Exercises in Advanced Risk and Portfolio Management
with Step-by-Step Solutions and Fully Documented Code

Attilio Meucci[1]
attilio_meucci@symmys.com

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Exercises and case studies for a rigorous approach to risk- and portfolio-management. These exercises stem from the review sessions of the six-day Advanced Risk and Portfolio Management bootcamp http://www.baruch.edu/math/arpm.

Solutions code at http://www.mathworks.com/matlabcentral/fileexchange/25010.

Contents include

- Advanced multivariate statistics; copula-marginal decomposition
- Annualization/projection (FFT, cumulants, simulations)
- Pricing: exact; first order (delta/duration); second order (gamma/convexity)
- Quest for invariance (stationarity, volatlity clustering, cointegration)
- Mutlivariate estimation
  - Non-parametric; MLE; shrinkage; random matrices; robust; Bayesian; missing data
  - Generalized hypothesis testing
- Dimension reduction
  - Statistical (principal components; factor analysis)
  - Cross-sectional / time-series factor models
  - Factors on Demand
- Risk management
  - VaR/CVaR (marginal Euler decomposition; extreme value theory; Cornish-Fisher; elliptical)
  - Generalized objectives (p&l, return, relative return, etc)
  - Stochastic dominance/utility theory
- Classical portfolio management: mean-variance
- Dynamic strategies (option replication, CPPI, utlity maximization)
- Advanced portfolio management
  - Robust optimization
  - Black-Litterman and beyond, fully flexible views

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1 Distributions

1.1 General

1.1.1 Invertible transformation of a random variable

Consider as in (T1.1) in the Technical Appendices at `symmys.com` > `Book` > `Downloads` the following transformation of the generic random variable X:

$$X \mapsto Y \equiv g(X), \tag{1}$$

where g is an increasing and thus invertible function.

Prove the following formulas:

$$F_Y(y) = F_X\left(g^{-1}(y)\right) \tag{2}$$

$$f_Y(y) = \frac{f_X\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)}. \tag{3}$$

$$Q_Y(p) = g\left(Q_X(p)\right) \tag{4}$$

See Section 1.1 in the Technical Appendices. In particular, for the cdf, by the definition of the cumulative distribution function F_Y we have:

$$\begin{aligned} F_Y(y) &\equiv \mathbb{P}\{Y \leq y\} = \mathbb{P}\{g(X) \leq y\} \\ &= \mathbb{P}\left\{X \leq g^{-1}(y)\right\} \\ &= F_X\left(g^{-1}(y)\right). \end{aligned} \tag{5}$$

For the pdf, by derivation of both sides the above result we obtain:

$$f_Y(y) = \frac{f_X\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} \tag{6}$$

For the quantile, as in the Technical Appendices consider the following series of identities:

$$F_Y\left(g\left(Q_X(p)\right)\right) = \mathbb{P}\left\{Y \leq g\left(Q_X(p)\right)\right\} = \mathbb{P}\left\{X \leq Q_X(p)\right\} = p, \tag{7}$$

By applying the definition of the quantile Q_Y to the above terms we obtain:

$$Q_Y(p) = g\left(Q_X(p)\right), \tag{8}$$

1.1.2 Affine transformation of a random variable

Consider as in (T1.12) in the Technical Appendices the following positive affine transformation of the generic random variable X:

$$X \mapsto Y \equiv g(X) \equiv m + sX, \tag{9}$$

where $s > 0$ and m is a generic constant.

Prove the following formulas:

$$
\begin{aligned}
f_Y(y) &= \frac{1}{s} f_X\left(\frac{y-m}{s}\right) && (10)\\
F_Y(y) &= F_X\left(\frac{y-m}{s}\right) && (11)\\
Q_Y(p) &= m + sQ_X(p) && (12)\\
\phi_Y(\omega) &= e^{i\omega m}\phi_X(s\omega) && (13)
\end{aligned}
$$

See Section 1.2 in the Technical Appendices.

1.1.3 Sum of random variables: characteristic function

Consider the random variable defined in distribution as

$$X \stackrel{d}{\equiv} Y + Z, \tag{14}$$

where Y and Z are independent.

Compute the characteristic function ϕ_X of X from the characteristic functions ϕ_Y of Y and ϕ_Z of Z.

$$
\begin{aligned}
\phi_X(\omega) &\equiv \mathrm{E}\left\{e^{i\omega X}\right\} = \mathrm{E}\left\{e^{i\omega(Y+Z)}\right\} = \mathrm{E}\left\{e^{i\omega Y}e^{i\omega Z}\right\} && (15)\\
&= \mathrm{E}\left\{e^{i\omega Y}\right\}\mathrm{E}\left\{e^{i\omega Z}\right\} = \phi_Y(\omega)\,\phi_Z(\omega)
\end{aligned}
$$

1.1.4 Sum of random variables: simulations

Consider a Student t-distributed random variable

$$X \sim \mathrm{St}\left(\nu, \mu, \sigma^2\right), \tag{16}$$

where $\nu \equiv 8$, $\mu \equiv 0$ and $\sigma^2 \equiv 0.1$. Consider an independent lognormal random variable

$$Y \sim \mathrm{LogN}\left(\mu, \sigma^2\right), \tag{17}$$

where $\mu \equiv 0.1$ and $\sigma^2 \equiv 0.2$. Consider the random variable defined as

$$Z \equiv X + Y. \tag{18}$$

Generate the script `S_NonAnalytical` in which you perform the following operations.

Generate a large ($\approx 10,000$ observations) sample `X` from (16), a sample `Y` of equal size from (17), sum them term by term (do not use loops) and obtain a large sample `Z` from (18).

Plot the sample `Z`. Do not join the observations (use the plot option `'.'` as in a scatterplot).

Plot the histogram of `Z`. Use `hist` and choose the number of bins appropriately.

Plot the empirical cdf of `Z`. Use `[f,z]=ecdf(Z)` and `plot(z,f)`.

Plot the empirical quantile of `Z`. Use `prctile`.

See the script `S_NonAnalytical`.

1.1.5 Fourier transformation

Prove that the Fourier transformation $(B.34)$ in Meucci (2005) is a linear operator, i.e. it satisfies $(B.24)$.

From the definition $(B.34)$

$$\begin{aligned} \mathcal{F}[u+v](\mathbf{y}) &\equiv \int_{\mathbb{R}^N} e^{i\mathbf{y}'\mathbf{x}} (u(\mathbf{x}) + v(\mathbf{x}))\, d\mathbf{x} \\ &= \int_{\mathbb{R}^N} e^{i\mathbf{y}'\mathbf{x}} u(\mathbf{x})\, d\mathbf{x} + \int_{\mathbb{R}^N} e^{i\mathbf{y}'\mathbf{x}} v(\mathbf{x})\, d\mathbf{x} \\ &\equiv \mathcal{F}[u](\mathbf{y}) + \mathcal{F}[v](\mathbf{y}) \end{aligned} \tag{19}$$

Also, from the definition $(B.34)$

$$\begin{aligned} \mathcal{F}[\alpha v] &\equiv \int_{\mathbb{R}^N} e^{i\mathbf{y}'\mathbf{x}} (\alpha v(\mathbf{x}))\, d\mathbf{x} \\ &= \alpha \int_{\mathbb{R}^N} e^{i\mathbf{y}'\mathbf{x}} v(\mathbf{x})\, d\mathbf{x} \equiv \alpha \mathcal{F}[v] \end{aligned} \tag{20}$$

Therefore $(B.24)$ is satisfied.

1.1.6 Convolution

Prove that the convolution $(B.43)$ in Meucci (2005) of two probability density functions is a probability density function.

Consider probability density functions f and h, i.e. two functions that satisfy (2.5)-(2.6). Consider their convolution $g \equiv f * h$, which from the definition $(B.43)$ reads explicitly:

$$g(\mathbf{x}) \equiv \int_{\mathbb{R}^N} f(\mathbf{y})\, h(\mathbf{x} - \mathbf{y})\, d\mathbf{y}. \tag{21}$$

Property (2.5) is satisfied because f and h are non-negative. As for (2.6)

$$\begin{aligned}\int_{\mathbb{R}^N} g(\mathbf{x})\, d\mathbf{x} &\equiv \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} f(\mathbf{y})\, h(\mathbf{x}-\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \\ &= \int_{\mathbb{R}^N} f(\mathbf{y})\left(\int_{\mathbb{R}^N} h(\mathbf{x}-\mathbf{y})\, d\mathbf{x}\right) d\mathbf{y} \\ &= \int_{\mathbb{R}^N} f(\mathbf{y})\, d\mathbf{y} = 1,\end{aligned} \tag{22}$$

where in the second to last row we used a change of variables $\mathbf{x} \to \mathbf{z} \equiv \mathbf{x} - \mathbf{y}$.

1.1.7 Raw moments to central moments

Consider the raw moments (1.47)

$$\widetilde{\mu}_X^{(n)} \equiv \mathrm{E}\{X^n\}, \quad n = 1, 2, \ldots; \tag{23}$$

and the central moments (1.48)

$$\mu_X^{(1)} \equiv \mu_X \equiv \widetilde{\mu}_X^{(1)}; \qquad \mu_X^{(n)} \equiv \mathrm{E}\{(X - \mu_X)^n\}, \quad n = 2, 3, \ldots. \tag{24}$$

Create a function `Raw2Central` that maps the first n raw moments into the first n central moments.

See function `Raw2Central`.

For $n > 1$, from the definition of central moment (24) and the binomial expansion we obtain

$$\begin{aligned}\mu_X^{(n)} &\equiv \mathrm{E}\{(X - \mu_X)^n\} \\ &= \mathrm{E}\left\{\sum_{k=0}^{n-1} (-1)^{n-k} \mu_X^{n-k} X^k + X^n\right\} \\ &= \sum_{k=0}^{n-1} (-1)^{n-k} \mu_X^{n-k}\, \mathrm{E}\left\{X^k\right\} + \mathrm{E}\left\{X^n\right\} \\ &= \sum_{k=0}^{n-1} (-1)^{n-k} \mu_X^{n-k} \widetilde{\mu}_X^{(k)} + \widetilde{\mu}_X^{(n)}.\end{aligned} \tag{25}$$

1.1.8 Central moments to raw moments

Consider the raw moments (1.47) in Meucci (2005)

$$\widetilde{\mu}_X^{(n)} \equiv \mathrm{E}\{X^n\}, \quad n = 1, 2, \ldots; \tag{26}$$

and the central moments (1.48) in Meucci (2005)

$$\mu_X^{(1)} \equiv \mu_X \equiv \widetilde{\mu}_X^{(1)}; \qquad \mu_X^{(n)} \equiv \mathrm{E}\{(X-\mu_X)^n\}, \quad n=2,3,\ldots. \tag{27}$$

Create a function `Central2Raw` that maps the first n central moments into the first n raw moments.

See function `Central2Raw`.
From (25) we obtain

$$\widetilde{\mu}_X^{(n)} = \mu_X^{(n)} + \sum_{k=0}^{n-1} (-1)^{n-k+1} \mu_X^{n-k} \widetilde{\mu}_X^{(k)}.$$

This is a recursive formula that we initiate as $\widetilde{\mu}_X^{(1)} = \mu_X^{(1)}$, which follows from (27).

1.2 Parametric

1.2.1 Normal

Consider as in (1.66) in Meucci (2005) a normal random variable

$$X \sim \mathrm{N}\left(\mu, \sigma^2\right). \tag{28}$$

Generate the script `S_NormalSample` in which you perform the following operations. Compute μ and σ^2 such that $\mathrm{E}\{X\} \equiv 3$ and $\mathrm{Var}\{X\} \equiv 5$.

From (1.71) $\mathrm{E}\{X\} = \mu$ and from (1.72) $\mathrm{Var}\{X\} = \sigma^2$. Notice that the MATLAB built-in functions take μ and $\sqrt{\sigma^2}$ as inputs.

Generate a large ($\approx 10,000$ observations) sample `X` from this distribution using `normrnd`.

In Figure 1, plot the sample. Do not join the observations (use the plot option `'.'` as in a scatterplot).

In Figure 2, plot the histogram. Use `hist` and choose the number of bins appropriately.

In Figure 3, plot the empirical cdf. Use `[f,x]=ecdf(X)` and `plot(x,f)`.

Superimpose (use `hold on`) the exact cdf as computed by `normcdf`. Use a different color.

In Figure 4, plot the empirical quantile. Use `prctile`.

Superimpose (use `hold on`) the exact quantile as computed by `norminv`. Use a different color.

See the script `S_NormalSample`.

1.2.2 Normal moments

Compute the central moments

$$\mu_X^{(n)} \equiv \mathrm{E}\left\{(X - \mathrm{E}\{X\})^n\right\} \tag{29}$$

of a normal distribution

$$X \sim \mathrm{N}\left(\mu, \sigma^2\right). \tag{30}$$

For a generic random variable X, the moment generating function

$$M_X(z) \equiv \int e^{zx} f_X(x)\, dx \tag{31}$$

is such that $D^n M_X(0) = \widetilde{\mu}_X^{(n)}$, where D is the derivation operator and

$$\widetilde{\mu}_X^{(n)} \equiv \mathrm{E}\{X^n\} \tag{32}$$

is the non-central moment. This follows from explicitly applying D on both sides of (31). The moment generating function is the characteristic function $\phi_X(\omega)$ defined in (1.12) in Meucci (2005) evaluated at $\omega \equiv z/i$

$$M_X(z) = \phi_X(z/i). \tag{33}$$

First we focus on the non-central moments of the standard normal distribution

$$Y \sim \mathrm{N}(0, 1). \tag{34}$$

From (1.69) in Meucci (2005) and (33) we obtain $M_Y(z) \equiv e^{z^2/2}$. Computing the derivatives

$$\begin{aligned}
D^0 M_Y(z) &= e^{z^2/2} \\
D^1 M_Y(z) &= z e^{\frac{1}{2}z^2} \\
D^2 M_Y(z) &= e^{\frac{1}{2}z^2} + z^2 e^{\frac{1}{2}z^2} \\
D^3 M_Y(z) &= z^3 e^{\frac{1}{2}z^2} + 3z e^{\frac{1}{2}z^2} \\
D^4 M_Y(z) &= 3e^{\frac{1}{2}z^2} + 6z^2 e^{\frac{1}{2}z^2} + z^4 e^{\frac{1}{2}z^2} \\
D^5 M_Y(z) &= 10z^3 e^{\frac{1}{2}z^2} + z^5 e^{\frac{1}{2}z^2} + 15z e^{\frac{1}{2}z^2} \\
D^6 M_Y(z) &= 15e^{\frac{1}{2}z^2} + 45z^2 e^{\frac{1}{2}z^2} + 15z^4 e^{\frac{1}{2}z^2} + z^6 e^{\frac{1}{2}z^2} \\
&\vdots
\end{aligned} \tag{35}$$

and evaluating in zero yields the result

$$\widetilde{\mu}_Y^{(n)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)!! & \text{if } n \text{ is even,} \end{cases} \tag{36}$$

where $n!! \equiv 1 \times 3 \times 5 \times \cdots \times n$.

Now we notice that

$$\mu_X^{(n)} = \widetilde{\mu}_{X-\mu}^{(n)} \tag{37}$$

which follows from (29), (32) and (1.71) in Meucci (2005). Furthermore

$$\widetilde{\mu}_{X-\mu}^{(n)} = \sigma^n \widetilde{\mu}_Y^{(n)} \tag{38}$$

because $X - \mu \stackrel{d}{=} \sigma Y$, as follows form (2.163) in Meucci (2005) applied to the univariate case. Therefore

$$\mu_X^{(n)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sigma^n (n-1)!! & \text{if } n \text{ is even} \end{cases} \tag{39}$$

1.2.3 Normal multivariate with matching moments

This exercise is discussed in greater depth and placed into a broader context in Meucci (2009c), freely available online at ssrn.com, which we follow below.

Consider as in (2.155) in Meucci (2005) a mutlivariate normal market

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{40}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are arbitrary.

Generate a large number of scenarios $\{\mathbf{x}_j\}_{j=1,\ldots,J}$ from the distribution (40) in such a way that the sample mean and covariance

$$\widehat{\boldsymbol{\mu}} \equiv \frac{1}{J}\sum_{j=1}^{J} \mathbf{x}_j, \quad \widehat{\boldsymbol{\Sigma}} \equiv \frac{1}{J}\sum_{j=1}^{J} (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})' \tag{41}$$

satisfy

$$\widehat{\boldsymbol{\mu}} \equiv \boldsymbol{\mu} \quad \widehat{\boldsymbol{\Sigma}} \equiv \boldsymbol{\Sigma}. \tag{42}$$

Hint. If you need to solve a Riccati equation

$$\boldsymbol{\Sigma} \equiv \mathbf{B}\widehat{\boldsymbol{\Sigma}}\mathbf{B}, \quad \mathbf{B} \equiv \mathbf{B}'. \tag{43}$$

you can follow Petkov, Christov, and Konstantinov (1991). First define the Hamiltonian matrix

$$\mathbf{H} \equiv \begin{pmatrix} \mathbf{0} & -\widehat{\mathbf{S}}_{\widetilde{\mathbf{y}}} \\ -\mathbf{S} & \mathbf{0} \end{pmatrix}. \tag{44}$$

Next perform its Schur decomposition

$$\mathbf{H} \equiv \mathbf{U}\mathbf{T}\mathbf{U}', \tag{45}$$

where $\mathbf{U}\mathbf{U}' \equiv \mathbf{I}$ and $\mathbf{T}$ is upper triangular with the eigenvalues of $\mathbf{H}$ on the diagonal sorted in such a way that the first N have negative real part and the remaining N have positive real part; the terms in this decomposition are similar in nature to principal components and are computed by MATLAB. Then the solution of the Riccati equation (43) reads

$$\mathbf{B} \equiv \mathbf{U}_{LL}\mathbf{U}_{UL}^{-1}, \tag{46}$$

where $\mathbf{U}_{UL}$ is the upper left $N \times N$ block of $\mathbf{U}$ and $\mathbf{U}_{LL}$ is the lower left $N \times N$ block of $\mathbf{U}$.

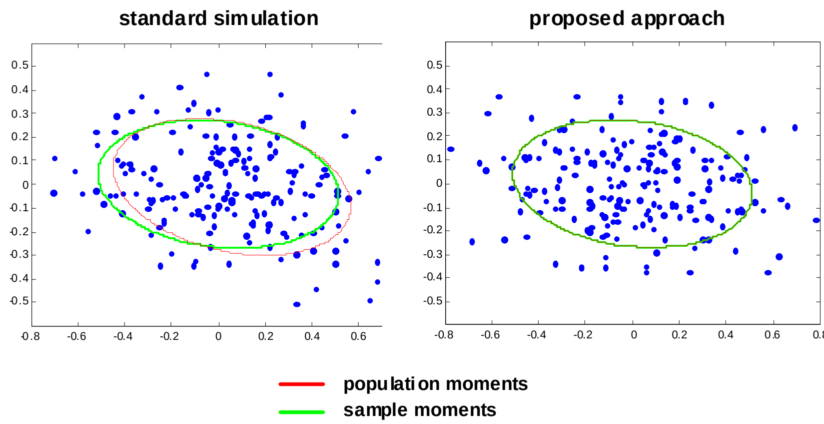

Figure 1: Sample and population moments coincide our approach

See the function MvnRnd, which takes the same inputs and yields the same output as the built-in MATLAB function mvnrnd.

First produce an auxiliary set of scenarios

$$\{\widetilde{\mathbf{y}}_j\}_{j=1,\ldots,\frac{J}{2}} \tag{47}$$

from the distribution $\mathrm{N}(\mathbf{0}, \mathbf{\Sigma})$. Then complement these scenarios with their opposite

$$\widetilde{\mathbf{y}}_j \equiv \begin{cases} \widetilde{\mathbf{y}}_j & \text{if } 1 \leq j \leq J/2 \\ -\widetilde{\mathbf{y}}_{j-\frac{J}{2}} & \text{if } J/2+1 \leq j \leq J. \end{cases} \tag{48}$$

These antithetic variables still represent the distribution $\mathrm{N}(\mathbf{0}, \mathbf{\Sigma})$, but they are more efficient they satisfy the zero-mean condition.

Next apply a linear transformation to the scenarios $\widetilde{\mathbf{y}}_j$, which again preserves normality:

$$\mathbf{y}_j \equiv \mathbf{B}\widetilde{\mathbf{y}}_j, \quad j = 1, \ldots, J. \tag{49}$$

For any choice of the invertible matrix $\mathbf{B}$, the sample mean is null. To determine $\mathbf{B}$ we impose that the sample covariance matches the desired covariance. Using the affine equivariance of the sample covariance which follows from (4.42), (4.36), (2.67) and (2.64) in Meucci (2005), we obtain the matrix Riccati equation (43).

With the solution (46) we can perform the affine transformation (49) and finally generate the desired scenarios

$$\mathbf{x}_j \equiv \boldsymbol{\mu} + \mathbf{y}_j, \quad j = 1, \ldots, J, \tag{50}$$

which satisfy (42), see Figure 1, where as in Meucci (2005) we represent the first two moments of a distribution in terms of an ellipsoid.

1.2.4 Student t

Generate the script `S_StudentTSample` in which you perform the following operations.

Consider a Student t random variable

$$X \sim \mathrm{St}\left(\nu, \mu, \sigma^2\right). \tag{51}$$

Knowing that $\sigma^2 \equiv 6$, compute ν and μ such that $\mathrm{E}\{X\} \equiv 2$ and $\mathrm{Var}\{X\} \equiv 7$. (Note: there is a typo in (1.90) in Meucci (2005). Check the "Errata" at `symmys.com` > `Book` > `Downloads`).

Generate a sample `X_a` of $T \equiv 10,000$ observations from (51) using the built-in t number generator.

Generate a sample `X_b` of $T \equiv 10,000$ observations from (51) using the normal number generator, the chi-square number generator and the following result:

$$X \stackrel{d}{=} \mu + \frac{Y}{\sqrt{Z/\nu}}, \tag{52}$$

where Y and Z are independent variables distributed as follows:

$$Y \sim \mathrm{N}\left(0, \sigma^2\right), \quad Z \sim \chi^2_\nu. \tag{53}$$

Generate a sample `X_c` of $T \equiv 10,000$ observations from (51) using the uniform generator number, `tinv` and (2.27) in Meucci (2005).

In a separate figure, `subplot` the histogram of the simulations of `X_a`, `subplot` the histogram of the simulations of `X_b` and `subplot` the histogram of the simulations of `X_c`.

Compute the empirical quantile functions of the three simulations corresponding to the confidence grid

$$\mathcal{G} \equiv \{0.01, 0.02, \ldots, 0.99\} \tag{54}$$

In a separate figure superimpose the plots of the above empirical quantiles, which should coincide. Use different colors.

See the script `S_StudentTSample`.

1.2.5 Lognormal

Consider as in (1.94) in Meucci (2005) a lognormal random variable

$$X \sim \mathrm{LogN}\left(\mu, \sigma^2\right). \tag{55}$$

Generate the script `S_LognormalSample` in which you perform the following operations.

Compute μ and σ^2 such that $\mathrm{E}\{X\} \equiv 3$ and $\mathrm{Var}\{X\} \equiv 5$.

From (1.98) and (1.99) we need to solve for μ and σ^2 the following system:

$$E = e^{\mu+\frac{\sigma^2}{2}} \tag{56}$$

$$V = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right) \tag{57}$$

or

$$2\ln(E) = 2\mu+\sigma^2 \tag{58}$$

$$\ln(V) = 2\mu+\sigma^2+\ln\left(e^{\sigma^2}-1\right) \tag{59}$$

Therefore

$$\ln\left(\frac{V}{E^2}\right) = \ln\left(e^{\sigma^2}-1\right) \tag{60}$$

or

$$\sigma^2 = \ln\left(1+\frac{V}{E^2}\right). \tag{61}$$

From (58) we then obtain:

$$\mu = \ln(E) - \frac{1}{2}\ln\left(1+\frac{V}{E^2}\right). \tag{62}$$

Notice that the MATLAB built-in functions take μ and $\sqrt{\sigma^2}$ as inputs.

Generate a large ($\approx 10,000$ observations) sample `X` from this distribution using `lognrnd`.

In Figure 1, plot the sample. Do not join the observations (use the plot option `'.'` as in a scatterplot).

In Figure 2, plot the histogram. Use `hist` and choose the number of bins appropriately.

In Figure 3, plot the empirical cdf. Use `[f,x]=ecdf(X)` and `plot(x,f)`.

Superimpose (use `hold on`) the exact cdf as computed by `logncdf`. Use a different color.

In Figure 4, plot the empirical quantile. Use `prctile`.

Superimpose (use `hold on`) the exact quantile as computed by `logninv`. Use a different color.

See the script S_LognormalSample.

1.2.6 Lognormal moments

Consider as in (1.94) in Meucci (2005) a lognormal random variable

$$X \sim \text{LogN}\left(\mu, \sigma^2\right). \tag{63}$$

Compute the raw moments

$$\mu_n \equiv \text{E}\left\{X^n\right\} \tag{64}$$

for all $n = 1, 2, \ldots$

From (1.94) in Meucci (2005)

$$X^n \stackrel{d}{=} e^{nY}, \tag{65}$$

where

$$Y \sim \text{N}\left(\mu, \sigma^2\right). \tag{66}$$

From (2.163) in Meucci (2005)

$$nY \sim \text{N}\left(n\mu, n^2\sigma^2\right) \tag{67}$$

Therefore

$$X^n \sim \text{LogN}\left(n\mu, n^2\sigma^2\right) \tag{68}$$

and the moments follow from (1.98) in Meucci (2005)

$$\mu_n = e^{n\mu + n^2\sigma^2/2} \tag{69}$$

1.2.7 Gamma versus chi-square

Consider as in (1.107) in Meucci (2005) a gamma-distributed random variable

$$X \sim \text{Ga}\left(\nu, \mu, \sigma^2\right). \tag{70}$$

We recall that such variable is defined in distribution as follows

$$X \stackrel{d}{\equiv} Y_1^2 + \cdots + Y_\nu^2, \tag{71}$$

where

$$Y_1 \stackrel{d}{\equiv} \cdots \stackrel{d}{\equiv} Y_\nu \sim \text{N}\left(\mu, \sigma^2\right) \tag{72}$$

are independent. For which values of ν, μ and σ^2 does this distribution coincide with the chi-square distribution with ten degrees of freedom?

For $\nu \equiv 10$, $\mu \equiv 0$ and $\sigma^2 \equiv 1$ we obtain $X \sim \chi^2_{10}$, see (1.109) in the textbook.

1.2.8 Wishart simulations

Consider the case $N \equiv 2$ of a Wishart distribution

$$\mathbf{W} \sim \mathrm{W}(\nu, \mathbf{\Sigma}), \tag{73}$$

where

$$\mathbf{\Sigma} \equiv \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{74}$$

Fix $\sigma_1 \equiv 1$ and $\sigma_2 \equiv 1$. Generate the script `S_Wishart`where you perform the following operations.

Set your inputs ρ and ν.

Generate a sample of size $J \equiv 10,000$ from $\mathrm{W}(\nu, \mathbf{\Sigma})$ using the equivalent stochastic representation (2.222) in Meucci (2005).

Positivity implies that $D \equiv \det(\mathbf{W})$ and $T \equiv \mathrm{tr}(\mathbf{W})$ are positive random variables, see (2.236), (2.237) in Meucci (2005). Plot the histograms of the realizations of D and T, which are approximations of the respective pdf's, to show that indeed these random variables are positive. Comment on whether this is also true for $\nu \equiv 1$.

Symmetry implies that a matrix is fully determined by the three non-redundant entries (W_{11}, W_{22}, W_{12}). Plot the 3-d scatter-plot of the realizations of W_{11} vs. W_{12} vs. W_{22} to show the Wishart cloud. Notice that as the degrees of freedom ν increases the clouds becomes less and less "wedgy". Eventually, it becomes a normal ellipsoid, in accordance with the central limit theorem.

Plot the separate histograms of the realizations of W_{11}, W_{12} and W_{22}.

From (2.230) in Meucci (2005) the marginal distributions of the diagonal elements of a Wishart matrix are gamma-distributed:

$$W_{nn} \sim \mathrm{Ga}(\nu, \Sigma_{nn}). \tag{75}$$

Superimpose the rescaled pdf (1.110) of the marginals of W_{11} and W_{22} to the respective histograms to show that histogram and gamma pdf coincide, see (T1.43) in the technical appendices at `www.symmys.com` > `Book` > `Downloads`

Compute and show on the command window the sample means, sample covariances, sample standard deviations and sample correlations.

Compute and show on the command window the respective analytical results (2.227) and (2.228) in Meucci (2005), making sure that they coincide.

See the script `S_Wishart`.

1.3 Special classes

1.3.1 Empirical

Derive expression (2.243) in Meucci (2005) for the characteristic function of the empirical distribution.

$$
\begin{aligned}
\phi_{i_T}(\boldsymbol{\omega}) &\equiv \mathrm{E}\left\{e^{i\boldsymbol{\omega}'\mathbf{X}}\right\} = \int_{\mathbb{R}^N} e^{i\boldsymbol{\omega}'\mathbf{x}} f_{i_T}(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\mathbb{R}^N} e^{i\boldsymbol{\omega}'\mathbf{x}} \left[\frac{1}{T}\sum_{t=1}^{T} \delta^{(\mathbf{x}_t)}(\mathbf{x})\right] d\mathbf{x} \\
&= \frac{1}{T}\sum_{t=1}^{T} \int_{\mathbb{R}^N} e^{i\boldsymbol{\omega}'\mathbf{x}} \delta^{(\mathbf{x}_t)}(\mathbf{x})\, d\mathbf{x}.
\end{aligned}
\tag{76}
$$

Using $(B.17)$ we obtain:

$$
\phi_{i_T}(\boldsymbol{\omega}) = \frac{1}{T}\sum_{t=1}^{T} e^{i\boldsymbol{\omega}'\mathbf{x}_t}. \tag{77}
$$

1.3.2 Order statistics

Replicate the exercise `S_OrderStatisticsPDFStudentT`assuming that the i.i.d. variables are lognormal instead of t-distributed.

Note. The figure generated by the script is 3-d: make sure to rotate the figure in order to appreciate the third dimension as in Figure 2.19 in Meucci (2005).

See the script `S_OrderStatisticsPDFLogn`.

1.3.3 Elliptical variables: radial-uniform representation

Generate a non-trivial 30×30 symmetric and positive matrix $\boldsymbol{\Sigma}$ and a 30-dim vector $\boldsymbol{\mu}$.

Generate $J \equiv 10,000$ simulations from a 30-dimensional elliptical random variable:

$$
\mathbf{X} \equiv \boldsymbol{\mu} + R\mathbf{A}\mathbf{U}. \tag{78}
$$

In this expression $\boldsymbol{\mu}$, R, $\mathbf{A}$, $\mathbf{U}$ are the terms of the radial-uniform decomposition, see (2.259) in Meucci (2005). In particular, set

$$
R \sim \mathrm{LogN}\left(\nu, \tau^2\right), \tag{79}
$$

where $\nu \equiv 0.1$ and $\tau^2 \equiv 0.04$.

See the script `S_EllipticalNDim`.

1.3.4 Elliptical markets and portfolios

Consider an elliptical market

$$\mathbf{M} \sim \mathrm{El}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_N\right), \tag{80}$$

Consider the objective $\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M}$, where $\boldsymbol{\alpha}$ is a generic vector of exposures. Prove that

$$\Psi_{\boldsymbol{\alpha}} \stackrel{d}{=} \mu_{\boldsymbol{\alpha}} + \sigma_{\boldsymbol{\alpha}} Z, \tag{81}$$

where

$$Z \sim \mathrm{El}\left(0, 1, g_1\right). \tag{82}$$

Write the expression for $\mu_{\boldsymbol{\alpha}}$ and $\sigma_{\boldsymbol{\alpha}}$.

From (2.270) in Meucci (2005) we obtain

$$\Psi_{\boldsymbol{\alpha}} \sim \mathrm{El}\left(\mu_{\boldsymbol{\alpha}}, \sigma^2_{\boldsymbol{\alpha}}, g_1\right), \tag{83}$$

where

$$\mu_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\boldsymbol{\mu} \tag{84}$$

$$\sigma^2_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} \tag{85}$$

From (2.258) in Meucci (2005)

$$\Psi_{\boldsymbol{\alpha}} \stackrel{d}{=} \mu_{\boldsymbol{\alpha}} + \sigma_{\boldsymbol{\alpha}} Z, \tag{86}$$

where Z is spherically symmetrical and therefore

$$Z \sim \mathrm{El}\left(0, 1, g_1\right). \tag{87}$$

2 Dependence

2.1 Correlation

2.1.1 Normal

Consider a bivariate normal random variable:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right). \tag{88}$$

Fix $\mu_1 \equiv 0$, $\mu_2 \equiv 0$, $\sigma_1 \equiv 1$, $\sigma_2 \equiv 1$ and generate a script that:
plots the correlation between X_1 and X_2 as a function of $\rho \in (-1, 1)$;

uses eig. to plot as a function of $\rho \in (-1,1)$ the condition ratio of $\mathbf{S}$, i.e. the ratio of the smallest eigenvalue of $\mathbf{S}$ over its largest eigenvalue:

$$\mathrm{CR}(\mathbf{S}) \equiv \lambda_2/\lambda_1 \tag{89}$$

Hint. See (4.115) in Meucci (2005). Remember the relationship between eigenvalues and ellipsoid

See the script `S_AnalyzeNormalCorrelation`.

2.1.2 Lognormal

Consider an N-variate lognormal random variable:

$$\mathbf{X} \sim \mathrm{LogN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) . \tag{90}$$

Generate a function `LogNormalParam2Statistics` that computes $\mathbf{m} \equiv \mathrm{E}\{\mathbf{X}\}$, $\mathbf{S} \equiv \mathrm{Cov}\{\mathbf{X}\}$ and $\mathbf{C} \equiv \mathrm{Cor}\{\mathbf{X}\}$ as functions of the generic inputs $\boldsymbol{\mu}, \boldsymbol{\Sigma}$.

Now focus on the bivariate case:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathrm{LogN}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right) . \tag{91}$$

Fix $\mu_1 \equiv 0$, $\mu_2 \equiv 0$, $\sigma_1 \equiv 1$, $\sigma_2 \equiv 1$ and use the above function to generate a script that:

uses `LogNormalParam2Statistics` to plot the correlation between X_1 and X_2 as a function of $\rho \in (-1,1)$ (notice that the correlation will not approach -1,why?);

uses eig. to plot as a function of $\rho \in (-1,1)$ the condition ratio of $\mathbf{S}$, i.e. the ratio of the smallest eigenvalue of $\mathbf{S}$ over its largest eigenvalue:

$$\mathrm{CR}(\mathbf{S}) \equiv \lambda_2/\lambda_1 \tag{92}$$

Hint. See (4.115) in Meucci (2005). Remember the relationship between eigenvalues and ellipsoid.

See the script `S_AnalyzeLognormalCorrelation`.

2.1.3 Independence versus no correlation

Consider a random vector $\mathbf{X} \equiv (X_1 \ldots, X_T)'$ that is t distributed

$$\mathbf{X} \sim \mathrm{St}(\nu, \mathbf{0}_T, \mathbf{I}_T) , \tag{93}$$

where $\mathbf{0}_T$ is a T-dimensional vector of zeros and $\mathbf{I}_T$ is the $T \times T$ identity matrix.

Compute the distribution of the marginals X_t, $t = 1, \ldots, T$.

Compute the correlation among each pair of entries.

Compute the distribution of

$$Y \equiv \sum_{t=1}^{T} X_t. \tag{94}$$

Now consider T i.i.d. t-distributed random variables:

$$\widetilde{X}_t \sim \mathrm{St}\left(\nu, 0, 1\right), \quad t = 1, \ldots, T \tag{95}$$

Compute the correlation among each pair of entries.
Consider the limit $T \to \infty$ and compute the distribution of

$$\widetilde{Y} \equiv \sum_{t=1}^{T} \widetilde{X}_t. \tag{96}$$

Comment on the difference between the distribution of Y versus the distribution of $\widetilde{Y}$.

From (2.194) in Meucci (2005) the marginals read:

$$X_t \sim \mathrm{St}\left(\nu, 0, 1\right), \quad t = 1, \ldots, T. \tag{97}$$

From (2.191) the cross-correlations read:

$$\mathrm{Cor}\left\{X_t, X_s\right\} = 0, \quad t \neq s. \tag{98}$$

From (2.195) we obtain

$$Y \equiv \sum_{t=1}^{T} X_t = \mathbf{1}'\mathbf{X} \sim \mathrm{St}\left(\nu, \mathbf{1}'\mathbf{0}_T, \mathbf{1}'\mathbf{I}_T\mathbf{1}\right). \tag{99}$$

Therefore

$$Y \sim \mathrm{St}\left(\nu, 0, T\right). \tag{100}$$

As far as $\widetilde{\mathbf{X}}$ is concerned, from (2.136) the cross-correlations read:

$$\mathrm{Cor}\left\{\widetilde{X}_t, \widetilde{X}_s\right\} = 0, \quad t \neq s. \tag{101}$$

From the central limit theorem and (1.90) in Meucci (2005) (fix the typo with the online "Errata" at `symmys.com` > `Book` > `Downloads`) we obtain:

$$\widetilde{Y} \to \mathrm{N}\left(0, \frac{\nu}{\nu - 2} T\right). \tag{102}$$

Both Y and $\widetilde{Y}$ are the sum of uncorrelated identically distributed t variables.

If the variables are independent, the CLT kicks in and the sum becomes normal. Note: this only holds for $\nu > 2$, otherwise the variance is not defined and the CLT does not hold. Indeed, if $\nu = 1$ we obtain the Cauchy distribution, which is stable: the sum of i.i.d. Cauchy variables is Cauchy. If the variables are jointly t, they cannot be independent, even if they are uncorrelated, recall the plot of the pdf of the Student t copula.

2.1.4 Correlation and location-dispersion ellipsoid

Consider the first diagonal entry and the off-diagonal entry (W_{11}, W_{12}) in Exercise 1.2.8. Define the following two rescaled variables:

$$
\begin{aligned}
X_1 &\equiv \frac{W_{11} - \mathrm{E}\{W_{11}\}}{\mathrm{Sd}\{W_{11}\}} && (103)\\
X_2 &\equiv \frac{W_{12} - \mathrm{E}\{W_{12}\}}{\mathrm{Sd}\{W_{12}\}} && (104)
\end{aligned}
$$

Simulate and scatter-plot a large number of joint samples of $\mathbf{X}$.

Superimpose to the above scatter-plot the plot of the location-dispersion ellipsoid of this variables. In order to do so, feed the function `TwoDimEllipsoid` with the real inputs $\mathrm{E}\{\mathbf{X}\}$ and $\mathrm{Cov}\{\mathbf{X}\}$ as they follow from the analytical results (2.227) and (2.228) in Meucci (2005), do not use the sample estimates from the simulations. Make sure that the ellipsoid suitably fits the simulation cloud.

Fix $\nu \equiv 15$ and $\sigma_1 \equiv \sigma_2 \equiv 1$ in (74). Plot the correlation $\mathrm{Cor}\{X_1, X_2\}$ as a function of $\rho \in (-1, +1)$. (Compare with the result of the previous point, which is a geometrical representation of the correlation).

See the scripts `S_WishartLocationDispersion`and `S_WishartCorrelation`.

Notice that the correlation can be computed analytically:

$$
\begin{aligned}
\mathrm{Cor}\{X_1, X_2\} &= \mathrm{Cov}\{X_1, X_2\} && (105)\\
&= \frac{1}{\sqrt{\mathrm{Var}\{W_{11}\}}\sqrt{\mathrm{Var}\{W_{12}\}}}\mathrm{Cov}\{W_{11}, W_{12}\}\\
&= \frac{1}{\sqrt{\nu 2\Sigma_{11}^2}\sqrt{\nu(\Sigma_{11}\Sigma_{22} + \Sigma_{12}\Sigma_{21})}}\nu(\Sigma_{11}\Sigma_{12} + \Sigma_{12}\Sigma_{11})\\
&= \frac{\sqrt{2}\rho}{\sqrt{1+\rho^2}}
\end{aligned}
$$

2.2 Copula

2.2.1 Normal copula pdf

Consider a generic N-variate normal random variable:

$$
\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \qquad (106)
$$

Generate the function `NormalCopulaPDF` which takes as input a generic value $\mathbf{u}$ in the N-dimensional unit hypercube as well as the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and outputs the pdf of the copula of $\mathbf{X}$ in $\mathbf{u}$.

Hint. See (2.30) in Meucci (2005). To generate a function use the following header:

`pdfu=NormalCopulaPDF(u,Mu,Sigma)`.
Then save the script as "`NormalCopulaPDF`". Since $\boldsymbol{\mu}$ is a generic $N \times 1$ vector and $\boldsymbol{\Sigma}$ is a generic symmetric and positive $N \times N$ matrix, you need the multivariate normal distribution functions. Use `mvnpdf` (see (2.156) in Meucci (2005)), `norminv`, and (2.30) in Meucci (2005).

Generate a script `S_DisplayNormalCopulaPDF` where you call the above function to evaluate the copula pdf at a select grid of bivariate values:

$$\mathbf{u} \in \mathcal{G} \equiv [0.05 : 0.05 : 0.95] \times [0.05 : 0.05 : 0.95] . \tag{107}$$

Pick $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ of your choice.

Hint. Calculate the pdf value on each grid point, which gives you a 19x19 matrix.

In a separate figure, plot the ensuing surface using `surf`.

See the script `S_DisplayNormalCopulaPDF`.

2.2.2 Normal copula cdf

Consider a bi-variate normal random variable:

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{108}$$

where

$$\boldsymbol{\mu} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\Sigma} \equiv \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}. \tag{109}$$

Pick ρ as you please, but make sure to play around with the values $\rho \equiv 0.99$, $\rho \equiv -0.99$ and $\rho \equiv 0$.

Generate a script `S_DisplayNormalCopulaCDF`where you evaluate the copula cdf at a select grid of bivariate values:

$$\mathbf{u} \in \mathcal{G} \equiv [0.05 : 0.05 : 0.95] \times [0.05 : 0.05 : 0.95] . \tag{110}$$

Do not call functions from within the script.

Hint. Calculate the cdf value on each grid point, which gives you a 19x19 matrix. Use (2.31) in Meucci (2005) and the built-in function `mvncdf`.

In a separate figure, plot the ensuing surface using `surf`.

See the script `S_DisplayNormalCopulaCDF`.

2.2.3 Lognormal copula pdf

Consider a generic N-variate lognormal random variable:

$$\mathbf{X} \sim \mathrm{LogN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{111}$$

Generate a function which takes as input a generic value $\mathbf{u}$ in the N-dimensional unit hypercube as well as the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and outputs the pdf of the copula of $\mathbf{X}$ in $\mathbf{u}$.

Generate a script where you call the above function to evaluate the copula pdf at a select grid of bivariate values:

$$\mathbf{u} \in \mathcal{G} \equiv [0.05 : 0.05 : 0.95] \times [0.05 : 0.05 : 0.95] . \tag{112}$$

In a separate figure, plot the ensuing surface.

Hint. Use (2.38) and (2.196) in Meucci (2005).

See the script `S_DisplayNormalCopulaPDF`.

2.2.4 Normal copula and given marginals

Generate the script `S_BivariateSample` in which you perform the following operations.

Generate a bivariate sample `X`, i.e. a $(T \equiv 10,000) \times (N \equiv 2)$matrix of joint observations, from a bivariate random variable $\mathbf{X}$whose distribution is defined as follows: the copula is the copula of a normal distribution with correlation $r \equiv -0.8$and the marginals are distributed as follows:

$$X_1 \sim \mathrm{Ga}\left(\nu_1, \sigma_1^2\right) \tag{113}$$

$$X_2 \sim \mathrm{LogN}\left(\mu_2, \sigma_2^2\right), \tag{114}$$

where $\nu_1 \equiv 9$, $\sigma_1^2 \equiv 2$, $\mu_2 \equiv 0$ and $\sigma_2^2 \equiv 0.04$.

Hints. You are asked to generate a bivariate sample, which has a marginal gamma distribution and a lognormal distribution but with a copula which is the same as the copula from a bivariate normal distribution. You will notice that the correlation of this normal distribution is r, but no other information is provided on the expected values or the standard deviations. Why? See (2.38) in Meucci (2005). Therefore, first generate a bivariate normal distribution sample with correlation r; then calculate its copula using (2.28) in Meucci (2005); finally remap it to the bivariate distribution you want using (2.34) in Meucci (2005).

In a separate figure, `subplot` the histogram of the simulations for X_1 and `subplot` the histogram of the simulations of X_2.

Comment on how these histograms, which represent the marginal pdf's of X_1 and X_2, change as the correlation r of the normal distribution varies.

In a separate figure, scatter-plot the simulations of X_1 against the respective simulations of X_2.

To visualize the joint pdf of X_1 and X_2, in a separate figure, plot the respective 3D-histogram.

In a separate figure, `subplot` the histogram of the grade of X_1 and `subplot` the histogram of the grade of X_2.

In a separate figure, scatter-plot the simulations of the grade of X_1 against the respective simulations of the grade of X_2.

To visualize the joint pdf of the grades of X_1 and X_2, in a separate figure use "hist3" to plot the respective 3D-histogram.

See the script S_BivariateSample.

2.2.5 FX copula-marginal factorization

Generate the script S_FXCopulaMarginalin which you perform the following operations.

Load from db_FX the daily observations of the foreign exchange rates USD/EUR, USD/GBP and USD/JPY. Define as variables the daily log-changes of the rates.

Represent the marginal distribution of the three variables in simulation and display the respective histograms

Represent the copula of the three variables in simulation and display the scatter-plot of the copula of all pairs of variables.

Hint. Applying the marginal cdf to the simulations of a random variable is equivalent to sorting

See the script S_FXCopulaMarginal.

2.2.6 Copula vs correlation

Consider a generic N-variate t random variable:

$$\mathbf{X} \sim \mathrm{St}\left(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{115}$$

Generate the function TCopulaPDF which takes as input a generic value $\mathbf{u}$ in the N-dimensional unit hypercube as well as the parameters ν, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and outputs the pdf of the copula of $\mathbf{X}$ in $\mathbf{u}$.

Hint. Use (2.30) and (2.188) in Meucci (2005) and the built-in functions tpdf and tinv. Notice that you will have to re-scale the built-in pdf and the built-in quantile of the standard t distribution.

Generate a script where you call the above function to evaluate the copula pdf at a select grid of bivariate values:

$$\mathbf{u} \in \mathcal{G} \equiv [0.05 : 0.05 : 0.95] \times [0.05 : 0.05 : 0.95]. \tag{116}$$

In a separate figure, plot the ensuing surface.

Comment on the (dis)similarities with the normal copula when $\nu \equiv 200$.

Comment on the (dis)similarities with the normal copula when $\nu \equiv 1$ and $\Sigma_{12} \equiv 0$. What is the correlation in this case?

See the script S_DisplayCopulaPDF.

From (2.191) in Meucci (2005), when the off-diagonal entries are null, the marginals are uncorrelated, if the correlation is defined, which is true only for

$\nu > 2$ (see fix in the Errata). Therefore, for $\nu > 2$ null correlation does not imply independence, because the pdf is clearly not flat as $\nu \to 2$. For $\nu \leq 2$ the correlation simply does not exist. However the co-scatter parameter Σ_{12} can be set to zero, but this does not imply independence because, again, the pdf is far from flat as $\nu \leq 2$.

2.2.7 Full codependence

Generate $J \equiv 10,000$ joint simulations for an $N \equiv 10$ -variate random variable $\mathbf{X}$ in such a way that each marginal is gamma-distributed

$$X_n \sim \mathrm{Ga}\,(n, 1)\,, \quad n = 1, \ldots, N, \tag{117}$$

and such that each two entries are fully codependent, i.e. the cdf of their copula is (2.106).

Hint. Use (2.34).

See the script `S_FullCodependence`.

3 Quest for invariance

3.1 Theory

3.1.1 Random walk

Generate a Merton jump-diffusion process X_t at discrete times with arbitrary parameters. What are the invariants in this process?

Hint. This is a random walk

See the script `S_DisplayJumpDiffusionMerton`.

The invariants are the non-overlapping changes in the process ΔX_t over arbitrary intervals. Notice that these are not the only invariants, as any i.i.d. shock used to generate the process is also an invariant. However, these invariants are directly observable and their distribution can be estimated with the techniques discussed in the course.

3.1.2 AR(1)

Generate an Ornstein-Uhlenbeck / AR(1) process.

Prove empirically that for small time intervals and/or low reversion parameters the Ornstein-Uhlenbeck process is a Brownian motion.

Prove analytically that for small time intervals and/or low reversion parameters the Ornstein-Uhlenbeck process is a Brownian motion.

See the script S_AutocorrelatedProcess.

Consider the solution of the Ornstein-Uhlenbeck process

$$X_{T+\tau} = m\left(1 - e^{-\theta\tau}\right) + e^{-\theta\tau} X_T + \epsilon_{T,\tau}, \tag{118}$$

where

$$\epsilon_{T,\tau} \sim \mathrm{N}\left(0, \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta\tau}\right)\right). \tag{119}$$

If $\theta\tau \approx 0$ we can write

$$X_{T+\tau} \approx X_T + m\theta\tau + \epsilon_{T,\tau}, \tag{120}$$

where

$$\epsilon_{T,\tau} \sim \mathrm{N}\left(0, \sigma^2\tau\right), \tag{121}$$

which is the standard arithmetic Brownian motion.

3.1.3 Volatility clustering

Generate a GARCH(1,1) process with arbitrary parameters. What are the invariants of this process?

See the script S_VolatilityClustering.

The invariants are the shocks in the volatility, which also directly drive the randomness of the process. Notice that these invariants are not directly measurable.

3.2 Empirical

3.2.1 Equity

Consider any of the daily time series P_t of the stock prices in the database DB_Equities. Consider the variables

$$X_t \equiv \frac{P_t}{P_{t-1}} \tag{122}$$

$$Y_t \equiv P_t - P_{t-1} \tag{123}$$

$$Z_t \equiv \left(\frac{P_t}{P_{t-1}}\right)^2 \tag{124}$$

$$W_t \equiv P_{t+1} - 2P_t + P_{t-1} \tag{125}$$

Determine which among X_t, Y_t, Z_t, W_t, can potentially be an invariant and which certainly cannot be an invariant, by computing the histogram from two sub-samples and by plotting the location-dispersion ellipsoid of a variable with its lagged value.

See the script S_EquitiesInvariants. The W_t's are clearly not invariants

3.2.2 Fixed income

Consider the time series of realizations of the yield curve in DB_FixedIncome.

Check whether the changes in yield curve for a given time to maturity are invariants using IIDAnalysis.

Check whether the changes in the logarithm of the yield curve for a given time to maturity are invariants using IIDAnalysis.

Changes in the yield curve and changes in the logarithm of the yield curve are approximately invariants, whereas the changes in the yield to maturity of a specific bond are not, see the script S_FixedIncomeInvariants

3.2.3 Derivatives

Consider the time series of daily realizations of the implied volatility surface in DB_Derivatives.

Check whether the weekly changes in implied volatility for a given level of moneyness and time to maturity are invariants using IIDAnalysis.

Check whether the weekly changes in the logarithm of the implied volatility for a given level of moneyness and time to maturity are invariants using IIDAnalysis.

Define the vector $\mathbf{Z}_t$ as the juxtaposition of all the entries of the logarithm of the implied volatility surface at time t. Fit the implied volatility data to a multivariate autoregressive process of order one:

$$\mathbf{Z}_{t+1} \equiv \widehat{\mathbf{a}} + \widehat{\mathbf{B}}\mathbf{Z}_t + \widehat{\boldsymbol{\epsilon}}_{t+1}, \tag{126}$$

where time is measured in weeks. Check whether the weekly residuals $\widehat{\boldsymbol{\epsilon}}_t$ are invariants using IIDAnalysis.

The weekly changes in (the logarithm of) the implied volatility are not invariants, because they display significant negative autocorrelation. On the other hand, the weekly residuals $\widehat{\boldsymbol{\epsilon}}_t$ are invariants. See the script S_DerivativesInvariants

3.2.4 Cointegration

Upload the database DB_SwapParRates of the daily series of a set of par swap rates.

Determine the (in-sample) decreasingly most cointegrated combination of the above par swap rates using principal component analysis.

Fit an AR(1) process to these combinations and compute the unconditional (long term, equilibrium) expectation and standard deviation. Plot the 1-z-score bands around the long-term mean to generate signals to enter or exit a trade.

Hint. See Meucci (2009b)

See the script `S_StatArbSwaps`.

4 Estimation

4.1 Non-parametric

4.1.1 Estimation of moment-based functional

Assume that the invariants X_t are distributed as a mixture. In other words, the pdf reads:

$$f_X \equiv \alpha f_Y + (1-\alpha) f_Z, \tag{127}$$

where $\alpha \in (0,1)$ and

$$\begin{aligned} f_Y &\Leftrightarrow \mathrm{N}\left(\mu_Y, \sigma_Y^2\right) && (128) \\ f_Z &\Leftrightarrow \mathrm{LogN}\left(\mu_Z, \sigma_Z^2\right). && (129) \end{aligned}$$

Consider the variable:

$$V \equiv \alpha Y + (1-\alpha) Z, \tag{130}$$

where Y and Z are independent and

$$\begin{aligned} Y &\sim \mathrm{N}\left(\mu_Y, \sigma_Y^2\right) && (131) \\ Z &\sim \mathrm{LogN}\left(\mu_Z, \sigma_Z^2\right). && (132) \end{aligned}$$

Is (127) the pdf of (130)? If so, prove it. If not, how do you compute the pdf of (130)?

Formula (127) is not the pdf of (130). You can see this in simulation. Alternatively, you can prove it by showing that the moments of X and the moments of V are different. For instance, denote

$$s_Y^2 \equiv \mathrm{E}\left\{Y^2\right\}, \quad s_Z^2 \equiv \mathrm{E}\left\{Z^2\right\}. \tag{133}$$

Then

$$\begin{aligned} \mathrm{E}\left\{X^2\right\} &\equiv \int u^2 f_X(u)\, du && (134) \\ &= \alpha \int u^2 f_Y(u)\, du + (1-\alpha) \int u^2 f_Z(u)\, du \\ &= \alpha s_Y^2 + (1-\alpha) s_Z^2. \end{aligned}$$

On the other hand

$$
\begin{aligned}
\mathrm{E}\left\{V^2\right\} &\equiv \mathrm{E}\left\{(\alpha Y + (1-\alpha) Z)^2\right\} \\
&= \mathrm{E}\left\{\alpha^2 Y^2 + 2\alpha (1-\alpha) Y Z + (1-\alpha)^2 Z^2\right\} \\
&= \alpha^2 \mathrm{E}\left\{Y^2\right\} + 2\alpha(1-\alpha) \mathrm{E}\left\{Y\right\} \mathrm{E}\left\{Z\right\} + (1-\alpha)^2 \mathrm{E}\left\{Z^2\right\} \\
&= \alpha^2 s_Y^2 + (1-\alpha)^2 s_Z^2 .
\end{aligned} \tag{135}
$$

Therefore (127) is not the pdf of (130). However, (127) is the pdf of a random variable, defined in distribution as follows:

$$
X \stackrel{d}{\equiv} BY + (1-B) Z. \tag{136}
$$

In this expression B is a Bernoulli variable:

$$
B \sim \mathrm{Ber}(\alpha). \tag{137}
$$

This is a discrete random variable that can only assume two values:

$$
B \equiv \begin{cases} 1 & \text{with probability } \alpha \\ 0 & \text{with probability } 1-\alpha \end{cases} . \tag{138}
$$

Therefore, the pdf of B reads:

$$
f_B = \alpha \delta^{(1)} + (1-\alpha) \delta^{(0)}, \tag{139}
$$

where $\delta^{(s)}$ is the Dirac delta centered in s, see Figure B.2 in the textbook.

When $B = 1$ in (136) the variable X will be normal as in (131), when $B = 0$ the variable X will be lognormal as in (132). Therefore, the pdf of X conditioned on B reads:

$$
f_{X|B}(x|0) = f_Z(x), \quad f_{X|B}(x|1) = f_Y(x). \tag{140}
$$

This two-step method gives rise to the pdf (127). To see this, as in (2.22) in Meucci (2005) the pdf of X can be written as the marginalization of the joint pdf of X and B:

$$
f_X(x) = \int f_{X,B}(x,b)\, db. \tag{141}
$$

As in (2.43) in Meucci (2005) the joint pdf of X and B can be written as the product of the conditional and the marginal:

$$
f_{X,B}(x,b) = f_{X|B}(x|b)\, f_B(b). \tag{142}
$$

Therefore

$$\begin{aligned}
f_X(x) &= \int f_{X|B}(x|b)\, f_B(b)\, db \qquad (143)\\
&= \int f_{X|B}(x|b)\left[\alpha\delta^{(1)}(b) + (1-\alpha)\,\delta^{(0)}(b)\right] db\\
&= \alpha\int f_{X|B}(x|b)\,\delta^{(1)}(b)\, db + (1-\alpha)\int f_{X|B}(x|b)\,\delta^{(0)}(b)\, db\\
&= \alpha f_{X|B}(x|1) + (1-\alpha)\, f_{X|B}(x|0)\\
&= \alpha f_Y(x) + (1-\alpha)\, f_Z(x).
\end{aligned}$$

As for the pdf of (130), it can be obtained as follows.

First we use $(T1.14)$ in the technical appendices at `symmys.com` > `Book` > `Downloads` with (1.67) and (1.95) in Meucci (2005) to compute the pdf of αY and $(1-\alpha)\,Z$:

$$f_{\alpha Y}(x) = \frac{1}{\alpha\sqrt{2\pi\sigma_Y^2}}\exp\left[-\frac{(x/\alpha - \mu_Y)^2}{2\sigma_Y^2}\right] \qquad (144)$$

$$f_{(1-\alpha)Z}(x) = \frac{1}{x\sqrt{2\pi\sigma_Z^2}}\exp\left[-\frac{(\ln(x/(1-\alpha)) - \mu_Z)^2}{2\sigma_Z^2}\right] \qquad (145)$$

Then we compute the characteristic functions of αY and $(1-\alpha)\,Z$ as in (1.14) in Meucci (2005) as the Fourier transform of the respective pdf's:

$$\phi_{\alpha Y} = \mathcal{F}\left[f_{\alpha Y}\right], \quad \phi_{(1-\alpha)Z} = \mathcal{F}\left[f_{(1-\alpha)Z}\right]. \qquad (146)$$

Then we compute the characteristic function of V:

$$\begin{aligned}
\phi_V(\omega) &\equiv \mathrm{E}\left\{e^{i\omega V}\right\} = \mathrm{E}\left\{e^{i\omega[\alpha Y + (1-\alpha)Z]}\right\}\\
&= \mathrm{E}\left\{e^{i\omega\alpha Y}\right\}\mathrm{E}\left\{e^{i\omega(1-\alpha)Z}\right\} \qquad (147)\\
&= \phi_{\alpha Y}(\omega)\,\phi_{(1-\alpha)Z}(\omega)\\
&= \mathcal{F}\left[f_{\alpha Y}\right](\omega)\,\mathcal{F}\left[f_{(1-\alpha)Z}\right](\omega).
\end{aligned}$$

Using $(B.45)$ in Meucci (2005) we can express the characteristic function of V in terms of the convolution $(B.43)$ of the pdf's and the Fourier transform:

$$\phi_V(\omega) = \mathcal{F}\left[f_{\alpha Y} * f_{(1-\alpha)Z}\right](\omega). \qquad (148)$$

Then we compute the pdf of V as in (1.15) in Meucci (2005) as the inverse Fourier transform of the characteristic function:

$$f_V = \mathcal{F}^{-1}\left[\phi_V\right]. \qquad (149)$$

Substituting (148) in (149) we finally obtain:

$$\begin{aligned} f_V &= \mathcal{F}^{-1}\left[\mathcal{F}\left[f_{\alpha Y} * f_{(1-\alpha)Z}\right]\right] \\ &= f_{\alpha Y} * f_{(1-\alpha)Z}. \end{aligned} \tag{150}$$

Assume that we are interested in the following moment-based functional:

$$G\left[f_X\right] \equiv \int_{\mathbb{R}} \left(x^2 - x\right) f_X\left(x\right) dx. \tag{151}$$

Compute (151) analytically as a function of the inputs α, μ_Y, σ_Y, μ_Z and σ_Z.

$$\begin{aligned} G\left[f_X\right] &\equiv \int_{\mathbb{R}} \left(x^2 - x\right) f_X\left(x\right) dx \\ &= \alpha \int_{\mathbb{R}} \left(x^2 - x\right) f_Y\left(x\right) dx + (1-\alpha) \int_{\mathbb{R}} \left(x^2 - x\right) f_Z\left(x\right) dx \\ &= \alpha \left(\mathrm{Var}\left\{Y\right\} + \left(\mathrm{E}\left\{Y\right\}\right)^2 - \mathrm{E}\left\{Y\right\}\right) \\ &\quad + (1-\alpha)\left(\mathrm{Var}\left\{Z\right\} + \left(\mathrm{E}\left\{Z\right\}\right)^2 - \mathrm{E}\left\{Z\right\}\right) \\ &\overset{(1.71)(1.72)(1.98)(1.99)}{=} \alpha\left(\mu_Y^2 + \sigma_Y^2 - \mu_Y\right) \\ &\quad + (1-\alpha)\left(e^{2\mu_Z + 2\sigma_Z^2} - e^{\mu_Z + \frac{\sigma_Z^2}{2}}\right) \end{aligned} \tag{152}$$

Build a function `QuantileMixture` that computes the quantile of (127) by linear interpolation/extrapolation of the respective cdf on a fine set of equally spaced points for generic values of $\alpha, \mu_Y, \sigma_Y, \mu_Z, \sigma_Z$. In order to use this function in the sequel, make sure the function can accept a vector of values $\mathbf{u}$ in $(0,1)$ as input, not just one value, thereby outputting the respective vector of values $\mathbf{x} \equiv Q_X\left(\mathbf{u}\right)$ in the domain of X.

Hint. Use the built-in cumulative distribution functions that correspond to (128) and (129).

Assume knowledge of the following parameters:

$$\alpha \equiv 0.8, \quad \sigma_Y \equiv 0.2, \quad \mu_Z \equiv 0, \quad \sigma_Z \equiv 0.15. \tag{153}$$

Set $\mu_Y \equiv 0.1$ and generate a sample of $T \equiv 52$ i.i.d. observations

$$i_T \equiv \left\{x_1, \ldots, x_T\right\} \tag{154}$$

from the distribution (127).

Hint. feed a uniform sample into the function `QuantileMixture`.

See the script S_GenerateMixtureSample.

Consider the following estimators:

$$\widehat{G}_a[i_T] \equiv (x_1 - x_T)\, x_2^2 \tag{155}$$

$$\widehat{G}_b[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T} x_t \tag{156}$$

$$\widehat{G}_c[i_T] \equiv 5. \tag{157}$$

Evaluate the performance of the estimators (155), (156) and (157) with respect to (151) as in the script S_Estimator by assuming (153) and by stress-testing the parameter μ_Y in the range $[0, 0.2]$.

See script S_EstimateMomentsComboEvaluation.

Compute the non-parametric estimator $\widehat{G}_d$ of (151) defined by (4.36). Assume knowledge of the parameters (153) and evaluate the performance of $\widehat{G}_d$ with respect to (151) as in the script S_EstimateExpectedValueEvaluation by stress-testing the parameter μ_Y in the range $[0, 0.2]$.

See script S_EstimateMomentsComboEvaluation.

The non-parametric estimator of

$$G[f_X] \equiv \int_{\mathbb{R}} \left(x^2 - x\right) f_X(x)\, dx \tag{158}$$

follows from (4.36) in Meucci (2005):

$$\begin{aligned}\widehat{G}_d(i_T) &\equiv \int_{\mathbb{R}} \left(x^2 - x\right) f_{i_T}(x)\, dx \\ &= \int_{\mathbb{R}} x^2 f_{i_T}(x)\, dx - \int_{\mathbb{R}} x f_{i_T}(x)\, dx.\end{aligned} \tag{159}$$

The second term is the sample mean (1.126) in Meucci (2005):

$$\widehat{m} \equiv \int_{\mathbb{R}} x f_{i_T}(x)\, dx = \frac{1}{T}\sum_{t=1}^{T} x_t \tag{160}$$

The first term is the sample non-central second moment:

$$\widehat{ns} \equiv \int_{\mathbb{R}} x^2 f_{i_T}(x)\, dx = \frac{1}{T}\sum_{t=1}^{T} x_t^2. \tag{161}$$

By applying (T1.39) in the technical appendices at symmys.com > Book > Download to (1.126) and (1.127) in Meucci (2005) we obtain

$$\widehat{ns} = \widehat{s}^2 + \widehat{m}^2, \tag{162}$$

where $\widehat{s}^2$ is the sample variance (1.127) in Meucci (2005):

$$\widehat{s}^2 \equiv \frac{1}{T}\sum_{t=1}^{T}(x_t - \widehat{m})^2 . \tag{163}$$

Therefore

$$\begin{aligned} \widehat{G}_d(i_T) &= \widehat{ns} - \widehat{m} \\ &= \widehat{s}^2 + \widehat{m}^2 - \widehat{m}. \end{aligned} \tag{164}$$

4.1.2 Estimation of quantile

Assume that we are interested in this functional:

$$G[f_X] \equiv (\mathcal{I}[f_X])^{-1}(p), \tag{165}$$

where $\mathcal{I}[\cdot]$ is the integration operator and $p \equiv 0.5$. Notice that the above is simply the quantile with confidence p, see (1.8) and (1.17) in Meucci (2005):

$$G[f_X] \equiv Q_X(p). \tag{166}$$

In particular, given that $p \equiv 0.5$, the above is the median.

Compute the non-parametric estimator $\widehat{q}_p$ of (165) defined by (4.36) in Meucci (2005). Assume knowledge of the parameters (153) and evaluate the performance of $\widehat{q}_p$ with respect to (165) as in the script `S_Estimator` by stress-testing the parameter μ_Y in the range $[0, 0.2]$.

Hint. Use the function `QuantileMixture`.

See script `S_EstimateQuantileEvaluation`.

From (4.39) in Meucci (2005), the non-parametric estimator of the median is the sample median (1.130) in Meucci (2005):

$$\widehat{G}_e(i_T) \equiv x_{[T/2]:T}. \tag{167}$$

Evaluate the performance of the estimator (156) with respect to (165) as in the script `S_Estimator` by stress-testing the parameter μ_Y in the range $[0, 0.2]$.

Hint. Use the function `QuantileMixture`.

See script `S_EstimateQuantileEvaluation`.

4.2 Maximum likelihood

4.2.1 Basics

Consider the time series of realizations of the random variable X in the database `DBTimeSeries`.

Check that the provided series represents the realizations of an invariant using `IIDAnalysis`.

Assuming that X is an invariant, let f_X denote the unknown pdf that represents the unknown distribution of each realization in the time series. Make the following assumption on the generating process for X, where we use the notation of (1.79) and (1.95) in Meucci (2005):

$$f_X \approx f_\theta \equiv \begin{cases} f^{\text{Ca}}_{\theta,\theta^2} & \text{for } \theta \in [-0.04, -0.01] \\ f^{\text{LogN}}_{\theta,(\theta-0.01)^2} & \text{for } \theta \in (\{0.02\} \cup \{0.03\}) \end{cases} \tag{168}$$

Compute numerically the maximum likelihood estimator $\widehat{\theta}_{ML}$ of θ.

Hint. Approximate the continuum $[-0.04, -0.01]$ with a fine set of equally spaced points; evaluate the (log-)likelihood for every value of θ.

Assume now that you are interested in the p-quantile of X for $p \equiv 1\%$, as defined in (1.17). Use the above result to compute the ML estimator $\widehat{q}_p^{ML}$ and compare this ML estimate with the non-parametric estimate of the quantile for the same confidence level.

Hint. As in (4.38) in Meucci (2005), the true quantile is a functional of the unknown distribution of X:

$$Q_p(X) \equiv q_p[f_X]. \tag{169}$$

Therefore, the ML estimator of the quantile is the functional applied to the ML-estimated distribution:

$$\widehat{q}_p^{ML} \equiv q_p\left[f_{\widehat{\theta}_{ML}}\right]. \tag{170}$$

On the other hand, as in (4.36) in Meucci (2005) the non-parametric quantile is the functional applied to the empirical pdf:

$$\widehat{q}_p^{NP} \equiv q_p[f_{i_T}], \tag{171}$$

see also Section 4.2.1.

See script `S_MLEbasics`.

4.2.2 MLE for univariate elliptical variables

Consider as in (1.28) in Meucci (2005) a symmetrical univariate random variable X. It is easy to check that such distributions are all and only the one-dimensional elliptical distributions. In other words, there exist two numbers μ and σ and a univariate function g such that:

$$X \sim \text{El}\left(\mu, \sigma^2, g\right). \tag{172}$$

Assume that you know the functional form of g. Adapt the proof in the technical appendix www.4.2 at `symmys.com > Book > Downloads` to compute the maximum likelihood estimators $\widehat{\mu}$ and $\widehat{\sigma}^2$ of μ and σ^2 respectively.

First of all we need two general results. Define

$$M_t^2 \equiv \omega^2 (x_t - \mu)^2 , \tag{173}$$

Then

$$\begin{aligned} \frac{\partial M_t^2}{\partial \mu} &= -2\omega^2 (x_t - \mu) & (174) \\ \frac{\partial M_t^2}{\partial \omega^2} &= (x_t - \mu)^2 & (175) \end{aligned}$$

Now assume that the distribution of the invariants is (172). To compute the ML estimators $\widehat{\mu}$ and $\widehat{\sigma}^2$ we have to maximize the likelihood function as in (4.66) in Meucci (2005), which after (4.74) reads:

$$\left(\widehat{\mu}\left[i_T\right], \widehat{\sigma}^2\left[i_T\right]\right) \equiv \underset{\mu,\sigma^2 \in \boldsymbol{\Theta}}{\operatorname{argmax}} \sum_{t=1}^{T} \ln\left(\frac{1}{\sqrt{\sigma^2}} g\left(\frac{(x_t - \mu)^2}{\sigma^2}\right)\right), \tag{176}$$

where the parameter set is

$$\boldsymbol{\Theta} \equiv \mathbb{R} \times \mathbb{R}^+. \tag{177}$$

We neglect the constraint that σ^2 be positive and verify ex-post that the unconstrained solution satisfies this condition. It is easier to compute the ML estimators of μ and $\omega^2 \equiv 1/\sigma^2$. The ML estimator of σ^2 is simply the inverse of the estimator of ω^2 by the invariance property (4.70) in Meucci (2005) of the ML estimators.

The log-likelihood reads:

$$\ln\left(f_{\boldsymbol{\theta}}\left(i_T\right)\right) = \frac{T}{2} \ln\left|\omega^2\right| + \sum_{t=1}^{T} \ln\left[g\left(M_t^2\right)\right]. \tag{178}$$

The first order conditions with respect to μ read:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \mu}\left[\ln\left(f_{\boldsymbol{\theta}}\left(i_T\right)\right)\right] & (179) \\ &= \frac{\partial}{\partial \mu}\left[\sum_{t=1}^{T} \ln f_{\boldsymbol{\theta}}\left(x_t\right)\right] \\ &= \frac{\partial}{\partial \mu}\left[\sum_{t=1}^{T} \ln\left[g\left(M_t^2\right)\right]\right] \\ &= \sum_{t=1}^{T} \frac{g'\left(M_t^2\right)}{g\left(M_t^2\right)} \frac{\partial M_t^2}{\partial \mu} = \sum_{t=1}^{T} w_t \omega^2 (x_t - \mu), \end{aligned}$$

where we used (174) and we defined:

$$w_t \equiv -2 \frac{g'\left(M_t^2\right)}{g\left(M_t^2\right)}. \tag{180}$$

The solution to this equations is

$$\widehat{\mu} = \frac{\sum_{t=1}^{T} w_t x_t}{\sum_{s=1}^{T} w_s}. \tag{181}$$

The first order conditions with respect to ω^2 read

$$\begin{aligned} 0 &= \frac{\partial \ln \left(f_{\boldsymbol{\theta}} \left(i_T \right) \right)}{\partial \omega^2} = \frac{\partial \sum_{t=1}^{T} \ln f_{\boldsymbol{\theta}} \left(x_t \right)}{\partial \omega^2} \\ &= \frac{T}{2} \frac{\partial \ln \left(\omega^2 \right)}{\partial \omega^2} + \sum_{t=1}^{T} \frac{g' \left(M_t^2 \right)}{g \left(M_t^2 \right)} \frac{\partial M_t^2}{\partial \omega^2} \\ &= \frac{T}{2} \frac{1}{\omega^2} - \frac{1}{2} \sum_{t=1}^{T} w_t \left(x_t - \mu \right)^2 , \end{aligned} \tag{182}$$

where in the last row we used (175).

Thus the solution to (182) reads:

$$\widehat{\sigma}^2 \equiv \frac{1}{\left(\widehat{\omega^2} \right)} = \frac{1}{T} \sum_{t=1}^{T} w_t \left(x_t - \widehat{\mu} \right)^2 \tag{183}$$

This number is positive and thus the unconstrained optimization is correct.

4.2.3 MLE for multivariate Student t distribution

Consider t-distributed invariants:

$$\mathbf{X} \sim \mathrm{St} \left(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma} \right). \tag{184}$$

Assume ν known and use (4.80)-(4.82) in Meucci (2005) to build a recursive routine that computes the ML estimates $\widehat{\boldsymbol{\mu}}_{ML}$ and $\widehat{\boldsymbol{\Sigma}}_{ML}$ of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ respectively.

Hint. The "pen & paper" part will lead you to the weights (4.80) in Meucci (2005) . First compute the generator g that appears in the weighting function (4.79) in Meucci (2005). Under the Student t assumption the pdf is (2.188) in Meucci (2005) and the generator follows accordingly. Now you can compute the weighting function (4.79) in Meucci (2005), namely:

$$w \left(z \right) \equiv -2 \frac{g' \left(z \right)}{g \left(z \right)}. \tag{185}$$

Finally, you can compute the weights (4.80) in Meucci (2005) in MATLAB.

Upload the database `DBUsSwapRates` of the daily time series of par 2yr, 5yr and 10yr swap rates. Compute the invariants relative to a daily estimation interval. Then use the above routine to estimate the expectation and the covariance relative to the 2yr and the 5yr rates under the assumption that $\nu \equiv 3$ and

$\nu \equiv 100$ respectively. Represent the two sets of expectations and the covariances in one figure in terms of the ellipsoid. Also scatter-plot the observations.

See script `S_FitSwapToT`.

First we have to compute the generator g that appears in the weighting function (4.79). Under the Student t assumption the pdf is (2.188). Thus, as in (2.188) the generator reads:

$$g(z) \equiv \frac{\Gamma\left(\frac{\nu+N}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{\frac{N}{2}}}\left(1+\frac{z}{\nu}\right)^{-\frac{\nu+N}{2}}. \tag{186}$$

Hence the weighting function (4.79) reads:

$$w(z) \equiv -2\frac{g'(z)}{g(z)} = \frac{\nu+N}{\nu+z}. \tag{187}$$

Therefore the weights (4.80) read:

$$w_t \equiv \frac{\nu+N}{\nu+(\mathbf{x}_t-\widehat{\boldsymbol{\mu}})'\,\widehat{\boldsymbol{\Sigma}}^{-1}(\mathbf{x}-\widehat{\boldsymbol{\mu}})}. \tag{188}$$

4.3 Shrinkage

4.3.1 Location

Fix $N \equiv 5$ and generate a N-dimensional location vector $\boldsymbol{\mu}$ and a $N \times N$ scatter matrix $\boldsymbol{\Sigma}$.

Consider a normal random variable:

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{189}$$

Generate a time series of $T \equiv 30$ observations from (189).

Build a function that computes the shrinkage estimator (4.138) in Meucci (2005).

See script `S_ShrinkageEstimators`.

4.3.2 Scatter

Build a function that computes the shrinkage estimator (4.160) in Meucci (2005).

See script `S_ShrinkageEstimators`.

4.3.3 Sample covariance and eigenvalue dispersion

Fix $N \equiv 50$, $\boldsymbol{\mu} \equiv \mathbf{0}_N$, $\boldsymbol{\Sigma} \equiv \mathbf{I}_N$. Reproduce the surface in Figure 4.15. You do not need to superimpose the true spectrum as in the figure

Hints: Determine a grid of values for the number of observations T in the time series. For each value of T
a) generate an i.i.d. time series

$$i_T \equiv \{\mathbf{x}_1, \ldots, \mathbf{x}_T\} \tag{190}$$

from

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{191}$$

b) compute the sample covariance $\widehat{\boldsymbol{\Sigma}}$.
c) perform the PC decomposition of $\widehat{\boldsymbol{\Sigma}}$ and store the sample eigenvalues (i.e. the sample spectrum)
d) perform a)-c) a large enough number of times (~ 100 times)
e) compute the average sample spectrum

See script `S_EigenvalueDispersion`.

4.4 Random matrix theory

4.4.1 Semi-circular law

Consider a $N \times N$ matrix $\mathbf{X}$ where for all $m, n = 1, \ldots, N$ the entries are i.i.d. $X_{mn} \sim f_X$, where f_X is a univariate distribution with expectation zero and standard deviation one.

Consider the symmetrized and rescaled matrix

$$\mathbf{Y} \equiv \frac{1}{\sqrt{8N}} (\mathbf{X} + \mathbf{X}'). \tag{192}$$

Consider the eigenvalues $\lambda_1, \ldots, \lambda_N$ of Y and the density that they define

$$h \equiv \frac{1}{N} \sum_{n=1}^{N} \delta^{(\lambda_n)}, \tag{193}$$

where δ is the Dirac delta (B.18) in Meucci (2005). Notice that, since (192) is random, so is the function (193).

According to random matrix theory, in some topology the following limit for the random function h holds

$$\lim_{N \to \infty} h = g, \tag{194}$$

where g is the rescaled upper semicircle function, defined for $\lambda \geq 0$ as follows

$$g(\lambda) \equiv \frac{2}{\pi} \sqrt{1 - \lambda^2}. \tag{195}$$

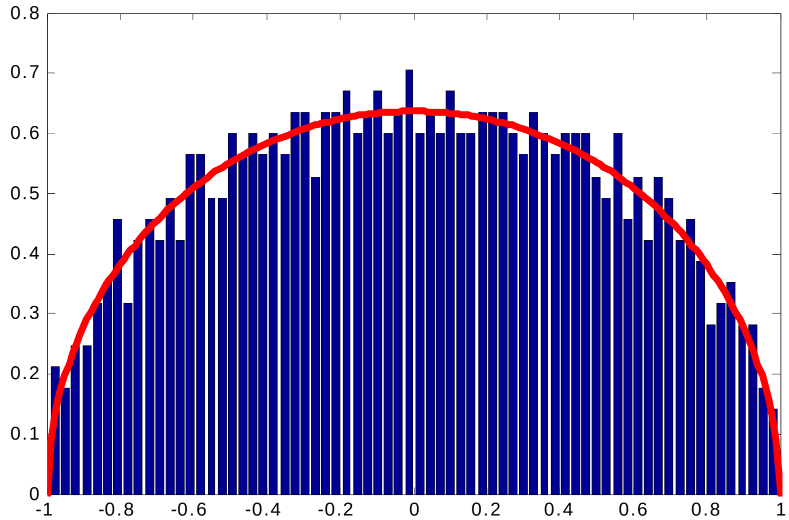

Figure 2: Semi-circle law for symmetric random matrices

Generate a script that shows (194) when the distribution f_X is standard normal, shifted/rescaled normal, and shifted/rescaled exponential.

Hint. Choose a large N and simulate (192) once. This is a realization of (192). Compute the realized eigenvalues and the respective realization of h defined in (193). Approximate h with a histogram. Show that the histogram looks similar to g defined in (195).

See the script `S_SemiCircular`.

4.4.2 Marchenko-Pastur limit

Consider a $T \times N$ matrix $\mathbf{X}$ where for all $m, n = 1, \ldots, N$ the entries are i.i.d. $X_{mn} \sim f_X$, where f_X is a univariate distribution with expectation zero and standard deviation one.

Consider the sample covariance estimator (4.42) in Meucci (2005)

$$\mathbf{Y} \equiv \frac{1}{T}\mathbf{X}'\mathbf{X}. \tag{196}$$

Consider the eigenvalues $\lambda_1, \ldots, \lambda_N$ of Y and the density that they define

$$h \equiv \frac{1}{N}\sum_{n=1}^{N} \delta^{(\lambda_n)}, \tag{197}$$

where δ is the Dirac delta $(B.18)$ in Meucci (2005). Notice that, since (196) is random, so is the function (197).

According to random matrix theory, in some topology the following limit for the random function h holds

$$\lim_{N \equiv qT \to \infty} h = g_q, \tag{198}$$

where the function g_q is defined as

$$g_q(\lambda) \equiv \frac{1}{2q\pi\lambda}\sqrt{\left(\overline{\lambda}_q - \lambda\right)\left(\lambda - \underline{\lambda}_q\right)}, \tag{199}$$

for $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$ where

$$\overline{\lambda}_q \equiv (1-\sqrt{q})^2, \quad \underline{\lambda}_q \equiv (1+\sqrt{q})^2. \tag{200}$$

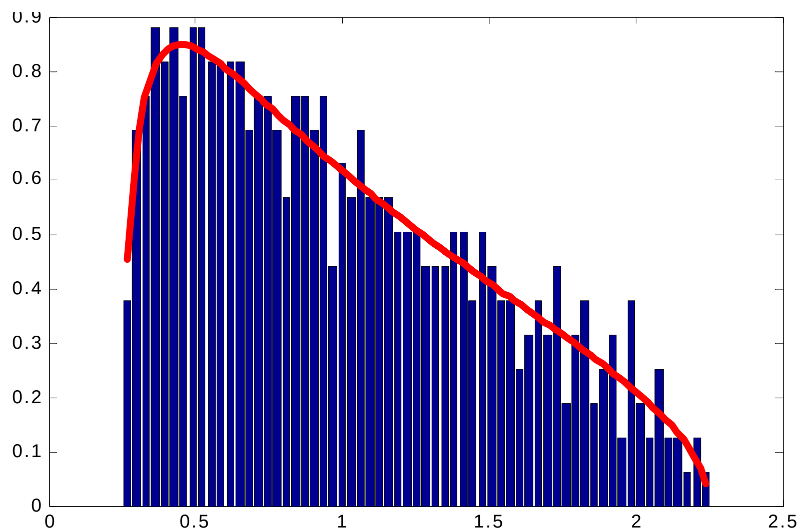

Figure 3: MP law for positive definite symmetric random matrices

Generate a script that shows (199) when the distribution f_X is standard normal, shifted/rescaled normal, and shifted/rescaled exponential.

Hint. Proceed as in Exercise 4.4.1.

See the script `S_PasturMarchenko`.

4.5 Robust

4.5.1 Influence function of sample mean

Adapt the proof in the technical appendix www.4.7 at `symmys.com` > `Book` > `Downloads` to the univariate case to compute the influence function of the sample mean.

Sample estimators of the unknown quantity $G[f_X]$ are by definition explicit functionals of the empirical pdf:

$$\widetilde{G}[f_{i_T}] \equiv G[f_{i_T}]. \tag{201}$$

Therefore from its definition (4.185) in Meucci (2005) the influence function reads:

$$\text{IF}\left(y, f_X, \widehat{G}\right) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(G\left[(1-\epsilon) f_X + \epsilon\delta^{(y)}\right] - G[f_X]\right), \tag{202}$$

where y is an arbitrary point. Now consider the function:

$$h_\epsilon \equiv (1-\epsilon) f_X + \epsilon\delta^{(y)}. \tag{203}$$

The influence function can be written:

$$\text{IF}\left(y, f_X, \widehat{G}\right) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(G[h_\epsilon] - G[h_0]\right) = \left.\frac{dG[h_\epsilon]}{d\epsilon}\right|_0. \tag{204}$$

Consider the functional associated with the sample mean $\widehat{\mu}$, which reads:

$$\widetilde{\mu}[h] \equiv \int_{\mathbb{R}} x h(x)\, dx. \tag{205}$$

From (204) the influence function reads:

$$\text{IF}(y, f, \widehat{\mu}) \equiv \left.\frac{d\widetilde{\mu}[h_\epsilon]}{d\epsilon}\right|_0. \tag{206}$$

First we compute:

$$\begin{aligned} \widetilde{\mu}[h_\epsilon] &\equiv \int_{\mathbb{R}} x h_\epsilon(x)\, dx \\ &= \int_{\mathbb{R}} x\left((1-\epsilon) f(x) + \epsilon\delta^{(y)}(x)\right) dx \\ &= (1-\epsilon)\int_{\mathbb{R}} x f(x)\, dx + \epsilon y \\ &= \mathrm{E}\{X\} + \epsilon\left(-\mathrm{E}\{X\} + y\right). \end{aligned} \tag{207}$$

From this and (206) we derive:

$$\text{IF}(y, f, \widehat{\mu}) = -\mathrm{E}\{X\} + y. \tag{208}$$

4.5.2 Influence function of sample variance

Adapt the proof in the technical appendix www.4.7 at `symmys.com` > `Book` > `Downloads` to the univariate case to compute the influence function of the sample variance.

Consider the functional associated with the sample variance $\widehat{\sigma}^2$, which reads:

$$\widetilde{\sigma}^2[h] \equiv \int_{\mathbb{R}} (x - \widetilde{\mu}[h])^2 h(x)\, dx. \tag{209}$$

From (204) the influence function reads:

$$\mathrm{IF}\left(y, f, \widehat{\sigma}^2\right) \equiv \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}\left(\widetilde{\sigma}^2[h_\epsilon] - \widetilde{\sigma}^2[h_0]\right) = \left.\frac{d\widetilde{\sigma}^2[h_\epsilon]}{d\epsilon}\right|_0 . \tag{210}$$

First we compute:

$$\begin{aligned}
\widetilde{\sigma}^2[h_\epsilon] &\equiv \int_{\mathbb{R}} (x - \widetilde{\mu}[h_\epsilon])^2 h_\epsilon(x)\, dx \\
&= \int_{\mathbb{R}} (x - \widetilde{\mu}[h_\epsilon])^2 \left((1-\epsilon) f(x) + \epsilon \delta^{(y)}(x)\right) dx \\
&= (1-\epsilon)\int_{\mathbb{R}} (x - \widetilde{\mu}[h_\epsilon])^2 f(x)\, dx + \epsilon\,(y - \widetilde{\mu}[h_\epsilon])^2
\end{aligned} \tag{211}$$

Deriving this expression with respect to ϵ we obtain:

$$\begin{aligned}
\mathrm{IF}\left(y, f, \widehat{\sigma}^2\right) &= \left.\frac{d\widetilde{\sigma}^2[h_\epsilon]}{d\epsilon}\right|_0 \\
&= -\int_{\mathbb{R}} (x - \widetilde{\mu}[h_0])^2 f(x)\, dx \\
&\quad + (1-0)\int_{\mathbb{R}} \left.\frac{d}{d\epsilon}\right|_0 (x - \widetilde{\mu}[h_\epsilon])^2 f(x)\, dx \\
&\quad + (y - \widetilde{\mu}[h_0])^2 \\
&\quad + 0 \times \left.\frac{d}{d\epsilon}\right|_0 (y - \widetilde{\mu}[h_\epsilon])^2
\end{aligned} \tag{212}$$

Using $\widetilde{\mu}[h_0] = \mathrm{E}\{X\}$ this means:

$$\begin{aligned}
\mathrm{IF}\left(y, f, \widehat{\sigma}^2\right) &= -\operatorname{Var}\{X\} \\
&\quad - \int_{\mathbb{R}} 2 \left.\frac{d\widetilde{\mu}[h_\epsilon]}{d\epsilon}\right|_0 (x - \mathrm{E}\{X\}) f(x)\, dx \\
&\quad + (y - \mathrm{E}\{X\})^2
\end{aligned} \tag{213}$$

Now using (208) we obtain:

$$\begin{aligned}
\mathrm{IF}\left(y, f, \widehat{\sigma}^2\right) &= -\operatorname{Var}\{X\} \\
&\quad - 2\int_{\mathbb{R}} (y - \mathrm{E}\{X\})(x - \mathrm{E}\{X\}) f(x)\, dx \\
&\quad + (y - \mathrm{E}\{X\})^2
\end{aligned} \tag{214}$$

The term in the middle is null. Therefore:

$$\mathrm{IF}\left(y, f, \widehat{\sigma}^2\right) = -\operatorname{Var}\{X\} + (y - \mathrm{E}\{X\})^2 \tag{215}$$

4.6 Bayesian

4.6.1 Prior on correlation

Assume that the returns $\mathbf{X}_t \equiv (X_{t,1}, X_{t,2}, X_{t,3})'$ on three stocks are jointly normal:

$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{216}$$

with null expectations and unit standard deviations:

$$\boldsymbol{\mu} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{217}$$

$$\boldsymbol{\Sigma} \equiv \begin{pmatrix} 1 & \theta_{12} & \theta_{13} \\ \theta_{12} & 1 & \theta_{23} \\ \theta_{13} & \theta_{23} & 1 \end{pmatrix} \tag{218}$$

In this situation the joint distribution of the returns is fully determined by three parameters:

$$\boldsymbol{\theta} \equiv (\theta_{12}, \theta_{13}, \theta_{23})'. \tag{219}$$

These parameters are constrained on a domain:

$$\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset (-1, 1) \times (-1, 1) \times (-1, 1). \tag{220}$$

Since $\boldsymbol{\Sigma}$ must be positive definite, the domain $\boldsymbol{\Theta}$ is a proper subset of $(-1, 1) \times (-1, 1) \times (-1, 1)$. For instance, $\theta \equiv -(.9, .9, .9)'$ is not a feasible value.

Assume an uninformative uniform prior for the correlations. In other words, assume that $\boldsymbol{\theta}$ is uniformly distributed on its domain:

$$\boldsymbol{\theta} \sim \mathrm{U}(\boldsymbol{\Theta}). \tag{221}$$

Generate $10,000$ simulations from (221).

Hint. Generate a uniform distribution on $(-1, 1)^3$ then discard the simulations such that $\boldsymbol{\Sigma}$ is not positive definite.

In three subplots plot the histograms of θ_{12}, θ_{13} and θ_{23} respectively, showing how the uniform prior implied non-uniform marginal distributions on each of the correlations.

See the script `S_CorrelationPriorUniform`.

4.6.2 Normal-Inverse-Wishart posterior

Create a function randNIW that takes as inputs a generic N-dimensional vector $\boldsymbol{\mu}_0$, a generic positive and symmetric $N \times N$ matrix $\boldsymbol{\Sigma}_0$, two positive scalars T_0 and ν_0 and the number of simulations J and outputs J independent simulations of the normal-inverse-Wishart distribution, as defined in (7.20)-(7.21) in Meucci (2005).

In a script S_AnalyzeNIWPriorPosterior upload the database DBUsSwapRates of daily USD swap rates and compute the daily rate changes from. Then compute T, $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$ (see Chapter 7 for the notation).

In the same script S_AnalyzeNIWPriorPosterior set $\nu_0 \equiv T_0 \equiv 52$ and define the prior parameters $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ arbitrarily. Use the function randNIW to generate $J \equiv 10^4$ scenarios of the prior (7.20)-(7.21) in Meucci (2005).

In the same script S_AnalyzeNIWPriorPosterior compute the parameters of the posterior distribution and use the function randNIW to generate $J \equiv 10^4$ scenarios of the posterior.

Specialize the script S_AnalyzeNIWPriorPosterior to the case $N \equiv 1$, i.e. only consider the first swap rate change.
In one figure, subplot the histogram of the marginal distribution of the prior of μ and superimpose the profile of its analytical pdf. Then subplot the histogram of the marginal distribution of the prior of $1/\sigma^2$ and superimpose the profile of its analytical pdf.
In a different figure, subplot the histogram of the marginal distribution of the posterior of μ and superimpose the profile of its analytical pdf. Then subplot the histogram of the marginal distribution of the posterior of $1/\sigma^2$ and superimpose the profile of its analytical pdf.
Check that (7.4) in Meucci (2005) holds by changing the relative weights of ν_0, T_0 with respect to T.

See script S_AnalyzeNIWPriorPosterior.

4.7 Missing data

4.7.1 EM algorithm

Consider the attached database db_HighYieldIndices of the daily time series of high-yield bond indices. As you will see, some observations are missing. Compute the time series of the daily compounded returns, replacing "NaN" for the missing observations.

Assume that the distribution of the daily compounded returns is normal

$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{222}$$

Estimate the parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ by means of the EM algorithm[2].

[2] Check the online Errata at symmys.com > Book > Downloads, as there is a typo in the first and second printing of the textbook (no typo from the third printing).

See script S_EMexampleHighYield.

4.8 Testing

4.8.1 Sample mean

Consider a time series of independent and identically distributed random variables

$$X_t \sim \mathrm{N}\left(\mu, \sigma^2\right), \quad t = 1, \ldots, T. \tag{223}$$

Consider the sample mean

$$\widehat{\mu} \equiv \frac{1}{T} \sum_{t=1}^{T} X_t. \tag{224}$$

Compute the distribution of $\widehat{\mu}$.

We can write

$$\left(\begin{array}{c} X_1 \\ \vdots \\ X_T \end{array} \right) \sim \mathrm{N}\left(\left(\left(\begin{array}{c} \mu \\ \vdots \\ \mu \end{array} \right), \left(\begin{array}{ccc} \sigma^2 & 0 & \cdots \\ 0 & \ddots & \\ \vdots & & \sigma^2 \end{array} \right) \right) \right). \tag{225}$$

From (2.163) in Meucci (2005), which we report here

$$\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \Rightarrow \mathbf{a} + \mathbf{B}\mathbf{X} \sim \mathrm{N}\left(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\right)$$

for any conformable vector and matrix $\mathbf{a}$ and $\mathbf{B}$ respectively, it follows

$$\widehat{\mu} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{T}\right). \tag{226}$$

What is the probability that the sample mean (224) exceed a given value $\widetilde{\mu}$?

$$\begin{aligned} \mathbb{P}\left\{\widehat{\mu} > \widetilde{\mu}\right\} &= 1 - \mathbb{P}\left\{\widehat{\mu} \leq \widetilde{\mu}\right\} \qquad (227) \\ &= 1 - \mathbb{P}\left\{ \frac{\widehat{\mu} - \mu}{\sqrt{\sigma^2/T}} \leq \frac{\widetilde{\mu} - \mu}{\sqrt{\sigma^2/T}} \right\} \end{aligned}$$

From (4.8.1) and (226) it follows

$$\frac{\widehat{\mu} - \mu}{\sqrt{\sigma^2/T}} \sim \mathrm{N}\left(0, 1\right), \tag{228}$$

therefore

$$\mathbb{P}\left\{\widehat{\mu} > \widetilde{\mu}\right\} = 1 - \Phi\left(\frac{\widetilde{\mu} - \mu}{\sqrt{\sigma^2/T}} \right), \tag{229}$$

where Φ denotes the cdf of the standard normal distribution.

4.8.2 p-value analytical

Consider a normal invariant

$$X_t \sim \mathrm{N}\left(\mu, \sigma^2\right) \tag{230}$$

in a time series of length T. Consider the ML estimator $\widehat{\mu}$ of the location parameter μ. Suppose that you observe a value $\widetilde{\mu}$ for the estimator. Assume that you believe that

$$\mu \equiv \mu_0, \quad \sigma^2 \equiv \sigma_0^2. \tag{231}$$

The *p-value* of $\widehat{\mu}$ for $\widetilde{\mu}$ under the hypothesis (231) is the probability of observing a value as extreme as the observed value:

$$p \equiv \mathbb{P}\left\{\widehat{\mu} \lesseqgtr \widetilde{\mu}\right\}. \tag{232}$$

Compute the expression of the p-value in terms of the cdf of the estimator.

From (4.102) in Meucci (2005)

$$\widehat{\mu} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{T}\right). \tag{233}$$

Therefore in the notation of (1.68) in Meucci (2005) we obtain

$$p \equiv \mathbb{P}\left\{\widehat{\mu} \leq \widetilde{\mu}\right\} = F^{\mathrm{N}}_{\mu_0, \sigma_0^2/T}\left(\widetilde{\mu}\right) \tag{234}$$

or

$$\begin{aligned} p &\equiv \mathbb{P}\left\{\widehat{\mu} \geq \widetilde{\mu}\right\} = 1 - \mathbb{P}\left\{\widehat{\mu} \leq \widetilde{\mu}\right\} \\ &= 1 - F^{\mathrm{N}}_{\mu_0, \sigma_0^2/T}\left(\widetilde{\mu}\right). \end{aligned} \tag{235}$$

4.8.3 t-test, location, analytical

Consider a normal invariant

$$X_t \sim \mathrm{N}\left(\mu, \sigma^2\right) \tag{236}$$

in a time series of length T. Consider the ML estimators $\widehat{\mu}$ and $\widehat{\sigma}^2$ of the location and scatter parameters μ and σ^2 respectively. The *t-statistic* for $\widehat{\mu}$ is defined as

$$\widehat{t}_{\mu_0} \equiv \frac{\widehat{\mu} - \mu_0}{\sqrt{\widehat{\sigma}^2 / (T-1)}}. \tag{237}$$

Compute the distribution of $\widehat{t}_{\mu}$.

Hint. Recall that if $Y_{\boldsymbol{\sigma}^2}$ and Z_ν are independent and such that

$$\begin{aligned} Y_{\boldsymbol{\sigma}^2} &\sim \mathrm{N}\left(0, \sigma^2\right) & (238) \\ \nu Z_\nu^2 &\sim \chi_\nu^2 & (239) \end{aligned}$$

then

$$X_{\nu,\boldsymbol{\sigma}^2} \equiv \frac{Y_{\boldsymbol{\sigma}^2}}{\sqrt{Z_\nu^2}} \sim \mathrm{St}\left(\nu, 0, \sigma^2\right). \tag{240}$$

From (4.103), (1.106) and (1.109) in Meucci (2005) it follows

$$(T-1)\left(\frac{T}{T-1}\frac{\widehat{\sigma}^2}{\sigma^2}\right) \sim \chi_{T-1}^2. \tag{241}$$

From (233) we obtain

$$\sqrt{\frac{T}{\sigma^2}}\left(\widehat{\mu}-\mu\right) \sim \mathrm{N}\left(0,1\right). \tag{242}$$

Furthermore, from the proof in Appendix www.4.3 we derive that $\widehat{\mu}$ and $\widehat{\sigma}^2$ are independent. From (240), we obtain

$$\begin{aligned} \widehat{t}_\mu &\equiv \frac{\widehat{\mu}-\mu}{\sqrt{\widehat{\sigma}^2/\left(T-1\right)}} & (243) \\ &= \sqrt{\frac{T}{\sigma^2}}\left(\widehat{\mu}-\mu\right)\frac{1}{\sqrt{\frac{T\widehat{\sigma}^2}{(T-1)\sigma^2}}} \stackrel{d}{=} \frac{Y_1}{\sqrt{Z_{T-1}^2}}. \end{aligned}$$

Therefore

$$\widehat{t}_\mu \sim \mathrm{St}\left(T-1,0,1\right). \tag{244}$$

You would like to ascertain whether it is possible that

$$\mu = \mu_0 \tag{245}$$

for an arbitrary value μ_0 in (236). How can you asses if the hypothesis (245) is acceptable?

First, compute the distribution of (237) under (245). Then compute the realization $\widetilde{t}_{\mu_0}$ of (237). In the notation (1.87) in Meucci (2005) we obtain:

$$\begin{aligned} \mathbb{P}\left\{\widehat{t}_{\mu_0} \leq \widetilde{t}_{\mu_0}\right\} &= F_{T-1,0,1}^{\mathrm{St}}\left(\widetilde{t}_{\mu_0}\right) & (246) \\ \mathbb{P}\left\{\widehat{t}_{\mu_0} \geq \widetilde{t}_{\mu_0}\right\} &= 1 - F_{T-1,0,1}^{\mathrm{St}}\left(\widetilde{t}_{\mu_0}\right). & (247) \end{aligned}$$

Therefore, if t_α is so small or so large that either probabilities are too small, then (245) is very unlikely.

4.8.4 t-test, factor loadings, analytical

Consider two jointly normal invariants

$$\begin{pmatrix} X_t \\ F_t \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mu_X \\ \mu_F \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_F \\ \rho\sigma_X\sigma_F & \sigma_F^2 \end{pmatrix} \right). \tag{248}$$

Consider a conditional model of the kind (4.88) in Meucci (2005):

$$X_t = \alpha + \beta f_t + U_t, \tag{249}$$

where

$$U_t | f_t \sim \mathrm{N}\left(0, \sigma^2\right). \tag{250}$$

What is the conditional model (249)-(250) ensuing from (248)?

See (2.173) and/or (3.130)-(3.131) in Meucci (2005) to derive

$$\alpha \equiv \mu_X - \rho\frac{\sigma_X}{\sigma_F}\mu_F \tag{251}$$

$$\beta \equiv \rho\frac{\sigma_X}{\sigma_F} \tag{252}$$

$$\sigma^2 \equiv \sigma_X^2\left(1-\rho^2\right). \tag{253}$$

Consider the conditional model (249)-(250) for the invariants. Compute the ML estimators of the factor loadings $\left(\widehat{\alpha}, \widehat{\beta}\right)$ given the observations

$$i_T \equiv \{x_1, f_1, \ldots, x_T, f_T\}. \tag{254}$$

Hint. Define $\mathbf{f}_t' \equiv (1, f_t)$ and

$$\widehat{\mathbf{\Sigma}}_{XF} \equiv \frac{1}{T}\sum_{t=1}^{T} x_t \mathbf{f}_t' \tag{255}$$

$$\widehat{\mathbf{\Sigma}}_{F} \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{f}_t \mathbf{f}_t'. \tag{256}$$

Follow the proof of (4.126) in Meucci (2005) to derive

$$\left(\widehat{\alpha}, \widehat{\beta}\right) = \widehat{\mathbf{\Sigma}}_{XF}\widehat{\mathbf{\Sigma}}_{F}^{-1}, \tag{257}$$

Compute the joint distribution of the ML estimators of the factor loadings $\left(\widehat{\alpha}, \widehat{\beta}\right)$ under the conditional model (249)-(250).

Follow the proof of (4.129) in Meucci (2005) to derive in terms of a (degenerate) matrix-valued normal distribution

$$\left(\widehat{\alpha}, \widehat{\beta}\right) \sim \mathrm{N}\left(\left(\alpha, \beta\right), \frac{\sigma^2}{T}, \widehat{\boldsymbol{\Sigma}}_F^{-1}\right). \tag{258}$$

From (2.180) in Meucci (2005) we then obtain

$$\left(\begin{array}{c} \widehat{\alpha} \\ \widehat{\beta} \end{array}\right) \sim \mathrm{N}\left(\left(\begin{array}{c} \alpha \\ \beta \end{array}\right), \frac{\sigma^2}{T}\widehat{\boldsymbol{\Sigma}}_F^{-1}\right). \tag{259}$$

Compute the distribution of the ML estimator $\widehat{\sigma}^2$ of the dispersion parameter that appears in (250).

Follow the proof of (4.130) and use (2.230) in Meucci (2005) to derive

$$T\widehat{\sigma}^2 \sim \mathrm{Ga}\left(T-2, \sigma^2\right). \tag{260}$$

Compute the distribution of the *t-statistic* for $\widehat{\alpha}$

$$\widehat{t}_\alpha \equiv \sqrt{T-2}\frac{(\widehat{\alpha}-\alpha)}{\sqrt{\widehat{\sigma}^2\widehat{\sigma}_\alpha^2}} \tag{261}$$

and the distribution of the *t-statistic* for $\widehat{\beta}$

$$\widehat{t}_\beta \equiv \sqrt{T-2}\frac{\widehat{\beta}-\beta}{\sqrt{\widehat{\sigma}^2\widehat{\sigma}_\beta^2}}, \tag{262}$$

where $\widehat{\sigma}_\alpha^2$ is the north-west entry of $\widehat{\boldsymbol{\Sigma}}_F^{-1}$ and $\widehat{\sigma}_\beta^2$ is its south-east entry.

From (260), (1.106) and (1.109) in Meucci (2005) it follows

$$(T-2)\left(\frac{T}{T-2}\frac{\widehat{\sigma}^2}{\sigma^2}\right) \sim \chi^2_{T-2}. \tag{263}$$

From (259) we obtain

$$\sqrt{\frac{T}{\widehat{\sigma}_\alpha^2\sigma^2}}\left(\widehat{\alpha}-\alpha\right) \sim \mathrm{N}\left(0,1\right), \tag{264}$$

and similarly for β. Furthermore, follow the proof in Appendix www.4.4 to derive that $\left(\widehat{\alpha}, \widehat{\beta}\right)$ and $\widehat{\sigma}^2$ are independent. Using (240), we obtain

$$\begin{aligned} \widehat{t}_\alpha &\equiv \sqrt{T-2}\frac{(\widehat{\alpha}-\alpha)}{\sqrt{\widehat{\sigma}^2\widehat{\sigma}_\alpha^2}} \\ &= \frac{\sqrt{\frac{T}{\widehat{\sigma}_\alpha^2\sigma^2}}\,(\widehat{\alpha}-\alpha)}{\sqrt{\frac{T}{T-2}\frac{\widehat{\sigma}^2}{\sigma^2}}} \end{aligned} \tag{265}$$

Then from (240) we derive

$$\widehat{t}_\alpha \stackrel{d}{=} \frac{Y_1}{\sqrt{Z_{T-2}^2}} \sim \mathrm{St}\left(T-2, 0, 1\right), \tag{266}$$

You would like to ascertain whether it is possible that

$$\alpha = \alpha_0 \tag{267}$$

for an arbitrary value α_0 in (249), typically $\alpha_0 \equiv 0$. How can you asses if the hypothesis (267) is acceptable?

First, compute the distribution of (261) under (267). Then compute the realization $\widetilde{t}_{\alpha_0}$ of (261). In the notation (1.87) in Meucci (2005) we obtain:

$$\mathbb{P}\left\{\widehat{t}_{\alpha_0} \leq \widetilde{t}_{\alpha_0}\right\} = F_{\nu,0,1}^{\mathrm{St}}\left(\widetilde{t}_{\alpha_0}\right) \tag{268}$$

$$\mathbb{P}\left\{\widehat{t}_{\alpha_0} \geq \widetilde{t}_{\alpha_0}\right\} = 1 - F_{\nu,0,1}^{\mathrm{St}}\left(\widetilde{t}_{\alpha_0}\right). \tag{269}$$

Therefore, if t_α is so small or so large that either probabilities are too small, then (267) is very unlikely.

More in general, consider a conditional model of the kind (4.88) in Meucci (2005)

$$X_t|\mathbf{f}_t = \mathbf{b}'\mathbf{f}_t + U_t, \tag{270}$$

where $\mathbf{f}_t$ is a set of K factors that include a constant and

$$U_t|\mathbf{f}_t \sim \mathrm{N}\left(0, \sigma^2\right). \tag{271}$$

Consider the ML estimator $\widehat{\mathbf{b}}$ of $\mathbf{b}$. Compute the distribution of the F-*statistic* for the linear assumptions $\mathbf{A}$

$$\widehat{F}_{\mathbf{b}} \equiv (T-2)\,\frac{\left(\widehat{\mathbf{b}}-\mathbf{b}\right)'\mathbf{A}'\mathbf{A}\left(\widehat{\mathbf{b}}-\mathbf{b}\right)}{\widehat{\sigma}^2} \tag{272}$$

and use this result to asses if a hypothesis

$$\mathbf{A}\mathbf{b} = \mathbf{a} \tag{273}$$

is acceptable for arbitrary conformable matrix/vector pairs $\mathbf{A}$ and $\mathbf{a}$.

4.8.5 Generalized t-tests, simulations

Consider a joint model for the invariants: for $t = 1, \ldots, T$. The marginals are

$$\begin{aligned} X_t &\sim \text{LogN}\left(\mu_X, \sigma_X^2\right), \quad (274) \\ F_t &\sim \text{Ga}\left(\nu_F, \sigma_F^2\right); \quad (275) \end{aligned}$$

and the copula is the copula of the diagonal entries of Wishart distribution

$$\mathbf{W}_t \sim \text{W}\left(\nu_W, \boldsymbol{\Sigma}_W\right). \quad (276)$$

Consider the coefficients that define the regression line (3.127)

$$\widetilde{X}_t \equiv \alpha + \beta F_t. \quad (277)$$

Compute the non-parametric estimators $\left(\widehat{\alpha}, \widehat{\beta}\right)$ of the regression coefficients.

From (4.52) in Meucci (2005) they read as in (257).

Are $\left(\widehat{\alpha}, \widehat{\beta}\right)$ the maximum-likelihood estimators of the regression coefficients?

No, because the regression model ensuing from (274)-(276) is not conditionally normal as in (249)-(250).

Generate arbitrary values for the parameters in (274)-(276) and for the number of observations T and compute in simulation the distribution of the statistic

$$\widehat{G}_\alpha \equiv \sqrt{T-2}\frac{(\widehat{\alpha} - \alpha)}{\sqrt{\widehat{\sigma}^2 \widehat{\sigma}_\alpha^2}} \quad (278)$$

and the distribution of the statistic

$$\widehat{G}_\beta \equiv \sqrt{T-2}\frac{\widehat{\beta} - \beta}{\sqrt{\widehat{\sigma}^2 \widehat{\sigma}_\beta^2}}. \quad (279)$$

Compare the empirical distribution of (278) with the analytical distribution of (261) as well as the empirical distribution of (279) with the analytical distribution of (262) and comment.

See script `S_TStatApprox`. The distribution of (278) is very similar to that of (261) even for relatively small values of T. The same holds for the distribution of (279) as compared to that of (262).

5 Projection and pricing

5.1 Projection of skewness, kurtosis, and all standardized summary statistics

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010a), freely available online at ssrn.com.

Consider an invariant, which we denote by X_t, refer to the "quest for invariance" in Meucci (2005), Meucci (2009a) and Meucci (2010c). Assume that we have properly estimated the distribution of such one-period invariant. Then we can compute the expectation (1.25) in Meucci (2005)

$$\mu_X \equiv \mathrm{E}\{X\}; \tag{280}$$

the standard deviation (1.42) in Meucci (2005)

$$\sigma_X \equiv \sqrt{\mathrm{E}\{(X-\mu_X)^2\}}; \tag{281}$$

the skewness (1.49) in Meucci (2005)

$$sk_X \equiv \mathrm{E}\{(X-\mu_X)^3\}/\sigma_X^3; \tag{282}$$

the kurtosis (1.51) in Meucci (2005)

$$ku_X \equiv \mathrm{E}\{(X-\mu_X)^4\}/\sigma_X^4; \tag{283}$$

and in general n-th standardized summary statistics

$$\gamma_X^{(n)} \equiv \mathrm{E}\{(X-\mu_X)^n\}/\sigma_X^n, \quad n \geq 3. \tag{284}$$

Consider the projected invariant, defined as the sum of k intermediate single-period invariants

$$Y = X_1 + \cdots + X_k. \tag{285}$$

Such rule applies e.g. to the compounded return (3.11) in Meucci (2005), but not to the linear return (3.10) in Meucci (2005), see also Meucci (2005) for this pitfall.

Project the single-period statistics (280)-(284) to the arbitrary horizon k, i.e. compute the first n standardized summary statistics for the projected invariant Y

$$\mu_Y, \sigma_Y, sk_Y, ku_Y, \gamma_Y^{(5)}, \ldots, \gamma_Y^{(n)} \tag{286}$$

from the first n single-period statistics for the single-period invariant X

$$\mu_X, \sigma_X, sk_X, ku_X, \gamma_X^{(5)}, \ldots, \gamma_X^{(n)}. \tag{287}$$

Hints.

Use the central moments, see (1.48) in Meucci (2005)

$$\mu_X^{(1)} \equiv \mu_X; \qquad \mu_X^{(n)} \equiv \mathrm{E}\{(X-\mu_X)^n\}, \quad n = 2, 3, \ldots; \tag{288}$$

the non-central, or raw, moments, see (1.47) in Meucci (2005)

$$\widetilde{\mu}_X^{(n)} \equiv \mathrm{E}\{X^n\}, \quad n = 1, 2, \ldots; \tag{289}$$

and the cumulants

$$\kappa_X^{(n)} \equiv \left. \frac{d^n \ln\left(\mathrm{E}\left\{e^{zX}\right\}\right)}{dz^n} \right|_{z=0}, \quad n = 1, 2, \ldots. \tag{290}$$

Then use recursively the identity

$$\kappa_X^{(n)} = \widetilde{\mu}_X^{(n)} - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_X^{(k)} \widetilde{\mu}_X^{(n-k)}, \tag{291}$$

see Kendall and Stuart (1969).

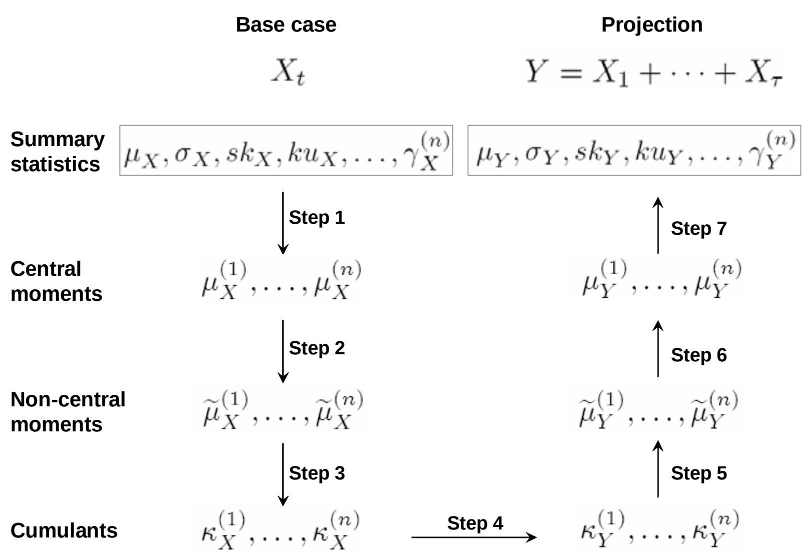

Figure 4: Projection of summary statistics to an aribtrary horizon

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See script `S_ProjSummStats`. The steps involved are the following, refer to Figure 4.

- Step 0. We collect the first n statistics (280)-(284) of the invariant X_t

$$\mu_X, \sigma_X, sk_X, ku_X, \gamma_X^{(5)}, \ldots, \gamma_X^{(n)}. \tag{292}$$

- Step 1. We compute from (292) the central moments $\mu_X^{(1)}, \ldots, \mu_X^{(n)}$ of X_t. To do so, notice from the definition of central moments (288) that $\mu_X^{(2)} \equiv \sigma_X^2$ and that from (284) we obtain

$$\mu_X^{(n)} = \gamma_X^{(n)} \sigma_X^n, \quad n \geq 3. \tag{293}$$

- Step 2. We compute from the central moments $\mu_X^{(1)}, \ldots, \mu_X^{(n)}$ the non-central moments $\widetilde{\mu}_X^{(1)}, \ldots, \widetilde{\mu}_X^{(n)}$ of X_t, see Exercise 1.1.8.
- Step 3. We compute from the non-central moments $\widetilde{\mu}_X^{(1)}, \ldots, \widetilde{\mu}_X^{(n)}$ the cumulants $\kappa_X^{(1)}, \ldots, \kappa_X^{(n)}$ of X_t. To do so, we start from $\kappa_X^{(1)} = \widetilde{\mu}_X^{(1)}$: this follows from the Taylor approximations $\mathrm{E}\left\{e^{zX}\right\} \approx \mathrm{E}\left\{1 + zX\right\} = 1 + z\widetilde{\mu}_X^{(1)}$ for any small z and $\ln(1 + x) \approx x$ for any small x, and from the definition of the first cumulant in (290). Then we apply recursively the identity (291).
- Step 4. We compute from the cumulants $\kappa_X^{(1)}, \ldots, \kappa_X^{(n)}$ of X_t the cumulants $\kappa_Y^{(1)}, \ldots, \kappa_Y^{(n)}$ of the projection $Y \equiv X_1 + \cdots + X_\tau$. To do so, we notice that for any independent variables $X_1, \ldots, X_\tau$ we have $\mathrm{E}\left\{e^{z(X_1 + \cdots + X_\tau)}\right\} = \mathrm{E}\left\{e^{zX_1}\right\} \cdots \mathrm{E}\left\{e^{zX_\tau}\right\}$. Substituting this in the definition of the cumulants (290) we obtain

$$\kappa_{X_1 + \cdots + X_\tau}^{(n)} = \kappa_{X_1}^{(n)} + \cdots + \kappa_{X_\tau}^{(n)}. \tag{294}$$

In particular, since X_t is an invariant, all the X_t's are identically distributed. Therefore the projected cumulants read

$$\kappa_Y^{(n)} = \tau \kappa_X^{(n)}. \tag{295}$$

Step 5. We compute from the cumulants $\kappa_Y^{(1)}, \ldots, \kappa_Y^{(n)}$ the non-central moments $\widetilde{\mu}_Y^{(1)}, \ldots, \widetilde{\mu}_Y^{(n)}$ of Y. To do so, we use recursively the identity

$$\widetilde{\mu}_Y^{(n)} = \kappa_Y^{(n)} + \sum_{k=1}^{n-1} \tbinom{n-1}{k-1} \kappa_Y^{(k)} \widetilde{\mu}_Y^{(n-k)}, \tag{296}$$

which follows from applying (291) to Y and rearranging the terms.
- Step 6. We compute from the non-central moments $\widetilde{\mu}_Y^{(1)}, \ldots, \widetilde{\mu}_Y^{(n)}$ the central moments $\mu_Y^{(1)}, \ldots, \mu_Y^{(n)}$ of Y. To do so, see Exercise 1.1.7.
- Step 7. We compute from the central moments $\mu_Y^{(1)}, \ldots, \mu_Y^{(n)}$ the standardized summary statistics

$$\mu_Y, \sigma_Y, sk_Y, ku_Y, \gamma_Y^{(5)}, \ldots, \gamma_Y^{(n)} \tag{297}$$

of the projected multi-period invariant Y, by applying to Y the definitions (280)-(284).

5.2 Multivariate square-root rule

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010g), freely available online at ssrn.com.Scatter-plot the differences

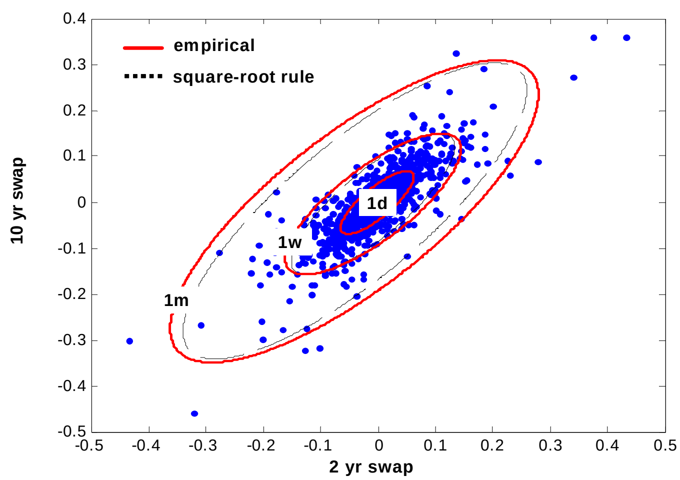

Figure 5: Swap rate changes are approximately invariants

over a time $\widetilde{\tau}$ = one day of the two-year versus the ten-year point of the swap curve from the database `DB_swaps`.

Compute the sample covariance $\widetilde{\mathbf{\Sigma}}$ between these two series. Next, represent geometrically $\widetilde{\mathbf{\Sigma}}$ by means of its location dispersion ellipsoid, which is the smallest ellipsoid in Figure 5.

Then consider the empirical covariance $\mathbf{\Sigma}$ at different horizons of τ = one week and τ = one month respectively. Represent all of these covariances by means of their location-dispersion ellipsoids, which we plot in the figure as solid red lines.

Finally compare these ellipsoids with the suitable multiple $\widetilde{\mathbf{\Sigma}}\tau/\widetilde{\tau}$ as in As in (3.76) in Meucci (2005)

of the daily ellipsoid, which we plot as dashed ellipsoids.

See from the figure that the solid and the dashed ellipsoids are comparable and thus the swap rate changes are approximately invariants: the volatilities increase according to the square-root rule and the correlation is approximately constant.

See the script `S_MultiVarSqrRootRule`.

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5.3 Stable invariants

Assume that the distribution of the market invariants at the estimation horizon $\widetilde{\tau}$ is multivariate Cauchy

$$\mathbf{X}_{\widetilde{\tau}} \sim \mathrm{Ca}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{298}$$

Assume that the invariants satisfy the "accordion" property (3.60) in Meucci (2005).

Prove that the distribution of the market invariants at *any* generic investment horizon τ is Cauchy.

Hint. Like the normal distribution, the Cauchy distribution is stable. Use the characteristic function (2.210) in Meucci (2005) to represent this distribution at any horizon.

Draw your conclusions on the propagation law of risk in terms of the modal dispersion (2.212) in Meucci (2005).

Hint. Notice that the covariance is not defined.

From (3.64) and (2.210) in Meucci (2005) we obtain:

$$\begin{aligned} \phi_{\tau}\left(\boldsymbol{\omega}\right) &= \left(\phi_{\widetilde{\tau}}\left(\boldsymbol{\omega}\right)\right)^{\frac{\tau}{\widetilde{\tau}}} \\ &= \left(e^{i\boldsymbol{\omega}'\boldsymbol{\mu}-\sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}}}\right)^{\frac{\tau}{\widetilde{\tau}}} = e^{i\boldsymbol{\omega}'\boldsymbol{\mu}\frac{\tau}{\widetilde{\tau}}-\sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}\left(\frac{\tau}{\widetilde{\tau}}\right)^{2}\boldsymbol{\omega}}}. \end{aligned} \tag{299}$$

Therefore

$$\mathbf{X}_{\tau} \sim \mathrm{Ca}\left(\frac{\tau}{\widetilde{\tau}}\boldsymbol{\mu}, \frac{\tau^{2}}{\widetilde{\tau}^{2}}\boldsymbol{\Sigma}\right). \tag{300}$$

In particular, from (2.212) we obtain

$$\mathrm{MDis}_{\tau}\left\{\mathbf{X}\right\} = \frac{\tau^{2}}{\widetilde{\tau}^{2}}\,\mathrm{MDis}_{\widetilde{\tau}}\left\{\mathbf{X}\right\} \tag{301}$$

Therefore, the propagation law for risk is linear in the horizon, instead of being proportional to the square root of the horizon.

5.4 Equities

5.4.1 Random walk (linear vs. compounded returns)

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010d), freely available online at ssrn.com.

Assume that the compounded returns (3.11) in Meucci (2005) of a given stock are market invariants, i.e. they are i.i.d. across time. Consider an estimation interval of one week $\widetilde{\tau} \equiv 1/52$ (time is measured in years). Assume that the distribution of the returns is normal:

$$C_{t,\widetilde{\tau}} \sim \mathrm{N}\left(0, \sigma^{2}\widetilde{\tau}\right), \tag{302}$$

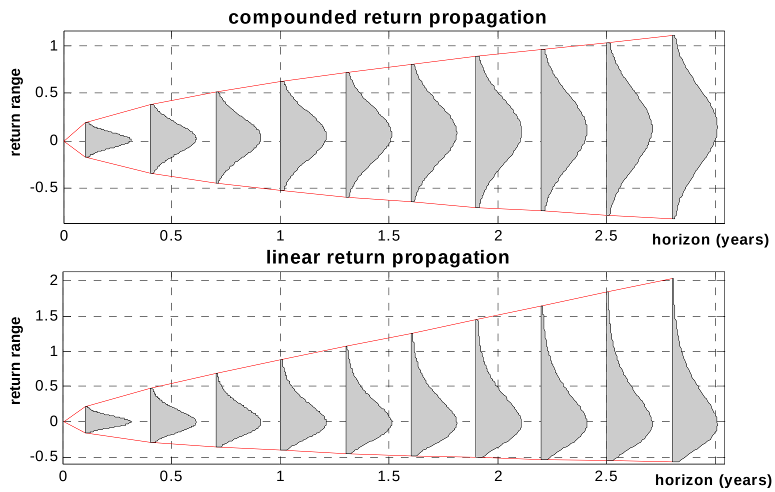

Figure 6: Linear and compounded return difference increases with investment horizon

where $\sqrt{\sigma^2} \equiv 0.4$. Assume that the stock currently trades at the price $P_T \equiv 1$. Fix a generic horizon τ.

Compute and plot the analytical pdf of the price $P_{T+\tau}$.

See the script `S_EquityProjectionPricing` and `S_LinVsLogReturn`.

As in (3.74) in Meucci (2005)

$$C_{t,\tau} \sim \mathrm{N}\left(0, \sigma^2 \tau\right). \tag{303}$$

Therefore, as in (3.87)-(3.88)-(3.92) in Meucci (2005)

$$P_{T+\tau} \sim \mathrm{LogN}\left(\ln\left(P_T\right), \sigma^2 \tau\right). \tag{304}$$

The pdf follows from (1.95) in Meucci (2005).

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Simulate the compounded return at the investment horizon and map these simulations into simulations of the price $P_{T+\tau}$ at the generic horizon τ.

Superimpose the rescaled histogram from the simulations of $P_{T+\tau}$ to show that they coincide.

Hint. Use ($T1.43$) in the technical appendix at `symmys.com` > `Book` > `Downloads` for the rescaling.

Compute analytically the distribution of the first-order Taylor approximation of the pricing function around zero and superimpose this pdf to the above plots. Notice how the approximation is good for short horizons and bad for long horizons.

See the script `S_EquityProjectionPricing`.

The first order Taylor approximation reads:

$$\begin{aligned} P_{T+\tau} &= P_T e^{C_{T+\tau,\tau}} \\ &\approx P_T\left(1+C_{T+\tau,\tau}\right) \\ &= P_T + P_T C_{T+\tau,\tau}. \end{aligned} \tag{305}$$

Therefore from (303) we obtain:

$$P_{T+\tau} \sim \mathrm{N}\left(P_T, P_T^2\sigma^2\tau\right). \tag{306}$$

5.4.2 Multivariate GARCH

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010b), freely available online at ssrn.com.

Upload the daily prices time series of a set of stocks from the database `DB_Equities.mat`. Define the k-period compounded return

$$\mathbf{X}_t^{(k)} \equiv \ln\left(\mathbf{P}_{t+k} - \mathbf{P}_t\right), \tag{307}$$

where $\mathbf{P}_t$ are the prices at time t of the securities, see (3.11) in Meucci (2005)

Assume a diagonal-vech GARCH(1,1) process for the one-period compounded returns

$$\mathbf{X}_t^{(1)} = \boldsymbol{\mu} + \sqrt[U]{\mathbf{H}_t}\boldsymbol{\epsilon}_t. \tag{308}$$

In this expression $\sqrt[U]{\mathbf{S}}$ denotes the upper triangular Cholesky decomposition of the generic symmetric and positive matrix $\mathbf{S}$; ϵ_t are normal invariants

$$\boldsymbol{\epsilon}_t \sim \mathrm{N}\left(\mathbf{0}, \mathbf{I}\right); \tag{309}$$

and the scatter matrix $\mathbf{H}_t$ has the following dynamics

$$\mathbf{H}_t = \mathbf{C} + \mathbf{A}\circ\boldsymbol{\Sigma}_{t-1} + \mathbf{B}\circ\mathbf{H}_{t-1}, \tag{310}$$

where $\circ$ is the element-by-element Hadamard product and

$$\boldsymbol{\Sigma}_t \equiv \left(\mathbf{X}_t^{(1)} - \boldsymbol{\mu}\right)\left(\mathbf{X}_t^{(1)} - \boldsymbol{\mu}\right)'. \tag{311}$$

Estimate the parameters $(\boldsymbol{\mu}, \mathbf{C}, \mathbf{A}, \mathbf{B})$ we use the methodology in Ledoit, Santa-Clara, Wolf (2003) "Flexible multivariate GARCH modeling with an application to international stock markets", Review of Economics and Statistics 85, 735-747.

Then use the estimated parameters to simulate the distribution of the T-day linear return.

See the script `S_ProjectNPriceMvGARCH.m`.

Assume it is now time $t = 0$. We are interested in the T-horizon compounded return

$$\mathbf{X}_0^{(T)} = \sum_{t=1}^{T} \mathbf{X}_t^{(1)} = T\boldsymbol{\mu} + \sum_{t=1}^{T} \sqrt[U]{\mathbf{H}_t}\boldsymbol{\epsilon}_t \tag{312}$$

We use Monte Carlo scenarios to represent this distribution, proceeding as follows.

First, we generate J independent scenarios for $\boldsymbol{\epsilon}_1$ from the multivariate distribution (309). For each scenario we generate a return scenario for the first period return $\mathbf{X}_1^{(1)}$ according to (308), i.e.

$$\mathbf{X}_1^{(1)} = \boldsymbol{\mu} + \sqrt[U]{\mathbf{H}_1}\epsilon_1. \tag{313}$$

where the matrix $\mathbf{H}_1$ is an outcome of the above estimation step. Then for each scenario we update the next-step scatter matrix $\mathbf{H}_2$ according to (310). Next, we generate J independent scenarios for $\boldsymbol{\epsilon}_2$ from the multivariate distribution (309) and we generate return scenarios for the second-period returns $\mathbf{X}_2^{(1)}$ according to (308). We proceed iteratively until the scenarios for all the entries in (312) have been generated.

Then the linear return distribution follows from the pricing equation $R = e^X - 1$.

5.5 Fixed income

5.5.1 Normal invariants

Assume that the weekly changes in yield to maturity are fully codependent, i.e. co-monotonic. In other words, assume that the copula of any pairs of weekly yield changes is (2.106) in Meucci (2005). Also, assume that they have the following marginal distribution:

$$Y_t^{(\upsilon)} - Y_{t-\tau}^{(\upsilon)} \sim \mathrm{N}\left(0, \left(\frac{20 + 1.25\upsilon}{10,000}\right)^2\right), \tag{314}$$

where υ denotes the generic time to maturity (measuring time in years) and τ is one week.

Restrict your attention to bonds with times to maturity 1, 5, 10, 52 and 520 weeks, and assume that the current yield curve, as defined in (3.30) in Meucci (2005) is flat at 4%.

Produce joint simulations of the five bond prices at the investment horizon τ of one week.

What are the analytical marginal distributions of the five bond prices at the investment horizon τ of one week?

Produce joint simulations of the five bond linear returns from today to the investment horizon τ of one week.

What are the analytical marginal distributions of the five bond linear returns at the investment horizon τ of one week?

Comment on why the return on the price of a bond cannot be an invariant.

Hints

1. Since the market is fully codependent you will only need one uniformly generated sample

2. You will need the quantile function to generate simulations. Compute the quantile function using `interp1`, the linear interpolation/extrapolation of the cdf.

3. For a generic bond with time to maturity υ from the decision date T the expiry date is $E \equiv T + \upsilon$. As in (3.81) in Meucci (2005) the price at the investment horizon of that bond reads:

$$Z_{T+\tau}^{(T+\upsilon)} = Z_T^{(T+\upsilon-\tau)} \exp\left(-(\upsilon-\tau)\,\Delta_\tau Y^{(\upsilon-\tau)}\right). \tag{315}$$

In other words, the price is determined by the market invariant, a random variable, and the known price of a different bond with shorter time to maturity.

4. The distribution of the 4-week-to-maturity bond at the 4-week-horizon is degenerate, i.e. its pdf is the Dirac delta, because the outcome is deterministic. Make sure that your outcome is consistent with this statement.

5. See hints for Exercise 5.5.2

See the script `S_BondProjectionPricingNormal`.

From (315) the bond price reads

$$Z_{T+\tau}^{(T+\upsilon)} = e^X, \tag{316}$$

where

$$X \equiv \ln\left(Z_T^{(T+\upsilon-\tau)}\right) - (\upsilon-\tau)\,\Delta_\tau Y^{(\upsilon-\tau)}. \tag{317}$$

From (314)

$$\Delta_\tau Y^{(\upsilon)} \sim \mathrm{N}\left(0, \sigma_\upsilon^2\right), \tag{318}$$

where

$$\sigma_\upsilon^2 \equiv \left(\frac{20 + 1.25\upsilon}{10,000}\right)^2. \tag{319}$$

Therefore

$$X \sim \mathrm{N}\left(\ln\left(Z_T^{(T+\upsilon-\tau)}\right), (\upsilon-\tau)^2\,\sigma_{\upsilon-\tau}^2\right), \tag{320}$$

which with (316) implies

$$Z_{T+\tau}^{(T+\upsilon)} \sim \mathrm{LogN}\left(\ln\left(Z_T^{(T+\upsilon-\tau)}\right), (\upsilon-\tau)^2\,\sigma_{\upsilon-\tau}^2\right). \tag{321}$$

From (3.10) in Meucci (2005) the linear return from the current time T to the investment horizon $T+\tau$ of a bond that matures at $E \equiv T + \upsilon$ is defined as

$$L_{T+\tau,\tau}^{(T+\upsilon)} \equiv \frac{Z_{T+\tau}^{(T+\upsilon)}}{Z_T^{(T+\upsilon)}} - 1. \tag{322}$$

Proceeding as above

$$L_{T+\tau,\tau} = e^{Y} - 1, \tag{323}$$

where

$$Y \equiv \ln\left(Z_T^{(T+\upsilon-\tau)}\right) - \ln\left(Z_T^{(T+\upsilon)}\right) - (\upsilon - \tau)\,\Delta_\tau Y^{(\upsilon-\tau)}, \tag{324}$$

and thus

$$1 + L_{T+\tau,\tau}^{(T+\upsilon)} \sim \mathrm{LogN}\left(\ln\left(Z_T^{(T+\upsilon-\tau)}\right) - \ln\left(Z_T^{(T+\upsilon)}\right), (\upsilon-\tau)^2\,\sigma_{\upsilon-\tau}^2\right), \tag{325}$$

or $L_{T+\tau,\tau}^{(T+\upsilon)}$ is a shifted lognormal random variable.

Notice that for $\upsilon \equiv \tau$ the above distribution is degenerate, i.e. deterministic, whereas any estimation would have yielded a non-degenerate distribution.

5.5.2 Student t invariants

Consider a fixed-income market, where the changes in yield-to-maturity, or rate changes, are the market invariants.

Assume that the weekly changes in yield for all maturities are fully codependent, i.e. co-monotonic. In other words, assume that the copula of any pairs of weekly yield changes is (2.106) in Meucci (2005). Assume that the marginal distributions of the weekly changes in yield for all maturities are:

$$\Delta_{\widetilde{\tau}} Y^{(\upsilon)} \sim \mathrm{St}\left(\nu, \mu, \sigma_\upsilon^2\right). \tag{326}$$

In this expression υ denotes the time to maturity (in years) and

$$\nu \equiv 8, \quad \mu \equiv 0, \quad \sqrt{\sigma_\upsilon^2} \equiv \left(20 + \frac{5}{4}\upsilon\right) \times 10^{-4}. \tag{327}$$

Consider bonds with current times to maturity 4, 5, 10, 52 and 520 weeks, and assume that the current yield curve, as defined in (3.30) in Meucci (2005) is flat at 4% (measuring time in years).

Use the function `ProjectionT` that takes as inputs the estimation parameters of the t-distributed invariants and the horizon-to-estimation ratio $\tau/\widetilde{\tau}$ to compute the cdf of the invariants at the investment horizon τ. You do not need to know how this function works. Make sure you properly compute the necessary inputs (see hints below).

Use the cdf obtained above to generate a joint simulation of the bond prices at the investment horizon τ of four weeks.

Plot the histogram of the *linear* returns $L_{T+\tau,\tau}$ of each bond over the investment horizon, where the linear return is defined consistently with (3.10) in Meucci (2005) as follows:

$$L_{t,\tau} \equiv \frac{Z_t^{(E)}}{Z_{t-\tau}^{(E)}} - 1. \tag{328}$$

Notice that the long-maturity (long duration) bonds are much more volatile than the short maturity (short duration) bonds.

Hints

Suppose today is Nov.1 2006 and we hold a zero-coupon bond that matures in 10 weeks, i.e., it matures on Jan.15. 2007.

We are interested in the value of the bond after 4 weeks, i.e., Dec.1 2006.

Recall from (3.30) in Meucci (2005) that the value of a zero-coupon bond is fully determined by its yield to maturity.

At the 4-week investment horizon (Dec.1 2006) our originally 10-week bond will be a 6-week bond. Therefore its price will be fully determined by the value of the 6-week yield to maturity on Dec.1 2006. For instance, if the 6-week yield to maturity on Dec.1 2006 is 4.1%, then in (3.30) we have $\upsilon \equiv 6/52$ and $Y^{(6/52)}_{12/1/2006} \equiv 0.041$. Therefore the bond price on Dec.1 2006 will be:

$$\begin{aligned} Z^{1/15/2007}_{12/1/2006} &= \exp\left(-\frac{6}{52} Y^{(6/52)}_{12/1/2006}\right) \\ &= e^{-\frac{6}{52}\times 0.041} \approx 0.99528. \end{aligned} \tag{329}$$

To summarize, in order to price the 10-week bond at the 4-week investment horizon we need the distribution of the 6-week yield to maturity on Dec.1 2006.

In order to proceed, we recall that in the zero-coupon bond world, the invariants are the non-overlapping changes in yield to maturity, for any yield to maturity, see the textbook from pp.109 to pp.113. In particular, from (3.31) in Meucci (2005), the following four random variables are i.i.d.:

$$X_1 \equiv Y^{(6/52)}_{11/08/2006} - Y^{(6/52)}_{11/1/2006} \tag{330}$$

$$X_2 \equiv Y^{(6/52)}_{11/15/2006} - Y^{(6/52)}_{11/08/2006} \tag{331}$$

$$X_3 \equiv Y^{(6/52)}_{11/22/2006} - Y^{(6/52)}_{11/15/2006} \tag{332}$$

$$X_4 \equiv Y^{(6/52)}_{12/1/2006} - Y^{(6/52)}_{11/22/2006} \tag{333}$$

Notice that we can express the random variable $Y^{(6/52)}_{12/1/2006}$ in (329) as follows:

$$Y^{(6/52)}_{12/1/2006} = Y^{(6/52)}_{11/1/2006} + X_1 + X_2 + X_3 + X_4. \tag{334}$$

Substituting this in (329) we obtain

$$\begin{aligned} Z^{1/15/2007}_{12/1/2006} &= e^{-\frac{6}{52}\left(Y^{(6/52)}_{11/1/2006} + X_1 + X_2 + X_3 + X_4\right)} \\ &= Z^{12/15/2006}_{11/1/2006} e^{-6/52(X_1+X_2+X_3+X_4)}, \end{aligned} \tag{335}$$

where in the last row we used (3.30) in Meucci (2005) again. This expression is (315).

The term $Z^{12/15/2006}_{11/1/2006}$ is the current value of a 6-week zero-coupon bond, which is known. Indeed, using the information that the curve is currently flat at 4% we obtain:

$$Z^{12/15/2006}_{11/1/2006} = e^{-\frac{6}{52} Y^{(6/52)}_{11/1/2006}} = e^{-\frac{6}{52}\times 0.04} \approx 0.99540. \tag{336}$$

We are left with the problem of projecting the invariant, i.e. computing the distribution of $X_1+X_2+X_3+X_4$, and pricing it, i.e. computing the distribution of $e^{-6/52(X_1+X_2+X_3+X_4)}$ in (335).

To project the invariant we need to compute the distribution of the sum four independent t variables

$$X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} X_3 \stackrel{d}{=} X_4 \sim \mathrm{St}\left(\nu, \mu, \sigma^2_{6/52}\right), \tag{337}$$

where the parameters follow from (326). This is the FFT algorithm provided in `ProjectionT`. The pricing is then performed by Monte Carlo simulations.

As for the linear returns, these are the return on our bond over the investment horizon. Therefore (328) reads:

$$L_{12/1/2006, \frac{4}{52}} \equiv \frac{Z^{(1/15/2007)}_{12/1/2006}}{Z^{(1/15/2007)}_{11/1/2006}} - 1. \tag{338}$$

See the script `S_BondProjectionPricingT`.

5.6 Derivatives

Consider a market of call options on the S&P 500, with current time to maturity of 100, 150, 200, 250, and 300 days and strikes equal 850, 880, 910, 940, and 970 respectively. Assume that the investment horizon is 8 weeks.

Consider the time series of the underlying and the implied volatility surface provided in `DB_ImplVol`. Fit a joint normal distribution to the weekly invariants, namely the log-changes in the underlying and the residuals from a vector autoregression of order one in the log-changes in the implied volatilities surface $\boldsymbol{\sigma}_t$.

$$\begin{pmatrix} \ln S_{t+\tau} - \ln S_t \\ \ln \boldsymbol{\sigma}_{t+\tau} - \ln \boldsymbol{\sigma}_t \end{pmatrix} \sim \mathrm{N}\left(\tau\boldsymbol{\mu}, \tau\boldsymbol{\Sigma}\right) \tag{339}$$

Hint. See Exercise 3.2.3: here we are assuming for simplicity that the invariants are the log-changes in the implied volatility surface instead of the the residuals from a vector autoregression of order one on the same variables. Notice how (339) represents a special case of Exercise 3.2.3, which we assume here for simplicity.

Generate simulations for the invariants and jointly project underlying and implied volatility surface to the investment horizon.

Price the above simulations through the full Black-Scholes formula at the investment horizon, assuming a constant risk-free rate at 4%.

Hint. You need to interpolate the surface at the proper strike and time to maturity, which at the horizon has shortened.

Compute the joint distribution of the linear returns of the call options, as represented by the simulations: the current prices of the options can be obtained similarly to the prices at the horizon by assuming that the current values of underlying and implied volatilities are the last observations in the database.

For each call option, plot the histogram of its distribution at the horizon and the scatter-plot of its distribution against the underlying.

Verify what happens as the investment horizon shifts further in the future.

See the script `S_CallsProjectionPricing`.

6 Dimension reduction

6.1 "Pure residual" models: duration/curve attribution

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com.

Consider as market $\mathbf{X}_t$ the weekly linear returns of $N \equiv 3$ bonds, net of the "carry", or "roll-down", i.e. the deterministic component as in (3.108) and comments thereafter in Meucci (2005). Consider as exogenous factors $\mathbf{F}_t$ the weekly changes in $K \equiv 9$ key rates of the government curve: 6 months, one year, 2, 3, 5, 7 10, 20 and 30 years. Consider as exogenous loadings $\mathbf{B}_t$ the key rate durations of these bonds, where $\mathbf{B}_t$ is a $N \times K = 3 \times 9$ matrix.

Consider the pure residual factor model

$$\mathbf{X}_t \equiv \mathbf{B}_t \mathbf{F}_t + \mathbf{U}_t. \tag{340}$$

Find in the database `db_BondAttribution` the time series over the year 2009 of the above variables. Model the joint distribution of the yet-to-be realized factors and residuals by means of the empirical distribution

$$f_{\mathbf{F}_{T+!},\mathbf{U}_{T+!}} \equiv \frac{1}{T} \sum_{t=1}^{T} \delta^{(\mathbf{f}_t,\mathbf{u}_t)}, \tag{341}$$

see (4.35) in Meucci (2005).

Compute the correlation among residuals and between factors and residuals according to the empirical distribution (341) and verify that (340) is not a systematic-plus-idiosyncratic model, see Figure 7.

See script `S_PureResidualBonds`

6.2 "Time series" or "macroeconomic" factor models

6.2.1 Unconstrained time series correlations and r-square

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com.

Consider the approximation $\mathbf{Y}$ provided to the market $\mathbf{X}$ by a given model

$$\mathbf{X} \equiv \mathbf{Y} + \mathbf{U}, \tag{342}$$

Correlations rate changes/residuals

	Δ 6m	Δ 1y	Δ 2y	Δ 3y	Δ 5y	Δ 7y	Δ 10y	Δ 20y	Δ 30y
Bond 1	23%	15%	32%	36%	37%	33%	40%	44%	47%
Bond 2	-25%	-25%	-9%	-11%	-29%	-32%	-31%	-23%	-12%
Bond 3	-31%	-34%	3%	4%	-4%	-7%	-5%	1%	10%

Correlations residuals/residuals

	Bond 1	Bond 2	Bond 3
Bond 1	100%	28%	43%
Bond 2	28%	100%	47%
Bond 3	43%	47%	100%

Figure 7: Pure residual model typical in fixed-income is not-systematic plus-idiosyncratic

where $\mathbf{U}$ is the residual that the model fails to approximate. To evaluate the goodness of a model, we introduce the generalized r-square as in Meucci (2010f)

$$R^2_{\mathbf{W}}\{\mathbf{Y},\mathbf{X}\} \equiv 1 - \frac{\operatorname{tr}(\operatorname{Cov}\{\mathbf{W}(\mathbf{Y}-\mathbf{X})\})}{\operatorname{tr}(\operatorname{Cov}\{\mathbf{WX}\})}. \tag{343}$$

Consider now a linear factor model

$$\mathbf{Y} \equiv \mathbf{BF} \tag{344}$$

where the factors $\mathbf{F}$ are imposed exogenously. Then each choice of $\mathbf{B}$ gives rise to a different model

Determine analytically the expressions for the optimal $\mathbf{B}$ that maximize the r-square (343). Then compute the residuals $\mathbf{U}$. Are the residuals correlated with the factors $\mathbf{F}$? Are the residuals idiosyncratic?

What is the r-square provided by the optimal optimal $\mathbf{B}$?

The solution reads

$$\mathbf{B}^* \equiv \operatorname{Cov}\{\mathbf{X},\mathbf{F}\}\operatorname{Cov}\{\mathbf{F}\}^{-1}, \tag{345}$$

The residuals and the factors are uncorrelated but the residuals are not idiosyncratic because their correlations with each other are not null. The r-square provided by the model with loadings (345) is

$$R^2_{\mathbf{W}} = \frac{\operatorname{tr}\left(\operatorname{Cov}\{\mathbf{WX},\mathbf{F}\}\operatorname{Cov}\{\mathbf{F}\}^{-1}\operatorname{Cov}\{\mathbf{F},\mathbf{WX}\}\right)}{\operatorname{tr}(\operatorname{Cov}\{\mathbf{WX}\})}. \tag{346}$$

See all the proofs in Meucci (2010f).

6.2.2 Unconstrained time series industry factors

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com. Also, compare with Exercise 6.2.1 and Exercise 6.2.3.

Consider the $n = 1, \ldots, N \approx 500$ stocks in the S&P 500 and the $k = 1, \ldots, K \equiv 10$ industry indices. We intend to build a factor model

$$\mathbf{X} = \mathbf{a} + \mathbf{B}\mathbf{F} + \mathbf{U}, \tag{347}$$

where $\mathbf{X} \equiv (X_1, \ldots, X_N)'$ are the yet to be realized returns of the stocks over next week; $\mathbf{a} \equiv (a_1, \ldots, a_N)'$ are N constants; $\mathbf{F} \equiv (F_1, \ldots, F_K)'$ are the factors, i.e. the yet to be realized returns of the industry indices over next week; $\mathbf{B}$ is a $N \times K$ matrix of coefficients that transfers the randomness of the factors into the randomness of the risk drivers; and $\mathbf{U} \equiv (U_1, \ldots, U_N)'$ are defined as the N residuals that make (347) an identity.

Upload the database of the weekly stock returns $\{\mathbf{x}_t\}_{t=1,\ldots,T}$, and the database of the simultaneous weekly indices returns $\{\mathbf{f}_t\}_{t=1,\ldots,T}$. Model the joint distribution of $\mathbf{X}$ and $\mathbf{F}$ by means of the empirical distribution

$$f_{\mathbf{X},\mathbf{F}} \equiv \frac{1}{T} \sum_{t=1}^{T} \delta^{(\mathbf{x}_t, \mathbf{f}_t)}, \tag{348}$$

where $\delta^{(\mathbf{y})}$ denotes the Dirac-delta, which concentrates a unit probability mass on the generic point $\mathbf{y}$.

Compute the optimal loadings $\mathbf{B}^*$ in (347) that give the factor model the highest generalized multivariate distributional r-square as in Meucci (2010f) (you will notice that the weights are arbitrary)

$$\mathbf{B}^* \equiv \underset{\mathbf{B}}{\operatorname{argmax}}\, R^2_{\mathbf{W}} \{\mathbf{B}\mathbf{F}, \mathbf{X}\}. \tag{349}$$

Compute the correlations of the residuals with the factors and verify that it is null. Then compute the correlations of the residuals with each other and verify that it is not null, i.e. the residuals are not idiosyncratic.

Hint: the optimal loadings turn out to be the standard multivariate OLS, see the proof in Meucci (2010f).

See script `S_TimeSeriesIndustries`

6.2.3 Generalized time-series industry factors

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com. Consider the same framework as in Exercise 6.2.2. Compute as in that exercise the optimal loadings $\mathbf{B}^*$ in (347) that give the factor model the highest constrained generalized multivariate distributional r-square defined in Meucci (2010f)

$$\mathbf{B}^* \equiv \underset{\mathbf{B} \in \mathcal{C}}{\operatorname{argmax}}\, R^2_{\mathbf{W}} \{\mathbf{B}\mathbf{F}, \mathbf{X}\}. \tag{350}$$

In this expression, assume that the constraints $\mathcal{C}$ are the following: all the loadings are bound from below by $\underline{B} \equiv 0.8$ and from above by $\overline{B} \equiv 1.2$ and the market-capitalization weighted sum of the loadings be one

$$0.8 \leq B_{n,k} \leq \overline{B}, \quad n = 1, \ldots, N, k = 1, \ldots, K \tag{351}$$

$$\sum_{n=1}^{N} M_n B_{n,k} \equiv 1. \tag{352}$$

Proxy the market capitalization as equal weights for this exercise.

Then compute the correlations of the residuals among each other and with the factors and verify that neither is null. In other words, the model is not of systematic-plus-idiosyncratic type.

See script `S_TimeSeriesConstrainedIndustries`

6.2.4 Analysis of residual

Consider a market of N stocks, where each stock $n = 1, \ldots, N$ trades at time t at the price $P_{t,n}$. Consider as interpretation factors the linear returns on a set of K indices, such as GICS sectors, where each index $k = 1, \ldots, K$ quotes at time t at the price $S_{t,k}$.

As in Black-Scholes, assume that stocks and indices follow a geometric Brownian motion, i.e.

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{F} \end{pmatrix} \sim \mathrm{N}\left(\tau \begin{pmatrix} \boldsymbol{\mu}_X \\ \boldsymbol{\mu}_F \end{pmatrix}, \tau \begin{pmatrix} \boldsymbol{\Sigma}_X & \boldsymbol{\Sigma}_{XF} \\ \boldsymbol{\Sigma}'_{XF} & \boldsymbol{\Sigma}_F \end{pmatrix}\right), \tag{353}$$

where

$$X_n \equiv \ln\left(\frac{P_{T+\tau,n}}{P_{T,n}}\right) \tag{354}$$

$$F_k \equiv \ln\left(\frac{S_{T+\tau,k}}{S_{T,k}}\right). \tag{355}$$

We want to represent the linear returns on the securities

$$\mathbf{R} = e^{\mathbf{X}} - \mathbf{1} \tag{356}$$

in terms of the explanatory factors

$$\mathbf{Z} = e^{\mathbf{F}} - \mathbf{1} \tag{357}$$

by means of a linear model

$$\mathbf{R} \equiv \mathbf{B}\mathbf{Z} + \mathbf{U}. \tag{358}$$

Compute the expression of $\mathbf{B}$ that minimizes the generalized r-square, the expression of the covariance $\boldsymbol{\Sigma}_Z$ of the explanatory factors and the expression of the covariance $\boldsymbol{\Sigma}_U$ of the residuals.

From (3.121) the optimal loadings read:

$$\mathbf{B} \equiv \mathbb{E}\left\{\mathbf{R}\mathbf{Z}'\right\}\left(\mathbb{E}\left\{\mathbf{Z}\mathbf{Z}'\right\}\right)^{-1}. \tag{359}$$

Also from (2.219)-(2.220) in general for a log-normal variable

$$\mathbf{Y} \sim \operatorname{LogN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{360}$$

then

$$\begin{aligned} \mathbb{E}\{\mathbf{Y}\} &= e^{\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)}. && (361) \\ \mathbb{E}\left\{\mathbf{Y}\mathbf{Y}'\right\} &= \left(e^{\left(\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)\right)} e^{\left(\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)\right)'}\right) \circ e^{\boldsymbol{\Sigma}}, && (362) \end{aligned}$$

where $\circ$ denotes the term-by-term Hadamard product.

Since

$$1+\begin{pmatrix} \mathbf{R} \\ \mathbf{Z} \end{pmatrix} \sim \operatorname{LogN}\left(\tau\begin{pmatrix} \boldsymbol{\mu}_X \\ \boldsymbol{\mu}_F \end{pmatrix}, \tau\begin{pmatrix} \boldsymbol{\Sigma}_X & \boldsymbol{\Sigma}_{XF} \\ \boldsymbol{\Sigma}'_{XF} & \boldsymbol{\Sigma}_F \end{pmatrix}\right), \tag{363}$$

using (361), (362) and the property

$$\begin{aligned} \mathbb{E}\{(\mathbf{Y}-\mathbf{1})\} &= \mathbb{E}\{\mathbf{Y}\}-\mathbf{1} && (364) \\ \mathbb{E}\left\{(\mathbf{Y}-\mathbf{1})(\mathbf{Y}-\mathbf{1})'\right\} &= \mathbb{E}\left\{\mathbf{Y}\mathbf{Y}'\right\}+\mathbf{1}\mathbf{1}'-\mathbb{E}\left\{\mathbf{Y}\mathbf{1}'\right\}-\mathbb{E}\left\{\mathbf{1}\mathbf{Y}'\right\}. && (365) \end{aligned}$$

we can easily compute all the terms $\mathbb{E}\{\mathbf{R}\}$, $\mathbb{E}\left\{\mathbf{R}\mathbf{R}'\right\}$, $\mathbb{E}\{\mathbf{Z}\}$, $\mathbb{E}\left\{\mathbf{Z}\mathbf{Z}'\right\}$ and $\mathbb{E}\left\{\mathbf{R}\mathbf{Z}'\right\}$.

Therefore we obtain the loadings (359). The covariance of the explanatory factors then follows from

$$\boldsymbol{\Sigma}_Z = \mathbb{E}\left\{\mathbf{Z}\mathbf{Z}'\right\} - \mathbb{E}\{\mathbf{Z}\}\,\mathbb{E}\{\mathbf{Z}'\} \tag{366}$$

and similarly the covariance of the returns follows from

$$\boldsymbol{\Sigma}_R = \mathbb{E}\left\{\mathbf{R}\mathbf{R}'\right\} - \mathbb{E}\{\mathbf{R}\}\,\mathbb{E}\{\mathbf{R}'\} \tag{367}$$

Generate randomly the parameters of the distribution (353). Then generate a large number of Monte Carlo scenarios from (353) and verify that the sample covariances of explanatory factors and linear returns match with (366) and (367).

Estimate the loadings by OLS of the Monte Carlo simulations i.e. the sample counterpart of (359). Compute the residuals, and their sample covariance $\widehat{\boldsymbol{\Sigma}}_R$ and verify that $\widehat{\boldsymbol{\Sigma}}_R$ is not diagonal and that

$$\widehat{\boldsymbol{\Sigma}}_R \neq \widehat{\mathbf{B}}\widehat{\boldsymbol{\Sigma}}_Z\widehat{\mathbf{B}}' + \widehat{\boldsymbol{\Sigma}}_U. \tag{368}$$

Now repeat the experiment assuming that one of the factor is non-random, by setting the respective volatility to zero in (353) and verify that

$$\widehat{\boldsymbol{\Sigma}}_R \equiv \widehat{\mathbf{B}}\widehat{\boldsymbol{\Sigma}}_Z\widehat{\mathbf{B}}' + \widehat{\boldsymbol{\Sigma}}_U. \tag{369}$$

In this case the most explanatory interpretation reads

$$\mathbf{R} \equiv \mathbf{a} + \mathbf{B}\mathbf{Z} + \mathbf{U}, \tag{370}$$

where the OLS loadings are

$$\mathbf{B} \equiv \boldsymbol{\Sigma}_{RZ}\boldsymbol{\Sigma}_{ZZ}^{-1} \tag{371}$$

and

$$\mathbf{a} \equiv \mathbb{E}\{\mathbf{R}\} - \mathbf{B}\mathbb{E}\{\mathbf{Z}\} \tag{372}$$

See script `S_ResidualAnalysisTheory`

Now repeat the experiment assuming that all the factors are random, but enforcing $\mathbb{E}\{\mathbf{Z}\} \equiv \mathbf{0}$ by suitably setting the drift $\boldsymbol{\mu}_F$ in (353) as a function of the diagonal of $\boldsymbol{\Sigma}_F$ and verify again that

$$\widehat{\boldsymbol{\Sigma}}_R \equiv \widehat{\mathbf{B}}\widehat{\boldsymbol{\Sigma}}_Z\widehat{\mathbf{B}}' + \widehat{\boldsymbol{\Sigma}}_U. \tag{373}$$

See script `S_ResidualAnalysisTheory`

Important: notice that under no circumstance is the residual covariance matrix $\widehat{\boldsymbol{\Sigma}}_U$ diagonal. This issue is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com.

6.3 "Cross-section" or "fundamental" factor models

6.3.1 Unconstrained cross-section correlations and r-square

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com.

Consider the approximation $\mathbf{Y}$ provided to the market $\mathbf{X}$ by a given model

$$\mathbf{X} \equiv \mathbf{Y} + \mathbf{U}, \tag{374}$$

where $\mathbf{U}$ is the residual that the model fails to approximate. To evaluate the goodness of a model, we introduce the generalized r-square as in Meucci (2010f)

$$R^2_{\mathbf{W}}\{\mathbf{Y},\mathbf{X}\} \equiv 1 - \frac{\operatorname{tr}(\operatorname{Cov}\{\mathbf{W}(\mathbf{Y}-\mathbf{X})\})}{\operatorname{tr}(\operatorname{Cov}\{\mathbf{W}\mathbf{X}\})}. \tag{375}$$

Consider now a linear factor model

$$\mathbf{Y} \equiv \mathbf{B}\mathbf{F} \tag{376}$$

where the loadings $\mathbf{B}$ are exogenously chosen, but the factors $\mathbf{F}$ are left unspecified. Then each choice of $\mathbf{F}$ gives rise to a different model. Assume that the factors are a linear function of the market

$$\mathbf{F} \equiv \mathbf{G}\mathbf{X}. \tag{377}$$

Determine analytically the expressions for the optimal $\mathbf{G}$ that maximize the r-square (375). Then compute the factors $\mathbf{F}$, the explained market $\mathbf{Y}$ and the residual $\mathbf{U}$. Is the market $\mathbf{Y}$ explained by the factors orthogonal to the residual $\mathbf{U}$? Is the explained market $\mathbf{Y}$ correlated with the residual $\mathbf{U}$? Are the residual idiosyncratic?

The solution reads

$$\mathbf{G}^* = \left(\mathbf{B}'\mathbf{\Phi}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{\Phi}, \tag{378}$$

where $\mathbf{\Phi} \equiv \mathbf{W}'\mathbf{W}$. Then r-square provided by this solution reads

$$R^2_{\mathbf{W}} = \frac{\operatorname{tr}\left(\operatorname{Cov}\left\{\mathbf{W}\mathbf{B}\mathbf{F}\right\}\right)}{\operatorname{tr}\left(\operatorname{Cov}\left\{\mathbf{W}\mathbf{X}\right\}\right)}, \tag{379}$$

see the proof in Meucci (2010f).

With (378) the factors (377) read

$$\mathbf{F} \equiv \left(\mathbf{B}'\mathbf{\Phi}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{\Phi}\mathbf{X} \tag{380}$$

and the model-recovered market is

$$\mathbf{Y} \equiv \mathbf{P}\mathbf{X}, \tag{381}$$

where

$$\mathbf{P} \equiv \mathbf{B}\left(\mathbf{B}'\mathbf{\Phi}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{\Phi} \tag{382}$$

is a projection operator. Indeed, it is easy to check that $\mathbf{P}^2 = \mathbf{P}$. The linear assumption (377) gives rise to the residuals

$$\mathbf{U} \equiv \mathbf{P}^{\perp}\mathbf{X}, \tag{383}$$

where

$$\mathbf{P}^{\perp} \equiv \mathbf{I} - \mathbf{B}\left(\mathbf{B}'\mathbf{\Phi}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{\Phi} \tag{384}$$

is the projection in the space orthogonal to the span of the model-recovered market. Therefore, the recovered market and the residuals live in orthogonal spaces. However, the residuals and the factors are not uncorrelated and the residuals are not idiosyncratic, see also the discussion in Meucci (2010f).

6.3.2 Unconstrained cross-section industry factors

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com. Also, compare with Exercise 6.3.1 and Exercise 6.3.3.

Consider the $n = 1, \ldots, N \approx 500$ stocks in the S&P 500 and the $k = 1, \ldots, K \equiv 10$ industry indices. We intend to build a factor model

$$\mathbf{X} = \mathbf{a} + \mathbf{BF} + \mathbf{U}, \tag{385}$$

where $\mathbf{X} \equiv (X_1, \ldots, X_N)'$ are the yet to be realized returns of the stocks over next week; $\mathbf{a} \equiv (a_1, \ldots, a_N)'$ are N constants; $\mathbf{F} \equiv (F_1, \ldots, F_K)'$ are the factors, i.e. the yet to be realized random variables; $\mathbf{B}$ is a $N \times K$ matrix of coefficients that transfers the randomness of the factors into the randomness of the risk drivers and that is imposed exogenously; and $\mathbf{U} \equiv (U_1, \ldots, U_N)'$ are defined as the N residuals that make (347) an identity.

Upload the database of the matrix $\mathbf{B}$ of dummy exposures of each stock to its industry. Upload the weekly stock returns $\{\mathbf{x}_t\}_{t=1,\ldots,T}$. Model the distribution of $\mathbf{X}$ by means of the empirical distribution

$$f_{\mathbf{X}} \equiv \frac{1}{T} \sum_{t=1}^{T} \delta^{(\mathbf{x}_t)}, \tag{386}$$

where $\delta^{(\mathbf{y})}$ denotes the Dirac-delta, which concentrates a unit probability mass on the generic point $\mathbf{y}$.

Define the cross-sectional factors as linear transformation of the market $\mathbf{F} \equiv \mathbf{GX}$. Compute the optimal coefficients $\mathbf{G}^*$ that give the factor model the highest generalized multivariate distributional r-square defined in Meucci (2010f)

$$\mathbf{G}^* \equiv \underset{\mathbf{G}}{\operatorname{argmax}}\, R^2_{\mathbf{W}} \{\mathbf{BGX}, \mathbf{X}\}, \tag{387}$$

In this expression assume that the r-square weights matrix $\mathbf{W}$ to be diagonal and equal to the inverse of the standard deviation of each stock return.

Then compute the correlations of the residuals among each other and with the factors and verify that neither is null. In other words, the model is not of systematic-plus-idiosyncratic type.

Hint: the optimal loadings turn out to be the standard multivariate weighted-OLS, see the proof in Meucci (2010f).

See script `S_CrossSectionIndustries`

6.3.3 Generalized cross-section industry factors

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com. Consider the same framework as in Exercise 6.3.2.

Compute similarly to (387) the optimal coefficients $\mathbf{G}^*$ that give the cross-sectional industry factors the highest generalized multivariate distributional r-square defined in Meucci (2010f), but, unlike in Exercise 6.3.2, add a set of constraints $\mathcal{C}$

$$\mathbf{G}^* \equiv \underset{\mathbf{B} \in \mathcal{C}}{\operatorname{argmax}}\, R^2_{\mathbf{W}} \{\mathbf{BGX}, \mathbf{X}\}. \tag{388}$$

In this expression, assume that the constraints $\mathcal{C}$ are that the factors $\mathbf{F} \equiv \mathbf{GX}$ be uncorrelated with the overall market

$$\mathcal{C} : \mathbf{G} \operatorname{Cov}\{\mathbf{X}\} \mathbf{m} \equiv \mathbf{0}, \tag{389}$$

where you can assume the market weights $\mathbf{m}$ to be equal weights for this exercise.

Then compute the correlations of the residuals among each other and with the factors and verify that neither is null. In other words, the model is not of systematic-plus-idiosyncratic type.

See script `S_CrossSectionConstrainedIndustries`

6.3.4 Comparison cross-section with time-series industry factors

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com. Consider the time-series industry factors constructed in Exercise 6.2.2 and the cross-section industry factors constructed in Exercise 6.3.2. Verify that the time series factors are highly correlated with their cross-sectional counterparts.

See script `S_TimeSeriesVsCrossSectionIndustries` in the "cross-section" folder.

6.3.5 Correlation factors-residual: normal example

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com.

Consider an N-dimensional normal market

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{390}$$

and assume that we want to explain it through a linear model

$$\mathbf{X} \equiv \mathbf{b}F + \mathbf{U}, \tag{391}$$

where $\mathbf{b}$ is a given vector of loadings, F is a yet-to-be defined explanatory factor, and $\mathbf{U}$ are residuals that make (391) hold.

Choose N and generate arbitrarily the parameters in (390) and the vector of loadings in (391). Then generate a large number of simulations from (390). Define the factor F through cross-sectional regression and compute the residuals $\mathbf{U}$. Then show that factor and residual are correlated:

$$\operatorname{Cor}\{F, \mathbf{U}\} \neq \mathbf{0}. \tag{392}$$

See script `S_FactorResidualCorrelation`

6.4 "Statistical" approach: principal component analysis

6.4.1 Matrix algebra

Consider a $N \times N$ matrix of the form

$$\mathbf{\Sigma} \equiv \mathbf{E\Lambda E}', \tag{393}$$

where $\mathbf{\Lambda}$ is diagonal and $\mathbf{E}$ is invertible.

Prove that $\mathbf{\Sigma}$ is symmetric, see definition $(A.51)$ in Meucci (2005).

$$\mathbf{\Sigma}' \equiv \left(\mathbf{E\Lambda E}'\right)' = \mathbf{E\Lambda}'\mathbf{E}' = \mathbf{E\Lambda E}' = \mathbf{\Sigma} \tag{394}$$

Prove that $\mathbf{\Sigma}$ is positive if and only if all the diagonal elements of $\mathbf{\Lambda}$ are positive, see definition $(A.52)$ in Meucci (2005).

For any $\mathbf{v}$ there exists one and only one $\mathbf{w} \equiv \mathbf{E}'\mathbf{v}$ and $\mathbf{w} \equiv \mathbf{0} \iff \mathbf{v} \equiv \mathbf{0}$. Assume that all the diagonal elements of $\mathbf{\Lambda}$ are positive and $\mathbf{v} \neq \mathbf{0}$. Then:

$$\begin{aligned} \mathbf{v}'\mathbf{\Sigma v} &\equiv \mathbf{v}'\mathbf{E\Lambda E}'\mathbf{v} = \mathbf{w}'\mathbf{\Lambda w} \\ &= \sum_{n=1}^{N} w_n^2 \lambda_n > 0. \end{aligned} \tag{395}$$

Similarly, from the above identities, if $0 < \mathbf{v}'\mathbf{\Sigma v}$ for any $\mathbf{v} \neq \mathbf{0}$, then each λ_n has to be positive.

6.4.2 Location-dispersion ellipsoid and geometry

Consider the ellipsoid

$$\mathcal{E}_{\mathbf{m},\mathbf{S}} \equiv \left\{ \mathbf{x} \in \mathbb{R}^N \text{ such that } (\mathbf{x} - \mathbf{m})' \, \mathbf{S}^{-1} (\mathbf{x} - \mathbf{m}) \leq 1 \right\}. \tag{396}$$

What is the geometrical interpretation of $\mathbf{m}$?

What is the geometrical interpretation of the eigenvectors of $\mathbf{S}$?

What is the geometrical interpretation of the eigenvalues of $\mathbf{S}$?

What is the statistical interpretation of (396)?

Hint. This is a trick question.

The vector $\mathbf{m}$ represents the center of ellipsoid.

The eigenvectors are the directions of the principal axes of the ellipsoid.

The square root of the eigenvalues are the length of the principal axes of the ellipsoid.

There is no statistical interpretation, as long as $\mathbf{m}$ and $\mathbf{S}$ are not the expected value and the covariance matrix respectively of a multivariate distribution.

6.4.3 Location-dispersion ellipsoid and statistics

Generate $J \equiv 10,000$ simulations from a bi-variate log-t variable:

$$\ln(\mathbf{X}) \sim \mathrm{St}(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{397}$$

where $\nu \equiv 40$, $\boldsymbol{\mu} \approx \mathbf{0.5}$ and $\mathrm{diag}(\boldsymbol{\Sigma}) \approx \mathbf{0.01}$ (you can choose the off-diagonal element).

Hint. You will have to shift and rescale the output of `mvtrnd`.

Consider the generic versor in the plane:

$$\mathbf{e}_\theta \equiv \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{398}$$

Consider the random variable $Z_\theta \equiv \mathbf{e}'_\theta \mathbf{X}$, namely the projection of $\mathbf{X}$ on the direction $\mathbf{e}_\theta$. Compute and plot the sample standard deviation σ_θ of Z_θ as a function of $\theta \in [0, \pi]$ (select a grid of 100 points).

Show in a figure that the minimum and the maximum of σ_θ are provided by versors parallel to the principal axes of the ellipsoid defined by the sample mean $\mathbf{m}$ and the sample covariance $\mathbf{S}$ as plotted by the function `TwoDimEllipsoid`.

Compute the radius r_θ, i.e. the distance between the surface of the ellipsoid and the center of the ellipsoid along the direction of the versor as a function of $\theta \in [0, \pi]$ (select a grid of 100 points).

Hint. To compute r_θ notice that it satisfies:

$$(r_\theta \mathbf{e}_\theta)' \mathbf{S}^{-1} (r_\theta \mathbf{e}_\theta) = 1. \tag{399}$$

In a separate figure superimpose the plot of σ_θ and the plot of r_θ, showing that the minimum and the maximum of σ_θ (i.e. the minimum and the maximum volatility), correspond to the the minimum and the maximum of r_θ respectively (i.e. the length of the smallest and largest principal axis). Notice that the radius equals the standard deviation *only* on the principal axes.

See the script `S_MaxMinVariance`.

6.4.4 PCA derivation, correlations and r-square

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com.

Consider the approximation $\mathbf{Y}$ provided to the market $\mathbf{X}$ by a given model

$$\mathbf{X} \equiv \mathbf{Y} + \mathbf{U}, \tag{400}$$

where $\mathbf{U}$ is the residual that the model fails to approximate. To evaluate the goodness of a model, we introduce the generalized r-square as in Meucci (2010f)

$$R^2_{\mathbf{W}}\{\mathbf{Y}, \mathbf{X}\} \equiv 1 - \frac{\mathrm{tr}(\mathrm{Cov}\{\mathbf{W}(\mathbf{Y} - \mathbf{X})\})}{\mathrm{tr}(\mathrm{Cov}\{\mathbf{W}\mathbf{X}\})}. \tag{401}$$

Consider now a linear factor model

$$\mathbf{Y} \equiv \mathbf{B}\mathbf{F} \tag{402}$$

where the factors are extracted by linear combinations from the market

$$\mathbf{F} \equiv \mathbf{G}\mathbf{X} \tag{403}$$

Then each choice of $\mathbf{B}$ and $\mathbf{G}$ gives rise to a different model $\mathbf{Y}$. Determine analytically the expressions for the optimal $\mathbf{B}$ and $\mathbf{G}$ that maximize the r-square (401) and verify that they are the principal components of the matrix $\mathrm{Cov}\{\mathbf{W}\mathbf{X}\}$. What is the r-square provided by the optimal optimal $\mathbf{B}$ and $\mathbf{G}$? Then compute the residuals $\mathbf{U}$. Are the residuals correlated with the factors $\mathbf{F}$? Are the residuals idiosyncratic?

First, we perform the spectral decomposition of the covariance matrix

$$\mathrm{Cov}\{\mathbf{W}\mathbf{X}\} \equiv \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}'. \tag{404}$$

In this expression $\boldsymbol{\Lambda}$ is the diagonal matrix of the decreasing, positive eigenvalues of the covariance:

$$\boldsymbol{\Lambda} \equiv \mathrm{diag}\left(\lambda_1^2, \ldots, \lambda_N^2\right); \tag{405}$$

and $\mathbf{E}$ is the juxtaposition of the respective eigenvectors, which are orthogonal and of length 1 and thus $\mathbf{E}\mathbf{E}' = \mathbf{I}_N$:

$$\mathbf{E} \equiv \left(\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(N)}\right), \tag{406}$$

Next, we define a $N \times K$ matrix as the juxtaposition of the first K eigenvectors

$$\mathbf{E}_K \equiv \left(\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(K)}\right). \tag{407}$$

Then optimal $\mathbf{B}$ and $\mathbf{G}$ read

$$\mathbf{B}^* = \mathbf{W}^{-1}\mathbf{E}_K, \quad \mathbf{G}^* \equiv \mathbf{E}_K'\mathbf{W}. \tag{408}$$

The r-square (387) provided by the principal component solution (408) reads

$$R_{\mathbf{W}}^2 = \frac{\sum_{k=1}^K \lambda_k^2}{\sum_{n=1}^N \lambda_n^2}. \tag{409}$$

The residuals are not correlated with the factors $\mathbf{F}$ but they are correlated with each other and therefore they are not idiosyncratic. See all the proofs in Meucci (2010f).

6.4.5 PCA and projection

Prove that the PCA-recovered market (3.160) in Meucci (2005) represents the orthogonal projection of the original market onto the hyperplane generated by the first K eigenvectors stemming from the expectation of the original market.

See the Technical Appendix www.3.5.

6.4.6 PCA of two-point swap curve

Upload the database `DB_Swap2y4y` of the 2yr and 4yr daily par swap rates.

Plot the "current curve" i.e. the rates at the last observation date as functions of the respective maturities ("points on the curve").

Find the two invariants for the two rate series for a weekly estimation horizon (one week = five days). Use the `RunAnalysis` test.

Scatter plot the time series of one invariant against the time series of the other invariant.

Superimpose the two-dimensional location-dispersion ellipsoid determined by sample mean and sample covariance of the invariants (set the scale factor equal to 2): you should see a highly elongated ellipsoid.

Perform the spectral decomposition of the sample covariance of the invariants.

Plot the two entries of the first eigenvector (first factor loadings) as a function of the respective points on the curve.

Superimpose the plot of the two entries of the second eigenvector (second factor loadings) as a function of the respective points on the curve.

Compute and plot a one-standard-deviation effect and a minus-one-standard-deviation effect of the first factor on the current curve as in the case study in the textbook.

Compute and plot a one-standard-deviation effect and a minus-one-standard-deviation effect of the second factor on the current curve, as in the case study in the textbook.

Compute the generalized R^2 provided by the first factor.

Compute the generalized R^2 provided by both factors.

See the script `S_SwapPCA2Dim`.

6.4.7 Eigenvectors for Toeplitz structure

Write a script that generates a $N \times N$ Toeplitz matrix with structure as in (3.209)-(3.214)-(3.222) in Meucci (2005), i.e.

$$S_{j,j+k} \equiv r^{|k|}, \tag{410}$$

where $0 < r < 1$.

Hint. Use the command `diag`.

Show in a figure that the eigenvectors have a Fourier basis structure as in (3.217) in Meucci (2005).

See the script `S_Toepliz`.

6.4.8 Generalized principal component analysis

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com. Also, compare with Exericse 6.2.4. ???

6.5 "Statistical" approach: factor analysis puzzle

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010f), freely available online at ssrn.com, to which we refer for the explanation of the puzzle below.

Consider a N-dimensional market that is normally distributed

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{411}$$

Choose an arbitrary dimension N and generate $\boldsymbol{\mu}$ arbitrarily. Then generate $\boldsymbol{\Sigma}$ as follows

$$\boldsymbol{\Sigma} \equiv \mathbf{B}\mathbf{B}' + \boldsymbol{\Delta}^2 \tag{412}$$

where $\mathbf{B}$ is an arbitrary matrix $N \times K$, with $K < N$ and $\boldsymbol{\Delta}^2$ is an arbitrary diagonal matrix.

Generate a large number of scenarios $\{\mathbf{x}_j\}_{j=1,\ldots,J}$ from the distribution (411) with matching sample first and second moment $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$, as in Exercise 1.2.3.

Run the built-in MATLAB function `factoran`, which outputs the estimated values of $\widehat{\mathbf{B}}$ and $\widehat{\boldsymbol{\Delta}}$ as well as the hidden factors $\{\mathbf{f}_j\}_{j=1,\ldots,J}$.

Verify that the factor analysis routine works well, i.e.

$$\widehat{\boldsymbol{\Sigma}} \equiv \widehat{\mathbf{B}}\widehat{\mathbf{B}}' + \widehat{\boldsymbol{\Delta}}^2. \tag{413}$$

Compute the residuals $\mathbf{u}_j \equiv \mathbf{x}_j - \mathbf{B}\mathbf{f}_j$ and their sample covariance.

Verify that the residuals are not idiosyncratic, in contradiction with the very principles of factor analysis.

See the script `S_FactorAnalysisNotOK`.

6.6 "Factors on Demand"

6.6.1 Horizon effect

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010c) and Meucci (2010b), both freely available online at ssrn.com.

Consider as in Exercise 6.2.4 a market of N stocks, where each stock $n = 1, \ldots, N$ trades at time t at the price $P_{t,n}$. Consider as interpretation factors the linear returns on a set of K indices, such as GICS sectors, where each index $k = 1, \ldots, K$ quotes at time t at the price $S_{t,k}$.

As in Black-Scholes, assume that stocks and indices follow a geometric Brownian motion, i.e.

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{F} \end{pmatrix} \sim \mathrm{N}\left(\tau \begin{pmatrix} \boldsymbol{\mu}_X \\ \boldsymbol{\mu}_F \end{pmatrix}, \tau \begin{pmatrix} \boldsymbol{\Sigma}_X & \boldsymbol{\Sigma}_{XF} \\ \boldsymbol{\Sigma}'_{XF} & \boldsymbol{\Sigma}_F \end{pmatrix}\right), \tag{414}$$

where

$$X_n \equiv \ln\left(\frac{P_{T+\tau,n}}{P_{T,n}}\right) \tag{415}$$

$$F_k \equiv \ln\left(\frac{S_{T+\tau,k}}{S_{T,k}}\right). \tag{416}$$

In particular, assume that the compounded returns are generated by the linear model

$$\mathbf{X} \equiv \tau\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{D}\mathbf{F} + \boldsymbol{\epsilon}, \tag{417}$$

where $\mathbf{D}$ is a constant matrix of loadings,

$$\begin{pmatrix} \mathbf{F} \\ \boldsymbol{\epsilon} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \tau\boldsymbol{\Sigma}_F & \mathbf{0} \\ \mathbf{0} & \tau\boldsymbol{\Sigma}_\epsilon \end{pmatrix}\right), \tag{418}$$

and $\boldsymbol{\Sigma}_\epsilon$ is diagonal. Notice that (417)-(418) is a specific case of, and fully consistent with, the more general formulation (414). The specification (417) is the "estimation" side of the model, i.e. the model that would be fitted to the empirical observations.

We want to represent the linear returns on the securities

$$\mathbf{R} = e^{\mathbf{X}} - \mathbf{1} \tag{419}$$

in terms of the explanatory factors

$$\mathbf{Z} = e^{\mathbf{F}} - \mathbf{1} \tag{420}$$

by means of a linear model

$$\mathbf{R} \equiv \mathbf{a} + \mathbf{B}\mathbf{Z} + \mathbf{U}. \tag{421}$$

The specification (421) is the interpretation side of the model, i.e. the model that would be used for portfolio management applications, such as hedging or style analysis.

Upload $\boldsymbol{\mu}_{\mathbf{X}}$, $\mathbf{D}$, $\boldsymbol{\Sigma}_F$ and $\boldsymbol{\Sigma}_\epsilon$ from `db_LinearModel` and study the relationship between the constant $\tau\boldsymbol{\mu}_{\mathbf{X}}$ in (417) and the intercept $\mathbf{a}$ in (421) and the relationship between the loadings $\mathbf{D}$ in (417) and the loadings $\mathbf{B}$ in (421).

See script `S_HorizonEffect`

In particular, in the simple bi-variate case and rescaling for simplicity such that $P_t = S_t = 1$ the returns are shifted multivariate lognormal

$$\left(\begin{array}{c} R_P^{(t)} \\ R_S^{(t)} \end{array} \right) \sim \text{LogN} \left(t \left[\begin{array}{c} \mu_X \\ \mu_F \end{array} \right], t \left[\begin{array}{cc} \sigma_X^2 & \rho_{X,F}\sigma_X\sigma_F \\ \rho_{X,F}\sigma_X\sigma_F & \sigma_F^2 \end{array} \right] \right) - \mathbf{1}, \tag{422}$$

We recall from (2.219)-(2.220) in Meucci (2005) that in general if

$$\mathbf{Y} \sim \text{LogN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{423}$$

then

$$\mathrm{E}\{\mathbf{Y}\} = e^{\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)}. \tag{424}$$

$$\mathrm{E}\{\mathbf{Y}\mathbf{Y}'\} = \left(e^{\left(\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)\right)}e^{\left(\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)\right)'}\right) \circ e^{\boldsymbol{\Sigma}} \tag{425}$$

Also

$$\operatorname{Cov}\{\mathbf{Y}\} = \mathrm{E}\{\mathbf{Y}\mathbf{Y}'\} - \mathrm{E}\{\mathbf{Y}\}\mathrm{E}\{\mathbf{Y}'\}. \tag{426}$$

From (3.127) in Meucci (2005) the beta in (421) simplifies as

$$\beta \equiv \frac{\operatorname{Cov}\left\{R_P^{(t)}, R_S^{(t)}\right\}}{\operatorname{Var}\left\{R_S^{(t)}\right\}}. \tag{427}$$

Therefore

$$\beta = \frac{\left(e^{\widetilde{\mu}_X t + \widetilde{\sigma}_X^2 t/2 + \widetilde{\mu}_F t + \widetilde{\sigma}_F^2 t/2}\right)\left(e^{\widetilde{\rho}_{X,F}\widetilde{\sigma}_X\widetilde{\sigma}_F t} - 1\right)}{e^{2\widetilde{\mu}_F t + \widetilde{\sigma}_F^2 t}\left(e^{\widetilde{\sigma}_F^2 t} - 1\right)}. \tag{428}$$

Is $\mathbf{U}$ idiosyncratic?

It is not, and the longer the horizon, the more pronounced this effect, see script `S_HorizonEffect`.

6.6.2 No-Greek hedging

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010c), see also Meucci (2010b), both freely available online at ssrn.com.

Consider the market of call options on the S&P 500 described in Exercise 5.6, namely call options on the S&P 500, with current time to maturity of 100, 150, 200, 250, and 300 days and strikes equal 850, 880, 910, 940, and 970 respectively.

Consider the time series of the underlying and the implied volatility surface provided in `DB_ImplVol`. Fit a joint normal distribution to the weekly invariants, namely the log-changes in the underlying and the residuals from a vector autoregression of order one in the log-changes in the implied volatilities surface $\boldsymbol{\sigma}_t$.

$$\begin{pmatrix} \ln S_{t+\tau} - \ln S_t \\ \ln \boldsymbol{\sigma}_{t+\tau} - \ln \boldsymbol{\sigma}_t \end{pmatrix} \sim \mathrm{N}\left(\tau\boldsymbol{\mu}, \tau\boldsymbol{\Sigma}\right) \tag{429}$$

Assume that the investment horizon is 8 weeks. We want to represent the linear returns on the options $\mathbf{R}_C$ in terms of the linear returns R of the underlying S&P 500 by means of a linear model

$$\mathbf{R}_C \equiv \mathbf{a} + \mathbf{b}R + \mathbf{U}. \tag{430}$$

Notice that the specification (430) is the interpretation side of a "factors on demand" model.

Generate joint simulations for $\mathbf{R}_C$ and R as in Exercise 5.6 and scatter-plot the results. Then compute $\mathbf{a}$ and $\mathbf{b}$ by OLS.

Compute the cash and underlying amounts necessary to hedge $\mathbf{R}_C$ based on the delta of the Black-Scholes formula and compare with $\mathbf{a}$ and $\mathbf{b}$.

Repeat the above exercise when the investment horizon shifts further or closer in the future.

See script `S_FODHedgeOptions`.

To compute the hedge, consider the risk-neutral pricing equation for a generic option (not necessarily a call option)

$$\Delta O - \delta \Delta S \approx C r \Delta t, \tag{431}$$

where O is the option price; S is the underlying value; r is the risk-free rate; δ is the "delta"

$$\delta \equiv \frac{\partial O}{\partial S}; \tag{432}$$

and C is the cash amount:

$$C \equiv O - \delta S. \tag{433}$$

Then

$$\frac{\Delta O}{O} \approx \frac{C}{O} r \Delta t + \frac{\delta}{O} S \frac{\Delta S}{S} \tag{434}$$

or

$$R_O \approx a + bR, \tag{435}$$

where

$$a \equiv \frac{C}{O} r \Delta t, \quad b \equiv \frac{\delta}{O} S. \tag{436}$$

6.6.3 Selection heuristics

This exercise is discussed in greater depth in the context of Factors on Demand in Meucci (2010c), freely available online at ssrn.com.

Consider the linear return R from the current date to the investment horizon of a portfolio and a pool of N generic risk factors $\{Z_n\}_{n=1,\ldots,N}$, such as the returns of hedging instruments for hedging purposes or the returns of style indices for style analysis.

As prescribed by the Factors on Demand approach, we want to express the portfolio return in a dominant-plus-residual way as a linear combination of only the best K out of the N factors

$$R = \sum_{k \in C_K} d_k Z_k + \eta, \tag{437}$$

where C_K is a subset of $\{1, \ldots, N\}$ of cardinality K.

Define the optimal exposures $\mathbf{d}$ in (437) to maximize the r-square (3.116) in Meucci (2005).

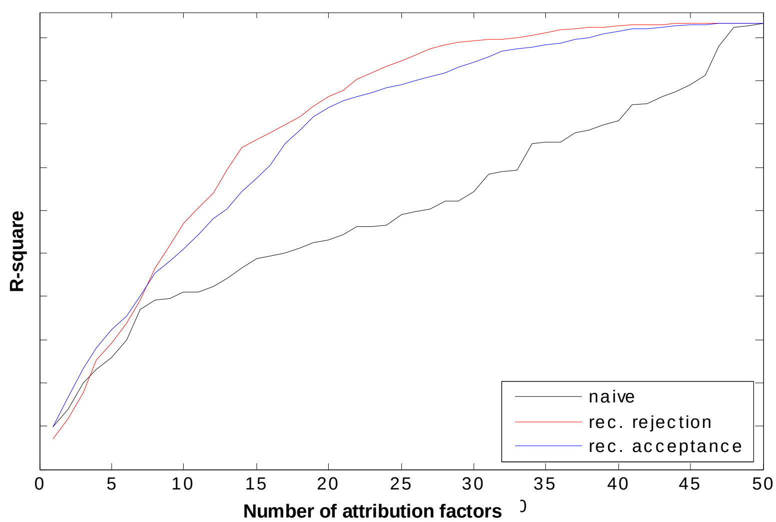

Figure 8: Performance of top-down FoD attribution as function of the number of factors

Generate arbitrarily the parameters of the joint distribution $f_{R,\mathbf{Z}}$ of the portfolio return and the factors necessary to maximize the r-square of the fit and generate a script that compares three approaches.

- A naive approach that ranks the r-square provided by each single factor and collects the K with the most explanatory power.

- The recursive rejection routine in Section 3.4.5 in Meucci (2005) to solve heuristically the above problem by eliminating the factors one at a time starting from the full set.

- The recursive acceptance routine, which is the same as the above recursive rejection, but it starts from the empty set, instead of from the full set.

See script S_SelectionHeuristicsFoD.

7 Risk Management

7.1 Investor's objectives

7.1.1 General

Assume a bivariate market, where the prices at the investment horizon (P_1, P_2) have the following marginal distributions:

$$P_1 \sim \mathrm{Ga}\left(\nu_1, \sigma_1^2\right) \tag{438}$$

$$P_2 \sim \mathrm{LogN}\left(\mu_2, \sigma_2^2\right). \tag{439}$$

Assume that the copula is lognormal, i.e. the grades (U_1, U_2) of (P_1, P_2) have the following joint distribution (not a typo, why?):

$$\begin{pmatrix} \Phi^{-1}(U_1) \\ \Phi^{-1}(U_2) \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), \tag{440}$$

where Φ denotes the cdf of the standard normal distribution. Assume that the current prices are $p_1 \equiv \mathrm{E}\{P_1\}$ and $p_2 \equiv \mathrm{E}\{P_2\}$.

Fix arbitrary values for the parameters $\left(\nu_1, \sigma_1^2, \mu_2, \sigma_2^2\right)$ and compute the current prices.

Consider the following allocation $\alpha_1 \equiv 1$, $\alpha_2 \equiv 2$. Simulate the distribution of the objective of an investor who is interested in final wealth.

Consider the previous allocation. Simulate the distribution of the objective of an investor who is interested in the P&L.

Consider the previous allocation and the following benchmark $\beta_1 \equiv 2$, $\beta_2 \equiv 1$. Simulate the distribution of the objective of an investor who is interested in beating the benchmark.

See the script S_InvestorsObjective.

7.2 Dominance

7.2.1 Strong dominance

Consider a bi-variate market $\mathbf{M}$. Define a market distribution and two portfolios such that one portfolio strongly dominates the other.

Consider the market $M_1 \equiv 1$ and

$$M_2 = \begin{cases} 2 \text{ with probability } 0.3 \\ 3 \text{ with probability } 0.7 \end{cases} \tag{441}$$

The allocation $\boldsymbol{\alpha} \equiv (0, 1)$ strongly dominates the allocation $\boldsymbol{\alpha} \equiv (1, 0)$.

7.2.2 Weak dominance

Consider a bi-variate market $\mathbf{M}$. Define a market distribution and two portfolios such that one portfolio weakly dominates the other.

Since strong dominance implies weak dominance, see the previous example.

7.3 Utility

7.3.1 Certainty equivalent interpretation

Express the meaning/intuition of the certainty equivalent of an allocation.

Hint. See somewhere on p.262.

The certainty-equivalent of an allocation is the risk-free amount of money that would make the investor as satisfied as the given risky allocation.

7.3.2 Certainty equivalent computation

Consider the copula in Exercise 7.1.1, but replace the marginal distributions as follows:

$$P_1 \sim \mathrm{N}\left(\mu_1, \sigma_1^2\right) \tag{442}$$

$$P_2 \sim \mathrm{N}\left(\mu_2, \sigma_2^2\right). \tag{443}$$

Consider the case where the objective is final wealth. Consider an exponential utility function:

$$u(\psi) \equiv a - be^{-\frac{\psi}{\zeta}}, \tag{444}$$

where $b > 0$.

Compute analytically the certainty equivalent as a function of a generic allocation vector (α_1, α_2). What is the effect of a and b?

Consider the utility function (444). As in (5.92) in Meucci (2005) expected utility reads:

$$\mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\} \equiv a - b\,\mathrm{E}\left\{e^{-\frac{\Psi_{\boldsymbol{\alpha}}}{\zeta}}\right\} = a - b\phi_{\Psi_{\boldsymbol{\alpha}}}\left(\frac{i}{\zeta}\right), \tag{445}$$

where ϕ denotes the characteristic function (1.12) of the objective. The inverse of (444) is:

$$\psi \equiv u^{-1}\left(\widetilde{u}\right) = -\zeta \ln\left(\frac{a - \widetilde{u}}{b}\right). \tag{446}$$

Therefore the certainty equivalent reads:

$$\begin{aligned} \mathrm{CE}\left(\boldsymbol{\alpha}\right) &\equiv u^{-1}\left(\mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\}\right) \\ &= -\zeta \ln\left(\phi_{\Psi_{\boldsymbol{\alpha}}}\left(\frac{i}{\zeta}\right)\right), \end{aligned} \tag{447}$$

as in (5.94) in Meucci (2005). The certainty equivalent is not affected by a and b. In other words, the certainty equivalent is not affected by positive affine transformations of the utility function.

To compute the certainty equivalent as a function of the allocation vector we recall from Exercise 7.1.1 that lognormal and normal copulas are the same, and we notice that normal marginals with a normal copula give rise to a normal joint distribution:

$$\mathbf{P} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{448}$$

where

$$\boldsymbol{\mu} \equiv \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} \equiv \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{449}$$

Therefore as in (5.144) we obtain:

$$\mathrm{CE}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'\boldsymbol{\mu} - \frac{1}{2\zeta}\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}. \tag{450}$$

Is the certainty equivalent corresponding to (444) positive homogeneous? If so, compute the contribution to the certainty equivalent from the two securities as defined by Euler's formula.

The certainty equivalent corresponding to (444) is not positive homogeneous. This is not surprising: indeed the class of utility functions that give rise to positive homogenous certainty equivalents is the power class, see (5.114) in Meucci (2005).

7.3.3 Arrow-Pratt aversion and prospect theory

Consider the prospect-theory utility function:

$$u(\psi) \equiv a + b\,\mathrm{erf}\left(\frac{\psi - \psi_0}{\sqrt{2\eta}}\right), \tag{451}$$

where $b > 0$.

Plot the utility function for different values of η and ψ_0.

Compute the Arrow-Pratt risk aversion (5.121) in Meucci (2005) implied by the utility (451).

Deriving $(B.75)$ in Meucci (2005), we obtain:

$$\frac{d}{dx}\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}. \tag{452}$$

Hence

$$u'(\psi) \equiv b\sqrt{\frac{2}{\pi\eta}}e^{-\left(\frac{\psi-\psi_0}{\sqrt{2\eta}}\right)^2} \tag{453}$$

and

$$u''(\psi) \equiv -\frac{2b}{\eta}\frac{1}{\sqrt{\pi}}e^{-\left(\frac{\psi-\psi_0}{\sqrt{2\eta}}\right)^2}\left(\frac{\psi-\psi_0}{\sqrt{2\eta}}\right). \tag{454}$$

Therefore

$$A(\psi) \equiv \frac{-u''(\psi)}{u'(\psi)} = \frac{\psi-\psi_0}{\eta},. \tag{455}$$

For the interpretation of this result see (5.124) in Meucci (2005) and comments thereafter.

7.4 VaR

7.4.1 VaR in elliptical markets

Consider an N-dimensional market that as in (2.144) in Meucci (2005) is uniformly distributed on an ellipsoid (surface and internal points):

$$\mathbf{M} \sim \mathrm{U}(\mathcal{E}_{\boldsymbol{\mu},\boldsymbol{\Sigma}}). \tag{456}$$

Write the quantile index $\mathrm{Q}_c(\boldsymbol{\alpha})$ of the objective (5.10) as defined in (5.159) in Meucci (2005) as a function of the allocation.

We can represent (456) in the notation (2.268) in Meucci (2005) as follows:

$$\mathbf{M} \sim \mathrm{El}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_N^{\mathrm{U}}\right), \tag{457}$$

where g_N^{U} is provided in (2.263). From (2.270) in Meucci (2005) we obtain

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M} \sim \mathrm{El}\left(\boldsymbol{\alpha}'\boldsymbol{\mu}, \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}, g_1^{\mathrm{U}}\right) \tag{458}$$

Therefore

$$\Psi_{\boldsymbol{\alpha}} \overset{d}{=} \boldsymbol{\alpha}'\boldsymbol{\mu} + \sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}X \tag{459}$$

where

$$X \equiv \mathrm{El}\left(0, 1, g_1^{\mathrm{U}}\right), \tag{460}$$

for some generator g_1^{U} induced by the N-dimensional uniform distribution.

Therefore

$$\mathrm{Q}_c(\boldsymbol{\alpha}) \equiv Q_{\Psi_{\boldsymbol{\alpha}}}(1-c) = Q_{\boldsymbol{\alpha}'\boldsymbol{\mu}+\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}X}(1-c). \tag{461}$$

Using $(T1.16)$ in the technical appendices at `symmys.com` > `Book` > `Downloads` we obtain:

$$\mathrm{Q}_c(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'\boldsymbol{\mu} + \sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}\gamma_c, \tag{462}$$

where the scalar

$$\gamma_c \equiv Q_X(1-c) \tag{463}$$

can be evaluated numerically.

Use the above results to factor $\mathrm{Q}_c(\boldsymbol{\alpha})$ in terms of its marginal contributions.
Hint. Compare with (5.189) in Meucci (2005).

Deriving (462) we obtain

$$\frac{\partial\,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}}=\boldsymbol{\mu}+\frac{\gamma_c}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}}\boldsymbol{\Sigma}\boldsymbol{\alpha}. \tag{464}$$

Therefore the marginal contributions $\mathbf{C}$ read:

$$\begin{aligned}\mathbf{C} &\equiv \operatorname{diag}(\boldsymbol{\alpha})\frac{\partial\,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}} \\ &= \operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{\mu}+\frac{\gamma_c}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}}\operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{\Sigma}\boldsymbol{\alpha}.\end{aligned} \tag{465}$$

It is immediate to check that

$$\mathrm{Q}_c(\boldsymbol{\alpha})\equiv\sum_{n=1}^{N}C_n, \tag{466}$$

see (5.67) and (5.190) in Meucci (2005).

Consider the case $N\equiv 3$. Generate randomly the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Generate a sample of $J\equiv 1,000$ simulations of the market (456).

Generate a random allocation vector $\boldsymbol{\alpha}$. Set $c\equiv 0.95$ and compute $\mathrm{Q}_c(\boldsymbol{\alpha})$ as the sample counterpart of (5.159) in Meucci (2005).

Compute the marginal contributions to $\mathrm{Q}_c(\boldsymbol{\alpha})$ from each security in terms of the empirical derivative of $\mathrm{Q}_c(\boldsymbol{\alpha})$:

$$\frac{\partial\,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial\alpha_n}\approx\frac{\mathrm{Q}_c\left(\boldsymbol{\alpha}+\epsilon\boldsymbol{\delta}^{(n)}\right)-\mathrm{Q}_c(\boldsymbol{\alpha})}{\epsilon}, \tag{467}$$

where $\mathrm{Q}_c(\mathbf{x})$ is calculated as in the previous point; $\boldsymbol{\delta}^{(n)}$ is the Kronecker delta (A.15) in Meucci (2005); and ϵ is a small number, as compared with the average size of the entries of $\boldsymbol{\alpha}$.

Display the result using the built-in plotting function `bar`.

Use the result above to compute $\mathrm{Q}_c(\boldsymbol{\alpha})$ in a different way, i.e. semi-analytically.

Hint. You will have to compute the quantile of the standardized univariate generator, use the simulations generated above.

Use the previous results to compute the marginal contributions to $\mathrm{Q}_c(\boldsymbol{\alpha})$ from each security. Display the result using the built-in plotting function `bar`.

See the script `S_VaRContributionsUniform`. In particular, to generate J scenarios from (469) you can use the following approach. Consider the uniform distribution on the N-dimensional hypercube:

$$\mathbf{X}\sim\mathrm{U}([-1,1]\times\cdots\times[-1,1]). \tag{468}$$

The entries of $\mathbf{X}$ are independent and therefore (468) can easily be simulated. Now consider the uniform distribution on the N-dimensional unit hypersphere:

$$\mathbf{Y} \sim \mathrm{U}\left(\mathcal{E}_{\mathbf{0},\mathbf{I}}\right). \tag{469}$$

To generate a sample of size J from (469) generate a sample of size $\widetilde{J}$ from (468). Then use

$$\mathbf{Y} \stackrel{d}{=} \mathbf{X}|\ \|\mathbf{X}\| \leq 1. \tag{470}$$

To set the number of simulations $\widetilde{J}$ use (A.78) in Meucci (2005). To generate a sample of size J from (456) apply (2.270) in Meucci (2005) to the sample from (469).

To generate a sample of size J from (469) more efficiently you can proceed as follows (courtesy Xiaoyu Wang, CIMS-NYU). In this function, we represent (469) as in (2.259)-(2.260) in Meucci (2005):

$$\mathbf{Y} \stackrel{d}{=} R\mathbf{U}. \tag{471}$$

In this expression $\mathbf{U}$ is uniform on the surface of the unit sphere and R is a suitable radial distribution independent of $\mathbf{U}$.

To generate J scenarios of $\mathbf{U}$, you can use

$$\mathbf{U} \stackrel{d}{=} \mathbf{Z}/\left\|\mathbf{Z}\right\|, \tag{472}$$

where

$$\mathbf{Z} \sim \mathrm{N}\left(\mathbf{0}_{N\times 1}, \mathbf{I}_{N\times N}\right); \tag{473}$$

this follows from the last expression in (2.260) and the fact that the normal distribution is elliptical.

To generate J scenarios of R, notice that, for a given radius r, the radial density must be proportional to r^{N-1}. Indeed, the infinitesimal volume surrounding the surface of the sphere of radius r is proportional to r^{N-1}. Therefore, pinning down the normalization constant, we obtain:

$$f_R\left(r\right) = \frac{r^{N-1}}{N-1}. \tag{474}$$

From (474), the radial cdf reads:

$$F_R\left(r\right) = r^N. \tag{475}$$

Inverting the cdf, we obtain the quantile function:

$$Q_R\left(u\right) = u^{1/N}. \tag{476}$$

Hence from (2.25)-(2.26) in Meucci (2005) we obtain:

$$R \stackrel{d}{=} W^{1/N}, \tag{477}$$

where $W \sim \mathrm{U}\left([0,1]\right)$ is uniform on the unit interval and is independent of $\mathbf{U}$.

To generate J scenarios from (471) it now suffices to multiply the scenarios for $\mathbf{U}$ by the respective scenarios for R.

7.4.2 Cornish-Fisher approximation of VaR

Assume that the investor's objective is lognormally distributed:

$$\Psi_{\boldsymbol{\alpha}} \sim \text{LogN}\left(\mu_{\boldsymbol{\alpha}}, \sigma^2_{\boldsymbol{\alpha}}\right), \tag{478}$$

where $\mu_{\boldsymbol{\alpha}} \equiv 0.05$ and $\sigma_{\boldsymbol{\alpha}} \equiv 0.05$.

Plot the true quantile-based index of satisfaction $\mathrm{Q}_c(\alpha)$ against the Cornish-Fisher approximation (5.179) in Meucci (2005) as a function of the confidence level $c \in (0,1)$.

See script `S_CornishFisher`.

7.4.3 Extreme value theory approximation of VaR

Assume that the objective is t distributed:

$$\Psi_{\boldsymbol{\alpha}} \sim \text{St}\left(\nu, \mu_{\boldsymbol{\alpha}}, \sigma^2_{\boldsymbol{\alpha}}\right), \tag{479}$$

where $\nu \equiv 7$, $\mu_{\boldsymbol{\alpha}} \equiv 1$, $\sigma^2_{\boldsymbol{\alpha}} \equiv 4$.

Plot the true quantile-based index of satisfaction $\mathrm{Q}_c(\boldsymbol{\alpha})$ for $c \in [0.950, 0.999]$.

Hint. Use the built-in function `tinv`.

Generate Monte Carlo simulations from (479) and superimpose the plot of the sample counterpart of $\mathrm{Q}_c(\alpha)$ for $c \in [0.950, 0.999]$.

Consider the threshold:

$$\widetilde{\psi} \equiv \mathrm{Q}_{0.95}(\boldsymbol{\alpha}). \tag{480}$$

Superimpose the plot of the EVT fit (5.186) in Meucci (2005) for $c \in [0.950, 0.999]$.

Hint. Estimate the parameters ξ and v using the built-in function `xi_v = gpfit(Excess)`, where `Excess` are the realizations of the random variable

$$Z \equiv \widetilde{\psi} - \Psi_{\boldsymbol{\alpha}} | \Psi_{\boldsymbol{\alpha}} \leq \widetilde{\psi}. \tag{481}$$

Indeed the cdf of Z satisfies

$$\begin{aligned}
F_Z(z) &\equiv \mathbb{P}\{Z \leq z\} \\
&= \mathbb{P}\left\{\widetilde{\psi} - \Psi_{\boldsymbol{\alpha}} \leq z | \Psi_{\boldsymbol{\alpha}} \leq \widetilde{\psi}\right\} \\
&= \mathbb{P}\left\{\Psi_{\boldsymbol{\alpha}} \geq \widetilde{\psi} - z | \Psi_{\boldsymbol{\alpha}} \leq \widetilde{\psi}\right\} \\
&= 1 - \mathbb{P}\left\{\Psi_{\boldsymbol{\alpha}} \leq \widetilde{\psi} - z | \Psi_{\boldsymbol{\alpha}} \leq \widetilde{\psi}\right\} \\
&\equiv 1 - L_{\widetilde{\psi}}(z),
\end{aligned} \tag{482}$$

where in the last row we used (5.182) in Meucci (2005). From (5.184) in Meucci (2005) we obtain

$$F_Z(z) \approx G_{\xi,v}(z), \tag{483}$$

which is the expression that `gpfit` attempts to fit.

See script `S_EVT`.

7.5 Expected shortfall

7.5.1 Expected shortfall in elliptical markets

Assume that the market is multivariate t distributed:

$$\mathbf{M} \sim \mathrm{St}\left(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{484}$$

Write the expected shortfall $\mathrm{ES}_c\left(\boldsymbol{\alpha}\right)$ defined in (5.207) in Meucci (2005) as a function of the allocation.

From (484) and (2.195) in Meucci (2005) we obtain

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M} \sim \mathrm{St}\left(\nu, \alpha'\boldsymbol{\mu}, \alpha'\boldsymbol{\Sigma}\alpha\right), \tag{485}$$

or

$$\Psi_{\boldsymbol{\alpha}} \stackrel{d}{=} \boldsymbol{\alpha}'\boldsymbol{\mu} + \sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}X, \tag{486}$$

where

$$X \sim \mathrm{St}\left(\nu, 0, 1\right). \tag{487}$$

From (5.207) in Meucci (2005) we obtain

$$\begin{aligned} \mathrm{ES}_c\left(\boldsymbol{\alpha}\right) &= \frac{1}{1-c}\int_0^{1-c} Q_{\Psi_{\boldsymbol{\alpha}}}\left(s\right) ds \\ &= \frac{1}{1-c}\int_0^{1-c} \left[\boldsymbol{\alpha}'\boldsymbol{\mu} + \sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}Q_X\left(s\right)\right] ds. \end{aligned} \tag{488}$$

and thus

$$\mathrm{ES}_c\left(\boldsymbol{\alpha}\right) = \boldsymbol{\alpha}'\boldsymbol{\mu} + \sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}\zeta_c, \tag{489}$$

where

$$\zeta_c \equiv \frac{1}{1-c}\int_0^{1-c} Q_X\left(s\right) ds. \tag{490}$$

This scalar can be evaluated as the numerical integral of the quantile function of the standard univariate t distribution.

Use the previous results to factor the $\mathrm{ES}_c\left(\boldsymbol{\alpha}\right)$ in terms of its marginal contributions.

Hint. Compare with (5.236) in Meucci (2005).

Deriving (489) we obtain

$$\frac{\partial\,\mathrm{ES}_c\left(\boldsymbol{\alpha}\right)}{\partial\boldsymbol{\alpha}} = \boldsymbol{\mu} + \frac{\zeta_c}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}}\boldsymbol{\Sigma}\boldsymbol{\alpha}. \tag{491}$$

Therefore the marginal contributions $\mathbf{C}$ read:

$$\begin{aligned} \mathbf{C} &\equiv \operatorname{diag}(\boldsymbol{\alpha}) \frac{\partial \operatorname{ES}_c(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \\ &= \operatorname{diag}(\boldsymbol{\alpha}) \boldsymbol{\mu} + \frac{\zeta_c}{\sqrt{\boldsymbol{\alpha}' \boldsymbol{\Sigma} \boldsymbol{\alpha}}} \operatorname{diag}(\boldsymbol{\alpha}) \boldsymbol{\Sigma} \boldsymbol{\alpha}. \end{aligned} \tag{492}$$

It is immediate to check that

$$\operatorname{ES}_c(\boldsymbol{\alpha}) \equiv \sum_{n=1}^{N} C_n, \tag{493}$$

see see (5.67) in Meucci (2005).

Assume $N \equiv 40$ and $\nu \equiv 5$. Generate randomly the parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and the allocation $\boldsymbol{\alpha}$. Then generate $J \equiv 10,000$ Monte Carlo scenarios from the market distribution (484).

Generate a random allocation vector $\boldsymbol{\alpha}$. Set $c \equiv 0.95$ and compute $\operatorname{ES}_c(\boldsymbol{\alpha})$ as the sample counterpart of (5.208) in Meucci (2005).

Compute the marginal contributions to $\operatorname{ES}_c(\boldsymbol{\alpha})$ from each security as the sample counterpart of (5.238) in Meucci (2005). Display the result in a subplot using the built-in plotting function `bar`.

Use the previous results to compute $\operatorname{ES}_c(\boldsymbol{\alpha})$ in a different way, i.e. semi-analytically. Never at any stage use simulations.

Hint. Use the numerical integration function quad applied to the built-in quantile function `tinv`.

Compute the marginal contributions to $\operatorname{ES}_c(\boldsymbol{\alpha})$ from each security using previous results. Never at any stage use simulations. Display the result in a second subplot using the built-in plotting function `bar`.

See script `S_ESContributionsT`.

7.5.2 Expected shortfall and linear factor models

Assume a linear factor model for the market

$$\mathbf{M} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}, \tag{494}$$

where $\mathbf{B}$ is a $N \times K$ matrix with entries of the order of the unit; $\mathbf{F}$ is a K-dimensional vector; $\mathbf{U}$ is a N-dimensional vector; and

$$\begin{pmatrix} \ln \mathbf{F} \\ \ln(\mathbf{U} + \mathbf{a}) \end{pmatrix} \sim \operatorname{St}(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{495}$$

where $\boldsymbol{\mu} \equiv \mathbf{0}$, $\mathbf{a}$ is such that $\operatorname{E}\{\mathbf{U}\} \equiv \mathbf{0}$ and

$$\boldsymbol{\Sigma} \equiv \begin{pmatrix} \epsilon \boldsymbol{\Sigma}^f & \mathbf{0} \\ \mathbf{0} & \epsilon^2 \boldsymbol{\Sigma}^u \end{pmatrix}, \tag{496}$$

with $\mathbf{\Sigma}^f$ a correlation matrix and $\mathbf{\Sigma}^u$ a Toepliz correlation matrix

$$\Sigma^u_{n,m} \equiv e^{-\gamma|n-m|}, \tag{497}$$

with $\epsilon \ll 1$ and γ arbitrary.

Assume $N \equiv 30$, $K \equiv 10$ and $\nu \equiv 10$. Generate randomly the parameters in $\mathbf{\Sigma}$ and the allocation $\boldsymbol{\alpha}$. Then generate $J \equiv 10,000$ Monte Carlo scenarios from the market distribution (494).

Set $c \equiv 0.95$ and compute $\mathrm{ES}_c(\boldsymbol{\alpha})$ as the sample counterpart of (5.208) in Meucci (2005).

Compute the K marginal contributions to $\mathrm{ES}_c(\boldsymbol{\alpha})$ from each factor and the one aggregate contribution from all the residuals, as the sample counterpart of (5.238) in Meucci (2005) adapted to the factors. Display the result in a subplot using the built-in plotting function `bar`.

Hint. Represent the objective as a linear function

$$\Psi \equiv \boldsymbol{\beta}\mathbf{F} + u. \tag{498}$$

See script `S_ESContributionsFacts`.

8 Static portfolio management

8.1 Mean-variance

8.1.1 Mean-variance pitfalls: two-step approach

Assume a market of $N \equiv 4$ stocks and all possible zero-coupon bonds. The weekly compounded returns of the stocks are market invariants with the following distribution:

$$\mathbf{C}_{t,\widetilde{\tau}} \sim \mathrm{N}(\mu, \mathbf{\Sigma}). \tag{499}$$

Estimate the matrix $\mathbf{\Sigma}$ and the vector $\boldsymbol{\mu}$ from the time series of weekly prices in the attached database `StockSeries`. To do this, shrink the sample mean as in (4.138) in Meucci (2005), where the target is the null vector and the shrinkage factor is set as $\alpha \equiv 0.1$. Similarly, shrink as in (4.160) in Meucci (2005) the sample covariance to a suitable multiple of the identity by a factor $\alpha \equiv 0.1$.

Assume that the weekly changes in yield to maturity for the bond market are fully codependent, i.e. co-monotonic. In other words, assume that the copula of any pairs of weekly yield changes is (2.106) in Meucci (2005). Also, assume that they have the following marginal distribution:

$$\Delta_{\widetilde{\tau}} Y^{(\upsilon)} \sim \mathrm{N}\left(0, \left(\frac{20 + 1.25\upsilon}{10,000}\right)^2\right), \tag{500}$$

where υ denotes the generic time to maturity (measuring time in years).

Assume that the bonds and the stock market are independent. Assume that the current stock prices are the last set of prices in the time series. Restrict your attention to bonds with times to maturity 4, 5, 10, 52 and 520 weeks, and assume that the current yield curve, as defined in (3.30) in Meucci (2005) is flat at 4%.

Produce joint simulations of the four stock and five bond prices at the investment horizon τ of four weeks.

Assume that the investor considers as his market one single bond with time to maturity $\upsilon \equiv$ five weeks and all the stocks.

Determine numerically the mean-variance inputs, namely expected prices and covariance of prices (*not* returns).

Determine analytically the mean-variance inputs, namely expected prices and covariance of prices (*not* returns) and compare with their numerical counterpart.

Assume that the investor's objective is final wealth. Suppose that his budget is $w \equiv 100$. Assume that the investor cannot short-sell his securities, i.e. the allocation vector cannot include negative entries.

Compute the mean-variance efficient frontier as represented by a grid of 40 portfolios whose expected values are equally spaced between the expected value of the minimum variance portfolio and the largest expected value among the portfolios composed of only one security.

Assume that the investor's satisfaction is the certainty equivalent associated with an exponential function

$$u(\psi) \equiv -e^{-\frac{1}{\zeta}\psi}, \tag{501}$$

where $\zeta \equiv 10$.

Compute the optimal allocation according to the two-step mean-variance framework.

Hints

- Do not use portfolio weights and returns. Instead, use number of securities and prices.
- Given the no-short-sale constraint, the minimum variance portfolio cannot be computed analytically, as in (6.99) in Meucci (2005): use `quadprog` to compute it numerically.
- Given the no-short-sale constraint, the frontier cannot be computed analytically, as in (6.97)-(6.100) in Meucci (2005): use `quadprog` to compute it numerically.

See the script `S_MVOptimization`.

8.1.2 Mean-variance pitfalls: horizon effect

Consider the stock market described in the previous exercise and the investment horizon τ of one day.

Determine analytically the mean-variance inputs in terms of weights and returns, namely expected linear returns and covariance of linear returns and compare with their numerical counterpart.

Assume that the investor's objective is final wealth. Suppose that the budget is $b \equiv 100$. Assume that the investor cannot short-sell the securities, i.e. the allocation vector cannot include negative entries.

Compute the mean-variance efficient frontier in terms of portfolio weights as represented by a grid of 40 portfolios whose expected linear returns are equally spaced between the expected value of the linear return on the minimum variance portfolio and the largest expected value among the linear returns of all the securities.

Assume that the investor's satisfaction is the certainty equivalent associated with an exponential function

$$u(\psi) \equiv -e^{-\frac{1}{\zeta}\psi}, \tag{502}$$

where $\zeta \equiv 10$.

Compute the optimal allocation according to the two-step mean-variance framework.

Repeat the above steps an the investment horizon τ of four years

See the script `S_MVHorizon`.

8.1.3 Benchmark driven allocation

Consider the market described in Exercise 8.1.1, namely one single bond with time to maturity $\upsilon \equiv$ five weeks and all the stocks.

Produce joint simulations of the four stock and the one bond prices at the investment horizon τ of four weeks.

Determine numerically the mean-variance inputs in terms of weights and returns, namely expected linear returns and covariance of linear returns.

Determine analytically the mean-variance inputs in terms of weights and returns, namely expected linear returns and covariance of linear returns and compare with their numerical counterpart.

Assume first that the investor's objective is final wealth. Suppose that the budget is $b \equiv 100$. Assume that the investor cannot short-sell the securities, i.e. the allocation vector cannot include negative entries.

Compute the mean-variance efficient frontier in terms of portfolio weights as represented by a grid of 40 portfolios whose expected linear returns are equally spaced between the expected value of the linear return on the minimum variance portfolio and the largest expected value among the linear returns of all the securities.

Now assume that the investor's objective is wealth relative to an equal-weight benchmark, i.e. a benchmark that invests an equal amount of money in each security in the market.

Compute the mean-variance efficient frontier in terms of portfolio weights as represented by a grid of 40 portfolios whose expected linear returns are equally spaced between the expected value of the linear return on the minimum variance portfolio and the largest expected value among the linear returns of all the securities.

Project the two efficient frontiers computed above in the expected outperformance/tracking error plane.

For the benchmark-driven investor, the objective is (6.170) in Meucci (2005) or

$$\Phi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}' \mathbf{K} \mathbf{P}_{T+\tau}, \tag{503}$$

where

$$\mathbf{K} \equiv \mathbf{I} - \frac{\mathbf{p}_T \boldsymbol{\beta}'}{\boldsymbol{\beta}' \mathbf{p}_T} \tag{504}$$

and $\boldsymbol{\beta}$ is the benchmark.

In particular, notice that $\mathbf{K}$ is singular: the columns of $\mathbf{K}$ span a vector space of dimension $N-1$. Any vector orthogonal to all the column of $\mathbf{K}$ is spanned by the benchmark. Therefore $\Phi_{\boldsymbol{\beta}} = \boldsymbol{\beta}' \mathbf{K} \mathbf{P}_{T+\tau} \equiv 0$, which implies that the benchmark has zero expected outperformance and zero tracking error, see Figure 6.21 in Meucci (2005).

For a budget b, the return-based objective is defined as:

$$\begin{aligned} \frac{\Phi_{\boldsymbol{\alpha}}}{b} &= \frac{\boldsymbol{\alpha}' \mathbf{K} \mathbf{P}_{T+\tau}}{\boldsymbol{\alpha}' \mathbf{p}_T} = \frac{\boldsymbol{\alpha}' \mathbf{P}_{T+\tau}}{\boldsymbol{\alpha}' \mathbf{p}_T} - \frac{\boldsymbol{\alpha}' \mathbf{p}_T \boldsymbol{\beta}' \mathbf{P}_{T+\tau}}{(\boldsymbol{\alpha}' \mathbf{p}_T)(\boldsymbol{\beta}' \mathbf{p}_T)} \\ &= \left[\frac{\boldsymbol{\alpha}' \mathbf{P}_{T+\tau}}{\boldsymbol{\alpha}' \mathbf{p}_T} - 1 \right] - \left[\frac{\boldsymbol{\beta}' \mathbf{P}_{T+\tau}}{\boldsymbol{\beta}' \mathbf{p}_T} - 1 \right] \\ &= L_{\boldsymbol{\alpha}} - L_{\boldsymbol{\beta}}, \end{aligned} \tag{505}$$

where $L_{\boldsymbol{\alpha}}$ and $L_{\boldsymbol{\beta}}$ are the linear returns of the portfolio and the benchmark, respectively.

Given that the constraints are linear, we resort to the dual formulation of the return-based mean-variance problem, which reads

$$\begin{aligned} \boldsymbol{\alpha}(e) &\equiv \underset{\substack{\boldsymbol{\alpha}' \mathbf{p}_T \equiv b \\ \boldsymbol{\alpha} \geq \mathbf{0} \\ \mathrm{E}\{L_{\boldsymbol{\alpha}} - L_{\boldsymbol{\beta}}\} = e}}{\operatorname{argmin}} \left\{ \mathrm{Var}\left\{ L_{\boldsymbol{\alpha}} - L_{\boldsymbol{\beta}} \right\} \right\} \\ &= \underset{\substack{\boldsymbol{\alpha}' \mathbf{p}_T \equiv b \\ \boldsymbol{\alpha} \geq \mathbf{0} \\ \mathrm{E}\{L_{\boldsymbol{\alpha}} - L_{\boldsymbol{\beta}}\} = e}}{\operatorname{argmin}} \left\{ \mathrm{Var}\left\{ L_{\boldsymbol{\alpha}} \right\} + \mathrm{Var}\left\{ L_{\boldsymbol{\beta}} \right\} - 2\,\mathrm{Cov}\left\{ L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}} \right\} \right\} \\ &= \underset{\substack{\boldsymbol{\alpha}' \mathbf{p}_T \equiv b \\ \boldsymbol{\alpha} \geq \mathbf{0} \\ \mathrm{E}\{L_{\boldsymbol{\alpha}}\} = e}}{\operatorname{argmin}} \left\{ \frac{1}{2} \mathrm{Var}\left\{ L_{\boldsymbol{\alpha}} \right\} - \mathrm{Cov}\left\{ L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}} \right\} \right\} \end{aligned} \tag{506}$$

Using (T6.85) in the technical appendices at `symmys.com` > `Book` > `Downloads` we obtain in terms of the portfolio weights $\mathbf{w}$, the benchmark weights $\mathbf{w}_b$, and the securities returns $\mathbf{L}$

$$\mathbf{w}(e) \equiv \underset{\substack{\mathbf{w}' \mathbf{1} \equiv 1 \\ \mathbf{w} \geq \mathbf{0} \\ \mathbf{w}' \mathrm{E}\{\mathbf{L}\} = e}}{\operatorname{argmin}} \left\{ \frac{1}{2} \mathbf{w}' \,\mathrm{Cov}\left\{ \mathbf{L} \right\} \mathbf{w} - \mathbf{w}_b' \,\mathrm{Cov}\left\{ \mathbf{L} \right\} \mathbf{w} \right\}. \tag{507}$$

This is the input for the relative optimization in the script `S_MVBenchmark`.

8.1.4 Mean-variance for derivatives

Consider a market of at-the-money call options on the underlyings whose daily time series are provided in the file `DB` (30, 91 and 182 are the time to expiry). Assume that the investment horizon is two days.

Fit a normal distribution to the invariants, namely the log-changes in the underlying in the file `DB` and the log-changes in the respective at-the-money implied volatilities in the file `DB`.

Project this distribution analytically to the horizon.

Generate simulations for the sources of risk, namely underlying prices and implied volatilities at the horizon.

Price the above simulations through the full Black-Scholes formula, assuming no skewness correction for the implied volatilities and a constant risk-free rate at 4%.

Compute the distribution of the linear returns, as represented by the simulations: the current prices of the options can be obtained similarly to the prices at the horizon by assuming that the current values of underlying and implied volatilities are the last observations in the database.

Compute numerically the mean-variance inputs.

Compute the mean-variance efficient frontier in terms of relative weights, assuming the standard long-only and full investment constraints.

Plot the efficient frontier in the plane of weights and standard deviation.

Hint. Compare with Exercise 5.6: here we are making unrealistic dimension reduction assumptions on the dynamics of the implied volatility surface.

See the script `S_MVCalls`.

9 Dynamic strategies

The exercises is this section are placed into a broader context in Meucci (2010e), freely available online at ssrn.com.

Assume there are two securities: a risk free asset whose value evolves deterministically with an exponential growth at a constant rate r

$$\ln D_{t+\delta t} = \ln D_t + r\delta t, \tag{508}$$

and a risky asset whose value P_t follows a geometric Brownian motion with drift μ and volatility σ

$$\ln P_{t+\delta t} = \ln P_t + (\mu - \frac{\sigma^2}{2})\delta t + \sigma\sqrt{\delta t} Z_t, \tag{509}$$

where $Z_t \sim \mathrm{N}(0,1)$ are independent across non-overlapping time steps.

Assume the current time is $t \equiv 0$ and the investment horizon is $t \equiv \tau$. Assume there is an initial budget

$$S_0 \text{ given.} \tag{510}$$

Consider a strategy that rebalances between the two assets throughout the investment period $[0, \tau]$

$$(\alpha_t, \beta_t)_{t \in [0,\tau]}, \tag{511}$$

where α_t denotes the number of units of the risky asset and β_t denotes the number of units of the risk-free asset. The value of the strategy is

$$S_t = \alpha_t P_t + \beta_t D_t, \tag{512}$$

and the strategy must be self-financing, i.e. whenever a rebalancing occurs

$$(\alpha_t, \beta_t) \mapsto (\alpha_{t+\delta t}, \beta_{t+\delta t}) \tag{513}$$

the following must hold true

$$S_{t+\delta t} \equiv \alpha_t P_{t+\delta t} + \beta_t D_{t+\delta t} \equiv \alpha_{t+\delta t} P_{t+\delta t} + \beta_{t+\delta t} D_{t+\delta t}, \tag{514}$$

The strategy (511) is determined equivalently by the weights

$$w_t \equiv \frac{\alpha_t P_t}{S_t}, \quad u_t \equiv \frac{\beta_t D_t}{S_t}. \tag{515}$$

Prove that the self-financing constraint (514) is equivalent to the weight of the risk-free asset being equal to

$$u_t \equiv 1 - w_t$$

and that therefore the whole strategy if fully determined by the free evolution of the weight w_t.

We denote by (w_t, u_t) the pre-trade weights and by $(\widetilde{w}_t, \widetilde{u}_t)$ the post-trade weights. Dividing both sides of (514) by the value of the strategy we obtain

$$\begin{aligned} w_{t+\delta t} + u_{t+\delta t} &= \frac{\alpha_t P_{t+\delta t} + \beta_t D_{t+\delta t}}{S_{t+\delta t}} \\ &= \frac{\alpha_t P_{t+\delta t} + \beta_t D_{t+\delta t}}{\alpha_t P_{t+\delta t} + \beta_t D_{t+\delta t}} = 1, \end{aligned} \tag{516}$$

and

$$\begin{aligned} \widetilde{w}_{t+\delta t} + \widetilde{u}_{t+\delta t} &= \frac{\alpha_{t+\delta t} P_{t+\delta t} + \beta_{t+\delta t} D_{t+\delta t}}{S_{t+\delta t}} \\ &= \frac{\alpha_{t+\delta t} P_{t+\delta t} + \beta_{t+\delta t} D_{t+\delta t}}{\alpha_{t+\delta t} P_{t+\delta t} + \beta_{t+\delta t} D_{t+\delta t}} = 1. \end{aligned} \tag{517}$$

9.1 Buy & hold

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010e), freely available online at ssrn.com.

Consider the buy & hold strategy, that invests the budget (510) in the two securities and never reallocates throughout the investment period $[0, \tau]$.

$$\alpha_t \equiv \alpha, \quad \beta_t \equiv \beta. \tag{518}$$

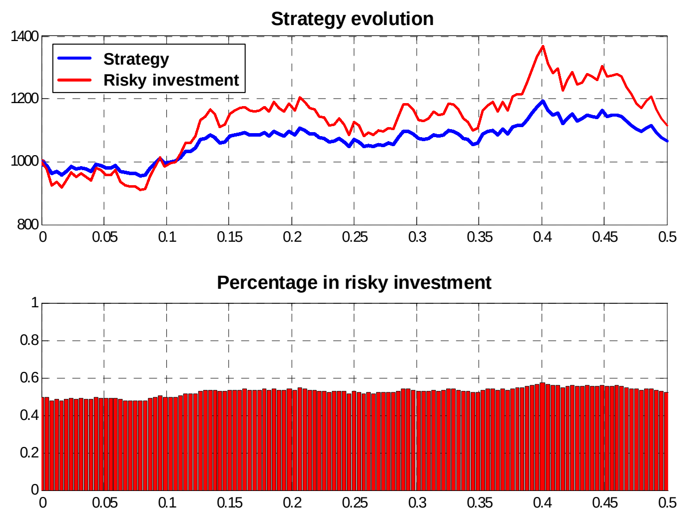

Figure 9: Buy & hold strategy: one path

Generate the deterministic exponential growth dynamics (508) at equally spaced time intervals τ.

Generate a large number of Monte Carlo paths from the geometric Brownian motion (509) at equally spaced time intervals $[0, \delta t, 2\delta t, \ldots, \tau]$.

Plot one path for the value of the risky asset $\{P_t\}_{t=0,\delta t,\ldots,\tau}$, and overlay the respective path $\{S_t\}_{t=0,\delta t,\ldots,\tau}$ for the value of the buy & hold strategy (518), see Figure 9

In a separate figure, plot the evolution of the portfolio weight (515) of the risky asset $\{w_t\}_{t=0,\delta t,\ldots,\tau}$ on that path, see Figure 9.

In a separate figure, scatter-plot the final payoff of the buy & hold strategy (518) over the payoff of the risky asset, and verify that the profile is linear, see Figure 10.

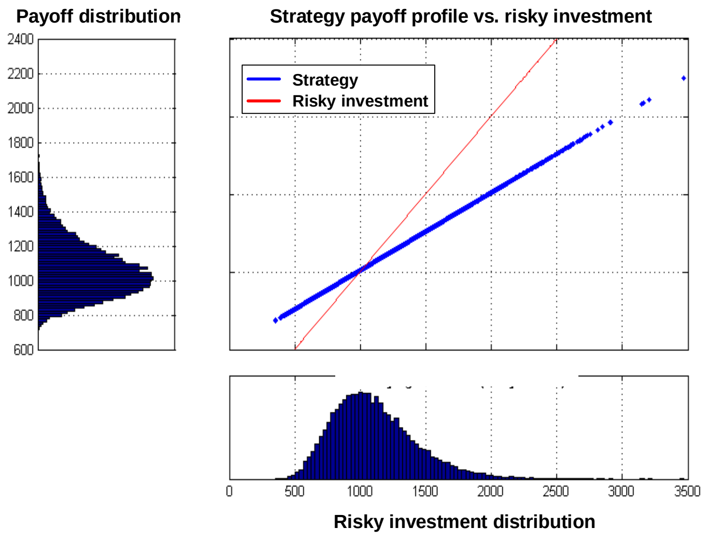

Figure 10: Buy & hold strategy: final payoff in many paths

See the script `S_BuyNHold`.

9.2 Utility maximization

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010e), freely available online at ssrn.com.

Consider the constant weight strategy, that invests the budget (510) in the two securities and keeps the weight of the risky security constant throughout the investment period $[0, \tau]$.

$$w_t \equiv w. \tag{519}$$

Prove that this strategy maximizes the expected final utility

$$w^*_{(\cdot)} \equiv \underset{S_0, w_{(\cdot)} \in \mathcal{C}}{\operatorname{argmax}} \left(\mathrm{E}\left\{ u\left(S_\tau\right)\right\}\right), \tag{520}$$

where u is the power utility function

$$u(s) = \frac{s^\gamma}{\gamma}, \tag{521}$$

with $\gamma < 1$.

Hint. As proved in Meucci (2010e) the strategy evolves as

$$\frac{dS_t}{S_t} = \left(r + w_t\left(\mu - r\right)\right) dt + w_t \sigma dB_t. \tag{522}$$

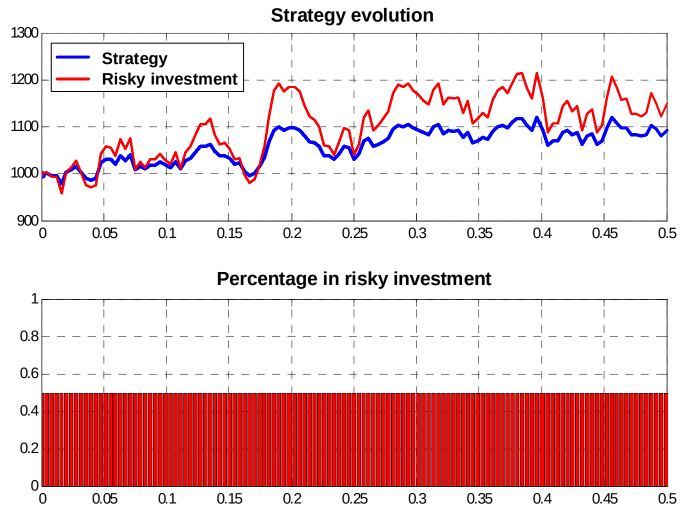

Figure 11: Constant weight dynamic strategy: one path

and the final value is lognormally distributed

$$S_\tau = S_0 e^{Y_{w(\cdot)}}, \tag{523}$$

where Y is a normal

$$Y_{w(\cdot)} \sim \mathrm{N}\left(m_{w(\cdot)}, s^2_{w(\cdot)}\right) \tag{524}$$

with expected value

$$m_{w(\cdot)} \equiv r\tau + \int_0^\tau \left((\mu - r) w_t - \frac{\sigma^2}{2} w_t^2\right) dt \tag{525}$$

and variance

$$s^2_{w(\cdot)} = \int_0^\tau \sigma^2 w_t^2 dt. \tag{526}$$

From (523) we obtain

$$\mathrm{E}\left\{u\left(S_\tau\right)\right\} = \frac{S_0^\gamma}{\gamma} \mathrm{E}\left\{e^{\gamma Y_{w(\cdot)}}\right\}. \tag{527}$$

Since $Y_{w(\cdot)}$ is normally distributed with expected value (525) and variance (526), it follows that $e^{\gamma Y}$ is lognormally distributed and thus from (1.98) in Meucci (2005) we obtain

$$\mathrm{E}\left\{e^{\gamma Y_{w(\cdot)}}\right\} = e^{\gamma\left[m_{w(\cdot)} + \frac{\gamma}{2} s^2_{w(\cdot)}\right]}. \tag{528}$$

Therefore, substituting (525) and (526), the optimal strategy (520) solves

$$w^*_{(\cdot)} \equiv \underset{w_{(\cdot)}}{\operatorname{argmax}} \left\{ \int_0^\tau \left(w_t \left(\mu - r\right) - w_t^2 \frac{\sigma^2}{2} \left(1 - \gamma\right) \right) dt \right\}. \tag{529}$$

The solution to this problem is the value that maximizes the integrand at each time. Therefore, the solution is the constant

$$w^*_{(\cdot)} \equiv w \equiv \frac{1}{1-\gamma} \frac{\mu - r}{\sigma^2}. \tag{530}$$

Generate the deterministic exponential growth dynamics (508) at equally spaced time intervals τ.

Generate a large number of Monte Carlo paths from the geometric Brownian motion (509) at equally spaced time intervals $[0, \delta t, 2\delta t, \ldots, \tau]$.

Plot one path for the value of the risky asset $\{P_t\}_{t=0,\delta t,\ldots,\tau}$, and overlay the respective path $\{S_t\}_{t=0,\delta t,\ldots,\tau}$ for the value of the constant weight strategy (519), see Figure 11.

In a separate figure, plot the (non) evolution of the portfolio weight (515) of the risky asset $\{w_t\}_{t=0,\delta t,\ldots,\tau}$ on that path, see Figure 11.

In a separate figure, scatter-plot the final payoff of the constant weight strategy (519) over the payoff of the risky asset, and verify that the profile is concave, see Figure 12.

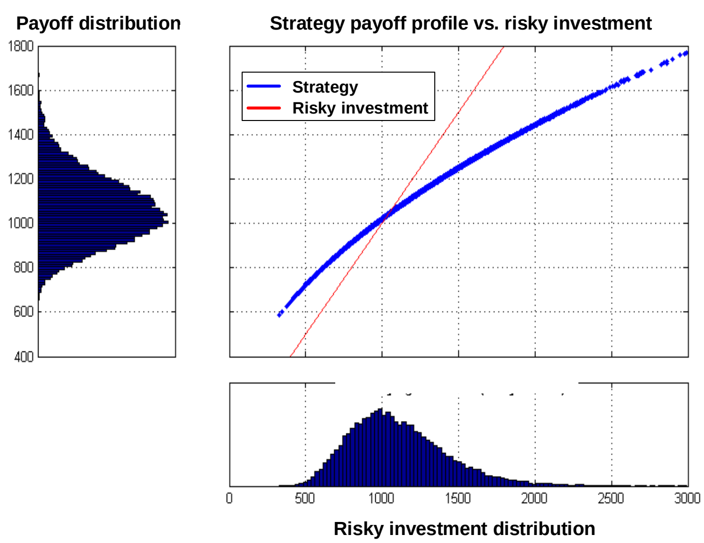

Figure 12: Constant weight dynamic strategy: final payoff in many paths

See the script S_UtlityMax.

9.3 Constant proportion portfolio insurance (CPPI)

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010e), freely available online at ssrn.com.

Consider the CPPI strategy, that invests the budget (510) as follows. First we specify a deterministically increasing floor F_t that satisfies the budget constraint

$$F_0 \leq S_0 \tag{531}$$

and grows to a guaranteed value H at the horizon

$$F_t \equiv He^{-r(\tau-t)}, \quad t \in [0, \tau]\,. \tag{532}$$

At all times t, for any level of the strategy S_t there is an excess cushion

$$C_t \equiv \max\left(0, S_t - F_t\right). \tag{533}$$

According to the CPPI, a constant multiple m of the cushion is invested in the risky asset, therefore obtaining the dynamic strategy's weight for the risky asset

$$\underline{w} \leq w_t \equiv \frac{mC_t}{S_t} \leq \overline{w}, \tag{534}$$

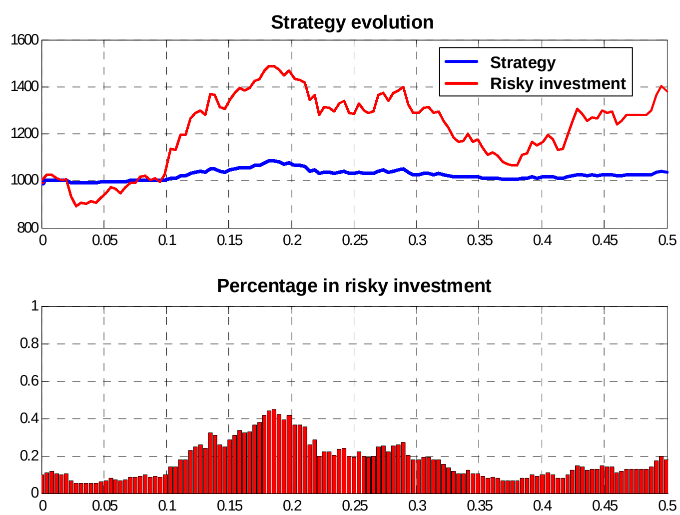

Figure 13: CPPI: one path

Generate the deterministic exponential growth dynamics (508) at equally spaced time intervals τ.

Generate a large number of Monte Carlo paths from the geometric Brownian motion (509) at equally spaced time intervals $[0, \delta t, 2\delta t, \ldots, \tau]$.

Plot one path for the value of the risky asset $\{P_t\}_{t=0,\delta t,\ldots,\tau}$, and overlay the respective path $\{S_t\}_{t=0,\delta t,\ldots,\tau}$ for the value of the CPPI strategy (534), see Figure 13.

In a separate figure, plot the evolution of the portfolio weight (515) of the risky asset $\{w_t\}_{t=0,\delta t,\ldots,\tau}$ on that path, see Figure 13.

In a separate figure, scatter-plot the final payoff of the CPPI strategy (534) over the payoff of the risky asset, and verify that the profile is convex, see Figure 14.

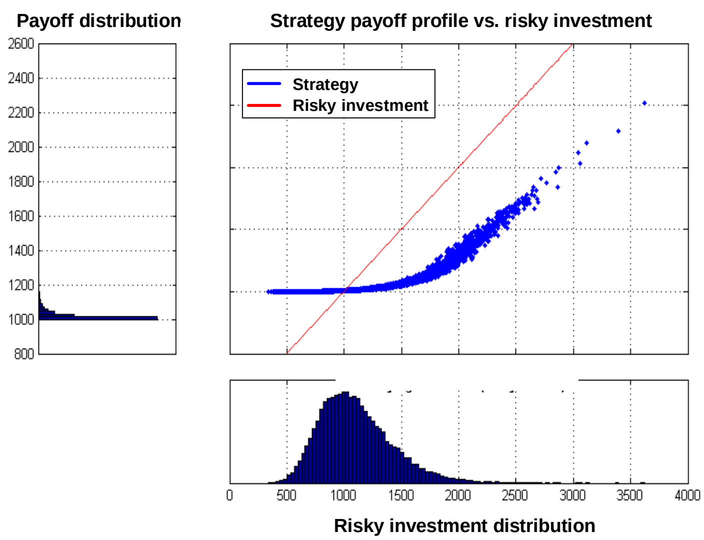

Figure 14: CPPI: final payoff in many paths

See the script S_CPPI.

9.4 Option based portfolio insurance (OBPI)

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010e), freely available online at ssrn.com.

Consider an arbitrary payoff at the investment horizon as a function of the risky asset

$$S_\tau = s(P_\tau). \tag{535}$$

and assume that you can compute the solution $G(t,p)$ of the following partial differential equation

$$\frac{\partial G}{\partial t}+\frac{\partial G}{\partial p}r+\frac{1}{2}\frac{\partial^2 G}{\partial p^2}\sigma^2 p^2-Gr=0, \tag{536}$$

with boundary condition

$$G(\tau,p)\equiv s(p), \quad p>0, \tag{537}$$

Prove that a strategy that invests an initial budget

$$S_0\equiv G(0,P_0) \tag{538}$$

and allocates dynamically the following weight in the risky asset

$$w_t\equiv\frac{P_t}{S_t}\frac{\partial G(t,P_t)}{\partial P_t}, \tag{539}$$

provides the desired payoff (535).

Hint. As proved in Meucci (2010e) the strategy evolves as

$$\frac{dS_t}{S_t}=(r+w_t(\mu-r))\,dt+w_t\sigma dB_t. \tag{540}$$

We want to prove that the following identity holds at all times, and in particular at $t\equiv\tau$

$$S_t\equiv G(t,P_t), \quad t\in[0,\tau]. \tag{541}$$

Indeed, using Ito's rule on $G(t,P_t)$, where P_t follows the geometric Brownian motion (509) yields

$$\begin{aligned} dG_t &= \frac{\partial G}{\partial t}dt+\frac{\partial G}{\partial P_t}dP_t+\frac{1}{2}\frac{\partial^2 G}{\partial P_t^2}(dP_t)^2 \\ &= \frac{\partial G}{\partial t}dt+\frac{\partial G}{\partial P_t}(\mu P_t dt+\sigma P_t dB)+\frac{1}{2}\frac{\partial^2 G}{\partial P_t^2}\sigma^2 P_t^2 dt \\ &= \left(\frac{\partial G}{\partial t}+\frac{\partial G}{\partial P_t}\mu P_t+\frac{1}{2}\frac{\partial^2 G}{\partial P_t^2}\sigma^2 P_t^2\right)dt+\frac{\partial G}{\partial P_t}\sigma P_t dB_t. \end{aligned} \tag{542}$$

Also, from (540) and (539) we obtain

$$\begin{aligned} dS_t &= S_t r dt+S_t w_t\left(\frac{dP_t}{P_t}-rdt\right) \\ &= S_t r dt+P_t\frac{\partial G}{\partial P_t}\left(\frac{dP_t}{P_t}-rdt\right) \\ &= S_t r dt+\frac{\partial G}{\partial P_t}(\mu P_t dt+\sigma P_t dB_t-P_t r dt) \end{aligned} \tag{543}$$

Therefore

$$\begin{aligned} d\left(G_t - S_t\right) &= \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial P_t}\mu P_t + \frac{1}{2}\frac{\partial^2 G}{\partial P_t^2}\sigma^2 P_t^2\right) dt + \frac{\partial G}{\partial P_t}\sigma P_t dB_t \\ &\quad - S_t r dt - \frac{\partial G}{\partial P_t}\left(\mu P_t dt + \sigma P_t dB_t - P_t r dt\right) \\ &= \left(\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial P_t^2}\sigma^2 P_t^2 - S_t r + \frac{\partial G}{\partial P_t} P_t r\right) dt \end{aligned} \tag{544}$$

Using (536) we finally obtain

$$\frac{d\left(G_t - S_t\right)}{\left(G_t - S_t\right)} = r dt, \tag{545}$$

Which means

$$\left(G_\tau - S_\tau\right) = \left(G_0 - S_0\right) e^{r\tau}. \tag{546}$$

Since from (538) we have $G_0 - S_0 = 0$, it follows that (541) holds true.

Assume that the payoff (535) is that of a call option with strike K

$$s\left(p\right) \equiv \max\left(0, p - K\right)$$

In this context the partial differential equation (536) was solved in Black and Scholes (1973)

$$G\left(t, p\right) = p\Phi\left(d_1\right) - e^{-r(\tau - t)} K\Phi\left(d_2\right), \tag{547}$$

where Φ is the cumulative distribution for the standard normal distribution and

$$d_1\left(t, p\right) \equiv \frac{1}{\sigma\sqrt{\tau - t}}\left(\ln\left(\frac{p}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\left(\tau - t\right)\right) \tag{548}$$

$$d_2\left(t, p\right) \equiv d_1\left(t, p\right) - \sigma\sqrt{\tau - t}. \tag{549}$$

From the explicit analytical expression (547) we can derive the expression for the weight (539) of the risky asset

$$w_t = \frac{P_t}{S_t}\Phi\left(d_1\left(t, P_t\right)\right). \tag{550}$$

Generate the deterministic exponential growth dynamics (508) at equally spaced time intervals τ.

Generate a large number of Monte Carlo paths from the geometric Brownian motion (509) at equally spaced time intervals $[0, \delta t, 2\delta t, \ldots, \tau]$.

Plot one path for the value of the risky asset $\{P_t\}_{t=0,\delta t,\ldots,\tau}$, and overlay the respective path $\{S_t\}_{t=0,\delta t,\ldots,\tau}$ for the value of the option replication strategy (550), see Figure 15.

In a separate figure, plot the evolution of the portfolio weight of the risky asset $\{w_t\}_{t=0,\delta t,\ldots,\tau}$ on that path, see Figure 15.

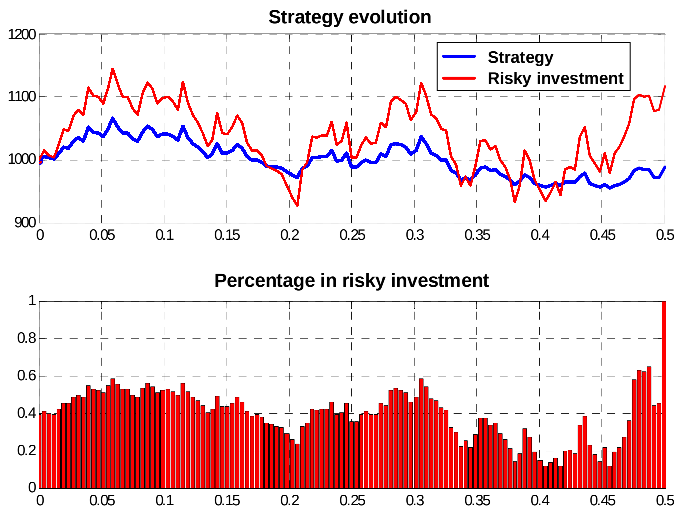

Figure 15: Dynamic replication of call option profile: one path

In a separate figure, scatter-plot the final payoff of the option replication strategy (550) over the payoff of the risky asset, and verify that it matches the option payoff, see Figure 16

See the script `S_OptionReplication`.

10 Estimation risk

10.1 General

10.1.1 Opportunity cost

Replicate the evaluation of the "best performer" allocation (8.42) in Meucci (2005) and described in Figure 8.2 in Meucci (2005). You do not need to draw the figure, as long as you correctly compute the number $\overline{\mathcal{S}}(\boldsymbol{\theta})$, as well as the distributions $\mathcal{S}_{\boldsymbol{\theta}}\left(\boldsymbol{\alpha}\left[I_T^{\boldsymbol{\theta}}\right]\right)$, $\mathcal{C}_{\boldsymbol{\theta}}^{+}\left(\boldsymbol{\alpha}\left[I_T^{\boldsymbol{\theta}}\right]\right)$ and $\mathrm{OC}_{\boldsymbol{\theta}}\left(\boldsymbol{\alpha}\left[I_T^{\boldsymbol{\theta}}\right]\right)$, as function of $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

See the script `S_EvaluationGeneric`.

10.2 Robust

10.2.1 Robust mean-variance for derivatives

Consider the market of call options in Exercise 8.1.4.

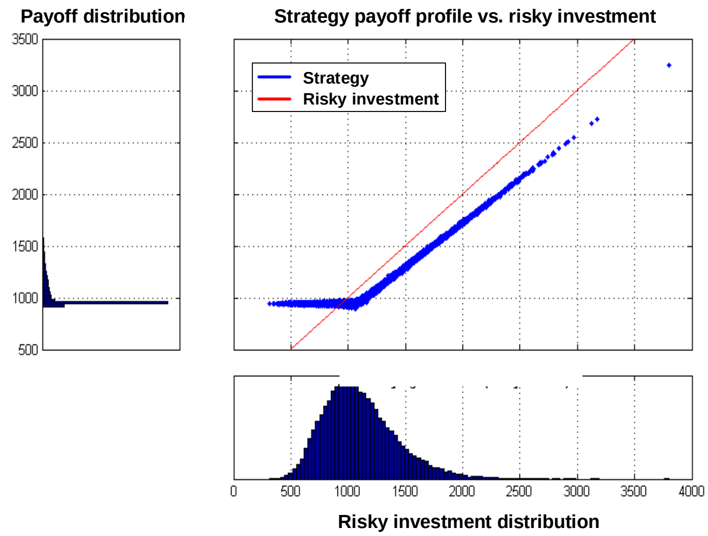

Figure 16: Dynamic replication of call option profile: final payoff in many paths

Consider a small ellipsoidal neighborhood of the expected linear returns, as determined by reasonable $\mathbf{T}$, $\mathbf{m}$, and q^2, see (9.118) in Meucci (2005), and assume no uncertainty in the estimation of the covariance of the linear returns, see (9.119) in Meucci (2005), and set up the robust optimization problem (9.117) in the form of SOCP, see comments after (9.130).

Compute the robust mean-variance efficient frontier in terms of relative weights, assuming the standard long-only and full investment constraints. Then plot the efficient frontier in the plane of weights and standard deviation.

Hint. Use the CVX package, located at www.stanford.edu/~boyd/cvx/

See the script `S_MVCallsRobust`.

Note: this script uses the CVX package. Before running the script, please download and install the package from `www.stanford.edu/~boyd/cvx`, following the instructions from Appendix A in `cvx\_usrguide.pdf`."

10.3 Black-Litterman & beyond

10.3.1 Black-Litterman

Assume as in Black-Scholes that the compounded returns are normally distributed

$$\ln\left(\mathbf{P}_{T+\tau}\right) - \ln\left(\mathbf{p}_T\right) \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \tag{551}$$

Compute the inputs of the mean-variance approach, namely expectations and covariances of the linear returns

Also from (2.219)-(2.220) in Meucci (2005) for a log-normal variable

$$\mathbf{Y} \sim \text{LogN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{552}$$

then

$$\begin{aligned} \mathbb{E}\{\mathbf{Y}\} &= e^{\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)}. & (553) \\ \mathbb{E}\{\mathbf{YY}'\} &= \left(e^{\left(\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)\right)}e^{\left(\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}(\Sigma)\right)'}\right) \circ e^{\boldsymbol{\Sigma}}, & (554) \end{aligned}$$

where $\circ$ denotes the term-by-term Hadamard product.

Since

$$1 + \mathbf{R} \sim \text{LogN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{555}$$

using (361), (362) and the property

$$\begin{aligned} \mathbb{E}\{(\mathbf{Y}-\mathbf{1})\} &= \mathbb{E}\{\mathbf{Y}\} - \mathbf{1} & (556) \\ \mathbb{E}\{(\mathbf{Y}-\mathbf{1})(\mathbf{Y}-\mathbf{1})'\} &= \mathbb{E}\{\mathbf{YY}'\} + \mathbf{11}' - \mathbb{E}\{\mathbf{Y1}'\} - \mathbb{E}\{\mathbf{1Y}'\}. & (557) \end{aligned}$$

we can easily compute the expectations $\mathbb{E}\{\mathbf{R}\}$ and the second moments $\mathbb{E}\{\mathbf{RR}'\}$. The covariance then follows from

$$\mathbb{C}ov\{\mathbf{R}\} = \mathbb{E}\{\mathbf{RR}'\} - \mathbb{E}\{\mathbf{R}\}\mathbb{E}\{\mathbf{R}'\}. \tag{558}$$

Upload the database `CovNRets` that contains estimates for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Use the results from the previous point to compute and plot numerically the efficient frontier under the constraints that the weights sum to one and that they are non-negative.

Construct a pick matrix that sets views on the spread between the compounded return of the first and the last security. Set a one-standard deviation bullish view on that spread. Use the market-based Black-Litterman formula (9.44) in Meucci (2005) to compute the normal parameters that reflect those views. Map the results into expectations and covariances for the linear returns. Compute and plot numerically the efficient frontier under the same constraints as above

See script `S_BLmBasic`.

10.3.2 Beyond Black-Litterman

Consider the market prior

$$X \overset{d}{=} BZ_{-1} + (1-B)Z_1 \tag{559}$$

where

$$Z_{-1} \sim \text{N}(-1, 1), \quad Z_1 \sim \text{N}(1, 1); \tag{560}$$

B is Bernoulli with $\mathbb{P}\{B=1\} \equiv 1/2$; and all the variables are independent.

Compute and plot the posterior market distribution that is the most consistent with the view

$$\widetilde{\mathbb{E}}\{X\} \equiv 0.5. \tag{561}$$

Hint. Use the package "Entropy Pooling: Fully Flexible Views and Stress-testing" available at www.mathworks.com/matlabcentral/fileexchange/21307.

See script `S_EntropyView`.

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