UNIT2. PARTIAL DERIVATIVES

L8. Level curves; partial derivatives; tangent plane approximation

-Plotting graphs of functions of 2 variables

-Using slices by the coordinate planes

-Computer Plots

-Contour Plot

slice the graph by horizontal planes $z = c$

-Partial Derivatives

$$f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f\left(x_0 + \Delta x, y_0\right) - f\left(x_0, y_0\right)}{\Delta x}$$

How to compute: treat x as variable, y as constant.

L9. Max-min problems; least squares

-Linear Approximation

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

- f_x and f_y give slopes of two lines tangent to the graph:

$$y = y_0, z = z_0 + f_x\left(x_0, y_0\right)\left(x - x_0\right)$$

$$x = x_0, z = z_0 + f_y\left(x_0, y_0\right)\left(y - y_0\right)$$

-the equation of the tangent plane to the graph:

$$z = z_0 + f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right)$$

-Min/Max Problems

-Critical Point: $\left(x_0, y_0\right)$ where $f_x = 0$ and $f_y = 0$

-Local min / Local max / Saddle

-Least-Squares Interpolation

-data points: $\left(x_1, y_1\right), \left(x_2, y_2\right), \ldots, \left(x_n, y_n\right)$

-interpolation line: $y = ax + b$ (**least-squares** line or the **regression** line)

-sum of squares of deviations: $D = \sum_{i=1}^{n} \left(y_i - \left(a x_i + b \right) \right)^2$

(assumed Gaussian error distribution)

(sum only positive quantities)

(weights more heavily the larger deviations)

-make D a minimum: $\dfrac{\partial D}{\partial a} = \sum_{i=1}^{n} 2 \left(y_i - a x_i - b \right) \left(-x_i \right) = 0$

$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2 \left(y_i - a x_i - b \right)(-1) = 0$$

-linear equations $\left(\sum x_i^2 \right) a + \left(\sum x_i \right) b = \sum x_i y_i$

$$\left(\sum x_i \right) a + nb = \sum y_i$$

or $\bar{s}a + \bar{x}b = \dfrac{1}{n} \sum x_i y_i$

$$\bar{x}a + b = \bar{y}$$

-Exponential laws (e.g. Moore's Law)

$$y = ce^{ax}$$

-taking logarithms: $\ln y = \ln c + ax$ (setting $b = \ln c$, linear)

L10. Second derivative test; boundaries and infinity

-$w = ax^2 + bxy + cy^2$ ($a \neq 0$)

$$w = a \left(x^2 + \frac{b}{a} xy \right) + cy^2 = a \left(x + \frac{b}{2a} y \right)^2 + \left(c - \frac{b^2}{4a} \right) y^2$$

$$= \frac{1}{4a}\left(4a^2\left(x + \frac{b}{2a}y\right)^2 + \left(4ac - b^2\right)y^2\right)$$

- if $4ac - b^2 > 0$, same signs, if $a > 0$ then minimum, if $a < 0$ then maximum

- if $4ac - b^2 < 0$, opposite signs, saddle

- if $4ac - b^2 = 0$, degenerate case

-Related to Quadratic Formula: $w = y^2\left(a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c\right)$

- If $b^2 - 4ac < 0$ then no roots, so $at^2 + bt + c$ has a constant sign (min or max)

- If $b^2 - 4ac > 0$ then $at^2 + bt + c$ crosses zero and changes sign (saddle)

-Second Derivative Test

$A = f_{xx}\left(x_0, y_0\right), B = f_{xy}\left(x_0, y_0\right), C = f_{yy}\left(x_0, y_0\right)$

- if $AC - B^2 > 0$ then: if $A > 0$ (or C), local min; if $A < 0$, local max.

- if $AC - B^2 < 0$ then saddle.

- if $AC - B^2 = 0$ then can't conclude.

-Quadratic Approximation

$$\Delta f \simeq f_x\left(x - x_0\right) + f_y\left(y - y_0\right) + \frac{1}{2}f_{xx}\left(x - x_0\right)^2 + f_{xy}\left(x - x_0\right)\left(y - y_0\right) + \frac{1}{2}f_{yy}\left(y - y_0\right)^2$$

$$\ldots\frac{1}{2}f_{yy}\left(y - y_0\right)^2$$

At a critical point, $f_x = f_y = 0$

$$\Delta f \simeq \frac{A}{2}\left(x - x_0\right)^2 + B\left(x - x_0\right)\left(y - y_0\right) + \frac{C}{2}\left(y - y_0\right)^2$$

$$B^2 - 4\left(\frac{A}{2}\cdot\frac{B}{2}\right) = B^2 - AC$$

NOTE: the global min/max of a function is not necessarily at a critical point! Need to check boundary / infinity.

L11. Differentials; chain rule

-More tools to study functions

-Differentials

-implicit differentiation

$$f = f(x, y, z) \qquad \rightarrow$$

$$df = f_x dx + f_y dy + f_z dz\,.$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

-Chain Rule

$$x = x(t), y = y(t), z = z(t)$$

$$\frac{df}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt}$$

-Application of Chain Rule

-Justification of Product and quotient formulas for derivatives

$$f = uv, u = u(t), v = v(t)$$

$$d(uv)/dt = f_u u' + f_v v' = vu' + uv'$$

-More Variables

$$w = f(x, y), x = x(u, v), y = y(u, v)$$

$$dw = f_x dx + f_y dy$$

$$= f_x\left(x_u du + x_v dv\right) + f_y\left(y_u du + y_v dv\right)$$

$$= \left(f_x x_u + f_y y_u\right)du + \left(f_x x_v + f_y y_v\right)dv$$

-Identifying coefficients of du and dv

$$\partial f/\partial u = f_x x_u + f_y y_u$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

<u>L12. Gradient; directional derivative; tangent plane</u>

-Gradient

chain rule: $\dfrac{dw}{dt} = w_x\dfrac{dx}{dt} + w_y\dfrac{dy}{dt} + w_z\dfrac{dz}{dt}$

-Vector Notation

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt}$$

(Velocity Vector: $\dfrac{d\vec{r}}{dt} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$)

-Definition: Gradient Vector

$$\nabla w = \langle w_x, w_y, w_z \rangle$$

-Theorem: ∇w **is perpendicular to the level surfaces/curves** $w = constant$

e.g. $w = ax + by + cz$

$w = constant$ are planes with normal vector $\nabla w = \langle a, b, c \rangle$

-Directional Derivatives

$\dfrac{dw}{ds}_{|\hat{u}} = \nabla w \cdot \dfrac{d\vec{r}}{ds} = \nabla w \cdot \hat{u}$ (also true for $dx, dy, dz \ldots$)

-Geometric Interpretation

$$dw/ds = \nabla w \cdot \hat{u} = |\nabla w| \cos\theta$$

Maximal for $\cos\theta = 1$, when $\hat{u}$ is in direction of ∇w

s.t. direction of ∇w is that of fastest increase of w

$$\left|\nabla w\right| = \frac{dw}{ds \mid \hat{u} = \mathrm{dir}\,(\nabla w)}$$

L13. Lagrange multipliers

-Problem: $f(x, y, z)$ **min/max when variables are constrained by an equation** $g(x, y, z) = c$

-Method: at the minimum, the level curves are tangent to each other, so **the normal vectors** ∇f **and** ∇g **are parallel**

$$\nabla f = \lambda \nabla g \qquad \rightarrow \qquad \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = c \end{cases} \qquad (\lambda\text{: multiplier})$$

++ $g = c$ **投影到** f **上的几何形的最小值**

L14. Non-independent variables

-Problem: $f(x, y, z)$ where $g(x, y, z) = c$

-Observation I: if $g(x, y, z) = c$ then can think of $z = z(x, y)$. What are $\partial z/\partial x, \partial z/\partial y$?

-Method: $g(x, y, z) = c \Rightarrow g_x dx + g_y dy + g_z dz = 0$

If y held fixed, get $g_x dx + g_z dz = 0 \Rightarrow$

$dz = -g_x/g_z dx,$ and $\partial z/\partial x = -g_x/g_z$

-Notation:

$$\left(\frac{\partial f}{\partial u}\right)_v = deriv./u \text{ with } v \text{ held fixed}$$

-e.g. $A = \dfrac{1}{2}ab\sin\theta$ and $a = b\cos\theta$ find $\left(\dfrac{\partial A}{\partial \theta}\right)_a$

-Method 0: Substitution, Solve b w.r.t. a, θ (not always possible)

-Method 1:

1. write dA in terms of $da, db, d\theta$
2. $a = constant \Rightarrow da = 0$
3. differentiate constraint $\Rightarrow$ solve for db in terms of $d\theta$

4. plug into dA, get answer

-Method 2: (chain rule)

$$\left(\frac{\partial A}{\partial \theta}\right)_a = A_\theta \left(\frac{\partial \theta}{\partial \theta}\right)_a + A_a \left(\frac{\partial \alpha}{\partial \theta}\right)_a + A_b \left(\frac{\partial b}{\partial \theta}\right)_a$$