UNIT3. DOUBLE INTEGRALS AND LINE INTEGRALS IN THE PLANE

L16. Double integrals

-Geometric Interpretation

double integral $\iint_R f(x,y)dA$ = volume below graph $z = f(x,y)$ over plane region R

-Method (Iterated Integral)

By taking slices: $S(x) =$ area of the slice by a plane parallel to yz -plane

$$\text{volume } = \int_{x_{\min}}^{x_{\max}} S(x)dx, \quad \text{and for given } x, S(x) = \int_{y_{min}(x)}^{y_{max}(x)} f(x,y)dy$$

$$\iint_R f(x,y)dA = \int_{x_{\min}}^{x_{\max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)dy \right] dx$$

-Exchanging order of Integration

e.g. $\int_0^1 \int_0^2 dxdy = \int_0^2 \int_0^1 dydx$ (since regions is a rectangle)

e.g. $\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y} dydx = \int_0^1 \int_{y^2}^{y} \frac{e^y}{y} dxdy$ (region: $x < y < \sqrt{x}$ for $0 \le x \le 1$)

L17. Double integrals in polar coordinates; applications

-Integration in Polar Coordinates ($x = r\cos\theta, y = r\sin\theta$ **)**

-Area Element: $dA = r\,dr\,d\theta$

e.g. $\iint_{x^2+y^2\le 1, x\ge 0, y\ge 0} \left(1 - x^2 - y^2\right) dxdy = \int_0^{\pi/2} \int_0^1 \left(1 - r^2\right) rdrd\theta$

-Applications

-Area of region R: $\iint_R 1dA$ (solid disk with unit thickness)

-Mass: $M = \iint_R \delta dA$ (δ: density - mass per unit area)

-Centroid: $\bar{x} = \dfrac{1}{mass} \iint_R x\delta dA, \quad \bar{y} = \dfrac{1}{mass} \iint_R y\delta dA$

-Moment of Inertia: $I_0 = \iint_R r^2 \delta dA$

<u>L18. Change of variables</u>

-Change of Variables

<u>Motivation: to simplify either integrand or bounds of integration</u>

e.g. area of ellipse with semiaxes a and b : setting $u = x/a, v = y/b$

$$\iint_{(x/a)^2+(y/b)^2<1} dxdy = \iint_{u^2+v^2<1} ab\, dudv = ab \iint_{u^2+v^2<1} dudv = \pi ab$$

-Scale Factor

approximation formula $\Delta u \approx u_x \Delta x + u_y \Delta y, \Delta v \approx v_x \Delta x + v_y \Delta y$

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

-Definition

Jacobian: $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$dudv = |J| dxdy$$

-Remark

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

L19. Vector fields and line integrals in the plane

-Vector Field

$\overrightarrow{F} = M\hat{\imath} + N\hat{\jmath},$ where $M = M(x, y), N = N(x, y)$:

at each point in the plane we have a vector $\overrightarrow{F}$ which depends on x, y

e.g. wind flow; velocity in a fluid $\overrightarrow{v}$; force field $\overrightarrow{F}$

-Work and line integrals

$$W = \int_C \overrightarrow{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \left(\overrightarrow{F} \cdot \frac{d\vec{r}}{dt} \right) dt$$

-Notation: $\overrightarrow{F} = \langle M, N \rangle,$ **and** $d\vec{r} = \langle dx, dy \rangle$

$$= \int_C M dx + N dy$$

-Geometric approach: $d\vec{r} = \hat{\boldsymbol{T}} ds$

($s =$ arclength, $\hat{\boldsymbol{T}} =$ unit tangent vector to trajectory)

$$= \int_C \overrightarrow{F} \cdot \hat{\boldsymbol{T}} ds$$

(when the field and the curve are relatively simple and have a geometric relation)

L20. Path independence and conservative fields

-Fundamental theorem of calculus for line integrals

If $\overrightarrow{F}$ is a gradient field, $\overrightarrow{F} = \nabla f = f_x \hat{\boldsymbol{\imath}} + f_y \hat{\boldsymbol{\jmath}}$ (f: "potential function")

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$$

Equivalent: $\displaystyle\int_C f_x dx + f_y dy = \int_C df = f\left(P_1\right) - f\left(P_0\right)$

-Proof

$$\int_C \nabla f \cdot d\vec{r} = \int_{t_0}^{t_1} \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \right) dt = \int_{t_0}^{t_1} \frac{d}{dt}(f(x(t), y(t))dt$$
$$= [f(x(t), y(t))]_{t_0}^{t_1} = f\left(P_1\right) - f\left(P_0\right)$$

WARNING: this is only true for gradient fields!

-Equivalent Properties

1. $\overrightarrow{F}$ is conservative ($\int_C \overrightarrow{F} \cdot d\vec{r} = 0$ for any closed curve C)

2. $\int F \cdot d\vec{r}$ is path independent (same work if same end points)

3. $\overrightarrow{F}$ is a gradient field: $\overrightarrow{F} = \nabla f = f_x \hat{\imath} + f_y \hat{\jmath}$.

4. $Mdx + Ndy$ is an exact differential ($= f_x dx + f_y dy = df$)

L21. Gradient fields and potential functions

-Test for Gradient Field

if $\overrightarrow{F} = M\hat{\boldsymbol{\imath}} + N\hat{\jmath}$ is a gradient field, $M = f_x,\ N = f_y,\ soN_x = f_{yx} = f_{xy} = M_y$

Conversely, if $\overrightarrow{F}$ is defined and differentiable at every point of the plane, and $N_x = M_y$

$\Rightarrow \quad \overrightarrow{F} = M\hat{\boldsymbol{\imath}} + N\hat{\jmath}$ is a gradient field.

-Potential Function f

-Method 1: line integrals

e.g. $f\left(x_1, y_1\right) - f(0,0) = \int_C \overrightarrow{F} \cdot d\vec{r}$ ***independent of path***

$$\int_{(0,0)}^{(x_1,y_1)} = \int_{(0,0)}^{(x_1,0)} + \int_{(x_1,0)}^{(x_1,y_1)}$$

-Method 2: antiderivatives

e.g. $f_x,\ f_y$

$$f = \int f_x dx + g(y) \equiv f^{(x)} + g(y) \quad \Rightarrow \quad f_y = f_y^{(x)} + g'(y)$$

compare $f_y^{(x)} + g'(y)$ with f_y to get $g'(y)$ thus $g(y)$

-Curl

Failure of conservativeness is given by the $curl$ of $\overrightarrow{F}$:

Definition: $\text{curl}(\overrightarrow{F}) = N_x - M_y$

-Interpretation

for a velocity field, curl $\overrightarrow{v}$ = (twice) angular velocity of the rotation component of the motion

e.g. $\overrightarrow{v} = <-y, x>$, angular velocity is 1, curl = 2

For a force field, curl $\overrightarrow{F}$ = torque exerted on a test mass, measures how F imparts rotation motion

<u>L22. Green's theorem</u>

-Green's Theorem

If C is a positively oriented closed curve enclosing a region R,

and $\overrightarrow{F}$ defined and differentiable in R:

$$\oint_C \overrightarrow{F} \cdot d\vec{r} = \iint_R \text{curl } \overrightarrow{F} dA \quad \Leftrightarrow$$

$$\oint_C M dx + N dy = \iint_R \left(N_x - M_y\right) dA$$

(reduce a complicated line integral to an easy $\iint$)

-Application

proof of the test for gradient field

$$N_x = M_y \quad \Rightarrow \quad \oint_C \overrightarrow{F} \cdot d\vec{r} = \iint_R \text{curl } \overrightarrow{F} dA = \iint_R 0 dA = 0 \quad \Rightarrow conservative$$

-Proof

see Lecture note

-e.g. Planimeter

L23. Flux; normal form of Green's theorem

-Flux (another line integral)

$$\int_C \overrightarrow{F} \cdot \hat{\boldsymbol{n}} ds$$

$\hat{n}$ = normal vector to C, rotated $90°$ clockwise from $\hat{T}$

-Physical Interpretation

if $\overrightarrow{F}$ is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through C per unit time.

-Calculation e.g. $\overrightarrow{F} = P\hat{\imath} + Q\hat{\jmath}$

$$\hat{n} ds = < dy, -dx >$$

$$\int_C \overrightarrow{F} \cdot \hat{\boldsymbol{n}} ds = \int_C \langle P, Q\rangle \cdot \langle dy, -dx\rangle = \int_C -Qdx + Pdy$$

-Green's theorem for flux (Green's theorem in normal form (vs. tangential))

If C encloses R counterclockwise, and $\overrightarrow{F} = P\hat{\imath} + Q\hat{\jmath}$

$$\oint_C \overrightarrow{F} \cdot \hat{\boldsymbol{n}} ds = \iint_R \operatorname{div}(\overrightarrow{F}) dA,$$ where $\operatorname{div}(\overrightarrow{F}) = P_x + Q_y$ is the divergence of $\overrightarrow{F}$

(Note: the counterclockwise orientation of C means that we count flux of $\overrightarrow{F}$ out of R through C.)

-Proof (using Green's theorem for potential)

$$\oint_C \overrightarrow{F} \cdot \hat{\boldsymbol{n}} ds = \oint_C -Qdx + Pdy = \iint_R \left(P_x - \left(-Q_y\right)\right) dA = \iint_R \operatorname{div}(\overrightarrow{F}) dA$$

-Physical Interpretation of Divergence

1. measures how much the flow is "expanding"

2. “source rate” e.g. amount of fluid added to the system per unit time and per unit area

L24. Simply connected regions; review

-Simply Connected

a region R in the plane is simply connected if, given any closed curve in R, its interior region is entirely contained in R.

If domain, where F is defined and differentiable, is simply connected, then can always apply Green’s theorem