Axiomatic Measures of Assortative Matching: Theoretical and Empirical Analyses

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Abstract

There is an active debate about the direction of change in assortative matching on education in the US, because different measures yield different conclusions. I take an axiomatic approach—starting with properties a measure should satisfy—to find appropriate measures of assortativity. I find that aggregate likelihood ratio (Eika, Mogstad, and Zafar, 2019) and normalized trace (proportion of pairs of like types) carry cardinal meanings. They also naturally extend to markets with singles and one-sided markets. The relation induced by odds ratio (Chiappori, Costa-Dias, and Meghir, 2022) is the unique total order on two-type markets that satisfies marginal independence (Edwards, 1963), but it has ordinal meaning only and does not have a multi-type extension. Unfortunately, there is no total order on multi-type markets that satisfies monotonicity and the additional requirement of robustness to categorization (i.e., assortativity order holds regardless of categorization of the types); positive quadrant dependence order (Anderson and Smith, 2022) is the unique partial order that satisfies them. I apply the measures to shed light on the evolution of educational assortativity in the US and other countries.

Keywords: assortative matching, axiomatic approach

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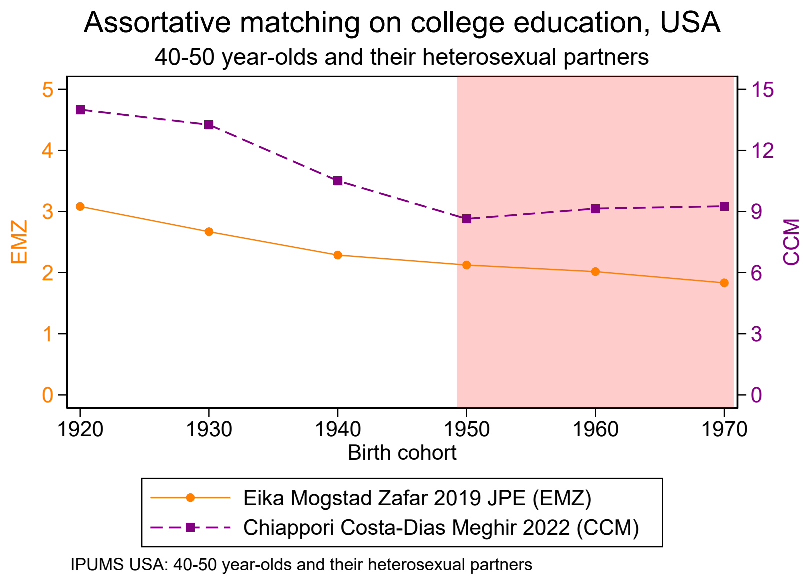
1 Introduction

There is an active debate about the direction of change in assortative matching on education in the US in recent decades. Figure 1 shows that based on different measures it can be concluded to have decreased (Eika, Mogstad, and Zafar, 2019, henceforth EMZ) or increased (Chiappori, Costa-Dias, and Meghir, 2022, henceforth CCM). All the measures used are "reasonable," but different measures yield different conclusions. Drawing a definitive conclusion is especially difficult given that educational composition differs across time and location.

Figure 1: Assortative matching on college education in the US



Motivated by the specific empirical debate, I would like to address the general theoretical question: How do I rank and/or quantify the assortativity of matching markets with potentially different distributions of educational and socioeconomic characteristics?

Rather than starting with specific measures and justifying them, I take an axiomatic approach—starting with a set of properties measures should satisfy—to discover the appropriate measure(s) of assortative matching. I start with two sets of basic axioms: (i) notions of *equivalence* (what I mean by two markets are equally assortative)—*scale invariance* (invariance to scaling the market size) and *side* and *type symmetry* (invariance to swapping sides and types, respectively)—and (ii) different notions of *monotonicity* (what I mean by one market is more assortative than another)—*diagonal* and *off-diagonal monotonicity* (assortativity increases when more like types match and decreases when unlike types match) and *marginal monotonicity* (defining assortativity order on markets with the same marginal distributions).

Besides the basic axioms, I state more crucial axioms that will characterize different measures (*characterization axioms*). First, I define *decomposability* axioms: The assortativity measure for any market is a weighted sum or average of the measures for submarkets decomposed from the market. The aggregate likelihood ratio (EMZ)—the excess likelihood of like types pairing up relative to the hypothetical likelihood under random matching—is the unique measure (up to linear transformation) that satisfies the basic axioms plus *random decomposability*, i.e., the weight to sum is the hypothetical proportion of pairs of like types under random matching (Theorem 1). The normalized trace—the proportion of pairs of like types—emerges as the unique measure (up to linear transformation) that satisfies the basic axioms plus *population decomposability*, i.e, the weight to average is the hypothetical proportion of pairs of like types under random matching (Theorem 2) or population size (Theorem 2). In addition, these measures have natural extensions to two-sided markets with singles, one-sided markets (e.g., homosexual matching markets), and multi-type markets (Propositions 1, 2, and 3).

In two-type markets, the odds ratio (CCM)—the ratio of the odds to match with the same type and with a different type—is the unique measure (up to monotonic transformation) that satisfies the basic axioms plus *marginal independence* (Edwards, 1963) (Theorem 3). Because all measures that are monotonic transformations of the odds ratio (e.g., Yule's Q, Yule's Y, and log odds ratio) provide the same ordering of markets in terms of assortativity, the result clarifies that the odds ratio carries ordinal meaning only. In addition, the odds ratio does not have a natural extension to multi-type markets and should be treated as a local measure of assortativity.

In general, in markets with more than two types (e.g., there are 14 detailed education categories in the US census), *robustness to categorizations* is a desired property: Assortativity comparison should be robust to how the education levels are categorized into two or several groups. I show that there exists no total order that satisfies the minimal set of axioms of monotonicity and robustness to categorizations. As a result, I must resort to partial orders. With three ordered types, I show that the positive quadrant dependence (PQD) order on markets with the same marginal distributions (Anderson and Smith, 2022, henceforth AS) is the unique partial order that satisfies monotonicity and robustness to categorizations.

Using the axiomatized measures, I discuss the evolution and magnitude of assortativity on education in the US. I find that in recent decades, assortative matching on college has decreased under aggregate likelihood ratio (EMZ) and normalized trace, and the decrease was more prominent when considering singles as part of the market. In contrast, it has increased under odds ratio (CCM); the likelihood ratio for college and that for noncollege (EMZ) provide conflicting conclusions and should be used with caution. The empirical analyses also motivate a more careful consideration of what constitutes a marriage market.

The rest of the paper is organized as follows. Section 2 presents the binary-type setup, gives examples of measures that have been used in the literature, and describes basic axioms. Section 3 presents characterization results for binary-type markets and discusses extensions. Section 4 presents the results for multi-type markets. Section 5 presents US evidence, and Section 6 concludes.

2 Binary types

Each man and woman possesses a trait (e.g., college education or any other socioeconomic trait) and can be one of two types, high θ_1 and low θ_2. Consider the matching between men and women that is described by matrix $M = (a, b, c, d)$:

$$M = \begin{array}{ccc} m\backslash w & \theta_1 & \theta_2 \\ \theta_1 & a & b \\ \theta_2 & c & d \end{array}.$$

A cell describes the mass of pairs between a specific combination of types of men and women. To recover the mass of individuals of a specific gender and type, I add up a column or a row; for example, there is mass $a + b$ of high-type men. The marginal distribution for men is then described by $(a + b, c + d)$ and that for women is $(a + c, b + d)$. Assume a full support of types on both sides of the market: $(a+b)(a+c)(b+d)(c+d) > 0$. I refer to M as matching, matrix, or market, and let $|M| \equiv a + b + c + d$ denote its population size.

There are a few matching arrangements that will appear repeatedly. A *strictly positive assortative matching (PAM)* has $b = c = 0$ and should be judged the most assortative by any reasonable measure. Any matching with $bc = 0$ should be considered as *positive assortative*, because the maximal feasible mass of pairs of like types has been reached. Analogously, a *strictly negative assortative matching (NAM)* has $a = d = 0$ and should be the least assortative by any reasonable measure; a matching with $ad = 0$ should be considered as *negative assortative*, because the maximal feasible mass of pairs of unlike types has been reached. *Random matching* is reached when all individuals are uniformly randomly matched. For market $M = (a, b, c, d)$, the hypothetical random matching is

$$R(M) \equiv \begin{array}{c|c} \frac{a+b}{|M|}\frac{a+c}{|M|}|M| & \frac{a+b}{|M|}\frac{b+d}{|M|}|M| \\ \hline \frac{a+c}{|M|}\frac{c+d}{|M|}|M| & \frac{c+d}{|M|}\frac{b+d}{|M|}|M| \end{array}.$$

Objective. Most ideally, I would like to find an *index*, i.e., a function $I : \mathbb{R}^4_+ \to \mathbb{R} \cup$

$\{-\infty, +\infty\}$, that measures the matching assortativity. Alternatively, I resort to find a *total order* $\succeq$ on the set of markets, $\mathbb{R}^4_+$. At the least, I would like to find a *partial order* on the set of markets. Total and partial orders have ordinal meanings only. An index I has an ordinal interpretation, because real numbers are endowed with a natural order. In addition, an index may also have a cardinal interpretation.

2.1 Examples of measures

I start by introducing the measures. They can be roughly divided into three categories: (i) measures that relate to the proportion of pairs of like types, (ii) measures that are compared to the hypothetical benchmark of random matching, (iii) measures that induce the same order as the odds ratio. Readers may skip ahead to the axioms; these measures serve to provide some concrete ideas before introducing the abstract axioms. In the parentheses, I provide the papers that have used/defined the measures.

2.1.1 Proportion of pairs of like types

Normalized trace equals 1 if the matching is positive assortative, equals 0 if the matching is negative assortative, and equals the proportion of like types in the market when it is neither positive assortative nor negative assortative.

(NT) Normalized trace

$$I_{tr}(a,b,c,d) = \begin{cases} 1 & \text{if } bc = 0 \\ \frac{a+d}{a+b+c+d} \in (0,1) & \text{if } abcd \neq 0 \\ 0 & \text{if } ad = 0 \end{cases}.$$

A linear transformation of this measure is

(MD) Minimum distance $I_{MD}(a,b,c,d) = 2I_{tr}(M) - 1$ that takes the range of $[-1, +1]$.

2.1.2 Normalization based on random matching

(LR) Likelihood ratio (EMZ); homogamy rate (Ciscato, Galichon, and Goussé, 2020)

$$I_{L1}(M) \equiv \frac{\text{observed } \#\theta_1\theta_1}{\text{random baseline}} = \frac{a}{|M|} \Big/ \frac{a+b}{|M|}\frac{a+c}{|M|} = \frac{a(a+b+c+d)}{(a+b)(a+c)}.$$

The likelihood ratio based on θ_1 compares the realized likelihood of high-type pairs ($\frac{a}{a+b+c+d}$) to the benchmark of the hypothetical likelihood of high-type pairs under purely random matching ($\frac{a+b}{a+b+c+d}\frac{a+c}{a+b+c+d}$). The likelihood ratio based on θ_2 is analogously defined:

$$I_{L2}(M) \equiv \frac{\text{observed } \#\theta_2\theta_2}{\text{random baseline}} = \frac{d}{|M|} \bigg/ \frac{d+b}{|M|}\frac{d+c}{|M|} = \frac{d(a+b+c+d)}{(d+b)(d+c)}.$$

One issue with type-dependent likelihood ratio is that it requires the choice of a type as a benchmark; in other words, it fails type symmetry. Empirically, I will see that the likelihood ratio based on different type will yield opposite conclusions on the direction of change in assortative matching.

The aggregate likelihood ratio is a weighted average of the type-specific likelihood ratios, where the weights are the expected mass of pairs of like types under random matching.

(ALR) Aggregate likelihood ratio (EMZ)

$$\begin{aligned} I_L(M) &\equiv \frac{(a+b)(a+c)}{(a+b)(a+c)+(d+b)(d+c)} I_{L1}(M) + \frac{(d+b)(d+c)}{(a+b)(a+c)+(d+b)(d+c)} I_{L2}(M) \\ &= \frac{\text{observed } \#\theta_1\theta_1 + \#\theta_2\theta_2}{\text{random baseline}} = \frac{a+d}{|M|} \bigg/ \left(\frac{a+b}{|M|}\frac{a+c}{|M|} + \frac{d+b}{|M|}\frac{d+c}{|M|} \right) \end{aligned}$$

Simplified, it is the ratio between the observed mass of couples of all like types and the counterfactual mass of pairs if individuals matched randomly.

In addition, correlation, and its squared expression (Spearman's rank correlation) as well as pure-random normalization all normalize random matching to zero.

(Corr) Correlation (Ciscato, Galichon, and Goussé, 2020; Hou et al., 2022)

$$I_{Corr}(M) = \frac{ad-bc}{\sqrt{(a+b)(c+d)(a+c)(b+d)}}.$$

A measure that generates the same order is its square, which is called chi-square or Spearman's rank correlation:

$$I_\chi(M) = [I_{Corr}(M)]^2 = \frac{(ad-bc)^2}{(a+b)(c+d)(a+c)(b+d)}.$$

Shen (2020) demonstrates numerically the peril of using only normalization compared to random matching, and advocates the pure-random normalization.

(PR) Pure-random normalization (Liu and Lu, 2006; Shen, 2020)

$$I_{MD}(M) = \frac{ad-bc}{(\max\{b,c\}+d)(a+\max\{b,c\})}.$$

2.1.3 Odds ratio and its monotonic transformations

The *odds ratio* is originally used to quantify the strength of the association between two events. In matching, the odds ratio is interpreted as the ratio of the odds marrying an educated versus uneducated woman for an educated man and the same odds for an uneducated woman, or equivalently (due to symmetry), the ratio of the odds of marrying an educated versus uneducated man for an educated woman and the same for an uneducated woman.

(OR) odds ratio (CCM); cross-ratio (Edwards, 1963)

$$I_O(M) \equiv \frac{a}{b} / \frac{c}{d} = \frac{a}{c} / \frac{b}{d} = \frac{ad}{bc}.$$

The index ranges from 0 to $+\infty$. There are a few monotonic transformations of the odds ratio that are used in the literature. For our purpose, they all provide the same ranking of assortativity over all matchings.

(LOR) Log odds ratio; log cross-ratio (Siow, 2015) $I_o(M) \equiv \ln I_O(M)$.

(Q) Yule's Q; Coefficient of association (Yule, 1900) $I_Q(M) \equiv \frac{I_O(M)-1}{I_O(M)+1}$. It is $+1$ when PAM, -1 when NAM, and 0 when random matching (uncorrelated).

(Y) Yule's Y; Coefficient of colligation (Yule, 1912) $I_Y(M) \equiv \frac{\sqrt{I_O(M)}-1}{\sqrt{I_O(M)+1}}$.

2.2 Basic axioms

I start with the following two sets of basic axioms that a measure should satisfy. They give ways to describe (i) when two markets are equally assortative and (ii) when one market is more assortative than another. For generality, whenever possible, I define axioms that an order—rather than an index—should satisfy. In those cases, an index is said to satisfy an axiom if the total order induced by the index satisfies it.

Definition 1. *I say that total order $\succeq$ is **induced by** index I if for all markets M and M', $I(M) > I(M') \Leftrightarrow M \succ M'$ and $I(M) = I(M') \Leftrightarrow M \sim M'$. Two indices I and I' are **order-equivalent** if the total orders $\succeq_I$ and $\succeq_{I'}$ they induce are equivalent.*

2.2.1 Equivalence axioms (EQV)

I start with axioms that define two equally assortative markets. First, when a matching market is doubled or scaled up or double by a constant without changing the relative composition, the assortativity should not be evaluated as changed.

[SInv] Scale Invariance. *For all M and $\lambda > 0$, $M \sim \lambda \cdot M$.*

Note that given SInv, we can transform the matrix into a contingency table by dividing each element by $\lambda = |M|$.

Next, the market should be considered equally assortative if we swap the labels of the two types, which is essentially swapping high-high (a) and low-low (d) pairs and swapping high-low (b) and low-high (c) pairs.

[TSym] Type Symmetry. *The market is equally assortative when types are switched:*

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \sim \begin{array}{c|c} d & c \\ \hline b & a \end{array}.$$

Neither should changing the sides of the markets affect the measure of assortativity. Effectively, it switches proportion of high-low (b) and low-high (c) pairs.

[SSym] Side Symmetry. *The market is equally assortative when sides are switched:*

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \sim \begin{array}{c|c} a & c \\ \hline b & d \end{array}.$$

Most of the measures I consider will satisfy these three axioms, and they are basic axioms that an appropriate measure should satisfy, so I will refer to all three together as the **equivalence axioms (EQV)**.

2.2.2 Monotonicity axioms

Next, I define notions of monotonicity that specify when one market is more assortative than another. I define two notions of monotonicity, one based on comparisons of markets of different marginal distributions and the other based on comparisons of markets of the same marginals. First, when pairs of like types (i.e., the terms on the diagonal of the matrix) increase, the matching becomes more assortative.

[DMon] Diagonal Monotonicity. *For all M and $\epsilon > 0$,*

$$\begin{array}{c|c} a+\epsilon & b \\ \hline c & d \end{array} \succeq \begin{array}{c|c} a & b \\ \hline c & d \end{array} \text{ and } \begin{array}{c|c} a & b \\ \hline c & d+\epsilon \end{array} \succeq \begin{array}{c|c} a & b \\ \hline c & d \end{array},$$

where the equalities hold if and only if $bc = 0$.

DMon implies the following property that defines PAM.

⇒**[PAM] Positive Assortative Matching.** *For any positive symbols,*

$$\begin{array}{c|c} a & 0 \\ \hline 0 & d \end{array} \sim \begin{array}{c|c} a' & b' \\ \hline 0 & d' \end{array} \sim \begin{array}{c|c} a'' & 0 \\ \hline c'' & d'' \end{array} \succ \begin{array}{c|c} a''' & b''' \\ \hline c''' & d''' \end{array}.$$

Note that the qualifier "if and only if $bc = 0$" is the key for the implication. In the extension in the next section, I will explore a modification of DMon and its implications on PAM.

I define analogously define off-diagonal monotonicity: When pairs of unlike types (i.e., the terms off the diagonal of the matrix) increase, the matching becomes less assortative.

[ODMon] Off-Diagonal Monotonicity. *For all M and $\epsilon > 0$,*

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \succeq \begin{array}{c|c} a & b+\epsilon \\ \hline c & d \end{array} \text{ and } \begin{array}{c|c} a & b \\ \hline c & d \end{array} \succeq \begin{array}{c|c} a & b \\ \hline c+\epsilon & d \end{array},$$

where the equalities hold if and only if $ad = 0$.

Analogous to the implication of DMon on PAM, ODMon implies the following property that defines NAM.

⇒**[NAM] Negative Assortative Matching.** *For any positive symbols,*

$$\begin{array}{c|c} 0 & b \\ \hline c & 0 \end{array} \sim \begin{array}{c|c} a' & b' \\ \hline c' & 0 \end{array} \sim \begin{array}{c|c} 0 & b'' \\ \hline c'' & d'' \end{array} \prec \begin{array}{c|c} a''' & b''' \\ \hline c''' & d''' \end{array}.$$

Again, the qualifier "if and only if $ad = 0$" is the key for the implication. In the extension in the next section, I will explore a modification of ODMon and its implication on NAM.

Now, I consider a notion of monotonicity that compares markets with the same marginal distributions of men and women. The markets that share the same marginal distributions as (a, b, c, d) are $(a+\epsilon, b-\epsilon, c-\epsilon, d+\epsilon)$, essentially a one-parameter family.

[MMon] Marginal Monotonicity (CCM). *Consider two markets M and M' with the same marginal distributions (i.e., $a+c = a'+c'$, $a+b = a'+b'$, $d+b = d'+b'$, and $d+c = d'+c'$). $M \succ M'$ if and only if $a > a'$ (equivalently, $b < b'$, $c < c'$, or $d > d'$).*

Claim 1. *DMon and ODMon imply MMon.*

Proof of Claim 1. Suppose $M = (a, b, c, d)$ and $M' = (a', b', c', d')$ have the same marginal distributions, and suppose $a > a'$. Market M' can be represented as

$$M' = \frac{a-(a-a') \;\big|\; b+(a-a')}{c+(a-a') \;\big|\; d-(a-a')} \prec \frac{a \;\big|\; b+(a-a')}{c+(a-a') \;\big|\; d} \prec \frac{a \;\big|\; b}{c \;\big|\; d} = M,$$

where the first $\prec$ follows from DMon and the second $\prec$ follows from ODMon. □

Conversely, however, MMon does not necessarily imply DMon or ODMon, because MMon solely specifies relations for markets with the same marginal distributions and by itself has no implications for markets with different marginal distributions.

Table 1 summarizes whether the measures satisfy the basic equivalence and monotonicity axioms. (Type-specific) likelihood ratio does not satisfy type symmetry, because relabeling the types would change the measure (a point first highlighted by CCM). The few measures that are based on random matching—aggregate likelihood ratio, aggregate homogamy rate, correlation, and pure-random normalization—do not satisfy DMon and/or ODMon. In particular, when a and/or d is relatively small, diagonal monotonicity tends to fail. Normalized trace and odds ratio are the only measures that satisfy all six basic axioms.

Table 1: Do the measures satisfy the basic equivalence and monotonicity axioms?

	equivalence axioms			monotonicity axioms		
	SInv	TSym	SSym	DMon	ODMon	MMon
Normalized trace	✓	✓	✓	✓	✓	✓
Likelihood ratio (EMZ)	✓	X	✓	✓	✓	✓
Aggregate likelihood ratio (EMZ)	✓	✓	✓	X	X	✓
Correlation	✓	✓	✓	X	X	✓
Pure-random normalization	✓	✓	✓	✓	X	✓
Odds ratio (CCM)	✓	✓	✓	✓	✓	✓

3 Characterization results for binary types

As Table 1 shows, several measures satisfy all the basic equivalence and monotonicity axioms. Besides likelihood ratio, all the measures listed satisfy the equivalence axioms and at least one notion of monotonicity, marginal monotonicity. Hence, the basic axioms are not sufficient to distinguish the measures and provide a definitive answer to what measure to use.

3.1 Characterization axioms

I will need additional axioms to distinguish and characterize the different measures. I will introduce what I call "characterization axioms" such that a measure will be the unique one that satisfies the basic axioms and an axiom that essentially characterizes the special property of the measure.

The first set of characterization axioms will bestow the measures cardinal interpretations. Namely, the assortativity measure of a market will be a weighted sum or average of the measures of submarkets decomposed from the original market. Depending on the weights used, I will have different decomposability axioms that correspond to different measures.

3.1.1 Random decomposability and aggregate likelihood ratio

The aggregate likelihood ratio is characterized by a decomposability axiom in which the weight to sum is the expected mass of randomly matched pairs of like types.

[RDec] Random Decomposability. For all markets $M \gg 0$ and $M' \gg 0$,

$$I(M+M') = \frac{r(M)}{r(M+M')} I(M) + \frac{r(M')}{r(M+M')} I(M'),$$

where $r(M)$ indicates the expected mass of pairs of like types under random matching in market M:

$$r(M) \equiv \frac{a+b}{|M|}\frac{a+c}{|M|}|M| + \frac{d+b}{|M|}\frac{d+c}{|M|}|M| = \frac{(a+b)(a+c)+(d+b)(d+c)}{a+b+c+d}.$$

Theorem 1. *An index satisfies EQV, MMon, and RDec if and only if it is proportional to aggregate likelihood ratio.*

I make two comments. First, note that ALR does not satisfy DMon or ODMon, so I do not have a characterization result that links those axioms and ALR. Second, note that the class of measures that satisfy the axioms in Theorem 1 must be a constant multiple of ALR. The class has to be a constant multiple of ALR because the weights $r(M)/r(M+M')$ and $r(M')/r(M+M')$ do not necessarily add up to be 1. Nonetheless, this characterization result gives ALR a cardinal interpretation.

3.1.2 Population decomposability and normalized trace

To address the two comments above, if I consider (i) a measure that satisfies DMon and ODMon and (ii) an axiom that involves weighted averages that add up to 1, then I have the following characterization axiom.

[RPDec] Random Population Decomposability. For all $M \gg 0$ and $M' \gg 0$,

$$I(M + M') = \frac{r(M)}{r(M) + r(M')} I(M) + \frac{(M')}{r(M) + r(M')} I(M').$$

Another reasonable weight to average is the population size of submarkets.

[PDec] Population Decomposability. For all markets $M \gg 0$ and $M' \gg 0$,

$$I(M + M') = \frac{|M|}{|M| + |M'|} I(M) + \frac{|M'|}{|M| + |M'|} I(M').$$

While, as argued above, MMon (combined with the basic axioms) cannot imply DMon and ODMon, MMon and PDec, combined with SInv and TSym, imply DMon and ODMon.

Claim 2. *SInv, TSym, MMon, and PDec imply DMon and ODMon.*

I show in the theorem below that RPDec and PDec are characterization axioms for NT. That is, NT is the unique index, up to linear transformation, that satisfies PDec and the basic equivalence and monotonicity axioms. Because Claims 1 and 2 combine to imply the equivalence of the two notions of monotonicity—DMon+ODMon and MMon—given PDec, the characterization result holds for both notions of monotonicity.

Theorem 2. *An index satisfies EQV, MMon (or DMon and ODMon), and PDec (or RPDec) if and only if it is a linear transformation of normalized trace.*

The index is uniquely determined if the range is specified: When the range is $[0, 1]$, the unique index is normalized trace; When the range is $[-1, +1]$, the unique index is MD.

3.1.3 Marginal independence and odds ratio

I now provide the characterization result regarding marginal independence (Edwards, 1963) and the odds ratio (CCM). Marginal independence of an order requires that multiplying any row or any column of any market does not change the assortativity order.

[MInd] Marginal Independence (Edwards, 1963). *For all $M \gg 0$ and $\lambda > 0$,*

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \sim \begin{array}{c|c} \lambda a & \lambda b \\ \hline c & d \end{array} \sim \begin{array}{c|c} a & b \\ \hline \lambda c & \lambda d \end{array} \sim \begin{array}{c|c} \lambda a & b \\ \hline \lambda c & d \end{array} \sim \begin{array}{c|c} a & \lambda b \\ \hline c & \lambda d \end{array}.$$

Note that while decomposability axioms contain both ordinal and cardinal contents of assortative matching, in contrast, MInd is an ordinal property. Nonetheless, MInd is a strong condition. MInd implies SInv, TSym, and SSym. MInd and MMon together imply DMon and ODMon.

Claim 3. *MInd and MMon imply DMon and ODMon.*

In fact, more strongly, under MInd, MMon, DMon, and ODMon are equivalent.

Claim 4. *Suppose an order or index satisfies MInd. It satisfies DMon if and only if it satisfies ODMon if and only if it satisfies MMon.*

Note that an index and its monotonic transformation are order-equivalent. CCM argue to use the odds ratio primarily because it satisfies MInd. While CCM show that the odds ratio satisfies MInd and MMon, I show that the odds ratio is the unique total order that satisfies MInd and MMon (which together imply the basic axioms).

Theorem 3. *A total order satisfies MMon and MInd if and only if it is the order induced by the odds ratio. Equivalently, an index satisfies MMon and MInd if and only if it is a monotonic transformation of the odds ratio.*

3.2 Extensions

3.2.1 Singles

Consider the markets with singles. Expand the market without singles by adding a row and a column to indicate the singles. Let μ_{i0} denote the mass of single type-θ_i men and M_{0j} the mass of single type-θ_j women:

$$M = \begin{array}{cccc} m\backslash w & \theta_1 & \theta_2 & \emptyset \\ \theta_1 & \mu_{11} & \mu_{12} & \mu_{10} \\ \theta_2 & \mu_{21} & \mu_{22} & \mu_{20} \\ \emptyset & \mu_{01} & \mu_{02} & 0 \end{array}.$$

If I do not consider singles, then the following three markets have the same assortativity because they are all perfectly positive assortative.

$$M_1 = \begin{array}{cccc} m\backslash w & \theta_1 & \theta_2 & \emptyset \\ \theta_1 & 100 & 0 & 50 \\ \theta_2 & 0 & 100 & 0 \\ \emptyset & 50 & 0 & 0 \end{array}; \quad M_2 = \begin{array}{cccc} m\backslash w & \theta_1 & \theta_2 & \emptyset \\ \theta_1 & 100 & 0 & 0 \\ \theta_2 & 0 & 100 & 0 \\ \emptyset & 0 & 0 & 0 \end{array}; \quad M_3 = \begin{array}{cccc} m\backslash w & \theta_1 & \theta_2 & \emptyset \\ \theta_1 & 150 & 0 & 0 \\ \theta_2 & 0 & 100 & 0 \\ \emptyset & 0 & 0 & 0 \end{array}.$$

Arguably, M_2 is more assortative than M_1 because there are no singles who could have matched with each other; M_3 is more assortative than M_1 because unmatched individuals in M_1 are assortatively matched in M_3.

[SMon] Singles Monotonicity. *Consider $M = (\mu_{ij})$ and $M' = (\mu'_{ij})$. If $\mu_{i0} > \mu'_{i0}$ for an $i \neq 0$ and $\mu_{jk} = \mu'_{jk}$ for any other combination of j and k, $M \succ M'$.*

Proposition 1. *(i) An index on two-type markets with singles satisfies EQV, MMon, SMon, and RDec if and only if it is a multiple of aggregate likelihood ratio.*

(ii) An index on two-type markets with singles satisfies EQV, MMon (or DMon and ODMon), SMon, and PDec (or RPDec) if and only if it is a linear transformation of normalized trace.

In this case, $\widetilde{I}_{tr}(M_1) = 400/500 = 4/5$ and $\widetilde{I}_{tr}(M_2) = \widetilde{I}_{tr}(M_3) = 1$.

3.2.2 One-sided markets

Consider that there is a one-sided market in which anyone can match with anyone, for example, a homosexual marriage market. With natural modifications of the axioms, I can show that ALR and NT continue to be appropriate measures in this market, even when there are singles and multiple types. The appendix states the modifications of the axioms and the general results with singles and multiple types. The odds ratio does not have an extension to one-sided markets.

Proposition 2. *(i) An index on one-sided markets satisfies EQV, MMon, and RDec if and only if it is a multiple of aggregate likelihood ratio.*

(ii) An index on one-sided markets satisfies EQV, MMon (or DMon and ODMon), and PDec (or RPDec) if and only if it is a linear transformation of normalized trace.

4 Multiple types

Consider market $M = (\mu_{ij})_{i,j\in\{1,\cdots,n\}}$ for $n \geq 2$.

4.1 Extensions from two-type markets

We have some straightforward extensions from the two-type markets. Aggregate likelihood ratio and normalized trace can be naturally extended to the multi-type markets, but the odds ratio does not have a natural extension.

(ALR) Aggregate likelihood ratio for multi-type markets

$$I_L(M) \equiv \frac{tr(M)/|M|}{\sum_{i=1}^n \left(\frac{\sum_{j=1}^n \mu_{ij}}{|M|}\right)\left(\frac{\sum_{j=1}^n \mu_{ji}}{|M|}\right)} = \frac{|M|(\sum_{i=1}^n \mu_{ii})}{\sum_{i=1}^n(\sum_{j=1}^n \mu_{ij})(\sum_{j=1}^n \mu_{ji})}.$$

(NT) Normalized trace for multi-type markets

$$I_{tr}(M) = \begin{cases} 1 & \textit{if } \mu_{ij}\mu_{ji} = 0 \forall i \neq j \\ 0 & \textit{if } tr(M) \equiv \sum_{i=1}^n \mu_{ii} = 0 \\ tr(M)/|M| & \textit{otherwise.} \end{cases}$$

Theorems 1 and 2 can be naturally extended so that aggregate likelihood ratio and normalized trace are the unique indices that satisfy the sets of axioms stated in those theorems. Hence, ALR and NT can still be used to measure the assortativity of multi-type markets.

Proposition 3. *(i) An index on multi-type markets satisfies EQV, MMon, and RDec if and only if it is a multiple of aggregate likelihood ratio.*

(ii) An index on multi-type markets satisfies EQV, MMon (or DMon and ODMon), and PDec (or RPDec) if and only if it is a linear transformation of normalized trace.

4.2 Robustness to categorizations

Consider three types θ_i, θ_j, and θ_k. When two types θ_i and θ_j merge so that the three categories are partitioned to $\{\{i,j\},\{k\}\}$, the market given this categorization becomes

$$M|_{(\{1,2\}\{3\})} = \begin{pmatrix} M_{11}+M_{12}+M_{21}+M_{22} & M_{13}+M_{23} \\ M_{31}+M_{32} & M_{33} \end{pmatrix}.$$

I say that M is more assortative than M' if and only if $M_{\{i,j\}}$ is more assortative than $M'_{\{i,j\}}$ for all i and all $j \neq i$. With the normalized trace measure, for M and M' such that $|M| = |M'|$ and $\text{tr}(M) = \text{tr}(M')$, M is more assortative than M' if and only if $\mu_{12}+\mu_{21} \geq \mu'_{12}+\mu'_{21}$, $\mu_{13}+\mu_{31} \geq \mu'_{13}+\mu'_{31}$, and $\mu_{23}+\mu_{32} \geq \mu'_{23}+\mu'_{32}$.

Theorem 4. *There is no total order that satisfies MMon (or DMon and ODMon) and RCat.*

A counterexample suffices for the claim.

Proof of Theorem 4. Consider markets

$$M = \begin{array}{c|c|c} 1/9 & 1/9 & 1/9 \\ \hline 1/9 & 1/9 & 1/9 \\ \hline 1/9 & 1/9 & 1/9 \end{array} \text{ and } M' = \begin{array}{c|c|c} 1/9-\epsilon & 1/9+\epsilon & 1/9 \\ \hline 1/9+\epsilon & 1/9 & 1/9-\epsilon \\ \hline 1/9 & 1/9-\epsilon & 1/9+\epsilon \end{array} = M + \begin{array}{c|c|c} -\epsilon & +\epsilon & 0 \\ \hline +\epsilon & 0 & -\epsilon \\ \hline 0 & -\epsilon & +\epsilon \end{array}$$

When I group θ_1 and θ_2,

$$M|_{(\{1,2\}\{3\})} = \begin{array}{c|c} 4/9 & 2/9 \\ \hline 2/9 & 1/9 \end{array} \text{ and } M'|_{(\{1,2\}\{3\})} = \begin{array}{c|c} 4/9+\epsilon & 2/9-\epsilon \\ \hline 2/9-\epsilon & 1/9+\epsilon \end{array} = M|_{(\{1,2\}\{3\})} + \begin{array}{c|c} +\epsilon & -\epsilon \\ \hline -\epsilon & +\epsilon \end{array}.$$

When I group θ_2 and θ_3,

$$M|_{(\{1\}\{2,3\})} = \begin{array}{c|c} 1/9 & 2/9 \\ \hline 2/9 & 4/9 \end{array} \text{ and } M'|_{(\{1\}\{2,3\})} = \begin{array}{c|c} 1/9-\epsilon & 2/9+\epsilon \\ \hline 2/9+\epsilon & 4/9-\epsilon \end{array} = M|_{(\{1\}\{2,3\})} + \begin{array}{c|c} -\epsilon & +\epsilon \\ \hline +\epsilon & -\epsilon \end{array}.$$

By MMon,

$$M|_{(\{1,2\}\{3\})} \prec M'|_{(\{1,2\}\{3\})} \text{ and } M|_{(\{1\}\{2,3\})} \succ M'|_{(\{1\}\{2,3\})}.$$

Hence, there does not exist a total order that satisfies MMon and RCat. $\square$

I must resort to partial orders to find a measure that satisfies RCat.

> **[RCat] Robustness to Categorizations $\mathcal{C}$.** Let $\mathcal{C}$ denote the collection of categorizations of types to be considered. Let $M|_C$ denote the market under categorization $C \in \mathcal{C}$. $M \succeq M'$ if and only if $M|_C \succeq M'|_C$ for any categorization $C \in \mathcal{C}$, and $M \succ M'$ if and only if $M|_C \succ M'|_C$ for any categorization $C \in \mathcal{C}$.

Depending on the set of categorization I consider, I have the following distinction between symmetric categorization and comprehensive categorization.

Definition 2. Symmetric categorizations *collect the categorizations that are symmetric across sides. Namely, for three types, $\mathcal{C} = \mathcal{C}_s^2$, where*

$$\mathcal{C}_s = \{(\{1\}\{2\}\{3\}), (\{1,2\}\{3\}), (\{1\}\{2,3\}), (\{1\}\{2,3\})\}.$$

Comprehensive categorizations *collect all categorizations, including side-asymmetric ones. Namely, for three types, $\mathcal{C} = \mathcal{C}_m \times \mathcal{C}_w$.*

For a market with ordered types, define

$$\mu_{\leq x, \leq y} \equiv \sum_{i \leq x} \sum_{j \leq y} \mu_{ij}.$$

(PQD) Positive quadrant dependence order on ordered types (AS). *Consider markets M and M' with the same marginal distributions and ordered types $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_N$. I say that M strictly PQD-dominates M', $M \succ_{PQD} M'$, if $\mu_{\leq x, \leq y} \geq \mu'_{\leq x, \leq y}$ for all $x \leq N$ and $y \leq N$ and at least one of the inequalities holds strictly.*

(SPQD) Symmetric positive quadrant dependence order on ordered types. *Consider markets M and M' with the same marginal distributions and ordered types $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_N$. I say that M strictly SPQD-dominates M', $M \succ_{SPQD} M'$, if $\mu_{\leq x, \leq x} \geq \mu'_{\leq x, \leq x}$ for all $x \leq N$ and at least one of the inequalities holds strictly.*

Note that M PQD-dominates M' implies that M SPQD-dominates M'.

Theorem 5. *A partial order on multi-type markets satisfies MMon and Robustness to Symmetric Categorizations if and only if it is the SPQD order.*

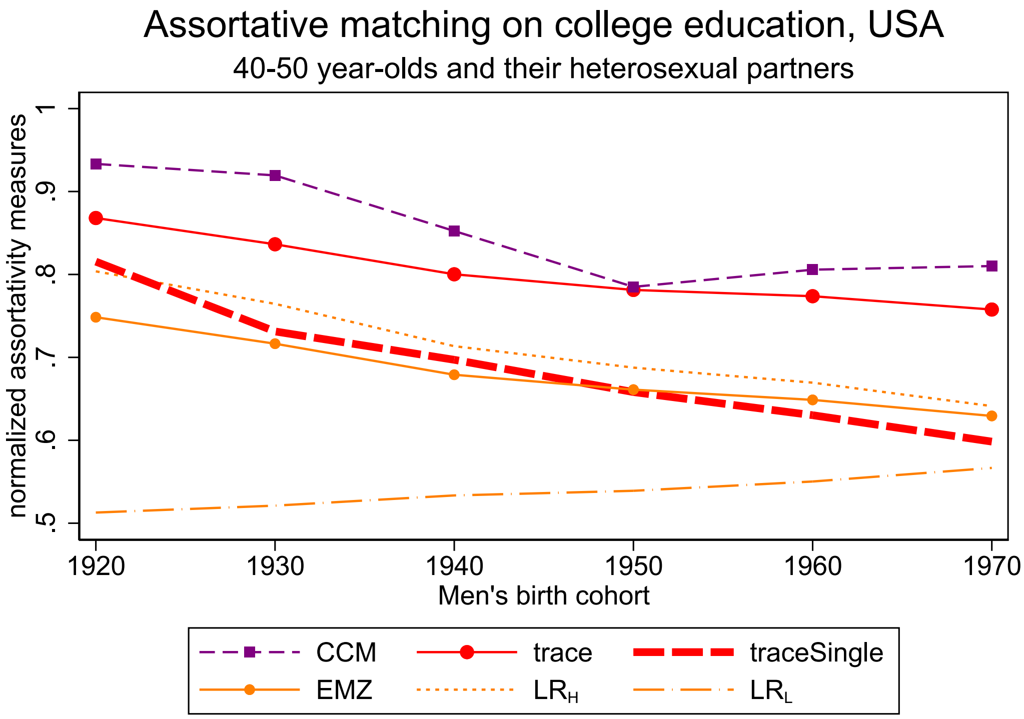
Proposition 4. *A partial order on multi-type markets satisfies MMon and Robustness to Comprehensive Categorizations if and only if it is the PQD order.*

5 US and global evidence

I use the axiomatized measures to study the patterns of assortative matching on education in the US and other countries across different birth cohorts. All data are from IPUMS-USA and IPUMS-International. Figure 2 shows changes in the degree of assortative matching under different measures. I consider men of different decal birth cohorts and their spouses forming a marriage market; more on this restriction later. I define highly educated individuals as those who finished four-year college. Under aggregate likelihood ratio (EMZ) and normalized trace, assortative matching has decreased across all decades. The decrease has been even steeper under normalized trace with singles. In contrast, if I the odds ratio, I would draw the conclusion that assortative matching has increased in recent decades. In addition, note that the likelihood ratio based on a specific education type provides conflicting conclusions: The likelihood ratio based on high education shows decreasing assortativity over time, but that based on low education shows the opposite conclusion of an increasing assortativity over time.

Because ALR and NT both have cardinal interpretations, I can use them to consider assortative matching of submarkets, i.e., states. Figure B1 shows the results for a few states.

Figure 2: Assortative matching of heterosexual couples on college education in the US

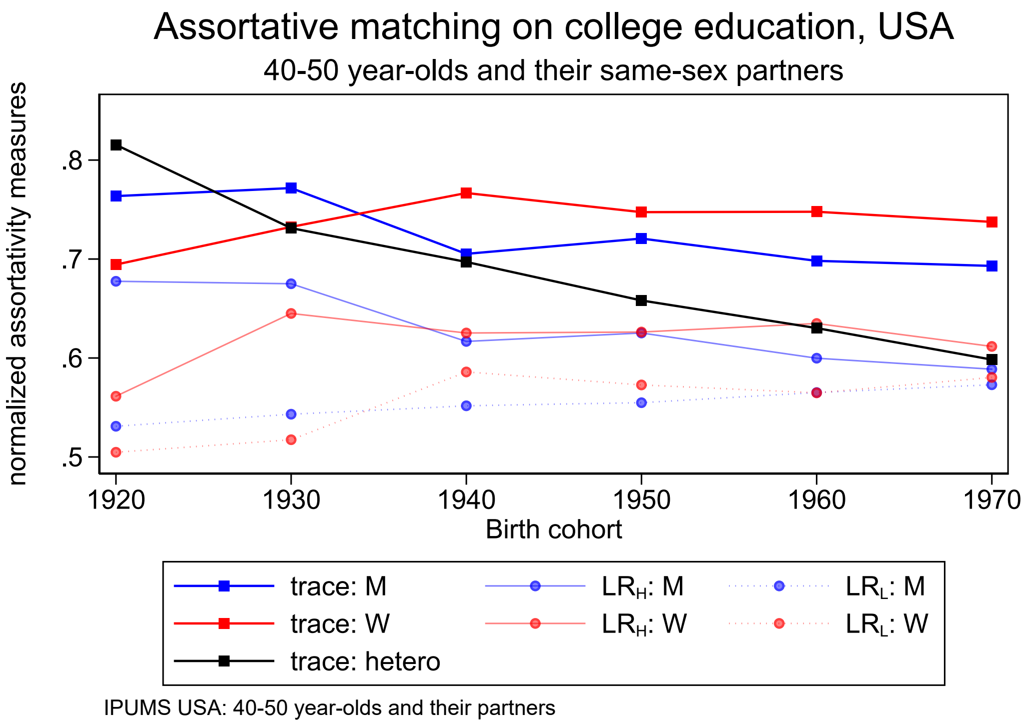


Generally, the patterns continue to hold when I consider each state. The odds ratio shows more fluctuations across states. I can apply the same measures to provide global evidence; Figure B2 illustrates the assortative matching pattern in a few neighboring countries of the US.

Figure 3 shows the evolution of assortative matching of homosexual couples in the US. The assortativity has remained steady over time for both male and female same-sex couples, and it is higher than that of heterosexual couples, when I compare the measure of normalized trace.

These empirical analyses motivate the question: What is a marriage market? There are a few ways I can restrict the sample to (i) 40-50 year-old individuals and their spouses, (ii) 40-50 year-old men and their wives, (iii) 40-50 year-old women and their husbands, (iv) all those of various birth cohorts who marry in the same year/decade, or (v) couples who cohabit and marry. The definition of a marriage market is outside the scope of this paper, but is empirically relevant. I compute the measures under these alternative definitions of marriage market, and the conclusions are broadly consistent.

Figure 3: Assortative matching of homosexual couples on college education in the US



6 Conclusion

Motivated by the recent debate on the direction of change of assortative matching on education in the US, I study the appropriate measures of assortative matching. Rather than starting with and justifying any particular measures, I take an axiomatic approach to characterize the appropriate measures—indices, total orders, and partial orders—of matching markets that satisfy a set of justifiable properties. The axioms I consider can be classified as equivalence axioms (those that specify equally assortative markets), monotonicity axioms (those that specify orders of assortativity on markets with or without the same marginal distributions), and characterization axioms (those that characterize the measures).

In summary, I provide axioms to support existing indices, total orders, and partial orders, as well as their modifications. First, the proportion of like types (normalized trace) and its comparison to that under counterfactual random matching (aggregate likelihood ratio) are the indices that are supported by the decomposability axioms that provide cardinal interpretations: A market's assortativity measure is weighted sums or averages of those of its decomposed submarkets. They have natural extensions to markets with singles, one-sided markets, and multi-type markets. The odds ratio is the unique total order on two-type markets that satisfies equivalence axioms, marginal monotonicity, and marginal independence; however, it does not have an extension to markets with singles, one-sided markets, or multi-type markets. When I consider robustness to categorizations, no total order (hence, no index) satisfies it and monotonicity axioms; I must resort to partial orders. The positive quadrant dependence order and its variants are the unique partial orders that satisfy categorization robustness.

I propose a new perspective to think about the different measures on assortative matching. Additional axioms may be proposed as well to further generalize the results. For example, the diagonal monotonicity axiom essentially gives full credit to pairs of like types and no credit to pairs of unlike types, regardless of how "distant" these types are (e.g., PhD-HS pairs are treated as the same as college-HS pairs in counting assortativity). Additional generalizations of the basic axioms will bring additional insights. In addition, it may be fruitful to take the axiomatic approach to study measures of homophily in many-to-one and many-to-many matching markets and networks. I hope that the current results open the door and build the foundation for further research.

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A Omitted proofs and details

A.1 Omitted proofs for characterization results

Proof of Theorem 1. It is straightforward to check that ALR satisfies the axioms, so it remains to show the other direction. I first show that any index I that satisfies SInv, TSym, SSym, MMon, and RDec is proportional to ALR. Consider $M = (a, b, c, d)$. By TSym,

$$I(M) = I\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I\begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{1}$$

By RDec,

$$I\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} \cdot r\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} = I\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot r\begin{pmatrix} a & b \\ c & d \end{pmatrix} + I\begin{pmatrix} d & c \\ b & a \end{pmatrix} \cdot r\begin{pmatrix} d & c \\ b & a \end{pmatrix},$$

which, by (1), is simplified to

$$I\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} \frac{|M|^2 + |M|^2}{2|M|} = 2I(M)r(M) \Rightarrow I(M) = \frac{1}{2}\frac{|M|}{r(M)} I\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix}. \tag{2}$$

Because for any $\epsilon < \min\{a+d, b+c\}$,

$$\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} = \begin{pmatrix} a+d-\epsilon & \epsilon \\ \epsilon & a+d-\epsilon \end{pmatrix} + \begin{pmatrix} \epsilon & b+c-\epsilon \\ b+c-\epsilon & \epsilon \end{pmatrix},$$

by RDec, for any $\epsilon < \min\{a+d, b+c\}$,

$$|M| \cdot I\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} = (a+d) \cdot I\begin{pmatrix} a+d-\epsilon & \epsilon \\ \epsilon & a+d-\epsilon \end{pmatrix} + (b+c) \cdot I\begin{pmatrix} \epsilon & b+c-\epsilon \\ b+c-\epsilon & \epsilon \end{pmatrix}.$$

Plugging in the expression of $I(M)$ in (2), I get, for any $\epsilon < \min\{a+d, b+c\}$,

$$I(M) = \frac{1}{2r(M)}\left[(a+d) \cdot I\begin{pmatrix} a+d-\epsilon & \epsilon \\ \epsilon & a+d-\epsilon \end{pmatrix} + (b+c) \cdot I\begin{pmatrix} \epsilon & b+c-\epsilon \\ b+c-\epsilon & \epsilon \end{pmatrix}\right].$$

Take $\epsilon \to 0$ and by SInv and $I(0, 1, 1, 0) = 0$, I have

$$I(M) = \frac{1}{2}\frac{a+d}{r(M)} I\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, any index that satisfies the aforementioned axioms is proportional to $(a+d)/r(M)$, the aggregate likelihood ratio. $\square$

Proof of Claim 2. Consider $M=(a,b,c,d)\gg 0$ and $M'=(a',b,c,d)\gg 0$, where $a'>a$. I want to show that $I(M')>I(M)$. By TSym,

$$I(a,b,c,d)=I(d,c,b,a).$$

By PDec,

$$\frac{1}{2}I(a,b,c,d)+\frac{1}{2}I(d,c,b,a)=I\left(\frac{1}{2}(a+d),\frac{1}{2}(b+c),\frac{1}{2}(b+c),\frac{1}{2}(a+d)\right).$$

By SInv,

$$I(a,b,c,d)=I\left(\frac{a+d}{a+b+c+d},\frac{b+c}{a+b+c+d},\frac{b+c}{a+b+c+d},\frac{a+d}{a+b+c+d}\right).$$

By the same sequence of arguments by TSym, PDec, and SInv,

$$I(a',b,c,d)=I\left(\frac{a'+d}{a'+b+c+d},\frac{b+c}{a'+b+c+d},\frac{b+c}{a'+b+c+d},\frac{a'+d}{a'+b+c+d}\right).$$

Note that the two matrices on the right-hand side of the two equations above have the same marginals (each row or column adds up to be 1). By MMon, $a'>a$ implies

$$I(M)=I(a',b,c,d)>I(M')=I(a,b,c,d).$$

Hence, DMon is proved. ODMon can be shown analogously. $\square$

Proof of Claim 5. Consider $M=(a,x,x,d)$, and consider $M_1=(a/2,x-\lambda/2,\lambda/2,d/2)$ and $M_2=(a/2,\lambda/2,x-\lambda/2,d/2)$ for some $\lambda\in(0,2x)$. Note that $M_1+M_2=M$ and M_1 and M_2 have the same total mass. Hence, by PDec,

$$2I(M)=I(M_1)+I(M_2).$$

By SSym, $I(M_1)=I(M_2)$. Hence, $I(M)=I(M_1)=I(M_2)$. By SInv,

$$I(M_1)=I(2M_1)=I(a,2x-\lambda,\lambda,d).$$

Hence, for all $\lambda \in (0, 2x)$,

$$I(a, 2x - \lambda, \lambda, d) = I(a, 2x - \lambda, \lambda, d).$$

Hence, I have shown that $I(a, b, c, d) = I(a, b', c', d)$ whenever $b + c = b' + c'$. By the same logic, and by SSym and TSym, $I(a, b, c, d) = I(a', b, c, d')$ whenever $a + d = a' + d'$. I, by DMon, is strictly increasing in $a + d$ whenever $bc \neq 0$, and, by ODMon, is strictly decreasing in $b + c$ whenever $ad \neq 0$. □

Claim 5. *Any index that satisfies SInv, TSym, SSym, DMon, ODMon, and PDec is an increasing function of $a + d$ and a decreasing function of $b + c$.*

In fact, the proof uses a weaker version of PDec, which I call the *Equal Weights* property: a market's assortativity is the average of the assortativity of two equally sized markets decomposed from the market. I state this axiom in the appendix.

[EW] Equal Weights. *For $M = (a, b, c, d) \gg 0$ and $M' = (a', b', c', d') \gg 0$ such that $|M| = |M'|$,*

$$I(M + M') = \frac{1}{2} I(M) + \frac{1}{2} I(M').$$

With EW substituted in for PDec, Claim 5 can be restated as:

Claim 5'. *Any index that satisfies SInv, TSym, SSym, DMon, ODMon, and EW is an increasing function of $a + d$ and a decreasing function of $b + c$.*

Proof of Theorem 2. I first show that any index I that satisfies SInv, TSym, SSym, DMon, ODMon, and PDec is order-equivalent to, i.e., a monotonic transformation of, NT. Consider $M = (a, b, c, d) \gg 0$ and $M' = (a', b', c', d') \gg 0$. If (i) $a + d > a' + d'$ and $b + c \leq b' + c'$ or (ii) $a + d \leq a' + d'$ and $b + c > b' + c'$, then, by Claim 5, (i) $I(M) > I(M')$ or (ii) $I(M) < I(M')$, respectively. Suppose $a + d > a' + d'$ and $b + c > b' + c'$. Define

$$M'' = (a'', b'', c'', d'') = M' \cdot (b + c)/(b' + c').$$

By definition of M'', $b'' + c'' = b + c$, and $a'' + d'' = (a' + d') \cdot (b + c)/(b' + c')$. By SInv of I, $I(M'') = I(M')$. The comparison of $a + d$ and $a'' + d''$ pins down the ordinal assortativity relation between M and M'. That is,

$$\frac{a + d}{a'' + d''} = \frac{a + d}{b + c} \Bigg/ \frac{a' + d'}{b' + c'} > 1 \Leftrightarrow I(M) > I(M').$$

When $a+d < a'+d'$ and $b+c < b'+c'$, I can similarly pin down the ordinal assortativity relation between M and M'. Note that for any $M=(a,b,c,d) \gg 0$,

$$I_{tr}(M) = \frac{(a+d)}{(a+b)+(c+d)} = \frac{(a+d)-(b+c)}{(a+b)+(c+d)} + 1 = \frac{\frac{a+d}{b+c}-1}{\frac{a+d}{b+c}+1}.$$

Hence, $I(M) > I(M')$ if and only if $I_{tr}(M) > I_{tr}(M')$.

Any nonlinear transformation of NT would violate PDec or RPDec, so any index that satisfies the stated axioms must be not only a monotonic transformation but also a linear transformation of NT. □

Proof of Claim 3. Suppose $M=(a,b,c,d)$ and $M'=(a',b',c',d')$ have the same marginal distributions and $M \succ M'$. By MInd,

$$\begin{aligned}(a,b,c,d) &\sim (a,b,c\frac{c'}{c},d\frac{c'}{c}) \sim (a,b\frac{c}{c'}\frac{d'}{d},c',d\frac{c'}{c}\frac{c}{c'}\frac{d'}{d}) \sim (a\frac{b'}{b}\frac{c'}{c}\frac{d}{d'},b\frac{b'}{b}\frac{c'}{c}\frac{d}{d'}\frac{c}{c'}\frac{d'}{d},c',d') \\ &\sim (a'\frac{a}{a'}\frac{b'}{b}\frac{c'}{c}\frac{d}{d'},b',c',d').\end{aligned}$$

Let $\delta \equiv \frac{a}{a'}\frac{b'}{b}\frac{c'}{c}\frac{d}{d'}$. By MMon, $a>a'$, $b<b'$, $c>c'$, and $d>d'$. Hence, $\delta>1$. $(a\delta,b',c',d') \sim (a',b'/\delta,c',d') \sim (a',b',c'/\delta,d') \sim (a',b',c',d'\delta) \succ (a',b',c',d')$ for all a',b',c',d' implies DMon and ODMon. □

Proof of Claim 4. Suppose order $\succeq$ satisfies MMon. Then I have for any δ,

$$(a,b,c,d) \succ (a-\delta,b+\delta,c+\delta,d+\delta).$$

By (repeatedly applying) MInd,

$$\begin{aligned}(a-\delta,b+\delta,c+\delta,d+\delta) &\sim (a,b+\delta,(c+\delta)\frac{a}{a-\delta},d-\delta) \\ &\sim (a,b,(c+\delta)\frac{a}{a-\delta},(d-\delta)\frac{b}{b+\delta}) \\ &\sim (a,b,c,d\frac{a-\delta}{a}\frac{b}{b+\delta}\frac{c}{c+\delta}\frac{d-\delta}{d}) \equiv (a,b,c,d\lambda(\delta)),\end{aligned}$$

where $\lambda(\delta)$ is strictly smaller than 1 for any $\delta>0$ and is a continuously decreasing in δ. With similar transformations, I have

$$\begin{aligned}&(a-\delta,b+\delta,c+\delta,d+\delta) \\ \sim\ &(a,b,c,d\lambda(\delta)) \sim (a\lambda(\delta),b,c,d) \sim (a,b/\lambda(\delta),c,d) \sim (a,b,c/\lambda(\delta),d).\end{aligned}$$

Hence, for any $\delta > 0$, DMon holds:

$$(a, b, c, d) \succ (a, b, c, d\lambda(\delta)) \sim (\lambda(\delta)a, b, c, d),$$

and ODMon holds:

$$(a, b, c, d) \succ (a, b/\lambda(\delta), c, d) \sim (a, b, c/\lambda(\delta), d).$$

Reversely,

$$(a, b, c, d) \succ (a, b, c, d\lambda(\delta)) \sim (\lambda(\delta)a, b, c, d)$$

for any δ implies

$$(a, b, c, d) \succ (a, b/\lambda(\delta), c, d) \sim (a, b, c/\lambda(\delta), d)$$

for any δ, and

$$(a, b, c, d) \succ I(a - \delta, b + \delta, c + \delta, d - \delta)$$

for any δ, where MInd is repeatedly used again. $\square$

Proof of Theorem 3. It is straightforward to check that the odds ratio and its monotonic transformation satisfy DMon, ODMon, and MInd.

It remains to show that there does not exist an index I that satisfies DMon, ODMon, and MInd, but is not order-equivalent to Yule's Q. Suppose by way of contradiction that such an index exists. Then, I must have for some $M = (a, b, c, d)$ and $M' = (a', b', c', d')$, one of the following four cases occurs: (i) $I(M) > I(M')$ and $Q(M) < Q(M')$, (ii) $I(M) < I(M')$ and $Q(M) > Q(M')$, (iii) $I(M) = I(M')$ and $Q(M) \neq Q(M')$, and (iv) $I(M) \neq I(M')$ and $Q(M) = Q(M')$.

First, suppose that case (i) $I(M) > I(M')$ and $Q(M) < Q(M')$ occurs. Because $I(M) > I(M')$, by the implication of DM, $ad \neq 0$, and by the implication of ODM, $b'c' \neq 0$, and because $Q(M) < Q(M')$, similarly, $bc \neq 0$ and $a'd' \neq 0$. For each of the following steps, I invoke a part of MInd:

$$\begin{aligned}
I(a, b, c, d) &= I(\frac{b'}{b}a, \frac{b'}{b}b, c, d) \\
&= I(\frac{b'}{b}a\frac{a'}{a}\frac{b}{b'}, b', c\frac{a'}{a}\frac{b}{b'}, d) \\
&= I(a', b', c\frac{a'}{a}\frac{b}{b'} \cdot \frac{c'}{c}\frac{a}{a'}\frac{b'}{b}, d \cdot \frac{c'}{c}\frac{a}{a'}\frac{b'}{b}\frac{d'}{d'}) \\
&= I(a', b', c', d' \cdot \frac{ad}{bc}/\frac{a'd'}{b'c'}).
\end{aligned}$$

By premise,

$$I(a', b', c', d' \cdot \frac{ad}{bc} / \frac{a'd'}{b'c'}) > I(a', b', c', d').$$

By DM, this implies

$$\frac{ad}{bc} > \frac{a'd'}{b'c'}.$$

However, this implies $Q(M) > Q(M')$, which contradicts the premise that $Q(M) < Q(M')$.

For each of the four possibilities, if no cell of matrices M and M' is zero, using the same logic as above, a contradiction can be derived.

Suppose there is a cell that is zero. Suppose $I(M) = I(M')$. If $bc = 0$, then $I(M) = I(a, 0, 0, d) = I(M')$, then $b'c' = 0$. In this case, by DM, $Q(M) = Q(M')$. If $ad = 0$ instead, then $I(M) = I(0, b, c, 0) = I(M')$ implies $a'd' = 0$. In this case, by ODM, $Q(M) = Q(M')$. Hence, whenever there is a cell with zero in one of the matrix, $I(M) = I(M')$ and $Q(M) = Q(M')$, excluding all four cases from happening. $\square$

A.2 Extension: A stricter notion of perfect assortativity

Suppose I desire a stricter notion of perfect assortativity by specifying that a market is perfectly positive assortative only if there is no pair of different types and is perfectly negative assortative only if there is no pair of like types.[1] Namely,

⇒**[PAM'] Strict Positive Assortative Matching.** *For any positive symbols,*

$$\begin{array}{c|c} a & 0 \\ \hline 0 & d \end{array} \succ \begin{array}{c|c} a' & b' \\ \hline 0 & d' \end{array}, \begin{array}{c|c} a'' & 0 \\ \hline c'' & d'' \end{array}, \begin{array}{c|c} a''' & b''' \\ \hline c''' & d''' \end{array}.$$

⇒**[NAM'] Strict Negative Assortative Matching.** *For any positive symbols,*

$$\begin{array}{c|c} 0 & b \\ \hline c & 0 \end{array} \prec \begin{array}{c|c} a' & b' \\ \hline c' & 0 \end{array}, \begin{array}{c|c} 0 & b'' \\ \hline c'' & d'' \end{array}, \begin{array}{c|c} a''' & b''' \\ \hline c''' & d''' \end{array}.$$

The following modifications of DMon and ODMon will imply PAM' and NAM', and they will characterize a slightly modified version of the normalized trace.

[1] For example, men and women could have chosen their education level to start, and any gender difference in education distribution and consequent mass of pairs of unlike education can be interpreted as an undesirable outcome. The level of nonassortativity can also be interpreted as the level of frictions in the market.

[DMon'] Strict Diagonal Monotonicity.
For all $\epsilon > 0$,

$$(a+\epsilon, b, c, d) \succeq (a, b, c, d) \text{ and } (a, b, c, d+\epsilon) \succeq (a, b, c, d),$$

where the equalities hold if and only if $b = c = 0$.

[ODMon'] Strict Off-Diagonal Monotonicity.
For all $\epsilon > 0$,

$$(a, b+\epsilon, c, d) \succeq (a, b, c, d) \text{ and } (a, b, c+\epsilon, d) \succeq (a, b, c, d),$$

where the equalities hold if and only if $a = d = 0$.

Arguably, in certain situations, this "stricter" notion of perfect assortative matching is needed. For example, when there is a pre-matching investment stage in which individuals can choose their matching types, although individuals may achieve the most possible pairs of like types, the fact that there is an imbalance in the types of agents indicates that there is coordination failure in the investment stage. Hence, there may be a need to distinguish the assortativity of $(a, 0, 0, d)$ and $(a, b, 0, d)$, where $b > 0$.

[PDec'] Generalized Population Decomposability.
For any $M = (a, b, c, d)$ and $M' = (a', b', c', d')$,

$$I(M + M') = \frac{|M|}{|M + M'|} I(M) + \frac{|M'|}{|M + M'|} I(M').$$

(NT') Continuous normalized trace

$$I'_{tr}(M) = \frac{\mathrm{tr}(M)}{|M|}.$$

Claim 6. *Any satisfies EQV, DMon', ODMon', and PDec' if and only if it is a linear transformation of NT'.*

A.3 Omitted proofs for multi-type markets

Proof of Theorem 5. First, I show that the PQD order satisfies MMon and RCat. Then, I complete the proof by showing that an order that satisfies MMon and RCat is the PQD order.

I prove that the PQD order satisfies MMon. Take two binary-type markets M and M' with $M \succ_{PQD} M'$. Then $M_{11} \geq M'_{11}$. Hence, monotonicity is satisfied. Second, take 3-type markets M and M' with $M \succ_{PQD} M'$. Let C_i denote the categorization such that all types weakly below i are grouped in one category and all types strictly above i are grouped in the other category. Explicitly, $C_1 = (\{1\}, \{2, 3\})$ and $C_2 = (\{1, 2\}, \{3\})$.

I prove that the PQD order satisfies RCat. Let Δ denote the 3-by-3 matrix $M - M'$. Each row and column of Δ adds up to be zero. Hence, $M \succ_{PQD} M'$ implies $\Delta_{11} \geq 0$, $\Delta_{\leq 1, \leq 2} \geq 0$, $\Delta_{\leq 2, \leq 1} \geq 0$ and $\Delta_{\leq 2, \leq 2} \geq 0$, with at least one strict inequality. Note that at least one of $\Delta_{11} \geq 0$ and $\Delta_{\leq 2, \leq 2} = \Delta_{11} + \Delta_{12} + \Delta_{21} + \Delta_{22} \geq 0$ holds with strict inequality: If both hold with equality, then $\Delta_{21} = \Delta_{12}$ and $\Delta_{\leq 1, \leq 2} = \Delta_{11} + \Delta_{12} \geq 0$ and $\Delta_{\leq 2, \leq 1} = \Delta_{11} + \Delta_{21} \geq 0$ must hold with equality, but this contradicts with the premise that one of the four inequalities must hold strictly. The fact that at least one of the two inequalities $\Delta_{11} \geq 0$ and $\Delta_{\leq 2, \leq 2} \geq 0$ must hold strictly implies that $M|_{C_1} \succeq_{PQD} M'|_{C_1}$ and $M|_{C_2} \succeq_{PQD} M'|_{C_2}$ with at least one strict relation.

Conversely, suppose that $M|_{C_1} \succ_{PQD} M'|_{C_1}$ and $M|_{C_2} \succ_{PQD} M'|_{C_2}$. Then, $\Delta_{11} > 0$, $\Delta_{33} > 0$, $\Delta_{11} + \Delta_{12} + \Delta_{21} + \Delta_{22} > 0$, and $\Delta_{22} + \Delta_{23} + \Delta_{32} + \Delta_{33} > 0$. It remains to show that $\Delta_{11} + \Delta_{12} \geq 0$ and $\Delta_{11} + \Delta_{21} \geq 0$. □

B Additional empirical analyses

Figure B1 considers assortative matching of heterosexual couples in different American states. Figure B2 considers assortative matching of heterosexual couples in different neighboring countries of the US.

Figure B1: Assortative matching of heterosexual couples on college education in the US

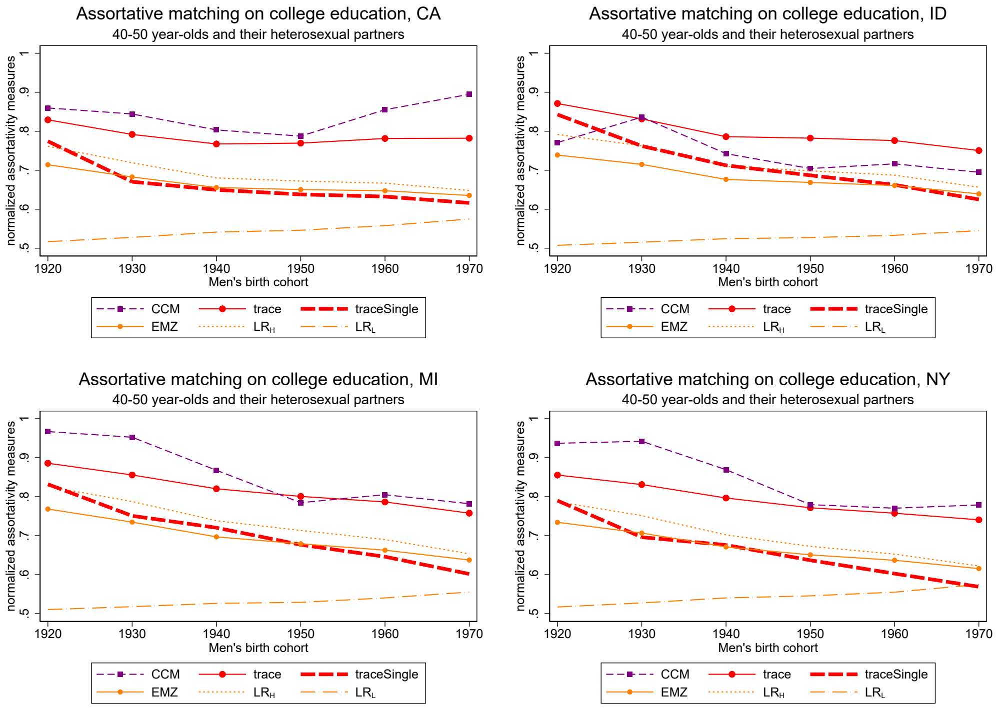

Figure B2: Assortative matching of heterosexual couples on college education in the world