18.02 Multivariable Calculus
Fall 2007

18.02 Practice Exam 2 A

Problem 1. (10 points: 5, 5) $(f_x, f_y) = (y - 4x^3, x) = (-3, 1)\big|_{(1,1)}$

Let $f(x, y) = xy - x^4$.

a) Find the gradient of f at $P : (1, 1)$.

b) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$. $\Delta w \approx -3\Delta x + \Delta y$

Problem 2. (20 points)

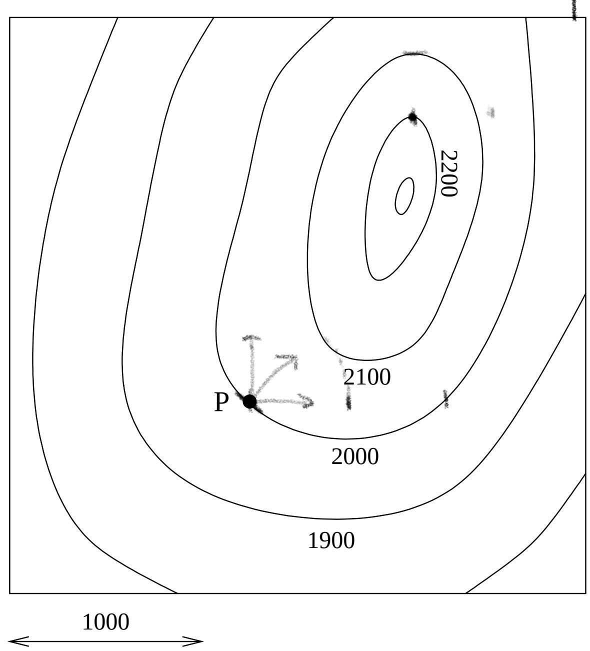
On the topographical map below, the level curves for the height function $h(x, y)$ are marked (in feet); adjacent level curves represent a difference of 100 feet in height. A scale is given.

a) Estimate to the nearest .1 the value at the point P of the directional derivative $\left(\dfrac{dh}{ds}\right)_{\hat{u}}$, where $\hat{u}$ is the unit vector in the direction of $\hat{\imath} + \hat{\jmath}$.

$\dfrac{100}{500} = 0.2$

b) Mark on the map a point Q at which $h = 2200$, $\dfrac{\partial h}{\partial x} = 0$ and $\dfrac{\partial h}{\partial y} < 0$. Estimate to the nearest .1 the value of $\dfrac{\partial h}{\partial y}$ at Q.

$\dfrac{-100}{300} \simeq -0.3$



Problem 3. (10 points)

Find the equation of the tangent plane to the surface $x^3y + z^2 = 3$ at the point $(-1, 1, 2)$.

$(f_x, f_y, f_z) = (3x^2y, x^3, 2z) = (3, -1, 4)\big|_{(-1,1,2)}$

$3x - y + 4z = c = 4\ \big|_{(-1,1,2)}$

$z = 1 - x^2 - y^2$ volume $= xyz = xy - x^3y - xy^3 = f(x,y)$

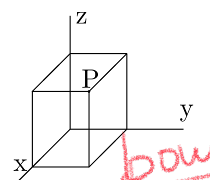
$$\begin{cases} f_x = y - 3x^2y - y^3 = 0 \to 3x^2 + y^2 = 1 \to 3 - 9y^2 + y^2 = 1 \to y^2 = \frac{1}{4} \to y = \frac{1}{2},\ x = \frac{1}{2} \\ f_y = x - x^3 - 3xy^2 = 0 \to x^2 + 3y^2 = 1 \end{cases}$$

$f_{xx} = -6xy = -\frac{6}{4}$ $f_{yy} = -6xy = -\frac{6}{4}$ $f_{xy} = 1 - 3x^2 - 3y^2 = -\frac{2}{4}$

$(-\frac{6}{4})(-\frac{6}{4}) - (-\frac{2}{4})^2 > 0$ $-\frac{6}{4} < 0$ max

Problem 4. (20 points: 5,5,5,5)

A rectangular box is placed in the first octant as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P : (x, y, z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. Which P gives the box of greatest volume?



a) Show that the problem leads one to maximize $f(x, y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f.

b) Find a critical point of f which lies in the first quadrant ($x > 0$, $y > 0$).

c) Determine the nature of this critical point by using the second derivative test.

boundary

d) Find the maximum of f in the first quadrant (justify your answer). $f_{max} = \frac{1}{4} - \frac{1}{16} - \frac{1}{16} = \frac{1}{8}$

Problem 5. (15 points)

$\frac{y}{x}$ $z = 2\lambda$ $\frac{x}{y}$

In Problem 4 above, instead of substituting for z, one could also use Lagrange multipliers to maximize the volume $V = xyz$ with the same constraint $x^2 + y^2 + z = 1$.

$x = y$ $x^2 = y^2 = \lambda$ $z = 2\lambda$

a) Write down the Lagrange multiplier equations for this problem.

$\lambda + \lambda + 2\lambda = 1$ $\lambda = \frac{1}{4}$

b) Solve the equations (still assuming $x > 0$, $y > 0$).

$x = y = \frac{1}{2}$

$$\begin{cases} V_x = yz = \lambda w_x = 2\lambda x \\ V_y = xz = \lambda w_y = 2\lambda y \\ V_z = xy = \lambda w_z = \lambda \\ x^2 + y^2 + z = 1 \end{cases}$$

Problem 6. (10 points)

Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.

$$\frac{\partial w}{\partial x} = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} = f_u y + \frac{f_v}{y}$$

$$\frac{\partial w}{\partial y} = f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} = f_u x - \frac{f_v x}{y^2}$$

Problem 7. (15 points)

Suppose that $x^2y + xz^2 = 5$, and let $w = x^3y$. Express $\left(\dfrac{\partial w}{\partial z}\right)_y$ as a function of x, y, z, and evaluate it numerically when $(x, y, z) = (1, 1, 2)$.

$d x^2y + d xz^2 = 2xy\,dx + z^2 dx + 2xz\,dz = 0$

$$\frac{\partial x}{\partial z} = -\frac{2xz}{2xy + z^2}$$

$$\left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial z} = -\frac{6x^3yz}{2xy + z^2} = -\frac{12}{2+4}\Big|_{(1,1,2)} = -2$$