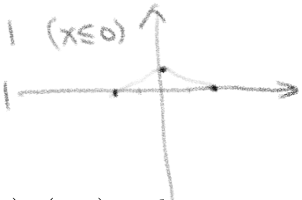
18.02 Multivariable Calculus
Fall 2007

$\bar{y} = \frac{1}{2}\left(\int_{-2}^{0}\int_{0}^{\frac{1}{2}x+1} y\,dA + \int_{0}^{2}\int_{0}^{-\frac{1}{2}x+1} y\,dA\right)$ ✓

$\begin{cases} y = \frac{1}{2}x+1 & (x \le 0) \\ y = -\frac{1}{2}x+1 \end{cases}$



$\bar{x} = 0$ ✓

18.02 Practice Exam 3 A

Double Integral

1. Let $(\bar{x}, \bar{y})$ be the center of mass of the triangle with vertices at $(-2, 0)$, $(0, 1)$, $(2, 0)$ and uniform density $\delta = 1$.

a) (10) Write an integral formula for $\bar{y}$. Do not evaluate the integral(s), but write explicitly the integrand and limits of integration.

b) (5) Find $\bar{x}$.

$\int_0^1 \sin\theta\, r^4\,dr = \frac{1}{5}\sin\theta$

Polar Coord

2. (15) Find the polar moment of inertia of the unit disk with density equal to the distance from the y-axis.

$2\int_0^{\pi}\int_0^1 \sin\theta\, r^3\, r\,dr\,d\theta = 2\int_0^{\pi}\frac{1}{5}\sin\theta\,d\theta = \frac{2}{5}(-\cos\theta)\Big|_0^{\pi} = \frac{4}{5}$ ✓

Conservative Vector Field

3. Let $\vec{F} = (ax^2y + y^3 + 1)\hat{\mathbf{i}} + (2x^3 + bxy^2 + 2)\hat{\mathbf{j}}$ be a vector field, where a and b are constants.

a) (5) Find the values of a and b for which $\vec{F}$ is conservative.

$M_y - N_x = ax^2 + 3y^2 - 6x^2 - by^2 = 0$
$a = 6 \quad b = 3$ ✓

b) (5) For these values of a and b, find $f(x, y)$ such that $\vec{F} = \nabla f$.

$f = \int f_x\,dx = 2x^3y + xy^3 + x + g(y) \quad f_y = 2x^3 + 3xy^2 + g'(y) \quad g'(y) = 2 \quad g(y) = 2y + c \quad f = 2x^3y + xy^3 + x + 2y + c$ ✓

c) (5) Still using the values of a and b from part (a), compute $\int_C \vec{F} \cdot d\vec{r}$ along the curve C such that $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

$C: (-e^{\pi}, 0) \to (1, 0) \quad f(1,0) - f(-e^{\pi}, 0) = (1 + c) - (-e^{\pi} + c) = e^{\pi} + 1$ ✓

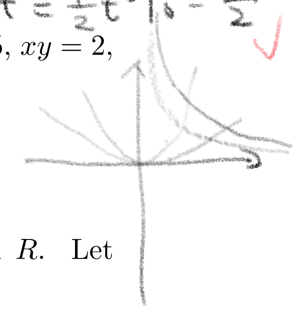
Line Integral: Parametric

4. (10) For $\vec{F} = yx^3\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}$, find $\int_C \vec{F} \cdot d\vec{r}$ on the portion of the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$.

$M_y = x^3 \quad N_x = 0 \quad$ let $x = t,\ y = t^2 \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 t^5\,dt + 2t^5\,dt = \frac{1}{2}t^6\Big|_0^1 = \frac{1}{2}$ ✓

5. Consider the region R in the first quadrant bounded by the curves $y = x^2$, $y = x^2/5$, $xy = 2$, and $xy = 4$.

$J = \begin{vmatrix} 2x/y & -x^2/y^2 \\ y & x \end{vmatrix} = 2x^2/y + x^2/y = 3x^2/y \quad dx\,dy = \frac{y}{3x^2}\,du\,dv = \frac{1}{3u}\,du\,dv$ ✓



Change of Variables

a) (10) Compute $dxdy$ in terms of $dudv$ if $u = x^2/y$ and $v = xy$.

b) (10) Find a double integral for the area of R in uv coordinates and evaluate it.

$\int_2^4\int_1^5 \frac{1}{3u}\,du\,dv = \int_2^4 \frac{1}{3}\ln u\Big|_1^5\,dv = \frac{2}{3}\ln 5$ ✓

Green Theorem

6. a) (5) Let C be a simple closed curve going counterclockwise around a region R. Let $M = M(x, y)$. Express $\oint_C M\,dx$ as a double integral over R.

$-\iint_R M_y\,dA$ ✓

b) (5) Find M so that $\oint_C M\,dx$ is the mass of R with density $\delta(x, y) = (x + y)^2$.

$-M_y = x^2 + 2xy + y^2 \quad M = -x^2y - xy^2 - \frac{1}{3}y^3 - c$ ✓

Green Normal Theorem

7. Consider the region R enclosed by the x-axis, $x = 1$ and $y = x^3$.

a) (5) Use the normal form of Green's theorem to find the flux of $\vec{F} = (1 + y^2)\hat{\mathbf{j}}$ out of R.

b) (5) Find the flux out of R through the two sides C_1 (the horizontal segment) and C_2 (the vertical segment).

c) (5) Use parts (a) and (b) to find the flux out of the third side C_3.

a. $M_x + N_y = 2y \qquad \int_0^1\int_0^{x^3} 2y\,dy\,dx = \int_0^1 x^6\,dx = \frac{1}{7}$ ✓

b. C_1: $-\int_0^1 1\,dx = -1 \qquad C_2$: 0 ✓

c. $C_1 + C_2 + C_3 = \frac{1}{7} \qquad C_3 = \frac{8}{7}$ ✓