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18.02 Multivariable Calculus
Fall 2007

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$5c_1 - \sin\phi d\phi = 3\rho^2 d\rho \quad 0 \leq \rho \leq 1 \quad 0 \leq \theta \leq \pi \quad F = -\hat{i} - \hat{j} - \hat{k}$
 $\hat{n} = \frac{\langle x, y, z \rangle}{\rho} \quad \vec{F} \cdot \hat{n} = \frac{z^2}{\rho} = \rho \cos^2\phi = \rho^2 \quad \vec{F} \cdot \hat{n} \, dS = \rho^2 \sin\phi d\phi d\theta \cdot \rho^2$
 $= -3\rho^4 d\rho d\theta$

$2a.3 \quad -\frac{8}{3} \cos^3\phi \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = 2\sqrt{3}$
 $\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 8 \sin\phi \cos^2\phi d\phi d\theta = 4\pi\sqrt{3}$ ✓

Problem 1. (15) a) Show that the vector field
 $\mathbf{F} = \langle e^x yz, e^x z + 2yz, e^x y + y^2 + 1 \rangle$
 is conservative.
 b) By a systematic method, find a potential for \mathbf{F} .
 c) Show that the vector field $\mathbf{G} = \langle y, x, y \rangle$ is not conservative.

18.02 – Practice Exam 4A
1a. $X_y = e^x z = Y_x = e^x z$
 $Y_z = e^x + 2y = Z_y = e^x + 2y$
 $Z_x = e^x y = X_z = e^x y$ ✓

1b. $f = e^x yz + y^2 z + z$ ✓
1c. $Y_z = 0 \neq Z_y = 1$ ✓

Problem 2. (20) Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 4$, lying in $x^2 + y^2 > 1$, which is to say outside the cylinder of radius one with axis the z -axis.
 a) Compute the flux outward through S of the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$.
 b) Show that the flux of this vector field through any part of the cylindrical surface is zero.
 c) Using the divergence theorem applied to \mathbf{F} , compute the volume of the region between S and the cylinder.

Divergence Theorem ✓
2b. $\hat{n} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} \quad \vec{F} \cdot \hat{n} = xy - xy + 0 = 0$ ✓
2c. $\operatorname{div} \mathbf{F} = 0 + 0 + 1 = 1 \quad \operatorname{div} \mathbf{F} \cdot \text{Volume} = 4\pi\sqrt{3} - 0 \quad V = 4\pi\sqrt{3}$ ✓

Problem 3. (20) Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 2$ lying in $z > 1$. Orient S upwards and give its bounding circle, C , lying in $z = 1$ the compatible orientation.
 a) Parametrize C and use the parametrization to evaluate the line integral
 $\oint_C \vec{F} \cdot d\vec{l} = \iiint \operatorname{curl} \vec{F} d\vec{S} = 0$ ✓ **Polar Coordinates**
 $I = \int_C xz dx + ydy + zdz = \int_C \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} (z - w\sin\theta \cos\theta) = 0$ ✓
 b) Compute the curl of the vector field $\mathbf{F} = xz\mathbf{i} + y\mathbf{j} + yk$. $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle 1, x, 0 \rangle$ ✓
 c) Write down a flux integral through S which can be computed using the value of I .

Problem 4. (15) Use the divergence theorem to compute the flux of $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ outwards across the closed surface $x^4 + y^4 + z^4 = 1$. $\nabla \cdot \vec{F} = 0 \quad \iint_S \vec{F} \cdot d\vec{S} = \iiint_R \nabla \cdot \vec{F} dv = 0$ ✓

Problem 5. (15) Consider the surface S given by the equation
 $\iiint_R \operatorname{div} \vec{F} dR = \iint_S \vec{F} \cdot d\vec{S} \quad z = (x^2 + y^2 + z^2)^2$
 a) Show that S lies in the upper half space ($z \geq 0$). $z = \rho^4 \geq 0$ ✓
 b) Write out the equation for the surface in spherical polar coordinates.
 c) Using the equation obtained in part b), give an iterated integral, with explicit integrand and limits of integration, which gives the volume of the region inside this surface. Do not evaluate the integral.

Integral in 3D
 $5c.2 \quad V = \int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt[4]{1}} -3\rho^4 d\rho d\theta d\phi$?
 $x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$
 $\cos\phi = \rho^4 \quad \cos\phi = \rho^3$

Problem 6. (15) Let S be the part of the surface $z = xy$ where $x^2 + y^2 < 1$. Compute the flux of $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ upward across S by reducing the surface integral to a double integral over the disk $x^2 + y^2 < 1$.

Flux
 $\hat{n} ds = \langle -y, -x, 1 \rangle dx dy \quad x = r \sin\theta \quad y = r \cos\theta$
 $\vec{F} \cdot \hat{n} ds = (-y^2 - x^2 + xy) dx dy \quad dx dy = r dr d\theta$
 $\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (-r^2 + r^2 \sin\theta \cos\theta) r dr d\theta$

Polar coordinate substitution