18.06SC Unit 3 Exam

1 (34 pts.) (a) If a square matrix A has all n of its *singular values* equal to 1 in the SVD, what basic classes of matrices does A belong to? (~~Singular,~~ symmetric, orthogonal, positive definite or semidefinite, diagonal)

(b) Suppose the (orthonormal) columns of H are eigenvectors of B:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad H^{-1} = H^{\mathrm{T}}$$

The eigenvalues of B are $\lambda = 0, 1, 2, 3$. Write B as the product of 3 specific matrices. Write $C = (B + I)^{-1}$ as the product of 3 matrices.

(c) Using the list in question (a), which basic classes of matrices do B and C belong to? (Separate question for B and C)

a. $A = U\Sigma V^T = UV^T \quad A^T = VU^T \quad A^TA = VU^TUV^T = I$

(V, U : orthogonal)

$\rightarrow A$: orthogonal

b. i $B = H \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{D} H^T$

$B + I = HDH^T + HIH^T = H\begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & 4 \end{bmatrix}H^T$

ii $C = (B+I)^{-1} = H\begin{bmatrix} 1 & & & 0 \\ & \frac{1}{2} & & \\ & & \frac{1}{3} & \\ 0 & & & \frac{1}{4} \end{bmatrix}H^T$

c. i B : symetric, singular, semidefinite

ii C : symetric, positive definite

2 (33 pts.) (a) Find three eigenvalues of A, and an eigenvector matrix S:

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Explain why $A^{1001} = A$. Is $A^{1000} = I$? Find the three diagonal entries of e^{At}.

(c) The matrix $A^{\mathrm{T}}A$ (for the same A) is

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}.$$

How many eigenvalues of $A^{\mathrm{T}}A$ are positive? zero? negative? (Don't compute them but explain your answer.) Does $A^{\mathrm{T}}A$ have the same eigenvectors as A?

a. $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 & 4 \\ 0 & -\lambda & 5 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (\lambda+1)\lambda(\lambda-1) = 0$

i $\lambda = -1, 0, 1$ ✓

ii $S = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

b. $\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = S\Lambda S^{-1}$ $A^n = S\Lambda^n S^{-1}$

i $\Lambda^{1001} = \Lambda \rightarrow A^{1001} = A$ ✓

ii $\Lambda^{1000} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow A^{1000} = S\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}S^{-1} \neq I$ ✓

iii $e^{At} = Se^{\Lambda t}S^{-1} = S\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{t} \end{bmatrix}S^{-1} = \begin{bmatrix} e^{-t} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{t} \end{bmatrix}$ ✓

c. i A: rank 2 $\rightarrow$ one 0 eigenvalue ✓

trace > 0, det > 0 $\rightarrow$ two positive ✓

ii different ✓

3 (33 pts.) Suppose the n by n matrix A has n orthonormal eigenvectors $q_1, \ldots, q_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus $Aq_j = \lambda_j q_j$.

(a) What are the eigenvalues and eigenvectors of A^{-1}? *Prove that your answer is correct.*

(b) Any vector b is a combination of the eigenvectors:

$$b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n \,.$$

What is a quick formula for c_1 using orthogonality of the q's?

(c) The solution to $Ax = b$ is also a combination of the eigenvectors:

$$A^{-1} b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n \,.$$

What is a quick formula for d_1? You can use the c's even if you didn't answer part (b).

a. eigenvalues $1/\lambda_j$ ✓ ..vectors q_j ✓

proof: $q_j = \lambda_j A^{-1} q_j \rightarrow \underline{A^{-1} q_j = \frac{1}{\lambda_j} q_j}$

b. $q_1^T b = c_1$ ✓

c. $\frac{1}{\lambda_j} q_j = A^{-1} \, q_j$

$c_1 A^{-1} q_1 = d_1 q_1 = \frac{c_1}{\lambda_1} q_1 \rightarrow d_1 = \frac{c_1}{\lambda_1}$ ✓