APPLYING COPULA FUNCTION TO RISK MANAGEMENT

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Abstract

This paper is part of the author's Ph. D. Thesis "Extreme Value Theory and coherent risk measures: applications to risk management".

The copula function describes the dependence structure of a multivariate random variable. In this paper, it is used as a practical and flexible instrument to generate Monte Carlo scenarios of risk factor returns. These risk factors affect the value of a credit or market portfolio. In fact, many of the models commonly used assume a multinormal distribution of such risk factor returns (or log-returns). This hypothesis underestimates the probability that a catastrophic event, such as a simultaneous slump of equity prices or the joint default of several counterparties in a credit portfolio, might occur. This kind of event worries both risk managers and supervisors. Our goal is to show that the use of a copula function different from the Gaussian copula can model such extreme events effectively.

Keywords: Copula, Dependence Structure, Distribution Function, Expected Shortfall, Latent Variable, Profit and Loss Distribution, Risk Factor, Tail Dependence, Value-at-Risk.

Introduction

The problem of measuring the risk of an asset portfolio may be divided into two main stages:

- the modelling of the joint evolution of **risk factor**[1] returns affecting portfolio profit and loss distribution during a specified holding period[2];

- the modelling of the impact of changes in risk factor returns on the value of portfolio instruments[3] using **pricing models**.

In this work, the focus is on the first stage of the problem. We describe in a practical way how to simulate, from a multivariate distribution, the risk factors. In order to do so, one has to know the dependence structure of risk factor returns.

Let $\mathbf{X}=(X_1, ..., X_n)$ be the random vector of the n risk factor returns which affect portfolio value, with marginal distribution functions (d.f.) $F_1, ..., F_n$.

The multivariate d.f., $F(x_1,...,x_n) = P[X_1 \leq x_1,..., X_n \leq x_n]$, completely determines the dependence structure of random returns $X_1, ..., X_n$. However, its analytic representation is often too complex, and so it is practically impossible to estimate and to use it in simulation models. The most common methodologies in portfolio risk measurement use the **multivariate Gaussian distribution** to simulate risk factor returns because it is easy to implement. Unfortunately, it is known from empirical evidence that it is inadequate to fit real data.

The use of **copula function** allows us to overcome the issue of estimating the multivariate d.f. by splitting it into two parts:

- determine the margins $F_1, ..., F_n$, which represent the distribution of each risk factor, and estimate their parameters, fitting the available data by choosing from all useful statistical methods[4];

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[1] e.g. , exchange rates, interest rates, stock indexes, commodity prices, and risk factors affecting the credit state of the counterparty.
[2] Usually it ranges from one day to two weeks for market risk management, while it is one year for credit risk management.
[3] Such as options, swaps, bonds, equities, etc.
[4] e.g., method of moments, maximum likelihood, etc.

- determine the dependence structure of the random variables X_1, ..., X_n, specifying a meaningful copula function.

The quality of the selection of the analytical form of margins and copula must be checked with backtesting techniques. The goal is the choice of margins and copula which better represent the real profit and loss distribution.

The paper is organized as follows. In section one, we give a definition of copula. We then define the concept of tail dependence in section two. In section three we describe the elliptical copulas focusing on the Gaussian copula and the t-Student copula. This family of copulas is very useful in many risk management applications because it is easy to implement to generate Monte Carlo scenarios. In section four, we present some possible applications of copula function to risk management. In section five, we apply some models, based on the use of different types of elliptical copulas, to assess the risk of a portfolio of ten Italian equities. The results are compared with empirical data in an attempt to determine the most adequate model. Finally, in section six, elliptical copulas are used to estimate the risk of a credit portfolio of 30 loans.

1. Definition of copula function

An n-dimensional copula[5] is a multivariate d.f. , C, with uniform distributed margins in [0,1] (U(0,1)) and the following properties:

1. $C: [0,1]^n \rightarrow [0,1]$;

2. C is grounded and n-increasing;

3. C has margins C_i which satisfy $C_i(u) = C(1, ..., 1, u, 1, ..., 1) = u$ for all $u \in [0,1]$.

It is obvious, from the above definition, that if F_1, ..., F_n are univariate distribution functions, $C(F_1(x_1), ..., F_n(x_n))$ is a multivariate d.f. with margins F_1, ..., F_n because $u_i = F_i(x_i)$, i = 1, ..., n, is a uniform random variable. Copula functions are a useful tool to construct and simulate multivariate distributions.

The following theorem is known as **Sklar's Theorem**. It is the most important theorem about copula functions because it is used in many practical applications.

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[5] The original definition is given by Sklar (1959).

Theorem[6]: *Let F be an n-dimensional d.f. with continous margins F_1, ..., F_n. Then it has the following unique copula representation:*

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)) .$$ (1)

From Sklar's Theorem we see that, for continous multivariate distribution functions, the univariate margins and the multivariate dependence structure can be separated. The dependence structure can be represented by a proper copula function. Moreover, the following corollary is attained from (1).

Corollary: *Let F be an n-dimensional d.f. with continous margins F_1, ..., F_n and copula C (satisfying (1)). Then, for any $u=(u_1,...,u_n)$ in $[0,1]^n$:*

$$C(u_1,...,u_n) = F\left(F_1^{-1}(u_1),...,F_n^{-1}(u_n)\right) ,$$ (2)

where F_i^{-1} is the generalized inverse of F_i.

An example of copula function is the **Gumbel** copula, endowed with the following analytic expression:

$$C_\beta^{Gu}(u_1,...,u_n) = \exp\left\{-\left[(-\log u_1)^{1/\beta} +...+(-\log u_n)^{1/\beta}\right]^\beta\right\} ,$$

where $0 < \beta \le 1$ is the parameter expressing the amount of dependence among the marginal components; $\beta = 1$ gives independence, while the limit for $\beta \to 0$ leads to perfect dependence.

Another trivial example is the copula of independent random variables. It takes the form:

$$C_{ind}(u_1,...,u_n) = u_1 \cdot ... \cdot u_n .$$

2. Tail dependence

The concept of tail dependence is relevant for the study of dependence between extreme values. Let $(X,Y)^T$ be a vector of continuous random variables with marginal distribution functions F and G. Trivially speaking, tail dependence expresses the probability of having a high (low) extreme value of Y given that a high (low) extreme value of X has occured.

The analytic form for the **coefficient of upper tail dependence** is

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[6] For the proof, see Sklar (1996).

$$\lim_{u \to 1} P\{Y > G^{-1}(u) | X > F^{-1}(u)\} = \lambda_u$$

provided that the limit $\lambda_u \in [0,1]$ exists. If $\lambda_u \in (0,1]$, the random variables X and Y are asymptotically dependent in the upper tail; on the contrary, if $\lambda_u = 0$, they are asymptotically independent in the upper tail.

An alternative definition, from which it is seen that the concept of tail dependence is a copula property, is the following[7]: let C be a bivariate copula. If C is such that

$$\lim_{u \to 1}(1 - 2u + C(u,u))/(1-u) = \lambda_u$$

exists, then C has upper tail dependence (independence) if $\lambda_u \in (0,1]$ ($\lambda_u = 0$).

The **coefficient of lower tail dependence** can be defined in a similar way: if the limit $\lim_{u \to 0} C(u,u)/u = \lambda_L$ exists, then C has lower tail dependence (independence) if $\lambda_L \in (0,1]$ ($\lambda_L = 0$).

3. Elliptical copulas

The class of elliptical distributions provides useful examples of multivariate distributions because they share many of the tractable properties of the multivariate normal distribution. Furthermore, they allow to model multivariate extreme events and forms of non-normal dependencies. Elliptical copulas are simply the copulas of elliptical distributions. Simulation from elliptical distributions is easy to perform. Therefore, as a consequence of Sklar's Theorem[8], the simulation of elliptical copulas is also easy.

3.1. Normal copula

The Gaussian (or normal) copula is the copula of the multivariate normal distribution. In fact, the random vector $\mathbf{X}=(X_1,\ldots,X_n)$ is multivariate normal iff:

1) the univariate margins F_1, …, F_n are Gaussians;

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[7] For further details about its derivation, see in Embrechts, Lindskog and McNeil (2001).
[8] See (1) e (2).

2) the dependence structure among the margins is described by a unique copula function C (the normal copula) such that[9]:

$$C_R^{Ga}(u_1,...,u_n) = \Phi_R\left(\phi^{-1}(u_1),...,\phi^{-1}(u_n)\right), \qquad (3)$$

where Φ_R is the standard multivariate normal d.f. with linear correlation matrix R and ϕ^{-1} is the inverse of the standard univariate Gaussian d.f.

Multivariate normal is commonly used in risk management applications to simulate the distribution of the n risk factors affecting the value of the trading book (market risk) or the distribution of the n systematic factors[10] influencing the value of the credit worthiness index of a counterparty[11] (credit risk).

If n=2, expression (3) can be written as:

$$C_R^{Ga}(u,v) = \int_{-\infty}^{\phi^{-1}(u)} \int_{-\infty}^{\phi^{-1}(v)} \frac{1}{2\pi(1-R_{12}^2)^{1/2}} \exp\left\{-\frac{s^2 - 2R_{12}st + t^2}{2(1-R_{12}^2)}\right\} ds\,dt \ ,$$

where R_{12} is simply the linear correlation coefficient between the two random variables.

The bivariate Gaussian copula does not have upper tail dependence if $R_{12} < 1$[12]. Furthermore, since elliptical distributions are symmetric, the coefficient of upper and lower tail dependence are equal. Hence, Gaussian copulas do not even have lower tail dependence.

To simulate random variables from the Gaussian copula, one can use the following procedure. If the matrix R is positive definite, then there are some $n \times n$ matrix A such as $R = AA^T$. It is also assumed that the random variables Z_1, ..., Z_n are independent standard normal. Then, the random vector $\mu + AZ$ (where $Z = (Z_1,...,Z_n)^T$ and the vector $\mu \in R^n$) is multinormally distributed with mean vector μ and covariance matrix R.

The matrix A can be easily determined with the Cholesky decomposition of R. This decomposition is the unique lower-triangular matrix L such as $LL^T = R$.

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[9] As one can easily deduce from (2).
[10] Credit drivers.
[11] The value of such index allow us to simulate counterparty default or nondefault over the holding period.
[12] For the proof, see Example 3.4 in Embrechts, Lindskog and McNeil (2001).

Hence, one can generate random variates from the n-dimensional Gaussian copula running the following algorithm:

- find the Cholesky decomposition A of the matrix R;
- simulate n independent standard normal random variates $\mathbf{z}=(z_1,\ldots,z_n)^T$;
- set $\mathbf{x}=A\mathbf{z}$;
- determine the components $u_i = \phi(x_i), i = 1,\ldots,n$;
- the vector $(u_1, \ldots, u_n)^T$ is a random variate from the n-dimensional Gaussian copula, C_R^{Ga}.

We have used the previous algorithm to plot Figure 1.

3.2. t-Student copula

The copula of the multivariate t-Student distribution is the t-Student copula. Let $\mathbf{X}$ be a vector with an n-variate t-Student distribution with v degrees of freedom, mean vector μ (for $v > 1$) and covariance matrix $\frac{v}{v-2}\Sigma$ (for $v > 2$)[13]. It can be represented in the following way:

$$\mathbf{X} \overset{\mathrm{d}}{=} \mu + \frac{\sqrt{v}}{\sqrt{S}}\mathbf{Z} \ , \tag{4}$$

where $\mu \in R^n$, $S \sim \chi_v^2$ and the random vector $\mathbf{Z} \sim N_n(\mathbf{0},\Sigma)$ are independent.

The copula of vector $\mathbf{X}$ is the t-Student copula with v degrees of freedom. It can be analytically represented in the following way:

$$C_{v,R}^t(\mathbf{u}) = t_{v,R}^n(t_v^{-1}(u_1),\ldots,t_v^{-1}(u_n)) \ , \tag{5}$$

where $R_{ij} = \Sigma_{ij} / \sqrt{\Sigma_{ii}\Sigma_{jj}}$ for $i,j \in \{1,\ldots,n\}$ and where $t_{v,R}^n$ denotes the multivariate d.f. of the random vector $\sqrt{v}\mathbf{Y}/\sqrt{S}$, where the random variable $S \sim \chi_v^2$ and the random vector $\mathbf{Y}$[14] are independent. t_v denotes the margins[15] of $t_{v,R}^n$.

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[13] If $v \leq 2$, then the covariance matrix is not defined.
[14] $\mathbf{Y}$ has an n-dimensional normal distribution with mean vector $\mathbf{0}$ and covariance matrix R.

For n=2, the t-Student copula has the following analytic form:

$$C_{v,R}^t(u,v) = \int_{-\infty}^{t_v^{-1}(u)} \int_{-\infty}^{t_v^{-1}(v)} \frac{1}{2\pi(1-R_{12}^2)^{1/2}} \left\{ 1 + \frac{s^2 - 2R_{12}st + t^2}{v(1-R_{12}^2)} \right\}^{-(v+2)/2} dsdt \ ,$$

where R_{12} is the linear correlation coefficient of the bivariate t-Student distribution with v degrees of freedom, if $v > 2$.

Unlike the Gaussian copula, it can be shown[16] that the t-Student copula has upper tail dependence[17]. As one would expect, such dependence is increasing in R_{12} and decreasing in v. Therefore, the t-Student copula is more suitable than the Gaussian copula to simulate events like stock market crashes[18] or the joint default in most of the counterparties in a credit portfolio.

To simulate random variates from the t-Student copula, $C_{v,R}^t$, we can use the following algorithm, which is based on equation (4):

- find the Cholesky decomposition, A, of R;

- simulate n independent random variates $\mathbf{z}=(z_1,...,z_n)^T$ from the standard normal distribution;

- simulate a random variate, s, from χ_v^2 distribution, independent of $\mathbf{z}$;

- determine the vector $\mathbf{y}=A\mathbf{z}$;

- set $\mathbf{x} = \dfrac{\sqrt{v}}{\sqrt{s}}\mathbf{y}$;

- determine the components $u_i = t_v(x_i)$, $i = 1,...,n$;

- the resultant vector is: $(u_1,...,u_n)^T \sim C_{v,R}^t$.

We have used the above algorithm (with n=2) to plot Figures 2, 3 and 4.

[15] All the margins are equally distributed.
[16] For the proof, see Embrechts, Lindskog and McNeil (2001).
[17] Because the t-Student copula is radial symmetric, it has equal lower tail dependance.
[18] That is a sharp drop of the prices in most of the stocks traded in a Stock Exchange.
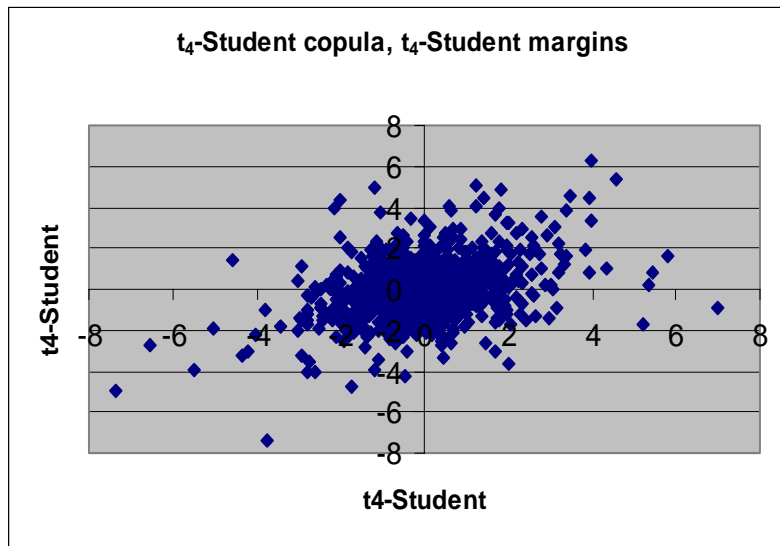
Figure 1. *1000 Monte Carlo simulations of bivariate random variates (x_1, x_2) with normal copula ($R_{12}=0.3$) and standard normal margins.*
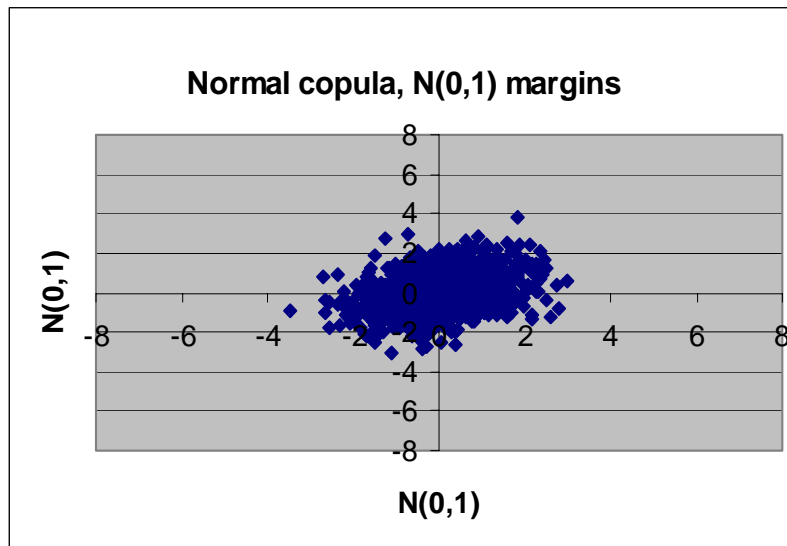

Figure 2. *1000 Monte Carlo simulations of bivariate random variates (x_1, x_2) with t_4-Student copula ($R_{12}=0.3$) and standard normal margins.*
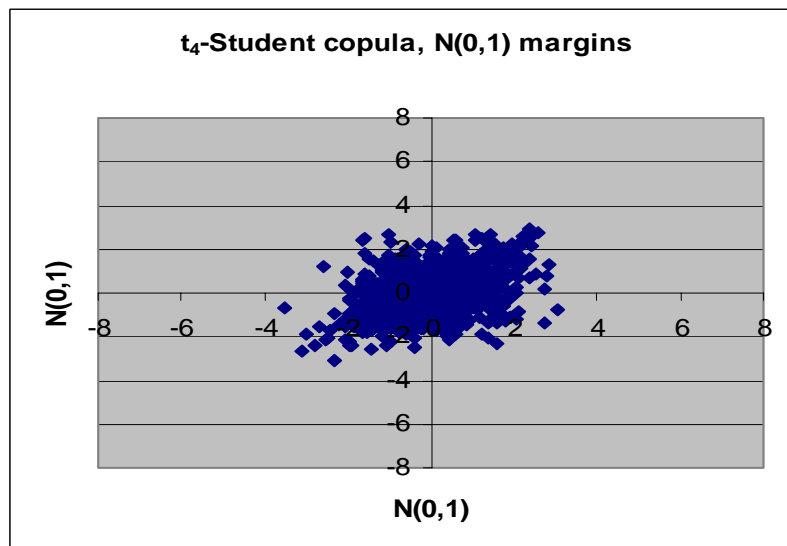
Figure 3. *1000 Monte Carlo simulations of bivariate random variates (x_1, x_2) with t_4-Student copula (R_{12}=0.3) and normalized t_4-Student margins.*

Figure 4. *1000 Monte Carlo simulations of bivariate random variates (x_1, x_2) with t_2-Student copula (R_{12}=0.3) and normalized t_2-Student margins.*
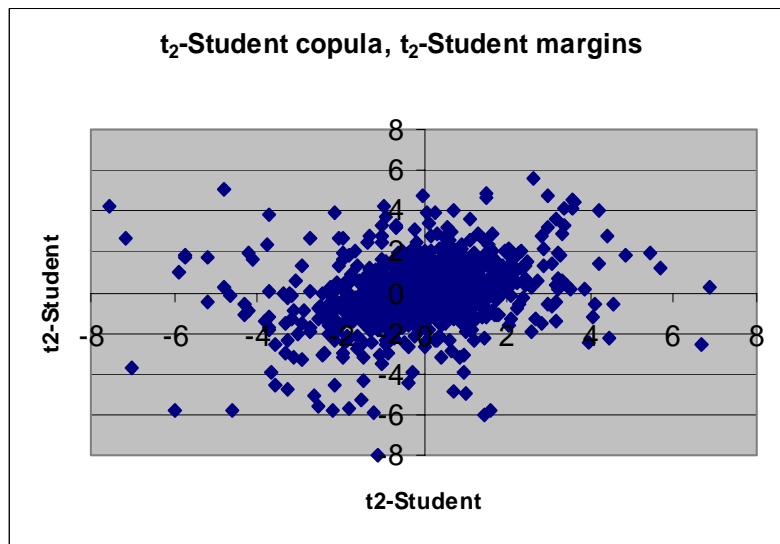
Viewing the previous figures, we can see the differences among the different hypothesis of bivariate distribution, under the same linear correlation coefficient (R_{12}=0.3). Particularly, in Figures 1 and 2 one can compare the Gaussian and the t_4-Student copulas, under the same margins (standard normal): because of its tail dependence, the t-Student copula produces a scatter plot more extended to the top right corner and the bottom left corner than the normal copula[19]. Because of the joint effect of copula function and margins with fat tails, the plot in Figure 3 is even more scattered. The scatter plot in Figure 4 is produced from a distribution with an even stronger dependence structure and margins with even fatter tails (t_2-Student) as the number of degrees of freedom is lower.

4. Applying copula to risk management

a) Market risk.

We suppose standing at time t and estimating a risk measure[20] of a trading book using Monte Carlo simulation. The holding period is (t,t+1). We consider the random variable X_{t+1} with a d.f. F, describing the portfolio profit and loss distribution in (t,t+1). Let $\mathbf{Y}_{t+1}$=($Y_{1,t+1}$, $Y_{2,t+1}$, …, $Y_{d,t+1}$) be the random vector of underlying d risk factor[21] returns[22] affecting portfolio value in t+1 and therefore the value of the random variable X_{t+1}. In other words, the following relation holds:

$$X_{t+1}=f(\mathbf{Y}_{t+1})=f(Y_{1,t+1},Y_{2,t+1},...,Y_{d,t+1}) ,$$

where f is a function[23] linking the risk factor returns to portfolio value and therefore to X_{t+1}.

To estimate the distribution of the random variable X_{t+1} we require a historical time series with m data, $\mathbf{Y}_{t-m+1},…,\mathbf{Y}_t$, for the risk factor returns.

The core activity of the estimate process is Monte Carlo simulation of n scenarios for the random vector $\mathbf{Y}_{t+1}$. In order to do this, it is necessary to assume a particular analytic form for the multivariate d.f. of $\mathbf{Y}_{t+1}$. The

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[19] In other words, the probability of having joint high (or low) values in both the random variables using the t-Student copula is higher than using the Gaussian copula.
[20] e.g. Value-at-Risk (VaR) or Expected Shortfall.
[21] i.e., interest rates, exchange rates, stock indexes.
[22] Always over the holding period (t,t+1).
[23] This function is known as the mapping (or pricing).

multinormal distribution is a standard assumption in many risk management applications. In fact, such distribution is easy to implement in a simulation model. Unfortunately, empirical evidence proved that the multivariate normal is inadequate because it underestimates both the thickness of the tails of the margins and their dependence structure[24].

In our opinion, a more effective model assumes any distributional form for the marginal distributions of $Y_{i,t+1}$, i=1,…,d, and their dependence structure described by an appropriate copula function. The margins and copula parameters are assessed from historical data.

When we have generated n scenaros for risk factor returns[25], we can easily determine n realisations of the random variable $X_{t+1}=f(\mathbf{Y}_{t+1})$. So we can build the simulated profit and loss distribution. We now have the following options to estimate risk measures (VaR or Expected Shortfall with a given confidence level):

- the n scenarios of X_{t+1} are ordered by growing loss values[26]; VaR is estimated by cutting the simulated distribution at the selected probability level; Expected Shortfall is the expected value of the losses in the worst scenarios[27];

- a suitable parametric univariate distribution is fitted to simulated data; VaR and Expected Shortfall are calculated from this distribution;

- estimate the tail distribution of X_{t+1} using the superior techniques of Extreme Value Theory and calculate VaR and Expected Shortfall at high confidence level.

b) Credit Risk.

The aim is the estimate of credit portfolio risk measures using a **latent variable model**[28] over a time horizon $[0,T]$[29]. We consider a portfolio of m

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[24] i.e., in the case of a market crash, all the stock prices falling down simultaneously. A methodology assuming the multinormal hypothesis does not model effectively such eventuality.

[25] n variates of the random vector $\mathbf{Y}_{t+1}$.

[26] Profits are assumed as negative losses.

[27] The number of worst scenarios derives from the selected probability level.

[28] Such kind of models are based on the **structural model** builds on the pioneering work of Merton (1974).

[29] Typically T=1 year.

counterparties. Let the random variable Y_i be a **default indicator** with the following analytical form:

$$Y_i = \begin{cases} 0 \text{ if counterparty i does not default in } [0,T] \\ 1 \text{ if counterparty i defaults in } [0,T] \end{cases}, \quad i=1, \ldots, m \,.$$

The latent variable models use the following pair of vectors to simulate a counterparty default or non default[30] over the time horizon [0,T]:

1. let $\mathbf{X}=(X_1,\ldots,X_m)$ be the random vector of the **latent variables**. In the Merton approach, such a vector is interpreted as counterparty asset values[31] over the time horizon [0,T];

2. let $\mathbf{D}=(D_1,\ldots,D_m)$ be the deterministic vector of the **threshold levels**. In the Merton approach, they represent the values of counterparty liabilities[32].

The pair $(X_i, D_i)_{i=1,\ldots,m}$ denotes a latent variable model for the binary random vector $\mathbf{Y}=(Y_1,\ldots,Y_m)$ if the following relations are true:

$$\begin{aligned} Y_i = 1 &\iff X_1 \le D_i \\ Y_i = 0 &\iff X_i > D_i \end{aligned}, \quad i=1,\ldots,m \,.$$

The traditional portfolio credit risk models[33] implicitly assume a multinormal distribution for the vector $\mathbf{X}$. In some recent academic papers[34], it has been stressed how such models can heavily underestimate the probability of many joint defaults and, therefore the whole portfolio risk.

Such a limit is overcome by a model assuming any kind of univariate d.f., F_i, for each component X_i of the vector $\mathbf{X}$, and a copula function (different from the Gaussian copula) describing their dependence structure[35].

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[30] That is if Y_i=1 or Y_i=0.
[31] Since asset returns are not directly observable, CreditMetrics[TM] of J.P. Morgan chose equity returns as a proxy.
[32] In KMV[TM], the thresholds represent the values of the short-term debt plus half the long-term debt.
[33] i.e., CreditMetrics or KMV.
[34] e.g., see Frey and McNeil (2001).
[35] e.g. the t-Student copula with a proper number of degrees.

5. Estimating risk measures for a portfolio of Italian equities with copulas

In this section, we perform an application of the model described in section 4a.

We estimate 99% VaR and Expected Shortfall for a portfolio of ten Italian equities[36] (d=10) over a one day holding period, using Monte Carlo simulation. We have assessed the model parameters using a four year historical time series[37] of the daily closing prices and returns for ten equities (a series of m=1012 data). Table 1 shows the closing prices[38] in t, P_{it}, the expected returns and the return variances. Table 2 shows the empirical asset return correlation matrix[39]. Portfolio value in t is[40]:

$$P_t = \sum_{i=1}^{10} P_{it} = 31,3821 \, \text{Euro} \quad .$$

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[36] See Table 1.
[37] From the 25th September 1997 to the 24th September 2001.
[38] In Euro.
[39] The correlation matrix actually used in the simulations is obtained as follows: the data are mapped to empirical uniforms and transformed with the inverse function of the Gaussian distribution. The correlation is then computed for the transformed data.
[40] We consider a portfolio with one unit of each asset.

Table 1. *Asset prices in t, expected returns and return variances.*

Asset (i)	Price, P_{it}	Expected return, r_i	Variance, σ_i^2
1 – Banca di Roma	1,95	-0,00024	0,000687
2 – Seat	0,651	0,001215	0,001088
3 – L'Espresso	1,975	0,001112	0,000971
4 – Autostrade	6,05	0,001286	0,000485
5 – AEDES	2,17	0,002933	0,002143
6 – TIM	4,78	0,000602	0,000669
7 – ENI	11,82	0,000218	0,000359
8 - FinMeccanica	0,635	0,000881	0,000855
9 – Olivetti	0,955	0,001406	0,000977
10 - Impregilo	0,3961	-0,00021	0,000539

Table 2. *Asset return correlation matrix.*

	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10
R1	1									
R2	0,1968	1								
R3	0,2536	0,4929	1							
R4	0,3258	0,1864	0,2689	1						
R5	0,1233	0,2538	0,2472	0,1353	1					
R6	0,3379	0,4345	0,4149	0,3407	0,1858	1				
R7	0,3524	0,1222	0,1646	0,2801	0,0716	0,2895	1			
R8	0,2900	0,3298	0,3345	0,3026	0,1843	0,3962	0,2187	1		
R9	0,3219	0,3888	0,3702	0,2967	0,1630	0,5241	0,1691	0,4002	1	
R10	0,3057	0,2000	0,2320	0,3706	0,1293	0,3432	0,2467	0,3658	0,3114	1

We assume standing at time t. We have simulated n=1000 Monte Carlo scenarios for asset returns, R_{ij}, i=1,…,10; j=1,…,1000, over the time horizon (t,t+1] (one day), assuming different analytic forms for the margins and the copula function describing their dependence structure. Using such scenarios, the portfolio has been revalued at time t+1:

$$P_{j,t+1} = \sum_{i=1}^{10} P_{it}(1 + R_{ij}) \ , \qquad j=1,\ldots,1000.$$

We express portfolio losses[41] in each scenario j as:

$$P_t - P_{j,t+1} = \sum_{i=1}^{10} \left[P_{it} - P_{it}(1 + R_{ij}) \right] = -\sum_{i=1}^{10} P_{it} R_{ij} \ , \qquad j=1,\ldots,1000.$$

So, we obtain the simulated portfolio profit and loss distribution. The calculus of 99% VaR and Expected Shortfall of this distribution is trivial. As already mentioned, such risk measures have been estimated making different hypothesis about the analytic forms of the marginal distribution functions of asset returns and the copula function explaining their dependence structure. The results are shown in Table 3 and compared with 99% VaR and Expected Shortfall obtained from empirical data. The goal is finding the model that better represents empirical evidence.

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[41] Possibile profits are considered as negative losses.

Table 3. *99% VaR and Expected Shortfall (ES)[42] estimated from historical data and from 1000 Monte Carlo (MC) scenarios under different hypotheses of marginal return distributions and their copula.*

Empirical distribution (1012 historical scenarios of asset returns)
VaR(99%)=1,2809 Euro (4,08%); ES(99%)=1,5662 Euro (4,99%)
1000 MC simulations, normal margins and copula
VaR(99%)=1,1213 Euro (3,57%); ES(99%)=1,3127 Euro (4,18%)
Valuation error: VaR -12,46%; ES -16,19%
1000 MC simulations, t_{50}-Student margins and copula
VaR(99%)=1,1531 Euro (3,67%); ES(99%)=1,3778 Euro (4,39%)
Valuation error: VaR -9,98%; ES -12,03%
1000 MC simulations, t_{20}-Student margins and copula
VaR(99%)=1,2629 Euro (4,02%); ES(99%)=1,5401 Euro (4,91%)
Valuation error: VaR -1,41%; ES -1,67%
1000 MC simulations, t_{10}-Student margins and copula
VaR(99%)=1,3022 Euro (4,15%); ES(99%)=1,6478 Euro (5,25%)
Valuation error: VaR +1,66%; ES +5,21%
1000 MC simulations, t_4-Student margins and copula
VaR(99%)=1,6065 Euro (5,12%); ES(99%)=2,4869 Euro (7,92%)
Valuation error: VaR +25,42%; ES +58,79%
1000 MC simulations, t_2-Student margins and copula
VaR(99%)=4,2247 Euro (13,46%); ES(99%)=8,6535 Euro (27,57%)
Valuation error: VaR +229,82%; ES +452,52%

From Table 3, we can immediately see that the hypothesis of normal asset return marginal distributions and normal copula, commonly used in risk

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[42] The per cent ratio between VaR (or Expected Shortfall) and portfolio value is showed in brackets.

management applications, underestimates portfolio 99% VaR and Expected Shortfall calculated with empirical data. Precisely, the VaR underestimate is slightly below 12.5%, while the ES underestimate is above 16%. On the other hand, the hypothesis of t-Student margins and copula, when the number of degrees of freedom is below ten, heavily overestimates portfolio risk. In fact, such a hypothesis produces an excessive overestimate of the tail thickness of profit and loss distribution for a market portfolio, as one can immediately see from Figure 5, 6, 7, and 8.

The hypothesis that better approximate empirical evidence seems to be the t-Student with a number of degrees of freedom between 10 and 20 for the margins and the copula. In fact, in using t_{20}-Student, portfolio VaR and ES are underestimated by about 1.5%, while in using t_{10}-Student there is a slight overestimate of the two risk measures.
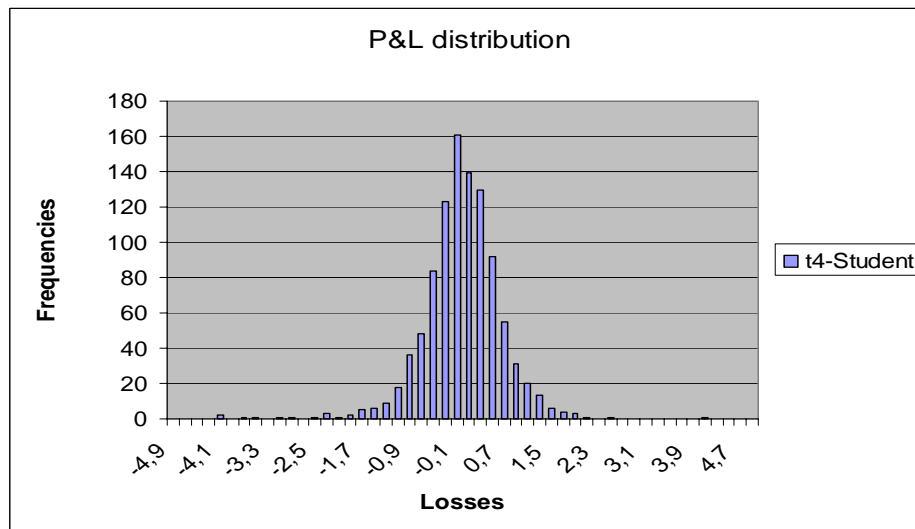
Figure 5. *Portfolio profit and loss frequency distributions under three different assumptions for asset return marginal distributions and copula function (Gaussian, t_4-Student, and t_2-Student).*
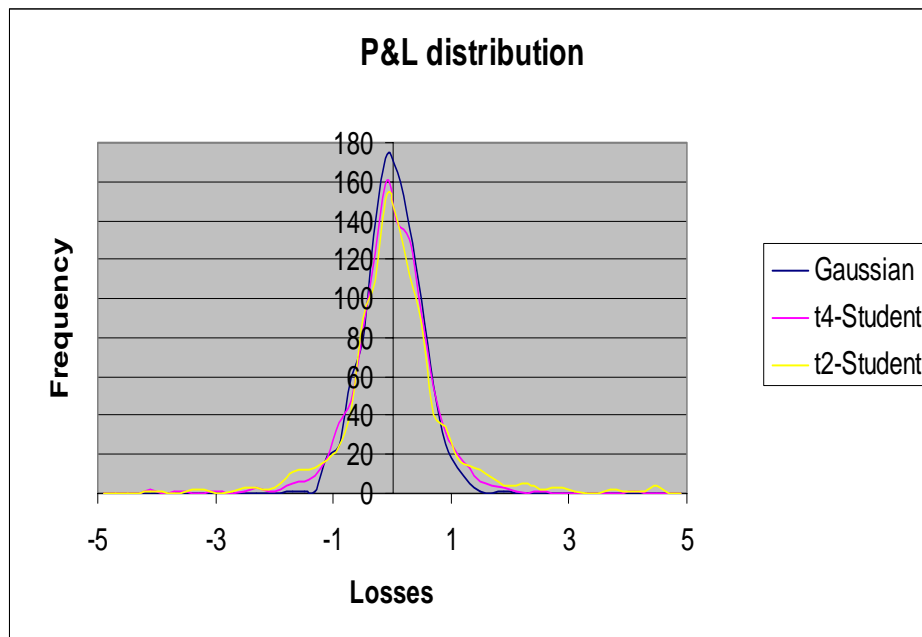
Figure 6. *Portfolio profit and loss frequency distribution obtained from 1000 MC simulations with Gaussian margins and copula.*
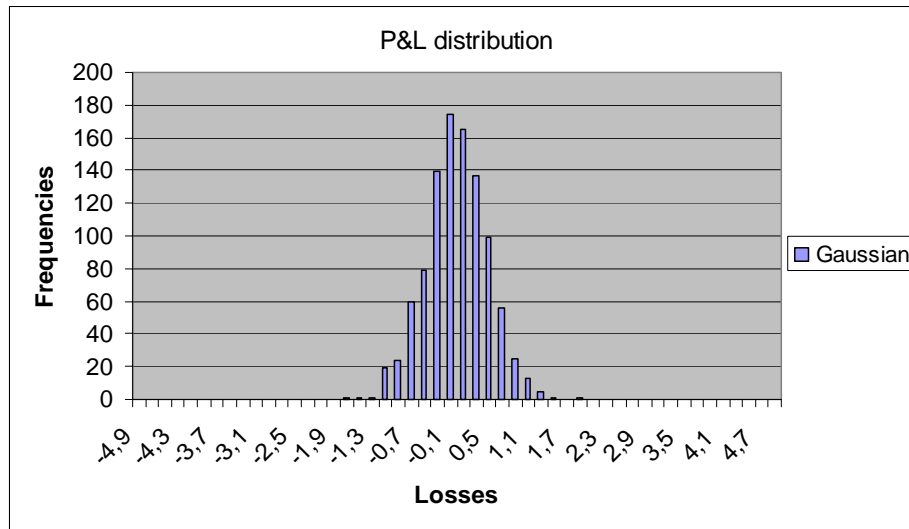
Figure 7. *Portfolio profit and loss frequency distribution obtained from 1000 MC simulations with t₄-Student margins and copula.*

Figure 8. *Portfolio profit and loss frequency distribution obtained from 1000 MC simulations with t₂-Student margins and copula.*
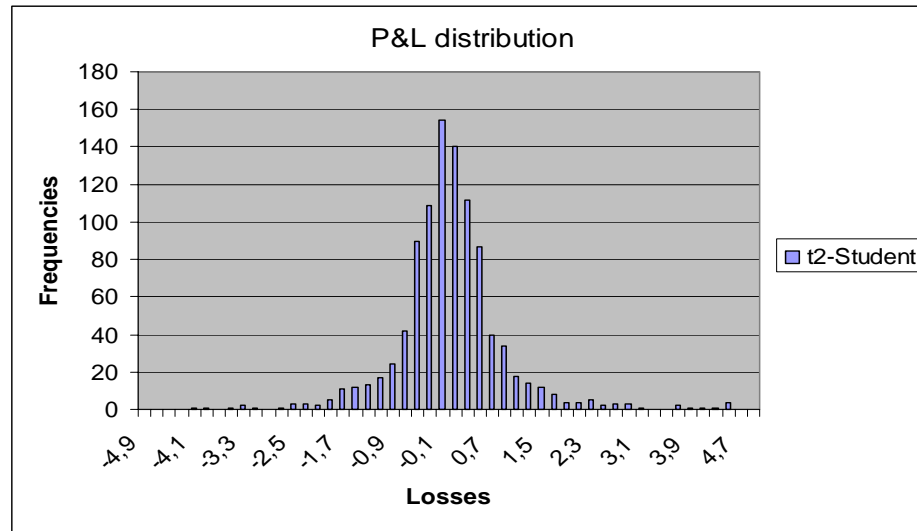
6. An application of copula function to portfolio credit risk measurement

The application in this section has been performed following the model described in section 4b.

We consider a generic portfolio of m=30 loans with exposure values between 31,250 Euro and 88,750 Euro. The total portfolio exposure is 1,607,500 Euro. The 30 loans are classified in 5 different **credit states**: A, B, C, D, and E. The loans with a better credit worthiness are in credit state A. A loan's credit worthiness gets worse up to credit state E. We suppose that all the loans in the same credit state have the same yearly probability of default (see Table 4).

Table 4. *Credit states and probability of default.*

Credit state	Probability of default
A	0.8%
B	2.5%
C	5%
D	7,5%
E	15%

The model assumes a constant recovery rate of 0.2 for all the loans.

The latent variables joined to the 30 loans are simulated assuming different analytic forms for the marginal distributions F_i, i=1,2,…,30, and the copula[43]. The thresholds, D_i, i=1,2,…,30, have been calculated inverting the distribution functions of the latent variables in a point equivalent to the probabilty of default of each loan (PD_i): $D_i=F_i^{-1}(PD_i)$. In this way, the probability that the latent variable value in a scenario falls below the threshold is exactly the probabilty of default of the loan.

The latent variables are simulated assuming a constant linear correlation coefficient of 0.2 between any pair[44]. The holding period is assumed to be one year.

A scenario is a determination of the vector composed of the 30 latent variables[45]. A counterparty is in default if its latent variable value, X_i, falls below the threshold, D_i. In this case, the loss is equal to the exposure value multiplied by one minus the recovery rate. On the contrary, the counterparty loss is assumed to be zero. Portfolio losses in a scenario are calculated by adding each counterparty loss. Generating n scenarios[46], we can easily obtain the portfolio loss distribution. In Figure 9, two different loss distributions, over

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[43] We have used the Gaussian copula and the t-Student copula with different degrees of freedom.
[44] One can observe that 0.2 is the asset correlation value assumed in the new Basle Committee proposal (January 2001) to calculate the risk weight coefficients in the **Internal Ratings Based** (IRB) approach. Remember that, in the Merton model, the latent variables are the asset returns of any counterparty.
[45] One latent variable for each loan (or counterparty).
[46] In our application n=1000.

a yearly holding period, are shown. The first is obtained assuming a multinormal distribution for the latent variables[47]; the second assuming t_2-Student margins and copula.

Figure 9. *Portfolio loss distributions calculated with 1000 Monte Carlo scenarios under the hypothesis of, respectevely, Gaussian and t_2-Student margins and copula for the latent variables.*

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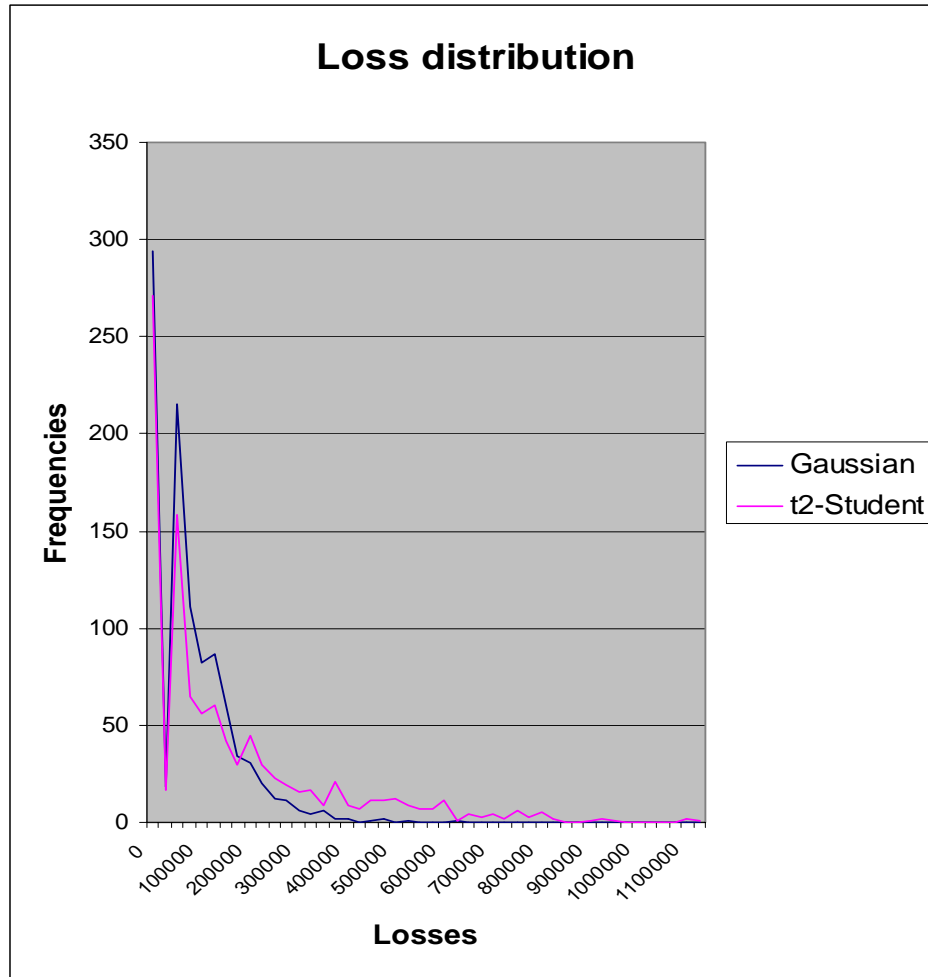
[47] This is the hypothesis made in most of the common portfolio credit risk internal models.

We can immediately see how the tail of portfolio loss distribution is remarkably longer and fatter under the hypothesis of t_2-Student margins and copula of the latent variables. Therefore, the multinormal distribution does not capture extreme risk. Extreme risk in loan portfolios is the risk of many joint defaults over the specified time horizon. On the contrary, events of this kind are effectively captured using the t-Student copula.

In Table 5 we show the values of some portfolio risk measures[48]. These measures are assessed assuming different analytical forms for the marginal distributions of the latent variables and the copula representing their dependence structure[49].

Table 5. *Portfolio risk measures (expressed in Euro) assessed with 1000 Monte Carlo scenarios, assuming different hypotheses for the margins and the copula of the latent variables (Gaussian and t-Student with, respectively, 20, 10, 4 and 2 degrees of freedom).*

	Gaussian	t_{20}-Student	t_{10}-Student	t_4-Student	t_2-Student
EL	71.141	71.180	80.377	107.848	147.707
ML(99%)	349.000	372.500	472.000	615.500	787.500
C-VaR(99%)	277.858	301.320	391.622	507.651	639.792
ES(99%)	433.150	464.250	556.500	783.250	936.050

We can immediately see that tail risk measures (ML, C-VaR and ES) are drastically higher under the t_2-Student hypothesis than under the Gaussian hypothesis. In fact, the normal copula does not have tail dependence and so it is not able to capture the event of many joint counterparty defaults. Unfortunately, it is just this kind of event that worries bank risk managers and supervisors. From this point of view, a model based on the t-Student copula is more effective.

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[48] That is the Expected Loss (EL), 99% Maximum Loss (ML), 99% Credit-VaR (C-VaR), and 99% Expected Shortfall (ES). We remember that Credit-VaR is the Maximum Loss, that is a quantile of loss distribution, minus the Expected Loss.
[49] It is an extension of the application developed in Frey and McNeil (2001). They determined the distribution of the number of joint counterparty defaults. All the counterparties belonged to the same credit state.

Concluding remarks

In this work we have described some possible uses of copula functions in risk management applications. Copula function represents the dependence structure of a random variable vector. For instance, these random variables stand for the asset returns of a portfolio over a specified time horizon. We have proved how some kind of copula functions[50] are easy to implement in Monte Carlo simulation models to estimate risk measures.

We have performed a practical application with a portfolio of ten Italian equities. We have proved that the common hypothesis of multinormal distribution for asset returns (or risk factor returns) underestimates the VaR and the Expected Shortfall of a market portfolio. We have performed a Monte Carlo simulation, modelling asset returns using fat tail marginal distributions and a copula function with tail dependence[51]. In this way, we have obtained a more accurate estimate for the two risk measures.

We have also presented an application of copula function to model portfolio credit risk. We have proved that a methodology using a multivariate Gaussian distribution of the latent variables[52] does not capture the risk of many joint counterparty defaults. On the contrary, events of this kind are effectively captured using the t-Student copula to describe the dependence structure of the latent variables. Therefore, the t-Student copula can be very useful to model the extreme risk that worries risk managers and supervisors.

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[50] Precisely the elliptical copulas.
[51] The t-Student copula.
[52] CreditMetrics and KMV use this hypothesis.

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