UNIT1. VECTORS AND MATRICES

L1. Dot Product

$\overrightarrow{A} = < a_1, a_2, a_3 > = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$|\overrightarrow{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

-Dot Product

Def: $\overrightarrow{A} \cdot \overrightarrow{B} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$ (scalar)

$\overrightarrow{A} \cdot \overrightarrow{B} = \overrightarrow{B} \cdot \overrightarrow{A}$

Theorem: geometrically, $\overrightarrow{A} \cdot \overrightarrow{B} = |\overrightarrow{A}||\overrightarrow{B}|\cos\theta$

L2. Determinants; cross product

-Area

$$Area\ of\ triangle = \frac{1}{2}|\overrightarrow{A}||\overrightarrow{B}|\sin\theta$$

$$\overrightarrow{A}' = \overrightarrow{A}\ rotated\ 90°\ conterclockwise \qquad \theta' = \pi/2 - \theta \qquad \rightarrow$$

$$= \frac{1}{2}|\overrightarrow{A}'||\overrightarrow{B}|\cos\theta' = \frac{1}{2}\overrightarrow{A}' \cdot \overrightarrow{B}$$

$$if\ \overrightarrow{A} = < a_1, a_2 >,\ \overrightarrow{A}' = < -a_2, a_1 > \qquad \rightarrow$$

$$= \frac{1}{2}(a_1 b_2 - a_2 b_1)$$

-Determinant

Def: $det(\overrightarrow{A}, \overrightarrow{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

Geometrically: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm$ area of parallelogram

$$\det(\vec{A}, \vec{B}, \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

-Cross-product

Def: $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\boldsymbol{\imath}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\jmath} \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Geometrically: $|\vec{A} \times \vec{B}| =$ area of space parallelogram with sides $\vec{A}, \vec{B}$

-Triple product

volume of parallelepiped = area(base) $\cdot$ height $= |\vec{B} \times \vec{C}|(\vec{A} \cdot \hat{n}),$ where $\hat{n} = \vec{B} \times \vec{C} / |\vec{B} \times \vec{C}|$

So volume $= \vec{A} \cdot (\vec{B} \times \vec{C}) = \det(\vec{A}, \vec{B}, \vec{C})$

L3. Matrices; inverse matrices

-Matrices

Often quantities are related by linear transformations

e.g. $\begin{cases} u_1 = 2x_1 + 3x_2 + 3x_3 \\ u_2 = 2x_1 + 4x_2 + 5x_3 \\ u_3 = x_1 + x_2 + 2x_3 \end{cases}$

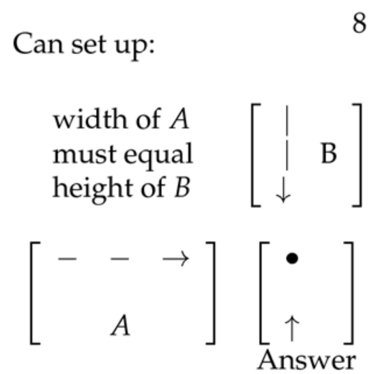
$$\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$AX = U$$

-Matrix Product

matrix product = dot product between rows of A and columns of X. (here we multiply a 3x3 matrix by a column vector = 3x1 matrix)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{bmatrix} = \begin{bmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$



-width of A must equal height of B

$(AB)X = A(BX)$

-Identity Matrix

$$IX = X \qquad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

-Rotation

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -y \\ x \end{bmatrix} \qquad R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad R^2 = -I$$

-Inverse Matrix

Def: Inverse of a matrix A (necessarily square) is a matrix M = A^{-1} such that AM = MA = I_n.

$$AX = B \quad \to \quad A^{-1}(AX) = A^{-1}B \quad \to \quad X = A^{-1}B$$

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$$

1. matrix of minors; 2. cofactors; 3. transpose; 4. divide by det(A)

L4. Square systems; equations of planes

plane through $P_0 = (2,1,-1)$ with normal vector $\boldsymbol{N} = \langle 1,5,10 \rangle$

$$\boldsymbol{N} \cdot \overrightarrow{P_0P} = 0 \Leftrightarrow (x-2) + 5(y-1) + 10(z+1) = 0,$$

$$x + 5y + 10z = -3$$

(coefficients: normal vector, -3: plugin P_0)

-Geometric Interpretation of 3x3 systems

-***<u>the intersection of 3 planes</u>***

$AX = B \qquad \rightarrow \qquad X = A^{-1}B$

If the line $\mathscr{P}_1 \cap \mathscr{P}_2$ is contained in or paralell to $\mathscr{P}_3$

$\det(A) = 0 \qquad \rightarrow \qquad A$ is invertible

$\rightarrow \qquad \det\left(\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3\right) = 0$

$\rightarrow$ the parallelepiped formed by the $\boldsymbol{N}_i$ has no area

-Homogeneous systems

$AX = 0 \qquad \rightarrow \qquad X = 0$

if $A = 0 \qquad \rightarrow$ infinitely many solutions

if infinitely solutions: $N_1 \times N_2$

<u>L5. Parametric equations for lines and curves</u>

-Parametric equation: as trajectory of a moving point

e.g. $Q_0 = (-1,2,2), Q_1 = (1,3,-1)$

$Q(t): \qquad Q_0 \ at \ t = 0; \ Q_1 \ at \ t = 1$

$\overrightarrow{v} = \overrightarrow{Q_0Q_1}; \ \overrightarrow{Q_0Q(t)} = t\overline{Q_0Q_1}$

$\langle x+1, y-2, z-2 \rangle = t\langle 2,1,-3 \rangle$

$$\begin{cases} x(t) = -1 + 2t \\ y(t) = 2 + t \\ z(t) = 2 - 3t \end{cases}$$

-General parametric curves

L6. Velocity, acceleration; Kepler's second law

-Kepler's second law

$$\text{area } \approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \overrightarrow{v}|\Delta t \qquad \rightarrow$$

$$\frac{d}{dt}(\text{ area }) = \frac{1}{2}|\vec{r} \times \overrightarrow{v}| \text{ is constant} \qquad +$$

$$\frac{d}{dt}(\vec{r} \times \overrightarrow{v}) = \frac{d\vec{r}}{dt} \times \overrightarrow{v} + \vec{r} \times \frac{d\overrightarrow{v}}{dt} = \overrightarrow{v} \times \overrightarrow{v} + \vec{r} \times \overrightarrow{a} = \vec{r} \times \overrightarrow{a} \qquad \rightarrow$$

Kepler's law $\Leftrightarrow \vec{r} \times \overrightarrow{v} =$ constant $\Leftrightarrow \vec{r} \times \overrightarrow{a} = 0 \Leftrightarrow \overrightarrow{a} // \vec{r} \Leftrightarrow$ the force $\overrightarrow{F}$ is central