Quant Nugget 1

Square-Root Rule, Covariances and Ellipsoids

How to Analyze and Visualize the Propagation of Risk[1]

Attilio Meucci

attilio_meucci@symmys.com

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The square-root rule of risk growth reads: "volatility increases as the square root of the investment horizon". This rule is based on the assumption that the non-overlapping compounded returns of a security are independently and identically (i.i.d.) distributed across time, i.e. they are market "invariants", see Meucci (2005)

$$R_t \equiv \ln(P_{t+1}) - \ln(P_t) \sim \text{i.i.d.}, \tag{1}$$

where P_t is the price of the security at time t.

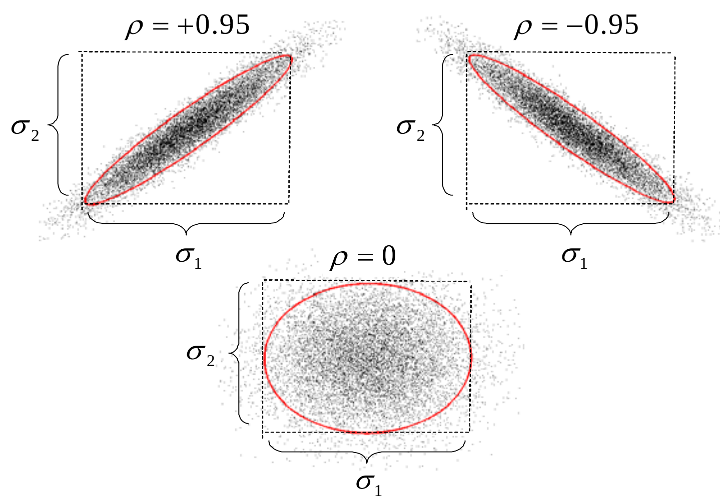

Figure 1: Visualizing covariance matrices: the location-dispersion ellipsoid

To derive the square root rule, we notice that the weekly returns are the sum of five daily returns, the monthly returns are the sum of twenty-two daily

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returns, and more in general the j-horizon return is the sum of j non-overlapping short-horizon returns:

$$\ln(P_{t+j}) - \ln(P_t) = R_t + \cdots + R_{t+j-1}. \tag{2}$$

If the short-horizon returns are invariants and we denote by σ^2 their variance, the variance $\widetilde{\sigma}^2(j)$ of the j-horizon return satisfies

$$\widetilde{\sigma}^2(j) \equiv \mathrm{Var}\{R_t + \cdots + R_{t+j-1}\} = j\sigma^2, \tag{3}$$

and thus the volatility, as represented by the square root of the variance, satisfies the square-root rule.

This simple result generalizes to a multivariate framework. Consider the daily returns on two stocks and their covariance matrix

$$\mathbf{\Sigma} \equiv \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix}, \tag{4}$$

where ρ is the correlation and σ_1 and σ_2 are the respective volatilities. To visualize the covariance matrix, we can use the location-dispersion ellipsoid, see Figure 1 and Meucci (2005), for more details: the size of the ellipsoid represents the volatilities, whereas the shape and the orientation represent the correlation.

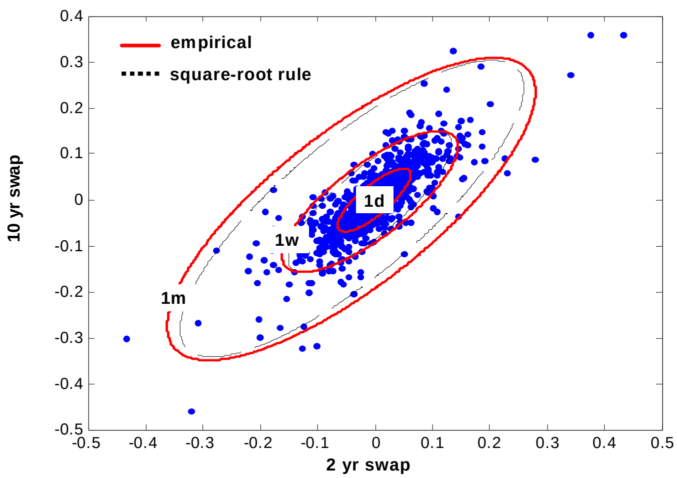

Figure 2: Swap rate changes are approximately invariants (data source: Bloomberg)

Now consider the covariance of the long-horizon (weekly, monthly, etc.) returns which, as in (2), are the sum of j (five, twenty-two, etc.) non-overlapping daily returns

$$\widetilde{\mathbf{\Sigma}}(j) \equiv \begin{pmatrix} \widetilde{\sigma}_1^2(j) & \widetilde{\sigma}_1(j)\widetilde{\sigma}_2(j)\widetilde{\rho}(j) \\ \widetilde{\sigma}_1(j)\widetilde{\sigma}_2(j)\widetilde{\rho}(j) & \widetilde{\sigma}_2^2(j) \end{pmatrix}. \tag{5}$$

If the returns are invariants, from the same argument as (3) it follows that the covariance grows proportionally to the horizon:

$$\widetilde{\mathbf{\Sigma}}(j) = j\mathbf{\Sigma}. \tag{6}$$

Since the variances grow proportionally to the horizon, i.e. $\widetilde{\sigma}_1^2(j) = j\sigma_1^2$ and $\widetilde{\sigma}_2^2(j) = j\sigma_2^2$, then if the returns are invariants their correlation is constant at any horizon, i.e. $\widetilde{\rho}(j) \equiv \rho$.

The above results do not always, or only, apply to the compounded returns. For *any* variables such as (1), which a) are invariants and b) can be expressed as differences, volatility propagates as the square root of the horizon and correlation is constant. This relationship is a powerful tool to test whether a given set of risk factors are market invariants.

For instance, the returns of near-maturity bonds are not invariants, so the square-root rule does not apply. On the other hand, consider the swap market. In Figure 2 we scatter-plot the daily differences of the two-year versus the ten-year point of the swap curve. Then we compute the sample covariance $\mathbf{\Sigma}$ between these two series. Next, we represent geometrically $\mathbf{\Sigma}$ by means of its location dispersion ellipsoid, which is the smallest ellipsoid in Figure 2. Then we consider the empirical covariance $\widetilde{\mathbf{\Sigma}}$ at different horizons of one week and one month respectively. We represent all of these covariances by means of their location-dispersion ellipsoids, which we plot in the figure as solid red lines. Finally we compare these ellipsoids with the suitable multiple $j\mathbf{\Sigma}$ of the daily ellipsoid, which we plot as dashed ellipsoids. We see from the figure that the solid and the dashed ellipsoids are comparable and thus the swap rate changes are approximately invariants: the volatilities increase according to the square-root rule and the correlation is approximately constant. To download the fully commented code used to generate this example refer to Meucci (2009).

References

Meucci, A., 2005, *Risk and Asset Allocation* (Springer).

——— , 2009, Exercises in advanced risk and portfolio management - with step-by-step solutions and fully documented code, *Free E-Book* available at http://ssrn.com/abstract=1447443.