UNIT4. TRIPLE INTEGRALS AND SURFACE INTEGRALS IN 3-SPACE

L25. Triple integrals in rectangular and cylindrical coordinates

$$\iiint_R f dV$$

e.g. $z = x^2 + y^2$ **and** $z = 4 - x^2 - y^2$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dzdydx$$

-Polar Coordinates: $x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$

(Cylindrical Coordinates)

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dzrdrd\theta$$

e.g. volume of region where $z > 1 - y$ and $x^2 + y^2 + z^2 < 1$

-Bounds

$$1 - y < z < \sqrt{1 - x^2 - y^2}$$

$$(1-y)^2 < 1 - x^2 - y^2 \quad \Rightarrow \quad -\sqrt{2y - 2y^2} < x < \sqrt{2y - 2y^2}$$

$$2y - 2y^2 > 0 \quad \Rightarrow \quad 0 < y < 1$$

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dzdxdy$$

L26. Spherical coordinates; surface area

-Spherical Coordinates (ρ, ϕ, θ **)**

ρ = radius, ϕ = angle down from z-axis

-Formulas

$$z = \rho\cos\phi, r = \rho\sin\phi, x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$$

-Volume element

$$dV = \rho^2\sin\phi d\rho d\phi d\theta$$

-Last e.g.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{1}{\sqrt{2}}}^{1}\sec\phi^2\sin\phi d\rho d\phi d\theta$$

-Application to gravitation

$$|\overrightarrow{F}| = \frac{G\Delta Mm}{\rho^2}, \mathrm{dir}(\overrightarrow{F}) = \frac{\langle x,y,z\rangle}{\rho}, \text{ i.e. } \overrightarrow{F} = \frac{G\Delta Mm}{\rho^3}\langle x,y,z\rangle$$

$$\overrightarrow{F} = \iiint_R \frac{Gm\langle x,y,z\rangle}{\rho^3}\delta dV, \quad \text{i.e. } z \text{ -component is} \quad F_z = Gm\iiint_R \frac{z}{\rho^3}\delta dV$$

<u>L27. Vector fields in 3D; surface integrals and flux</u>

-Vector fields in space

$\overrightarrow{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{\boldsymbol{k}},$ where P, Q, R are functions of x, y, z

Force field: Gravitational force; Electric field; Magnetic field; Velocity fields (fluid flow)

Gradient field: Temperature gradient; Pressure gradient

-Flux

$$\text{Flux} = \iint \overrightarrow{F}\cdot\hat{\boldsymbol{n}}dS = \iint \overrightarrow{F}\cdot d\overrightarrow{S}$$

($d\overrightarrow{S}$ is often easier to compute than $\hat{\boldsymbol{n}}$ and dS)

-Setup of dS and selection of two variables to describe the surface

1. $z = a$: $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{k}}, dS = dxdy$

2. $sphere$: $\hat{\boldsymbol{n}} = \dfrac{1}{a}\langle x,y,z\rangle, dS = a^2\sin\phi d\phi d\theta$

3. $cylinder: \quad \hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, 0\rangle, dS = a\,dz\,d\theta$

4. $z = f(x,y): \quad \overrightarrow{S} = \hat{\boldsymbol{n}}\,dS = \left\langle -f_x, -f_y, 1 \right\rangle dx\,dy$

L28. Flux (cont.); Divergence theorem

- S: graph of $z = f(x,y)$

$$\Delta\overrightarrow{S} = \left\langle \Delta x, 0, \Delta x f_x \right\rangle \times \left\langle 0, \Delta y, \Delta y f_y \right\rangle = \left(\Delta x \left\langle 1, 0, f_x \right\rangle\right) \times \left(\Delta y \left\langle 0, 1, f_y \right\rangle\right)$$

$$= \Delta x \Delta y \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \left\langle -f_x, -f_y, 1 \right\rangle \Delta x \Delta y$$

$$d\overrightarrow{S} = \pm \left\langle -f_x, -f_y, 1 \right\rangle dx\,dy$$

$$\hat{\boldsymbol{n}} = \mathrm{dir}(d\overrightarrow{S}) = \frac{\left\langle -f_x, -f_y, 1 \right\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}; \quad dS = |d\overrightarrow{S}| = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

-Parametric Surfaces

e.g. $\vec{r} = \vec{r}(u, v)$

$$\Delta\overrightarrow{S} = \pm\left(\frac{\partial \vec{r}}{\partial u}\Delta u\right) \times \left(\frac{\partial \vec{r}}{\partial v}\Delta v\right), \quad d\overrightarrow{S} = \pm\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) du\,dv$$

-Implicit surfaces

$$g(x, y, z) = 0$$

$$\mathbf{N} = \nabla g$$

$$dS = \frac{|\mathbf{N}|}{|\mathbf{N} \cdot \hat{\boldsymbol{k}}|} dA$$

$$\hat{\boldsymbol{n}}\,dS = \frac{|\mathbf{N}|\hat{\boldsymbol{n}}}{|\mathbf{N} \cdot \hat{\boldsymbol{k}}|} dx\,dy = \pm \frac{\mathbf{N}}{\mathbf{N} \cdot \hat{\boldsymbol{k}}} dx\,dy$$

-Divergence Theorem (Gauss-Green Theorem)

If S is ***a closed surface*** bounding a region D, with normal pointing outwards, and $\vec{F}$ vector field defined and differentiable over all of D, then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F} dV, \quad \text{where} \quad \operatorname{div}(P\hat{\imath} + Q\hat{\jmath} + R\hat{\boldsymbol{k}}) = P_x + Q_y + R_z$$

$\operatorname{div} \vec{F}$ = source rate = flux generated per unit volume

L29. Divergence theorem (cont.): applications and proof

-Del operator

$$\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$$

$\nabla f = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle =$ gradient

$\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z =$ divergence

-Proof of divergence theorem

Simplification: $\iint_S R\hat{\boldsymbol{k}} \cdot d\vec{S} = \iiint_D R_z dV$ (D: vertically simple)

R.H.S.: $$\iiint_D R_z dV = \iint_U \left(\int_{z_1(x,y)}^{z_2(x,y)} R_z dz \right) dx dy$$

$$= \iint_U \left(R\left(x, y, z_2(x, y)\right) - R\left(x, y, z_1(x, y)\right) dx dy\right.$$

L.H.S.: Top: $$d\vec{S} = \left\langle -\partial z_2/\partial x, -\partial z_2/\partial y, 1 \right\rangle dx dy$$

$$\iint_{\text{top}} R\hat{\boldsymbol{k}} \cdot d\vec{S} = \iint R\left(x, y, z_2(x, y)\right) dx dy$$

Bottom: $$d\vec{S} = -\left\langle -\partial z_1/\partial x, -\partial z_1/\partial y, 1 \right\rangle dx dy$$

$$\iint_{\text{bottom}} R\hat{\boldsymbol{k}} \cdot d\vec{S} = \iint -R\left(x, y, z_1(x, y)\right) dx dy$$

-Diffusion Equation (heat equation)

governs motion of smoke in (immobile) air

dye in solution

$$\frac{\partial u}{\partial t} = k\nabla^2 u = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$ (∇^2: Laplacian)

$u(x, y, z, t) =$ concentration of smoke (amount of smoke per unit)

$\mathbf{F} =$ flow of smoke

1. $\mathbf{F} = -k\nabla u$

2. $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \iiint_D \operatorname{div}\mathbf{F}dV =$ amount of smoke through S per unit time

$=$ variation of amount of smoke in D per unit time $= -\frac{d}{dt}\iiint_D u\,dV$

$= -\iiint_D \frac{\partial}{\partial t}u\,dV$

$\Rightarrow \quad \partial u/\partial t = -\operatorname{div}\mathbf{F} = +k\operatorname{div}(\nabla u) = k\nabla^2 u$

<u>L30. Line integrals in space, curl, exactness and potentials</u>

-Line integrals in space

Force field $\mathbf{F} = \langle P, Q, R\rangle,$ curve C in space, $d\vec{r} = \langle dx, dy, dz\rangle$

Work $= \int_C \mathbf{F}\cdot d\vec{r} = \int_C Pdx + Qdy + Rdz$

-e.g. $\mathbf{F} = \langle yz, xz, xy\rangle$. $\quad C : x = t^3, y = t^2, z = t. 0 \le t \le 1$

$$\int_C \mathbf{F}\cdot d\vec{r} = \int_C yzdx + xzdy + xydz = \int_0^1 6t^5 dt = 1$$

<u>(In general, express (x, y, z) in terms of a single parameter: 1 degree of freedom)</u>

-Gradient fields

$$\mathbf{F} = \langle P, Q, R \rangle = \left\langle f_x, f_y, f_z \right\rangle$$

$$\Rightarrow \quad f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy}$$

$$\Rightarrow \quad P_y = Q_x, P_z = R_x, Q_z = R_y$$

$$\int_{P_0}^{P_1} \nabla f \cdot d\vec{r} = f\left(P_1\right) - f\left(P_0\right)$$

-Potential function

e.g. $f_x = 2xy, f_y = x^2 + z^3, f_z = 3yz^2 - 4z^3$

$$f_x = 2xy \Rightarrow f(x, y, z) = x^2 y + g(y, z)$$

$$f_y = x^2 + g_y = x^2 + z^3 \Rightarrow g_y = z^3 \Rightarrow g(y, z) = yz^3 + h(z)$$

$$\Rightarrow \quad f = x^2 y + yz^3 + h(z)$$

$$f_z = 3yz^2 + h'(z) = 3yz^2 - 4z^3 \Rightarrow h'(z) = -4z^3 \Rightarrow h(z) = -z^4 + c$$

$$\Rightarrow \quad f = x^2 y + yz^3 - z^4 + c$$

-Curl: encodes by how much $\mathbf{F}$ fails to be conservative

$$\operatorname{curl}\langle P, Q, R \rangle = \left(R_y - Q_z\right) \hat{\boldsymbol{\imath}} + \left(P_z - R_x\right) \hat{\jmath} + \left(Q_x - P_y\right) \hat{\boldsymbol{k}}$$

Recall: $\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z = \operatorname{div} \mathbf{F}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \operatorname{curl} \mathbf{F}$$

-Interpretation of curl for velocity fields

e.g. $\boldsymbol{v} = \langle -\omega y, \omega x, 0 \rangle$

$$\nabla \times \boldsymbol{v} = \ldots = 0\hat{\imath} + 0\hat{\jmath} + (\omega + \omega)\hat{\boldsymbol{k}} = 2\omega \hat{\boldsymbol{k}}$$

length of curl = twice angular velocity

direction = axis of rotation

curl measures the rotation component of a complex motion

L31. Stokes' theorem

-Stokes' Theorem

if C is a closed curve, and S ***any*** surface bounded by C

$$\oint_C \mathbf{F} \cdot d\vec{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dS = \iint_S \operatorname{curl} \mathbf{F} d\overrightarrow{S}$$

-Stokes vs. Green

If S is a portion of xy-plane bounded by a curve C counterclockwise

$$\oint_C \mathbf{F} \cdot d\vec{r} = \int_C Pdx + Qdy = \iint_S \left(Q_x - P_y\right) dxdy \qquad \text{(Green)}$$

$$= \iint_S \nabla \times \mathbf{F} \cdot \hat{\boldsymbol{k}} dxdy = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}} dS \qquad \text{(Stokes)}$$

-Remark

In Stokes' theorem we are free to choose any surface S bounded by C

L32. Stokes' theorem (cont.); review

-Stokes and path independence

Definition: region R is simply connected if every closed loop C inside R bounds some surface S inside R

e.g. space w/ z-axis removed, is NOT simply connected

space w/ origin removed, is simply connected

-Theorem

if $\nabla \times \mathbf{F} = 0$ in a simply connected region then $\mathbf{F}$ is conservative

so $\int \mathbf{F} \cdot d\vec{r}$ is path-independent and we can find a potential

-Stokes and surface independence

$$\oint_C \mathbf{F} \cdot d\vec{r} = \iint_{S_1} \text{curl}\, \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_2} \text{curl}\, \mathbf{F} \cdot \hat{\mathbf{n}} dS$$

-change orientation of S_2, then $S = S_1 - S_2$ is a closed surface

-apply divergence theorem: $\iint_S \text{curl}\, \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iiint_D \text{div}(\text{curl}\, \mathbf{F}) dV = 0$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

<u>L33. Topological considerations; Maxwell's equations</u>

-Applications of div and curl to physics

e.g. for uniform rotation about z-axis, $\boldsymbol{v} = \omega(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}})$, and $\nabla \times \boldsymbol{v} = 2\omega\hat{\boldsymbol{k}}$

Curl singles out the rotation component of motion

Div singles out the stretching component

-Interpretation of curl for force fields

$$\frac{\text{Force}}{\text{Mass}} = \text{acceleration} = \frac{d}{dt}(\text{ velocity })$$

$$\frac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \frac{d}{dt}(\text{ angular velocity})$$

s.t. $\left|\text{curl}\left(\frac{\text{Force}}{\text{Mass}}\right)\right| = \left|2\frac{\text{Torque}}{\text{Moment of inertia}}\right|$

-Div and curl of electrical field