18.01 Single Variable Calculus
Fall 2006

18.01 Exam 3

$= \frac{2}{5}$ ✓

Problem 1. (20 pts) Evaluate the following integrals

$a) \int_0^2 \frac{xdx}{(1+x^2)^2}$ $\quad \left(\frac{1}{1+x^2}\right)' = \frac{-2x}{(1+x^2)^2}$ $\quad F = \frac{-1}{2(1+x^2)}$ $\quad F(2)-F(0) = -\frac{1}{10} - \left(-\frac{1}{2}\right)$

$(\sin^7 x)' = 7\sin x \cos x$

$b) \int_{-\pi/2}^{\pi/2} \sin^6 x \cos x dx$ $\quad F = \frac{\sin^7 x}{7}$ $\quad F(\pi/2) - F(-\pi/2) = \frac{1}{7} - \left(-\frac{1}{7}\right)$

$= \frac{2}{7}$ ✓

Problem 2. (20 pts.) Find the following approximations to

$$\int_0^{\pi/2} \cos x dx \qquad \Delta x = \frac{\pi}{4}$$

$f(x_0) = 1$

$f(x_1) = \frac{1}{\sqrt{2}}$

$f(x_2) = 0$

(Do not give a numerical approximation to square roots; leave them alone.)

a) Using the upper Riemann sum with two intervals $\quad \frac{\pi}{4} + \frac{\pi}{4\sqrt{2}}$ ✓

b) Using the trapezoidal rule with two intervals $\quad \frac{\pi}{8} + \frac{\pi}{4\sqrt{2}} + 0$ ✓

c) Using Simpson's rule with two intervals $\quad \frac{\pi}{2}\left(\frac{1}{6} + \frac{\sqrt{2}}{3} + 0\right)$ ✓

Problem 3. (20 points) Find the volume of the solid of revolution formed by revolving the *y*-axis the region enclosed by $\sqrt{\frac{\pi}{2}}$

~~$\int_0^1 \pi x^2 dy = \int_0^1 \pi \arccos(y) dy$~~

$$y=\cos(x^2) \qquad \int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x \cdot y \cdot dx$$

and the *x*-axis (central hump, only). $\quad = \int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x \cos(x^2)\, dx$

$= \pi \sin(x^2)\Big|_0^{\sqrt{\frac{\pi}{2}}} = \pi.$ ✓

Problem 4. (20 points) Students studying for an exam get *x* hours of sleep in the two days leading up to the exam, where *x* is the range $0 \le x \le a$. The numbers of students who got between x_1 and x_2 hours of sleep in given by

a) $\int cx dx = \frac{c}{2}x^2$ $\quad \frac{\frac{c}{2}\left(\frac{a}{2}\right)^2}{\frac{c}{2}a^2} = \frac{1}{4}$ ✓

$$\int_{x_1}^{x_2} cxdx; 0 \le x_1 \le x_2 \le a \qquad \frac{100c}{a}x^2 \quad \frac{100c}{3a}x^3$$

a) What fraction o the student got less than a/2 hours of sleep?

b) Their scores are proportional to the amount of sleep they got: *S(x)* = 100 (*x/a*). Find the (correctly weighted) average score in the class.

$\int_0^a S(x)dy = \int_0^a 100(x/a)\cdot cx\, dx = \frac{100c}{3a}(a)^3 = \frac{100a^2c}{3}$

$\frac{100a^2c}{3} \Big/ \frac{a^2c}{2} = \frac{200}{3}$ ✓

Problem 5. (20 points) Let

$$F(x)=\int_0^x \sqrt{t}\sin t\,dt$$

$F'(x) = \sqrt{x}\sin x$

$F'(a)=0 \quad a = n\pi$ ✓

a) Find *F′(x)* for *x > 0* identify the points *a > 0* *F′(a) = 0*

✓

b) Decide whether F has a local maximum at the smallest critical point a > 0 that you found in part (a) by evaluating *F″*.

$F''(\pi) = -\sqrt{\pi}$

$F''(x) = \frac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x \qquad F''(2\pi) = \sqrt{\pi}$

c) Say whether *F(x)* is positive, negative or zero at each of the following points, and give a reason in each case.

$i)\ x=0$ $=0$ $\qquad \int_0^0 \cdots = 0$

$ii)\ x=\pi$ >0 $\qquad F'(x)>0\ (0<x<\pi)\quad F'(x)=0\ (x=0,\pi) + F(0)=0$

$iii)\ x=2\pi$ <0 $\qquad \boxed{\int_\pi^{2\pi}\sqrt{t}\sin t\,dt = -\int_\pi^{2\pi}\sqrt{t}\,|\sin t|\,dt}$

d) Use a change of the variable to express $G(x)=\int_0^x u^2\sin(u^2)du$ in terms of *F.*

$t = u^2 \qquad dt = 2u\,du$

$F(x) = \int_0^{\sqrt{x}} \sqrt{u^2}\sin u^2 \cdot 2u\,du$

$= \int_0^{\sqrt{x}} 2u^2 \sin(u^2)\,du$

$G(x) = \frac{1}{2}F(x^2)$ ✓

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