Estimation of operational risks using non-parametric approaches with an application to US business losses*

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Abstract

Following the recent global financial crisis, many banks and other businesses in the industrialized countries incurred notably heavy losses. As a consequence, reliable estimation of operational risk (OR) is becoming increasingly important to all internationally active banks and other financial institutions. The OR is the unexpected loss, which is the difference between the 99.9 per cent quantile and the mean of the loss distribution. This paper adapts non-parametric methods based on heavy-tailed distributions and constructs point and 95 per cent confidence interval (CI) estimates for ORs. The main advantage of these non-parametric methods is that there are no assumptions made about the shape of loss distributions and that data determines their shapes, providing robust estimates for ORs. Employing these methods, we construct point as well as interval estimates for ORs for US businesses. The noteworthy observation is that the CIs are asymmetric with huge upper bounds, highlighting the extent of uncertainties associated with the point estimates of ORs. The estimates of expected shortfalls lie within these intervals. The nonparametric methods introduced in this paper will have much wider applications, for example, in estimating another popular measure of risk, credit risk.

Keywords: Heavy-tailed distribution, tail index, confidence intervals, non-parametric method, empirical likelihood, data-tilting, bootstrap

JEL Classification: C13, C14, C46

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1 Introduction

Due to recent large losses incurred by banks and other financial institutions in many industrialized countries, measuring and managing operational risk (OR) have been given considerable attention in the recent literature. The OR is defined as the difference between the 99.9% quantile and the mean of the loss distribution. In other words, the OR is the unexpected loss measured with a high degree of confidence. In the document released recently, The Basel Committee on Banking Supervision [2010], BCBS (2010) hereafter, emphasized the need to come up with a reliable OR estimate, based on which the economic capital charge to cover operational losses is calculated. There are pros and cons associated with the currently widely used advanced measurement approach (AMA). Following Basel II (2003, 2005), academics and practitioners studied the applicability and flexibility of AMA to model the severity of operational losses, and made significant contributions to the development of AMA; more discussion on this is given in the next section. Furthermore, Dutta and Perry [2007] studied the AMA modeling extensively, fitted many parametric distributions for the loss severity, and found that the OR estimate is very sensitive to the choice of loss distribution and that g-and-h distribution is robust to various types of scenarios compared with popular exponential, gamma, generalized Pareto, log-logistic, truncated lognormal, Weibull and the generalized beta distributions.

According to Basel II (2003, 2005) guidelines, banks have already developed AMA with flexibility. However, regulators' concern is that although the flexibility is provided in the AMA, its OR estimates and the subsequent amount of capitals do not appear to result in effective OR management. This may be partly due to the use of diversity of modeling approaches, including distributional assumptions and other critical features of the models. BCBS (2010) provides the following guidelines, among others, on the improvement of AMA approach: (i) A bank must be able to demonstrate that its approach captures potentially severe 'tail' loss events; (ii) Whatever approach is used, a bank must demonstrate that its OR measure meets a soundness standard that results in the reliable estimation of a 99.9th percentile of the severity loss distribution; and (iii) Identification of loss severity probability distribution is important, and it should be well specified, reviewed and its parameters be estimated consistently, with the risk profile in the tail properly captured. Furthermore, banks should pay particular attention to the estimates of the kurtosis- and skewness- related parameters, which describe the tail region of the losses. Because of data scarcity, the estimates can be highly unstable. The banks are expected to put in place methodologies to reduce estimate variability and provide measures of the errors around these estimates such as confidence intervals (CI). A bank should envisage several distributions to be tailored to the different types and shapes of operational losses within the organization. This approach is consistent with that of loss distribution (LDA) models, where the shape of the data is a crucial driver in identifying whether distributions are lighter- or heavier-tailed ones.

In this respect, our paper makes a significant contribution on the improvements of AMA for OR estimation. We introduce and apply nonparametric methods for estimating OR, which do not assume any shape for the operational loss distribution and the data is allowed to determine its shape. Employing these methods, we estimate the points as well as 95% CIs for ORs of US businesses. By doing so, we provide robust and consistent estimates for OR, with its interval estimates indicating the level of uncertainties associated with the OR point estimates. It is well-known that any quantile based risk measures such as VaR and OR are not coherent risk measures. On the other hand, the expected shortfall (ES) is a coherent risk measure of OR, and we report the ES estimates of US business losses in this paper; more details are given in section 3.

The primary objective of this paper is to adapt nonparametric statistical methods and apply them to estimate ORs of US businesses. The specific objectives are to: (i) estimate the mean of the heavy-tailed distribution; (ii) construct point and interval estimates for the 99.9% quantiles of the heavy-tailed distribution; (iii) construct point and 95% CI estimates for ORs of US business lossess; and (iv) estimate the ES of operational losses. These methods are briefly discussed in sections 3 and 4 of this paper.

The nonparametric methods adapted by this paper are recent developments in the statistical and econometric literatures, and, to our knowledge, they have not been yet employed for estimating ORs. We employ a number of methods to construct 95% CI estimates for ORs. The empirical likelihood (EL) and data tilting methods that are specifically proposed for estimating CIs for the quantiles of heavy-tailed distributions; see [Peng and Qi, 2006] and [Hall and Yao, 2003] for details. We suggest the sub-sampling bootstrap method for constructing the CIs for ORs. Despite its simplicity, the normal approximation based method (NA) does not have the correct coverage probability even in large samples. In our experience, we find that this methods works better than the others in some cases; section 6 provides the details. The DT method is expected to be better than others for estimating CIs for ORs, because this method puts more weights to the tail, which is the most important region of the distribution for estimating ORs. The OR estimates that we provide in this paper are expected to be consistent with some of the guidelines given in BCBS (2010) on the improvement of AMA approach, in particular on modelling severity loss distribution, and hence on the subsequent OR estimates. In this paper, various event- and business line-type losses are assumed to be independent, which can relaxed and copula modelling be incorporated in the estimation of OR, which is left for the future work. See, for example, Chavez-Demoulin et al. [2006] and Rosenberg and Schuermann [2006] for applications of copulas for fitting joint parametric multivariate distributions.

As was mentioned before, traditionally, the distribution of operational severity losses is estimated by parametric distributions. Several studies employed semi-parametric methods and in such methods, a non-parametric distribution is fitted to small to medium losses and then a parametric distribution to the large losses. This method is known as EVT-POT method proposed by Chavez-Demoulin et al. [2006]. There are, however, several limitations with these approaches: (i) the mean of the distribution (i.e. the expected loss) is estimated as the sample average of the data, which is biased and inconsistent when the distribution is heavy-tailed, and this estimate can be improved by exploiting the tail properties of the heavy-tailed loss distribution; see Peng [2001] and section 3 of this paper for details; (ii) a major problem associated with parametric or semi-parametric approaches is that if the underlying true loss distribution is unwittingly misspecified, then the OR estimates would be inconsistent and hence statistical inference based on such estimates becomes unreliable; and (iii) to date, the CI estimate for the underlying OR has not been reported, and such an interval estimate would indicate the degree of uncertainty associated with the OR point estimate as indicated by the results of empirical applications presented in this paper. In the non-parametric approach, on the other hand, no assumption is made about the shape of the loss distribution and the data is allowed to determine its shape.

In section 2, we briefly discuss how the modeling and estimation methods for OR have evolved in the literature over time. In section 3, we outline methods for estimating the tail index, which determines heaviness of the tails of the loss distributions, and describe the method for estimating the mean of the right heavy-tailed loss distribution. Further in this section, we discuss estimation methods for 99.9% quantiles of heavy right tailed loss distribution. Next, in section 4 we describe methods for estimation of CIs and risk measures for heavy tailed distributions.

In section 5, we describe the operational loss data series used in this study, including the time span, data source, and the number of business lines and events used in this empirical application. The overall nature of business losses constituting business line/event type losses are discussed.

In section 6, we apply the methodologies discussed in sections 3 and 4 to construct the point and interval estimates for ORs of eight business lines/event types. To make the applications of the methodologies clear to the readers, we provide a step-by-step instructions for constructing point and interval estimates for OR. The estimates of ES are also reported and analyzed in this section. Section 7 concludes this paper.

2 A Brief Survey of the Literature

Measuring and managing OR have been receiving great attention by academics and practitioners over the past two decades. In the recent past, financial system has become susceptible to many types of operational losses, largely due to, among others, the following crucial developments: (i) globalization of international financial sectors has increased the complexity of services provided by these sectors, intensifying their exposure to OR events and leading to large losses; (ii) growing reliance on computers and electronic communications and online trading activities has led to frequent systems failures; (iv) interbank lending can result in insufficient bank liquidity and thus to inadequate allocation of capital and risk sharing between banks, leading to increased exposure to potentially contagious operational loss events; and (v) evolution of e-commerce exposes banks and other financial institutions to unknown risks, as well as increasing their exposure to fraud.

The Basel II (2003, 2005) Capital Accord, implemented in 2007, is based on a Three Pillars approach which ensures the soundness of the financial system. The Three Pillars includes minimum capital requirements, the supervisory review process, and market discipline. Basel II includes an explicit minimum capital charge for operational risk as part of Pillar 1. Three measurement methodologies are permitted to calculate the operational risk capital charge, with the advanced measurement approaches (AMA) being the most sophisticated one. AMA includes bank-specific risk measurement and management models. To encourage better risk management of the financial system, the Committee also published guidelines and principles for the estimation and management of OR. The primary aim of our paper is to make improvements to AMA method for estimating OR. Therefore, the literature related to this approach is reviewed in this paper.

Several previous studies found statistical uncertainty associated with the individual VaR estimate together with little knowledge on the interdependence between different classes implies a very high level of non-robustness of the final estimates. The relative scarcity of high impact low frequency events at the individual bank level has prompted the construction and use of external data bases. Empirical findings of de Fontnouvelle et al. [2005] led them to conclude that "supplementing internal data with external data on extremely large rare events could significantly improve banks' models of OR". In the Basel framework, OR is defined as the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events; see, for example, see Cruz [2002, 2004], King [2001] and McNeil and Embrechts [2005] for details on excellent expositions and applications.

Under the AMA approach, banks will have to integrate internal data with relevant external loss data, account for stress scenarios, and include in the modelling process factors which reflect the business environment and the internal control system; see EBK [2005]. Moreover, the resulting risk capital must correspond to a 99.9%-quantile (VaR) of the aggregated loss data over the period of a year.

A loss distribution approach (LDA) consists of three basic components. First, a loss frequency distribution is used to model the number of losses that may have occurred within a given time period (usually one year). Second, a loss severity distribution is used to model the dollar amount of the individual losses that occur during the one year period. Third, an aggregate loss distribution is used to model the aggregate loss amount that a firm will experience over the chosen horizon. For more discussion, refer to de Fontnouvelle et al. [2005], Klugman et al. [1998] and Embrechts et al. [2002]. The findings of these studies have several implications that are of immediate practical relevance: (1) the analysis indicates that reporting biases in external data are significant and can vary by both business line and loss event type, and failure to account for these biases could overstate the amount of OR that bank faces and could also distort the relative riskiness of various business lines; (2) the analysis indicates that external data can be an important supplement to banks' internal data. While many banks should have adequate data for modeling high frequency low severity operational losses, only a few will have sufficient internal data to estimate the tail properties of the very largest losses; and (3) external data are an important part of the effort to understand the distribution of these large losses, an ongoing effort at many major U.S. banks.

2.1 Recent Developments in Advanced Modelling Approaches

It is evident from the recent work that, under AMA, there are many parametric loss severity distributions that can be employed for estimating OR for the same institution, resulting in materially different capital estimates. Some of the one or two-parameter distributions that are widely used in modelling operational losses include exponential, gamma, generalized Pareto, log-logistic, truncated lognormal, and Weibull. Dutta and Perry [2007] conducted an extensive empirical investigation into operational loss severity data and found that there exist some methodologies that can yield consistent and plausible results across the different institutions in spite of each institution having different data characteristics; more discussion on this later. This finding is important for several reasons. First, it suggests that operational losses can be modeled and that there is indeed some regularity in loss data across institutions. Second, it lays out the foundation that while preserving the AMA's fexibility, one can still develop a benchmark model that can be used by both financial institutions and regulators.

In the recent literature, four-parameter distributions such as the Generalized Beta Distribution of Second Kind (GB2) and the g-and-h distribution, which have the property that many different distributions are nested in them, and they can be generated from from these two four- parameter-distributions for specific values of their parameters. For modeling with the g-and-h and GB2 and estimation of their parameters, each of these distributions has its own estimation procedure. Goodness-of-fit tests are important because they provide an objective means of ruling out some modeling techniques which if used could potentially contribute to cross-institution dispersion in capital charges. The statistical fit of each distribution other than g-and-h was tested using the formal statistical tests for specifications. For the g-and-h, its Quantile-Quantile (Q-Q) plot to compared with Q-Q plots of other distributions to assess its goodness-of-fit.

In an empirical study, Dutta and Perry [2007] found that applying different models to the same institution yielded vastly different capital estimates. In several cases, applying the same model to different institutions yielded very inconsistent and unreasonable estimates across institutions even when statistical goodness-of-fit tests were satisfied. In those situations, poor tail fitting was observed. This raises two primary questions regarding the models that imply realistic estimates only in a few situations: (1) Are these models even measuring risk properly for the cases when they do yield reasonable exposure estimates, or do models generate some reasonable estimates simply due to chance? (2) If an institution measures its exposure with one of these models and finds its risk estimate to be reasonable today, how reasonable will this estimate be over time? The g-and-h distribution capital estimates was found to provide estimates that are consistent and reasonable across institutions and at different units of measurement in comparison with the other techniques particularly extreme value theory (EVT) and its power law variants. There is no unambiguous answer to question (2) at this stage, and perhaps, our non-parametric methods studied in this paper could provide robust results.

The most important conclusion emerged from recent studies is that the OR loss data can be modeled in the loss distribution framework without trimming or truncating the data in an arbitrary or subjective manner. It is clear from the findings of recent studies that the problem of OR measurement is essentially a measurement and modeling of the tail structure. A distribution or method that is unable to model a variety of tail shapes is not expected to perform well in modeling the operational loss data and hence reliably estimating OR. At this stage, of all parametric distributions that have been widely used, the g-and h- is found to be the best parametric distribution for reliably modeling and estimating operational loss distributions, and hence ORs.

2.2 A Brief Summary of Basel III Guidelines

In the latest Consultative Document, The Basel Committee on Banking Supervision [2010] includes identifying the practical challenges associated with the successful development, implementation and maintenance of an AMA framework as mandatory. Given the continuing evolution of analytical approaches for estimating operational risk, the Committee has neither specified the approach nor distributional assumptions used to generate the operational risk measure for regulatory capital purposes. However, it has been argued that a bank must be able to demonstrate that its approach captures potentially severe 'tail' loss events. Whatever approach is used, a bank must demonstrate that its OR measure meets a soundness standard comparable to that of the internal ratings-based approach for credit risk (i.e. comparable to a one year holding period and a 99.9th percentile confidence interval). A bank should follow a well specified, documented and traceable process for the selection, update and review of probability distributions and the estimate of their parameters. This process should result in consistent and clear choices and be finalised to properly capture the risk profile in the tail. Severity distributions play a crucial role in AMA models. That the models are often medium/heavy tailed implies that the final outcome is significantly impacted by the chosen distribution. The choice of frequency distributions has a lesser impact on the final outcome. This is precisely the aim of this study in that we focus on non-parametric methods that specifically defined to capture the properties of right heavy-tailed distributions, which are known to adequately fit the severity losses.

BCBS (2010) emphasized the use of appropriate techniques for the estimation of the distributional parameters, and appropriate diagnostic tools for evaluating the quality of the fit of the distributions to the data, giving preference to those most sensitive to the tail. A bank should pay particular attention to the positive skewness of the data when selecting a severity distribution. In particular, when the data are medium/heavy tailed (therefore very dispersed in the tail), the use of empirical curves to estimate the tail region is an unacceptable practice due to the inability to extrapolate information beyond the last observable data point.

Banks benefit from the flexibility to explore different options for combining the data elements, such as internal loss data, external data and scenario analysis, within a model to produce reliable estimates for capital requirements. As was mentioned in the introduction, in this paper, we employ nonparametric methods based on heavy-tailed loss distribution and construct point as well as 95% CI estimates for underlying true ORs. As a consequence, the capital charges emanating from the OR estimates also could be reported with confidence intervals. Several previous studies and Basel III are concerned with reporting just the one and only one point estimate for OR, and banks are expected to conduct some robust checks, such as scenario analysis and aggregating internal and external loss data, in order to come up with a range of potential OR estimates that would ensure the robustness of the estimates. In what we do, the 95% interval estimate for OR would indicate the extent of uncertainty associated with the point estimate of OR.

3 Methodologies

In this section, we briefly discuss the statistical methods employed in this paper. First, we outline the estimators of the tail index and the method for estimating the mean of heavy right-tailed distributions. Then, we discuss the method of estimating the 99.9% quantile of heavy tailed distribution and the following four methods of estimating confidence intervals (CI) for quantiles: (i) normal approximation method (NA), (ii) empirical likelihood method (EL), (iii) data tilting method (DT), and (iv) sub-sample bootstrap method (BT).

3.1 A weighted average of Hill estimators

To define the well-known and widely used Hill index [Hill, 1975], consider the order statistics $X_{n,1} \leq X_{n,2} \leq \ldots \leq X_{n,n}$ of the *i.i.d.* random variables $X_1, ..., X_n$. For a selected threshold $X_{n,n-k+1}$(or u_n), there are only k extreme observations lie in the farthest right tail, and use these k number of observations to define the Hill index estimator as follows.

$$\gamma_k = \frac{1}{k}\sum_{i=1}^{k} ln(X_{n,n-i+1}) - ln(X_{n,n-k}) \tag{1}$$

The tail index α_k is defined as $\frac{1}{\gamma_k}$. The main idea of this method is that if a random variable has a Pareto distribution then the log of this variable will have an exponential distribution with the tail index as the parameter. The asymptotic normality of the Hill estimator was established by many authors. See, for example, Häusler and Teugels [1985], Beirlant and Teugels [1987], and Beirlant et al. [2004]. As was mentioned before, the accuracy of the Hill estimator depends on the sample size n, and the tail length k.

In order to improve the conventional Hill index estimator defined in (2), Huisman et al. [2001] proposed a tail index estimator which, in fact, is the weighted average of Hill estimators for various values of k. To outline this method, consider the following model:

$$\gamma(k) = \beta_0 + \beta_1 k + \epsilon(k), \tag{2}$$

where $k = 1, ..., \kappa$.

Traditionally, the optimum k is chosen first and then the tail index is estimated using (1). In the present method, however, the Hill index $\gamma(k)$ is estimated for every k from 1 to $n/2$, and the estimated tail index is used as the dependent variable in (2). The property of the tail index exploited in developing the new estimator is that the Hill index estimator $\gamma(k)$ is unbiased only when $k \to 0$. Therefore, based on this property, the estimated β_0 is an unbiased estimator of the tail index. However, the OLS estimate of tail index is inefficient due to two reasons: (i) the variance of

$\gamma(k)$ is not constant for different k, and thus the error term is heteroscedastic, and (ii) an overlapping data problem exists due to the way in which $\gamma(k)$ is constructed. To overcome these problems, Huisman et al. [2001] suggest to use a weighted least squares method, which is given as follows:

Multiplying (2) by the weight $w(k)$, we obtain,

$$\gamma(k)w(k) = \beta_0^{wls}w(k) + \beta_1^{wls}kw(k) + u(k) \tag{3}$$

where $w(k)$ is defined in a vector form as $W = (\sqrt{1}, \sqrt{2}, \ldots, \sqrt{\kappa})$ and $u(k) = \epsilon(k)w(k)$. The new tail index estimator $\hat{\beta}_0^{wls}$ is both unbiased and efficient. See Huisman et al. [2001] for details.

The estimator of k is chosen by minimising the distance between $\gamma(k)$ and $\hat{\beta}_0^{wls}$ which is obtained from (3). As a goodness of fit tests for k we use two graphical-based tools: the plot of the empirical mean excess function and the Hill plot, which are explained in the following subsection.

3.2 Mean excess function and Hill plot

The mean excess function (MEF) proposed by Davison and Smith [1990] is widely used in practice to identify whether data exhibits a heavy-tail behavior. The MEF of a r.v. X with finite expectation and the right endpoint x_F is defined as follows:

$$e(u) = E(X - u|X > u), \ 0 \leq u < x_F, \tag{4}$$

where the quantity $e(u)$ can be reffered to as the mean excess over the threshold u.

This plot is based on the following properties: if X has exponential distribution with parameter λ, $Exp(\lambda)$ then $e(u) = \lambda^{-1}$ for all positive threshold values. On the other hand, if X has normal distribution then the MEF tends to be zero and if the distribution os X is subexponential then the MEF tends to ∞. Further details on the properties of the MEF can be found in Embrechts et al. [1997].

The MEF $e(u)$ for a random variable X with Pareto distribution $G_{\xi,\beta}$ is given by:

$$e(u) = \frac{\beta + \xi u}{1 - \xi}, \; u \in D(\xi, \beta), \; \xi < 1, \tag{5}$$

hence $e(u)$ is linear in u. This suggests that the threshold should be selected so that $e_n(x)$ is approximately linear for $x \geq u$.

The empirical counterpart of the MEF, used in data analysis, is given as:

$$\hat{F}_u = \frac{\sum_{i=1}^{n} X_{i,n} \mathbf{I}_{X_{i,n} > u}}{\sum_{i=1}^{n} \mathbf{I}_{X_{i,n} > u}} - u. \tag{6}$$

where u is the threshold and $\mathbf{I}$ is an indicator function. This empirical version is plotted against the values of $u = x_{i,n}$ for $k = 1, ..., n-1$. We employ this plot for visual testing of the robustness of k, where k the tail length.

Another popular popular approach for choosing k in practice is the Hill plot. Traditionally, Hill estimators, found by inverting (1) and then plotting them together with the 95%-confidence bounds. Using both the ME plot and the Hill plot to select the optimal tail length is a subjective approach that does not always provide a unique choice. In this paper, we choose optimal k and then use both plots as tools for testing of the robustness of k.

3.3 Estimation of the mean of heavy tailed distributions

To define the mean estimate, consider the $X_1, ..., X_n$ $i.i.d.$ random variables with the common distribution function F with regularly varying tails and tail index $\alpha > 1$ that satisfies following conditions

$$\begin{array}{c} lim_{t \to \infty} \frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} = x^{-\alpha},\ x > 0, \\ lim_{t \to \infty} \frac{1-F(t)}{1-F(t)+F(-t)} = p \in [0,1]\,. \end{array} \tag{7}$$

where $\alpha = 1/\gamma$ and γ is defined in (1). When $1 < \alpha < 2$, then (7) implies that F is in the domain of attraction of a stable law, otherwise when $\alpha > 2$, then F is in the domain of attraction of a normal distribution.

To obtain a consistent estimator of the population mean for any $\alpha > 1$, we use the idea of Peng [2001] and partition the population mean $E(X)$ into two components as follows:

$$E(X) = \int\limits_0^1 F^-(u)du = \int\limits_0^{1-k/n} F^-(u)du + \int\limits_{1-k/n}^1 F^-(u)du := \mu_n^{(1)} + \mu_n^{(2)}, \tag{8}$$

where $F^-(s) := inf\,\{x : F(x) \geq s\}$, $0 < s < 1$, denotes the inverse function of F and the sample fraction of extremes k, which is the number of observations in the farthest right tail.

The adjusted mean is estimated as follows:

$$\hat{\mu}_{adj} = \mu_n^{(1)} + \mu_n^{(2)}, \tag{9}$$

The $\mu_n^{(1)}$ and $\mu_n^{(2)}$ are estimated separately. The first component $\mu_n^{(1)}$ is the simple average of observations in the middle part of the distribution, excluding the extreme k observations in the right tail:

$$\mu_n^{(1)} = \frac{1}{n-k}\sum_{i=1}^{n-k} X_{n,i}. \tag{10}$$

The second component $\hat{\mu}_n^{(2)}$ is estimated using EVT and the k estreme observations as:

$$\hat{\mu}_n^{(2)} = \frac{k}{n} X_{n,n-k+1}\frac{\hat{\alpha}_k}{\hat{\alpha}_k - 1} \tag{11}$$

See Peng [2001] for more details.

3.4 Estimation of quantiles of heavy-tailed distributions

The maximum likelihood method and the method of probability weighted moments are the most frequently used methods for estimating quantiles of heavy tailed distributions. Smith [1985] has shown that for the tail index greater than -0.5, the asymptotic properties such as consistency, efficiency and normality of the maximum likelihood estimates hold. Coles and Dixon [1999] have found that the poor small sample properties of the maximum likelihood estimators can be improved by penalising maximum likelihood estimators. The detailed description of the probability weighted moments method can be found in Beirlant et al. [2004] and the references therein.

Let $X_1, ..., X_n$ be *i.i.d.* observations, with a common distribution function F, and $X_{n,1} \leq ... \leq X_n, n$ are the order statistics. Assume F has the following form

$$1 - F(x) = cx^{-\alpha}\, for\, x > X_{n,n-k} \tag{12}$$

where α is an unknown tail index and $X_{n,n-k}$ is the threshold value, and the optimal k is estimated by using the method discussed in section 3.1.

We obtain $\hat{\alpha}_k$ from the weighted average of Hill estimators, then $\hat{c}$ is computed as $\hat{c}_k = \frac{k}{n} X^{\hat{\alpha}}_{n,n-k}$. For $p_n = 0.001$, an estimator of a 99.9% quantile is defined as x_p and can be estimated as follows:

$$\hat{x}_p = (\frac{p_n}{\hat{c}})^{-\frac{1}{\hat{\alpha}}} \tag{13}$$

We will use this maximum likelihood estimate of the quantile for constructing CIs.

4 Estimation of confidence intervals for the quantile of heavy-tailed distributions

In estimating quantiles of heavy tailed distributions, only a part of the upper order statistics is used. The efficient the high quantile estimation depends on the reliably estimating the sample fraction k, which we determine using the weighted average of Hill estimators outlined in section 2.1. Several studies have proposed methods to construct asymptotically valid CIs for the quantile of heavy tailed distribution, these being: (i) normal approximation (NA) method and (ii) EL based method [Peng and Qi, 2006], and (iii) data tilting method [Hall and Yao, 2003]. The NA method was found to have much smaller coverage probabilities than the nominal levels. In what follows, we discuss the procedures for estimating CIs for quantiles using the methods mentioned above.

To explain the methods for construction CI, let assume $p_n \in (0, 1)$ and then a

$100(1-p_n)\%$ quantile of F is

$$\hat{x}_p = (1-F)^{-}(p_n) \tag{14}$$

where $(\cdot)^{-}$ denotes the inverse function of $(\cdot)$.

Then, the likelihood function is

$$L(\alpha, c) = \prod_{i=1}^{n} (c\alpha X_i^{-\alpha-1})^{\delta_i} (1 - cX_{n,n-k}^{-\alpha})^{1-\delta_i} \tag{15}$$

where $\delta_i = \mathbf{I}(X_i > X_{n,n-k})$, and $\mathbf{I}$ is an indicator function, taking values 0 and 1.

The maximum likelihood estimators of c and α are obtained by maximising the likelihood function (15). In the following subsections we briefly discuss the methods for construction CIs for quantiles of heavy tailed distributions.

4.1 Normal approximation method

Here, we briefly discuss the NA method for constructing CI for high quantiles.

Based on the Theorem from Peng and Qi [2006]

$$\frac{\hat{\gamma}\sqrt{k}}{log(\frac{k}{np_n})} log\frac{\hat{x}_p}{x_p} \xrightarrow{d} N(0,1) \tag{16}$$

Therefore, a $(1-\alpha)\%$ CI for x_p is

$$I_\alpha^n = (\hat{x}_p e^{-z_\alpha \frac{log(\frac{k}{npn})}{(\hat{\gamma}\sqrt{k})}}, \hat{x}_p e^{z_\alpha \frac{log(\frac{k}{npn})}{(\hat{\gamma}\sqrt{k})}}) \tag{17}$$

where $P(|N(0,1)| \leq z_\alpha) = \alpha$. This CI for x_p has asymptotically correct coverage probability α. The details of coverage expansion for i_α^n can be found in Peng and Qi

[2006].

4.2 Empirical likelihood method

Here, we briefly discuss the EL method for constructing CI for high quantiles.

Let $\hat{\alpha}$ and $\hat{c}$ denote the maximum likelihood estimators of parameters α and c, respectively. Now, the corresponding maximum log-likelihood function is,

$$l_1 = max_{\alpha>0,\, c>0} logL(\alpha, c) = logL(\hat{\alpha},\, \hat{c}). \tag{18}$$

Then, maximise $logL(\alpha, c)$ subject to $\alpha > 0$, $c > 0$ and

$$\alpha log x_p + log(\frac{p_n}{c}) = 0 \tag{19}$$

and denote the corresponding log-likelihood function by $l_2(x_p)$. Where $p_n = 0.001$, and x_p is the corresponding 99.9% quantile of the distribution. It can be shown that

$$l_2(x_p) = logL(\bar{\alpha}, \bar{c}), \tag{20}$$

where

$$\bar{\alpha}(\lambda) = \frac{k}{\sum_{i=1}^{k}(logX_{n,n-i+1} - logX_{n,n-k}) + \lambda logX_{n,n-k} - \lambda log x_p} \tag{21}$$

and

$$\bar{c}(\lambda) = X_{n,n-k}^{\bar{\alpha}(\lambda)} \frac{k-\lambda}{n-\lambda} \tag{22}$$

and λ satisfies two conditions

$$\bar{\alpha}(\lambda)logx_p + log(\frac{p_n}{\bar{c}(\lambda)}) = 0, \tag{23}$$

$$\bar{\alpha}(\lambda) > 0 \, and \, \lambda < k. \tag{24}$$

Therefore, the log of the likelihood ratio multiplied by -2 takes the form

$$l(x_p) = -2(l_2(x_p) - l_1). \tag{25}$$

Peng and Qi [2006] showed that $l\left(x_p\right)$ at the true quantile $x_{p,0}$ has χ^2 distribution with one degree of freedom: $l(x_{p,0}) \xrightarrow{d} \chi^2_{(1)}$ and a 95% CI for x_p is $I^{lr}_{\alpha_{cr}} = \{x_p : l\left(x_p\right) \leq u_{\alpha_{cr}}\}$, where $u_{\alpha_{cr}}$ is the α_{cr}-level critical point of a $\chi^2_{(1)}$ distribution. See Peng and Qi [2006] for proofs.

4.3 Data tilting method

The DT method is a weighted EL ratio method. An advantage of this method is that it can allocate higher weights to the region of the distribution of interest, in this case, the tail part of the distribution. For any weights $q = (q_1, ..., q_n)$ such that $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = 1$, we solve

$$(\hat{\alpha}(q), \hat{c}(q)) = argmax_{(\alpha,c)} log((c\alpha X_i^{-\alpha-1})^{\delta_i}(1 - cX_{n,n-k}^{-\alpha})^{(1-\delta_i)}). \tag{26}$$

This will result in

$$\hat{\alpha}(q) = \frac{\sum_{i=1}^{n} q_i\delta_i}{\sum_{i=1}^{n} q_i\delta_i(logX_i - logX_{n,n-k})}, \tag{27}$$

$$\hat{c}(q) = X_{n,n-k}^{\hat{\alpha}(q)} \sum_{i=1}^{n} q_i \delta_i. \tag{28}$$

Now, we define function $D_p(q)$ as a measure of distance between q and uniform distribution $q_i = 1/n$ as follows

$$D_p(q) = \sum_{i=1}^{n} q_i log(nq_i) \tag{29}$$

We minimise the distance $D_p(q)$ subject to following constraints $q_i \geq 0$, $\sum_{i=1}^{n} q_i = 1$ and

$$\hat{\alpha}(q) log \frac{x_p}{X_{n,n-k}} = log \frac{\sum_{i=1}^{n} q_i \delta_i}{p_n}.$$

This minimization procedure results in solving $(2n)^{-1} L(x_p) = min_q D_p(q)$ that gives us following measures

$$A_1(\lambda_1) = 1 - \frac{n-k}{n} e^{-1-\lambda_1}$$

and

$$A_2(\lambda_1) = A_(\lambda_1) \frac{log(x_p/X_{n,n-k})}{log(A_1(\lambda_1)/(p_n)}.$$

Then, by method of Lagrange multipliers, we have $q_i = q_i(\lambda_1, \lambda_2) =$

$$\begin{cases} \frac{1}{n} e^{-1-\lambda_1}, \, if \, \delta_i = 0 \\ \frac{1}{n} e^{-1-\lambda_1 + \lambda_2 \left(\frac{log(x_p/X_{n,n-k})}{A_2(\lambda_1)} - \frac{1}{A_1(\lambda_1)} - \frac{A_1(\lambda_1) log(X_i/X_{n,n-k}) log(x_p/X_{n,n-k})}{A_2^2(\lambda_1)} \right)} \, if \, \delta_i = 1, \end{cases} \tag{30}$$

where λ_1 and λ_2 satisfy $\sum_{i=1}^{n} q_i = 1$ and

$$\hat{\gamma}(q) log \frac{x_p}{X_{n,n-k}} = log \frac{\sum_{i=1}^{n} q_i \delta_i}{p_n}. \tag{31}$$

Then, according to the Theorem from Peng and Qi [2006] there exists solution $((\hat{\lambda})_1(x_p), (\hat{\lambda})_2(x_p))$, such that, for $(\lambda_1, \lambda_2) = ((\hat{\lambda})_1(x_p), (\hat{\lambda})_2(x_p))$,

$$-log(1 + \frac{\sqrt{k}\sqrt{log(k/(np_n))}}{n-k} \leq 1 + \lambda_1 \leq -log(1 - \frac{\sqrt{k}\sqrt{log(k/(np_n))}}{n-k}, \tag{32}$$

$$|\lambda_2| \leq k^{-1/4} \frac{k/n}{log(k/(np_n))} \tag{33}$$

and $L(x_p, 0) \xrightarrow{d} \chi^2_{(1)}$ with $(\lambda_1, \lambda_2) = ((\hat{\lambda})_1(x_p, 0), (\hat{\lambda})_2(x_p, 0))$ in the definition of $L(x_p, 0)$.

Therefore, a CI for x_p is $I_\alpha^{dt} = \{x_p : l\,(x_p) \leq u_\alpha\}$, where u_α is the α-level critical point of a $\chi^2_{(1)}$ distribution.

4.4 Risk measures: Value at Risk and Expected Shortfall

The main characteristic of a risk measure for calculating operational risk capital charge is it's ability to provide a reasonably accurate measure for the capital charge. It is known that the popular quantile based risk measure such as VaR or OR may underestimate the risk of a losses with heavy tails. The OR at the 99.9 per cent confidence level is the difference between the upper 99.9 percentile and the mean of the loss distribution:

$$OR = \hat{x}_p - \hat{\mu}_{adj}$$

where $\hat{\mu}_{adj}$ and $\hat{x}_p$ are defined in equations (9) and (13) respectively.

By definition, OR measure is based on the quantile of the distribution, and disregards extreme losses beyond the OR. On the other hand, the ES at 99.9 per cent confidence level is defined as follows

$$ES = E\{X_i | X_i > OR\} \tag{34}$$

where OR is the 99.9% OR. The fact that this ES measure is even more sensitive to the shape of the tail region of the loss distribution makes ES more attractive to use in the context of heavy-tailed distributions.

In what follows, we state the four most important properties that a coherent risk measure should satisfy.

Consider a set V of real-valued random variables. A function $\rho : V$ is called a coherent risk measure if it satisfies the following four conditions:

1. monotonous: $X \in V, X \geq 0 \implies \rho(X) \leq 0$,
2. sub-additive: $X, Y, X + Y \in V \implies \rho(X + Y) \leq \rho(X) + \rho(Y)$,
3. positively homogeneous: $X \in V,\ hX \in V, \implies \rho(h, X) = h\rho(X)$,
4. translation invariant: $X \in V,\ a \in \mathbb{R}, \implies \rho(X + a) = \rho(X) - a$

A sub-additivity measure is a desirable property for calculating the capital adequacy requirements. The sub-additivity axiom does not hold for VaR and therefore it is not a coherent risk measure [Acerbi and Tasche, 2001]. On the other hand, ES satisfies the above four conditions, including the sub-additivity axiom, and hence it is a coherent risk measure. This means that if we compute a risk measure for a bank that has a number of branches, then the sum of individual ESs constitutes an upper bound for the ES of the bank. Since OR is calculated merely as a measure of quantile of the loss distribution, OR does not have this property.

5 Data

This paper analyses nominal and real operational losses exceeding $1mln. incurred by businesses in the United Stated between May 1984 and May 2008. The real losses are expressed in the January 1997 level consumer Price Index. The data spans a 24 year period.

Recently, Landau, the Deputy Governor of the Bank of France, has recognized that banks lack reliable data over sufficient period of time for estimating ORs for banks. BCBS (2010) recommends using external databases to improve the OR estimation process. In our case, there is no sufficient number of losses available for each of the business sectors. For this reason, we study the pooled data of all the available US business losses, categorised by Basel business lines and event types. The pie charts in Figures 1 and 2 show the breakdown of losses by business lines and event types respectively. Two business lines losses, *Trading and Sales* and *Commercial Banking* account for approximately 29 per cent and 28 per cent of losses respectively, classified by business lines. The *Internal Fraud* losses constitute the largest proportion (more than 80 per cent) of losses classified by the event types.

We have assumed all the losses occurred over the 24 years to be *i.i.d.*. We believe when more data becomes available the OR can be estimated for each sector separately, assuming dependence of losses. Moreover, one can construct quarterly or even monthly loss series and study the time series structure of these losses, before estimating the OR. This line of research is left as one of the topics for the further research.

The pie chart in Figure 3 illustrates the breakdown of *Employment PWS* event type losses by business sectors, including: "Automotive", "Corporation", "Energy Corporation", "Insurance Company", "Agency", "Full Service Bank", "Retail", "Corporate Conglomerate", "Brokerage", "Software Corporation" and others. It appears that the largest proportion of *Internal Fraud* losses incurred by the "Automotive" sector.

Table 1 reports descriptive statistics of these losses, representing the distributional properties of losses across five business lines and three event types, as well as the size of the loss at the 50^{th}, 75^{th}, and 95^{th} percentiles. As expected, the percentiles for the real losses are generally slightly lower compared to the nominal losses.

The *Commercial Banking* losses are smaller than the *Trading and Sales* losses at all three percentiles. Moreover, *Retail Banking* has the second largest loss at the 95^{th} percentile, while *Retail Brokerage* is the smallest losses. The *Internal Fraud* losses is very large compared with the other two event types. However, both *External Fraud* and *Employment PWS* losses appear to be very close to each other at all three percentiles. Furthermore, the main difference between the nominal losses and the real losses was note only at the 95 percentile. Generally, the size of the the real losses at the 95^{th} percentiles are smaller than those of the corresponding nominal losses.

For all types of losses the mean is much larger that the median, indicating that loss distributions are heavyly skewed to the right. The skewness for all types of losses is positive and above 5. The standard deviation ranges from 66 mln for the *External Fraud* losses to 443 for the *Trading and Sales* losses.

6 Estimation of operational risks and expected shortfalls

In this section, we apply the methodologies discussed in sections 3 and 4 to five business lines and three event type business losses for estimating their operational risks (ORs) and expected shortfalls (ESs). Since some of these methods are not straightforward to apply and it involves many stages, we illustrate the their applications to the the business line *Employment PWS* losses in a step-by-step manner, which will make the process easily understood by the readers. We then discuss the overall results.

6.1 An application to *Employment PWS losses*

The time series plot of the *Employment PWS* losses is presented in Figure 4. It is evident that many extremely large losses occurred during these 24 years with the largest loss occurred in 1996 . Moreover, the losses are irregularly spaced over the period 1994 to 2008, with the number of losses increased after 1998. The probability distribution function for the *Employment PWS* losses plotted in Figure 5 indicates that this distribution is heavily skewed to the right.

Now, we explain the estimation of OR for the *Employment PWS* losses step-by-step as follows:

1. Using equation (1), compute the Hill estimate $\hat{\gamma}_k$ for this loss data for $k = 1, ..., n/2\,(n = 237)$.

2. The next step is to choose the tail length k. To choose such a k we calculate the weighted average of Hill estimators using (3) and inverting the value which is found to be 1.3. Then, we find the optimal k that minimises the distance

between 1.3 and the Hill estimators for different values of k. Such a k is found to be 21.

3. Next, the MEF-plot for the data excluding the top 21 observations is given in Figure 6. We observe that the MEF has approximately linear pattern, indicating that the behavior of mean excesses at the selected threshold is reasonably stable.

4. Then, Using the bootstrap method compute 95% CI for the tail index as the percentiles of the sampling distribution presented in Figure 7. The CI estimate for the tail index is [1.1 , 1.9].

5. Using (9), the estimate of the adjusted mean $\hat{\mu}_{adj} = 41.8$, which is the sum of the sample average of non-tail observations computed using (10), $\hat{\mu}_n^{(1)} = 11.5$, and the mean for the right tail computed using (11), $\hat{\mu}_n^{(2)} = 30.3$. The estimated adjusted mean is about 40 % greater than the simple sample mean of the losses for this business line.

6. Using (13), the 99.9% quantile estimate, $\hat{x}_{(p_n=0.001)} = 2,475$.

7. The OR estimate is the difference between the quantile and the mean of the loss distribution, $OR = 2,445$. The ES estimate using (34) is $3,582$.

8. The 95 per cent EL CI for the quantile is estimated as follows:

   First, the asymptotic profile log-likelihood function is plotted in Figure 8. In this plot, the log-likelihood ratio $l\,(\mu)$, which is defined in (15), is plotted against a range of possible values of the population quantiles. Then, to construct the 95 % CI estimates: (i) draw a horizontal line $\chi^2_{(1)} = 3.84$ through the asymmetric profile log-likelihood curve, (ii) draw two vertical lines through the points of intersections A and B, and (iii) the lower and upper limits of the CI estimates for x_p are the points at which these vertical lines intersect the x_p axis. Now, it is easy to deduce that the EL CI estimate is $[1,327\,,\,5,580]$.

9. The 95 per cent DT CI is estimated similarly to the EL CI. The weights for adjusting the profile log-likelihood function are computed using equation (30). While the weights for the middle of the distribution are constant, the tail weights are not (Figure 9). The weighted asymptotic profile log-likelihood function is plotted in Figure 10. The estimate of DT CI is [1,385 , 5,029]. Note that this interval estimate is 17% narrower than the EL CI estimate. The width of the DT CI has been reduced by optimizing the weights given to the losses in the tail - the most important part in the estimation of OR. As a result, the DT CI is contained in the EL CI.

10. The 95 per cent NA CI is $[886\,,\,8,638]$ and the BT CI is $[978\,,\,6,668]$. It is obvious that the NA CI is too wide with the very small lower bound and very high upper bound. The BT CI is 36% narrower than the NA CI, however it is 25% wider than the EL CI and 36% wider than the DT CI.

Tables 2 and 3 present the estimated ORs together with their 95% CIs for $p_n = 0.001$, and ESs. The overall results are discussed in the following section.

6.2 Discussion of the results

In this section we discuss the results of the application of the methodologies described in Sections 3 and 4 for estimating ORs and ESs of eight business lines/event type losses. Further, we provide the details of difficulties encountered in applying some of the methods. Based on the results and our experience in this application, we make some recommendations for the use of methodologies that work well under various conditions for estimating ORs.

One of the most important parameters of heavy tailed distributions is the tail index. The degree of tail thickness increases as the tail index decreases. The second column of Table 2 reports the estimates of the tail index for losses classified by five business lines and three event types. All of the estimated tail index values are below two, indicating that all severity loss distributions have very heavy right tails. Moreover, it is equally imperative to know how much this tail index estimate can vary, as indicated by its standard deviation and/or interval estimate. In this paper, we estimated bootstrap CI for the tail index. The point estimates of the tail index and its CI estimates are reported in columns 3 and 4 of Table 2. The distribution of *Commercial Banking* business line losses has the smallest tail index estimate, while that of *Retail Banking* losses has the largest tail index estimate. On the other hand, for the *Internal Fraud* even type losses, the CI estimate of the tail index is the narrowest, whereas for the *Commercial Banking* losses it is the widest interval estimate.

The columns 4 and 5 of tables 1 and 2 report the sample averages and the estimates of adjusted means respectively. Based on these two estimates for the mean of the operational losses, the eight business line/event type losses constitute two groups. The group one consists of *Retail Brokerage*, *Commercial Banking*, *External Fraud*, and *Employment and PWS* losses for which the adjusted mean exceeds the sample mean. The group two, on the other hand, consists of *Trading and Sales*, *Retail Banking*, *Asset Management* and *Internal Fraud* losses for which the sample mean is greater than the adjusted mean estimate. This difference is largely due to the fact that for the first group of losses, the 95^{th} percentile losses reported in Table 1 are relatively smaller than that for the second group.

Table 3 reports the estimates of ORs and ESs together with four different CI estimates for ORs. The largest OR estimate is obtained for the *Commercial Banking* losses, which is \$7,451 mln. The estimate of ES is about 50% greater than that of OR counterpart. The *Commercial Banking* losses constitute the second largest proportion of all losses classified by business lines, which is 28%. Moreover, the losses of this business line have the second largest standard deviation and the maximum loss. The CIs for this OR are the widest.

The second largest OR estimate is obtained for the *Trading and Sales* losses, which is \$5,500 mln. The estimated ES is about 45% greater than the OR. The *Trading and Sales* losses constitute the largest proportion of all losses classified by business line, which is 29%. Moreover, the standard deviation of this business line losses is the largest, with the maximum of overall losses. The estimated CIs for this OR are the second widest. The DT CI is 20% narrower than the EL CI.

The third largest OR estimate is obtained for the *Internal Fraud* losses, which is \$4,151 mln. The estimated ES is about 40% greater than the OR. The *Internal Fraud* losses constitute by far the largest proportion of all losses classified by event type, which is 82% and these losses have the largest sample size. Moreover, the losses of this event type have the third largest standard deviation and the largest loss at the 95^{th} percentile. The estimated CIs for the OR are among the narrowest. The DT CI estimate is 10% narrower than the EL CI estimate. The BT CI is slightly wider than other CIs (Table 3).

The fourth largest OR estimate is obtained for the *Asset Management* losses, which is \$3,803 mln. The estimated ES is about 40% greater than the OR. The *Asset Management* losses have the smallest sample size. The lower bound of the CI for the tail index is one. The DT CI estimate is about 10% narrower than the EL CI estimate. Both the NA CI and the BT CI are much wider than other CIs and are not recommended to be used (Table 3). Again, for these losses, the DT CI is the recommended one for the OR.

The next largest OR estimate is obtained for the *Employment and PWS* losses, which is \$2,433 mln. The estimate of ES is about 40% greater than that of OR. The DT CI estimate is about 30% narrower than the EL CI estimate. The BT CI is slightly wider than NA CIs (Table 3).

The very next largest OR estimate is obtained for the *External Fraud* losses, which is \$2,264 mln. The estimated ES is about 45% greater than the OR. The *External Fraud* losses have relatively small sample size. The DT CI is about 20% narrower than the EL CI.

The second smallest OR estimate is obtained for the *Retail Brokerage* losses, which is \$1,394 mln. The estimate of ES is about 50% greater than the OR estimate. The lower limit of this CI for the tail index is below one. The DT CI estimate is slightly narrower than the EL CI estimate.

The smallest OR estimate is obtained for the *Retail Banking* losses, which is \$748 mln. The estimated ES is about 20% greater than the OR. The*Retail Banking* losses have the largest tail index estimate. The DT CI estimate is slightly narrower than the EL CI estimate.

Overall, when the tail index is near one with the wide CI and a small lower bound, the CIs for OR become very wide, especially for the NA CI and the BT CI. Moreover, the DT CIs are up to 20% narrower than their EL CI counterparts. As the tail index increases the estimated CIs for OR become narrower and, the difference between OR estimates and ES estimates decreases.

It appears that losses with a relatively small OR estimates, such as *Retail Banking* and the *Retail Brokerage* losses either the DT CI or EL CI can be used. The NA method is not recommended as it yields extremely wide CIs with the smallest lower bounds and highest upper bounds for some types of losses. The BT method yield CIs that are narrower than NA CIs, but wider than the EL-based CIs and can be used if needed; the coverage properties of BT CI estimates need to be assessed. Overall, the DT method works well for every type of loss, as is evident from our empirical application and the fact that this method has the correct coverage probability.

7 Conclusion

This paper employs non-parametric approaches specifically designed for heavy-tailed distributions and construct consistent and unbiased point and interval estimates for the OR, which is the unexpected operational losses for US businesses. Recently, The Basel Committee on Banking Supervision [2010] provides guidelines for improving AMA for modeling OR, emphasizing the need to come up with a reliable OR estimate, based on which the economic capital charge to cover operational losses is calculated. A consensus emerging from several academic studies and BCBS (2010) is the importance of adequately modeling the heavy tail nature of severity loss distributions in order to come up with reliable OR estimates. To this end, this paper makes a significant contribution on the improvement of AMA by adapting non-parametric methods for heavy-tailed distributions.

US business losses constituting five business lines and three event types are used in this paper for an empirical application. The point and 95% interval estimates for ORs were constructed by non-parametric methods. Furthermore, a coherent risk measure, expected shortfall (ES) was estimated for all eight types of losses. The point and/or interval estimators of the mean and quantiles of heavy tailed distributions are found to be very sensitive to the size of the tail index. In order to eliminate the small-sample bias present in the widely used Hill-index as a tail index estimator, this paper employs a weighted average of Hill estimators to ensure that the tail index estimator is reliable.

The estimates of 99.9 % ORs and ESs of five business lines and three event type losses range from \$748 mln to \$7,451 mln and from \$919 mln to \$11,491 mln, respectively. The *Commercial Banking* losses have the largest OR and ES, while the *Retail Banking* losses have the smallest ones. The CIs for the OR are asymmetric with very high upper limits, indicating the high degree of uncertainty associated with OR point estimates. The estimated CIs are extremely wide when the tail index is close to one (*Commercial Banking*) and relatively narrow when it is close to two (*Retail Banking*). The closer the tail index (inverse of the shape parameter) to one, the wider is the CI for OR. Both the EL and DT methods yield relatively narrow CIs. For all types of losses, the DT method is contained in EL CI. By construction, the DT method for interval estimation is the weighted average of EL method. In the estimation of OR, one would like more weights to be given to the tail part of the loss distribution, which we did in this application. As a consequence, DT interval estimates turned out to be the best, which we recommend for all type of severity losses.

The non-parametric methods adapted in this paper have much wider applications than the one illustrated in this paper. For example, these methods can be employed to estimate another currently popular risk measure, credit risk. These findings have implications for economic capital requirements and risk management.

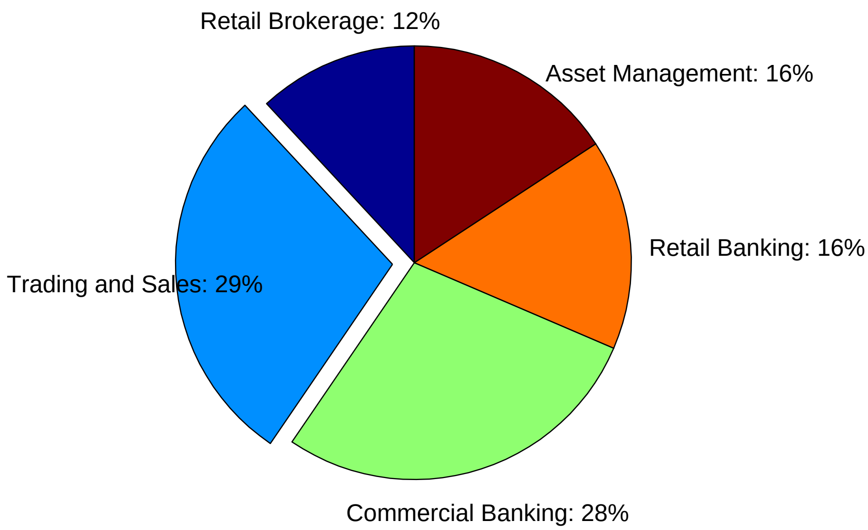

Figure 1: Pie chart of losses by five business lines.

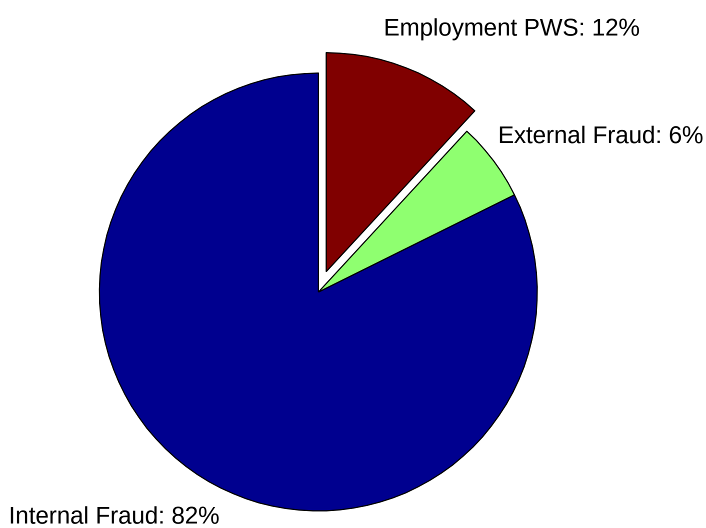

Figure 2: Pie chart of losses by three event types

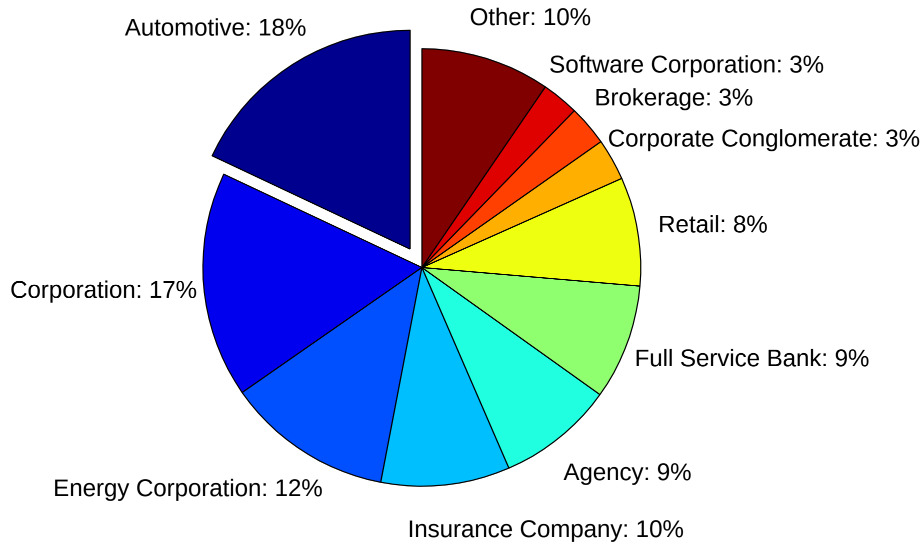

Figure 3: Pie chart for the *Employment PWS* losses by various business sectors

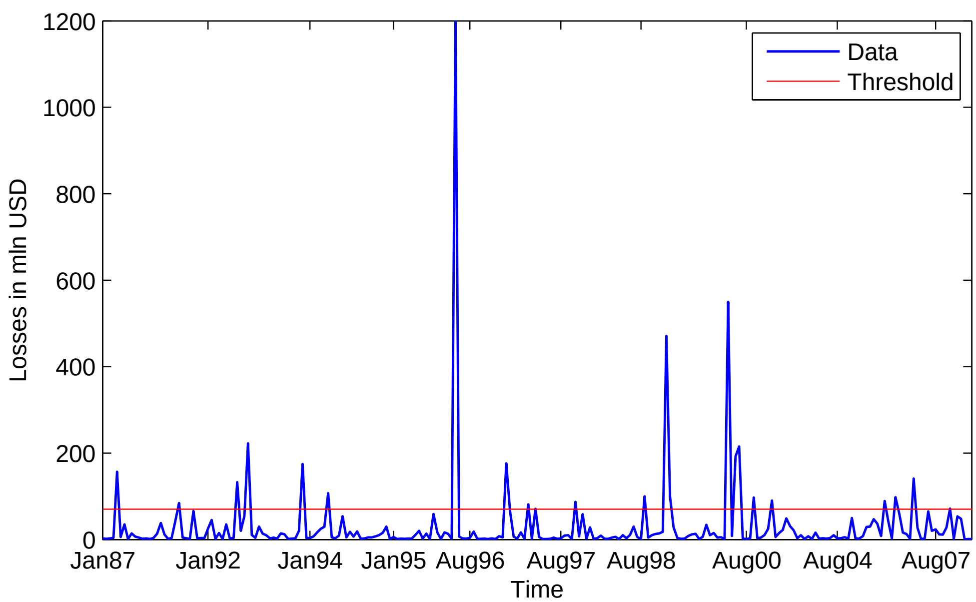

Figure 4: The time series plot of *Employment PWS* losses in mln USD. The threshold is $70 mln.

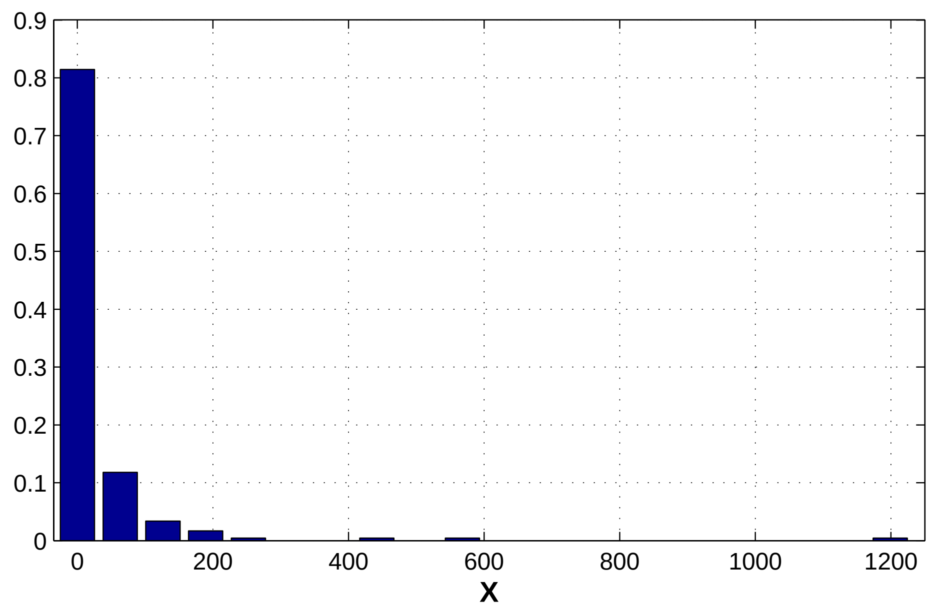

Figure 5: Histogram of the *Employment PWS* losses

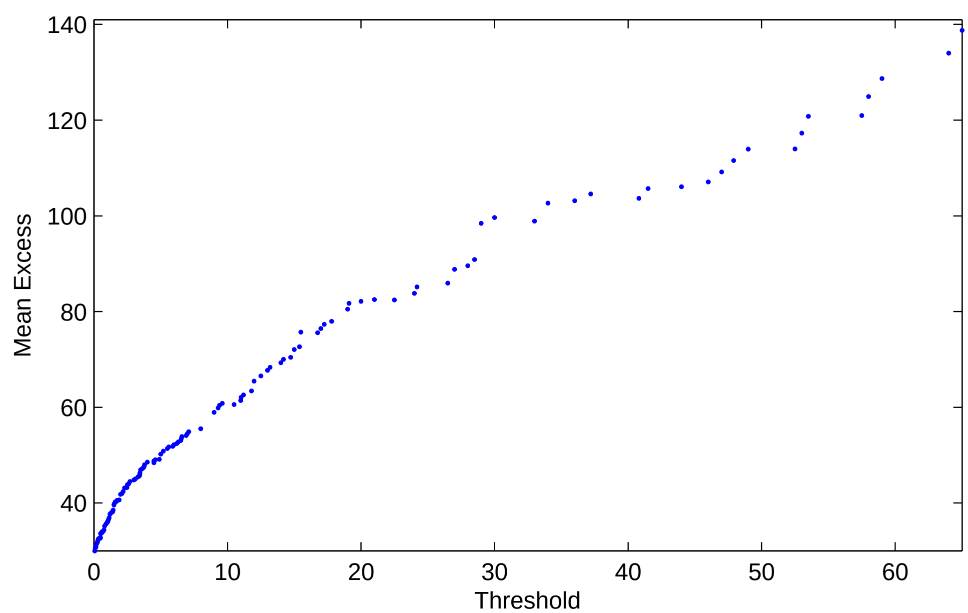

Figure 6: Mean excess function of the *Employment PWS* losses excluding losses exceeding $70 mln.

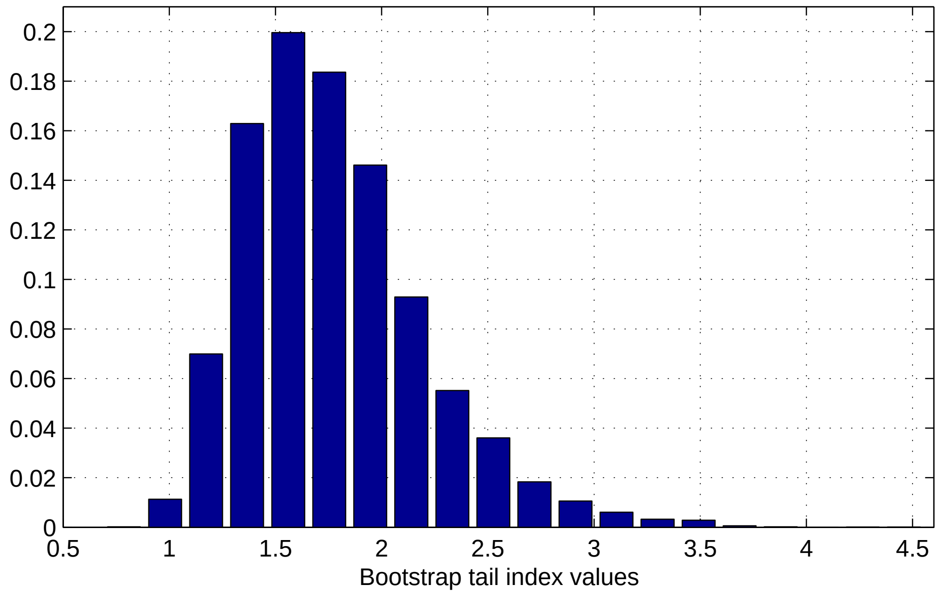

Figure 7: *Employment PWS* losses: bootstrap sampling distribution of the tail index estimate values (with 10,000 replications)

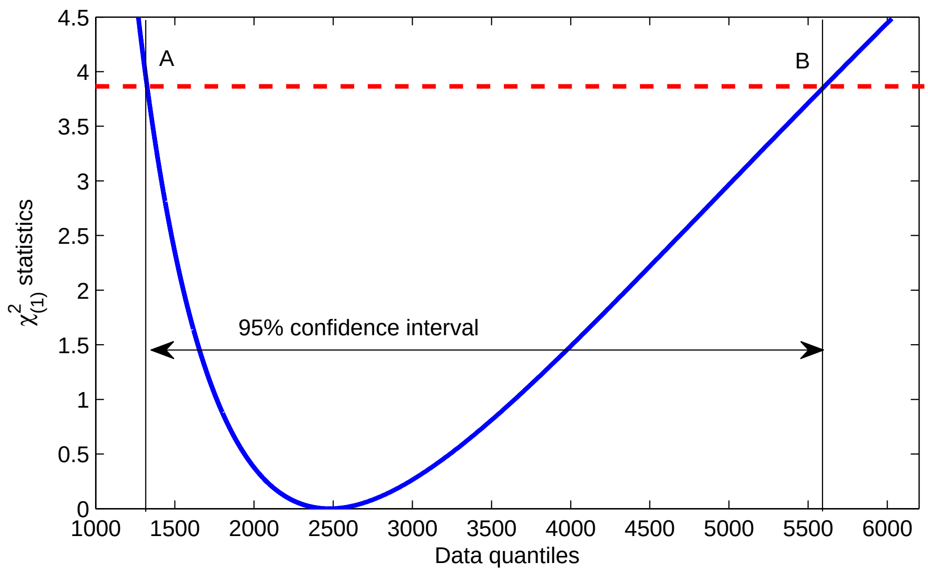

Figure 8: The profile log-likelihood function of the empirical likelihood method (section 4.2) for the *Employment PWS* losses. The 95% empirical likelihood confidence bound estimate is $[C_l,\, C_u] \equiv [1,327\,,\, 5,580]$

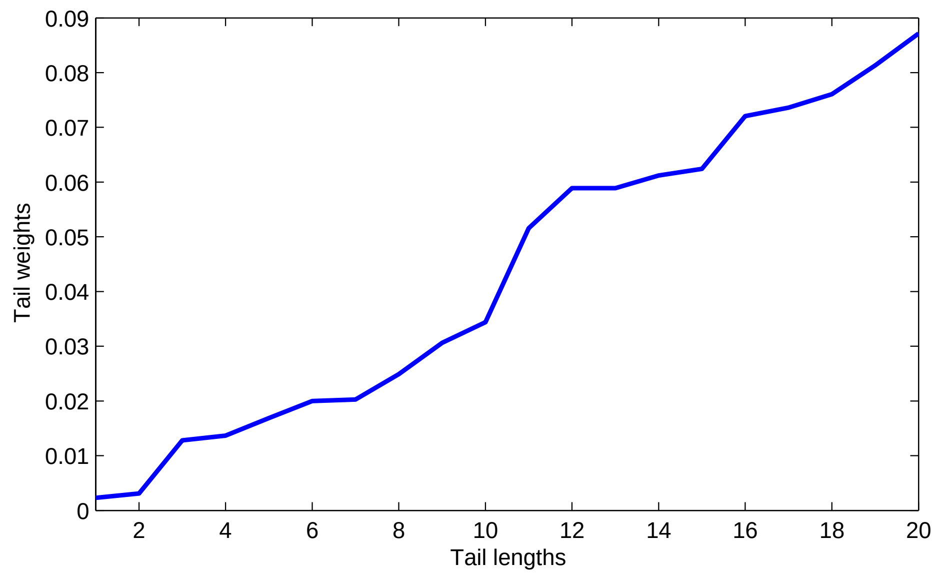

Figure 9: The tail weights for the data tilting method computed using equation (30) for the *Employment PWS* losses

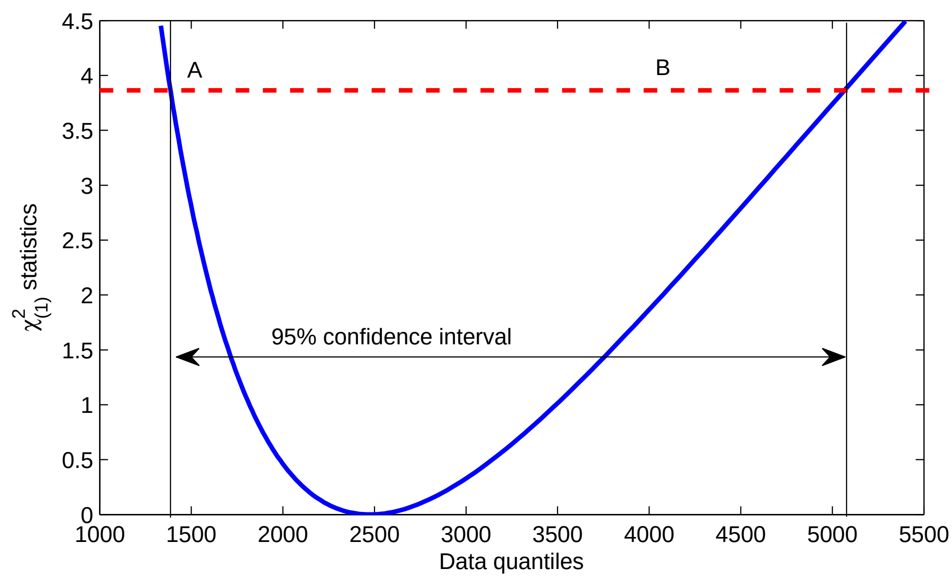

Figure 10: The profile log-likelihood function of the data tilting method (section 4.3) for the *Employment PWS* losses. The 95% data tilting confidence bound estimate is $[C_l,\, C_u] \equiv [1,385\,,\, 5,029]$

Table 1: Descriptive statistics for operational losses by Basel business lines and event types.

Loss type	Sample size	% of all losses	Sample mean	Median	75^{th}*perc.*	95^{th}*perc.*	St.dev.	Skewness	Max
Operational losses: Basel business line									
Retail Brokerage	198	12	34.1	3.6	14.0	98.2	154.3	7.7	1,499.0
Trading and Sales	141	29	114.9	14.0	50.3	399.0	443.1	7.4	4,399.0
Commercial Banking	176	28	90.4	9.4	40.9	223.7	376.2	7.3	3,599.0
Retail Banking	123	16	72.3	5.0	23.0	432.8	246.4	5.8	2,085.0
Asset Management	100	16	89.4	24.8	78.0	424.8	196.3	5.1	1,599.0
Operational losses: Basel event type									
Internal Fraud	490	82	99.6	10.7	42.0	504.0	330.1	6.1	3399.0
External Fraud	130	6	26.5	7.0	16.0	118.3	73.7	6.2	674.0
Employment PWS	237	12	29.7	5.9	21.4	103.7	96.0	8.9	1,199.0
Operational losses: Basel business line (in real terms)									
Retail Brokerage	191	12	33.6	3.0	12.3	86.3	162.6	8.1	1,570.1
Trading and Sales	141	30	111.1	15.7	49.7	363.0	430.4	7.5	4,278.0
Commercial Banking	177	28	84.0	9.1	37.2	200.1	342.2	7.0	3,165.2
Retail Banking	120	14	63.1	3.9	23.4	434.8	212.7	6.0	1,860.4
Asset Management	100	16	82.4	21.7	72.3	361.3	194.8	6.1	1,699.5
Operational losses: Basel event type (in real terms)									
Internal Fraud	501	82	91.0	9.2	41.0	450.8	297.3	6.0	2,665.0
External Fraud	131	6	25.3	5.8	14.9	140.1	66.2	5.1	528.3
Employment PWS	241	12	28.9	5.5	20.0	103.8	95.8	9.2	1,225.2

Table 2: Operational risk estimates.

Loss type	Threshold	Tail index	Tail index CI	μ_{adj}	Quantile
Operational losses: Basel business line					
Retail Brokerage	30.0	1.3	[0.8 1.7]	40.6	1,418.4
Trading and Sales	70.0	1.2	[0.8 1.6]	98.6	5,598.9
Commercial Banking	99.0	1.1	[0.8 1.8]	143.3	7,594.4
Retail Banking	49.0	1.8	[1.3 1.9]	24.4	772.6
Asset Management	44.0	1.3	[1.0 1.6]	76.2	3,879.3
Operational losses: Basel event type					
Internal Fraud	99.0	1.3	[1.1 1.5]	75.6	4,227.0
External Fraud	27.5	1.2	[1.0 1.6]	41.3	2,305.4
Employment PWS	70.4	1.3	[1.1 1.9]	41.8	2,475.0

The estimation results are reported only for the nominal losses. The difference between nominal and real losses is negligible; see section 5 for details.

Table 3: Confidence intervals for operational risks.

Loss type	OR	ES	NA CI	EL CI	DT CI	BT CI
Operational losses: Basel business line						
Retail Brokerage	1,394	1,982	[399 5,044]	[762 3,139]	[766 2,972]	[448 3,441]
Trading and Sales	5,500	8,067	[1,362 23,013]	[2,808 13,433]	[3,087 11,946]	[1,861 17,156]
Commercial Banking	7,451	11,491	[1,798 32,096]	[3,744 18,525]	[3,936 16,483]	[2,599 20,578]
Retail Banking	748	919	[257 2,322]	[457 1,565]	[482 1,506]	[607 2,101]
Asset Management	3,803	5,348	[875 9,715]	[2,030 8,651]	[2,157 8,136]	[1,501 9,338]
Operational losses: Basel event type						
Internal Fraud	4,151	5,730	[2,031 8,796]	[2,904 6,518]	[2,999 6,276]	[1,675 7,795]
External Fraud	2,264	3,400	[444 6,354]	[1,049 5,939]	[1,153 5,230]	[907 6,675]
Employment PWS	2,433	3,451	[886 8,638]	[1,327 5,580]	[1,385 5,029]	[978 6,668]

OR - Operational Risk,
ES - Expected Shortfall,
NA CI - Normal approximation based confidence interval estimate,
EL CI - Empirical likelihood confidence interval estimate,
DT CI - Data tilting confidence interval estimate,
BT CI - Sub-sample bootstrap confidence interval estimate.

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