L10. HOOKE'S LAW - SPRINGS - SIMPLE HARMONIC MOTION - PENDULUM - SMALL ANGLE APPROXIMATION

-Hooke's Law (ideal springs)

$$|F| \propto |x|$$

$F = -kx$ (k: spring constant, unit: N/m)

$$k = \frac{\Delta F}{\Delta x}$$

-Dynamic Equations of a Displaced Spring

$T = 2\pi\sqrt{\frac{m}{k}}$ (the period of oscillation)

-Derivation of the period

$$ma = -kx \;\rightarrow\; m\ddot{x} + kx = 0 \;\rightarrow\; \ddot{x} + \frac{k}{m}x = 0$$

$x = A\cos(\omega t + \phi)$ (The curve looks like a sine or a cosine function)

(A: the amplitude, ω: angular frequency)

$$\dot{x} = -A\omega\sin(\omega t + \phi)$$

$$\ddot{x} = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x$$

$$\therefore \omega^2 = \frac{k}{m} \qquad \rightarrow \qquad \omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} \qquad \rightarrow \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

(independent of amplitude and phase angle)

-Dynamic Equations of a Pendulum

$$m\ddot{x} = -T\sin\theta = -T\cdot\frac{x}{l} \qquad m\ddot{y} = T\cos\theta - mg$$

-small-angle approximations

$\theta < < 1$

-Consequence

1. $\cos\theta \approx 1$

2. excursion in y-direction is way smaller than in x-direction so that $\ddot{y} \approx 0$

$$0 = T - mg \qquad \rightarrow \qquad T = mg \qquad \rightarrow \qquad m\ddot{x} = -mg\frac{x}{l} \qquad \rightarrow$$

$$\ddot{x} + \frac{g}{l}x = 0$$ (simple harmonic oscillation)

$$\omega = \sqrt{\frac{g}{l}} \qquad T = 2\pi\sqrt{\frac{l}{g}}$$

L11. WORK - KINETIC ENERGY - POTENTIAL ENERGY - CONSERVATIVE FORCES - CONSERVATION OF MECHANICAL ENERGY - NEWTON'S UNIVERSAL LAW OF GRAVITATION

$$W_{AB} = \int_A^B F\mathrm{d}x$$ (work, unit: Nm, or J, “joule”)

$$F = ma \qquad F = mdv/dt \qquad dx = vdt \qquad \rightarrow$$

$$W_{AB} = \int_A^B m\frac{dv}{dt}vdt = \int_{v_A}^{v_B} mvdv \qquad \rightarrow$$

$$W_{AB} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

$$\frac{1}{2}mv^2 = KE = K$$ (kinetic energy)

$$W_{AB} = \mathrm{KE}_B - \mathrm{KE}_A$$ (work-energy theorem)

e.g.1 $W_{AB} = \int_A^B -mgdy = -mgh = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = -\frac{1}{2}mv_A^2$

$$mgh = \frac{1}{2}mv_A^2 \qquad \rightarrow \qquad h = \frac{v_A^2}{2g}$$

-Work Calculated in 3-Dimensions

$$W_{AB} = \int_A^B \vec{F} \cdot d\vec{r}$$

$$\vec{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z} \qquad d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z} \qquad \rightarrow$$

$$W_{AB} = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B F_x dx + \int_A^B F_y dy + \int_A^B F_z dz$$

$$= \frac{1}{2}m\left(v_{B_x}^2 - v_{A_x}^2\right) + \frac{1}{2}m\left(v_{B_y}^2 - v_{A_y}^2\right) + \frac{1}{2}m\left(v_{B_z}^2 - v_{A_z}^2\right) \qquad \rightarrow$$

$$W_{AB} = \frac{1}{2}m\left(v_B^2 - v_A^2\right)$$

-Gravity is a Conservative Force

$$W_{AB} = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B F_y dy = -mg(y_B - y_A) = -mgh$$

-When Gravity is the only Force

$$-mgh = -mg(y_B - y_A) = \mathrm{KE}_B - \mathrm{KE}_A \qquad \rightarrow$$

$$mgy_B + \mathrm{KE}_B = mgy_A + \mathrm{KE}_A$$

(mgy: gravitational potential energy, notation: PE, or U)

-What Matters is the Difference in Potential Energy

$$\mathrm{U}_B - \mathrm{U}_A = mgh$$

-A Roller Coaster

$$h \geq \frac{5}{2}R$$

-Newton's Law of Universal Gravitation

$$F = \frac{mMG}{r^2}$$

$$G = 6.67 \times 10^{-11} Nm^2/kg^2$$

-Gravitational Potential Energy

$$W_{WL} = \int_{\infty}^{R} \frac{mMG}{r^2} dr = -\frac{mMG}{r}\Big|_{\infty}^{R} = -\frac{mMG}{R}$$

L12. NON-CONSERVATIVE FORCES - RESISTIVE FORCES - AIR DRAG - TERMINAL VELOCITY

$$\overrightarrow{F}_{res} = -(k_1 v + k_2 v^2)\hat{v}$$

the k value depends on:

the shape of the object

the size of the object

the medium through which you move it

-Sphere

$$|F_{res}| = C_1 r v + C_2 r^2 v^2$$ (C1 unit: km/m/s, C2 unit: km/m^3)

first term: Viscous term, second term: Pressure term

(C1 is a strong function of temperature)

(there is a very strong correlation between the C2 and the density)

-Terminal Velocity

$$mg = C_1 r v + C_2 r^2 v^2$$

-Two Regimes and the Critical Velocity

Regime I: the viscous term is dominating

Regime II: the pressure term is dominating

-the Critical Velocity

$$C_1 r v = C_2 r^2 v^2$$

$$v_{crit} = \frac{C_1}{C_2 r}$$

-Regime I

$$v << v_{crit} \quad \rightarrow \quad mg = C_1 r v_{term} \quad \rightarrow \quad v_{term} = \frac{mg}{C_1 r}$$

$$m = \frac{4}{3}\pi\rho r^3 \quad \rightarrow \quad v_{term} \propto r^2$$

-Regime II

$$v >> v_{crit} \quad \rightarrow \quad mg = C_2 r^2 v_{term}^2 \quad \rightarrow$$

$$v_{term} = \sqrt{\frac{mg}{C_2 r^2}} \quad \rightarrow \quad v_{term} \propto \sqrt{r}$$

L13. POTENTIAL ENERGY - ENERGY CONSIDERATIONS TO DERIVE SIMPLE HARMONIC MOTION

-Potential Energy of the Spring

$$U = \frac{1}{2}kx^2 \qquad \frac{dU}{dx} = -F_x$$

-of the Gravitational Force

$$F = -\frac{dU}{dy} = -mg$$

-Parabolic Potential Energy Well ==> SHO

$$\frac{1}{2}kx_{max}^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 \quad \rightarrow$$

$$\frac{1}{2}\dot{x}^2 + \frac{k}{2m}x^2 = \frac{k}{2m}x_{max}^2 \quad \rightarrow \frac{d}{dt}(\frac{1}{2}\dot{x}^2 + \frac{k}{2m}x^2) = \frac{d}{dt}(\frac{k}{2m}x_{max}^2) \quad \rightarrow$$

$$\dot{x}\ddot{x} + \frac{k}{m}x\dot{x} = 0 \quad \rightarrow \ddot{x} + \frac{k}{m}x = 0$$ (a simple harmonic oscillation: SHO)

-Circular Potential Energy Well ==> SHO

$$U = mgy = mgR(1 - \cos\theta) \qquad v(\theta) = R\frac{d\theta}{t}$$

$$\mathrm{ME} = \frac{1}{2}mR^2\dot{\theta}^2 + mgR(1 - \cos\theta)$$

-small angle approximation

$$\cos\theta = 1 - \frac{\theta^2}{2}$$

$$\mathrm{ME} = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mgR\theta^2 \qquad \rightarrow 0 = mR^2\dot{\theta}\ddot{\theta} + mgR\theta\dot{\theta} \qquad \rightarrow$$

$$\ddot{\theta} + \frac{g}{R}\theta = 0$$ (a simple harmonic oscillation: SHO)

L14. ESCAPE VELOCITIES - BOUND AND UNBOUND ORBITS - CIRCULAR ORBITS - VARIOUS FORMS OF ENERGY - POWER

-Escape Velocity

$$\frac{1}{2}mv_{escape}^2 = \frac{mM_{\oplus}G}{R_{\oplus}} \qquad \rightarrow \qquad v_{escape} = \sqrt{\frac{2M_{\oplus}G}{R_{\oplus}}}$$

-Circular Orbits

$$m\frac{v^2}{R} = \frac{mM_{\oplus}G}{R^2} \qquad \rightarrow \qquad v = \sqrt{\frac{M_{\oplus}G}{R}}$$

$$T = \frac{2\pi R}{v} = 2\pi R\sqrt{\frac{R}{M_{\oplus}G}} = \frac{2\pi R^{3/2}}{\sqrt{M_{\oplus}G}}$$

-Total Mechanical Energy

$$\mathrm{ME} = \frac{1}{2}mv^2 - \frac{mMG}{r} = -\frac{mMG}{2r} \qquad \rightarrow$$

$$\mathrm{ME} = \frac{1}{2}U = -\,\mathrm{KE}$$

-Power

$$P = \frac{dW}{dt} = \frac{\vec{F}\cdot d\vec{r}}{dt} = \vec{F}\cdot\vec{v}$$ (unit: joules per second, or W)

-Heat

$Q = mC\Delta T$

(C: specific heat, unit: 1 calorie per gram per degree centigrade)

L15. MOMENTUM - CONSERVATION OF MOMENTUM - CENTER OF MASS

$\vec{p} = m\vec{v}$ (p: momentum, unit: kg·m/s)

-Conservation of Momentum

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

$$\vec{p}_{total} = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_i + \ldots \quad \rightarrow$$

$$\frac{d\vec{p}_{total}}{dt} = \vec{F}_{1,net} + \vec{F}_{2,net} + \ldots + \vec{F}_{i,net} + \ldots = \vec{F}_{net,external}$$

-Center of Mass of a System

$$M_{total}\vec{r}_{cm} = \sum_i m_i\vec{r}_i \quad \rightarrow \quad \frac{d}{dt}(M_{total}\vec{r}_{cm}) = \frac{d}{dt}(\sum_i m_i\vec{r}_i) \quad \rightarrow$$

$$M_{total}\vec{v}_{cm} = \sum_i m_i\vec{v}_i \quad \rightarrow \quad \vec{p}_{total} = M_{total}\vec{v}_{cm}$$

$$\frac{d}{dt}(\vec{p}_{total}) = \frac{d}{dt}(M_{total}\vec{v}_{cm}) \quad \rightarrow \quad \vec{F}_{net,external} = M_{total}\vec{a}_{cm}$$

L16. COLLISIONS - ELASTIC AND INELASTIC - CENTER OF MASS FRAME OF REFERENCE

$m_1v_1 = m_1v_1' + m_2v_2'$

$K + Q = K'$

$Q > 0$: superelastic collision

$Q = 0$: elastic collision

$Q < 0$: inelastic collision

-Elastic Collision

when $v_2 = 0$

$$\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$$

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 \qquad v_2' = \frac{2m_1}{m_1 + m_2}v_1$$

-Center of Mass (CM) Frame of Reference (Q=0)

$$m_1u_1 + m_2u_2 = 0 = m_1u_1' + m_2u_2' \qquad +$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2 \qquad \rightarrow$$

$$u_1' = -u_1 \qquad u_2' = -u_2$$

-the Velocity of the Center of Mass

$$M_{total}\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2$$

$$v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

$$\overrightarrow{u}_1 = \overrightarrow{v}_1 - \overrightarrow{v}_{cm} \qquad \overrightarrow{u}_2 = \overrightarrow{v}_2 - \overrightarrow{v}_{cm}$$

-1D Inelastic Collision and Internal Energy

$$m_1v_1 = (m_1 + m_2)v' \qquad \rightarrow$$

$$K' - K = -\frac{1}{2}\frac{m_1m_2}{m_1 + m_2}v_1^2$$

$$u_1 = v_1 - v_{cm} = \frac{m_2}{m_1 + m_2}v_1 \qquad u_2 = 0 - v_{cm} = -\frac{m_1}{m_1 + m_2}v_1 \qquad \rightarrow$$

$$K = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}v_1^2$$

<u>L17. IMPULSE - ROCKETS</u>

$$\vec{I} = \int_0^{\Delta t} \overrightarrow{F}dt \qquad \text{(I: impulse, unit, same as momentum)}$$

$$\overrightarrow{F} = m\overrightarrow{a} = \frac{d\overrightarrow{p}}{dt} \qquad \rightarrow$$

$$= \int_{p_i}^{p_f} d\overrightarrow{p} = \overrightarrow{p}_f - \overrightarrow{p}_i$$