A Note on using Normalized Power Prior for Reliability Lifetime Data with R and Stan

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1 Model Description

In order to better understand failure mode and life characteristics of products like cellphone, light bulb, etc., more failures need to be observed in life testing. However, in traditional life data analysis, it is often difficult to obtain the sufficient time-to-failure data under normal operating conditions. In this case an accelerated life testing (ALT) is used to force these products to fail earlier than they would under normal conditions. The operating conditions, normally known as the stress levels, might be temperatures, air pressures, etc. It is also noted that more failures are typically associated with dramatic increase in experimental cost, since units will be defected after testing. That's why in these ALT experiments, sample sizes are not large, and therefore the confidence intervals of the lifetime is not narrow.

In this note, we show that the *normalized power prior*, an informative prior based on historical data, can be utilized to calibrate the parameter estimates. We refer readers to our recent paper (Ye et al. 2021) for details of the methodology, and the seminar paper (Ibrahim et al. 2015) for more information. Such estimates may serve as the reference values with narrower credible intervals. The key idea is to incorporate the information from the past experiments. Such experiments may not strictly based on the same batch of the product, but should be in principle based on the same or similar model series under the same brand. Another advantage of using this approach is, we show that it can also be effective even under different types of the design.

Two classical design frameworks that are commonly used are constant stress test (Figure 1) and step-stress test (Figure 2). Throughout this article, we use $k \in \mathbb{N}^+$ to denote the number of stress levels, and assume $x_1 < x_2 < \cdots < x_k$ are the ordered stress levels to be used in the experiment. The constant stress test is the simplest case. For stress level $i = 1, 2, \ldots, k$, a total of N_i test units is placed on the i^{th} stress level x_i. At each stress level, we run the test until time τ_i. Assume we observe n_i failures at each stress level so finally the $\sum_{i=1}^k N_i - n_i$ surviving items are (type-I) censored. A constant stress test is widely used in some historical experiments (where in our example, we treat these data as the historical data D_0).

Suppose we observe the number of failures at each stress level $\boldsymbol{n} = (n_1, n_2, \ldots, n_k)$, with their corresponding lifetime $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)$, where $\boldsymbol{y}_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$. In general, we can assume that the individual lifetime Y of a test unit follows a distribution from a log-location-scale family (for instance, exponential, Weibull and lognormal are commonly used cases), with $F_i(\cdot)$ and $f_i(\cdot)$ denote the individual cdf and pdf of the lifetime. The likelihood under constant-stress test can be expressed as

$$L(\boldsymbol{\theta}|\boldsymbol{y}, \boldsymbol{n}) = \left\{ \prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!} [1 - F_i(\tau_i)]^{N_i - n_i} \right\} \prod_{i=1}^{k} \prod_{j=1}^{n_i} f_i(y_{i,j})$$

In our example, we assume an exponential model. Then, for each stress x_i, we assume the lifetime cumulative distribution is the exponential distribution with scale parameter θ_i written as

$$F_i(t) = 1 - \exp(-t/\theta_i), \text{ for } i = 1, 2, \ldots, k.$$

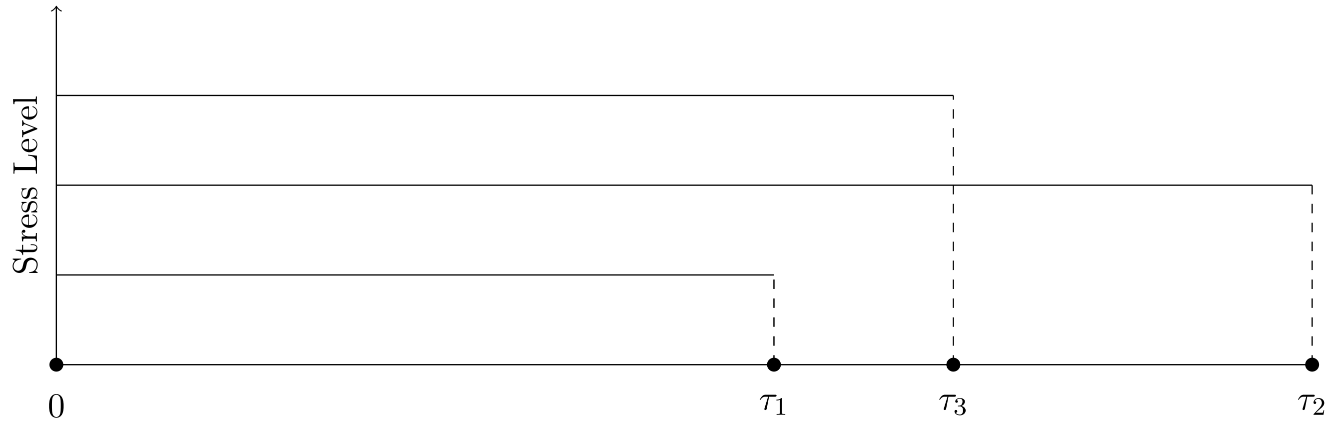


Figure 1: Constant-stress accelerated life testing

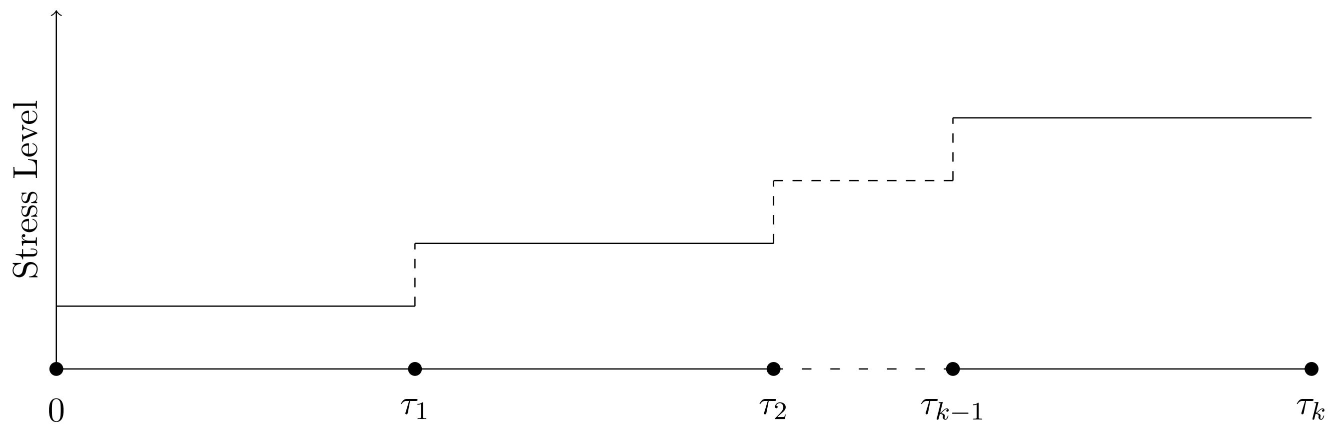
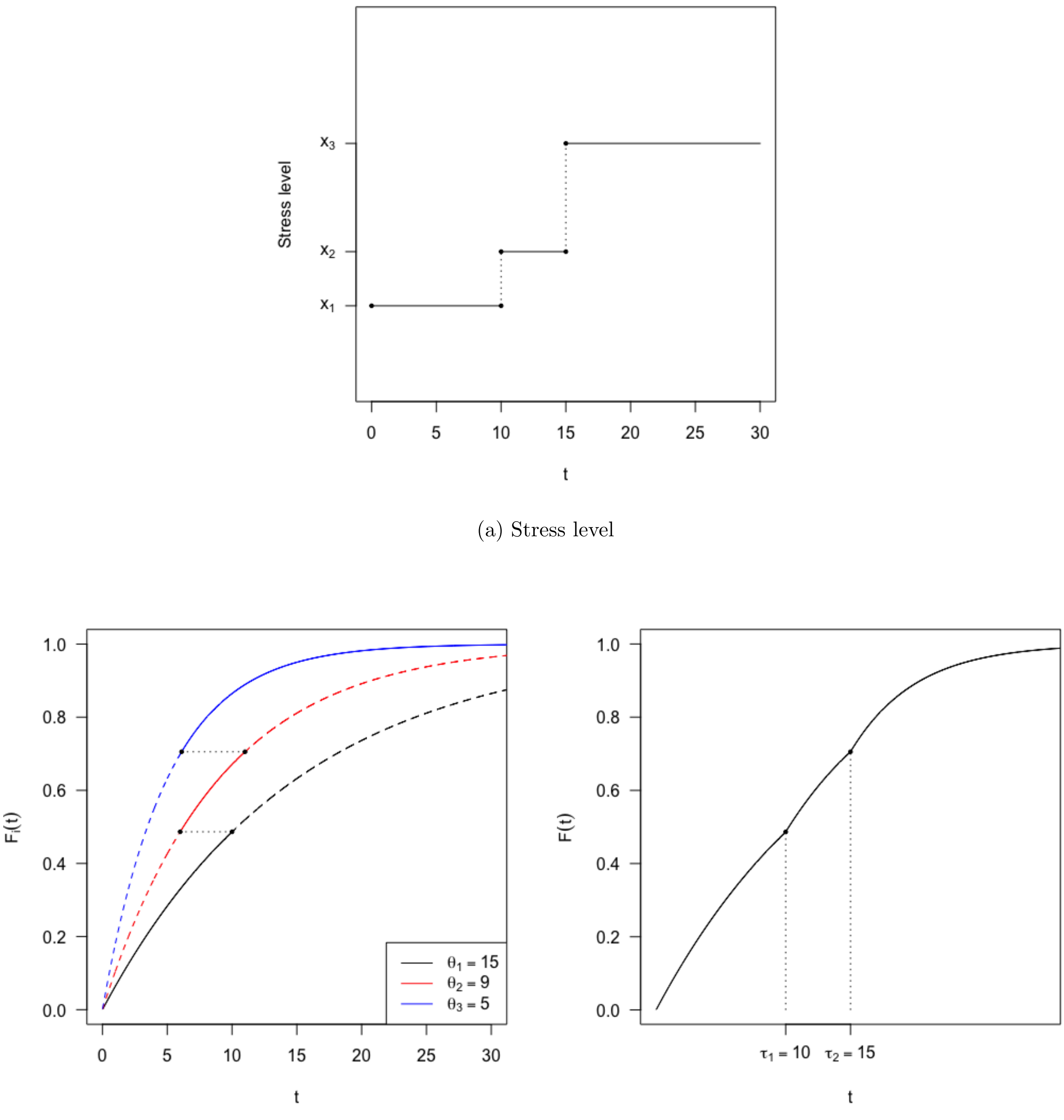


Figure 2: Step-stress accelerated life testing

In a step-stress test, a total of N test units is initially placed on test at the first stress level and run until time τ_1. At that point there are n_1 items failed. The test continues with an increased stress level on the remaining $N - n_1$ items until τ_2, and so on. Finally at time τ_{k-1} the test increases to the k^{th} stress level and at time τ_k all surviving $N - \sum_{i=1}^{k} n_i$ items are censored, and thereby the life-test is terminated. The time point $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k)$ is chosen by design and considered to be fixed, under which the censoring scheme is also type-I.

In such a step-stress test, we further assume that the data come from a cumulative exposure model, which is one of the most prominent models in the analysis of the step-stress test data. The cumulative exposure assumption is made in order to connect the failure characteristics across different stress levels. It maintains the continuity of the total lifetime CDF where the stress level is changed by shifting the individual stress CDF to the right horizontally, see Figure 3. Similar to a constant stress test, suppose we observe the number of failures at each stress level $\boldsymbol{n} = (n_1, n_2, \ldots, n_k)$, with their corresponding lifetime $\boldsymbol{y}_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$. Under the cumulative exposure model, the cdf and pdf of the overall lifetime of the test unit under the k-level step-stress ALT is given by

$$F(t) = \begin{cases} G_i(t) = 1 - \exp\left(-(t-\tau_{i-1})/\theta_i + \sum_{j=1}^{i-1}(\tau_j - \tau_{j-1})/\theta_j\right) & \tau_{i-1} \le t \le \tau_i \quad \text{for } i = 1, 2, \cdots, k-1 \\ G_k(t) = 1 - \exp\left(-(t-\tau_{k-1})/\theta_k + \sum_{i=1}^{k-1}(\tau_i - \tau_{i-1})/\theta_k\right) & \tau_i \le t < \infty \quad \text{for } i = k \end{cases},$$



(a) Stress level

(b) Each stress level lifetime CDF

(c) Total lifetime CDF

Figure 3: 3-level step-stress ALT with $\tau_1 = 10$ and $\tau_2 = 15$

and

$$f(t) = \begin{cases} g_i(t) = \dfrac{1}{\theta_i} \exp\left(-(t - \tau_{i-1})/\theta_i + \displaystyle\sum_{j=1}^{i-1} (\tau_j - \tau_{j-1})/\theta_j \right) & \tau_{i-1} \le t \le \tau_i \quad \text{for } i = 1, 2, \cdots, k-1 \\ g_i(t) = \dfrac{1}{\theta_k} \exp\left(-(t - \tau_{k-1})/\theta_k + \displaystyle\sum_{i=1}^{k-1} (\tau_i - \tau_{i-1})/\theta_k \right) & \tau_i \le t < \infty \quad \text{for } i = k \end{cases},$$

respectively.

Therefore the likelihood of observed data $\boldsymbol{n}$ and $\boldsymbol{y}$ is given by

$$L(\boldsymbol{\theta}|\boldsymbol{y},\boldsymbol{n}) = \frac{N!}{(N-\sum_{i=1}^{k} n_i)!}\left\{\prod_{i=1}^{k}\left[\prod_{j=1}^{n_i} g_i(y_{i,j})\right]\right\}[1-G_k(\tau_k)]^{N-\sum_{i=1}^{k} n_i}$$

If the Engineer's interest is to make inference on the lifetime at any specified stress level, it is also assumed that at stress level x_i, the mean time to failure of a subject, θ_i, has a log-linear relationship with stress level, specified as

$$\log\theta_i = \alpha + \beta x_i,$$

for which the regression parameters (α, β) need to be calibrated.

Now we finished the introduction of the model. To illustrate that the power priors can be used to adaptively combine a constant stress test with a step-stress test, we use an example that the historical data comes from a constant stress test and the current test comes from a step-stress test. This is more common in practice.

2 Data Description and Analysis

Let us denote $s(t)$ to be the specified function of stress loading in time for the ALT under consideration. We define s_H to be the upper bound of stress level and s_U to be the normal use-stress level. Then, the stress loading is standardized as

$$x(t) = \frac{s(t)-s_U}{s_H-s_U}, \qquad t \geq 0$$

such that the range of $x(t)$ is between 0 and 1 inclusive. In our example, we standardized the stress levels as per the formula above.

In our example, the historical data D_0 is a time-constrained failure data from $n = 30$ solar lighting devices on a simple constant-stress ALT. The stress levels here are operating temperatures in practice. Since one may expect an elongated lifetime under normal temperature, the accelerate life testing is used. The design factors for historical data includes: Temperature 1 ($x_{0_1} = 0.1$), $\tau_{0_1} = 15$; $N_{0_1} = 20$, $n_{0_1} = 10$, $c_{0_1} = 10$; Temperature 2 ($x_{0_2} = 0.6$), $\tau_{0_2} = 10$; $N_{0_2} = 10$, $n_{0_2} = 8$, $c_{0_2} = 2$.

The current Data D comes from a three-level step-stress test that was conducted to assess the reliability characteristics of the solar lighting device of the same model but comes from different a batch. The standardized stress loading was $x_1 = 0.1$, $x_2 = 0.5$, and $x_3 = 0.9$. It is assumed that at any constant temperature, the device lifetime is exponentially distributed. The stress change time points were 15 (in hundred hours) and 20 (in hundred hours), with the censoring time point at 25 (in hundred hours) so $\boldsymbol{\tau} = (15, 20, 25)$. The resulting dataset consists of $(n_1, n_2, n_3) = (11, 8, 6)$ failures out of the 30 testing units. Details of the failure times are shown in Table 1.

Below we show the model fitting results in Table 2. The normalizing factor $C(\delta)$ is calculated via the algorithm described in Ye et al. (2021). The uniform initial prior is used for δ, and a normal initial prior is used for α and β, with mean 0 and standard deviation 25. The trace plots and ACF plots suggest good convergence of the MCMC; See Figure 4. Using the normalized power prior, the posterior mean of δ is about 0.593, which indicates on average, 60% of the historical information is utilized in the prior elicitation. Since the normalized power prior is capable for adaptive borrowing, the result is not bad. It suggests that D_0 and D are quite compatible and overall homogeneous. Compared to no borrowing, the credible intervals are narrower. Though detail interpretations may require professional's opinion, we can consider the quantities obtained based on the normalized power prior as the reference values in practice.

Table 1: *Dataset of solar lighting device ALT.*

Stress Levels in D_0	Lifetimes in the Historical Data					
$x_{0_1} = 0.1$	0.611	2.337	5.461	5.726	7.138	9.832
	10.021	12.136	12.375	13.996		
$x_{0_2} = 0.6$	0.081	0.828	1.116	1.839	5.043	6.717
	7.258	8.472				
Stress Levels in D	**Lifetimes in the Current Data**					
$x_1 = 0.1$	1.515	2.225	4.629	4.654	6.349	8.003
	8.262	10.416	11.381	12.433	14.755	
$x_2 = 0.5$	15.164	15.355	15.953	16.654	16.735	18.796
	19.248	19.295				
$x_3 = 0.9$	20.110	20.318	21.228	21.543	22.227	24.541

Table 2: *Summary of study results. We use posterior mean of the corresponding parameters as the estimates, and the empirical* 95% *credible interval based on the empirical highest posterior density are reported.*

Method	$\hat{\alpha}$ (95% CI)	$\hat{\beta}$ (95% CI)	$\hat{\delta}$ (95% CI)
$\delta = 0$	3.692 (3.091, 4.332)	-2.280 (-3.423, -1.092)	-
$\delta = 1$	3.530 (3.040, 4.002)	-2.228 (-3.135, -1.135)	-
NPP	3.571 (3.061, 4.112)	-2.230 (-3.231, -1.176)	0.593 (0.141, 0.999)

3 Coding Tips

In `Stan`, when sample from a posterior with fixed δ power prior, one can simply use

```
target += delta*loglik0+loglik;
```

at the end of the model block as the defined target. (note that in Stan the "target" does not include the prior, since priors are defined directly following the "model".

Then, when calculating $C(\delta)$ on selected grids, one should first define an outer loop for δ, then in the end use

```
target += delta[i]*loglik0;
```

`Stan` will recognize and return a matrix-valued sample for each parameter, and this can be further paralleled to increase the computational speed. In the end, one needs to define a function to interpolate the δ based on the sampling grids. We design another program in `Stan` using the output from the previous steps, with a defined simple interpolation function as shown in the source program `StanALTRandPath.stan`. In our example, when 201 grids are used, the overall timing is less than 1 minute to obtain the Table 2 on a 2.3 GHz 8-Core Intel Core i9 Macbook pro. All the computer code in `R` and `Stan` are available at `https://github.com/hanzifei/Code_NPP_JSPI`.

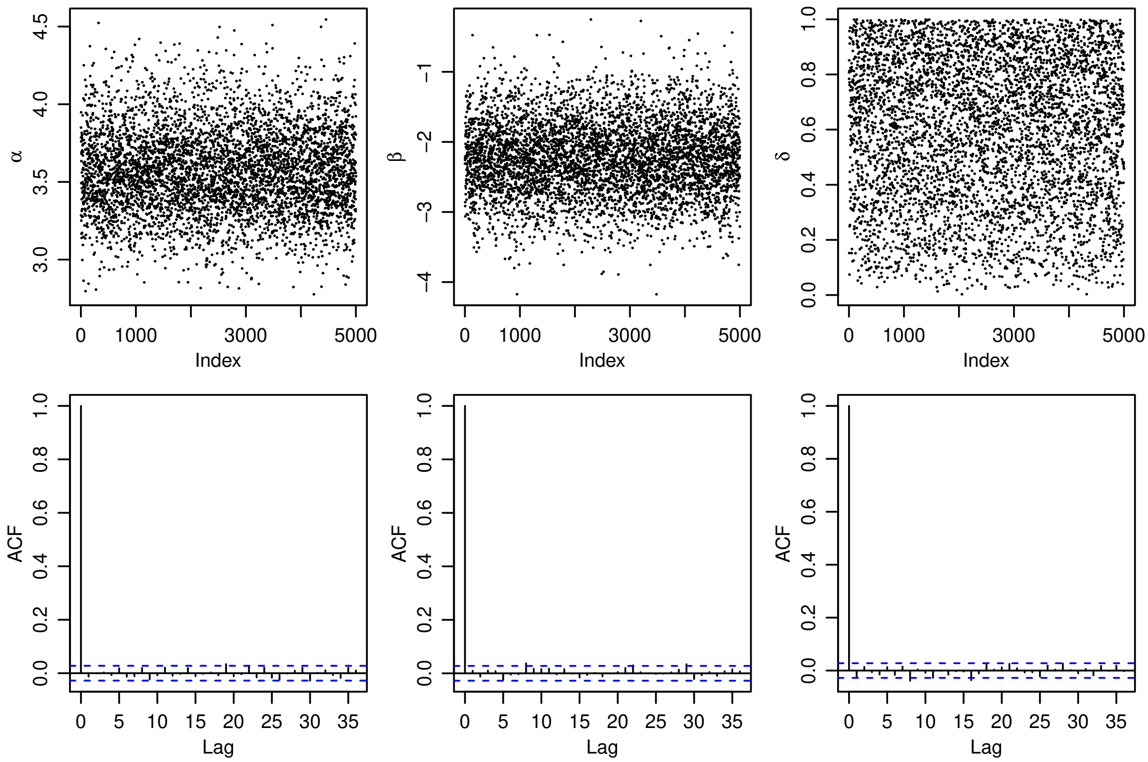


Figure 4: *The model fitting results*

References

[1] Ye, K., Han, Z., Duan, Y. and Bai, T. (2021). Normalized Power Prior Bayesian Analysis. *Journal of Statistical Planning and Inference*, in revision.

[2] Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine*, 34:3724–3749.