

INVESTOR'S OBJECTIVES EVALUATION: UTILITY

Risk and Asset Allocation - Springer – symmys.com

Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

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$$\Psi_{\alpha} = \alpha' M \quad (5.10)$$

$$\alpha \mapsto E \{ u (\Psi_{\alpha}) \} \equiv \int_{\mathbb{R}} u (\psi) f_{\Psi_{\alpha}} (\psi) d\psi, \quad (5.90)$$

$$\alpha \mapsto \mathcal{S} (\alpha) \quad (5.48)$$

$$E \{ u (\Psi_{\alpha}) \} = -\phi_{\Psi_{\alpha}} \left(\frac{i}{\zeta} \right) \quad (5.92)$$

$$u (\psi) \equiv -e^{-\frac{1}{\zeta} \psi}, \quad (5.91)$$

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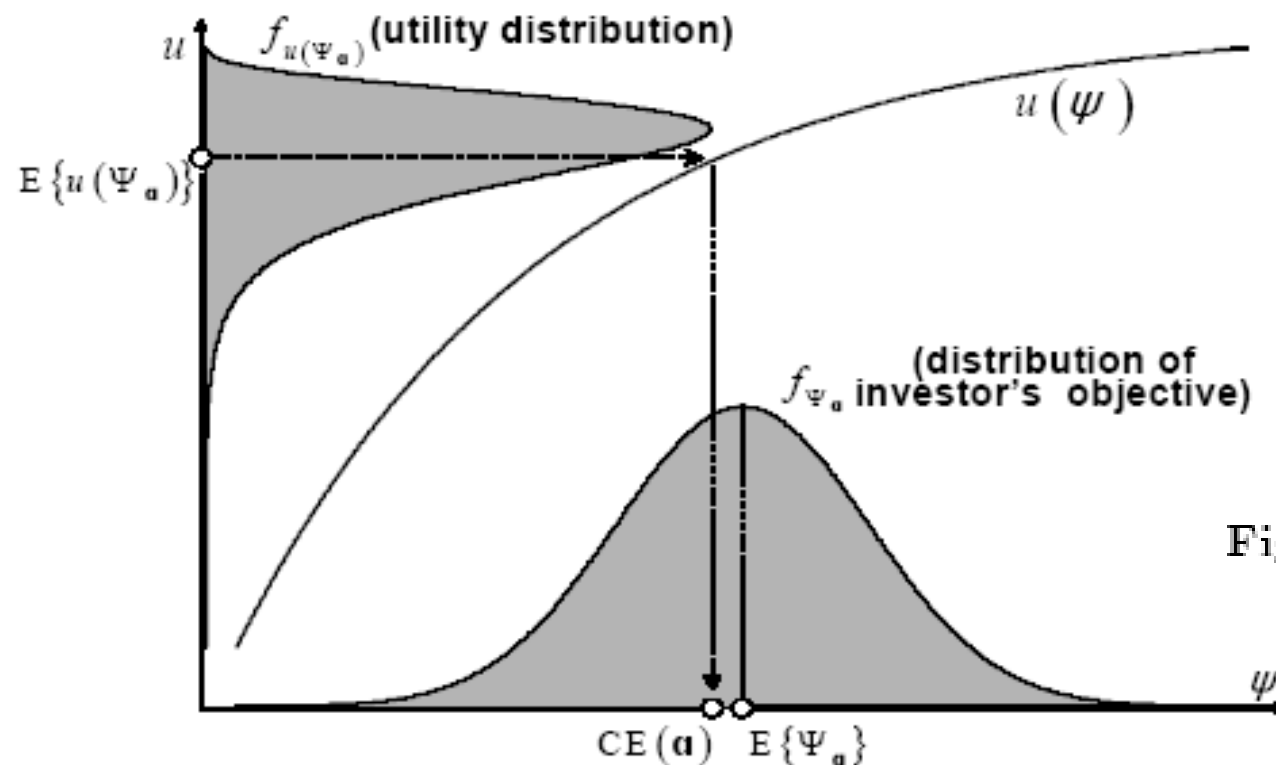


Fig. 5.8.

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$$M \equiv P_{T+\tau} \sim N(\mu, \Sigma) \quad (5.143)$$

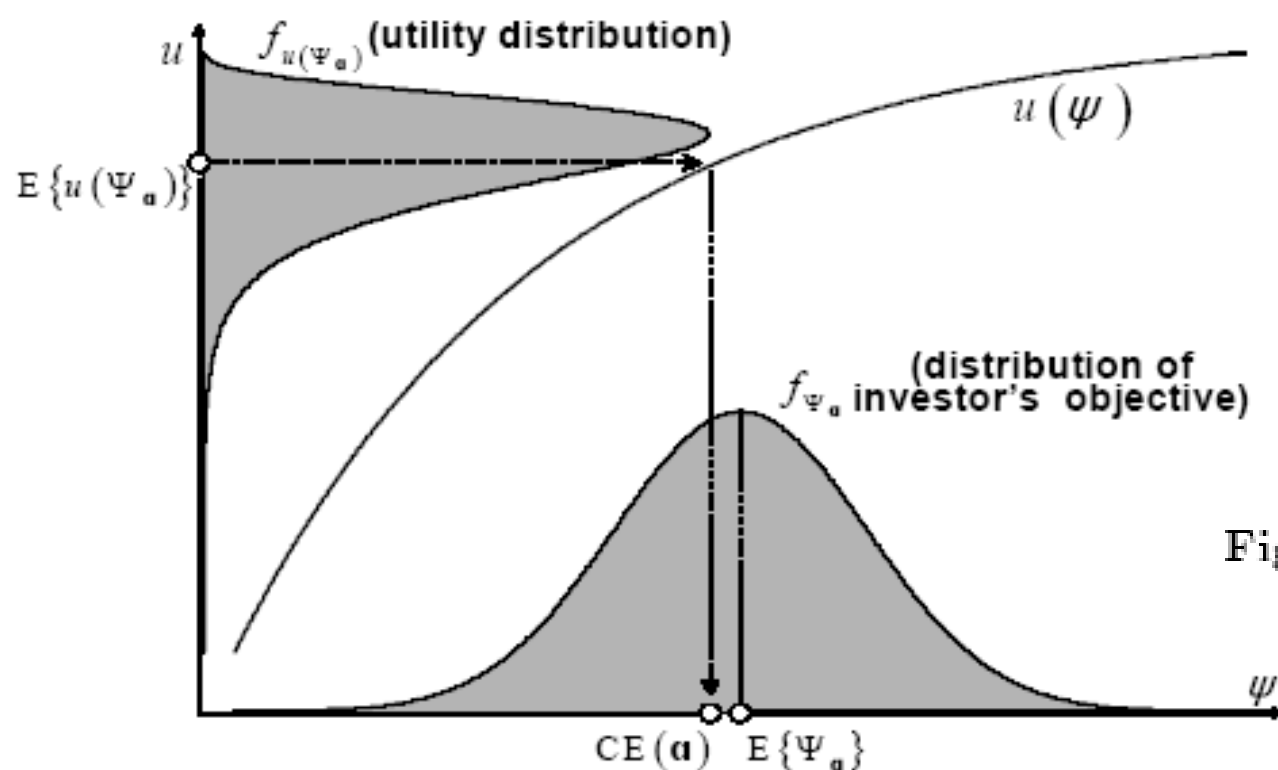


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- Estimability

$$\alpha \mapsto \Psi_{\alpha} \mapsto f_{\Psi_{\alpha}} \mapsto \text{CE}(\alpha) \quad (5.96)$$

- Sensibility

$$\Psi_{\alpha} \geq \Psi_{\beta} \text{ in all scenarios } \Rightarrow \text{CE}(\alpha) \geq \text{CE}(\beta) \quad (5.100) \quad \Leftarrow \quad \mathcal{D}u \geq 0, \quad (5.98)$$

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- Consistence with stochastic dominance

$$Q_{\Psi_{\alpha}}(p) \geq Q_{\Psi_{\beta}}(p) \text{ for all } p \in (0, 1) \Rightarrow \text{CE}(\alpha) \geq \text{CE}(\beta) \quad (5.109) \quad \Leftarrow \quad \mathcal{D}u \geq 0, \quad (5.98)$$

$$\text{consistence with higher order dominance} \quad \Leftarrow \quad (-1)^k \mathcal{D}^k u \leq 0, \quad k = 1, 2, \dots, q. \quad (5.111)$$

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- Constancy

$$\Psi_{\mathbf{b}} \equiv \psi_{\mathbf{b}} \Rightarrow \text{CE}(\mathbf{b}) = \psi_{\mathbf{b}} \quad (5.112)$$

$$\text{CE}(\alpha) = \alpha' \mu - \frac{\alpha' \Sigma \alpha}{2\zeta} \quad (5.144)$$

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$$\boxed{\Psi_{\alpha} = \alpha' M}_{(5.10)} \quad \alpha \mapsto \text{CE}(\alpha) \equiv u^{-1}(\mathbb{E}\{u(\Psi_{\alpha})\}) \quad (5.93)$$

- Positive homogeneity

$$\text{CE}(\lambda \alpha) = \lambda \text{CE}(\alpha) \quad (5.113)$$

$$\Leftarrow \quad u(\psi) \equiv \psi^{1-\frac{1}{\gamma}} \quad (5.114)$$

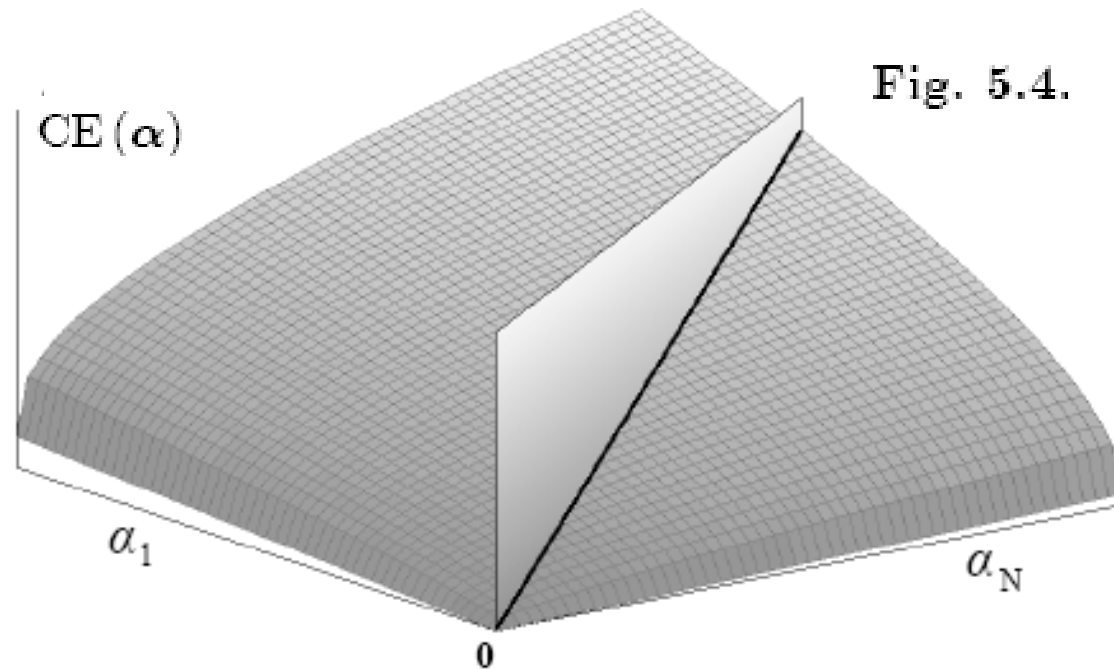


Fig. 5.4.

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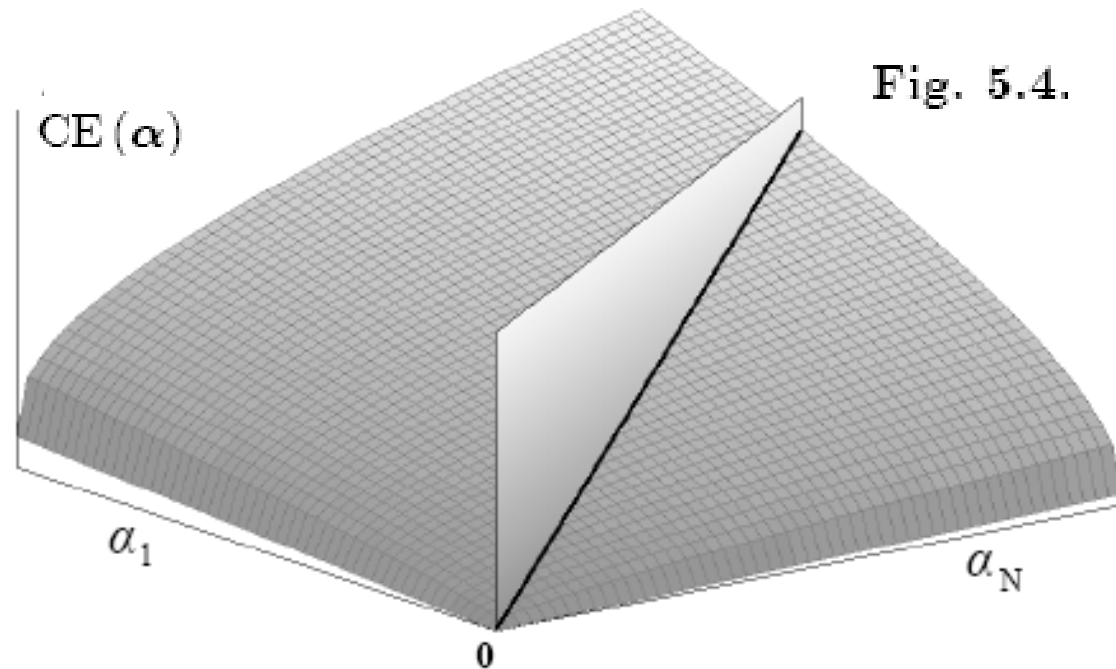
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Euler:

$$\text{CE}(\alpha) = \sum_{n=1}^N \alpha_n \left[\mathbb{E} \left\{ M_n (\alpha' M)^{-\frac{1}{\gamma}} \right\} (\text{CE}(\alpha))^{\frac{1}{\gamma}} \right] \quad (5.152)$$

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- **Translation invariance**

$$\Psi_b \equiv 1 \Rightarrow \text{CE}(\alpha + \lambda b) = \text{CE}(\alpha) + \lambda \quad (5.115) \quad \Longleftarrow \quad u(\psi) \equiv -e^{-\frac{1}{\zeta}\psi} \quad (5.91)$$

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- ~~Concavity~~

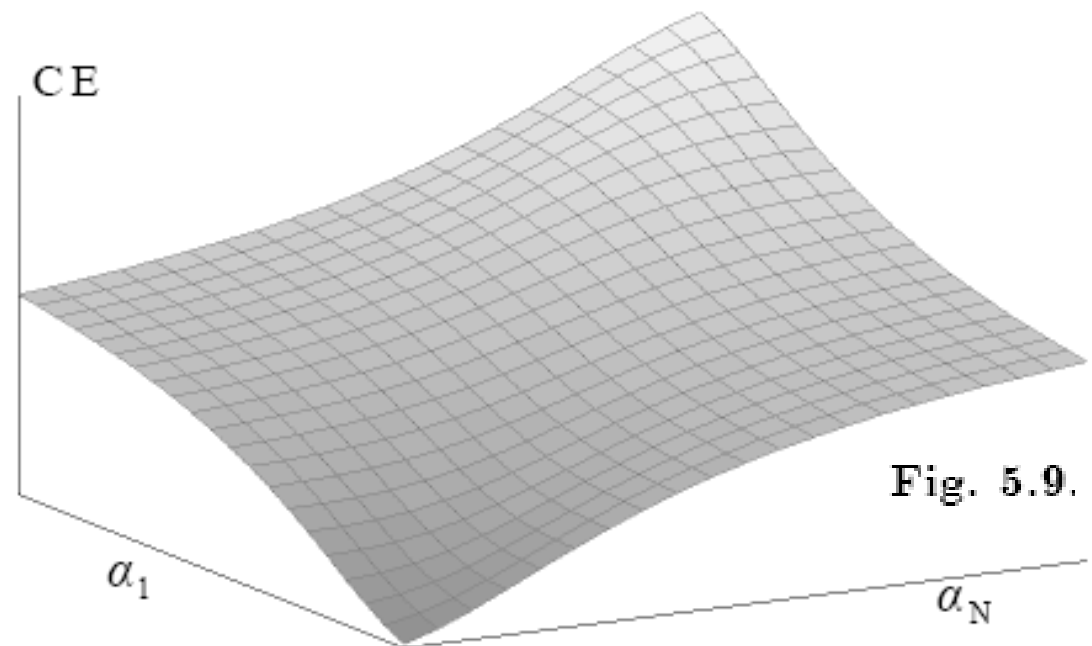


Fig. 5.9.

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- ~~super-additivity~~
- ~~Co-monotonic additivity~~
- ~~Concavity~~

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- ~~super-additivity~~

- ~~Co-monotonic additivity~~

- ~~Concavity~~

- Risk aversion/propensity/neutrality

$$\text{RP}(\alpha) = \mathbb{E}\{\Psi_{\alpha}\} - \text{CE}(\alpha) \quad (5.119) \quad u \text{ concave} \Leftrightarrow \text{RP}(\alpha) \geq 0 \quad (5.120)$$

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$$\boxed{\Psi_{\alpha} = \alpha' M} \quad (5.10) \quad \alpha \mapsto \text{CE}(\alpha) \equiv u^{-1}(\mathbb{E}\{u(\Psi_{\alpha})\}) \quad (5.93)$$

$$\text{RP}(\alpha) = \mathbb{E}\{\Psi_{\alpha}\} - \text{CE}(\alpha) \quad (5.119) \quad \Rightarrow \quad \text{RP}(\alpha) \approx \frac{1}{2} A(\mathbb{E}\{\Psi_{\alpha}\}) \text{Var}\{\Psi_{\alpha}\} \quad (5.122)$$

$$A(\psi) \equiv -\frac{\mathcal{D}^2 u(\psi)}{\mathcal{D}u(\psi)} \quad (5.121)$$

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$$A(\psi) \equiv \frac{\psi}{\gamma\psi^2 + \zeta\psi + \eta} \quad (5.132) \quad \left\{ \begin{array}{l} \eta \equiv 0 \\ \text{HARA} \end{array} \right\} \left\{ \begin{array}{ll} \zeta > 0 \text{ and } \gamma \equiv 0 & u(\psi) = -e^{-\frac{1}{\zeta}\psi} \quad (5.133) \\ \zeta > 0 \text{ and } \gamma \equiv -1 & u(\psi) = \psi - \frac{1}{2\zeta}\psi^2 \quad (5.134) \\ \zeta \equiv 0 & \left\{ \begin{array}{ll} \gamma \geq 1 & u(\psi) \equiv \psi^{1-\frac{1}{\gamma}} \quad (5.135) \\ \gamma \rightarrow 1 & u(\psi) \equiv \ln \psi \quad (5.136) \\ \gamma \rightarrow \infty & u(\psi) \equiv \psi \quad (5.137) \\ \gamma \equiv 0 & u(\psi) \equiv \text{erf}\left(\frac{\psi}{\sqrt{2\eta}}\right) \quad (5.138) \end{array} \right. \end{array} \right.$$