Visualization of Complex Functions

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A complex function, in this article, is defined as a function whose domain or range are complex. Therefore, complex valued functions can be divided into the following three types:

$$f: \mathbb{R} \mapsto \mathbb{C}$$

$$f: \mathbb{C} \mapsto \mathbb{C}$$

$$f: \mathbb{C} \mapsto \mathbb{R}$$

where $\mathbb{R}$ represents the set of all real numbers and $\mathbb{C}$ represents the set of all complex numbers.

Normally, we would like to plot the graph of a function in either Cartesian plane or Cartesian space (three-dimensional space). A function (or an equation) is graphable in Cartesian plane if the relationship between real numbers x and y is explicitly, implicitly or parametrically defined.

Explicit function

$$y = f(x)$$

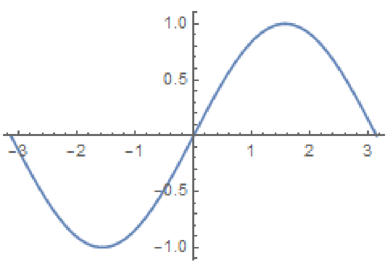

Figure 1

$$y = sin(x)$$

Implicit function

$$f(x, y) = 0$$

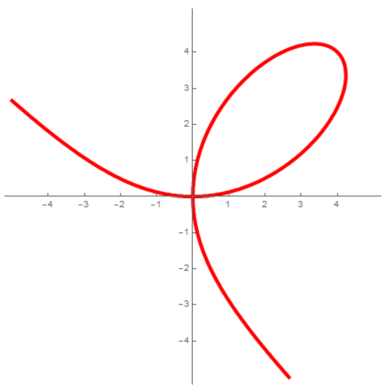

Figure 2

$$x^3 + y^3 - 8xy = 0$$

Parametric function

$$\begin{cases} y = f(t) \\ x = g(t) \end{cases}, t \in \mathbb{R}$$

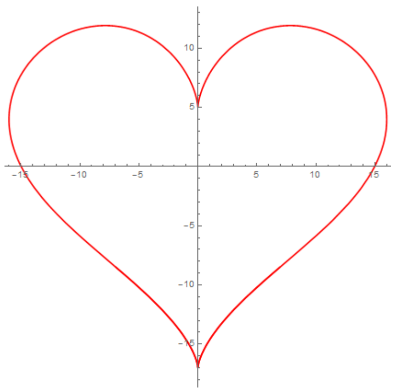

Figure 3

$$\begin{cases} x = \sin^3 t \\ y = 13\cos 3t - \cos 2t - 2\cos 3t - \cos 4t \end{cases}$$

Similarly, a function is graphable in Cartesian space if

$$z = f(x, y), or$$

$$f(x, y, z) = 0, or$$

$$\begin{cases} x = f(u, v) \\ y = g(u, v)\ where\ (u, v) \in \mathbb{R}^2 \\ z = h(u, v) \end{cases}$$

Explicit function

$$z = f(x, y)$$

Implicit function

$$f(x, y, z) = 0$$

Parametric function

$$\begin{cases} x = f(u, v) \\ y = g(u, v), (u, v) \in \mathbb{R}^2 \\ z = h(u, v) \end{cases}$$

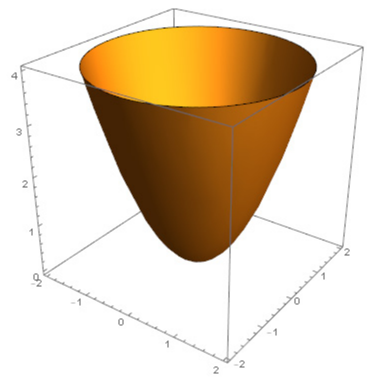

Figure 4

$$z = x^2 + y^2$$

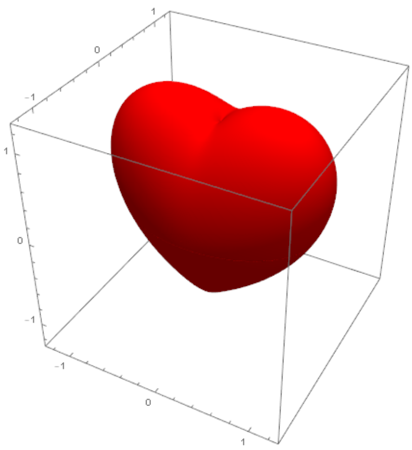

Figure 5

$$-x^2 z^3 - \frac{9y^2 z^3}{80} + \left(x^2 - 1 + \frac{9y^2}{4} + z^2\right)^3 = 0$$

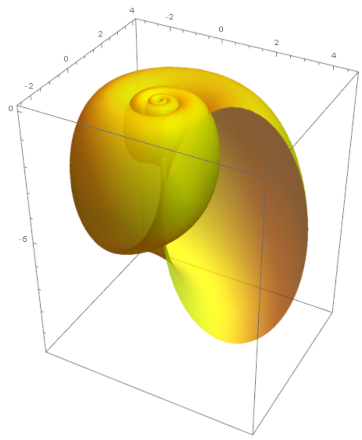

Figure 6

$$\begin{cases} x = 1.16^v \cos v\,(1 + \cos u) \\ y = -1.16^v \sin v\,(1 + \cos u) \\ z = -2 * 1.16^v (1 + \sin u) \end{cases}$$

Visualization of complex valued functions is based on an idea that there exists a mapping between $\mathbb{C}$ and $\mathbb{R}^2$, the cartesian plane.

$$\mathrm{c} \mapsto (\mathrm{x}, \mathrm{y}) \in \mathbb{R}^2, \forall \mathrm{c} = (\mathrm{x} + \mathrm{y}\mathrm{i}) \in \mathbb{C}$$

Or

$$\mathrm{c} \mapsto \big(\mathrm{Re}(z), \mathrm{Im}(\mathrm{z})\big) \in \mathbb{R}^2, \forall \mathrm{c} \in \mathbb{C}$$

(To avoid conflict between representations, as z is often used to denote the z value in Cartesian space or the complex number z. From now on, I use c instead of z to represent a complex number, while z is only used to denote the z value in a Cartesian space.)

$$f: \mathbb{R} \mapsto \mathbb{C}$$

Let's reconsider the first type of complex valued functions: $f: \mathbb{R} \mapsto \mathbb{C}$. They can be converted to $g: \mathbb{R} \mapsto \mathbb{R}^2$. Although g is three dimensional, it cannot be plotted in Cartesian space like the function $h: \mathbb{R}^2 \mapsto \mathbb{R}$

In this case, we can plot the real the imaginary part of f separately.

Let g and h be two functions that satisfy

$$\begin{cases} g(x) = Re\big(f(x)\big) \\ h(x) = Im\big(f(x)\big) \end{cases}$$

Now, $g(x)$ and $h(x)$ represent the correlation between the x and the real and imaginary part of f, respectively.

Plotting $f(x) = \sin((6+5i)x)$ using this approach would yield the following graph.

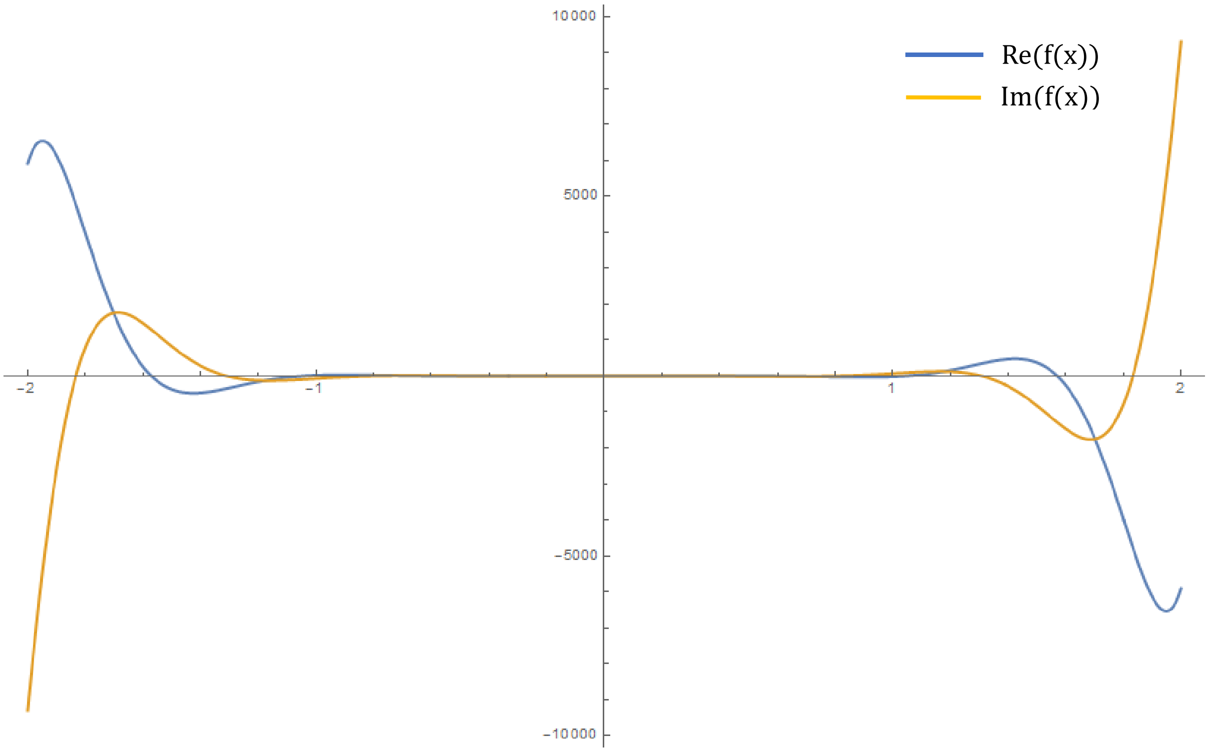

Figure 7 Real and imaginary part of $f(x) = \sin((6+5i)x)$

$f: \mathbb{C} \mapsto \mathbb{R}$

Then, let's look at the function $f: \mathbb{C} \mapsto \mathbb{R}$. It can be converted to $g: \mathbb{R}^2 \mapsto \mathbb{R}$, which is just a typical form of an explicitly defined function in Cartesian space.

Therefore, $z = g(x, y) = f(x + yi) \in \mathbb{R}$

Plotting $f(c) = \sin\left(\left|\sqrt{Re(c)} + Im(c)\right|\right), c \in \mathbb{C}$ with this approach would yield the following graph.

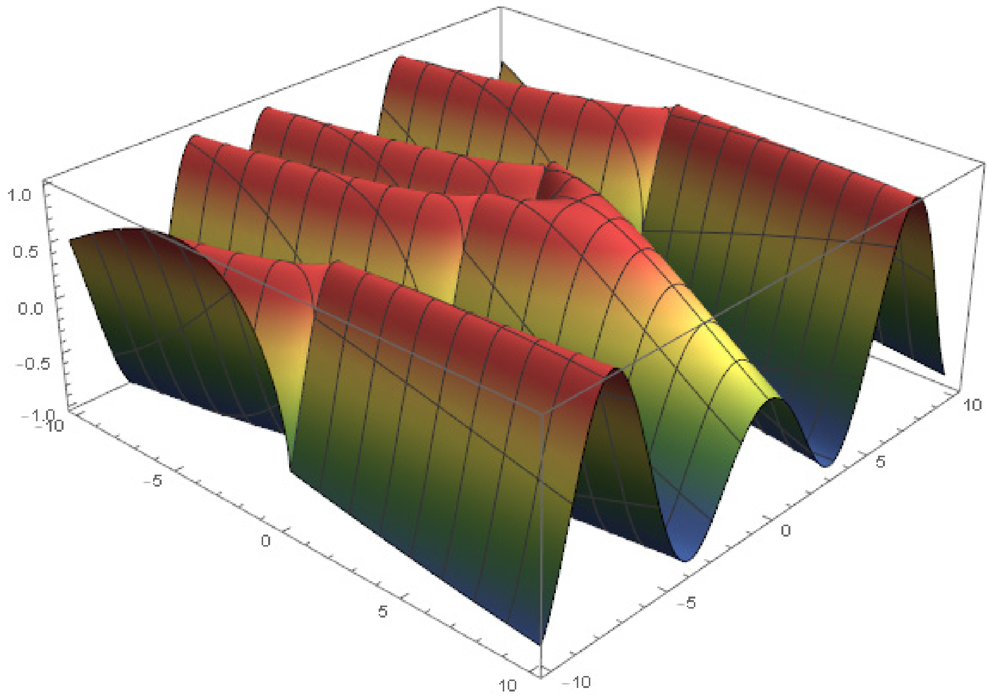

Figure 8 $f(c) = \sin\left(\left|\sqrt{Re(c)} + Im(c)\right|\right)$

Noticed that the colormap is applied only to make this graph more appealing.

$f: \mathbb{C} \mapsto \mathbb{C}$

Finally, let's look at the "pure" complex function $f: \mathbb{C} \mapsto \mathbb{C}$. It can be converted to $g: \mathbb{R}^2 \mapsto \mathbb{R}^2$, which unfortunately is a four-dimensional function. To graph it, we can separate the real and imaginary part of f.

Suppose that p and q are two functions such that

$$\begin{cases} p(x,\ y) = Re\big(f(x+yi)\big) \\ q(x,\ y) = Im\big(f(x+yi)\big) \end{cases}$$

Plotting p and q with respect to inputs (x, y) would yield the graph for the real and imaginary part of f.

Example: $f(c) = \ln(c), c \in \mathbb{C}$

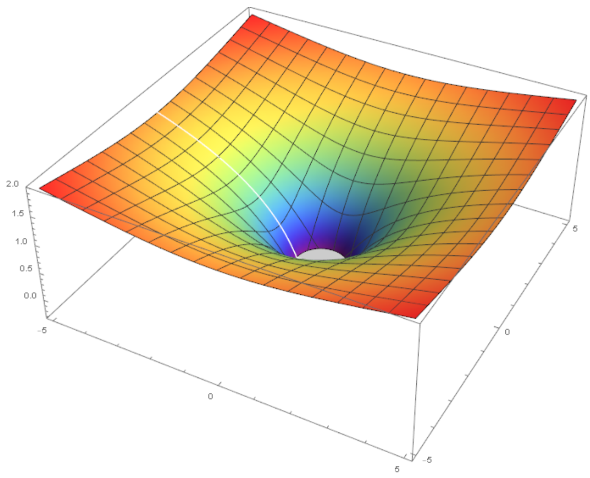

Figure 9 $p(x,\ y) = Re(\ln(c))$

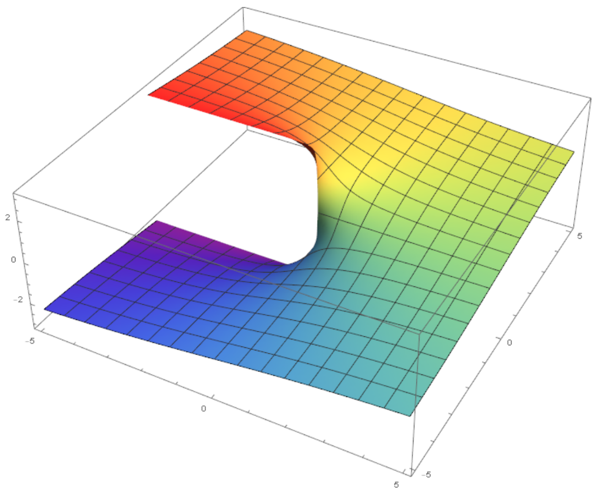

Figure 10 $q(x,\ y) = Im(\ln(c))$

Noticed that the colormap is applied only to make this graph more appealing.

Not satisfied to have separate graphs for a single complex function? Actually, there is a way to plot $f: \mathbb{C} \mapsto \mathbb{C}$ in one graph.

The color in figure 8~figure 10 provides intuitive visualization for the z value. A relatively low z value corresponds to blue, while a relatively high z value corresponds to red, if the “rainbow” colormap is applied.

Can we use the colormap to show one of the real or imaginary part of complex function? The answer is “not really”. This is because the colormap is finite, whereas the real or imaginary part of a complex function can spread across the whole real domain.

However, there is a way. Apart from the Cartesian form, a complex number can also be expressed in the polar form – the modulus-argument form. By such, the domain of the argument is limited to $(-\pi, \pi]$.

$$c = |c|e^{i \cdot \arg(c)} \in \mathbb{C}, \text{where } |c| \in \mathbb{R}, \arg(c) \in (-\pi, \pi]$$

Now, the argument, which is finite, can be mapped to a specific colormap, using any appropriate function. The most common used approach is to map the argument to the hue, in HSV (hue, saturation, value) color space. For more details, look up "HSV" in Wikipedia.

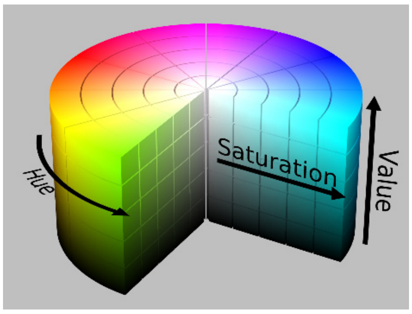

Figure 11 HSV color space

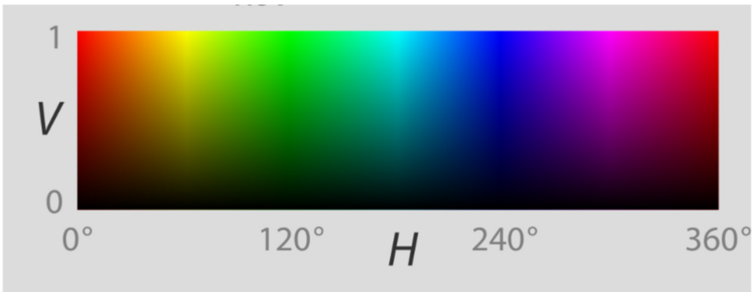

Figure 12 A map of hue (H) and value (V)

For the same function, $f(c) = \ln(c), c \in \mathbb{C}$, let $g(x,y)$ to be the modulus of $f(c)$ and $H(x,y)$ to be the argument of $f(c)$, where $c = x + yi$

$$\begin{cases} z = g(x,y) = |f(x+yi)| \\ H(x,y) = arg\big(f(x+yi)\big) + 180^{\circ} \end{cases}$$

Applying the color function, we obtained a single graph describing $f(c) = \ln(c)$

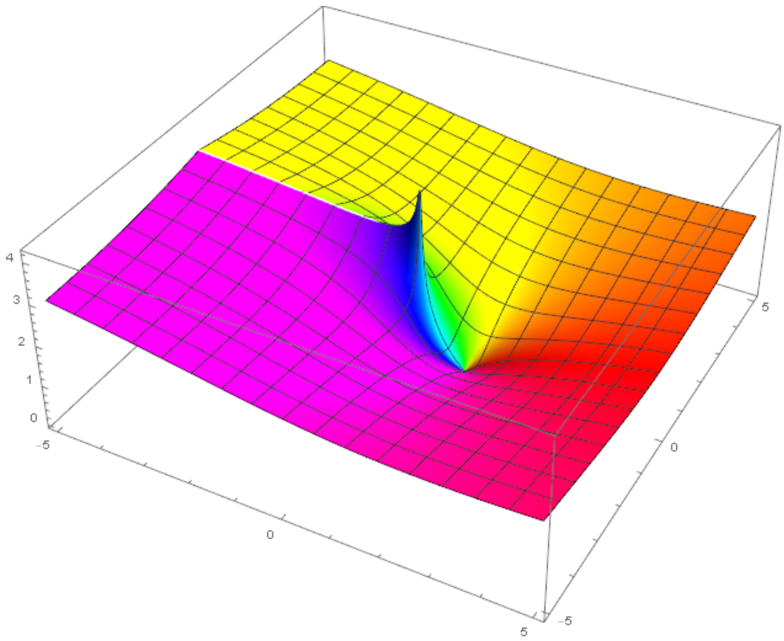

Figure 13 Modulus-argument graph of $f(c) = \ln(c)$