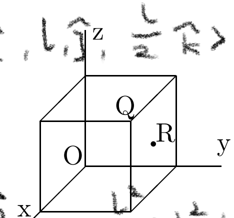
18.02 Multivariable Calculus
Fall 2007

18.02 Practice Exam 1 A

Problem 1. (15 points) 1.a. $\overrightarrow{OQ} = \langle l\hat{i}, l\hat{j}, l\hat{k}\rangle$ $\quad \overrightarrow{OR} = \langle \frac{l}{2}\hat{i}, l\hat{j}, \frac{l}{2}\hat{k}\rangle$



A unit cube lies in the first octant, with a vertex at the origin (see figure).

a) Express the vectors $\overrightarrow{OQ}$ (a diagonal of the cube) and $\overrightarrow{OR}$ (joining O to the center of a face) in terms of $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

b) Find the cosine of the angle between OQ and OR.

1.b. $\cos\theta = \dfrac{\overrightarrow{OQ}\cdot\overrightarrow{OR}}{|\overrightarrow{OQ}|\,|\overrightarrow{OR}|} = \dfrac{\frac{l^2}{2}+l^2+\frac{l^2}{2}}{\sqrt{3l^2\cdot\frac{3}{2}l^2}} = \dfrac{2\sqrt{2}}{3}$

Problem 2. (10 points) $\vec{V} = \dfrac{d\vec{R}}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + \hat{k}$

The motion of a point P is given by the position vector $\vec{R} = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + t\,\hat{k}$. Compute the velocity and the speed of P.

$|\vec{V}| = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$

Problem 3. (15 points: 10, 5)

3.a. $\text{min} = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -2 & -2 \\ -3 & -5 & -6 \end{bmatrix}$ $\quad \text{cof} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 2 \\ -3 & 5 & -6 \end{bmatrix}$ $\quad \text{adj} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$

a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ then $\det(A) = 2$ and $A^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & a & b \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$; find a and b. $\quad 2 \quad -3$

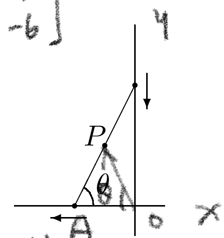
b) Solve the system $A\,X = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. $X = \frac{1}{2}\begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -6 \\ 8 \\ -8 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$

c) In the matrix A, replace the entry 2 in the upper-right corner by c. Find a value of c for which the resulting matrix M is not invertible. $1\times 1 + 3\times(-1) + c\times 2 = 2c - 2 = 0 \quad c = 1$

For this value of c the system $M\,X = 0$ has other solutions than the obvious one $X = 0$: find such a solution by using vector operations. (*Hint:* call U, V and W the three rows of M, and observe that $M\,X = 0$ if and only if X is orthogonal to the vectors U, V and W.)

$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{vmatrix} i & j & k \\ 1 & 3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = -3i + 3j - 6k = \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix}$

Problem 4. (15 points)



The top extremity of a ladder of length L rests against a vertical wall, while the bottom is being pulled away. Find parametric equations for the midpoint P of the ladder, using as parameter the angle θ between the ladder and the horizontal ground.

$\overrightarrow{OP} = \langle -\frac{\cos\theta\, L}{2}, \frac{\sin\theta\, L}{2}\rangle$

Problem 5. (25 points: 10, 5, 10) $\quad \overrightarrow{P_0P_1} = \langle -1, -1, 1\rangle \quad \overrightarrow{P_0P_2} = \langle 0, -2, 1\rangle$

5.a. $\overrightarrow{P_0P_1}\times\overrightarrow{P_0P_2} = \begin{vmatrix} i & j & k \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 1i + 1j + 2k$ $\quad$ area $= \dfrac{\sqrt{6}}{2}$

a) Find the area of the space triangle with vertices $P_0 : (2, 1, 0)$, $P_1 : (1, 0, 1)$, $P_2 : (2, -1, 1)$.

b) Find the equation of the plane containing the three points P_0, P_1, P_2. $\quad x + y + 2z = 3$

c) Find the intersection of this plane with the line parallel to the vector $\vec{V} = \langle 1, 1, 1\rangle$ and passing through the point $S : (-1, 0, 0)$.

$\langle x+1, y, z\rangle = t\langle 1, 1, 1\rangle \quad t - 1 + t + 2t = 3 \to t = 1$

$\begin{cases} x = t - 1 \\ y = t \\ z = t \end{cases}$

Problem 6. (20 points: 5, 5, 10)

a) Let $\vec{R} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be the position vector of a path. Give a simple intrinsic formula for $\dfrac{d}{dt}(\vec{R}\cdot\vec{R})$ in vector notation (not using coordinates).

$\dfrac{d}{dt}(\vec{R}\cdot\vec{R}) = \vec{R}\cdot\vec{V} + \vec{V}\cdot\vec{R} = 2\vec{R}\cdot\vec{V}$

b) Show that if $\vec{R}$ has constant length, then $\vec{R}$ and $\vec{V}$ are perpendicular. $\quad 2\vec{R}\cdot\vec{V} = 0 \quad \perp$

c) let $\vec{A}$ be the acceleration: still assuming that $\vec{R}$ has constant length, and using vector differentiation, express the quantity $\vec{R}\cdot\vec{A}$ in terms of the velocity vector only.

$\dfrac{d}{dt}(\vec{R}\cdot\vec{V}) = \vec{R}\cdot\vec{A} + \vec{V}\cdot\vec{V} = 0$

$\vec{R}\cdot\vec{A} = -\vec{V}\cdot\vec{V}$