18.06SC Final Exam

1 (4+7=11 pts.) Suppose A is 3 by 4, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad x_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Remembering that A is 3 by 4, find its row reduced echelon form R.

(b) Find the dimensions of all four fundamental subspaces $C(A)$, $N(A)$, $C(A^{\mathrm{T}})$, $N(A^{\mathrm{T}})$.

You have enough information to find bases for one or more of these subspaces—find those bases.

a. $\begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \rightarrow R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b. i $2, 2, 2, 1$

ii $N(A)$: x_1, x_2

$C(A^{T})$: $\begin{bmatrix} - & y_1 & - \\ - & y_2 & - \end{bmatrix} \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix} = 0$

$$\rightarrow \begin{cases} y_{11} + y_{12} + y_{13} = 0 \\ y_{21} + y_{22} + y_{23} = 0 \\ 2y_{11} + y_{12} - y_{14} = 0 \\ 2y_{21} + y_{22} - y_{24} = 0 \end{cases}$$

$$\rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

2 (6+3+2=11 pts.) (a) Find the inverse of a 3 by 3 upper triangular matrix U, with **nonzero** entries a, b, c, d, e, f. You could use cofactors and the formula for the inverse. Or possibly Gauss-Jordan elimination.

$$\text{Find the inverse of } U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

(b) Suppose the columns of U are eigenvectors of a matrix A. Show that A is also upper triangular.

(c) Explain why this U **cannot** be the same matrix as the first factor in the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$.

a. $\det U = adf$. $U^{-1} = \frac{1}{\det U} C^T$

$$= \begin{bmatrix} \frac{1}{a} & -\frac{b}{ad} & \frac{be-cd}{adf} \\ 0 & \frac{1}{d} & -\frac{e}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$$ ✓

b. $A = U\Lambda U^{-1}$ upper × diagonal × upper = upper ✓

c. not orthogonal ✓

3 (3+3+5=11 pts.) (a) A and B are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

$$\text{rank of } A \qquad\qquad \text{rank of the block matrix } \begin{bmatrix} A & B \end{bmatrix}$$

(b) Suppose $B = A^2$. How do those ranks compare? Explain your reasoning.

(c) If A is m by n of rank r, what are the dimensions of these nullspaces?

$$\text{Nullspace of } A \qquad\qquad \text{Nullspace of } \begin{bmatrix} A & A \end{bmatrix}$$

a. former $\leq$ latter ✓

b. equal. $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

rank of A^2 equals rank of A ... ✓

c. $n - r$ ✓ $\qquad 2n - r$ ✓

4 (3+4+5=12 pts.) Suppose A is a 5 by 3 matrix and Ax is never zero (except when x is the zero vector).

(a) What can you say about the columns of A?

(b) Show that $A^{\mathrm{T}}Ax$ is also never zero (except when $x = 0$) by explaining this key step:

If $A^{\mathrm{T}}Ax = 0$ then obviously $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$ and then (WHY?) $Ax = 0$.

(c) We now know that $A^{\mathrm{T}}A$ is invertible. Explain why $B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is a one-sided inverse of A (which side of A?). B is NOT a 2-sided inverse of A (*explain why not*).

a. indepent. ✓

b. $x^T A^T A x = (Ax)^T Ax = \|Ax\| = 0$ only when $Ax = 0$ ✓

c. i $BA = (A^TA)^{-1}A^TA = I$

left-sided inverse, ✓

ii A is not invertible. thus no 2-sided inverse matrix ✓

5 (5+5=10 pts.) If A is 3 by 3 symmetric positive definite, then $Aq_i = \lambda_i q_i$ with positive eigenvalues and orthonormal eigenvectors q_i.

Suppose $x = c_1q_1 + c_2q_2 + c_3q_3$.

(a) Compute $x^{\mathrm{T}}x$ and also $x^{\mathrm{T}}Ax$ in terms of the c's and λ's.

(b) Looking at the ratio of $x^{\mathrm{T}}Ax$ in part (a) divided by $x^{\mathrm{T}}x$ in part (a), what c's would make that ratio as large as possible? You can assume $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Conclusion: the ratio $x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ is a maximum when x is ______.

a. $x^T x = c_1^2 + c_2^2 + c_3^2$ ✓

$x^T A x = x^T(c_1\lambda_1 q_1 + c_2\lambda_2 q_2 + c_3\lambda_3 q_3)$

$= c_1^2\lambda_1 + c_2^2\lambda_2 + c_3^2\lambda_3$ ✓

b. i 0, 0, 1 ✓

ii $c_3 q_3$. ✓

6 (4+4+4=12 pts.) (a) Find a linear combination w of the linearly independent vectors v and u that is perpendicular to u.

(b) For the 2-column matrix $A = \begin{bmatrix} u & v \end{bmatrix}$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.

(c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v.

a. $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad (v-u) \perp u$ $\quad \times \quad w = v - \underbrace{cu}_{\text{projection}}$

b. $Q = \begin{bmatrix} \frac{u}{|u|} & \frac{v-u}{|v-u|} \end{bmatrix}$ $\quad R = \begin{bmatrix} |u| & |u| \\ 0 & |v-u| \end{bmatrix}$ $\quad \times$

c. $P = A(A^TA)^{-1}A^T$

$= QR(R^TQ^TQR)^{-1}R^TQ^T$

$= QR(R^TR)^{-1}R^TQ^T$

$= QRR^{-1}(R^T)^{-1}R^TQ^T$ ✓

$= QQ^T$

7 (4+3+4=11 pts.) (a) Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?

(c) Find the determinants of C and $C + I$ and $C + 2I$.

a. C: rank 4. trace $= 0$. det $= -1$ $\rightarrow \lambda^4 = -1$ ✗

C^2

b. $C^{-1} = C^T$ $\quad (C^2)^{-1} = (C^2)^T$ ✓

c. $\det(C) = -1$ ✓ $\quad \det(C+I) = 0$ ✓ $\quad \det(C+2I) = 15$ ✓

8 (4+3+4=11 pts.) Suppose a rectangular matrix A has independent columns.

(a) How do you find the best least squares solution $\widehat{x}$ to $Ax = b$? By taking those steps, give me a formula (letters not numbers) for $\widehat{x}$ and also for $p = A\widehat{x}$.

(b) The projection p is in which fundamental subspace associated with A? The error vector $e = b - p$ is in which fundamental subspace?

(c) Find by any method the projection matrix P onto the column space of A:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

a. $A^TA\hat{x} = A^Tb$

$\rightarrow \hat{x} = (A^TA)^{-1}A^Tb$

$\rightarrow p = A(A^TA)^{-1}A^Tb$ ✓

b. p in $C(A)$

e in $N(A^T)$ $(\perp C(A))$ ✓

c. $A = QR$ where $Q = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}$

$P = QQ^T = \frac{1}{10}\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix}$ ✓

9 (3+4+4=11 pts.) This question is about the matrices with 3's on the main diagonal, 2's on the diagonal above, 1's on the diagonal below.

$$A_1 = \begin{bmatrix} 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A_n = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}$$

(a) What are the determinants of A_2 and A_3?

(b) The determinant of A_n is D_n. Use cofactors of row 1 and column 1 to find the numbers a and b in the recursive formula for D_n:

$$(*) \qquad D_n = a\,D_{n-1} + b\,D_{n-2}\,.$$

(c) This equation $(*)$ is the same as

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}.$$

From the eigenvalues of that matrix, how fast do the determinants D_n grow? (If you didn't find a and b, say how you would answer part (c) for any a and b) For 1 point, find D_5.

a. $\det(A_2) = 7$ $\quad \det(A_3) = 21 - 6 = 15$

b. $D_n = 3 \cdot D_{n-1} - 2 \cdot 1 \cdot D_{n-2}$

$= 3D_{n-1} - 2D_{n-2}$

c. $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ $\quad |A - \lambda I| = \lambda(\lambda - 3) + 2 = (\lambda - 2)(\lambda - 1) = 0$

$\rightarrow \lambda_1 = 2 \quad x_1 = \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix} \quad \lambda_2 = 1 \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 x_1 + c_2 x_2 \rightarrow c_1 = \frac{2}{3} \quad c_2 = -1$

$u_n = \frac{2}{3} \cdot 2^n x_1 - x_2 = \begin{bmatrix} 2^{n+1} - 1 \\ 2^n - 1 \end{bmatrix} = \begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} \rightarrow D_5 = 63.$