An Application of the Black-Litterman Model with EGARCH-M-Derived Views for International Portfolio Management

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Abstract

This paper provides an application of the Black-Litterman (1991, 1992) methodology to portfolio management in a global setting. The novel feature of this paper relative to the extant literature on Black-Litterman methodology is that we use GARCH-derived views as an input into the Black-Litterman model. The returns on our portfolio surpass those of portfolios that rely on market equilibrium weights or Markowitz-optimal allocations. We thereby illustrate how the Black-Litterman model can be put to work in designing global investment strategies.

Keywords: Black-Litterman, GARCH, Global Portfolio Management.

JEL classification: G11, G15.

1 Introduction

This paper has several closely related objectives: to find an econometric model that accurately describes the dynamics of the excess returns on international portfolios and the dynamics of the returns volatility, to use the selected model in forecasting conditional variance, and to ultimately choose an optimal global portfolio using the Black-Litterman model. We believe that this approach provides a sufficiently sophisticated and consistent method for managing international portfolio risk and return. The results of this paper should be of interest to individuals and institutions participating in the international equity market, who wish to have access to useful predictions of future returns and volatilities and proven methodology for applying them in portfolio management.

The disparity between the traditional mean-variance portfolio selection theory and investment practice is quite striking (e.g., He and Litterman, 1999, Drobetz, 2001). This is primarily due to investment professionals' desire to create intuitive portfolio weights rather than to rely on the weights produced by an optimization routine. Black and Litterman (1991, 1992) made a long-awaited step in closing the gap between investment theory and practice. Their model incorporates investor views into the standard portfolio optimization theory. The results are portfolio weights that can be close to, or far from, the market equilibrium weights, depending on the investor's confidence in his or her views and willingness to assume risk in the portfolio.

This paper illustrates how the Black-Litterman model can be put to use in resolving questions regarding appropriate allocations in global portfolio management. We present time series of results, whereas most previous work gives one-period allocation examples.[1] The novel feature of the paper relative to the extant literature on Black-Litterman methodology is the use of GARCH-derived views as proxies for investor views in the Black-Litterman model. This approach may be superior to using subjective views of analysts, as General Autoregressive Conditional Heteroskedasticity (GARCH) models capture many regularities of stock returns in an elegant and systematic way. It is useful for expository purposes, since there is no model that effectively describes an investor's views.

Thus, as our main forecasting technique we use asymmetric GARCH models. These models are widely recognized as a sharp tool in predicting the dynamics of stock returns. In the international finance setting, as well as other financial applications, forecasts are aided by predicting volatility as a weighted average of a long-term trend, the forecast variance from past periods, and information observed in the previous periods. This approach is consistent with modelling so-called "volatility clustering," whereby large swings in returns are usually followed by further large changes, as well as other stylized facts about asset returns. Our data (described below) do exhibit such volatility clustering: (absolute level of) fluctuations between any two consecutive months are correlated with the adjacent periods.

Modelling and forecasting volatilities are important in several respects. First, the risk of participating in the international equity market can be assessed. Second, confidence intervals for equity return forecasts are most likely to be time-varying; disregarding this possibility would lead to inaccuracies in forecasts. Third, heteroskedastisity of the forecast errors can be handled properly in the context of the GARCH models, so that more efficient return estimates can be obtained. Finally, volatility forecasting is important in investment and risk management (Poon and Granger, 2003).

[1] The important exception is Bevan and Winkelmann (1998), who provide portfolio performance based on allocations established by including analyst "views."

Drobetz (2001) surveys the reasons why investment practitioners are dissatisfied with the traditional mean-variance approach to portfolio optimization. Among those reasons are unreasonable quantity and quality of required input data, large negative portfolio weights, and high sensitivity of weights to changes in expected returns. The Black-Litterman model copes with these deficiencies by combining equilibrium returns (or the "neutral view") with investor's subjective return views.

We use the results from the GARCH model as inputs to the Black and Litterman approach to establish an optimal global portfolio allocation on a rolling basis. The existing market capitalization weights are considered to be the global equilibrium, from which equilibrium expected returns are obtained through reverse optimization. Inputs of expected returns and standard deviations from the GARCH model establish the "investor view". The Black-Litterman approach provides a Bayesian methodology for combining the equilibrium expected returns and the investor views to produce a vector of expected returns. This expected return vector and the historical covariance matrix are then used to establish the Black-Litterman investment weights. A major benefit of the Black-Litterman approach is the ability to avoid excessive corner solutions that many optimization routines will generate when some assets have high expected return or low standard deviation estimates. The confidence in estimates determines the extent to which the model will deviate from the equilibrium weightings.

The next section reminds the reader about the stylized facts of asset returns; Section 3 briefly reviews GARCH models and presents a GARCH model that is used to forecast mean returns and conditional variance; the Black-Litterman methodology of combining equilibrium weights with investor's subjective views is outlined in Section 4; Section 5 describes the dataset; GARCH estimation and Black-Litterman portfolio weight calculations are presented in Sections 6 and 7; Section 8 concludes.

2 Empirical Regularities of Asset Returns

Asset returns are characterized by several stylized facts: "volatility clustering" (highly persistent periods of volatility and tranquility), excess kurtosis (i.e., thick tail distributions), asymmetry, mean reversion, autocorrelation in risk, time-varying volatility, leverage effects, inverse relationship between volatility and serial correlation, co-movements in volatilities across assets as well as across markets, and the effects of macroeconomic variables on volatility (Bollerslev, Chou and Kroner, 1992, Campbell and Hentschel, 1992). Combining the Exponential GARCH (Nelson, 1991) and GARCH-in-Mean (Engle, Lilien and Robins, 1987) models that we employ in this paper allows one to take into account most of these characteristics of financial data.

It is well known that most asset returns are characterized by time-varying volatility, or heteroskedastisity. Volatility is also conditional in that current observations depend on the observations of the (immediate) past. To that end, GARCH models allow for an explicit modelling of this relationship between future variances and past variances, as well as past variance forecasts. GARCH models also take into account the fact that asset returns are usually leptokurtic, i.e., they exhibit excess kurtosis (their probability distributions have fat tails). Finally, GARCH models take into consideration volatility clustering, another salient feature of financial time series (Bollerslev, Chou and Kroner, 1992). Volatility clustering, or persistence, occurs when large (in absolute value) changes in returns are followed by large changes, and small changes are followed by other small changes.

Indeed, in our data we observe that the (absolute level of) fluctuations between any two consecutive months are correlated with the adjacent periods. In other words, we observe volatility clusters, so there is a need to focus on modelling conditional variance (i.e., risk). GARCH models not only acknowledge heteroskedastisity, but allow for a rigorous modelling of this phenomenon. The GARCH model allows us to favorably combine theory and pragmatism when making forecasts and choosing an international portfolio. Our data display highly significant GARCH effects (identified by ARCH tests), which further supports the use of a GARCH model to forecast conditional volatility.

3 GARCH Modelling

Engle (2001) notes that GARCH models are useful when there is a need to analyze the size of the errors and to forecast volatility. As mentioned in the previous section, this is especially crucial in the presence of heteroskedasticity. In cross-sectional econometrics researchers have White's (1980) and other methods to cope with heteroskedasticity to calculate "robust standard errors." However, one should not straightforwardly apply those methods to time-series data. Rather, a researcher needs to be able to make inferences about the accuracy of the predictions of the model and explain what makes the variance of the error large. Volatility forecasts based on GARCH modelling are especially important in portfolio construction (Pojarliev and Polasek, 2003).

Rolling standard deviation — a weighted average of n most recent observations — is a natural way to forecast the variance of the return conditional on the past information. This ARCH model, which excludes less recent observations and estimates weights attached to the more recent data, was first proposed by Engle (1982). Bollerslev (1986) introduced the GARCH term so the weights never go to zero. The GARCH model offers a more parsimonious model that reduces the computational burden. It uses past variances and past variance forecasts to forecast future variances. The model is quite successful in predicting conditional variances (e.g., Bollerslev, Chou and Kroner, 1992, Bollerslev, Engle and Nelson, 1994).

It is beneficial to use GARCH models in many international finance applications primarily because GARCH models allow not only to forecast conditional means of asset returns, but also conditional variances. In this paper we combine (and extend) the ARCH-in-Mean model (Engle, Lilien and Robins, 1987) to relate the expected return on assets to the expected risk, and the Exponential GARCH model (Nelson, 1991) to allow for asymmetric shocks to volatility.

3.1 GARCH(1,1)

In general, a GARCH model can be represented by two equations — one for the conditional mean and the other for the conditional variance:

$$\begin{aligned} y_t &= x_t'\gamma + \varepsilon_t &(1)\\ \sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, &(2)\end{aligned}$$

where y_t is the dependent variable (i.e., excess return), x_t is a vector of exogenous variables, $\varepsilon_t \tilde{} N(0, \sigma_t^2)$ is an error term, and α, β, and γ are the coefficients to be estimated. The one-period ahead forecast variance σ_t^2 (conditional variance) depends on the mean (ω), news about volatility from the previous period (ε_{t-1}^2, the ARCH term), and last period's forecast variance (σ_{t-1}^2, the GARCH term). GARCH(1,1) refers to the presence of a first-order GARCH term (the first term in parentheses) and a first-order ARCH term (the second term in parentheses).[2]

3.2 EGARCH-M(1,1) with Regressors

For equities, it is often observed that downward movements in the market are followed by higher volatilities than upward movements of the same magnitude. Exponential GARCH (EGARCH) models are able to account for asymmetric shocks to volatility. ARCH-in-Mean models introduce the conditional variance into the mean equation. ARCH-M is used in financial applications where the expected return on an asset is related to the expected asset risk. The estimated coefficient on the expected risk is a measure of the risk-return trade-off. The higher the coefficient, the more favorable the compensation for risk-taking. Thus,

[2] A problem with this simplest specification is that the forecast variances must be constrained to be positive.

an attractive feature of the ARCH-M model is that it allows one to explicitly model the risk-return trade-off.

We extend the ARCH-M and EGARCH equations by including additional sets of regressors z_1 and z_2:

$$y_t = x_t'\gamma + \delta\hat{\sigma}_t^2 + \psi z_{1t} + \varepsilon_t \tag{3}$$

$$\log\sigma_t^2 = \omega + \beta\log\sigma_{t-1}^2 + \alpha\left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + \gamma\frac{\varepsilon_{t-1}}{\sigma_{t-1}} + \varphi z_{2t}. \tag{4}$$

This EGARCH-M(1,1) (Exponential GARCH-in-Mean) model, with additional regressors, is used to estimate expected returns and expected variances in this paper. The *log* implies that the leverage effect is exponential, thus forecasts of the conditional variance are guaranteed to be nonnegative. The EGARCH model is able to capture the empirical regularity that a negative shock leads to a relatively higher conditional variance than a positive shock of the same magnitude.[3]

4 Black-Litterman Global Asset Allocations

4.1 Discussion

In the investment industry, the Black-Litterman model has been used since 1990 by Goldman Sachs and is used to varying degrees by numerous quantitative investment shops. Software that implements the Black-Litterman model has been available to institutional investors for several years and is now being marketed to financial advisors. As a result, more investment managers, financial advisors, and sophisticated individual investors are being exposed to this methodology. The education efforts by the purveyors of these software programs and investment vehicles are on the increase.

The Black-Litterman asset allocation model utilizes CAPM theory (to establish equilibrium returns and risk), Sharpe's reverse optimization, Bayesian mixed estimation, and mean-variance optimization based on Markowitz (1952). Additional development of the model included elements of Black's global CAPM and the universal hedge ratio in a global model (Black 1989, 1990). Some less technical works that introduce the Black-Litterman model and attempt to make it more accessible to practitioners are Bevan and Winkleman (1998), He and Litterman (1999), Satchell and Scowcroft (2000), Drobetz (2001), and Idzorek (2005). Readers desiring a working knowledge of the model can reference those works.

[3] Readers interested in a more detailed presentation of GARCH models can refer to a survey by Bollerslev, Engle and Nelson (1994).

The fundamental idea behind the Black-Litterman model is that the equilibrium that exists in the financial markets, represented by the existing capitalization weights, serves as the basis for establishing an optimal allocation. These market capitalization weights are used to establish implied equilibrium expected returns. The views held by the investor regarding expected returns are an additional input to the asset allocation decision. In the research presented in this paper, an Exponential GARCH-in-Mean (EGARCH-M) model is used to provide the investor views and confidence measures for those views.

The Black-Litterman model has received limited attention in the academic literature for a variety of reasons, despite significant and growing use in the investment industry. A notable reason for this situation is that the Black-Litterman model has inputs which are established by investor views. The Black-Litterman model functions within a Bayesian framework in which these views are inputs to an optimization routine. In contrast, for the Markowitz framework, historical statistics are the only admissible inputs. Without constraints, traditional optimization methods can result in extreme allocations that are not intuitive or fail to be acceptable relative to investor's risk tolerance. A common response to shortcomings in standard portfolio theory is to impose constraints on the model-generated allocations, which results in globally optimal allocations only on rare occasions.

In the Black-Litterman model, when an allocation is established, investors can adjust the confidence in their views, based on the portfolio risk that would exist with the allocations. An adjustment in stated confidence in the investor view is a natural, admissible response to the investment risk in a Bayesian context. Testing hypotheses in this context, however, is convoluted by a joint hypothesis problem, which includes questions regarding the quality and validity of the investor views that are used in the model, and compounded by the opportunity for updating of the investor confidence parameter. As a result, none of the available literature presents a time-series of historical results that can be usefully compared to benchmark returns. In this paper, several simplifications are imposed in the model so Black-Litterman model returns can be reasonably compared to benchmark returns, including an example of the flexibility of this approach to portfolio selection.

4.2 Black-Litterman Equation

The formula given below is known as the Black-Litterman equation and represents the expected return vector that is established from a Bayesian mixing of the implied equilibrium excess return vector (Π) and the vector of investor's views (V):

$$E(R) = \left[(\tau\Omega)^{-1} + P'\Sigma^{-1}P\right]^{-1}\left[(\tau\Omega)^{-1}\Pi + P'\Sigma^{-1}V\right].$$

The Ω matrix is the covariance matrix of excess returns, and Σ is a diagonal matrix of error terms (or variances) of the views.[4] The effective weight placed on the views is set with the value of τ, which in most cases will be near zero. In the Black-Litterman equation, a lower value of τ gives greater weight to the implied equilibrium return vector Π.[5] The implied return vector $\Pi = \lambda\Omega^{-1}w_{mkt}$ represents the required (excess) returns that would clear the market, based on the given vector of market capitalization weights (w_{mkt}), the covariance matrix of excess returns (Ω), and the estimated coefficient of investor risk aversion (λ). The value of λ is an estimate of the required investor reward-to-risk, $\lambda = \left[E\left(R_m\right) - r_f\right]/\sigma_m^2$, and was set to 2.65.[6] The P matrix selects the assets for which views are imposed. We use the rolling 120-month historical covariance matrix as the estimate of Ω for each month's calculation of the Black-Litterman equation. The weight on each observation in the 120-month estimation period is equal.[7]

The investment weights from the Black-Litterman model are then established by reverse optimization utilizing the blended return vector and the covariance matrix of historical returns $W_i = \Omega^{-1}E\left(R\right)$. The proportional weights are established as follows:

$$w_i = \frac{W_i}{\sum\limits_{i=1}^{n} W_i}.$$

One of the difficulties in assessing the usefulness of the Black-Litterman model is the quality of the investor views that are a major component of the process. Any testing of Black-Litterman model portfolio results is jointly an assessment of the view inputs. The views used in this paper are generated from 120-month EGARCH-M models applied to each market's returns. The EGARCH-M models also generate the variances used in Σ. In contrast to most

[4] The EGARCH-M-estimated variances on the diagonal of Σ are conditional on the macroeconomic data included in their estimation. These variances, if estimated in GARCH model without the conditioning macroeconomic factors, would constitute historical variances calculated with a decay factor.

[5] If the investor states no views ($V = 0$), or $\tau = 0$, implying no confidence in the views, then $E\left(R\right) = \Pi$. This result would indicate that the investor would expect to obtain the expected returns from holding the market capitalization weighted portfolio.

[6] Drobetz (2001) used $\lambda = 3$ and Idzorek (2005) used $\lambda = 2.25$, based on a market risk premium of 7.5% and a standard deviation for the DJIA of 18.25%. Idzorek also estimated $\lambda = 2.62$ using an estimate for the market capitalization portfolio variance. For the world portfolio, we calculated $\lambda = 2.01$ for the period of January 1998 through December 2004.

[7] It is possible to use a decay factor to weigh more heavily on recent observations. Litterman and Winkelmann (1998) discuss covariance matrix estimation procedures and the similarity of GARCH-estimated variances to those calculated with a decay factor. Improved covariance matrix estimation may be obtained following Forbes and Rigobon (2002), who provide a model that corrects for an upward bias in correlations when market volatility is high and provides unconditional cross-market correlation coefficients. Ang and Chen (2002) develop a summary statistic that measures the degree of covariance asymmetry, correcting for conditioning biases.

discussions of the Black-Litterman model, which emphasize the use of relative views, our application uses absolute views on returns.

A description of the parameters and the data used in this paper are given in Table 1.[8] The Σ matrix is commonly referenced as reflecting the "confidence" in the views. Here we utilize one-period forward estimates of the variance from the EGARCH-M models (one estimate for each country, for each period) to provide the measure of confidence for each view.

4.3 Foreign Exchange Risk

The hedging of foreign exchange risk in an international portfolio is a significant decision that should be grounded in several practical and theoretical considerations. The decision to not hedge the currency exposure in this research project is discussed in this subsection. If purchasing power parity holds (and investors globally hold the same consumption basket), then the hedging of foreign currency risk would not be necessary, because the currency returns would exactly reflect inflation differences around the globe. Since purchasing power parity is known to not hold, hedging currency risk may be optimal.

Black (1989, 1990) shows that international investors holding the global market portfolio will hedge a portion of their foreign currency exposure and that the hedge ratio will be the same regardless of the investor's home country. This universal hedge ratio will fall between 0% and 100%, not at the extremes, due to Seigel's paradox, which highlights the fact that currency returns are not a zero-sum game between long and short positions. Black proposes a hedge ratio of approximately 70%. In Black and Litterman (1992), an 80% hedge ratio is applied, and the benefits of hedging are sizeable in a bond-only portfolio, at a 0.57% return advantage to the hedged portfolio, given a fixed risk level. For the equity-only portfolio, the return advantage for the hedged portfolio is only 0.08%, potentially insufficient to justify the cost in time and transaction costs.

Many individual investors and small portfolio shops combine the currency and equity allocation decisions, with no attempt to hedge the currency exposure. Possibly, they assume that currency fluctuations are random or mean-revert in a manner that makes the currency allocation irrelevant over an extended timeframe. Alternatively, an investor may believe that the combined allocation to equities and currencies can be managed more effectively than by making separate decisions. If investors do not hold the global market portfolio, the optimal hedge is not universal. In this paper, portfolio holdings are expected to deviate from the global market portfolio weightings. Given the limited expected benefits, potential costs,

[8]Since one view is expressed for each market, $n = k$.

and a decision that depends upon the investor's risk aversion, we have chosen not to hedge the foreign currency risk. This approach allows a focus on other salient components of the international asset allocation decision.

5 Data

We use 15 years of monthly data from January 1988 to December 2002 (180 months). In the GARCH regressions below, the dependent variable is (variance of) excess return on a country-specific index, constructed using one-month eurodollar rate as a risk-free return. We study the influence of a collection of macroeconomic factors upon the returns from an international equity index. The equity returns are from the Morgan Stanley Capital International (MSCI) World Index with gross dividends reinvested. Table 2 presents summary statistics for the country-specific returns. Fourteen countries reject normality based on Jarque-Bera statistic. Positive or negative skewness is observed for many markets. Excess kurtosis is indicated for most of the return series. For each country, the domestic market dividend yield is used as a local factor (Table 3). The monthly local market dividend yields are from MSCI.

The macroeconomic factors mimic those of many of the classic APT and multifactor pricing model investigations. The global factors included in the model are Production (the growth in industrial production for industrial countries), Inflation (industrial country inflation), DTWD (the return on the U.S. dollar index relative to major currencies), Premium (the difference in the yield on BAA and AAA bond indexes from Moody's), TED Spread (the difference in the three-month Eurodollar yield and the three-month treasury bill yield), Term (the difference in the 10-year treasury bond yield and the three-month treasury bond yield), and Oil (the percentage change in the world spot price of oil). Table 4 presents summary statistics for the global macroeconomic factors. The data for Production, Inflation, and Oil are from the International Financial Corporation IFS Database. The data for DTWD, Premium, TED Spread, and Term are from the FRED II database of the Federal Reserve Bank of St. Louis.

We assume these global macroeconomic APT factors can help to forecast returns and to identify periods of high risk for the global stock market. Research has found that global factors can be more significant than local factors in explaining developed market returns (Ferson and Harvey, 1994, 1995). Alternatively, local factors are more significant than global factors in explaining emerging market returns (Bekaert and Harvey, 1997, Bekaert, Erb, Harvey, Viskanta, 1998, and Harvey, 1995).

6 EGARCH-M Methodology

We run a 10-year rolling estimation starting in 1988 and ending in 2003 for 20 countries: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Hong Kong, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, UK and US. We proceed in several steps. First, we compute correlations of excess return to obtain 61 20×20 covariance matrices (from 1998:01 through 2003:01) and obtain historical standard deviations for each country. Second, we estimate the EGARCH for each country using the 10-year rolling window. We use EGARCH-M estimates (a) to compute the predicted excess returns for each country in each period and (b) to make one-step-ahead forecasts of conditional standard deviations.[9]

We estimate equations (3) and (4) jointly for the excess returns on an international portfolio using the maximum likelihood method assuming that the errors are conditionally normally distributed.[10] Different combinations of variables become significant for different countries and different periods, and we choose a unified framework for all countries and all periods. We find an econometric model that accurately describes the dynamics of levels and variances of returns, and use the selected model to forecast future returns and conditional variances. Our forecast of variances are based on 120 months, a sufficient number of periods.

For our regressors in the EGARCH-M(1,1) mean equation for excess returns we include country-specific Dividend Yield, Premium, Spread, Term and Oil. The variance equation regressors are Inflation, Dividend Yield, Premium, Spread and Term. This model was selected using the theoretical consideration about asset returns described above, as well as various statistics (e.g., adjusted R-squared, Akaike information criterion, Schwarz criterion, Log likelihood, Durbin-Watson statistic, F-statistic) and specification tests (e.g., the ARCH test and omitted variables test). An examination of the residuals was performed to fine-tune the model specification. Thus, every effort was made to be pragmatic and consistent in estimation for the entire sample. The factors that appear in our model are significant in most (but not all) regressions. However, an occasional insignificant coefficient does not pose

[9] This paper studies diversified (country and world) portfolios, so a GARCH model is appropriate; it may not have been appropriate for individual securities.

[10] The contribution to the log likelihood from excess return at date t is therefore

$$l_t = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\sigma_t^2 - \frac{1}{2}\left(y_t - x_t'\gamma\right)^2/\sigma_t^2,$$

where σ_t^2 is as defined in equation (2):

$$\sigma_t^2 = \omega + \alpha\left(y_{t-1} - x_{t-1}'\gamma\right)^2 + \beta\sigma_{t-1}^2.$$

a problem, as our goal at this stage is to forecast the conditional variance.

7 Black-Litterman Results

Table 5 summarizes the allocations and resulting returns of portfolios using market allocations and four different "confidence"-based Black-Litterman allocation series. Market allocations to the 20 countries resulted in an average monthly return of -0.393%, a compounded return of -0.505%, and a standard deviation of 5.05%. The largest loss was -13.74% and the highest return was 8.414%. As the τ measure of confidence in the "Black-Litterman implied returns" is increased, the value of the inclusion of the manager's view, proxied by the EGARCH-M inputs, is examined for repeated applications of the Black-Litterman model.

Portfolio risk can be estimated by taking a suggested allocation and applying it to the covariance matrix of the returns from the previous 120-month period. Using these inputs, the variance of the portfolio, $v = w'\Omega w$, is calculated for each month. This approach allows an investor to determine if the allocations result in a portfolio that carries more risk than is acceptable. If the portfolio risk is higher than acceptable, the investor can alter the confidence parameter (τ), until a risk-appropriate allocation is generated. This approach is superior to constrained optimization that sets allocation limits, generally where short sales ($w_i < 0$) and allocations over 100% ($w_i > 1$) are not allowed. The approach is consistent with a Bayesian framework (as in Black-Litterman) and results in an iterative decision process that can be quickly and effectively resolved. Ammann and Verhofen (2006) show two variance regimes exhibit clearly different optimal allocations, reflecting the benefits of variance-based allocation decisions.

At the highest level of confidence investigated, with $\tau = 0.01$, the risk of the portfolio is often above levels one may expect an investor would be willing to assume. The variance of the portfolios at different confidence levels on the implied returns (different values of τ) is presented in Figure 1. At $\tau = 0.01$, the variance is higher in most cases than for all other confidence levels. As τ is set lower, and the weighting on the market weights is given more importance, the risk of the portfolio trends lower in most months. The portfolio risk increased over the 5 years, with significant jumps (for $\tau = 0.01$ in particular) in August 1998, March 2001, and several months in late 2001 and throughout 2002. It appears that the model, with the GARCH inputs, anticipated highly risky investing periods as evidenced by market-weighted returns of -13.74% in August 1998, -6.88% in March 2001, -5.31% in August 2001, and -9.29% in September 2001. Additionally, June 2002–Dec 2002 had several months of significant losses. The 5.5% return in November 2002 was the only month with positive returns having been identified as having an expected portfolio variance over 0.2%

(with $v = 0.208\%$).

At $\tau = 0.01$, the mean return of 0.689%, standard deviation of 7.12%, and worst-case return of -23.85% reflect the aggressive allocations associated with this level of confidence in the implied returns. The monthly compound return was 0.417%. The risk and returns for $\tau = 0.01$ are the highest of the Black-Litterman portfolios. For $\tau = 0.005$, the mean return of 0.616%, standard deviation of 4.87%, and worst-case return of -11.51% reflect lower returns and lower risk associated with this level of confidence in the implied returns. The monthly compound return of 0.499%, however, is greater than for the $\tau = 0.01$ Black-Litterman portfolio. These differences indicate a significant reduction in the overall risk, as τ is set lower to reflect less confidence in investor (here GARCH-generated) views. For $\tau = 0.0025$ and $\tau = 0.001$, the mean returns fall, the standard deviation increases, and the compound returns decrease, with the results for $\tau = 0.001$ moving close to those for the market-weighted portfolio.

The portfolio returns indicate that the Black-Litterman model using the EGARCH inputs produces allocations with potentially sizeable benefits. Greater reliance on the implied returns in setting the allocations resulted in higher returns, while the risk is highest at $\tau = 0.01$, but lowest at $\tau = .005$. As related earlier, the most extreme allocations and the highest portfolio variance occur for the Black-Litterman allocations with $\tau = 0.01$. For ten months, the allocations resulted in predicted portfolio variances greater than 0.2%.

Assuming this level of risk to be unacceptable, a new portfolio is created for which it is assumed that an investment manager would choose to reduce the risk by adjusting τ downward when predicted portfolio variance exceeded 0.2%. All other months' allocations were set using $\tau = 0.01$ in this dynamic portfolio management approach. For the ten months of highest portfolio variance, τ was adjusted downward until $v < 0.2\%$ was achieved. For seven of the ten months, $\tau = 0.005$ was sufficient to provide $v < 0.2\%$. For July, September, and December 2002, $\tau = 0.0025$ was required to reduce the portfolio variance below 0.2%. The results for this risk-reduced Black-Litterman portfolio are impressive. The average monthly return is in excess of 1%, with a worst-case scenario return of -11.51%, a standard deviation of 5.38%, and a compound monthly return of 0.959%. This portfolio contains risk comparable to the market portfolio, but with significantly higher returns.

The risk-tailored portfolio produced the highest returns of the portfolios considered, with lower risk than the $\tau = 0.01$ "confidence" portfolio. The standard deviation was higher than the market-allocation portfolio (5.4% vs. 5.1%), however, with a considerable geometric return advantage (1.0% vs. -0.5%). In Table 6, summary statistics for results from these risk-tailored allocations are provided, in addition to the market-weighted allocation results, and results from allocations derived from traditional Markowitz optimization. The Markowitz-optimal allocations provide an additional benchmark for evaluating the Black-Litterman results. In Table 7, a summary of the portfolio risk (v) estimates is provided. Many of the concerns about Markowitz optimal allocations are displayed, as extreme positive (up to 181.2%) and negative (-433.5%) allocations result. The risk-tailored allocations from the Black-Litterman approach include less extreme maximum (99.1%) and negative (-13.0%) allocations. The risk-tailored portfolios generate an average 1.1% (1.0% compounded) monthly return, far in excess of the -0.1% (-0.8% compounded) average returns for the Markowitz-optimal allocations and the -0.4% (-0.5% compounded) average returns for the market-weighted allocations. The risk of the risk-tailored allocations (average $v = 0.151\%$) is considerably lower than for the Markowitz-optimal allocations (average $v = 0.709\%$). The standard deviation of returns generated by the risk-tailored approach ($\sigma = 5.4\%$) is much lower than from Markowitz optimization ($\sigma = 12.3\%$) and slightly higher than for the market-weighted allocations ($\sigma = 5.1\%$). The risk-reduction in the Black-Litterman model from using lower values of τ is confirmed by the portfolio risk estimates and the resulting standard deviations.

8 Conclusions

This paper provides an application of the Black-Litterman methodology to portfolio management in a global setting. The main value added of this paper stems from using EGARCH-M inputs instead of relying on financial analyst views, and the resulting validity of time-series results. As our results indicate, the returns on our portfolio surpass those of portfolios that rely on market equilibrium weights. We thereby illustrate how the Black-Litterman model can be put to work in designing global investment strategies.

Black-Litterman allows investors to take risk where they have views, with stronger views justifying more risk-taking (Bevan and Winkelmann, 1998). Thus, the output of the Black-Litterman model is a mixture of equilibrium (or neutral) returns and investor views. The results presented in this paper attest to the great potential of Black-Litterman methodology in generating global portfolios and contribute to a risk-based decision method for adjusting the confidence in the expected return inputs.

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Table 1: Black-Litterman Parameter Values

Parameter	Description (each month, $T = 120$)	Value/Data (each month)
τ	Value between 0 and 1, reflects confidence in views	0.01, 0.005, or 0.0025
Ω	$n \times n$ covariance matrix of excess returns	Rolling 120-month historical values
P	$k \times n$ matrix identifying the assets with views	Diagonal identity matrix; views imposed for all assets
Π	$n \times 1$ implied equilibrium return vector	$\Pi = \lambda \Omega w_{mkt}$, $\lambda = 2.65$ is the estimated investor risk aversion coefficient. w_{mkt} is the vector of month-beginning market capitalization weights
Σ	$k \times k$ diagonal covariance matrix of error terms for views	EGARCH-M estimates of variance using 120 months of data
V	$n \times 1$ vector of return views	EGARCH-M estimates of expected return using 120 months of data

Table 2: Returns on Country-Specific Indices: Summary Statistics (January 1988 – December 2002, $n = 180$ months)

Country	Mean	Median	Maximum	Minimum	St. Dev.	Skewness	Kurtosis	Jarque-Bera	Probability
Australia	0.004	0.003	0.169	−0.157	0.056	0.041	3.239	0.479	0.787
Austria	0.002	0.002	0.247	−0.240	0.068	0.071	4.589	19.093	0.000
Belgium	0.005	0.007	0.246	−0.191	0.051	−0.047	6.421	87.844	0.000
Canada	0.003	0.007	0.140	−0.222	0.051	−0.605	4.756	34.135	0.000
Denmark	0.006	0.010	0.141	−0.135	0.054	−0.111	2.928	0.405	0.817
Finland	0.009	0.000	0.328	−0.322	0.099	0.213	3.921	7.718	0.021
France	0.005	0.006	0.205	−0.158	0.058	−0.070	3.553	2.445	0.294
Germany	0.003	0.005	0.195	−0.245	0.064	−0.534	4.638	28.668	0.000
Greece	0.009	0.004	0.546	−0.230	0.111	1.624	8.576	312.316	0.000
Hong Kong	0.008	0.003	0.328	−0.293	0.083	0.303	4.760	25.972	0.000
Italy	0.002	0.000	0.214	−0.189	0.070	0.238	3.142	1.846	0.397
Japan	−0.005	−0.010	0.236	−0.201	0.070	0.387	3.516	6.491	0.039
Netherlands	0.006	0.008	0.129	−0.179	0.050	−0.825	4.656	40.996	0.000
New Zealand	0.000	0.000	0.269	−0.205	0.070	0.259	4.072	10.634	0.005
Norway	0.003	0.004	0.164	−0.282	0.067	−0.413	3.872	10.814	0.004
Portugal	−0.001	−0.002	0.277	−0.195	0.068	0.428	4.352	19.205	0.000
Spain	0.004	0.002	0.215	−0.221	0.065	−0.078	3.783	4.780	0.092
Sweden	0.007	0.009	0.224	−0.227	0.076	−0.204	3.454	2.791	0.248
UK	0.003	0.001	0.143	−0.105	0.047	0.214	3.098	1.440	0.487
US	0.006	0.009	0.110	−0.144	0.043	−0.428	3.382	6.579	0.037

Source: Morgan Stanley Capital International: MSCI Gross Index (Dividends Reinvested) in USD

Table 3: Local Market Dividend Yields: Summary Statistics (December 1987 – December 2002, $n = 181$ months)

Country	Mean	Maximum	Minimum	St. Dev.
Australia	3.865	6.800	2.769	0.915
Austria	1.829	2.752	0.968	0.469
Belgium	4.078	8.000	1.927	1.185
Canada	2.469	4.000	0.915	0.743
Denmark	1.615	2.670	0.884	0.371
Finland	1.883	4.361	0.462	0.810
France	2.858	4.300	1.402	0.664
Germany	2.821	4.900	1.632	0.726
Greece	4.137	12.320	1.300	2.234
Hong Kong	3.760	6.635	2.160	0.898
Italy	2.430	4.429	1.188	0.741
Japan	0.769	1.100	0.400	0.165
Netherlands	3.427	5.600	1.726	1.062
New Zealand	4.911	7.829	2.903	1.051
Norway	2.065	3.967	1.187	0.478
Portugal	2.641	4.930	0.860	0.869
Spain	3.330	6.300	1.447	1.263
Sweden	2.014	4.200	0.939	0.660
UK	3.818	6.000	2.096	1.046
US	2.426	4.000	1.109	0.897

Source: Morgan Stanley Capital International

Table 4: Global Macroeconomic Factors: Summary Statistics (December 1987 – December 2002, $n = 181$ months)

Factor	Mean	Maximum	Minimum	St. Dev.
Production	0.133	3.537	−3.215	0.811
Inflation	0.221	0.883	−0.223	0.186
US Index Return	0.070	6.218	−4.489	1.951
Premium	0.841	1.410	0.530	0.222
Spread	0.540	2.090	0.090	0.344
Term	1.711	3.760	−0.530	1.117
Oil	0.606	57.882	−21.740	8.800

The global factors are: the growth in industrial production for industrial countries ("production"), industrial country inflation ("inflation"), the return on the U.S. dollar index relative to major currencies ("US index return"), the difference in the yield on BAA and AAA bond indexes from Moody's ("premium"), the difference in the three-month Eurodollar yield and the three-month treasury bill yield ("spread"), the difference in the 10-year treasury bond yield and the three-month treasury bond yield ("term"), and the percentage change in the world spot price of oil ("oil"). The data for production, inflation, and oil are from the International Financial Corporation IFS Database. The data for the US index return, premium, spread, and term are from the FRED II database of the Federal Reserve Bank of St. Louis.

Table 5: Market and Black-Litterman Portfolio Allocations and Summary of Resulting Generated Returns

Country	Market Weights			Weight $\tau = 0.01$			Weight $\tau = 0.005$			Weight $\tau = 0.0025$			Weight $\tau = 0.001$		
	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max
Australia	1.4%	1.1%	1.9%	1.9%	−17.3%	17.4%	1.5%	−7.4%	6.1%	1.5%	−3.0%	3.8%	1.4%	−0.4%	2.4%
Austria	0.1%	0.1%	0.2%	−0.4%	−8.9%	19.3%	−0.1%	−2.8%	2.8%	0.0%	−1.4%	1.1%	0.1%	−0.5%	0.5%
Belgium	0.7%	0.5%	1.0%	0.2%	−50.3%	35.2%	−0.3%	−26.3%	4.4%	0.4%	−5.1%	2.6%	0.6%	−1.1%	1.6%
Canada	2.7%	2.2%	3.4%	4.7%	−12.6%	46.3%	3.6%	1.9%	11.3%	3.1%	2.3%	4.6%	2.8%	2.3%	3.5%
Denmark	0.4%	0.3%	0.5%	−0.1%	−54.1%	41.5%	−0.4%	−32.0%	4.9%	0.3%	−6.5%	2.7%	0.4%	−1.6%	1.3%
Finland	0.7%	0.4%	1.3%	−0.6%	−83.8%	26.3%	0.3%	−11.6%	3.8%	0.7%	−2.6%	2.5%	0.7%	−0.5%	1.7%
France	4.5%	3.4%	5.1%	5.5%	−3.7%	20.3%	4.9%	1.0%	8.0%	4.7%	3.0%	6.3%	4.5%	3.6%	5.3%
Germany	4.3%	3.3%	5.1%	1.7%	−67.9%	67.6%	3.4%	−24.8%	9.9%	4.1%	−2.9%	7.3%	4.2%	2.0%	5.7%
Greece	0.4%	0.2%	0.7%	−2.1%	−154.7%	17.5%	−0.4%	−15.5%	1.6%	0.2%	−5.4%	1.1%	0.3%	−1.6%	0.7%
Hong Kong	1.9%	1.1%	2.4%	0.8%	−29.1%	19.8%	2.0%	−6.2%	8.8%	1.9%	−1.8%	5.2%	1.9%	0.0%	3.1%
Italy	2.3%	1.8%	2.7%	−0.9%	−101.4%	34.0%	1.4%	−23.2%	9.2%	2.1%	−3.3%	5.9%	2.2%	0.3%	3.7%
Japan	11.3%	8.8%	14.4%	11.6%	−23.0%	96.6%	11.6%	6.4%	35.6%	11.2%	7.6%	16.6%	11.2%	8.7%	14.3%
Netherlands	2.2%	1.8%	2.7%	1.1%	−18.3%	12.9%	1.8%	−0.1%	5.0%	2.0%	1.1%	3.5%	2.1%	1.5%	2.9%
New Zealand	0.1%	0.1%	0.1%	−0.2%	−36.9%	28.5%	−0.2%	−8.8%	4.3%	0.0%	−3.4%	2.3%	0.1%	−1.1%	1.0%
Norway	0.2%	0.2%	0.3%	−2.0%	−55.6%	29.4%	−0.8%	−13.2%	4.4%	−0.1%	−5.0%	2.4%	0.1%	−1.6%	1.2%
Portugal	0.2%	0.2%	0.3%	−1.2%	−35.0%	12.1%	−0.5%	−9.4%	2.3%	−0.1%	−1.9%	1.3%	0.1%	−0.4%	0.7%
Spain	1.7%	1.3%	2.3%	0.9%	−26.1%	24.4%	1.1%	−15.8%	3.8%	1.6%	−1.8%	3.0%	1.7%	0.9%	2.5%
Sweden	1.1%	0.8%	1.5%	0.6%	−45.7%	35.8%	0.3%	−27.0%	4.4%	1.0%	−5.1%	2.7%	1.1%	−0.9%	1.7%
UK	9.3%	8.2%	10.8%	7.9%	−20.7%	37.8%	8.2%	−5.8%	11.8%	8.9%	6.2%	10.6%	9.2%	7.9%	10.4%
US	54.6%	50.7%	57.6%	70.5%	−167.2%	527.0%	62.7%	37.1%	203.9%	56.8%	43.1%	86.7%	55.3%	48.2%	64.0%
Min	0.1%	0.1%	0.1%	−2.1%	−167.2%	12.1%	−0.8%	−32.0%	1.6%	−0.1%	−6.5%	1.1%	0.1%	−1.6%	0.5%
Max	54.6%	50.7%	57.6%	70.5%	−3.7%	527.0%	62.7%	37.1%	203.9%	56.8%	43.1%	86.7%	55.3%	48.2%	64.0%

Portfolio Returns:	Market Weights	Weight $\tau = 0.01$	Weight $\tau = 0.005$	Weight $\tau = 0.0025$	Weight $\tau = 0.001$
Average	−0.4%	0.7%	0.6%	0.0%	−0.2%
St. dev.	5.1%	7.1%	4.9%	5.0%	5.1%
Min	−13.7%	−23.9%	−11.5%	−12.9%	−13.4%
Max	8.4%	11.0%	10.1%	9.4%	8.9%
Geometric returns	−0.5%	0.4%	0.5%	−0.1%	−0.4%

The top panel of the Table presents country allocations based on market weights and allocations produced from Black-Litterman optimization with confidence parameters of $\tau = 0.01$, 0.005, 0.025, and 0.001. The bottom panel of the Table provides summary statistics for the portfolio returns from the indicated allocations.

Table 6: Market, Markowitz, and Risk-Tailored Black-Litterman Portfolio Allocations and Summary of Resulting Generated Returns

Country	Market Weights			Markowitz Optimals			Risk-tailored τ		
	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max
Australia	1.4%	1.1%	1.9%	−7.0%	−33.8%	47.4%	1.8%	−0.7%	10.1%
Austria	0.1%	0.1%	0.2%	−18.2%	−96.1%	31.8%	−0.4%	0.1%	5.8%
Belgium	0.7%	0.5%	1.0%	93.0%	32.8%	181.2%	−0.1%	−5.1%	7.7%
Canada	2.7%	2.2%	3.4%	−27.4%	−76.8%	49.6%	3.9%	3.2%	6.4%
Denmark	0.4%	0.3%	0.5%	77.3%	53.5%	112.2%	0.4%	−6.5%	8.7%
Finland	0.7%	0.4%	1.3%	34.0%	11.5%	67.2%	0.6%	−6.6%	6.4%
France	4.5%	3.4%	5.1%	99.1%	38.8%	163.2%	5.0%	0.4%	11.0%
Germany	4.3%	3.3%	5.1%	58.5%	−9.7%	111.0%	3.8%	−3.9%	15.4%
Greece	0.4%	0.2%	0.7%	−5.2%	−24.0%	13.7%	0.0%	−5.5%	2.6%
Hong Kong	1.9%	1.1%	2.4%	9.9%	−3.1%	21.9%	2.1%	1.6%	19.8%
Italy	2.3%	1.8%	2.7%	−0.8%	−33.6%	3.1%	1.7%	−13.0%	15.2%
Japan	11.3%	8.8%	14.4%	14.7%	−9.0%	38.6%	10.5%	6.4%	18.6%
Netherlands	2.2%	1.8%	2.7%	−282.9%	−433.5%	−136.0%	1.5%	−1.2%	5.5%
New Zealand	0.1%	0.1%	0.1%	64.8%	17.5%	89.3%	−0.4%	−1.2%	7.7%
Norway	0.2%	0.2%	0.3%	−30.7%	−64.9%	6.0%	−1.0%	−5.5%	7.9%
Portugal	0.2%	0.2%	0.3%	10.8%	−15.5%	33.5%	−0.9%	−5.3%	4.3%
Spain	1.7%	1.3%	2.3%	−38.4%	−70.6%	1.3%	1.3%	−5.0%	5.4%
Sweden	1.1%	0.8%	1.5%	−42.1%	−74.1%	−17.6%	0.8%	−5.1%	6.3%
UK	9.3%	8.2%	10.8%	−20.4%	−140.1%	64.2%	8.2%	4.2%	15.1%
US	54.6%	50.7%	57.6%	118.2%	43.6%	177.9%	61.1%	46.2%	99.1%
Min	0.1%	0.1%	0.1%	−282.9%	−433.5%	−136.0%	−1.0%	−13.0%	2.6%
Max	54.6%	50.7%	57.6%	118.2%	53.5%	181.2%	61.1%	46.2%	99.1%
Portfolio Returns:									
Average		−0.4%			−0.1%			1.1%	
St. dev.		5.1%			12.3%			5.4%	
Min		−13.7%			−23.8%			−11.5%	
Max		8.4%			42.2%			11.0%	
Geometric returns		−0.5%			−0.8%			1.0%	

The top panel of the Table presents country allocations based on market weights, Markowitz optimal allocations, and allocations produced from Black-Litterman optimization with risk-tailored confidence parameters. For the risk-tailored allocations τ was selected to maintain portfolio risk below $v = 0.2\%$. The bottom panel of the Table provides summary statistics for the portfolio returns from the indicated allocations.

Table 7: Portfolio Risk

	Avg	Min	Max
Weight $\tau = 0.01$	0.329%	0.111%	6.840%
Weight $\tau = 0.005$	0.155%	0.110%	0.638%
Weight $\tau = 0.0025$	0.139%	0.111%	0.184%
Weight $\tau = 0.001$	0.138%	0.112%	0.174%
Risk-tailored	0.151%	0.111%	0.194%
Markowitz Optimals	0.709%	0.442%	1.240%

Summary statistics for estimated portfolio risks $v = w'\Omega w$.

Figure 1: Portfolio Risk (Estimate)

Time-series of portfolio risk estimates for confidence parameters of $\tau = 0.01, 0.005, 0.025$, and 0.001.