18.06SC Unit 2 Exam

1 (24 pts.) Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \pm 1$ (unit volume) ✓

(b) $\det\begin{bmatrix} q_1+q_2 & q_2+q_3 & q_3+q_1 \end{bmatrix} = \det[q_1\ q_2\ q_3] + \det[q_2\ q_3\ q_1] = \pm 2$ ✓

(c) $\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det\begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} = 1$ (2 unit volumes) ✓

2 (24 pts.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\widehat{C}$ and $\widehat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

a. let $A = \begin{bmatrix} 1 & -10 \\ \vdots & \vdots \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 10 \end{bmatrix}$ $x = \begin{bmatrix} C \\ D \end{bmatrix}$, then $Ax = b$ (where $b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$A^T A \hat{x} = A^T b \longrightarrow \begin{bmatrix} 21 & 0 \\ 0 & ? \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1/21 \\ 0 \end{bmatrix}$ ✓

b. basis: 2 column vectors of A. ✓

$[1\ 1\ 1\ 1\ 1\ -1\ -1\ -1\ -1\ -1\ 0\ -1\ -1\ -1\ -1\ -1\ 1\ 1\ 1\ 1\ 1]^T$ ✓

3 (9 + 12 + 9 pts.) The Gram-Schmidt method produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices Q and A.

(a) Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A.

(b) ***Is $P_Q = P_A$ and why ? What is P_Q times Q ? What is $\det P_Q$?***

(c) Suppose a_4 is a new vector and a_1, a_2, a_3, a_4 are independent. Which of these (if any) is the new Gram-Schmidt vector q_4 ? (P_A and P_Q from above)

$$\textbf{1.}\ \frac{P_Q a_4}{\|P_Q a_4\|} \qquad \textbf{2.}\ \frac{a_4 - \frac{a_4^T a_1}{a_1^T a_1} a_1 - \frac{a_4^T a_2}{a_2^T a_2} a_2 - \frac{a_4^T a_3}{a_3^T a_3} a_3}{\|\text{ norm of that vector }\|} \qquad \textbf{3.}\ \frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$$

a. $P_Q = Q(Q^TQ)^{-1}Q^T \qquad P_A = A(A^TA)^{-1}A^T$ ✓✗

b. i. yes. P_Q, P_A project vector to the same space. ✓

ii. $P_Q Q = Q$: column vectors of Q are in the space P_Q projects to ✓

iii. $\det P_Q Q = \det P_Q \det Q = \det Q$ ✗ $\quad$ P_Q is singular. so is Q.

$\rightarrow \det P_Q = 1$

c. 3 $\rightarrow$ it is orthogonal to column space of A

✓

4 (22 pts.) Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \end{bmatrix}$$

(a) The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial? **Explain your answer**.

(b) If those 9 numbers give the identity matrix I, what is $\det A$? Which values of $\times$ give $\det A = 0$?

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

a. 2 ✓ (maximum 2 X in the choice of distinct column/row)

b. i. $\det A = X - X^2 - X^2 - X^2 = X - 3X^2$ ✓

ii. $X = 0$ or $\frac{1}{3}$ ✓