18.01 Single Variable Calculus
Fall 2006

Problem 1. (10 pts.) Find the tangent line to $y = \frac{1}{3}x^2$ at x = 1

P1.
$y' = \frac{2}{3}x$
at $x = 1$
$y = \frac{1}{3}$
t.l. $y = \frac{2}{3}x - \frac{1}{3}$ ✓

Problem 2. Find the derivative of the following functions:

a. (7 pts.) $\frac{x}{\sqrt{1-x}}$

a. $f' = \frac{d}{dx}\, x\cdot(1-x)^{-\frac{1}{2}}$
$= (1-x)^{-\frac{1}{2}} + x\cdot(-\frac{1}{2})(1-x)\cdot(-1)$
$= \frac{x-x^2}{2} + \frac{1}{\sqrt{1-x}}$ ✓

b. (8 pts). $\frac{\cos(2x)}{x}$

b. $f' = \frac{d}{dx}\cos(2x)\cdot x^{-1}$
$= \frac{-2\sin x}{x} + \left(-\frac{\cos(2x)}{x^2}\right)$
$= \frac{2x\sin x + \cos(2x)}{-x^2}$ ✓

c. (5 pts). $e^{2f(x)} = g(x)$

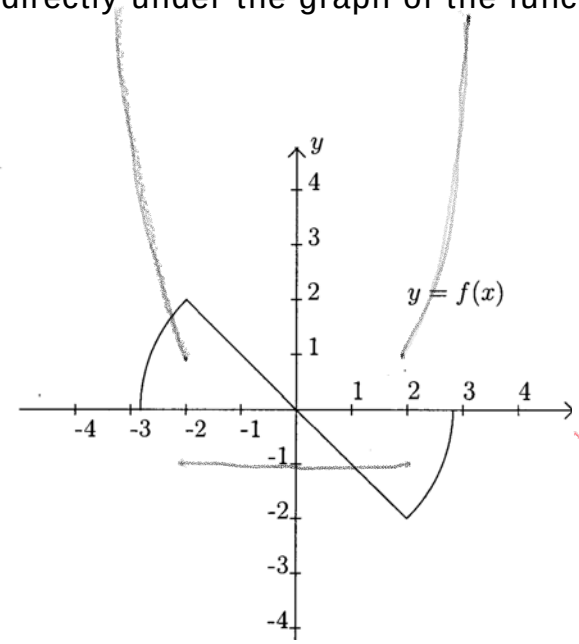
d. (5 pts.) $\ln(\sin x)$

Problem 3. (15 pts.) Find $\frac{dy}{dx}$ for the function for the function *y* defined implicitly by

$y^4 + xy = 4$ at x = 3, y = 1

$\frac{dy}{dx}\,4y^3 + y + \frac{dy}{dx}\,x = 0$ t.l. $\frac{dy}{dx} = -\frac{1}{7}$ ✓

Problem 4. (15 pts.) Draw the graph of the derivative of the function (qualitatively accurate) directly under the graph of the function.



✓

$y^2 + x^2 = n$
$\frac{dy}{dx}\,2y + 2x = 0$
$\frac{dy}{dx} = -\frac{x}{y}$

Problem 5. (15 pts) Let

$$f(x) = \begin{cases} ax + b & x < 1 \\ x^4 + x + 1 & x \geq 1 \end{cases}$$

Find all a and b such that the function f(x) is differentiable.

$f' = \begin{cases} a & \\ 4x^3 + 1 & \end{cases}$ s.t. $a + b = 3$, $a = 5$, $b = -2$ ✓

Problem 6. Evaluate these limits by relating them to a derivative.

a. (5 pts.) Evaluate $\lim_{x \to 0} \frac{(1+2x)^{10} - 1}{x}$

$\lim_{\Delta x \to 0} \frac{(1 + 2\Delta x)^{10} - 1}{\Delta x}$ → $(1+2\times0)^{10}$

$f(x) = x^{10}$, when $x=1$, $f' = 10x^9 = 10$ ✓

b. (5 pts.) Evaluate $\lim_{x \to 0} \frac{\sqrt{\cos x} - 1}{x}$ → $\sqrt{\cos(0)}$ ✗

Problem 7. (10 pts.) Derive the formula $\frac{d}{dx} a^x = M(a) a^x$ directly from the definition of the derivative, and identify $M(a)$ as a limit.

$$\frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(a^{\Delta x} - 1) a^x}{\Delta x}$$

$$= \underbrace{\left(\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} \right)}_{M(a)} \cdot a^x$$ ✓

$(\ln(\sin x))' = \frac{\cos(x)}{\sin(x)} = \cot(x)$ ✓

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$e^{2f(x)} = g(x)$

$2e^{f(x)} f'(x) = g'(x)$

$f'(x) = \frac{g'(x)}{2e^{f(x)}}$ ✓