UNIT1. VECTORS AND MATRICES

-Dot Product

Def: $\overrightarrow{A} \cdot \overrightarrow{B} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\overrightarrow{A}||\overrightarrow{B}|\cos\theta$

-Determinant 行列式

Def: $det(\overrightarrow{A}, \overrightarrow{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = \pm$ area of parallelogram

-Cross-product

Def: $\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\boldsymbol{\imath}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\jmath} \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

-Triple product

volume $= \overrightarrow{A} \cdot (\overrightarrow{B} \times \overrightarrow{C}) = \det(\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C})$

-Equations of planes

plane through $P_0 = (2,1,-1)$ with normal vector $\boldsymbol{N} = \langle 1,5,10 \rangle$

$\boldsymbol{N} \cdot \overrightarrow{P_0 P} = 0 \Leftrightarrow (x-2) + 5(y-1) + 10(z+1) = 0,$

$x + 5y + 10z = -3$

(coefficients: normal vector, -3: plugin P_0)

-Matrix Product (Square systems)

$AX = B \quad \rightarrow \quad A^{-1}(AX) = A^{-1}B \quad \rightarrow \quad X = A^{-1}B$

(see lecture notes)

UNIT2. PARTIAL DERIVATIVES

-Partial Derivatives

$$f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f\left(x_0 + \Delta x, y_0\right) - f\left(x_0, y_0\right)}{\Delta x}$$

-Gradient 梯度

$$\nabla w = \langle w_x, w_y, w_z \rangle$$

++ w 增长速度最快的方向

-Theorem: ∇w is perpendicular to the level surfaces/curves $w = constant$

e.g. $w = ax + by + cz$

$w = constant$ are planes with normal vector $\nabla w = \langle a, b, c \rangle$

-Directional Derivatives

$$\frac{dw}{ds}_{|\hat{u}} = \nabla w \cdot \frac{d\vec{r}}{ds} = \nabla w \cdot \hat{u}$$

-implicit differentiation 隐函数微分

$$f = f(x, y, z) \qquad \rightarrow$$

$$df = f_x dx + f_y dy + f_z dz\,.$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

-Chain Rule

$$x = x(t), y = y(t), z = z(t)$$

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

-More Variables

$$w = f(x, y), x = x(u, v), y = y(u, v)$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

-Linear Approximation

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

-the equation of the tangent plane to the graph:

$$z = z_0 + f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right)$$

-Min/Max Problems

-Critical Point: $\left(x_0, y_0\right)$ where $f_x = 0$ and $f_y = 0$

-Second Derivative Test

$$A = f_{xx}\left(x_0, y_0\right), B = f_{xy}\left(x_0, y_0\right), C = f_{yy}\left(x_0, y_0\right)$$

- if $AC - B^2 > 0$ then: if $A > 0$ (or C), local min; if $A < 0$, local max.

- if $AC - B^2 < 0$ then saddle.

- if $AC - B^2 = 0$ then can't conclude.

NOTE: the global min/max of a function is not necessarily at a critical point! Need to check boundary / infinity.

-Quadratic Approximation

$$\Delta f \simeq f_x\left(x - x_0\right) + f_y\left(y - y_0\right) + \frac{1}{2} f_{xx}\left(x - x_0\right)^2 + f_{xy}\left(x - x_0\right)\left(y - y_0\right)$$
$$+ \frac{1}{2} f_{yy}\left(y - y_0\right)^2$$

-Least-Squares Interpolation (fit data points to an interpolation line)

-data points: $\left(x_1, y_1\right), \left(x_2, y_2\right), \ldots, \left(x_n, y_n\right)$

-interpolation line: $y = ax + b$ (**least-squares** line or the **regression** line)

-sum of squares of deviations: $D = \sum_{i=1}^{n} \left(y_i - \left(a x_i + b\right)\right)^2$

(assumed Gaussian error distribution)

(sum only positive quantities)

(weights more heavily the larger deviations)

-make D a minimum: $$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2\left(y_i - a x_i - b\right)\left(-x_i\right) = 0$$

$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2\left(y_i - a x_i - b\right)(-1) = 0$$

-linear equations $$\left(\sum x_i^2\right) a + \left(\sum x_i\right) b = \sum x_i y_i$$

$$\left(\sum x_i\right) a + nb = \sum y_i$$

or $$\bar{s} a + \bar{x} b = \frac{1}{n} \sum x_i y_i$$

$$\bar{x} a + b = \bar{y}$$

-Lagrange multipliers

-Problem: $f(x, y, z)$ **min/max when variables are constrained by an equation** $g(x, y, z) = c$

the normal vectors ∇f and ∇g are parallel

$$\nabla f = \lambda \nabla g \quad \rightarrow \quad \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = c \end{cases}$$ (λ: multiplier)

++ $g = c$ 投影到 f 上的几何形的最小值

-Non-independent variables

-Problem: $f(x, y, z)$ where $g(x, y, z) = c$

-Notation:

$$\left(\frac{\partial f}{\partial u}\right)_v = deriv./u$$ with v held fixed

$$
\begin{aligned}
\left(\frac{\partial f}{\partial x}\right)_y &= A_x\left(\frac{\partial x}{\partial x}\right)_y + A_y\left(\frac{\partial y}{\partial x}\right)_y + A_z\left(\frac{\partial z}{\partial x}\right)_y \\
&= A_x + A_z\left(\frac{\partial z}{\partial x}\right)_y
\end{aligned}
$$

UNIT3. DOUBLE INTEGRALS AND LINE INTEGRALS IN THE PLANE

-Double integrals (Iterated Integral)

$$\iint_R f(x,y)dA = \int_{x_{\min}}^{x_{\max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)dy \right] dx$$

-Work and line integrals

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt$$

-Notation: $\vec{F} = \langle M, N \rangle,$ **and** $d\vec{r} = \langle dx, dy \rangle$

$$= \int_C Mdx + Ndy$$

-Fundamental theorem of calculus for line integrals

If $\vec{F}$ is a gradient field, $\vec{F} = \nabla f = f_x \hat{\imath} + f_y \hat{\jmath}$ (f: "potential function")

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$$

-Test for Gradient Field

if $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is a gradient field, $M = f_x,\ N = f_y,\ soN_x = f_{yx} = f_{xy} = M_y$

-Green's Theorem

-Curl

Failure of conservativeness is given by the $curl$ of $\vec{F}$:

Definition: $\mathrm{curl}(\vec{F}) = N_x - M_y$

++ $= Y_x - X_y$ **场的 Y项 随 x 的变化速率和 X项 随 y 的变化速率的差值**

If C is a positively oriented closed curve enclosing a region R,

and $\vec{F}$ defined and differentiable in R:

$$\oint_C \overrightarrow{F} \cdot d\vec{r} = \iint_R \text{curl } \overrightarrow{F} dA$$

-Flux (another line integral)

$$\int_C \overrightarrow{F} \cdot \hat{\boldsymbol{n}} ds$$

$\hat{n}$ = normal vector to C, rotated $90°$ clockwise from $\hat{T}$

-Green's theorem for flux (Green's theorem in normal form (vs. tangential))

-Divergence

$\text{div}(\overrightarrow{F}) = P_x + Q_y$ is the divergence of $\overrightarrow{F}$

1. measures how much the flow is "expanding"

2. "source rate": amount of fluid added to the system per unit time and per unit area

++ $= X_x + Y_y$ **<u>场的 X项 随 x 的变化速率和 Y项 随 y 的变化速率的和</u>**

If C encloses R counterclockwise, and $\overrightarrow{F} = P\hat{\imath} + Q\hat{\jmath}$

$$\oint_C \overrightarrow{F} \cdot \hat{\boldsymbol{n}} ds = \iint_R \text{div}(\overrightarrow{F}) dA$$

-Integration in Polar Coordinates ($x = r\cos\theta, y = r\sin\theta$ **)**

-Area Element: $dA = dl\, dr = r\, d\theta\, dr$ **(** dl**: differential of circumference)**

e.g. $$\iint_{x^2+y^2\le 1, x\ge 0, y\ge 0} \left(1 - x^2 - y^2\right) dxdy = \int_0^{\pi/2} \int_0^1 \left(1 - r^2\right) rdrd\theta$$

-Change of Variables

<u>Motivation: to simplify either integrand or bounds of integration</u>

e.g. area of ellipse with semiaxes a and b : setting $u = x/a, v = y/b$

$$\iint_{(x/a)^2+(y/b)^2<1} dxdy = \iint_{u^2+v^2<1} ab\, dudv = ab \iint_{u^2+v^2<1} dudv = \pi ab$$

-Scale Factor (Jacobian)

approximation formula $\Delta u \approx u_x \Delta x + u_y \Delta y, \Delta v \approx v_x \Delta x + v_y \Delta y$

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

-Definition

Jacobian: $J = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$dudv = |J| dxdy$$

-Exchanging order of Integration

e.g. $\displaystyle\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y} dydx = \int_0^1 \int_{y^2}^{y} \frac{e^y}{y} dxdy$ (region: $x < y < \sqrt{x}$ for $0 \leq x \leq 1$)

-Potential Function f

-Method 1: line integrals

e.g. $f\left(x_1, y_1\right) - f(0,0) = \displaystyle\int_C \overrightarrow{F} \cdot d\vec{r}$ ***independent of path***

$$\int_{(0,0)}^{(x_1,y_1)} = \int_{(0,0)}^{(x_1,0)} + \int_{(x_1,0)}^{(x_1,y_1)}$$

-Method 2: antiderivatives

e.g. $f_x,\ f_y$

$$f = \int f_x dx + g(y) \equiv f^{(x)} + g(y) \quad \Rightarrow \quad f_y = f_y^{(x)} + g'(y)$$

compare $f_y^{(x)} + g'(y)$ with f_y to get $g'(y)$ thus $g(y)$

UNIT4. TRIPLE INTEGRALS AND SURFACE INTEGRALS IN 3-SPACE

-Triple integrals

e.g. $z = x^2 + y^2$ **and** $z = 4 - x^2 - y^2$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz dy dx$$

-Vector fields in space

$\overrightarrow{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{\boldsymbol{k}},$ where P, Q, R are functions of x, y, z

-Del operator

$\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$

$\nabla f = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle =$ gradient

$\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z =$ divergence

-Flux

$$\text{Flux} = \iint \overrightarrow{F} \cdot \hat{\boldsymbol{n}} dS = \iint \overrightarrow{F} \cdot d\overrightarrow{S}$$

-Divergence Theorem (Gauss-Green Theorem)

If S is ***a closed surface*** bounding a region D, with normal pointing outwards,

and $\overrightarrow{F}$ vector field defined and differentiable over all of D, then

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_D \text{div } \overrightarrow{F} dV,$$ where $\text{div}(P\hat{\imath} + Q\hat{\jmath} + R\hat{\boldsymbol{k}}) = P_x + Q_y + R_z$

$$\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z = \text{div}\,\mathbf{F}$$

$\text{div } \overrightarrow{F}$ = source rate = flux generated per unit volume

-Line integrals in space

Force field $\mathbf{F} = \langle P, Q, R \rangle,$ curve C in space, $d\vec{r} = \langle dx, dy, dz \rangle$

Work $= \int_C \mathbf{F} \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz$

-Gradient fields

$$\mathbf{F} = \langle P, Q, R \rangle = \left\langle f_x, f_y, f_z \right\rangle$$

$$\Rightarrow \quad f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy} \quad \Rightarrow \quad P_y = Q_x, P_z = R_x, Q_z = R_y$$

-Curl: encodes by how much $\mathbf{F}$ fails to be conservative

$$\text{curl}\langle P, Q, R \rangle = \left(R_y - Q_z\right)\hat{\boldsymbol{\imath}} + \left(P_z - R_x\right)\hat{\jmath} + \left(Q_x - P_y\right)\hat{\boldsymbol{k}}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \text{curl}\,\mathbf{F}$$

<u>curl measures the rotation component of a complex motion</u>

-Stokes' Theorem

if C is a closed curve, and S ***<u>any</u>*** surface bounded by C

$$\oint_C \mathbf{F} \cdot d\vec{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\,dS = \iint_S \text{curl}\,\mathbf{F}\,d\overrightarrow{S}$$

Remark: In Stokes' theorem we are free to choose any surface S bounded by C

-Spherical Coordinates (ρ, ϕ, θ)

ρ = radius, ϕ = angle down from z-axis

-Formulas

$$z = \rho\cos\phi, r = \rho\sin\phi, x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

-Volume element

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

-Flux: Setup of dS and selection of two variables to describe the surface

1. $z = a$: $\hat{\boldsymbol{n}} = \pm \hat{\boldsymbol{k}}, dS = dxdy$

2. $sphere$: $\hat{\boldsymbol{n}} = \dfrac{1}{a}\langle x, y, z\rangle, dS = a^2 \sin\phi d\phi d\theta$

3. $cylinder$: $\hat{\boldsymbol{n}} = \dfrac{1}{a}\langle x, y, 0\rangle, dS = adzd\theta$

4. $z = f(x, y)$: $\overrightarrow{S} = \hat{\boldsymbol{n}}dS = \left\langle -f_x, -f_y, 1 \right\rangle dxdy$

5. $\vec{r} = \vec{r}(u, v)$: $d\overrightarrow{S} = \pm \left(\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v} \right) dudv$

6. $g(x, y, z) = 0$: $\mathbf{N} = \nabla g \quad \hat{\boldsymbol{n}}dS = \pm \dfrac{\mathbf{N}}{\mathbf{N} \cdot \hat{\boldsymbol{k}}} dxdy$

-Potential function

e.g. $f_x = 2xy, f_y = x^2 + z^3, f_z = 3yz^2 - 4z^3$

$$f_x = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z)$$

$$f_y = x^2 + g_y = x^2 + z^3 \Rightarrow g_y = z^3 \Rightarrow g(y, z) = yz^3 + h(z)$$

$$\Rightarrow \quad f = x^2y + yz^3 + h(z)$$

$$f_z = 3yz^2 + h'(z) = 3yz^2 - 4z^3 \Rightarrow h'(z) = -4z^3 \Rightarrow h(z) = -z^4 + c$$

$$\Rightarrow \quad f = x^2y + yz^3 - z^4 + c$$