Simulations with Exact Means and Covariances[1]

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Abstract

We present a simple method to generate scenarios from multivariate elliptical distributions where the sample mean and covariances match the respective population moments. This methodology easily applies to large numbers of scenarios and large-dimensional distributions. We show an application to the risk management of a book of options. Fully documented MATLAB code illustrating our approach can be downloaded from MATLAB Central File Exchange.

JEL Classification: C1, G11

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1 Introduction

In order to perform risk and portfolio management, we must represent the distribution of the risk factors that affect the market. The most flexible approach is in terms of scenarios and their probabilities, which includes historical scenarios, pure Monte Carlo, importance sampling, etc., see Glasserman (2004).

Here we present a simple method to generate scenarios from elliptical distributions with given sample correlations. This is very important in such applications as mean-variance portfolio optimization, which are heavily affected by incorrect representations of the first two moments.

The same problem has been tackled, among others, by Wedderburn (1975), Cheng (1985), Li (1992), Alexander, Ledermann, and Ledermann (2008). However, these approaches require handling matrices or loops of the same size as the number of scenarios: this quickly becomes intractable for large Monte Carlo simulations. Instead, our method is a multivariate generalization of the intuitive shift/rescaling that appears e.g. in Boyle, Broadie, and Glasserman (1995). This method amounts to solving a matrix Riccati equation independent of the number of scenarios.

2 Methodology

Consider a multivariate normal distribution

$$\mathbf{X} \sim \mathrm{N}(\mathbf{m}, \mathbf{S}), \tag{1}$$

where $\mathbf{m}$ is an arbitrary location parameter, $\mathbf{S}$ is an arbitrary scatter matrix. Consider the representation of this distribution in terms of the probability-scenario pairs $(p_j, \mathbf{x}_j)$, $j = 1, \ldots J$. Our aim is to ensure that

$$\widehat{\mathbf{m}}_x = \mathbf{m}, \quad \widehat{\mathbf{S}}_x \equiv \mathbf{S}, \tag{2}$$

where $\widehat{\mathbf{m}}_x$ and $\widehat{\mathbf{S}}_x$ denote the sample mean and sample covariance of $(p_j, \mathbf{x}_j)$. To do so, one can either constrain the probabilities p_j, or the scenarios $\mathbf{x}_j$. The former approach, pursued e.g. in Avellaneda (1999), D'Amico, Fusai, and Tagliani (2003), Glasserman and Yu (2005) and Meucci (2008), is very flexible, but for large-dimensional markets it becomes computationally challenging. Here, we choose the second route, which relies on the affine equivariance of the elliptical distributions.

First we produce an auxiliary set of scenarios

$$(\widetilde{p}_j, \widetilde{\mathbf{y}}_j), \quad j = 1, \ldots, \frac{J}{2} \tag{3}$$

from the distribution $\mathrm{N}(\mathbf{0}, \mathbf{S})$. Then we complement these scenarios with their opposite

$$(p_j, \widetilde{\mathbf{y}}_j) \equiv \begin{cases} (\widetilde{p}_j/2, \widetilde{\mathbf{y}}_j) & \text{if } j \leq J/2 \\ \left(\widetilde{p}_{j-\frac{J}{2}}/2, -\widetilde{\mathbf{y}}_{j-\frac{J}{2}}\right) & \text{if } j > J/2. \end{cases} \tag{4}$$

These antithetic variables still represent the distribution $\mathrm{N}(\mathbf{0}, \mathbf{S})$, but they are more efficient, see Boyle, Broadie, and Glasserman (1995), and they satisfy the zero-mean condition $\widehat{\mathbf{m}}_{\widetilde{y}} \equiv \mathbf{0}$.

Next we apply a linear transformation to the scenarios $\widetilde{\mathbf{y}}_j$, which again preserves normality:

$$\mathbf{y}_j \equiv \mathbf{B}\widetilde{\mathbf{y}}_j, \quad j = 1, \ldots, J. \tag{5}$$

For any choice of the invertible matrix $\mathbf{B}$, the sample mean is null: $\widehat{\mathbf{m}}_y \equiv \mathbf{0}$. To determine $\mathbf{B}$ we impose that the sample covariance $\widehat{\mathbf{S}}_y$ matches the desired covariance $\mathbf{S}$. Using the affine equivariance of the sample covariance, see e.g. Meucci (2005), we obtain a matrix Riccati equation

$$\mathbf{S} \equiv \mathbf{B}\widehat{\mathbf{S}}_{\widetilde{\mathbf{y}}}\mathbf{B}, \quad \mathbf{B} \equiv \mathbf{B}'. \tag{6}$$

To solve this equation we follow Petkov, Christov, and Konstantinov (1991). First we define the Hamiltonian matrix

$$\mathbf{H} \equiv \begin{pmatrix} \mathbf{0} & -\widehat{\mathbf{S}}_{\widetilde{\mathbf{y}}} \\ -\mathbf{S} & \mathbf{0} \end{pmatrix}. \tag{7}$$

Next we perform its Schur decomposition

$$\mathbf{H} \equiv \mathbf{U}\mathbf{T}\mathbf{U}', \tag{8}$$

where $\mathbf{U}\mathbf{U}' \equiv \mathbf{I}$ and $\mathbf{T}$ is upper triangular with the eigenvalues of $\mathbf{H}$ on the diagonal sorted in such a way that the first N have negative real part and the remaining N have positive real part; the terms in this decomposition are similar in nature to principal components and are computed by standard software packages, see e.g. Anderson, Bai, Bischof, Blackford, Demmel, Dongarra, Du Croz, Greenbaum, Hammarling, McKenney, and Sorensen (1999). Then the solution of the Riccati equation (6) reads

$$\mathbf{B} \equiv \mathbf{U}_{LL}\mathbf{U}_{UL}^{-1}, \tag{9}$$

where $\mathbf{U}_{UL}$ is the upper left $N \times N$ block of $\mathbf{U}$ and $\mathbf{U}_{LL}$ is the lower left $N \times N$ block of $\mathbf{U}$.

With the solution (9) we can perform the affine transformation (5) and finally generate the desired scenarios

$$\mathbf{x}_j \equiv \mathbf{m} + \mathbf{y}_j, \quad j = 1, \ldots, J, \tag{10}$$

which satisfy (2), see Figure 1, where as in Meucci (2005) we represent the first two moments of a distribution in terms of an ellipsoid.

Note that the steps (3)-(10) only require a few fractions of a second to run even for large problems, refer to `www.symmys.com` ⇒ Teaching ⇒ MATLAB for a fully functional implementation.

The present methodology is based on affine transformations as well as on the affine equivariance of the sample mean and covariance. Therefore, it extends straightforwardly to general elliptical distributions, such as the t.

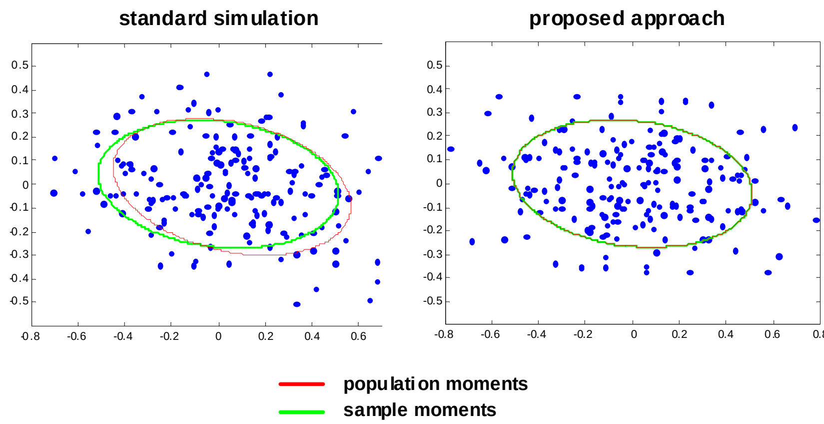

Figure 1: Sample and population moments coincide our approach

3 Applications

To illustrate the ubiquitous nature of normal simulations, in this section we apply our methodology to compute the VaR and its decomposition in a book of I plain vanilla call options, see also Meucci (2008). We denote by T the current time. We notice that the price $P_{T+\tau}$ of a call option at the investment horizon can be written in the format

$$P_{T+\tau} \equiv P\left(X_{T,y}, X_{T,\sigma}; \mathcal{I}_T\right), \tag{11}$$

where $(X_{T,y}, X_{T,\sigma})$ are risk factors and $\mathcal{I}_T$ represents currently available information. Indeed, consider the Black-Scholes pricing formula

$$C_{BS}\left(y, \sigma, K, \upsilon\right) \equiv y\Phi\left(d_1\right) - Ke^{-r\upsilon}\Phi\left(d_2\right), \tag{12}$$

where Φ is the cdf of the standard normal distribution and

$$d_1 \equiv \frac{-\ln\left(K/y\right) + \upsilon\left(r + \sigma^2/2\right)}{\sqrt{\sigma^2\upsilon}} \tag{13}$$

$$d_2 \equiv \frac{-\ln\left(K/y\right) + \upsilon\left(r - \sigma^2/2\right)}{\sqrt{\sigma^2\upsilon}}. \tag{14}$$

Then

$$P_{T+\tau} = C_{BS}\left(y_T e^{X_{T,y}}, h\left(y_T e^{X_{T,y}}, \sigma_T e^{X_{T,\sigma}}, K, T-\tau\right); K, T-\tau, r\right). \tag{15}$$

In this expression y_T is the current value and $X_{t,y} \equiv \ln\left(y_{t+\tau}/y_t\right)$ is the log-change of the underlying; σ_T is the current value and $X_{t,\sigma} \equiv \ln\left(\sigma_{t+\tau}/\sigma_t\right)$ is the log-change in $(T-\tau)$-expiry, at-the-money implied volatility; and h is a skew/smile map

$$h\left(y,\sigma;K,T\right)\equiv\sigma+\alpha\frac{\ln\left(y/K\right)}{\sqrt{T}}+\beta\left(\frac{\ln\left(y/K\right)}{\sqrt{T}}\right)^{2},\tag{16}$$

for coefficients α and β which depend on the underlying and are fitted empirically, similarly to Malz (1997). Clearly, (15) is in the format (11).

Consider a portfolio represented by the vector $\mathbf{w}$, whose generic i-th entry is the number of contracts in the respective call. The p&l then reads

$$H_{\mathbf{w}}\equiv\sum_{i=1}^{I}w_i\left(P_i\left(X_{T,y}^{(i)},X_{T,\sigma}^{(i)};\mathcal{I}_T\right)-p_{i,T}\right),\tag{17}$$

where $p_{i,t}$ denotes the currently traded price of the i-th call. In order to compute the VaR we need the distribution of the p&l (17) and to obtain the latter, we need the joint distribution of all the sources of risk $\left(X_{T,y}^{(i)},X_{T,\sigma}^{(i)}\right)$ in the portfolio.

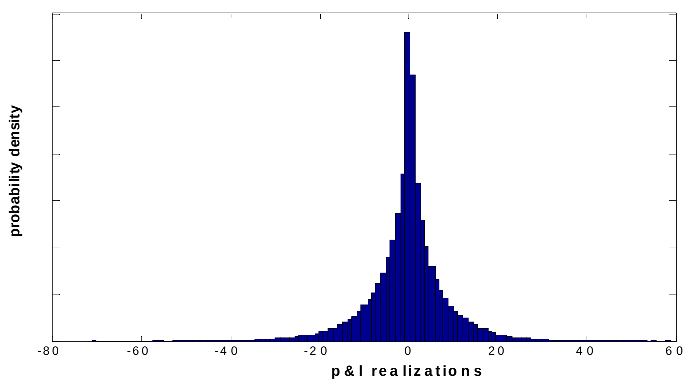

Figure 2: P&L of highly non-linear option book with normal copula

We realize that the sources of risk are approximately invariants, i.e. their joint distribution is independent and identical across time, and thus it does not depend on the specific time. We model this joint distribution by means of a multivariate normal copula with non-parametric marginals:

$$\left(\begin{array}{c}X_y^{(1)}\\X_\sigma^{(1)}\\\vdots\\X_y^{(I)}\\X_\sigma^{(I)}\end{array}\right)\overset{d}{=}\left(\begin{array}{c}\widehat{F}_{X_y^{(1)}}^{-1}\left(\Phi\left(Z_1^{(1)}\right)\right)\\\widehat{F}_{X_\sigma^{(1)}}^{-1}\left(\Phi\left(Z_2^{(1)}\right)\right)\\\vdots\\\widehat{F}_{X_y^{(I)}}^{-1}\left(\Phi\left(Z_1^{(I)}\right)\right)\\\widehat{F}_{X_\sigma^{(I)}}^{-1}\left(\Phi\left(Z_2^{(I)}\right)\right)\end{array}\right).\tag{18}$$

In this expression

$$\mathbf{Z} \sim \mathrm{N}\left(\mathbf{0}, \widehat{\mathbf{C}}\right), \tag{19}$$

for a suitably estimated correlation matrix $\widehat{\mathbf{C}}$; and $\widehat{F}_X$ denotes a suitable estimate of the marginal distribution of X. In particular, we estimate these marginals by a non-parametric kernel smoothing of the historical data. Then we use the cdf's $\widehat{F}_X$ to invert (18) for each entry in our joint time series of log price changes $X_{t,y}$ and the log volatility changes $X_{t,\sigma}$:

$$Z_t^{(i)} \equiv \Phi^{-1}\left(\widehat{F}_{X^{(i)}}\left(X_t^{(i)}\right)\right), \quad t = 1, \ldots, T, \tag{20}$$

where this simplified notation applies to both y and σ. Next, we fit the correlation (19) to the time series (20).

Now we can simulate the p&l distribution (17). First, we use our recipe to draw Monte Carlo scenarios from (19) in such a way that the sample mean is zero and the sample covariance exactly matches the estimated correlation matrix $\widehat{\mathbf{C}}$. Then we map those simulations into factor realizations using (18). Since the expression of the inverse cdf is not available analytically, we perform a linear interpolation of the cdf as in Meucci (2006). Next, those simulations are fed into the pricing functions that appear in (17), thereby generating a $J \times I$ panel $\mathcal{H}$ of joint p&l scenarios for the I options at the investment horizon. The portfolio p&l (17) is then represented by the simulations vector $\mathcal{H}_{\mathbf{w}} \equiv \mathcal{H}\mathbf{w}$. In Figure 2 we report this distribution in an example of a portfolio long-short twenty options.

In order to compute the VaR and its contributions from the different securities in the portfolio, first we express the former in terms of the latter

$$VaR \equiv \sum_{i=1}^{I} w_i \frac{\partial VaR}{\partial w_i}, \tag{21}$$

where the partial derivatives that appear in (21) can be expressed conveniently as in Hallerbach (2003) and Gourieroux, Laurent, and Scaillet (2000)

$$\frac{\partial VaR}{\partial \mathbf{w}} \equiv \mathbf{p} - \mathbb{E}\left\{\mathbf{P} | H_{\mathbf{w}} \equiv -VaR\right\}, \tag{22}$$

where $\mathbf{p}$ denote the current prices, which are known, and $\mathbf{P}$ the horizon prices, which are a random vector, as they appear in (17).

Then the expectations in (22) are approximated numerically as in Mausser (2003), see also Epperlein and Smillie (2006) and Meucci, Gan, Lazanas, and Phelps (2007)

$$\frac{\partial VaR}{\partial \mathbf{w}} \approx -\mathbf{k}' \widetilde{\mathcal{H}}. \tag{23}$$

In this expression $\widetilde{\mathcal{H}}$ is a $J \times I$ panel whose generic i-th column is the i-th column of the options p&l panel $\mathcal{H}$, sorted as the order statistics of the J-dimensional vector of the portfolio losses $-\mathcal{H}\mathbf{w}$; and $\mathbf{k}$ is a Gaussian smoothing kernel peaked

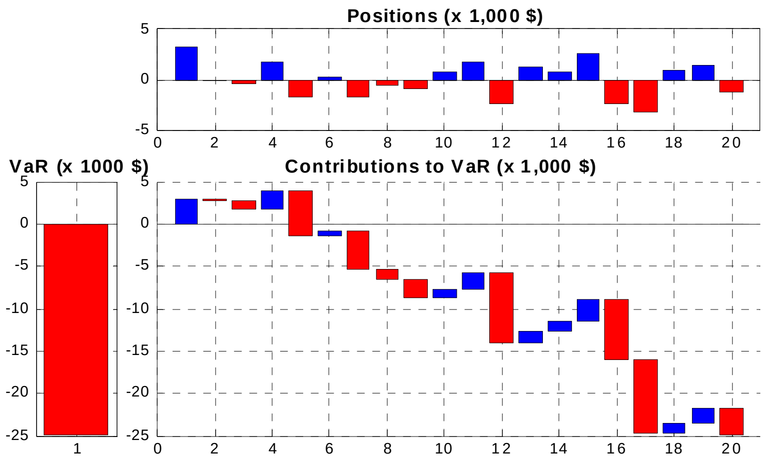

Figure 3: VaR of highly non-linear option book with normal copula

around the rescaled confidence level cJ. Finally, (21) yields the contributions from each option as well as the total VaR. The total VaR number then follows from (21).

In Figure 3 we report the total VaR in our example of a portfolio long-short twenty options, as well as its decomposition in terms of the contributions from each call.

The risk manager can now proceed to stress-test the correlation $\widehat{\mathbf{C}}$ and analyze the impact of the stress-test on the risk report such as Figure 3, confident that his assumptions will be faithfully reflected in the simulations.

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