $\widetilde{\sigma}_1(a_1)$ and $\widehat{\sigma}_1(a_1)$, the equilibrium probabilities that an unjustified player 1 will mimic a_1, are the same. □

C Frictional bluffing opportunities and extensions

C.1 Proof of Theorem 1 with frictional bluffing opportunities

Suppose an unjustified player 1 can challenge according to a Poisson rate $\rho_1 := \gamma_1$.

Claim 2. *Player 1's challenging strategy $\widehat{G}_1$ is continuous for $t \geqslant 0$.*

Proof. The claim holds by definition. □

Claim 3. *Upon being challenged, player 2 weakly prefers yielding to not yielding. Hence, player 2's payoff conditional on being challenged is $1 - a_1$.*

Proof. Suppose by contradiction player 2 strictly prefers not to yield when challenged. If player 1 challenges, he gets $1 - a_2 + (1 - \mu_2)w_1 D - c_1 D$, which is weakly smaller than $1 - a_2 + (w_1 - c_1)D$. By the assumption that $w_1 < c_1$ and $D > 0$, player 1's expected payoff from challenging is strictly smaller than $1 - a_2$, and he can guarantee that payoff by simply conceding. Hence, player 1 would choose not to challenge so that the bluffing rate is $\beta_1 = 0$. In this case, because any challenging player 1 is justified, player 2 would strictly prefer to yield to than to see the challenge, because $k_2 > 0$, contradicting that player 2 strictly prefers not to yield to a challenge. □

Claim 4. *F_1 and F_2 do not have an atom at the same time.*

Proof. If F_i jumps at time t, then player j receives a strictly higher expected utility by conceding an instant after time t (which guarantees a payoff strictly greater than $1-a_i$, if j yields if challenged) than by conceding at time t (which results in a payoff of $1 - a_i$). □

Claim 5. *F_2 has no atom at time $t > 0$.*

Proof. If F_2 has an atom at $t > 0$, then there exists an $\varepsilon > 0$ such that F_1 is constant in the time interval $(t - \varepsilon, t)$. Because F_1 has no atom at t when F_2 has an atom at t (by Claim 3), and because conditional on a challenge, player 2 gets a payoff of $1 - a_1$, player 2 strictly prefers yielding at $t - \varepsilon/2$ to yielding at t. □

Let $\inf \emptyset := \infty$. Define $\tau_i := \inf\{t \geqslant 0 | \mu_i(t) = 1\}$ as the earliest time for i's reputation to reach 1.

Claim 6. *$\tau_1 = \tau_2$: Players' reputations either never reach 1 or reach 1 at the same time.*

Proof. First, $\tau_1 \geqslant \tau_2$. Suppose otherwise: $\tau_1 < \tau_2$. Then τ_1 is finite. For time $t \in (\tau_1, \tau_2]$, player 2 strictly prefers conceding to waiting. Hence, it is a strictly dominated strategy to concede at time τ_2 than to concede at any time prior to $\tau_2 - \varepsilon$ for sufficiently small ε. In equilibrium, τ_2 cannot be the latest time of concession for player 2. If τ_2 is infinite, then the claim holds. If τ_2 is finite, then it is a strictly dominated strategy for player 1 to concede or challenge at any time strictly after τ_2. Hence, $\tau_1 = \tau_2$ in case $\tau_2 < \infty$. □

We denote by τ^* the common time for the two players' reputations to reach 1.

Case 1: $\tau^* < \infty$. Let μ_2^X be the reputation of player 2 such that player 1 is indifferent between conceding and waiting to challenge with rate ρ_1 while player 2 would yield to a challenge but would not voluntarily concede:

$$-r_1(1-a_2) + \rho_1(1-\mu_2^X - c_1)D = 0 \iff \mu_2^X = \mu_2^N - \lambda_2/\rho_1.$$

In other words, player 1's flow payoff from challenging is $\rho_1(1-\mu_2^X - c_1)D$. Let

$$t^* := \inf\{t : \mu_2(t) > \mu_2^X\}.$$

Because μ_2 is weakly increasing, for $t > t^*$, $\mu_2(t) > \mu_2^X$. Let μ_1^F be player 1's reputation such that if he challenged with rate ρ_1, his reputation would be v_1^* and player 2 would be indifferent between seeing and yielding to a challenge.

$$v_1^* = \frac{\gamma_1 \mu_1^F}{\gamma_1 \mu_1^F + \rho_1(1-\mu_1^F)} \iff \mu_1^F = \frac{\rho_1 v_1^*}{\rho_1 v_1^* + \gamma_1(1-v_1^*)}.$$

We assume $\rho_1 \geqslant \gamma_1$ so that $\mu_1^F \geqslant v_1^*$.

Claim 7. *F_1 does not have an atom at any time $t > t^*$.*

Proof. Suppose it does. Then, for some $\varepsilon > 0$, F_2 is constant for $(t-\varepsilon, t)$, and $t - \varepsilon > t^*$. Hence, $\mu_2(t)$ is constant and larger than μ_2^X in this interval. Hence, player 1's continuation payoff at $t - \varepsilon/2$ is strictly less than $1 - a_2$ (regardless of player 2's response to challenge). □

Claim 8. *For $t > t^*$, there is no interval where $F_i(t) < 1$ and is constant.*

Proof. Suppose $F_1(t)$ is constant on (t_1, t_2) and F_1 is increasing at t_2. Then F_2 is also constant on (t_1, t_2); otherwise player 2 would move the yield probability to earlier. Since F_2 does not have an atom at t_2 and since F_1 is increasing at t_2, player 1 is better off yielding earlier than at t_2. The proof regarding $F_2(t)$ is similar. □

Claim 9. *Players' reputations are continuous for $t > t^*$. This implies $\lim_{t \downarrow t^*} \mu_2(t) = \mu_2^X$ if $t^* > 0$.*

Proof. F_1 and F_2 are continuous and strictly increasing for $t > t^*$ (by Claim 8) and G_1 is continuous and weakly increasing by the premise that $\rho_1 < \infty$. Hence, the reputation $\mu_i(t)$ is continuous for $t > t^*$. □

Denote by t^N the time for player 2's reputation to reach μ_2^*; it is unique given that F_2 is strictly increasing (Claim 9). Denote by $\mu_1^N := \mu_1(t^N)$ player 1's reputation a time t^N.

Claim 10. *If $\mu_1^F < \mu_1(t) < \mu_1^N$ and $\mu_2(t) < \mu_2^*$, player 2 strictly prefers yielding to a challenge to conceding, and player 1's flow payoff of challenging is strictly higher than conceding.*

Proof. When $\mu_1(t) > \mu_1^F$, by challenging at any rate weakly below ρ_1, player 1's reputation strictly exceeds v_1^*, so it is a strictly dominant strategy for player 2 to yield to a challenge. Hence, it is a strictly dominant strategy for player 1 to challenge, and to challenge at the maximum rate ρ_1. □

Claim 11. *If $t^* > 0$, then $\lim_{t \downarrow t^*} \mu_1(t) < \mu_1^F$.*

Proof. If $\mu_1^N \leqslant \mu_1^F$, we are done, because $\mu_1(t) < \mu_1^N \leqslant \mu_1^F$ for all $t \in (t^*, t^N)$. Suppose $\mu_1^N > \mu_1^F$. When $\mu_1^F < \mu_1(t) < \mu_1^N$, player 1's reputation evolution is

$$\frac{\mu_1'(t)}{\mu_1(t)} = \lambda_1 + [1 - \mu_1(t)](\rho_1 - \gamma_1) \geqslant 0,$$

and player 2 concedes with overall rate $\lambda_2(t) = \lambda_2 - \rho_1\mu_2^* + \rho_1\mu_2(t)$ for player 1 to be indifferent between challenging and conceding (by Claim 8), and player 2's reputation evolution is

$$\frac{\mu_2'(t)}{\mu_2(t)} = \lambda_2 - \rho_1[\mu_2^* - \mu_(t)],$$

which is positive if $\mu_2(t) > \mu_2^X$ (see Lemma 1). Let t^F denote the time such that $\mu_1(t^F) = \mu_1^F$ and μ_2^F player 2's reputation at time t^F. Because reputations are continuous for $t > t^*$ (by Claim 9), $\mu_2^F > \mu_2^X$. By the continuity of players' reputations, player 1's reputation is monotonic if $\mu_1^F = \rho_1 v_1^* / [\rho_1 v_1^* + \gamma_1(1 - v_1^*)] > \widetilde{\mu}_1 = 1 - \lambda_1/\gamma_1$, where $v_1^* = 1 - k_2/(1 - w_1)$. □

Claim 12. $\mu_2(t) < \mu_2^X$ *for $t < t^*$.*

Proof. Suppose not. Then $\mu_2(t) = \mu_2^X$ for $t \in (t^* - \varepsilon, t^*)$ for some $\varepsilon > 0$. Hence, player 2 does not concede in the time interval. Because F_2 does not have a jump at t^* (by Claim 5), for player 1's payoff at $t \in (t^* - \varepsilon, t^*)$ to be at least $1 - a_2$, player 1 challenges at rate ρ_1. Hence, $\mu_1' \geqslant 0$ in this interval. For F_2 to be constant, F_1 needs a jump at t^*. Because $\lim_{t \downarrow t^*} \mu_1(t) < \mu_1^F$ (by Claim 11), $\mu_1(t) < \mu_1^F$ for $t \in (t^* - \varepsilon, t^*)$. Mimicking ρ_1, player 1 cannot have a reputation above v_1^*, and flow payoff will be low. □

Claim 13. *For $t < t^*$, F_1 is strictly increasing.*

Proof. Suppose F_1 is constant on (t_1, t_2) and then increasing at t_2. We know $t_2 \leqslant t^*$ (because starting at t^* we are on the reputation coevolution curve). Then, F_2 is also constant on (t_1, t_2), hence μ_2 is constant in that interval, and $\mu_2 < \mu_2^X$. Because F_2 does not have an atom (by Claim 5), first, $t_2 < t^*$, and second, player 1's payoff at $t \in (t_1, t_2)$ is equal to flow payoff from challenging and then yielding at t_2 (i.e., convex combination of flow payoffs from challenge and $1 - a_2$).

If $\mu_1(t) > \mu_1^F$ for some $t \in (t_1, t_2)$, then $\beta_1 = \rho_1$, because this generates the highest flow payoff for player 1. Because $\rho_1 \geqslant \gamma_1$, $\mu_1' \geqslant 0$, this means $\beta_1 = \rho_1$ for all $t' \in [t, t_2)$, and $\mu_1(t_2) > \mu_1^F$. But then, player 1 does not have an incentive to concede at t_2, contradicting the premise that F_1 is increasing at t_2.

If $\mu_1(t) = \mu_1^F$ for some $t \in (t_1, t_2)$, then $\beta_1 = \rho_1$ (otherwise the posterior upon challenge is strictly larger than v_1^* and player 1 strictly prefers challenging because $\mu_2 < \mu_2^X$). Then, $\mu_1' \geqslant 0$ and $\beta_1(t') = \rho_1$ for all $t' \in [t, t_2)$. Hence, $\mu_1(t_2) = \mu_1^F$. Because $\mu_2(t_2) < \mu_2^X$, $\beta_1(t_2) = \rho_1$, and because F_1 is increasing at t_2, for all $t \in (t_2, t_2 + \varepsilon)$, $\mu_1(t) > \mu_1^F$ and $\mu_2(t) < \mu_2^X$. However, in this case, it is a strictly dominant strategy for player 1 to challenge, which contradicts that F_1 is strictly increasing at t_2.

Hence, $\mu_1(t) < \mu_1^F$ for all $t \in (t_1, t_2)$. Hence, player 1 chooses $\beta_1 < \rho_1$ (since otherwise player 2 does not yield). Hence, player 2 weakly prefers not conceding in this interval. But then there is no flow payoff for player 1 with a positive probability, a contradiction to his payoff being at least $1 - a_2$. □

Claim 14. *For $t < t^*$, F_2 is strictly increasing.*

Proof. Suppose F_2 is constant on an interval. Because F_1 is strictly increasing (by Claim 13), player 1 is indifferent between conceding at any point in the interval. Because $\mu_2 < \mu_2^X$, if $\mu_1(t) \geqslant \mu_1^F$ in this interval at least once, then player 1 strictly prefers waiting for challenges to conceding. Hence, $\mu_1(t) < \mu_1^F$ for all t in the interval. Player 1 is indifferent between challenging and not challenging, so he is indifferent between conceding and waiting. Then this contradicts that F_1 is strictly increasing in the interval when F_2 is constant. □

Case 2: $\tau^* = \infty$. In this case, it must be the case that $t^* = \infty$. Otherwise, if $t^* < \infty$, then by Claim 9, it takes a finite time to reach reputation 1. In this case, player i's continuation payoff is at least $1 - a_j$, since otherwise i would concede with probability 1, which would be a contradiction to $\tau^* = \infty$, since the other player would also concede before infinity.

If $\mu_2(t) = \mu_2^X$ for $t \geqslant t_1$ for some $t_1 < \infty$, then for player 1 to not concede with probability 1 at any point, his continuation payoff is at least $1 - a_2$. This means that he challenges with rate ρ_1, $\mu_1' \geqslant 0$, and because player 2's continuation payoff is at least $1 - a_1$, player 1 is sometimes yielding, so eventually player 1's reputation reaches 1, contradiction.

Hence, $\mu_2(t) < \mu_2^X$ for all t. If $\mu_1(t) \geqslant \mu_1^F$ at some t, then $\beta_1 = \rho_1$ at t, hence $\mu_1' \geqslant 0$, and hence this leads to $\beta_1 = \rho_1$, which leads to $\mu_1' \geqslant 0$ for all $t' \geqslant t$. Because there is a positive probability that player 2 never yields, player 1 needs to concede occasionally, leading his reputation to 1, a contradiction.

If $\mu_1(t) < \mu_1^F$ for all t, then player 1 is never challenging with rate ρ_1, and the payoff from never challenging is strictly less than $1 - a_2$ when μ_2 is sufficiently close to $\lim_{t\to\infty} \mu_2(t)$ (in which player 2 concedes with a vanishingly small rate).

C.2 Separation of demand justifiability and commitment behavior

Player $2h$'s expected gain from seeing a challenge compared to yielding is

$$(v_{1H} + v_{1h})(1 - w_1)D + (v_{1L} + v_{1\ell})D - k_2 D > (1 - w_1 - k_2)D > 0.$$

Because $1 - w_1 > k_2$, $2h$ strictly prefers seeing a challenge to yielding. When $1h$ and 1ℓ have exited the game, player $2h$'s expected gain from waiting is

$$-r_2(1 - a_1) + \gamma_1 \psi_1 (1 - w_1 - k_2)D + \gamma_1 (1 - \psi_1)(1 - k_2)D = \gamma_1 (1 - \psi_1 w_1 - k_2)D - r_2(1 - a_1),$$

which is positive if $\lambda_1 < \gamma_1(1-\psi_1 w_1 - k_2)$. When $1h$ and 1ℓ remain in the game, they will concede with a positive probability and concede at the overall rate λ_1. Player $2h$ gets a strictly positive payoff from not conceding. Hence, player $2h$ never concedes.

Player 2ℓ's gain from seeing a challenge is

$$(v_{1L} + v_{1\ell})(1-w_1)D - k_2 D.$$

Since $1H$ and $1L$ challenge for sure and $1h$ is more likely to challenge than 1ℓ, $v_{1L} + v_{1\ell} \leqslant 1-\psi_1$. The gain is less than

$$[(1-\psi_1)(1-w_1) - k_2]D,$$

which is negative when $k_2 > (1-\psi_1)(1-w_1)$. Hence, 2ℓ does not see.

Player $1h$ is indifferent between challenging and conceding if player $2h$'s reputation μ_{2h} satisfies

$$(\mu_{2H} + \mu_{2h})w_1 + (\mu_{2L} + \mu_{2\ell}^h) - c_1 = 0,$$

which, because $2H$, $2L$, and $2h$ never concede, is substituted to

$$\left(1 - \frac{z_{2L}}{1-z_{2\ell}}\right)\mu_{2\ell}^h + \frac{z_{2L}}{1-z_{2\ell}} = \frac{c_1 - w_1}{1-w_1} \Longrightarrow \mu_{2\ell}^h = \frac{1 - \frac{z_{2L}}{1-z_{2\ell}}}{\frac{c_1-w_1}{1-w_1} - \frac{z_{2L}}{1-z_{2\ell}}},$$

which is smaller than one if

$$\frac{z_{2L}}{1-z_{2\ell}} < \frac{c_1 - w_1}{1-w_1}.$$

Consider phase 4: $\mu_{2\ell} < \mu_{2\ell}^h$. In equilibrium, $1h$ does not challenge and concedes at rate λ_1 and 2ℓ concedes at rate λ_2. Consider phase 3: $\mu_{2\ell} > \mu_{2\ell}^h$. In equilibrium, $1h$ challenges and concedes at rate λ_1 and 2 concedes at the overall rate

$$\lambda_2(t) = \lambda_2 - \gamma_1[(\mu_{2H} + \mu_{2h})w_1 + \mu_{2L} + \mu_{2\ell} - c_1].$$

Consider phase 2: For each player i, ih does not concede and player $i\ell$ concedes such that the overall concession rate is λ_i. Finally, in phase 1, player 1ℓ challenges at the full rate and concedes so that the overall concession rate is λ_1 and player 2ℓ concedes so that the overall rate

$$\lambda_2(t) = \lambda_2 - \gamma_1(\mu_{2\ell} + \mu_{2L}w_1 - c_1) < \lambda_2.$$

Player 1ℓ prefers challenge to no challenge if $\mu_{2\ell} > \mu_{2\ell}^\ell$, where

$$\mu_{2\ell}^\ell = c_1 - \mu_{2L} \cdot w_1.$$

Either 1ℓ or 1ℓ will concede with a positive probability so that the time that reputations μ_{1h} and $\mu_{2\ell}$ reach zero at the same time.

C.3 Two-sided ultimatum opportunities and single demand types

C.3.1 Formal description of the game

Let us formally describe the strategies and payoffs of the (unjustified) players. Let $\Sigma_i = (F_i, G_i, q_i)$ denote an unjustified player i's strategy, where $F_i(t)$ is player i's probability of conceding by time t, $G_i(t)$ is player i's probability of challenging by time t, and $q_i(t)$ is player i's probability of conceding to a challenge at time t. Restrict F_i and G_i to be right-continuous and increasing functions with $F_i(t) + G_i(t) \leqslant 1$ for every $t \geqslant 0$, and $q_i(t) \in [0,1]$ to be a measurable function. For $i = 1, 2$, player i's time-zero expected utility of conceding at time t is

$$\begin{aligned} U_i(t, q_i, \Sigma_j) \quad &= \quad W_i(t, q_i, \Sigma_j) + e^{-r_i t}(1-a_j)\Big[1-(1-z_j)F_j(t)-(1-z_j)G_j(t)-z_j(1-e^{-\gamma_j t})\Big] \\ &\quad + e^{-r_i t}(1-z_j)\Big[F_j(t)-F_j(t^-)\Big]\frac{a_i+1-a_j}{2}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} W_i(t, q_i, \Sigma_j) \quad &= \quad (1-z_j)\int_0^t a_i e^{-r_i s} dF_j(s) + z_j \int_0^t \Big\{1-a_j-\Big[1-q_i(s)\Big]k_i D\Big\} e^{-r_i s}\gamma_j e^{-\gamma_j s} ds \\ &\quad + (1-z_j)\int_0^t \Big\{1-a_j+\Big[1-q_i(s)\Big]\Big[(1-w_j)D-k_i D\Big]\Big\} e^{-r_j s} dG_i(s), \end{aligned}$$

and it is assumed that players equally divide their surplus if they concede simultaneously, which happens with probability zero in equilibrium. Player i's time-zero expected utility of challenging at time t is

$$\begin{aligned} &V_i(t, q_i, \Sigma_j) = \\ &W_i(t, q_i, \Sigma_j) + (1-z_j)[1-F_j(t)-G_j(t^-)]e^{-r_i t}[(1-q_j(t))w_i + q_j(t)]D \\ &+ \left[1-(1-z_j)F_j(t)-(1-z_j)G_j(t)-z_j\left(1-e^{-\gamma_j t}\right)\right] e^{-r_i t}(1-a_j-c_i D) + (1-z_j)\times \\ &[G_j(t)-G_j(t^-)]\left\{1-a_j+\frac{1}{2}[(1-q_i(s))(1-w_j)-k_i]D+\frac{1}{2}[(1-q_j(t))w_i+q_j(t)]D\right\} \end{aligned}$$

where it is assumed that players resolve the dispute in court and players are equally likely to be the challenger if they challenge simultaneously at time t, which happens with probability zero in equilibrium, and Player i's expected utility from strategy Σ_i is

$$u_i(\Sigma_i, \Sigma_j) = \int_0^\infty U_i(s, q_i, \Sigma_j) dF_i(s) + \int_0^\infty V_i(s, q_i, \Sigma_j) dG_i(s).$$

We again study the Bayesian Nash equilibria of this game. Let $\mu_i(t)$ denote the posterior belief (of player $j \neq i$) that player i is justified conditional on the game not ending by game time t. By Bayes' rule,

$$\mu_i(t) := \frac{z_i\left[1-\int_0^t \gamma_i e^{-\gamma_i s} ds\right]}{z_i\left[1-\int_0^t \gamma_i e^{-\gamma_i s} ds\right] + (1-z_i)\left[1-F_i(t^-)-G_i(t^-)\right]}.$$

Let $\nu_i(t)$ denote the posterior belief that player i is justified if player i challenges at time t. If G_i has an atom at t, then $\nu_i(t) = 0$. If G_i is differentiable at t, then

$$\nu_i(t) = \frac{\mu_i(t)\gamma_i}{\mu_i(t)\gamma_i + [1-\mu_i(t)]\beta_i(t)},$$

where $\chi_i(t)$ is the hazard rate of challenging for an unjustified player i,

$$\beta_i(t) = \frac{G_i'(t)}{1 - F_i(t^-) - G_i(t^-)}.$$

C.3.2 Equilibrium strategies and reputations in games with single demand types and slow ultimatum opportunity arrival for at least one player

Theorem 4. *Consider $B = \left(\{a_i, z_i, r_i, \gamma_i, c_i, k_i, w_i\}_{i=1}^2\right)$, a bargaining game with two-sided ultimatum opportunities and single demand types. If $\lambda_i \geqslant \gamma_i$ for some $i = 1, 2$, there exist finite times T and $T_1, T_2 \in [0, T)$ such that equilibrium strategies satisfy the following properties. For both $i = 1, 2$,*

1. *$\widehat{F}_i$ is strictly increasing in $(0, T)$ and constant for $t \geqslant T$;*
2. *$\widehat{F}_i$ is atomless in $(0, T]$ and at most one of the two has an atom at $t = 0$;*
3. *$\widehat{F}_i(T) + \widehat{G}_i(T_i) = 1$.*
4. *(a) $\widehat{G}_i$ is atomless, strictly increasing in $[0, T_i]$, and constant for $t \geqslant T_i$;*

   (b) For almost every $t \in [0, T]$, $\widehat{q}_i(t) \in (0, 1)$ if $t \in [0, T_i]$ and $\widehat{q}_i(t) = 1$ if $t \in (T_i, T]$;

Moreover, $\widehat{F}_i$ and $\widehat{G}_i$ are unique, and $\widehat{q}_i$ is unique almost everywhere for $t \leqslant T$.

Proof of Theorem 4. All the properties in the equilibrium characterization in the setting with one-sided ultimatum opportunities are satisfied. Therefore, we can derive the equilibrium strategies and reputations as follows.

Players' conceding strategies. In equilibrium, players concede at the same rates as in AG. Players are indifferent between conceding and waiting to concede the next instant. An unjustified player concedes at a rate $\kappa_i = \lambda_i/(1-\mu_i)$ to make the opposing unjustified player indifferent between conceding and not conceding, where $\lambda_i = r_j(1-a_i)/D$. **Player i's optimal yielding strategy.** An unjustified player i is indifferent between responding and yielding when player $j \neq i$ is believed to be justified with probability $\nu_j = 1 - k_i/(1-w) =: \nu_j^*$, strictly prefers to respond when $\nu_j < \nu_j^*$, and strictly prefers to yield when $\nu_j > \nu_j^*$.

Player i's optimal challenging strategy. We consider the optimal challenging strategy of an unjustified player i who believes that player $j \neq i$ is justified with probability μ_j and an unjustified player j yields to a challenge with probability q_j. An unjustified player i is indifferent between challenging and not challenging if $\mu_j = 1 - c_i/[q_j + (1-q_j)w]$. In particular, an unjustified player i strictly prefers not to challenge when $\mu_j < 1 - c_i =: \mu_j^*$.

Candidate equilibrium challenging and yielding strategies. If player j is justified with a probability more than μ_j^*, an unjustified player i strictly prefers not to challenge. If player j is justified with a probability less than μ_j^*, an unjustified player i must challenge at rate χ_j to make player i believe that a challenging player i is justified with probability $\nu_i^* := 1 - k_j/(1 - w_i)$:

$$\frac{\mu_i \gamma_i}{\mu_i \gamma_i + (1 - \mu_i)\chi_i} = \nu_i^* \Longrightarrow \chi_i(\mu_i) = \frac{1 - \nu_i^*}{\nu_i^*} \frac{\mu_i}{1 - \mu_i} \gamma_i.$$

If an unjustified player i challenges at a rate higher than the specified rate, then an unjustified player j is strictly better off responding than yielding to the challenge. If an unjustified player i challenges at a rate lower than the specified rate, then an unjustified player 2 is strictly worse off responding than yielding to the challenge. On the other hand, to make player i indifferent between challenging and not challenging, player j yields to a challenge with probability

$$q_j(\mu_j) = \frac{1}{1 - w_i} \left(\frac{k_i}{1 - \mu_j} - w_i \right).$$

Reputation in the challenge phase. When an unjustified player i challenges, player i's reputation follows the following Bernoulli differential equation:

$$\mu_i'(t) = (\lambda_i - \gamma_i)\mu_i(t) + \frac{\gamma_i}{\nu_i^*} \mu_i^2(t).$$

Reputation in the no-challenge phase. When an unjustified player i does not challenge, player i's reputation follows the following Bernoulli differential equation:

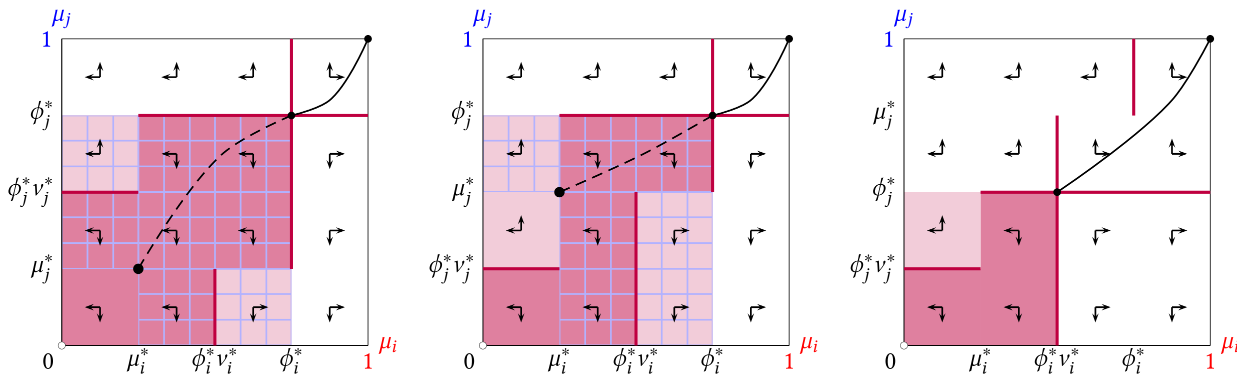
$$\mu_i'(t) = (\lambda_i - \gamma_i)\mu_i(t) + \gamma_i \mu_i^2(t).$$

Finite time. If $\lambda_i \geqslant \gamma_i$ for some $i = 1, 2$, then $\mu_i'(t) \geqslant \gamma_i z_i^2$ for all $\mu_i(t) \geqslant z_i$. Hence, $\tau < \infty$.

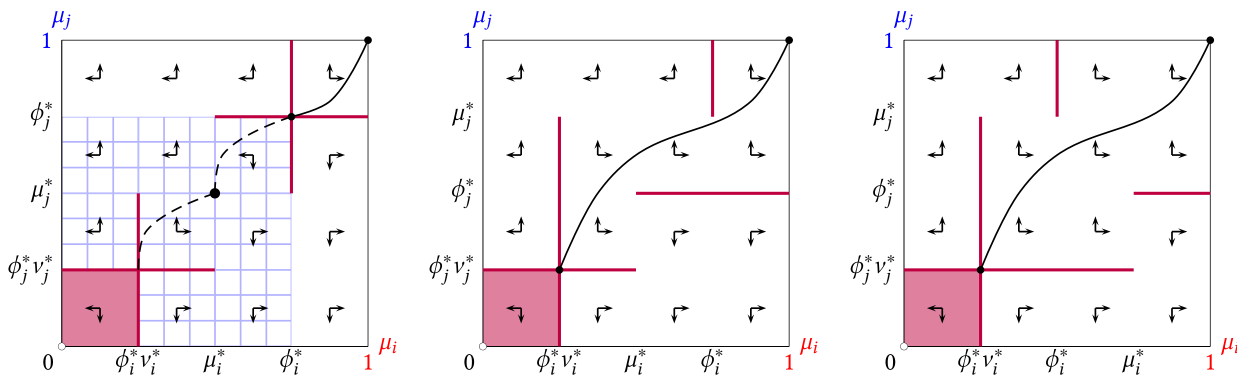
According to the differential equations characterizing the players' reputations, a reputation coevolution diagram can be uniquely drawn backwards from $(1, 1)$, the pair of terminal reputations. Hence, the strategies are uniquely pinned down as claimed. □

C.3.3 Equilibrium reputations and strategies in games with single demand types and fast ultimatum opportunity arrival for both players

The six reputation planes in Figure O1, with appropriate labeling of i and j as 1 and 2, cover all possible settings with (1) $\mu_1^* \in (0, \phi_1^* \nu_1^*)$, $\mu_1^* \in [\phi_1^* \nu_1^*, \phi_1^*]$, or $\mu_1^* \in (\phi_1^*, 1)$, and (2) $\mu_2^* \in (0, \phi_2^* \nu_2^*)$, $\mu_2^* \in [\phi_2^* \nu_2^*, \phi_2^*]$, or $\mu_2^* \in (\phi_2^*, 1)$. Each reputation plane has player i's reputation on the x-axis and player j's reputation on the y-axis, and is divided into sixteen regions with μ_k^*, $\phi_k^* \nu_k^*$, and ϕ_k^*, $k = i, j$, as dividing lines. Fix any of the sixteen regions. For any initial reputation vector in the region or on its boundary, if neither player concedes at time zero and both players follow the strategies specified above, the horizontal arrow in the region represents the direction of player i's reputation building, and the vertical arrow represents the direction of player j's reputation building, with the direction strict on the dividing lines unless the



(a) $\mu_i^* \in (0, \phi_i^* v_i^*)$ and $\mu_j^* \in (0, \phi_j^* v_j^*)$ (b) $\mu_i^* \in (0, \phi_i^* v_i^*)$ and $\mu_j^* \in [\phi_j^* v_j^*, \phi_j^*]$ (c) $\mu_i^* \in (0, \phi_i^* v_i^*)$ and $\mu_j^* \in (\phi_j^*, 1)$



(d) $\mu_i^* \in [\phi_i^* v_i^*, \phi_i^*]$ and $\mu_j^* \in [\phi_j^* v_j^*, \phi_i^*]$ (e) $\mu_i^* \in [\phi_i^* v_i^*, \phi_i^*]$ and $\mu_j^* \in (\phi_j^*, 1)$ (f) $\mu_i^* \in (\phi_i^*, 1)$ and $\mu_j^* \in (\phi_j^*, 1)$

Figure O1: Illustration of characterization of equilibrium reputations in bargaining games with single demand types and high ultimatum opportunity arrival rates $\gamma_1 > \lambda_1$ and $\gamma_2 > \lambda_2$.
The six figures, with appropriate labeling of i and j as 1 and 2, cover all possible settings with (1) $\mu_1^* \in (0, \phi_1^* v_1^*)$, $\mu_1^* \in [\phi_1^* v_1^*, \phi_1^*]$, or $\mu_1^* \in (\phi_1^*, 1)$, and (2) $\mu_2^* \in (0, \phi_2^* v_2^*)$, $\mu_2^* \in [\phi_2^* v_2^*, \phi_2^*]$, or $\mu_2^* \in (\phi_2^*, 1)$. Each of the six figures contains a reputation plane with player i's reputation on the x-axis and player j's reputation on the y-axis. Each reputation plane is divided into sixteen regions with μ_k^*, $\phi_k^* v_k^*$, and ϕ_k^*, $k = i, j$, as dividing lines. **(Finite-T equilibrium)** If the initial reputation vector lies in the white region and its boundary, there is a unique equilibrium, which is a finite-T equilibrium with players' reputations coevolving on the solid line to $(1, 1)$ after at most one player concedes at time zero. **(Type-1 infinite-T equilibrium)** If the initial reputation vector lies in the (lighter and darker) purple region and its boundary, there are many infinite-T equilibria for each of which at most one player concedes at time zero, the reputation vector after initial concession lies in the darker purple region and any darker purple lines on the boundary of the region, and players' reputations evolve to but never reach $(0, 0)$, or $(0, \omega)$ if $(0, \omega)$ lies on a darker purple line. **(Type-2 infinite-T equilibrium)** If the initial reputation vector lies in the crosshatched region excluding its boundary, there is an infinite-T equilibrium in which at most one player concedes at time zero, players' reputations after time zero coevolve on the dashed line to (μ_i^*, μ_j^*).

dividing line is darker purple.

Equilibrium reputations must eventually reach $(1, 1)$ in a finite-T equilibrium, or approach $(0, 0)$, approach $(0, \omega)$ if $(0, \omega)$ is on the purple line, or reach and stay at (μ_i^*, μ_j^*) in infinite-T equilibria. Using the directions of reputation building after initial concessions, if players follow specified equilibrium strategies, we can derive contradictions with the eventual reputation vector for any initial concession that is not part of any equilibrium. Hence, the directions of reputation building restrict candidate equilibrium reputation vectors immediately after initial concessions, and consequently initial concessions in equilibrium. The set of reputation vectors that can be equilibrium reputation vectors immediately after initial concessions is represented by a solid line, a darker purple region (and selective darker purple lines on its boundary), and a dashed line. Specifically, the solid line represents the collection of reputation vectors immediately after initial concessions that situates on the path to eventually reach $(1, 1)$ if players follow specified post-concession strategies, a darker purple region and its selected darker purple lines on its boundary represent the collection of reputation vectors that can be supported as equilibrium reputation vector immediately after initial concessions, and the dashed line, if it exists in a figure, collects the reputation vector that situates on a reputation coevolution curve that eventually reaches—increases or decreases to—(μ_i^*, μ_j^*) if players follow specified post-initial-concession strategies.

The equilibrium post-initial-concession strategies must be consistent with the reputation building specified by the solid line, the darker purple region and its appropriate boundary, and the dashed line, and the equilibrium initial concession. The initial concession by one player is part of an equilibrium as long as the posterior reputations after the initial concession lies on the lines or the darker purple region.