Attilio Meucci

# Review of Discrete and Continuous Processes in Finance

## Theory and Applications

- **Slides from paper**

**“Review of Discrete and Continuous Processes in Finance - Theory and Applications”**

**available at www.symmys.com > Research > Working Papers**

- **MATLAB code available at www.symmys.com > Teaching > MATLAB**

## advanced dynamics and invariants



$\mathbb{P}$ : **estimate the future**

$\mathbb{Q}$ : **interpolate the present**

| | discrete time ($\mathbb{P}$) | continuous time ($\mathbb{Q}$) |
|---|---|---|
| base case | random walk | Levy processes |
| autocorrelation | ARMA | Ornstein-Uhlenbeck |
| long memory | fractional integration | fractional Brownian motion |
| volatility clustering | GARCH | stochastic volatility<br>subordination |

# random walk



$$X_{t+1} = X_t + \boxed{\epsilon_{t+1}},$$

invariants (i.i.d.)

# random walk



$$X_{t+1} = X_t + \boxed{\epsilon_{t+1},}$$

invariants (i.i.d.)

**continuous variable**

$\epsilon_t \sim$ **stable, elliptical, log-distributions, etc.**

# random walk



$$X_{t+1} = X_t + \boxed{\epsilon_{t+1},}$$

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**continuous variable**

$$\epsilon_t \sim \mathrm{N}\left(\mu, \sigma^2\right)$$

# random walk



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invariants (i.i.d.)

**continuous variable**

$\epsilon_t \sim \mathrm{N}\left(\mu, \sigma^2\right)$

**discrete variable**

$\epsilon_t \sim$ **(generalized) Bernoulli**

# random walk

$$X_{t+1} = X_t + \boxed{\epsilon_{t+1},}$$

invariants (i.i.d.)

**continuous variable**

$$\epsilon_t \sim \mathrm{N}\left(\mu, \sigma^2\right)$$

**discrete variable**

$$\epsilon_t \sim \mathrm{Po}\left(\lambda; \Delta\right)$$

$$p_k \equiv \mathbb{P}\left\{\epsilon_t = k\Delta\right\} \equiv \frac{\lambda^k e^{-\lambda}}{k!}$$

# random walk



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**invariants (i.i.d.)**

**continuous variable**

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**generalized representations**

**mixture models:**
**prototype "regime shift"**

$$\epsilon_t \stackrel{d}{=} (1 - B_t) Y_t + B_t Z_t$$

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**conditional location-dispersion:**
**prototype “stochastic volatility”**

$$\epsilon_t \stackrel{d}{=} \mu_t + \sigma_t Z_t$$

**e.g. Student t:** $\mu_t \equiv \mu$ $\quad \nu / \sigma_t^2 \sim \chi^2_{\nu;}$ $\quad Z_t \sim \mathrm{N}\left(0, \sigma^2\right)$

# Levy processes



$$X_{t+1} = X_t + \boxed{\epsilon_{t+1},}$$

**invariants (i.i.d.)**

$$\boxed{\epsilon_t} \equiv X_t - X_{t-1} = \left(X_t - X_{t-\frac{1}{K}}\right) + \cdots + \left(X_{t-1+\frac{1}{K}} - X_{t-1}\right)$$

**infinitely divisible**

# Levy processes



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$$\epsilon_{t,\tau} \equiv B_t^{\mu,\sigma^2} - B_{t-\tau}^{\mu,\sigma^2} \sim \mathrm{N}\left(\mu\tau, \sigma^2\tau\right)$$

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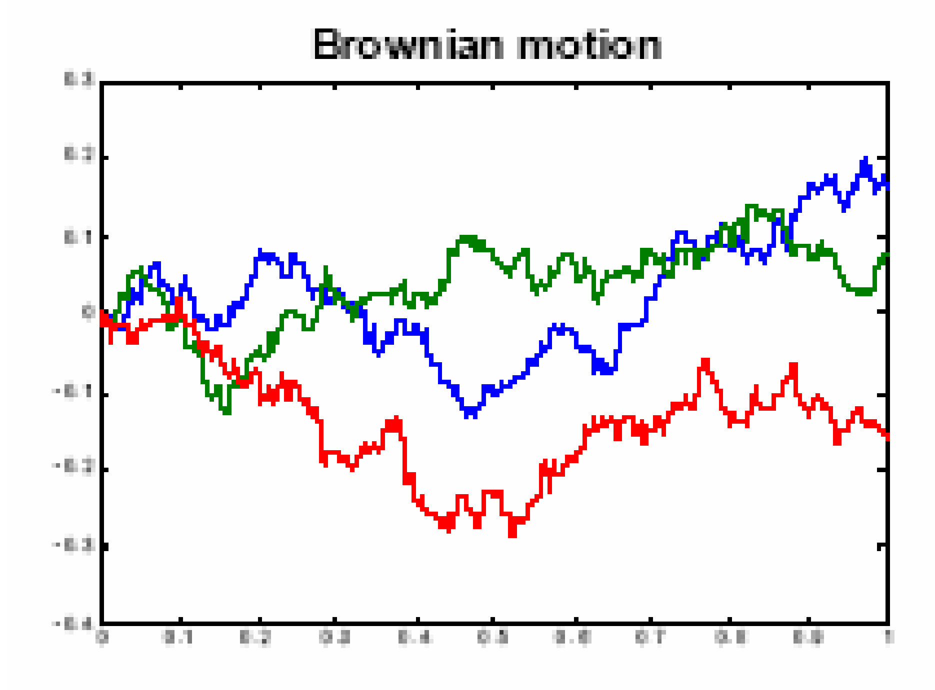

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\mathbb{P}\left\{\epsilon_{t,\tau} = 0\right\} &\approx 1 - \lambda\tau \\
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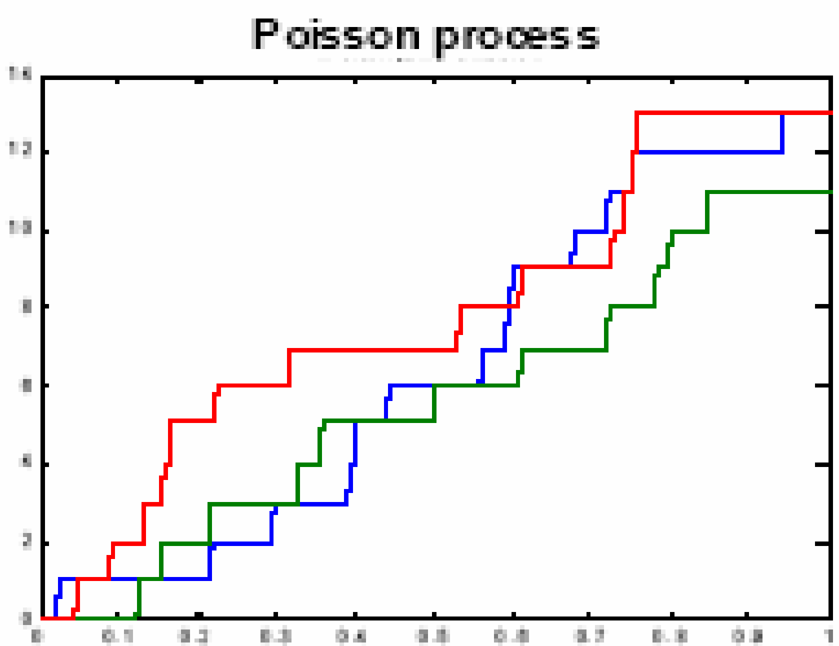

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mixture models:
prototype "regime shift"

$$\epsilon_t \stackrel{d}{=} (1 - B_t) Y_t + B_t Z_t$$

**"abridged" Levy-Khintchine**

$$X_t = B_t^{\mu,\sigma^2} + \int_{-\infty}^{+\infty} P_t^{\Delta,\lambda(\Delta)} d\Delta.$$

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**Black-Scholes** $dX_t = \mu dt + \sigma dB_t$

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**Black-Scholes** $\quad dX_t = \mu dt + \sigma dB_t$

**jump-diffusion** $\quad X_t = B_t^{\mu,\sigma^2} + \sum_{n=1}^{P_t^\lambda} Z_n, \qquad \lambda\left(\Delta\right) \equiv \lambda f_Z\left(\Delta\right)$

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**subordination**

$$X_t \stackrel{d}{=} B_{T_t}^{\mu,\sigma^2}$$

# random walk



invariants (i.i.d.)

$$X_{t+1} = X_t + \boxed{\epsilon_{t+1},}$$

# Levy processes



invariants (i.i.d.)

$$X_{t+1} = X_t + \epsilon_{t+1},$$

$$dX_t = \mu dt + \sigma \boxed{dB_t}$$

# autocorrrelation: ARMA



$$X_{t+1} = X_t + \epsilon_{t+1},$$

invariants (i.i.d.)

$$X_{t+1} = aX_t + \boxed{\epsilon_{t+1}} \quad |a| < 1$$

AR (1)

$$\mathrm{Cor}\left\{X_t, X_{t-\tau}\right\} = e^{(\ln a)\tau}$$

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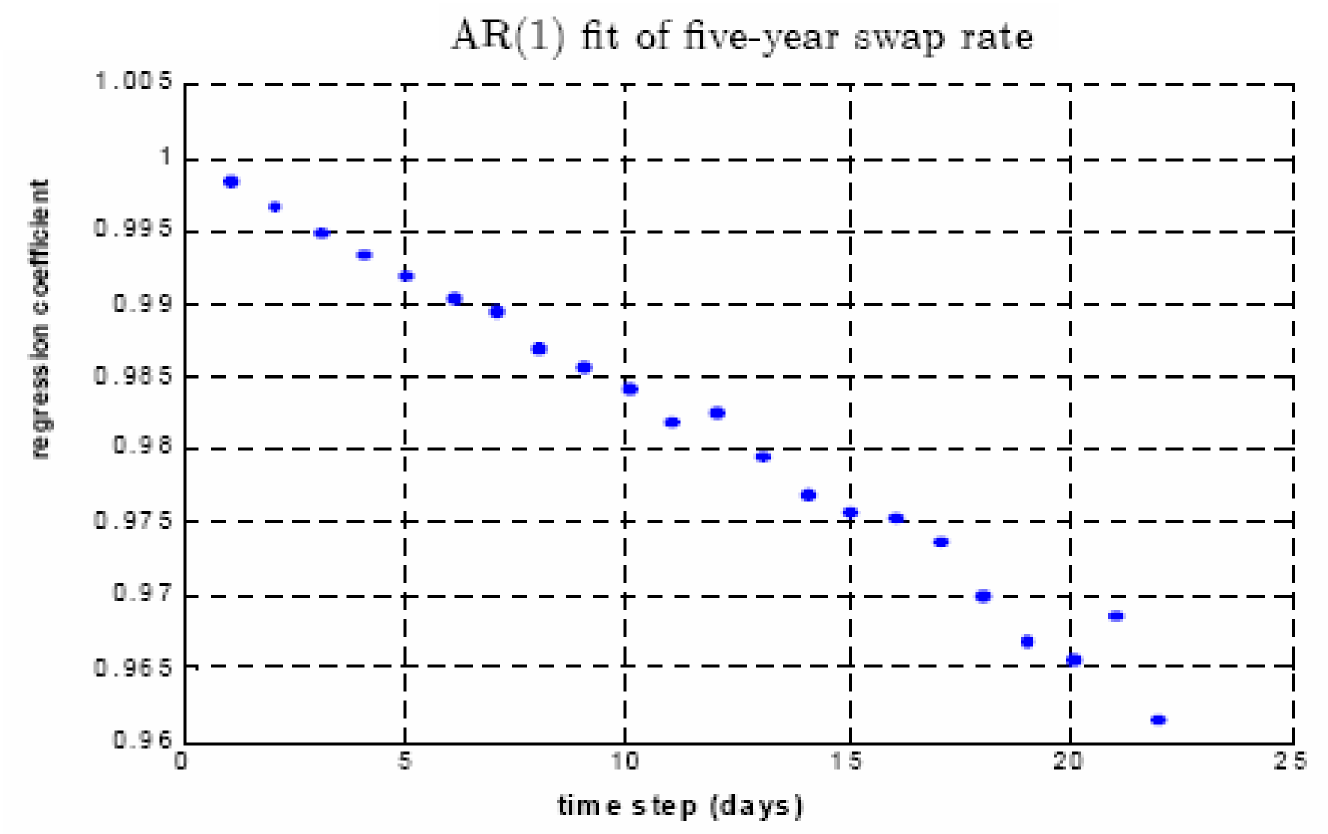

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$$\mathrm{Cor}\{X_t, X_{t-\tau}\} = e^{(\ln a)\tau}$$

invariants (i.i.d.)

$$\prod_{j=1}^{p} (1 - a_j \mathcal{L}) X_t = D_t + \prod_{j=1}^{q} (1 - b_j \mathcal{L}) \boxed{\epsilon_t,}$$

$$\mathcal{L}X_t \equiv X_{t-1},$$

$\mathrm{ARMA}(p, q).$

$$\mathrm{Cor}\{X_t, X_{t-\tau}\} \approx e^{-\gamma\tau}$$

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$\mathcal{L} X_t \equiv X_{t-1},$

ARMA$(p, q)$,

$$(1 - \gamma \mathcal{L})^{-1} \equiv \sum_{k=0}^{\infty} (\gamma \mathcal{L})^k$$

$$\operatorname{Cor}\{X_t, X_{t-\tau}\} \approx e^{-\gamma\tau}$$

ARMA$(\infty, 0)$

ARMA$(0, \infty)$

# autocorrrelation: ARMA



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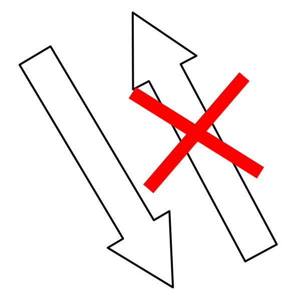

Wold theorem

$\mathrm{ARMA}(0, \infty)$

# autocorrelation: Ornstein-Uhlenbeck



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$$\text{Cor}\{X_t, X_{t-\tau}\} = e^{(\ln a)\tau}$$

$$dX_t = \mu dt + \sigma dB_t$$

invariants (i.i.d.)

$$dX_t = -\theta\,(X_t - m)\,dt + \sigma \boxed{dB_t.} \qquad \theta > 0.$$

# autocorrelation: Ornstein-Uhlenbeck



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$$X_t \stackrel{d}{=} m + e^{-\theta\tau}\left(X_{t-\tau} - m\right) + \epsilon_{t,\tau}$$

$$\epsilon_{t,\tau} \sim \mathrm{N}\left(0, \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta\tau}\right)\right)$$

# autocorrelation: Ornstein-Uhlenbeck



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$$X_t \stackrel{d}{=} m + e^{-\theta\tau} (X_{t-\tau} - m) + \epsilon_{t,\tau}$$

decay to equilibrium

$$\epsilon_{t,\tau} \sim \text{N}\left(0, \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta\tau}\right)\right)$$

# autocorrrelation: ARMA > long memory

$$X_{t+1} = X_t + \epsilon_{t+1},$$

$$X_{t+1} = aX_t + \epsilon_{t+1} \qquad |a| < 1 \qquad \text{AR (1)}$$

$$\text{Cor}\{X_t, X_{t-\tau}\} = e^{(\ln a)\tau}$$

$$\prod_{j=1}^{p} (1 - a_j \mathcal{L})\, X_t = D_t + \prod_{j=1}^{q} (1 - b_j \mathcal{L})\, \epsilon_t, \qquad \text{ARMA}(p, q).$$

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**Wold theorem**

$$\text{ARMA}(0, \infty)$$

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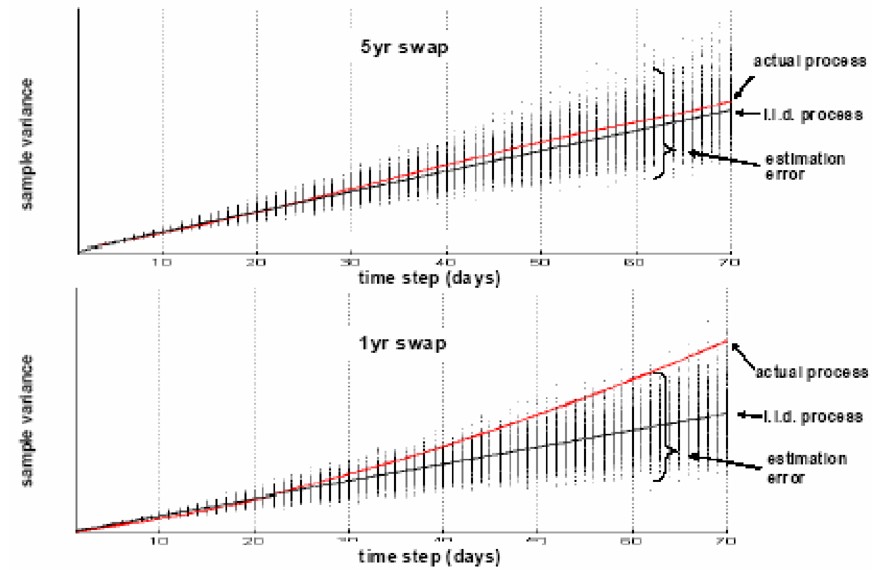
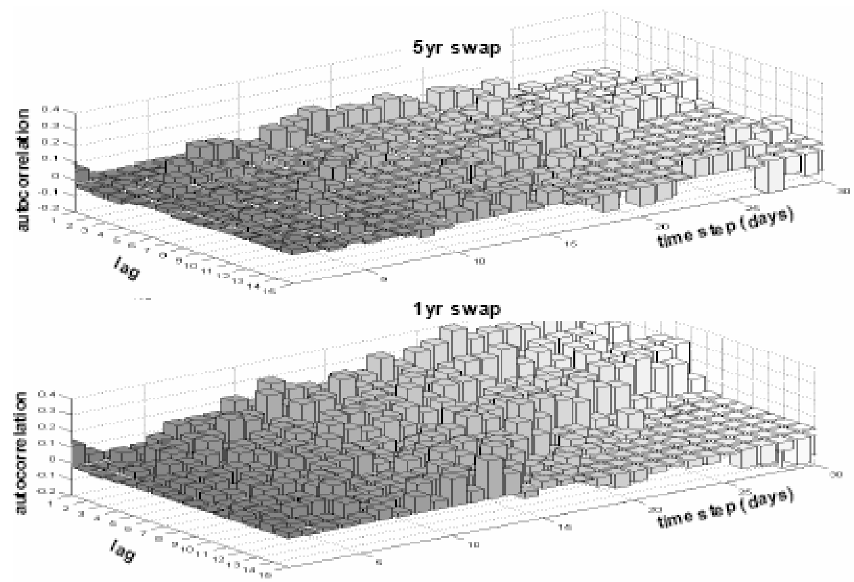
Wold theorem

$$\mathrm{ARMA}(0, \infty)$$



# long memory: fractional integration



$$X_{t+1} = X_t + \epsilon_{t+1},$$

$$X_{t+1} = X_t + \widetilde{\epsilon}_{t+1}.$$

invariants (i.i.d.)

$$\widetilde{\epsilon}_t = (1 - \mathcal{L})^{-d} \boxed{\epsilon_t.} \qquad 0 < d < 1/2$$

# long memory: fractional integration

$$X_{t+1} = X_t + \epsilon_{t+1},$$

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invariants (i.i.d.)

$$\widetilde{\epsilon}_t = (1 - \mathcal{L})^{-d} \boxed{\epsilon_t.} \qquad 0 < d < 1/2$$

$$(1 - \mathcal{L})^{-d} = \sum_{k=0}^{\infty} \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)} \mathcal{L}^k. \qquad \mathrm{ARMA}(0, \infty)$$

**(Wold theorem)**

$$\mathrm{Cor}\{\widetilde{\epsilon}_t, \widetilde{\epsilon}_{t-\tau}\} \approx \frac{\Gamma(1-d)}{\Gamma(d)} \tau^{2d-1}$$

# long memory: fractional Brownian motion



$$X_{t+1} = X_t + \epsilon_{t+1},$$

$$dX_t = \mu dt + \sigma dB_t$$

$$X_{t+1} = X_t + \widetilde{\epsilon}_{t+1}.$$

**invariants (i.i.d.)**

$$X_t \equiv \mu t + \sigma \int_{-\infty}^{t} \frac{(t-s)^d}{\Gamma(d+1)} \boxed{dB_s}.$$

$$\widetilde{\epsilon}_t = (1-\mathcal{L})^{-d} \epsilon_t. \qquad 0 < d < 1/2$$

$$(1-\mathcal{L})^{-d} = \sum_{k=0}^{\infty} \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)} \mathcal{L}^k.$$

$$\mathrm{Cor}\{\widetilde{\epsilon}_t, \widetilde{\epsilon}_{t-\tau}\} \approx \frac{\Gamma(1-d)}{\Gamma(d)} \tau^{2d-1}$$

# long memory: fractional Brownian motion

$$X_{t+1} = X_t + \epsilon_{t+1},$$

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invariants (i.i.d.)

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$$X_t = X_{t-\tau} + \widetilde{\epsilon}_{t,\tau},$$

$$\widetilde{\epsilon}_{t,\tau} \sim \mathrm{N}\left(\mu\tau, \sigma^2\tau^{2H}\right)$$

$$H \equiv d + \frac{1}{2}$$

Hurst coefficient

# long memory: fractional Brownian motion



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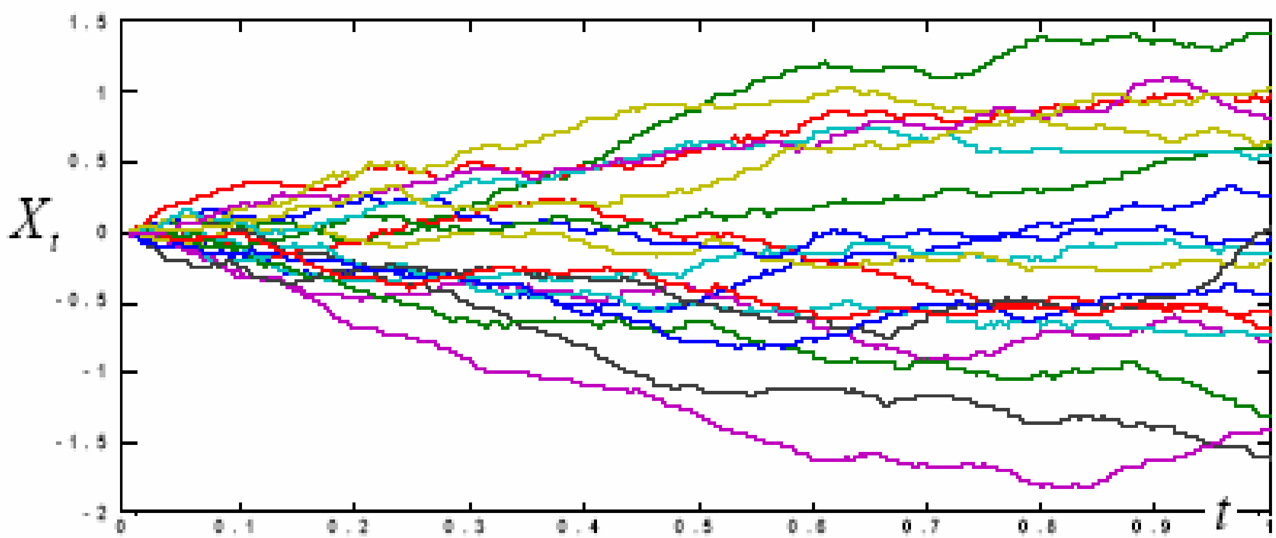
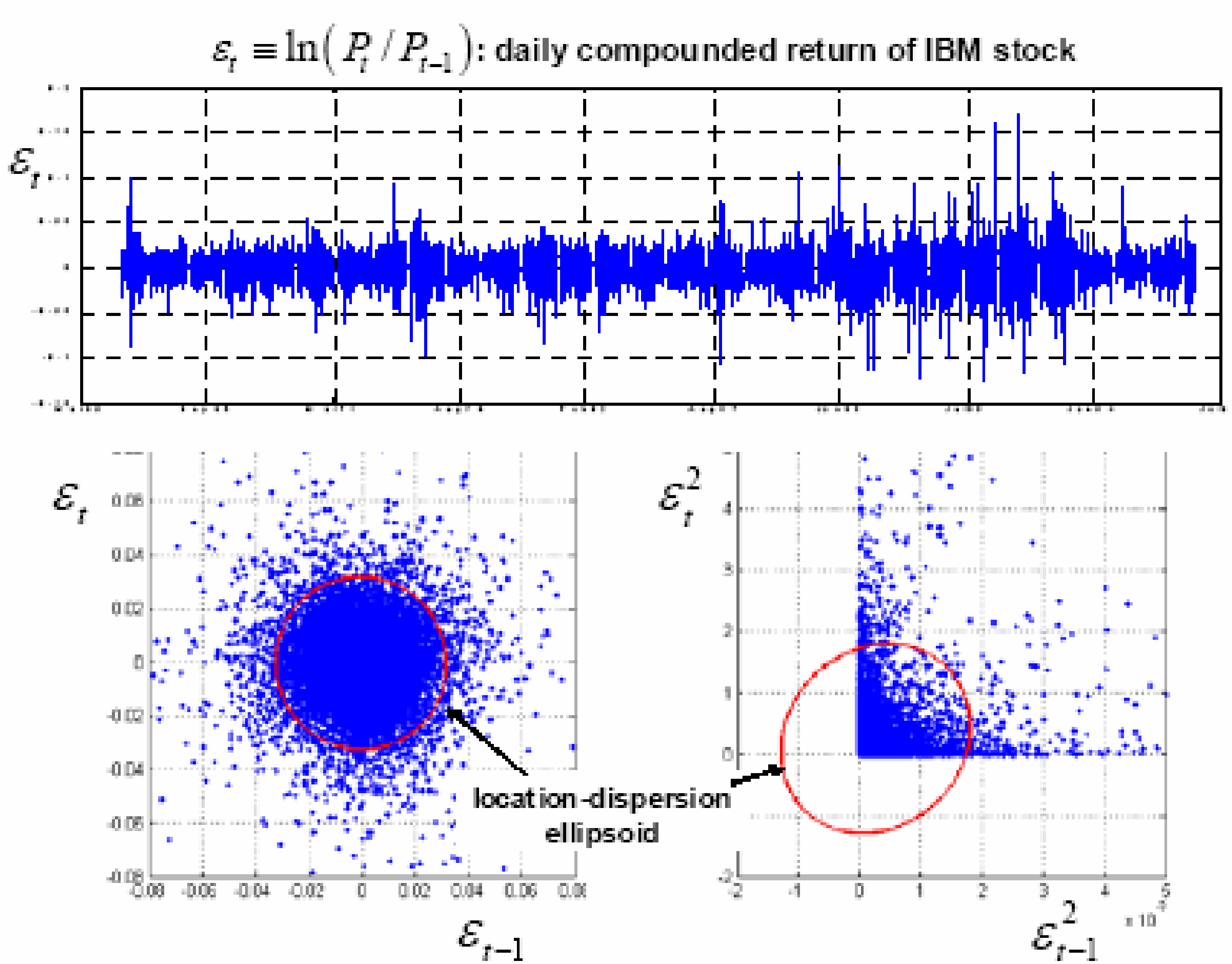


# volatility clustering





# volatility clustering: GARCH



$$X_{t+1} = X_t + \boxed{\epsilon_{t+1},}$$

invariants (i.i.d.)

$$\epsilon_t \stackrel{d}{=} \boxed{\mu_t} + \boxed{\sigma_t}\boxed{Z_t}$$

# volatility clustering: GARCH

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$$\epsilon_t \stackrel{d}{=} \mu + \sigma_t \boxed{Z_t},$$ **invariants (i.i.d.)**

$$\mathrm{GARCH}(p, q)$$

$$\sigma_t^2 = \sigma^2 + a\sigma_{t-1}^2 + b\boxed{Z_{t-1}^2}$$

$$0 < a + b < 1.$$

# volatility clustering: GARCH



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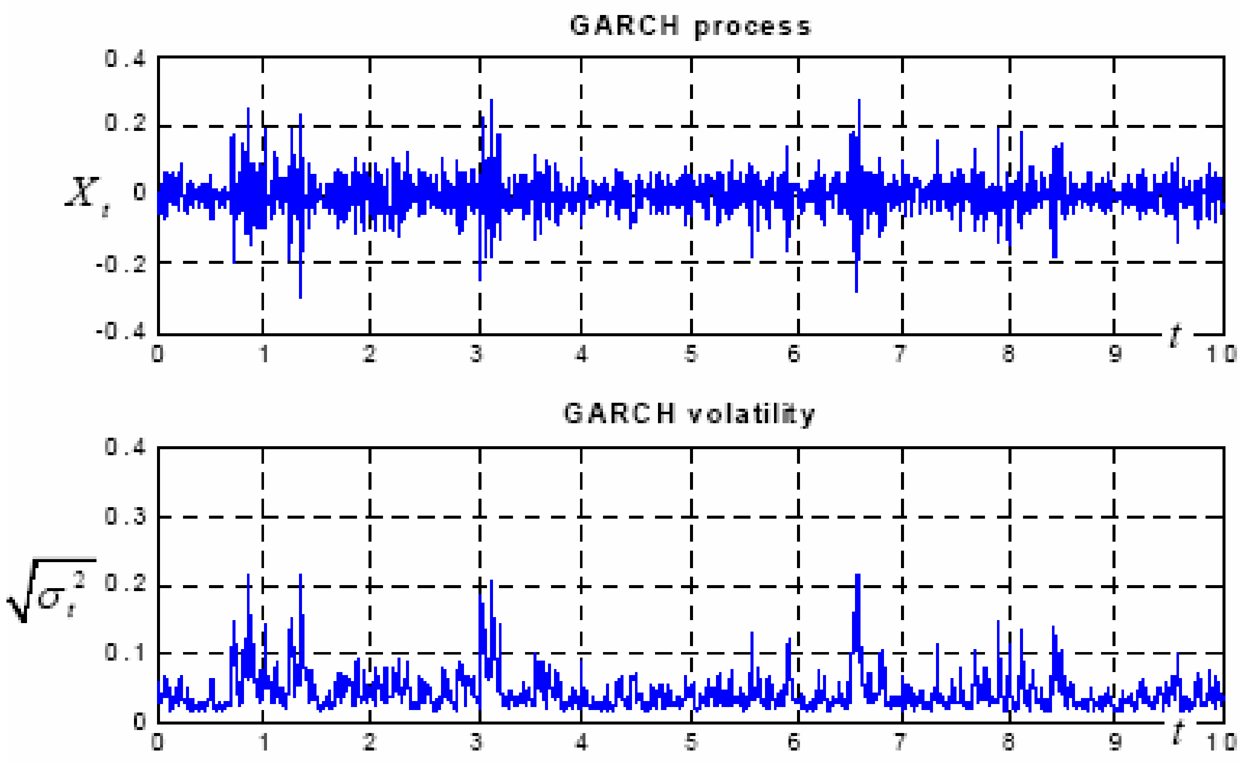

# volatility clustering: stochastic volatility

symmys.com > Research > Working Papers

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**invariants (i.i.d.)**

$$dX_t \equiv \boxed{\mu dt} + \boxed{\sigma dZ_t}$$

# volatility clustering: stochastic volatility



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$$dX_t \equiv \mu dt + \sigma_t \boxed{dZ_t.}$$

Heston

**invariants (i.i.d.)**

$$d\sigma_t^2 = -\kappa\left(\sigma_t^2 - \overline{\sigma}^2\right) dt + \lambda\sqrt{\sigma_t^2}\boxed{dB_t.}$$

$$2\kappa\overline{\sigma}^2 > \lambda^2$$

$$\mathrm{E}\{dB_t, dZ_t\} = \rho < 0.$$

**“leverage”**

# volatility clustering: stochastic volatility

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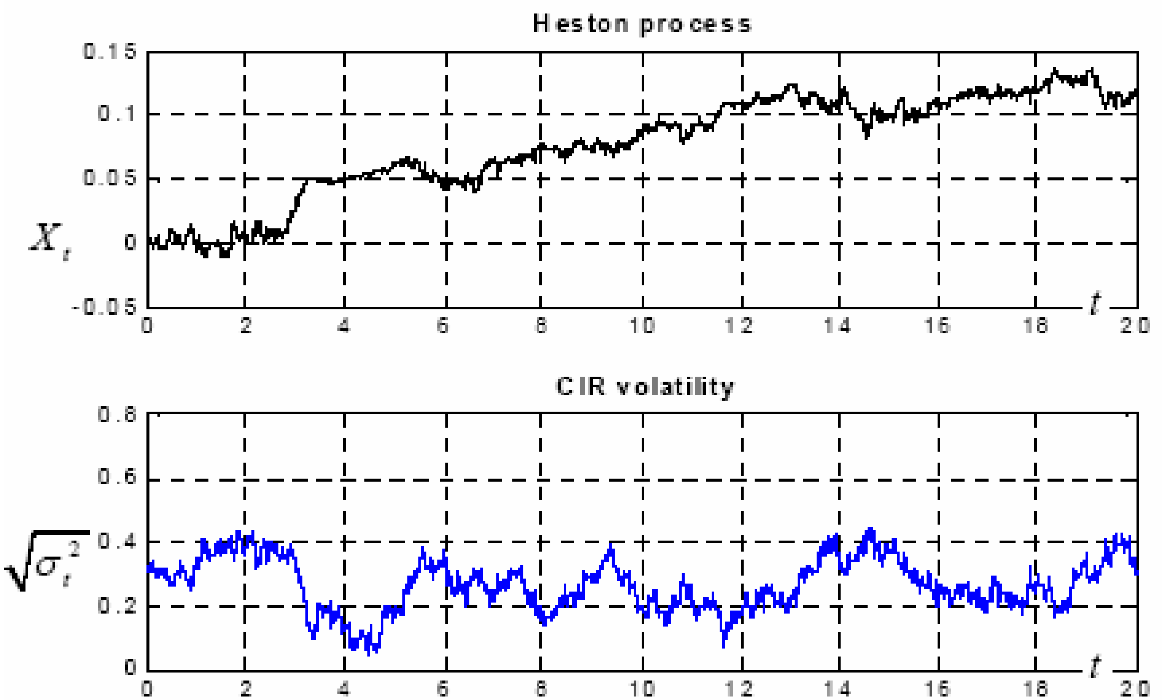


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# volatility clustering: subordination



$$X_{t+1} = X_t + \epsilon_{t+1},$$

$$dX_t \equiv \boxed{dL_t}$$

$$X_t \stackrel{d}{=} \boxed{B}^{\mu,\sigma^2}_{\boxed{T_t}}$$

invariants (i.i.d.)

# volatility clustering: subordination



$$X_{t+1} = X_t + \epsilon_{t+1},$$

$$dX_t \equiv dL_t$$

$$X_t \overset{d}{=} \boxed{B}^{\mu,\sigma^2}_{T_t}$$

$$T_t \equiv \int_0^t \dot{T}_s ds.$$

invariants (i.i.d.)

$$d\dot{T}_s = -\widetilde{\theta}\left(\dot{T}_s - 1\right) dt + \widetilde{\sigma}\sqrt{\dot{T}_s}\boxed{dZ_s}$$

# volatility clustering: subordination



$$X_{t+1} = X_t + \epsilon_{t+1},$$

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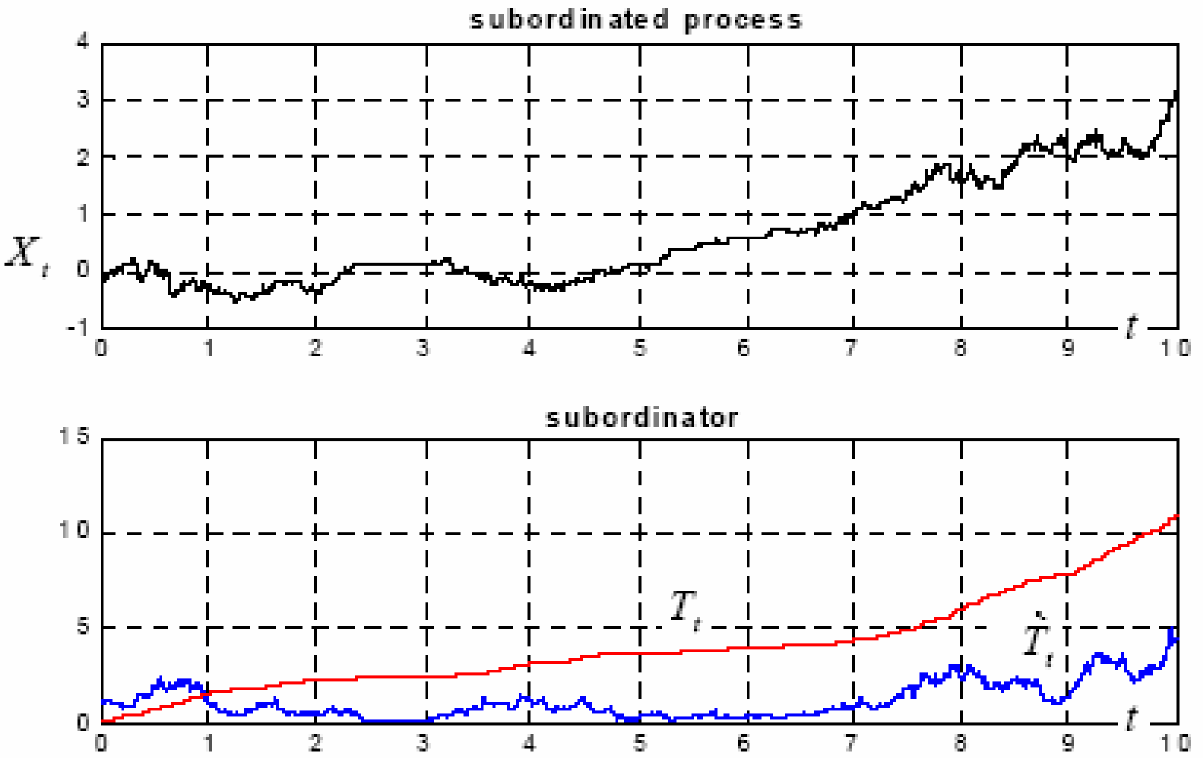


$$T_t \equiv \int_0^t \dot{T}_s ds.$$

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