18.02 Final Exam

No books, notes or calculators.
15 problems, 250 points.

$3y + 2z = D$
$= 7$ ✓

Useful formula: $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$

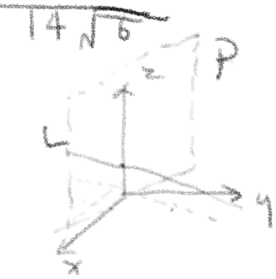
$\vec{P_1} \times \vec{P_2} = \begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ -2 & 0 & 0 \end{vmatrix} = \langle 0, 6, 4 \rangle$

Problem 1. (20 points)
a) (15 pts.) Find the equation in the form $Ax + By + Cz = D$ of the plane $\mathcal{P}$ which contains the line L given by $x = 1 - t$, $y = 1 + 2t$, $z = 2 - 3t$ and the point (- 1, 1, 2). $P_0 = (1, 1, 2)$ $\vec{P_1} = (-1, 2, -3)$ $\vec{P_2} = (-2, 0, 0)$

b) (5 pts.) Let $\mathcal{Q}$ be the plane $2x + y + z = 4$. Find the component of a unit normal vector for $\mathcal{Q}$ projected on a unit direction vector for the line L of part(a).

$\frac{(2, 1, 1)}{\sqrt{6}} \cdot \frac{(-1, 2, -3)}{\sqrt{14}} = \frac{-3}{2\sqrt{21}}$ ✓ $\frac{3(-1, 2, -3)}{14\sqrt{6}}$

Problem 2. (15 points)
Let L denote the line which passes through (0,0,1) and is parallel to the line in the xy-plane given by $y = 2x$.



a) (5 pts.) Sketch L and give its equation in vector-parametric form.

$\begin{cases} x = t \\ y = 2t \\ z = 1 \end{cases}$ ✓

b) (5 pts.) Let $\mathcal{P}$ be the plane which passes through (0,0,1) and is *perpendicular* to the line L of part(a). Sketch in $\mathcal{P}$ (above) and give its equation in point-normal form.

$\hat{n} = \langle 1, 2, 0 \rangle$
$P_0 = (0, 0, 1)$

c) (5 pts.) Suppose that the point $P \neq (0, 0, 1)$ lies on L. Write down the method or formula you would use to find the point P^* which is: (i) on L; (ii) the same distance away from the point (0,0,1) as P; and is (iii) on the *other* side of $\mathcal{P}$ from P.

$P^* = 2P_0 - P$ $P^* = (-t, -2t, 1)$

Problem 3. (20 points)

Given the 3×3 matrix: $A_a = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & a \end{bmatrix}$: $|A_2| = 3 + 0 + (-3) = 0$ ✓

a) (5 pts.) Let $a = 2$: show that $|A_2| = 0$

$\begin{vmatrix} i & j & k \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} = 3i + 5j - k$

b) (7 pts.) Find the line of solutions to $A_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{cases} x = 3t \\ y = 5t \\ z = -t \end{cases}$ ✓

c) (8 pts.) Suppose now that $a = 1$, and that $A_1^{-1} = \begin{bmatrix} * & * & * \\ -3 & p & 5 \\ * & * & * \end{bmatrix}$. Find p.

-4 ✓

Problem 4. (10 points)
Let $\mathbf{r}(t) = \langle \cos(e^t), \sin(e^t), e^t \rangle$.

a) (5 pts.) Compute and simplify the unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

b) (5 pts.) Compute $\mathbf{T}'(t)$

$\frac{\langle -\sin(e^t)e^t, \cos(e^t)e^t, e^t \rangle}{\sqrt{(\sin^2 + \cos^2)e^{2t} + e^{2t}}}$

$= \frac{1}{\sqrt{2}} \langle -\sin(e^t), \cos(e^t), 1 \rangle$ ✓

$T'(t) = \frac{1}{\sqrt{2}} \langle -\cos(e^t)e^t, -\sin(e^t)e^t, 0 \rangle$ ✓

$F_x = \frac{xz}{\sqrt{x^2+y}} = 1$ $\quad F_y = \frac{z}{2\sqrt{x^2+y}} + \frac{2}{z} = \frac{3}{2}$ $\quad F_z = \sqrt{x^2+y} - 2\frac{y}{z^2} = \frac{1}{2}$

Problem 5. (20 points)

Consider the function $F(x,y,z) = z\sqrt{x^2+y} + 2\,\dfrac{y}{z}$: $\quad x + \frac{3}{2}y + \frac{1}{2}z = C = \frac{13}{2}$

a) (10 pts.) The point $P_0 : (1,3,2)$ lies on the surface $F(x,y,z) = 7$. Find the equation of the tangent plane to the surface $F = 7$ at P_0.

b) (5 pts.) If starting at P_0 a small change were to be made in only *one* of the variables, which one would produce the largest change (in absolute value) in F ? If the change is this variable was of size 0.1, approximately how large would the change in F be ?

y, 0.15

c) (5 pts.) What distance from P_0 in the direction $\pm\langle -2, 2, -1\rangle$ will produce an approximate change in F of size 0.1 units, according to the (already computed) linearization of F?

$\hat{u} = \pm\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\rangle$ $\quad |\nabla w \cdot \hat{u}| = |(-\frac{2}{3}) + 1 + (-\frac{1}{6})| = \frac{1}{6}$ $\quad du = 0.6$

$F_u\, du = 0.1$

Problem 6. (15 points)

Let $f(x,y) = x + 4y + \dfrac{2}{xy}$.

$f_x = 1 - \frac{2}{x^2y} = 0$ $\quad f_y = 4 - \frac{2}{xy^2} = 0$

$x^2y = 2$ $\quad xy^2 = \frac{1}{2}$ $\quad x^2y = 4xy^2$ $\quad x = 4y$ $\quad \begin{cases} x = 2 \\ y = \frac{1}{2} \end{cases}$

a) (10 pts.) Find the critical point(s) of $f(x,y)$

local min.

b) (5 pts.) Use the second-derivative test to test the critical point(s) found in part(a).

$f_{xx} = \frac{4}{x^3y} = 1$ $\quad f_{yy} = \frac{4}{xy^3} = 16$ $\quad f_{xy} = \frac{2}{x^2y^2} = 2$ $\quad 16 - 4 > 0$ $\quad 1 > 0$

Problem 7. (10 points)

Let $\mathcal{P}$ be the plane with equation $Ax + By + Cz = D$ and $P_0 = (x_0, y_0, z_0)$ be a point which is *not* on $\mathcal{P}$.

Use the Lagrange multiplier method to *set up* the equations satisfied by the point (x,y,z) on $\mathcal{P}$ which is *closest* to P_0. (Do not solve.)

$f(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$

$$\begin{cases} 2(x-x_0) = \lambda A \\ 2(y-y_0) = \lambda B \\ 2(z-z_0) = \lambda C \\ D \end{cases}$$

Problem 8. (15 points)

a) (10 pts.) Let $F(x,y,z)$ be a smooth function of three variables for which $\nabla F(1,-1,\sqrt{2}) = \langle 1, 2, -2\rangle$.

$F_x \frac{dx}{d\phi} + F_y \frac{dy}{d\phi} + F_z \frac{dz}{d\phi} = \rho\cos\phi\cos\theta - \rho\cos\phi\cos\theta - \sqrt{2}\rho\sin\phi = -2$

Use the Chain Rule to evaluate $\dfrac{\partial F}{\partial \phi}$ at $(\rho, \phi, \theta) = (2, \frac{\pi}{4}, -\frac{\pi}{4})$.

(Use $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.)

b) (5 pts.) Suppose $f(x,y)$ is a smooth, non-constant function. Is it possible that, for *all* points (x,y), the gradient of f at the point (x,y) is equal to the vector $\langle -y, x\rangle$? Justify (briefly).

$f_x = -y$ $\quad f_y = x$

$f = -xy + g(y)$ $\quad f_y = -x + g' = x$

$g' = 2x$ $\quad g = 2xy + C$ (not a function of y)

not possible.

$\langle -y, x\rangle$ not conservative

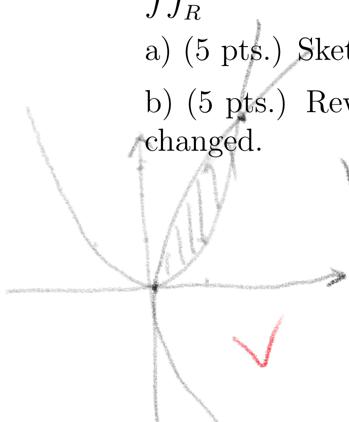
Problem 9. (10 points)

$$\iint_R f\, dA = \int_0^2 \int_{x^2}^{2\sqrt{2x}} f(x,y)\, dy\, dx\ .$$

a) (5 pts.) Sketch the region R.

b) (5 pts.) Rewrite the double integral as an iterated integral with the order interchanged.

$y = x^2 \sim 8x = y^2$ $\quad x : \frac{1}{8}y^2 \sim \sqrt{y}$ $\quad y : 0 \sim 4$



$$\int_0^4 \int_{\frac{1}{8}y^2}^{\sqrt{y}} f(x,y)\, dx\, dy$$

Problem 10. (15 points)

Set up the integral $\displaystyle\iint_R f(x,y)\,dA$ where R is the region bounded by the four curves $x^2y = 4$, $x^2y = 9$, $\frac{y}{x} = 1$, and $\frac{y}{x} = 2$ as a double integral in the variables $u = x^2y$ and $v = \frac{y}{x}$. (Your answer should be completely ready to integrate, once the function f is given.)

$g(u,v) = f(x,y)$

Note: the inverse transformation is given by $x = u^{\frac13}\,v^{-\frac13}$, $y = u^{\frac13}\,v^{\frac23}$.

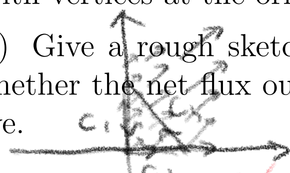
$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2xy & x^2 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 3y = 3u^{\frac13}v^{\frac23}$

$\int_4^9\int_1^2 g(u,v)\,\frac{1}{3u^{\frac13}v^{\frac23}}\,du\,dv$ ✓

Problem 11. (15 points)

$\mathbf{F}(x,y) = x\,(\mathbf{i} + \mathbf{j})$, and let C be the closed curve in the xy-plane formed by the triangle with vertices at the origin and the points (1,0) and (0,1).

a) (5 pts.) Give a rough sketch of the field $\mathbf{F}$ in the first quadrant, and use it to predict whether the net flux out of the region R = the interior of C will be positive or negative.



positive ✓

C_2 $\langle 0,-1\rangle\cdot\langle x,x\rangle = -x$

$\int_0^1 -x\,dx = -\frac12$

b) (5 pts.) Compute the flux integral $\displaystyle\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds$ directly.

C_3 $\langle\frac{1}{\sqrt2},\frac{1}{\sqrt2}\rangle\cdot\langle x,x\rangle = \sqrt2 x$

(Specify which orientation you are using for C.)

$\int_0^1 \sqrt2 x\cdot\sqrt2\,dx = 1$ $\qquad \oint_C F\cdot\hat n\,ds = \frac12$ ✓

c) (5 pts.) Compute the flux integral $\displaystyle\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds$ using the appropriate *double* integral. (Set up, then using short-cut is ok.)

$\oint_C F\cdot\hat n\,ds = \iint_S \operatorname{curl} F\,dA = \iint_S 1\,dA = \frac12$ ✓

Problem 12. (20 points)

Let G be the solid 3-D cone bounded by the lateral surface given by $z = 2\sqrt{x^2+y^2}$ and by the plane $z = 2$. The problem is to compute $\bar{z}$ = the z-coordinate of the center of mass of G, in the case where the density is equal to the height above the xy-plane.

$z = 2r$

$dV = r\,d\theta\,dr\,dz$ $\qquad \rho = z$

a) (5 pts.) Find the mass of G using cylindrical coordinates

a. $\int_0^{2\pi}\int_0^2\int_0^{\frac z2} r z\,dr\,dz\,d\theta$

inner $= \frac12 r^2 z\Big|_0^{\frac z2} = \frac{z^3}{8}$

mid $= \frac{z^4}{32}\Big|_0^2 = \frac12$

outer $= \frac12\theta\Big|_0^{2\pi} = \pi$ ✓

(b) (5 pts.) Set up the calculation for $\bar{z}$ using cylindrical coordinates (Answers should be ready to integrate out – but *do not evaluate.*)

(c) (10 pts.) Set up the calculation for $\bar{z}$ using spherical coordinates. (Answers should be ready to integrate out – but *do not evaluate.*)

b. $\frac{1}{\pi}\int_0^{2\pi}\int_0^2\int_0^{\frac z2} z^2 r\,dr\,dz\,d\theta$ ✓

$z = \rho\cos\phi = 2$ $\quad \rho = \frac{2}{\cos\phi}$

$\tan\phi = \frac{r}{z} = \frac12$ $\quad \phi = \arctan\frac12$

$\frac{1}{\pi}\int_0^{2\pi}\int_0^{\arctan\frac12}\int_0^{\frac{2}{\cos\phi}} \rho^4\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta$ ✓

$dV = \rho\,d\phi\cdot\rho\sin\phi\,d\theta\cdot d\rho = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Problem 13. (15 points)

$\mathbf{F}(x,y,z) = \left(y + y^2z\right)\mathbf{i} + (x - z + 2xyz)\,\mathbf{j} + \left(-y + xy^2\right)\mathbf{k}$

a) (3 pts.) Show that $\mathbf{F}(x,y,z)$ is a gradient field using the derivative conditions.

b) (10 pts.) Find a potential function $f(x,y,z)$ for $\mathbf{F}(x,y,z)$, using any *systematic* method. Show the method used and all work clearly.

$f(1,-1,2) - f(2,2,1) = 3 - 10 = -7$ ✓

c) (2 pts.) Find $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the straight line joining the points (2, 2, 1) and (1, -1, 2) (in that order), using as little computation as possible.

a. $X_y = 1 + 2yz = Y_x = 1 + 2yz$ $\qquad f = yx + y^2zx + g(y,z)$

$Y_z = -1 + 2xy = Z_y = -1 + 2xy$ $\qquad f_y = x + 2xyz + g(y,z)_y = x - z + 2xyz$

$Z_x = y^2 = X_z = y^2$ ✓ $\qquad f_z = y^2x + g(y,z)_z = -y + xy^2$

$g(y,z) = -zy + h(y) = g(y,z) = -yz + i(z)$ $\qquad f = xy + xy^2z - yz$ ✓

$\hat{n} = \frac{\langle x, y \rangle}{2}$ $\quad F \cdot \hat{n} = \frac{x^2}{2} = 2\cos^2\theta \quad dS = 2\,d\theta\,dz \quad \int_0^4 \int_0^{\frac{\pi}{2}} 4\cos^2\theta \, d\theta \, dz$

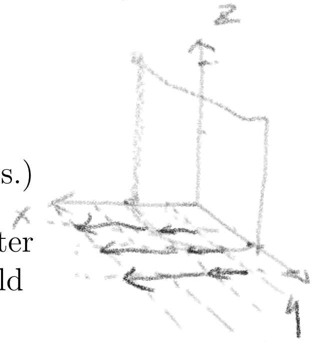
inner: $\int_0^{\frac{\pi}{2}} 2(1+\cos 2\theta)\,d\theta = 4\theta + 2\sin 2\theta \Big|_0^{\frac{\pi}{2}} = \pi$

outer: $2\pi z \Big|_0^4 = 4\pi$

Problem 14. (25 points)
In this problem S is the surface given by the quarter of the right-circular cylinder centered on the z-axis, of radius 2 and height 4, which lies in the first octant. The field $\mathbf{F}(x, y, z) = x\,\mathbf{i}$.

$\sin\theta\cos\theta$

a) (5 pts.) Sketch the surface S and the field $\mathbf{F}$. $\quad -\sin^2\theta + \cos^2\theta$
(Suggestion: use a coordinate system with y pointing out of the paper.)



b) (10 pts.) Compute the flux integral $\displaystyle\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$.
(Use the normal which points 'outward' from S, i.e. on the side away from the z-axis.)

c) (5 pts.) G be the 3D solid in the first octant given by the interior of the quarter cylinder defined above. Use the divergence theorem to compute the flux of the field $\mathbf{F} = x\,\mathbf{i}$ out of the region G.

$\iiint_G \operatorname{div}\vec{F}\,dV = \frac{\pi r^2}{4} h = 4\pi$

d) (5 pts.) The boundary surface of G is comprised of S together with four other faces. What is the flux outward through these four faces, and why ? Use the answers to parts(b) and (c), and also verify using the sketch of part(a).

0

Problem 15. (25 points)
$\mathbf{F}(x, y, z) = (yz)\,\mathbf{i} + (-xz)\,\mathbf{j} + \mathbf{k}$. Let S be the portion of surface of the paraboloid $z = 4 - x^2 - y^2$ which lies above the first octant; and let C be the closed curve $C = C_1 + C_2 + C_3$, where the curves C_1, C_2 and C_3 are the three curves formed by intersecting S with the xy, yz and xz planes respectively (so that C is the boundary of S). Orient C so that it is traversed CCW when seen from above in the first octant.

a) (15 pts.) Use Stokes' Theorem to compute the loop integral $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$ by using the *surface integral* over the capping surface S.

inner: $r^4 - 4r^2 \Big|_0^2 = 0$

$= \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \int_0^{\frac{\pi}{2}} \int_0^2 (4r^2 - 8)\, r\,dr\,d\theta \quad$ outer: 0

b) (10 pts.) Set up and evaluate the loop integral $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly by parametrizing each piece of the curve C and then adding up the three line integrals.

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & -xz & 1 \end{vmatrix} = \langle x, y, -2z \rangle$$

$x: 0 \sim \sqrt{4-y^2}$

$\vec{S} = \langle 2x, 2y, 1 \rangle\, dx\,dy \quad y: 0 \sim 2$

$\operatorname{curl}\vec{F} \cdot d\vec{S} = (2x^2 + 2y^2 - 2(4 - x^2 - y^2))\,dx\,dy$
$= (4x^2 + 4y^2 - 8)\,dx\,dy$
$= (4r^2 - 8)\, r\,d\theta\,dr$

$C_1: \begin{cases} x = 2\sin\theta \\ y = 2\cos\theta \\ z = 0 \end{cases}$

$\vec{F} \cdot d\vec{r} = 0$

$C_2 \begin{cases} x = 0 \\ y = t \\ z = 4 - t^2 \end{cases} \quad t: 0 \sim 2$

$\vec{F} \cdot d\vec{r} = -2t$

$C_3 \begin{cases} x = t \\ y = 0 \\ z = 4 - t^2 \end{cases} \quad t: 2 \sim 0$

$\vec{F} \cdot \vec{dr} = -2t$

$\int_{C_3} = -\int_{C_2} \quad \int_{C_1 + C_2 + C_3} = 0$