Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Quiz H | Fall 2010)

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.
2. **(10 points)** Find the probability that $Y \geq X$.
   (Let c be the answer to this question.)
3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
   (You may express your answer in terms of the constant c from the previous part.)
4. **(10 points)** Find the PDF of $Z = X + Y$.
5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.
6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.
7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.
8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \leq |X| \leq 1 + \delta) \approx \alpha\delta$, for some constant α. Find the value of α.
2. **(10 points)** Find the variance of S.
3. **(5 points)** Are N and S uncorrelated? Justify your answer.
4. **(5 points)** Are N and S independent? Justify your answer.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.

$$E[X-3Y] = E[X] - 3E[Y]$$
$$= 2 - \frac{1}{2} = 1.5$$
$$\text{var}[X-3Y] = \text{var}[X] + 9\,\text{var}[Y]$$
$$= \frac{4}{12} + 9 \cdot \frac{1}{4}$$

2. **(10 points)** Find the probability that $Y \geq X$.
   (Let c be the answer to this question.)

$$c = P(Y \geq X) = \int_0^4 f_X(z)\,(1 - F_Y(z))\,dz$$

$$= \int_0^4 \frac{1}{4} \cdot e^{-2z}\,dz$$

$$= \frac{1}{8}\, e^{-2z} \Big|_0^4 = \frac{1}{8}(1 - e^{-8})$$

3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
(You may express your answer in terms of the constant c from the previous part.)

$$f_{X,Y|Y\geq X} = \frac{f_X(x) f_Y(y)}{P(Y \geq X)} = \frac{\frac{1}{4} \cdot 2e^{-2y}}{c}$$

4. **(10 points)** Find the PDF of $Z = X + Y$.

if $z > 4$ $\quad f_Z(z) = \int_0^4 f_X(x) \cdot f_Y(z-x)\,dx$

$$= \int_0^4 \frac{1}{4} \cdot 2 e^{2x-2z}\,dx$$

$$= \frac{1}{4} e^{2x-2z} \Big|_0^4$$

$$= \frac{1}{4}\left(e^{8-2z} - e^{-2z}\right)$$

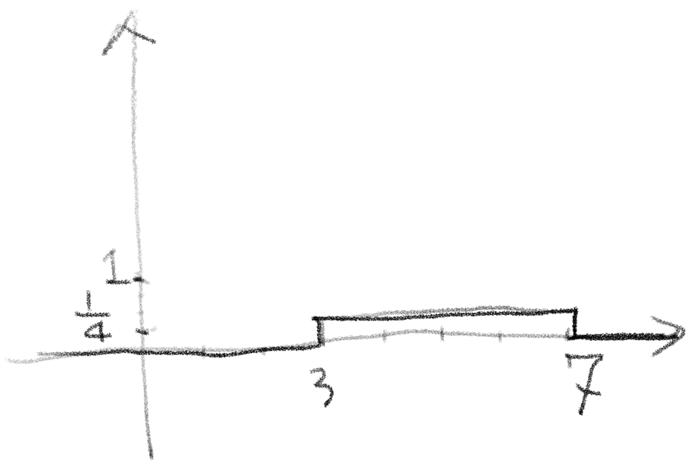
$$= \frac{e^8 - 1}{4e^{2z}}$$

if $z < 4$. $\ldots = \int_0^z \ldots$

$$= \frac{1}{4} e^{2x-2z} \Big|_0^z$$

$$= \frac{1 - e^{-2z}}{4}$$

5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.



6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.

$$E[Z] = E[X+Y] = E[X] + E[Y]$$
$$= 2 + E[Y]$$
$$E[Z \mid Y = y] = 2 + y$$
$$E[Z \mid Y] = 2 + Y$$

7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.

$$f_{Z,Y}(z,y) = f_Y(y) \cdot f_{Z|Y}(z|y)$$

$$= f_Y(y) \cdot f_X(z-y)$$

$$= \begin{cases} \frac{1}{2} e^{-2y}, & z-y \le 4 \\ 0, & \text{otherwize.} \end{cases}$$

8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

$$\frac{f_Y(3)}{f_Y(3) + f_Y(1)} = \frac{e^{-6}}{e^{-6} + e^{-2}}$$

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \le |X| \le 1+\delta) \approx \alpha\delta$, for some constant α. Find the value of α.

$X \sim N(0,3)$

$f_X(x) = \frac{1}{3\sqrt{2\pi}} e^{-x^2/18}$

$\alpha = f_X(1) + f_X(-1)$

$= \frac{2}{3\sqrt{2\pi}} e^{-1/18}$

2. **(10 points)** Find the variance of S.

$\mathrm{var}(S) = E[N]\,\mathrm{var}(X) + (E[X])^2\,\mathrm{var}(N)$

$= 2 \cdot 9 + 0 = 18$

3. **(5 points)** Are N and S uncorrelated? Justify your answer.

yes.

$$\text{cov}(S, N) = E[SN] - E[S]E[N]$$

$$= E[X] \cdot \text{something} - E[X] \cdot (E[N])^2$$

$$= 0.$$

4. **(5 points)** Are N and S independent? Justify your answer.

no. $f_{S|N}(s|1) = f_X(s)$

$f_{S|N}(s|2) = f_{X+X}(s) \neq f_X(s)$