Liquidity-Adjusted Portfolio Distribution and Liquidity Score[1]

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Abstract

We discuss a technique to adjust for liquidity risk the P&L of fully general, highly non-normal, multi-asset class portfolios. The liquidity adjustment reacts differently in different states of the market, such as high/low volatility, high/low performance, or high/low overall liquidity. Then we introduce a liquidity score measured in monetary terms based on the liquidity adjustment.

Our methodology relies on the scenarios-probabilities representation of the pure market risk and an analytical convolution of the liquidity adjustment. Therefore, we can adjust for liquidity also the distribution of highly non-linear, non-normal portfolios that contain derivatives.

We present an application to the stock market. Fully documented code is available at http://symmys.com/node/350.

JEL Classification: C1, G11

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1 Introduction

Liquidity risk represents one of the main challenges in buy-side quantitative finance. Loosely speaking, liquidity risk is the value loss of an asset with respect to a reference price, due to the action of trading the asset.

Across all asset classes, liquidity risk stems from the interaction of two sources: the shadow order book and the liquidation policy. The shadow order book represents the structure of supply and demand for an asset at any

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point in time. In the case of exchange-traded stocks, the shadow order book is actually the order book. More in general, the shadow order book is not directly observable. The liquidation policy is the amount of a given asset that the portfolio manager intends to trade within a given time frame.

Here we propose a technique to model liquidity risk which applies to all asset classes and accounts for all the features of the P&L distribution beyond mean and variance and thus in particular non-symmetrical tail events. Furthermore, in our approach the liquidity adjustment can depend on the specific market conditions: if at the investment horizon we are in a down markets, liquidity will be worse than if we are in an up market. Therefore the present framework is particularly suitable for risk management purposes. As a side product of the proposed approach, we introduce a natural definition of liquidity score measured in monetary units.

Liquidity risk has been addressed in a variety of contexts and possibly under different names in the financial literature. To mention only a few: axiomatic definition of price, see e.g. Cetin, R., and Protter (2004) and Acerbi and Scandolo (2007); optimal execution, see e.g. Bertsimas and Lo (1998), Almgren and Chriss (2000), Obizhaeva and Wang (2005), Gatheral, Schied, and Slynko (2011); cost-aware portfolio optimization, see e.g. Lobo, Fazel, and Boyd (2007), Lo, Petrov, and Wierzbicki (2003), Engle and Ferstenberg (2007).

The practical among the above approaches emphasize optimization, whether the optimization of a portfolio assuming a market impact function, or the optimization of market impact assuming a pre-specified optimal trade. In this context, the focus is on the mean and variance of the P&L. Furthermore, those approaches focus on specific asset classes, typically stocks.

The present approach seeks to model, not to optimize, the liquidity of a portfolio. The conceptually simple, yet computationally challenging task of optimizing a book based on the present approach is deferred to a separate study.

Furthermore, we emphasize that here we do not discuss how to estimate the current or past liquidity of a security or a portfolio, but rather we propose a methodology to model the impact that a given liquidation policy can have on the projected P&L of a portfolio, *provided* that the liquidity features of all the securities in the portfolio have been measured. The very definition of such liquidity measures is an open problem. In equities, measures such as the illiquidity ratio by Amihud (2002) have become standard. For bonds, we explore in Meucci and Pasquali (2010) a definition based on cluster analysis. In other asset classes, the measurement of liquidity is an active field of study.

This article is organized as follows. In Section 2 we introduce a formal theoretical framework to model liquidity risk, where liquidity risk is an adjustment to the pure market-risk component of a portfolio's P&L. In Section 3 we review the most flexible approach to model the pure market-risk component of the portfolio P&L, namely the scenario-probability representation. In Section 4 we introduce a first intuitive model for the liquidity adjustment, where such adjustment does not depend on the state of the market: the level of illiquidity faced in difficult or good times is the same. In Section 5 we refine the previous model, to account for liquidity adjustments that reflect the state of the market.

In Section 6 we discuss the liquidity score, which summarizes into one number the overall liquidity of the given portfolio and bears an interpretation in terms of liquidity-driven potential loss. In Section 7 we present a case study for a book of equities. Fully documented code for the general methodology and the case study is available at http://symmys.com/node/350.

2 Market risk and liquidity adjustment: theory

Consider a portfolio of N securities that will give rise to the projected P&L $\Pi \equiv (\Pi_1, \ldots, \Pi_N)'$ between the current time and the investment horizon. We assume as in Meucci (2011) that the stochastic behavior of the projected P&L is determined by $D \geq N$ risk drivers $Y \equiv (Y_1, \ldots, Y_D)'$. In other words, we assume the existence of pricing functions π_n that map the realizations of the risk drivers Y into the P&L Π_n for each position in the book

$$\Pi_n \equiv \pi_n (Y), \quad n = 1, \ldots, N, \tag{1}$$

This framework is completely general. For instance, in a book of options Y can represent the changes in all the underlyings and implied volatilities. In an equity portfolio, each Y_n is the log-price of the stock at the investment horizon. The pricing function (1) can be exact, or it can be approximated by a Taylor expansion whose coefficients are the well-known "deltas", "vegas", "gammas", "vannas", "volgas", etc.

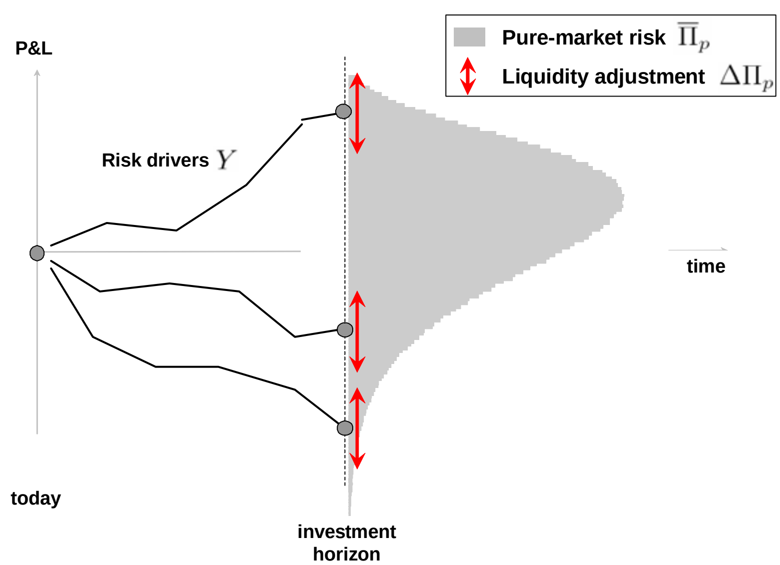

Figure 1: Liquidity risk is price uncertainty at the investment horizon, in addition to market risk

In standard risk and portfolio management applications the pricing function (1) is assumed deterministic. However, for a specific realization y of the risk drivers Y, there exists an uncertainty in the horizon price, and thus in the P&L, due to liquidity-related issues, which include transaction costs, market impact, and price uncertainty, see Figure 1.

Therefore, a more accurate model for the projected P&L of the securities contains two terms, a pure market risk component and a liquidity adjustment

$$\Pi_n = \overline{\Pi}_n + \Delta\Pi_n, \quad n = 1, \ldots, N. \tag{2}$$

The pure market-risk components $\overline{\Pi} \equiv \left(\overline{\Pi}_1, \ldots, \overline{\Pi}_N\right)'$ are deterministic functions of the risk factors as in (1)

$$\overline{\Pi}_n \equiv \pi_n\left(Y\right), \quad n = 1, \ldots, N. \tag{3}$$

The liquidity adjustments $\Delta\Pi \equiv \left(\Delta\Pi_1, \ldots, \Delta\Pi_N\right)'$ depend on two factors.

The first factor is the state of the market, contained in the risk drivers Y. For instance, in a down market, liquidity can suddenly decrease and thus the dispersion of the liquidity adjustment $\Delta\Pi_n$ increase.

The second factor affecting the liquidity adjustment is the action that the risk or portfolio manager intends to take at the investment horizon when the P&L is recorded, i.e. the liquidation policy. Suppose that we have a position of say, 10,000 shares of a stock. If at the investment horizon we plan to hold the position, for mark-to-market purposes it is sufficient to adjust the P&L as if selling the position at the bid price and thus $\Delta\Pi_n$ will be a modest negative number close to half the bid-ask spread. On the other hand, if at the horizon we intend to liquidate all the 10,000 shares, then the liquidity adjustment $\Delta\Pi_n$ should reflect the impact of the transaction on the price.

Once the single-security P&L liquidity-adjusted P&L's (2) are computed, they must be aggregated at the portfolio level. More precisely, consider a portfolio that contains the holdings $h \equiv \left(h_1, \ldots, h_N\right)'$, where by holdings we do not mean portfolio weights, but rather the number of shares for stocks, the number of contracts for futures, etc. The total portfolio P&L Π_h for the portfolio with holdings $h \equiv \left(h_1, \ldots, h_N\right)'$ is obtained by summing the contributions from each security. Similarly to (2), the projected portfolio P&L is the sum of two terms, a pure market risk component and a liquidity adjustment

$$\Pi_{h,\Delta h} = \overline{\Pi}_h + \Delta\Pi_{h,\Delta h}. \tag{4}$$

The pure market risk component is the weighed sum of the single-security P&L's

$$\overline{\Pi}_h \equiv \textstyle\sum_n h_n \overline{\Pi}_n, \tag{5}$$

where we normalized $\overline{\Pi}_n$ to represent the P&L generated by one unit $h_n \equiv 1$ of the n-th security. On the other hand, the liquidity adjustment is the simple sum of the respective contributions from each position

$$\Delta\Pi_{h,\Delta h} \equiv \textstyle\sum_n \Delta\Pi_{n,\Delta h}. \tag{6}$$

Notice that the notation emphasizes how the liquidity adjustments depend both on the asset (single security n or portfolio h) and on the planned policy $\Delta h \equiv (\Delta h_1, \ldots, \Delta h_N)'$.

The purpose of this note is to model and compute the distribution of the total projected portfolio P&L (4).

3 Market risk implementation: scenarios-probabilities distribution

The most flexible approach to model and implement the pure market risk component. Here we sketch the main features. We refer to Meucci (2010) for detailed discussion.

First, we represent the joint distribution of the risk drivers by means of scenario-probability pairs

$$f_Y \Leftrightarrow \{(y_{1,j}, \ldots, y_{D,j}), p_j\}_{j=1,\ldots,J} . \tag{7}$$

In this expression the generic j-th joint scenario $y_j \equiv (y_{1,j}, \ldots, y_{D,j})'$ has relative probability p_j is a coefficient that rescales the unit probability mass of y_j, and thus it represents the relative probability of y_j with respect to the other scenarios.

Starting from the scenario-probability representation for the risk drivers, the distribution of the pure market-risk component of the securities P&L (3) is obtained by re-pricing each scenario

$$f_{\overline{\Pi}_n} \Leftrightarrow \{\pi_{n,j}, p_j\}_{j=1,\ldots,J} , \tag{8}$$

where $\pi_{n,j} \equiv \pi_n (y_{1,j}, \ldots, y_{D,j})$.

Finally, from the securities P&L distribution (8) we immediately obtain the distribution of the market component for the whole portfolio, by aggregating each scenario

$$f_{\overline{\Pi}_h} \Leftrightarrow \{\pi_{h,j}, p_j\}_{j=1,\ldots,J} , \tag{9}$$

where $\pi_{h,j} \equiv \sum_n h_n \pi_{n,j}$.

This concludes the modeling and computation of the market risk component (5) of the portfolio P&L distribution. The simple matrix manipulations required to model the portfolio distribution (9) make the scenario-probability framework easy to implement. Furthermore, all distributional assumptions can be approximated by (7), and different pricing techniques can be utilized, from simple "Greek"-based Taylor approximations to exact full repricing. Therefore, the scenario-probability framework is not only flexible, but also very accurate. Finally, the scenario-probability framework allows for the Fully Flexible Probabilities approach, whereby the relative probabilities of each scenario are modified while keeping the scenarios fixed. This approach is a computationally light, yet powerful methodology to perform generalized stress-testing and robustness checks, see Meucci (2010).

4 Liquidity adjustment implementation: a simple model

In this section we make the simplifying assumption that the liquidity adjustment $\Delta\Pi_{h,\Delta h}$ in (4) is independent of the state of the market, as represented by the value of one or more of the risk drivers Y. We will relax this assumption in Section 5.

Under the above assumption, the yet to be determined liquidity adjustment $\Delta\Pi_{h,\Delta h}$ is independent of the pure market-risk term $\overline{\Pi}_h$ in (4).

In (9) we derived the pdf $f_{\overline{\Pi}_h}$ of the market-risk component. Suppose that we have computed the pdf $f_{\Delta\Pi_{h,\Delta h}}$ of the liquidity adjustment. Then, due to the independence of $\overline{\Pi}_h$ and $\Delta\Pi_{h,\Delta h}$, the pdf of the total portfolio P&L is the convolution of the market-risk component and the liquidity adjustment

$$f_{\Pi_{h,\Delta h}} = f_{\overline{\Pi}_h} \star f_{\Delta\Pi_{h,\Delta h}}, \tag{10}$$

where $(g \star h)(x) \equiv \int_{\mathbb{R}} g(z)\, h(x-z)\, dz$. The convolution (10) can be performed numerically using Fourier transform techniques, see e.g. Albanese, Jackson, and Wiberg (2004).

However, we can actually compute the convolution (10) analytically for any specification of the yet to be defined liquidity adjustment $f_{\Delta\Pi_{h,\Delta h}}$ because of the scenarios-probabilities representation of the market-risk component (9). As we show in the appendix

$$f_{\Pi_{h,\Delta h}}(x) = \textstyle\sum_j p_j f_{\Delta\Pi_{h,\Delta h}}(x - \pi_{h,j}). \tag{11}$$

This is the pdf of a mixture distribution.

To fully specify the P&L distribution (11), we must now model the distribution of the liquidity adjustment $f_{\Delta\Pi_{h,\Delta h}}$. For analytical tractability, let us assume that at the position level the liquidity adjustment is normally distributed

$$\Delta\Pi \sim \mathrm{N}\left(\mu(h,\Delta h), \sigma^2(h,\Delta h)\right), \tag{12}$$

In this expression $\mu \equiv (\mu_1, \ldots, \mu_N)'$ is a $N \times 1$ vector-valued function and $\sigma^2 \equiv \{\sigma^2{}_{m,n}\}_{m,n=1\ldots,N}$ is a $N \times N$ symmetric and positive matrix-valued function. These functions define the impact on the liquidity adjustment of the liquidation policy $\Delta h \equiv (\Delta h_1, \ldots, \Delta h_N)'$ that will take place at the investment horizon.

To illustrate the specification (12) in the stock market, we draw from the standard industry practice to model market impact

$$\mu_n(h,\Delta h) \equiv -a_n \cdot |\Delta h_n| - b \cdot s_n \cdot \left|\frac{\Delta h_n}{v_n}\right|^{\delta}. \tag{13}$$

In this expression a is approximately the bid-ask spread, $b \approx 10^{-6}$, s_n is an estimate of the market volatility of the position, v_n is the daily volume, $\delta \approx 1.5$, refer e.g. to Almgren, Thum, Hauptmann, and Li (2005).

For the covariance, we set the standard deviations as multiples of the expectations and we inherit the correlations from the pure market risk components. Therefore

$$\begin{aligned} \sigma_n (h, \Delta h) &\equiv c \cdot \mu_n (h, \Delta h) && (14) \\ \rho_{n,m} (h, \Delta h) &\equiv \rho_{n,m}, && (15) \end{aligned}$$

where $c \approx 5$ and $\rho_{n,m}$ is a sample estimate of the market correlation among the stock returns.

Given the joint distribution of the liquidity adjustment for all the securities (12), the liquidity correction at the portfolio level follows from the aggregation rule (6) and the aggregation property of the normal distribution

$$\Delta\Pi_{h,\Delta h} \sim \mathrm{N}\left(1'\mu_{h,\Delta h}, 1'\sigma^2_{h,\Delta h}1\right). \tag{16}$$

We can now substitute this expression in (11), obtaining the distribution of the total portfolio P&L, as represented by its cdf

$$F_{\Pi_{h,\Delta h}} (x) = \sum_j p_j \Phi(\frac{x - \pi_{h,j} - 1'\mu_{h,\Delta h}}{\sqrt{1'\sigma^2_{h,\Delta h}1}}), \tag{17}$$

where Φ denotes the cdf of the standard normal distribution.

From the cdf (17) we can compute VaR, CVaR and any other statistics or moments for the total portfolio P&L.

5 Liquidity adjustment implementation: a more accurate model

Let us now consider the more realistic case where the liquidity adjustment $\Delta\Pi_{h,\Delta h}$ in (4) does depend on the state of the market, which is represented by the value of one or more of the risk drivers Y. Then, as we show in the appendix, we obtain

$$f_{\Pi_{h,\Delta h}} (x) = \sum_j p_j f_{\Delta\Pi_{h,\Delta h}|y_j} (x - \pi_{h,j}). \tag{18}$$

This is the pdf of a conditional mixture, which generalizes to the state-dependent case the unconditional mixture (11).

Again, we must model the distribution of the liquidity adjustment $f_{\Delta\Pi_{h,\Delta h}|y_j}$, which this time is conditional on the market scenario y_j. The simplest approach is to adapt the normal unconditional model (12) and assume a conditionally normal distribution for the liquidity adjustments at the security level $\Delta\Pi_{h,\Delta h}|y \sim \mathrm{N}(\mu (h, \Delta h, y), \sigma^2 (h, \Delta h, y))$.

However, the only important feature of the normal model (12) is its aggregation property, which directly yields the liquidity adjustment for the portfolio P&L (16). Elliptical distributions are share the same aggregation properties as the normal distribution, but allow for much more flexibility in modelling tail behavior and codependency. Therefore, without penalizing tractability, we generalize the normal specification by assuming a conditional elliptical structure

$$\Delta\Pi_{h,\Delta h}|y \sim \mathrm{El}\left(\mu\left(h,\Delta h;y\right),\sigma^2\left(h,\Delta h;y\right),\varphi_N\right). \tag{19}$$

In this expression μ is the $N\times 1$ location vector; σ^2 is the $N\times N$ scatter matrix, and φ_N is the tail generator for dimension N. The parameters μ and σ^2 are respectively vector-valued and matrix-valued functions of both the liquidation policy and the state of the market, generalizing (12). We illustrate a practical example for this framework in Section 7, with a Student t specification for the liquidity adjustment.

Then, from the aggregation rule (6) and the properties of the elliptical family of distributions, the portfolio-level conditional liquidity adjustment is also elliptically distributed

$$\Delta\Pi_{h,\Delta h}|y \sim \mathrm{El}\left(1'\mu\left(h,\Delta h;y\right),1'\sigma^2\left(h,\Delta h;y\right)1,\varphi_1\right), \tag{20}$$

where φ_1 is the tail generator for dimension 1.

We can now substitute this expression in (18), obtaining the distribution of the total portfolio P&L, as represented by its cdf

$$F_{\Pi_{h,\Delta h}}\left(x\right) = \textstyle\sum_j p_j\varphi\left(\frac{x-\pi_{h,j}-\mu_{h,\Delta h;j}}{\sigma_{h,\Delta h;j}}\right). \tag{21}$$

In this expression φ denotes the univariate cdf of the standardized elliptical distribution with generator φ_1, which is computed numerically once and for all; and

$$\mu_{h,\Delta h;j} \equiv 1'\mu\left(h,\Delta h;y_j\right), \quad \sigma^2_{h,\Delta h;j} \equiv 1'\sigma^2\left(h,\Delta h;y_j\right)1. \tag{22}$$

From the total portfolio P&L cdf (21) we can easily compute the VaR, the CVaR all the moments and any statistics.

6 The liquidity score

A portfolio is liquid when the liquidity adjustment is minor. Since the liquidity adjustment only hits the left tail, it is natural to define a liquidity score as the percentage deterioration in the left tail, as measured by a standard measure such as the conditional value at risk with 95% confidence.

Accordingly, we define the liquidity score $\mathrm{ls}\left(h,\Delta h\right)$ of the book h and the liquidation policy Δh in terms of the difference between the pure market-risk CVaR $CVaR\left\{\overline{\Pi}_h\right\}$ and the total, liquidity-adjusted CVaR $CVaR\left\{\Pi_{h,\Delta h}\right\}$, normalized as a return

$$\mathrm{ls}\left(h,\Delta h\right) \equiv 1 - \frac{\left|CVaR\left\{\overline{\Pi}_h\right\} - CVaR\left\{\Pi_{h,\Delta h}\right\}\right|}{CVaR\left\{\overline{\Pi}_h\right\}}. \tag{23}$$

The liquidity score is always larger than zero and less than one. When the impact of liquidity is negligible, the liquidity score approaches the upper boundary of one.

7 Case study: liquidity management for equity portfolios

Here we consider a portfolio of stocks. In the stock market, the risk drivers are the log-prices of the stocks at the future investment horizon $Y_n \equiv \ln P_{n,T+\tau}$. Therefore the P&L pricing function (1) reads

$$\Pi_n = \pi_n(Y) = e^{Y_n} - p_{n,T}, \tag{24}$$

where $p_{n,T}$ is the price of the stock at the current time T, which is known.

We model the liquidity perturbation as a special case of the conditional elliptical assumption (19), namely the Student t distribution

$$\Delta\Pi_{h,\Delta h}|y \sim \mathrm{St}\left(\nu, \mu\left(h, \Delta h; y\right), \sigma^2\left(h, \Delta h; y\right)\right). \tag{25}$$

In particular, we set the degrees of freedom $\nu \approx 5$. The power-law tail decay of this Student t distribution with low degrees of freedom induces fat-tails in the liquidity adjustment.

To illustrate the specification of μ and σ^2 in (25), we generalize the market impact model for μ_n introduced in (13) to a state-dependent model as follows

$$\mu_n\left(h, \Delta h; y\right) \approx g\left(\textstyle\sum_n h_n \pi_n(y)\right) \mu_n\left(h, \Delta h\right). \tag{26}$$

In this expression the conditioning term $g(\ldots)$ is a function of the portfolio market P&L $\sum_n h_n \pi_n(y)$, which in turns depends on the state of the market y. The conditioning term is specified as

$$g(z) \equiv \frac{1}{\underline{z}} \min\left\{z, \underline{z}\right\}. \tag{27}$$

This function is always larger than and it increases when the P&L is below a given threshold $\underline{z}$. The rationale behind this choice is that liquidity hits the hardest when the portfolio is underperforming. In particular, if the threshold is $\underline{z} \equiv -\infty$, the liquidity adjustment will not depend on the state of the market. Alternative or additional conditioning variables can include an overall index of illiquidity such as Amihud (2002) for stocks or Meucci and Pasquali (2010) for bonds.

To complete the specification of the model (25) we set the scatter parameter σ^2 as in (14)-(15).

In Figure 2 we display the case of an equity portfolio with $N \equiv 40$ stocks, with average daily volume across all stocks of approximately 10^5 shares. We assume that the current positions h are long-short 1/3 of the total volume and that the liquidation policy Δh is the full amount of the book. The market risk portion is computed with $J \equiv 10^5$ scenarios.

In the top portion of the figure we set the threshold for the state-dependent conditioning term in the liquidity adjustment (27) as $\underline{z} \equiv -\infty$, which makes the liquidity adjustment not state-dependent. In the lower portion of the figure the

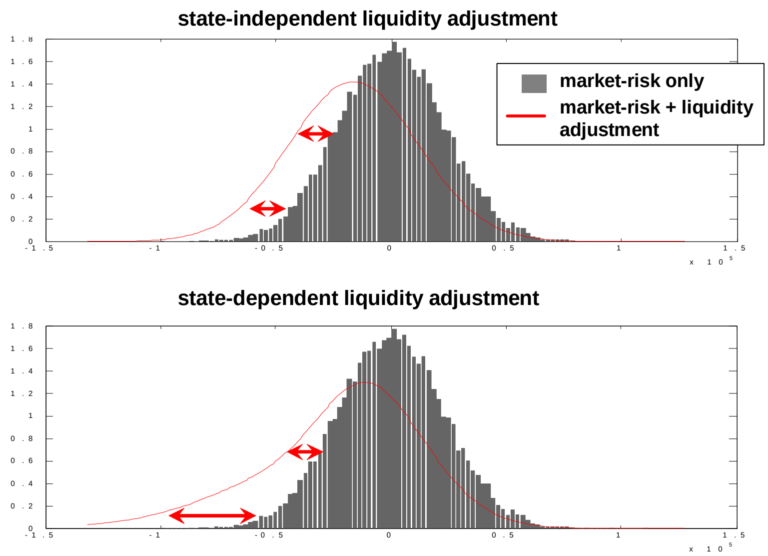

Figure 2: Distribution of portfolio P&L with liquidity adjustment

threshold is set as a negative one standard deviation event: the non-symmetric effect of the liquidity adjustment, which worsens as the portfolio degenerates, becomes apparent.

With these settings, the liquidity score (23) of the portfolio is $\text{ls}(h, \Delta h) \approx 70\%$. We refer to http://symmys.com/node/350 for fully documented MATLAB code and for more details.

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A Appendix

First, we represent the scenarios-probabilities joint distribution of the risk drivers (7) as follows

$$f_Y = \sum_j p_j \delta^{y_j}. \tag{28}$$

In this expression δ^{y_j} denotes the Dirac delta centered in the j-th joint scenario $y_j \equiv (y_{1,j}, \ldots, y_{D,j})$, i.e. a spike of unit probability mass in y_j; and p_j is a coefficient that rescales the unit probability mass of y_j, and thus it represents the relative probability of y_j with respect to the other scenarios. Then the joint distribution of the securities P&L's (8) is obtained by re-pricing each scenario

$$f_{\overline{\Pi}_n} = \sum_j p_j \delta^{\pi_{n,j}}. \tag{29}$$

Finally, the pure market-risk component for the whole portfolio (9) reads

$$f_{\overline{\Pi}_h} = \sum_j p_j \delta^{\pi_{h,j}}. \tag{30}$$

The convolution with a Dirac delta causes a shift $(f \star \delta^z)(x) = f(x - z)$. Thus for (10) we obtain

$$f_{\Pi_h}(x) = \sum_j p_j \left(\delta^{\pi_{h,j}} \star f_{\Delta\Pi_h}\right)(x) = \sum_j p_j f_{\Delta\Pi_h}(x - \pi_{h,j}). \tag{31}$$

As far as (18) is concerned, using the conditional-marginal representation of a generic pdf, we obtain

$$\begin{aligned} f_{\Pi_p}(x) &= \int f_{\Pi_p|y}(x) f_Y(y)\, dy \\ &= \int f_{\Delta\Pi_p|y+\sum_j h_n \pi_n(y)}(x) f_Y(y)\, dy \\ &= \int f_{\Delta\Pi_p|y}\left(x - \sum_j h_n \pi_n(y)\right) f_Y(y)\, dy. \end{aligned} \tag{32}$$

Then, substituting in this expression the scenario-probability representation of the risk drivers distribution (28), we obtain

$$\begin{aligned} f_{\Pi_h}(x) &= \int f_{\Delta\Pi_h|y}\left(x - \sum_j h_n \pi_n(y)\right) \sum_j p_j \delta^{y_j}(y)\, dy \\ &= \sum_j p_j f_{\Delta\Pi_h|y_j}\left(x - \sum_j h_n \pi_{n,j}\right) = \sum_j p_j f_{\Delta\Pi_h|y_j}(x - \pi_{h,j}). \end{aligned} \tag{33}$$