Attilio Meucci

FACTORS ON DEMAND

Building a Platform for Portfolio Managers Risk Managers and Traders

Estimation (RM) versus Attribution (PM)

$N \times 1$ $\mathbf{R}$ **Returns of N securities from today to investment horizon**

$N \times 1$ $\mathbf{w}$ **Weights of N securities in portfolio**

$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$ **Portfolio return from today to investment horizon**

Estimation (RM) versus Attribution (PM)

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$N \times 1$ $\mathbf{w}$ **Weights of N securities in portfolio**

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<u>RISK MANAGEMENT</u>: ESTIMATION

Compute risk of portfolio return $R_{\mathbf{w}}$

Returns covariances

$$(\mathrm{SDev}\{R_{\mathbf{w}}\})^2 = \mathbf{w}'\Sigma_R\mathbf{w}$$

Estimation (RM) versus Attribution (PM)

$N \times 1$ $\mathbf{R}$ Returns of N securities from today to investment horizon

$N \times 1$ $\mathbf{w}$ Weights of N securities in portfolio

$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$ Portfolio return from today to investment horizon

RISK MANAGEMENT: ESTIMATION

Compute risk of portfolio return $R_{\mathbf{w}}$

$$(\mathrm{SDev}\{R_{\mathbf{w}}\})^2 = \mathbf{w}' \underbrace{\Sigma_R}_{\text{Returns covariances}} \mathbf{w}$$

PORTFOLIO MANAGEMENT: ATTRIBUTION

Express portfolio return $R_{\mathbf{w}}$ as factors + residual :

$$R_{\mathbf{w}} = \sum_{k=1}^{K} \underbrace{d_{\mathbf{w},k}}_{\text{Exposure of portfolio to factor k}} \underbrace{Z_k}_{\text{factor k}} + \eta_{\mathbf{w}}$$

ESTIMATION VERSUS ATTRIBUTION

STANDARD APPROACH TO FACTOR MODELING

RATIONALE OF FACTORS ON DEMAND

IMPLEMENTATION OF FACTORS ON DEMAND

APPLICATIONS OF FACTORS ON DEMAND

REFERENCES

Appendix: factor models pitfalls

Standard Approach to Factor Modeling

BUILDING BLOCKS:

$\mathbf{R}$ $N \times 1$ **Returns of securities**

1) Structure R_n

return of n-th security

RISK MANAGEMENT: ESTIMATION

Compute risk of portfolio return $R_{\mathbf{w}}$

$$(\mathrm{SDev}\{R_{\mathbf{w}}\})^2 = \mathbf{w}'\underbrace{\Sigma_R}_{\text{Returns covariances}}\mathbf{w}$$

PORTFOLIO MANAGEMENT: ATTRIBUTION

Express portfolio return $R_{\mathbf{w}}$ **as factors + residual :**

$$R_{\mathbf{w}} = \sum_{k=1}^{K} \underbrace{d_{\mathbf{w},k}}_{\text{Exposure of portfolio to factor k}} \underbrace{Z_k}_{\text{factor k}} + \eta_{\mathbf{w}}$$

Standard Approach to Factor Modeling

BUILDING BLOCKS:

1) Structure R_n Z_k

k-th systematic factor

$\mathbf{R}$ $N \times 1$ **Returns of securities**

$\mathbf{Z}$ $K \times 1$ **Systematic factors**

RISK MANAGEMENT: ESTIMATION

Compute risk of portfolio return $R_{\mathbf{w}}$

Returns covariances

$$(\mathrm{SDev}\{R_{\mathbf{w}}\})^2 = \mathbf{w}'\Sigma_R\mathbf{w}$$

PORTFOLIO MANAGEMENT: ATTRIBUTION

Express portfolio return $R_{\mathbf{w}}$ **as factors + residual :**

Exposure of portfolio to factor k

$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

factor k

Standard Approach to Factor Modeling

BUILDING BLOCKS:

1) Structure

$$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n \quad \left\{ \begin{array}{lll} \mathbf{R} & N \times 1 & \text{Returns of securities} \\ \mathbf{D} & N \times K & \text{Exposures of returns to factors} \\ \mathbf{Z} & K \times 1 & \text{Systematic factors} \\ \boldsymbol{\eta} & N \times 1 & \text{Idiosyncratic shocks} \end{array} \right.$$

exposure of security n to systematic factor k

idiosyncratic shock for n-th security

RISK MANAGEMENT: ESTIMATION

Compute risk of portfolio return $R_{\mathbf{w}}$

Returns covariances

$$(\mathrm{SDev}\{R_{\mathbf{w}}\})^2 = \mathbf{w}'\Sigma_R\mathbf{w}$$

PORTFOLIO MANAGEMENT: ATTRIBUTION

Express portfolio return $R_{\mathbf{w}}$ as factors + residual :

Exposure of portfolio to factor k

$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

factor k

Standard Approach to Factor Modeling

BUILDING BLOCKS:

1) Structure $R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n$

$\mathbf{R}$	$N \times 1$	**Returns of securities**
$\mathbf{D}$	$N \times K$	**Exposures of returns to factors**
$\mathbf{Z}$	$K \times 1$	**Systematic factors**
$\boldsymbol{\eta}$	$N \times 1$	**Idiosyncratic shocks**

Independent

2) Structure is supported by Arbitrage Pricing Theory

RISK MANAGEMENT: ESTIMATION

Compute risk of portfolio return $R_{\mathbf{w}}$

Returns covariances

$$(\mathrm{SDev}\{R_{\mathbf{w}}\})^2 = \mathbf{w}' \Sigma_R \mathbf{w}$$

PORTFOLIO MANAGEMENT: ATTRIBUTION

Express portfolio return $R_{\mathbf{w}}$ **as factors + residual :**

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$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

factor k

BUILDING BLOCKS:

1) Structure $$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n$$

$\mathbf{R}$	$N \times 1$	Returns of securities
$\mathbf{D}$	$N \times K$	Exposures of returns to factors
$\mathbf{Z}$	$K \times 1$	Systematic factors
$\boldsymbol{\eta}$	$N \times 1$	Idiosyncratic shocks

Independent

2) Structure is supported by Arbitrage Pricing Theory

3) Structure implies efficient estimate of return distribution

Returns covariances $N \times N$

$$\boldsymbol{\Sigma}_R = \mathbf{D}\boldsymbol{\Sigma}_Z\mathbf{D}' + \operatorname{diag}\left(\boldsymbol{\sigma}_\eta^2\right)$$

Factors covariances $K \times K$

Idio variances $N \times 1$

Standard Approach to Factor Modeling

BUILDING BLOCKS:

1) Structure $$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n$$

- $\mathbf{R}$ — $N \times 1$ — **Returns of securities**
- $\mathbf{D}$ — $N \times K$ — **Exposures of returns to factors**
- $\mathbf{Z}$ — $K \times 1$ — **Systematic factors**
- $\boldsymbol{\eta}$ — $N \times 1$ — **Idiosyncratic shocks**

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2) Structure is supported by Arbitrage Pricing Theory

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RISK MANAGEMENT: ESTIMATION

Compute risk of portfolio return $R_{\mathbf{w}}$

$$(\mathrm{SDev}\{R_{\mathbf{w}}\})^2 = \mathbf{w}'\Sigma_R \mathbf{w}$$

$$[\mathbf{D}\Sigma_Z\mathbf{D}' + \mathrm{diag}(\boldsymbol{\sigma}_\eta^2)]$$

PORTFOLIO MANAGEMENT: ATTRIBUTION

Express portfolio return $R_{\mathbf{w}}$ **as factors + residual**

$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

$$w_1 d_{1,k} + \cdots + w_N d_{N,k}$$

$$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n$$

Estimation
Securities returns

Aggregation →

Estimation
Portfolio return

Same model

Attribution
Securities returns

Aggregation →

Attribution
Portfolio return

RISK MANAGEMENT: ESTIMATION

Compute risk of portfolio return $R_{\mathbf{w}}$

$$(\mathrm{SDev}\{R_{\mathbf{w}}\})^2 = \mathbf{w}'\Sigma_R \mathbf{w}$$

$$\Sigma_R = [\mathbf{D}\Sigma_Z\mathbf{D}' + \mathrm{diag}(\boldsymbol{\sigma}_\eta^2)]$$

PORTFOLIO MANAGEMENT: ATTRIBUTION

Express portfolio return $R_{\mathbf{w}}$ **as factors + residual**

$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

$$d_{\mathbf{w},k} = w_1 d_{1,k} + \cdots + w_N d_{N,k}$$

BUILDING BLOCKS:

1) Structure $$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n.$$

$\mathbf{R}$	$N \times 1$	Returns of securities
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BUILDING BLOCKS:

1) Structure

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$\boldsymbol{\eta}$	$N \times 1$	**Idiosyncratic shocks**

Independent

2) Structure is supported by Arbitrage Pricing Theory

BUILDING BLOCKS:

3) Structure implies efficient estimate of ~~return~~ distribution

QUEST FOR INVARIANCE:

Risk drivers $\mathbf{X}$ determine returns distribution

Rationale of Factors on Demand

log-return

Equities: $R = \exp(X) - 1$

government curve changes

spread changes

Bonds: $R = \frac{P(X_1, X_2; \theta)}{P_0} - 1$

log-return of underlying

changes in implied volatility surface

Derivatives: $R = \frac{BS(X_1, X_2; \theta)}{P_0} - 1$

QUEST FOR INVARIANCE:

Risk drivers $\mathbf{X}$ determine returns distribution

SYSTEMATIC + IDIOSYNCRATIC RETURNS

1) Structure $$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n$$

$\mathbf{R}$	$N \times 1$	Returns of securities	
$\mathbf{D}$	$N \times K$	Exposures of returns to factors	
$\mathbf{Z}$	$K \times 1$	Systematic factors	Independent
$\boldsymbol{\eta}$	$N \times 1$	Idiosyncratic shocks	

2) Structure is supported by Arbitrage Pricing Theory

3) Structure implies efficient estimate of return distribution

DOMINANT + RESIDUAL RISK DRIVERS

1) Risk drivers $\mathbf{X}$ determine returns distribution

2) Structure $$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

$\mathbf{X}$	$N \times 1$	Risk drivers
$\mathbf{B}$	$N \times K$	Loadings
$\mathbf{F}$	$K \times 1$	Dominant risk factors
$\mathbf{U}$	$N \times 1$	Residuals

3) Structure implies efficient estimate of risk drivers distribution

A - QUEST FOR INVARIANCE

Estimate dominant factors + residual for risk drivers $\mathbf{X}$

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

B - NON-LINEAR PRICING

From risk-drivers $\mathbf{X}$ to returns $\mathbf{R}$

C - AGGREGATION

From securities returns $\mathbf{R}$ to portfolio return $R_{\mathbf{w}}$

Rationale of Factors on Demand

A - QUEST FOR INVARIANCE

Estimate dominant factors + residual for risk drivers $\mathbf{X}$

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

B - NON-LINEAR PRICING

From risk-drivers $\mathbf{X}$ to returns $\mathbf{R}$

C - AGGREGATION

From securities returns $\mathbf{R}$ to portfolio return $R_{\mathbf{w}}$

Conditional link

D - RISK MANAGEMENT

Compute risk of portfolio return $R_{\mathbf{w}}$

E - PORTFOLIO MANAGEMENT

Attribute portfolio return $R_{\mathbf{w}}$ to dominant factors + residual

$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Rationale of Factors on Demand

A - QUEST FOR INVARIANCE

Estimate dominant factors + residual for risk drivers $\mathbf{X}$

B - NON-LINEAR PRICING

From risk-drivers $\mathbf{X}$ to returns $\mathbf{R}$

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From securities returns $\mathbf{R}$ to portfolio return $R_{\mathbf{w}}$

D - RISK MANAGEMENT

Compute risk of portfolio return $R_{\mathbf{w}}$

E - PORTFOLIO MANAGEMENT

Attribute portfolio return $R_{\mathbf{w}}$ to dominant factors + residual

Estimation Factor Model

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

Conditional link

Attribution Factor Model

$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Rationale of Factors on Demand

A - QUEST FOR INVARIANCE

Estimate dominant factors + residual
for risk drivers $\mathbf{X}$

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

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From risk-drivers $\mathbf{X}$ to returns $\mathbf{R}$

C - AGGREGATION

From securities returns $\mathbf{R}$ to portfolio return $R_{\mathbf{w}}$

D - RISK MANAGEMENT

Compute risk of portfolio return $R_{\mathbf{w}}$

Conditional link

2 of 3

E - PORTFOLIO MANAGEMENT

Attribute portfolio return $R_{\mathbf{w}}$
to dominant factors + residual

$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Rationale of Factors on Demand

A - QUEST FOR INVARIANCE

Estimate dominant factors + residual
for risk drivers $\mathbf{X}$

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

B - NON-LINEAR PRICING

From risk-drivers $\mathbf{X}$ to returns $\mathbf{R}$

C - AGGREGATION

From securities returns $\mathbf{R}$ to portfolio return $R_{\mathbf{w}}$

Bottom-up
(from securities
to portfolio)

D - RISK MANAGEMENT

Compute risk of portfolio return $R_{\mathbf{w}}$

Conditional link

E - PORTFOLIO MANAGEMENT

Attribute portfolio return $R_{\mathbf{w}}$
to dominant factors + residual

$$R_{\mathbf{w}} = \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Top-down
(portfolio
specific)

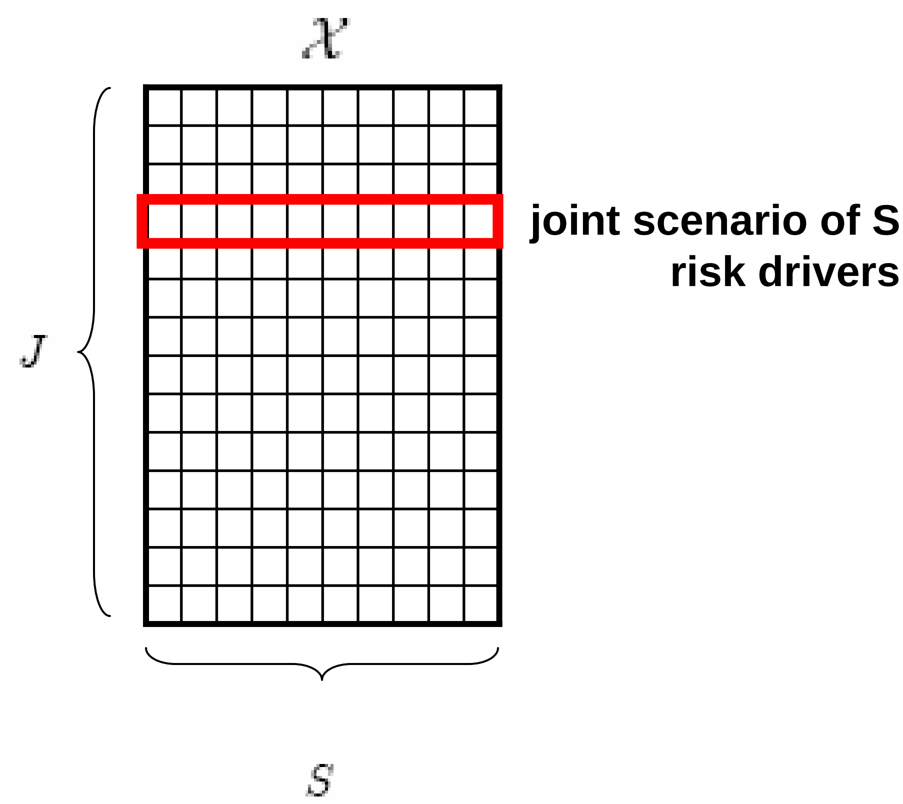
STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation

$\mathbf{X}$

$J \times S$ **scenarios**

$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$



STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation

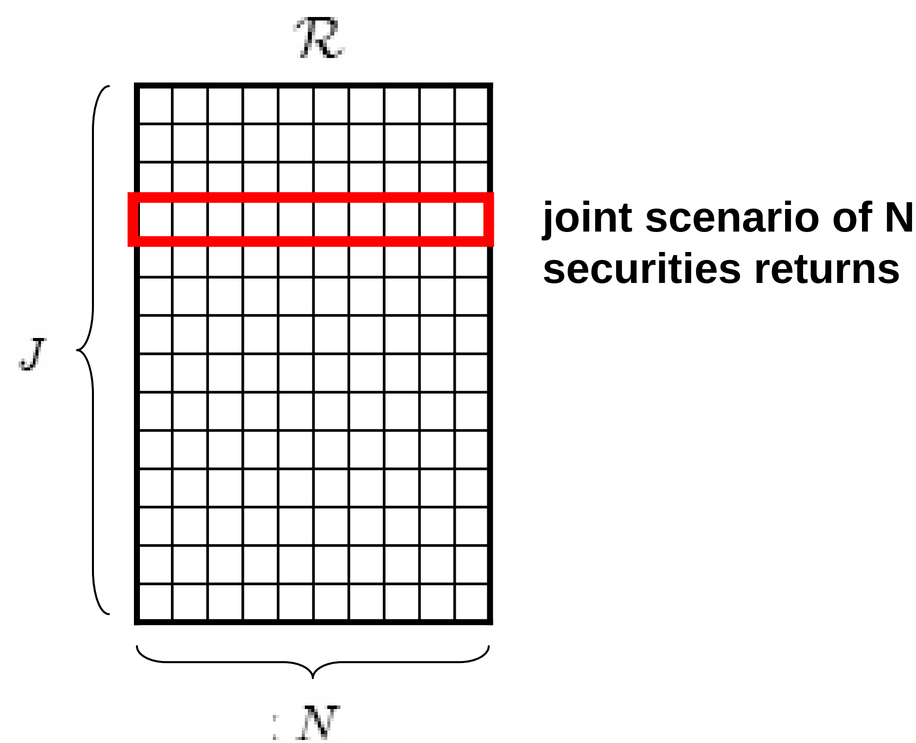
$\mathbf{X}$

$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$ ← $J \times S$ **scenarios**

2: Pricing (e.g. Black-Scholes formula)

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$ ← $J \times N$ **scenarios**



STAGE A: RISK MANAGEMENT

1: Risk drivers Estimation (e.g. changes of impl. vol.)

$$\mathbf{X}$$

$$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$$

$J \times S$ **scenarios**

2: Pricing (e.g. Black-Scholes formula)

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

$$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$$

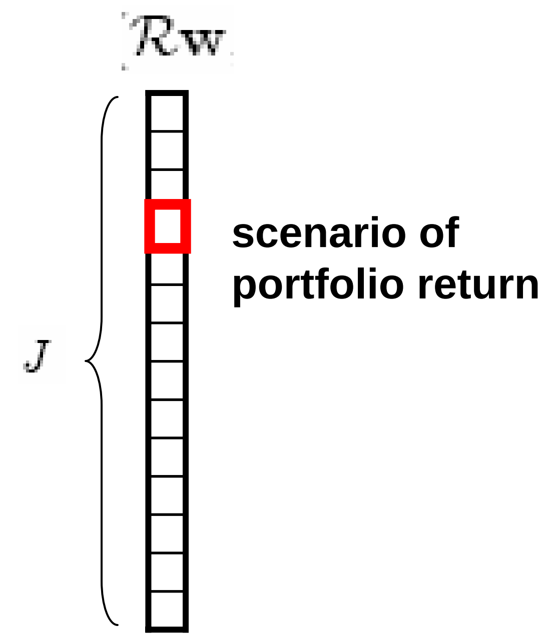
$J \times N$ **scenarios**

3: Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}\mathbf{w}$$

$J \times 1$ **scenarios**



Implementation Steps of Factors on Demand

STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation

$\mathbf{X}$

$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$ ← $J \times S$ **scenarios**

2: Pricing (e.g. Black-Scholes formula)

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

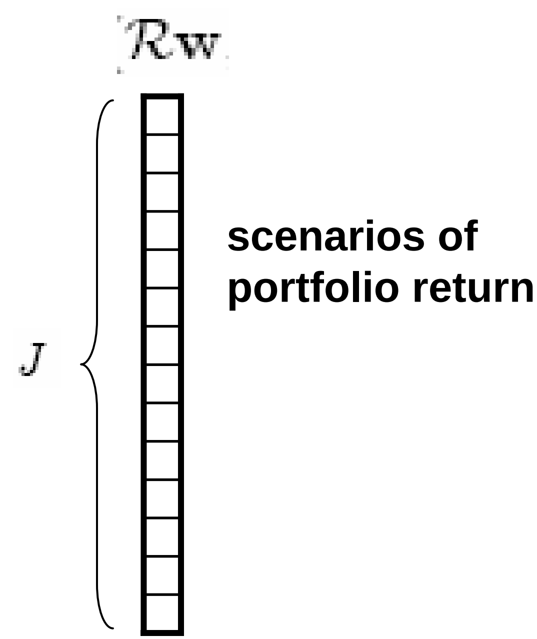
$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$ ← $J \times N$ **scenarios**

3: Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}_{\mathbf{w}}$ ← $J \times 1$ **scenarios**

SDev, VaR, CVaR,
Contributions, …



Implementation Steps of Factors on Demand

STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)

Estimation

$\mathbf{X}$ $\quad J \times S$ **scenarios**

$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$

2: Pricing (e.g. Black-Scholes formula)

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$ $\quad J \times N$ **scenarios**

3: Aggregation

$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$ $\quad J \times 1$ **scenarios**

$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}_{\mathbf{w}}$

SDev, VaR, CVaR, Contributions, …

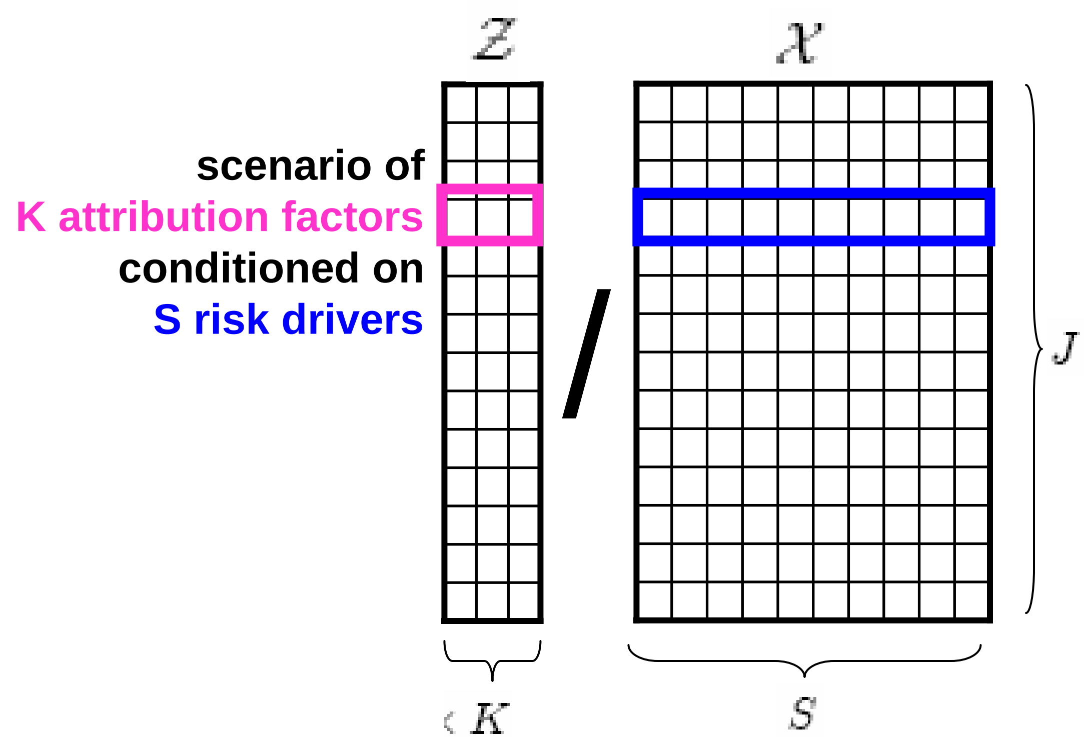
STAGE B: PORTFOLIO MANAGEMENT

4: Attribution factors (e.g. fundamental factors)

Conditional link

$\mathbf{Z}$ $\quad J \times K$ **conditional scenarios**

$f_{\mathbf{Z}|\mathbf{x}} \Leftrightarrow \mathcal{Z}|_{\mathcal{X}}$.



Implementation Steps of Factors on Demand

STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation

$\mathbf{X}$

$f_{\mathbf{X}} \Leftrightarrow \mathcal{X} \leftarrow J \times S$ **scenarios**

2: Pricing (e.g. Black-Scholes formula)

$$R_n = g_n(X_1, \ldots, X_S)$$

$f_{\mathbf{R}} \Leftrightarrow \mathcal{R} \leftarrow J \times N$ **scenarios**

3: Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}_{\mathbf{w}} \leftarrow J \times 1$ **scenarios**

SDev, VaR, CVaR, Contributions, …

STAGE B: PORTFOLIO MANAGEMENT

4: Attribution factors (e.g. fundamental factors)
Conditional link

$\mathbf{Z}$

$f_{\mathbf{Z}|\mathbf{x}} \Leftrightarrow \mathcal{Z}|_{\mathcal{X}} \leftarrow J \times K$ **conditional scenarios**

5: Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \, \mathrm{E}\left\{\left(R_{\mathbf{w}} - \mathbf{d}'\mathbf{Z}\right)^2\right\}$$

top-down exposures

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Implementation Steps of Factors on Demand

STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation

$\mathbf{X}$

$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$ ← $J \times S$ **scenarios**

2: Pricing (e.g. Black-Scholes formula)

$R_n = g_n(X_1, \ldots, X_S)$

$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$ ← $J \times N$ **scenarios**

3: Aggregation

$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$

$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}\mathbf{w}$ ← $J \times 1$ **scenarios**

SDev, VaR, CVaR,
Contributions, …

STAGE B: PORTFOLIO MANAGEMENT

4: Attribution factors (e.g. fundamental factors)
Conditional link

$\mathbf{Z}$

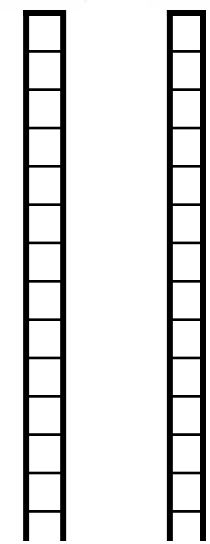
$f_{\mathbf{Z}|\mathbf{X}} \Leftrightarrow \mathcal{Z}|_{\mathcal{X}}$ ← $J \times K$ **conditional scenarios**

5: Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \operatorname{E}\left\{ \left(R_{\mathbf{w}} - \mathbf{d}'\mathbf{Z} \right)^2 \right\}$$

$$\Updownarrow \qquad \Updownarrow$$

$$\mathcal{R}\mathbf{w} \qquad \mathcal{Z}\mathbf{d}$$



scenarios of $R_{\mathbf{w}}$

scenarios of $\mathbf{d}'\mathbf{Z}$

Implementation Steps of Factors on Demand

STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation

$\mathbf{X}$

$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$ ← $J \times S$ **scenarios**

2: Pricing (e.g. Black-Scholes formula)

$$R_n = g_n(X_1, \ldots, X_S)$$

$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$ ← $J \times N$ **scenarios**

3: Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}\mathbf{w}$ ← $J \times 1$ **scenarios**

SDev, VaR, CVaR, Contributions, …

STAGE B: PORTFOLIO MANAGEMENT

4: Attribution factors (e.g. fundamental factors)
Conditional link

$\mathbf{Z}$

$f_{\mathbf{Z}|\mathbf{x}} \Leftrightarrow \mathcal{Z}|\mathcal{X}$. ← $J \times K$ **conditional scenarios**

5: Attribution

top-down exposures

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{E}} \left[(\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d})^2 \right] \right\}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Implementation Steps of Factors on Demand

STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation

$\mathbf{X}$ — $J \times S$ **scenarios**

$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$

2: Pricing (e.g. Black-Scholes formula)

$$R_n = g_n(X_1, \ldots, X_S)$$

$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$ — $J \times N$ **scenarios**

3: Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$J \times 1$ **scenarios**

$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}\mathbf{w}$

SDev, VaR, CVaR,
Contributions, …

STAGE B: PORTFOLIO MANAGEMENT

4: Attribution factors (e.g. fundamental factors)
Conditional link

$\mathbf{Z}$ — $J \times K$ **conditional scenarios**

$f_{\mathbf{Z}|\mathbf{x}} \Leftrightarrow \mathcal{Z}|_{\mathcal{X}}$

5: Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \hat{\mathrm{E}}\left[(\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d})^2 \right] \right\}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Exposures, Hedging,
Contributions from factors, …

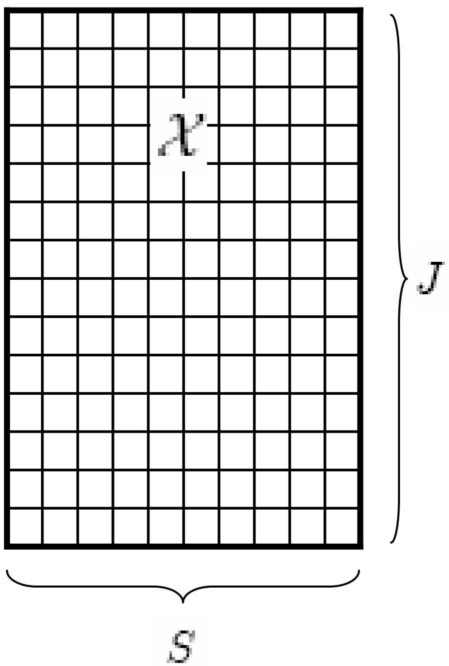
Implementation Steps of Factors on Demand

Risk drivers (e.g. changes of impl. vol.)

Estimation

$\mathbf{X}$

$$f_{\mathbf{X}} \Leftrightarrow \mathcal{X}$$

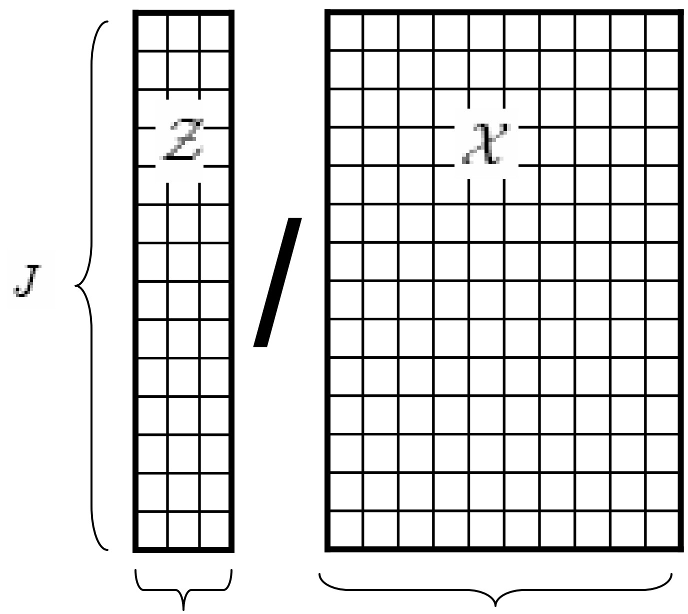


Attribution factors (e.g. fundamental factors)

Conditional link

$\mathbf{Z}$

$$f_{\mathbf{Z}|\mathbf{x}} \Leftrightarrow \mathcal{Z}|_{\mathcal{X}}.$$



Implementation Steps of Factors on Demand

<u>Risk drivers</u> (e.g. changes of impl. vol.)

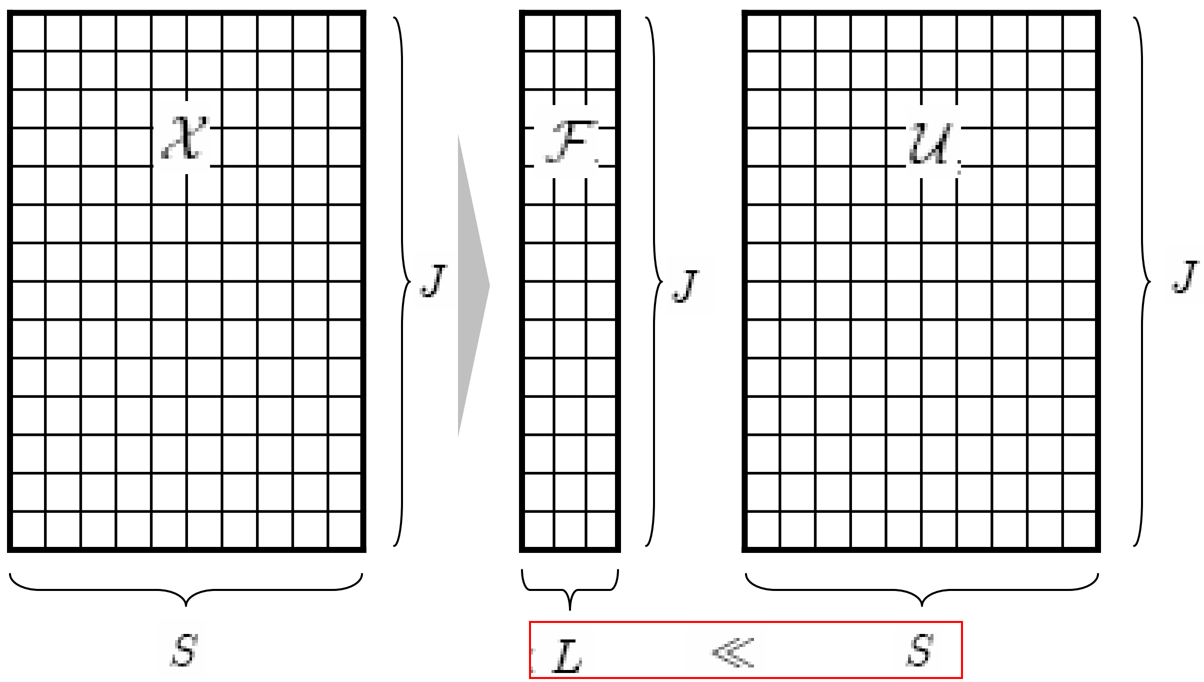
<u>Estimation – Dimension reduction</u>

$$\mathbf{X} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}$$ (e.g. PCA)

$$f_{\mathbf{X}} \Leftrightarrow \mathcal{X} \equiv \mathcal{F}\mathbf{B}' + \mathcal{U}$$

high quality,
copula matched
scenarios

low burden
scenarios



Implementation Steps of Factors on Demand

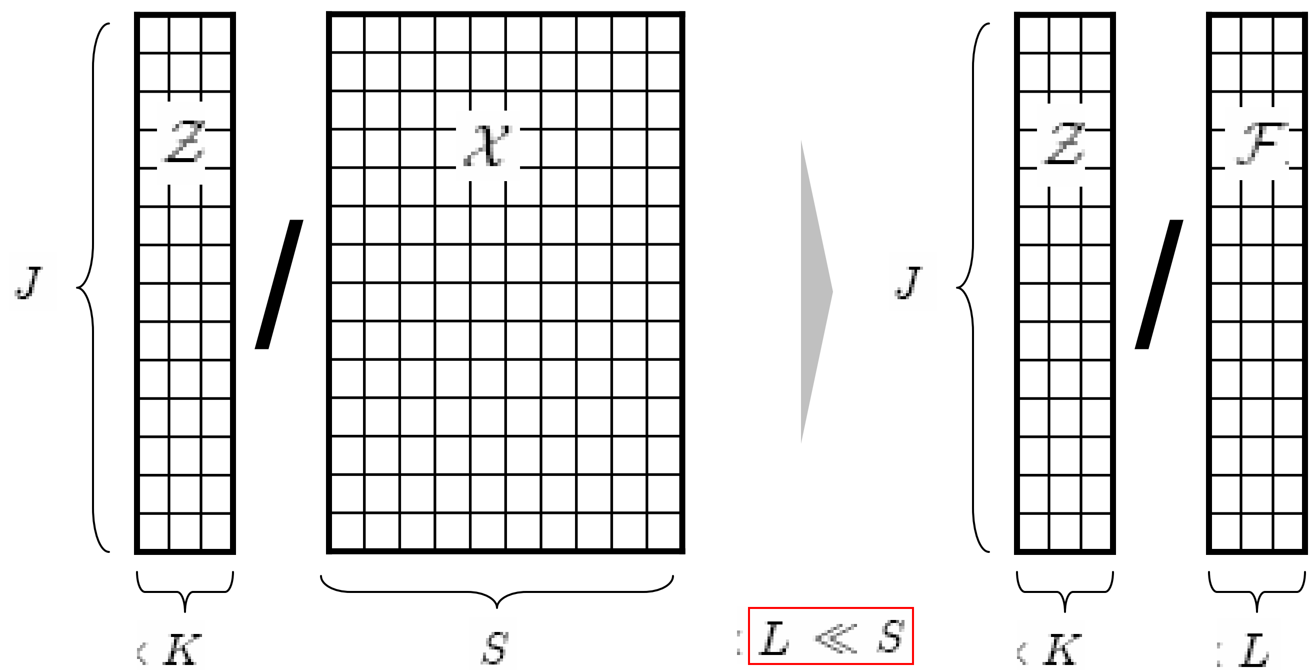
Attribution factors (e.g. fundamental factors)

Conditional link

$\mathbf{Z}$

$$f_{\mathbf{Z}|\mathbf{x}} = f_{\mathbf{Z}|\mathbf{f}} \Leftrightarrow \mathcal{Z}|_{\mathcal{X}} = \mathcal{Z}|_{\mathcal{F}}$$

high quality
conditional copula
matched scenarios



Implementation Steps of Factors on Demand

STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation – Dimension reduction (e.g. PCA)

$$\mathbf{X}$$

$$f_{\mathbf{X}} \Leftrightarrow \boxed{\mathcal{X} \equiv \mathcal{F}\mathbf{B}' + \mathcal{U}}$$ estimation FM

2: Pricing (e.g. Black-Scholes formula)

$$R_n = g_n(X_1, \ldots, X_S)$$

$$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$$

3: Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}\mathbf{w}$$

SDev, VaR, CVaR, Contributions, …

STAGE B: PORTFOLIO MANAGEMENT

4: Attribution factors (e.g. fundamental factors)
Conditional link

$$\mathbf{Z}$$

$$f_{\mathbf{Z}|\mathbf{x}} = f_{\mathbf{Z}|\mathbf{f}} \Leftrightarrow \mathcal{Z}|_{\mathcal{X}} = \mathcal{Z}|_{\mathcal{F}}$$

5: Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \hat{\mathrm{E}}\left[(\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d})^2 \right] \right\}$$

$$\boxed{R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}}$$ attribution FM

Exposures, Hedging, Contributions from factors, …

Implementation Steps of Factors on Demand

STAGE A: RISK MANAGEMENT

1: Risk drivers (e.g. changes of impl. vol.)
Estimation – Dimension reduction (e.g. PCA)

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$$f_{\mathbf{X}} \Leftrightarrow \mathcal{X} \equiv \mathcal{F}\mathbf{B}' + \mathcal{U}$$

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$$R_n = g_n(X_1, \ldots, X_S)$$

$$f_{\mathbf{R}} \Leftrightarrow \mathcal{R}$$

3: Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$$f_{R_{\mathbf{w}}} \Leftrightarrow \mathcal{R}\mathbf{w}$$

SDev, VaR, CVaR, Contributions, … ← **SAME**

STAGE B: PORTFOLIO MANAGEMENT

4: Attribution factors (e.g. hedging instruments)
Conditional link

$\widetilde{\mathbf{Z}}$

$$f_{\widetilde{\mathbf{Z}}|\mathbf{x}} = f_{\widetilde{\mathbf{Z}}|\mathbf{f}} \Leftrightarrow \widetilde{\mathcal{Z}}|_{\mathcal{X}} = \widetilde{\mathcal{Z}}|_{\mathcal{F}}$$

5: Attribution

$$\widetilde{\mathbf{d}}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \hat{\mathrm{E}}\left[(\mathcal{R}\mathbf{w} - \widetilde{\mathcal{Z}}\mathbf{d})^2 \right] \right\}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} \widetilde{d}_{\mathbf{w},k} \widetilde{Z}_k + \widetilde{\eta}_{\mathbf{w}}$$

Exposures, Hedging, Contributions from factors, … ← **DIFFERENT**

Risk drivers

$$\mathbf{X} \equiv \mathbf{BF} + \mathbf{U}$$

Pricing

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

Aggregation

$$R_{\mathbf{w}} = \bar{\mathbf{w}}'\mathbf{R}$$

Risk management

Vol, VaR, CVaR,
Contributions, …

Attribution factors

$$\mathbf{Z}$$

Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \hat{\mathrm{E}} \left[\left(\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d} \right)^2 \right] \right\}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Portfolio management

Exposures, Hedging,
Contributions from factors, …

Risk drivers

$$\mathbf{X} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}$$

Principal component analysis - Random matrix theory

Pricing

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

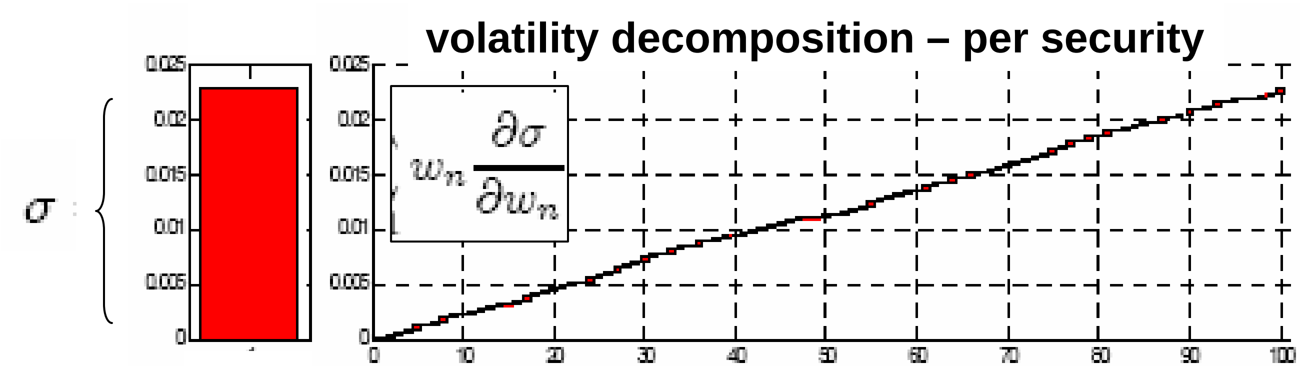
Risk management

Vol, VaR, CVaR, Contributions, …

Risk drivers

$$\mathbf{X} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}$$

Principal component analysis -
Random matrix theory



Risk management

$$\sigma = \sum_{n=1}^{N} w_n \frac{\partial \sigma}{\partial w_n}$$

 Applications – Risk Mgmt. vs. Portfolio Mgmt.

Attribution factors

$\mathbf{Z}$

GICS Industry index returns

Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{E}} \left[\left(\mathcal{R} \mathbf{w} - \mathcal{Z} \mathbf{d} \right)^2 \right] \right\}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Portfolio management

Exposures, Hedging,
Contributions from factors, …

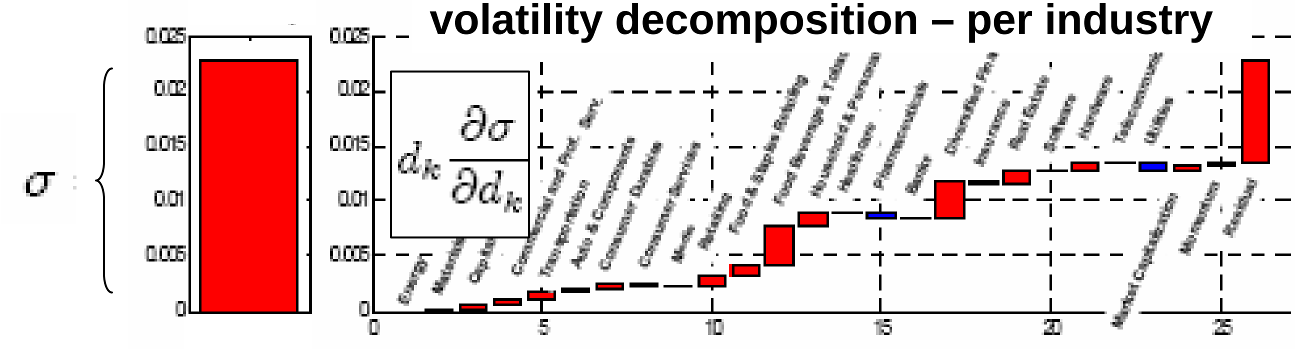
Applications – Risk Mgmt. vs. Portfolio Mgmt.

Attribution factors

$\mathbf{Z}$

GICS Industry index returns

volatility decomposition – per industry



Portfolio management

$$\sigma = \sum_{k=1}^{K+1} d_k \frac{\partial \sigma}{\partial d_k}$$

Risk drivers

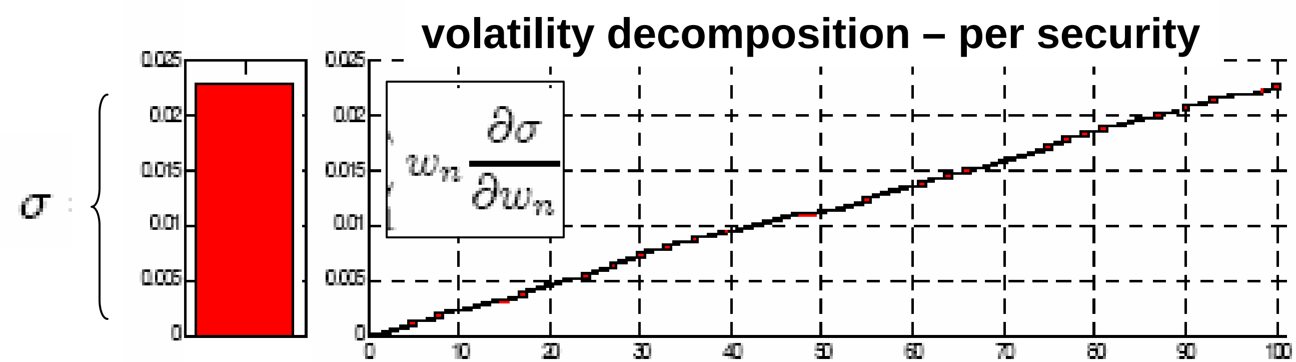
$$\mathbf{X} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}$$

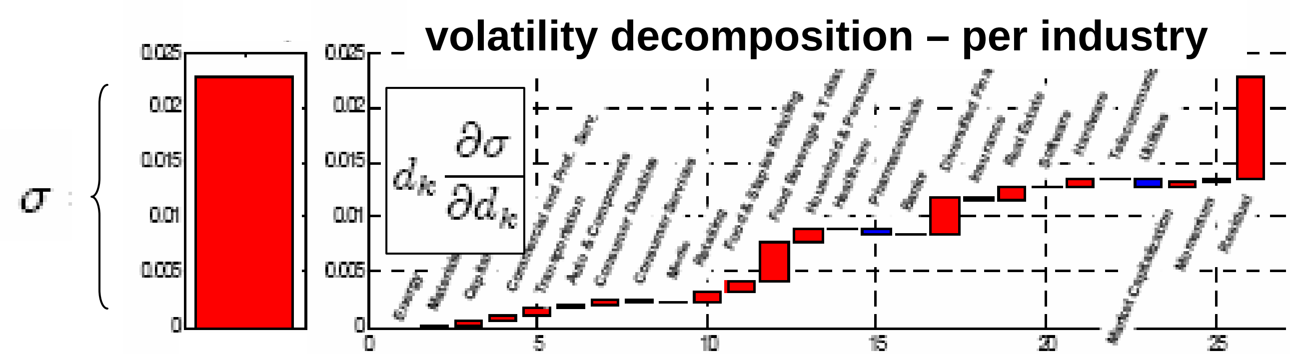
Principal component analysis - Random matrix theory

Attribution factors

$$\mathbf{Z}$$

GICS Industry index returns





Portfolio management analysis consistent with risk management numbers

Risk drivers

Granular regional equity factor model

$$\mathbf{R}^{(\alpha)} \equiv \mathbf{B}^{(\alpha)}\mathbf{F}^{(\alpha)} + \mathbf{U}^{(\alpha)}$$

(e.g. US financial, US utilities,…)

$$\vdots$$

$$\mathbf{R}^{(\omega)} \equiv \mathbf{B}^{(\omega)}\mathbf{F}^{(\omega)} + \mathbf{U}^{(\omega)}.$$

(e.g. UK financial, UK utilities,…)

Risk drivers

Granular regional equity factor model

$$\mathbf{R}^{(\alpha)} \equiv \mathbf{B}^{(\alpha)}\mathbf{F}^{(\alpha)} + \mathbf{U}^{(\alpha)}$$

(e.g. US financial, US utilities,…)

$$\vdots$$

$$\mathbf{R}^{(\omega)} \equiv \mathbf{B}^{(\omega)}\mathbf{F}^{(\omega)} + \mathbf{U}^{(\omega)}.$$

(e.g. UK financial, UK utilities,…)

Attribution factors

Coarse global factors

$$\mathbf{Z} \equiv \mathbf{A}\begin{pmatrix} \mathbf{F}^{(\alpha)} \\ \vdots \\ \mathbf{F}^{(\omega)} \end{pmatrix}$$

(e.g. global financial, global utilities,…)

Applications – Global vs. Regional Equity Model

Risk drivers

Granular regional equity factor model

$$\mathbf{R}^{(\alpha)} \equiv \mathbf{B}^{(\alpha)}\mathbf{F}^{(\alpha)} + \mathbf{U}^{(\alpha)}$$

(e.g. US financial, US utilities,…)

$$\vdots$$

$$\mathbf{R}^{(\omega)} \equiv \mathbf{B}^{(\omega)}\mathbf{F}^{(\omega)} + \mathbf{U}^{(\omega)}.$$

(e.g. UK financial, UK utilities,…)

Attribution factors

Coarse global factors

$$\mathbf{Z} \equiv \mathbf{A} \begin{pmatrix} \mathbf{F}^{(\alpha)} \\ \vdots \\ \mathbf{F}^{(\omega)} \end{pmatrix}$$

(e.g. global financial, global utilities,…)

Aggregation

$$R_{\mathbf{w}} = \bar{\mathbf{w}}'\mathbf{R}$$

Attribution

Coarse global factor equity factor model

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

No need for inconsistent estimates of regional <u>and</u> global models

Risk drivers

$$\mathbf{X} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}$$

Arbitrary estimation criterion

Pricing

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

Aggregation

$$R^{t}_{\mathbf{w}} = \mathbf{w}_t'\mathbf{R}$$

Portfolio at current time *t*

Applications – Point in Time Style Analysis

Risk drivers

$$\mathbf{X} \equiv \mathbf{BF} + \mathbf{U}$$

Arbitrary estimation criterion

Attribution factors

$$\mathbf{Z}$$

Returns of style indices

Pricing

$$R_n = g_n(X_1, \ldots, X_S)$$

Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\boxed{\mathbf{d}'\mathbf{1}=1, \mathbf{d} \geq \mathbf{0}}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{E}} \left[\left(\mathcal{R}\mathbf{w}_t - \mathcal{Z}\mathbf{d} \right)^2 \right] \right\}$$

Sum-to-one, long-only

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Aggregation

$$R_{\mathbf{w}}^{t} = \mathbf{w}_t' \mathbf{R}$$

Portfolio at current time *t*

Risk drivers

$$\mathbf{X} \equiv \mathbf{BF} + \mathbf{U}$$

Arbitrary estimation criterion

Attribution factors

$$\mathbf{Z}$$

Returns of style indices

Pricing

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

Attribution

$$\mathrm{d}_{\mathbf{w}} \equiv \underset{\boxed{\mathbf{d}'\mathbf{1}=1,\,\mathbf{d}\geq\mathbf{0}}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{E}}\left[\left(\mathcal{R}\mathrm{w}_t - \mathcal{Z}\mathrm{d}\right)^2\right]\right\}$$

Sum-to-one, long-only

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Aggregation

$$R_{\mathbf{w}}^{t} = \mathbf{w}_t'\mathbf{R}$$

Portfolio at current time *t*

Point-in-time, non-lagging, non spurious style analysis

Risk drivers

$$\mathbf{X} \equiv \mathbf{BF} + \mathbf{U}$$

Arbitrary estimation criterion

Attribution factors

$$\mathbf{Z} \equiv (\mathbf{R}'\mathbf{w}_1, \ldots, \mathbf{R}'\mathbf{w}_K)'$$

Returns of basis of portfolios

Pricing

$$R_n = g_n(X_1, \ldots, X_S)$$

Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{E}} \left[(\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d})^2 \right] \right\}$$

Unconstrained

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

Risk drivers

$$\mathbf{X} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}$$

Arbitrary estimation criterion

Attribution factors

$$\mathbf{Z} \equiv (\mathbf{R}'\mathbf{w}_1, \ldots, \mathbf{R}'\mathbf{w}_K)'$$

Returns of basis of portfolios

Pricing

$$R_n = g_n (X_1, \ldots, X_S)$$

Attribution

$$\mathbf{d}_{\mathbf{w}} = \left(\mathbf{W}'\widehat{\boldsymbol{\Sigma}}\mathbf{W}\right)^{-1} \mathbf{W}'\widehat{\boldsymbol{\Sigma}}\mathbf{w}$$

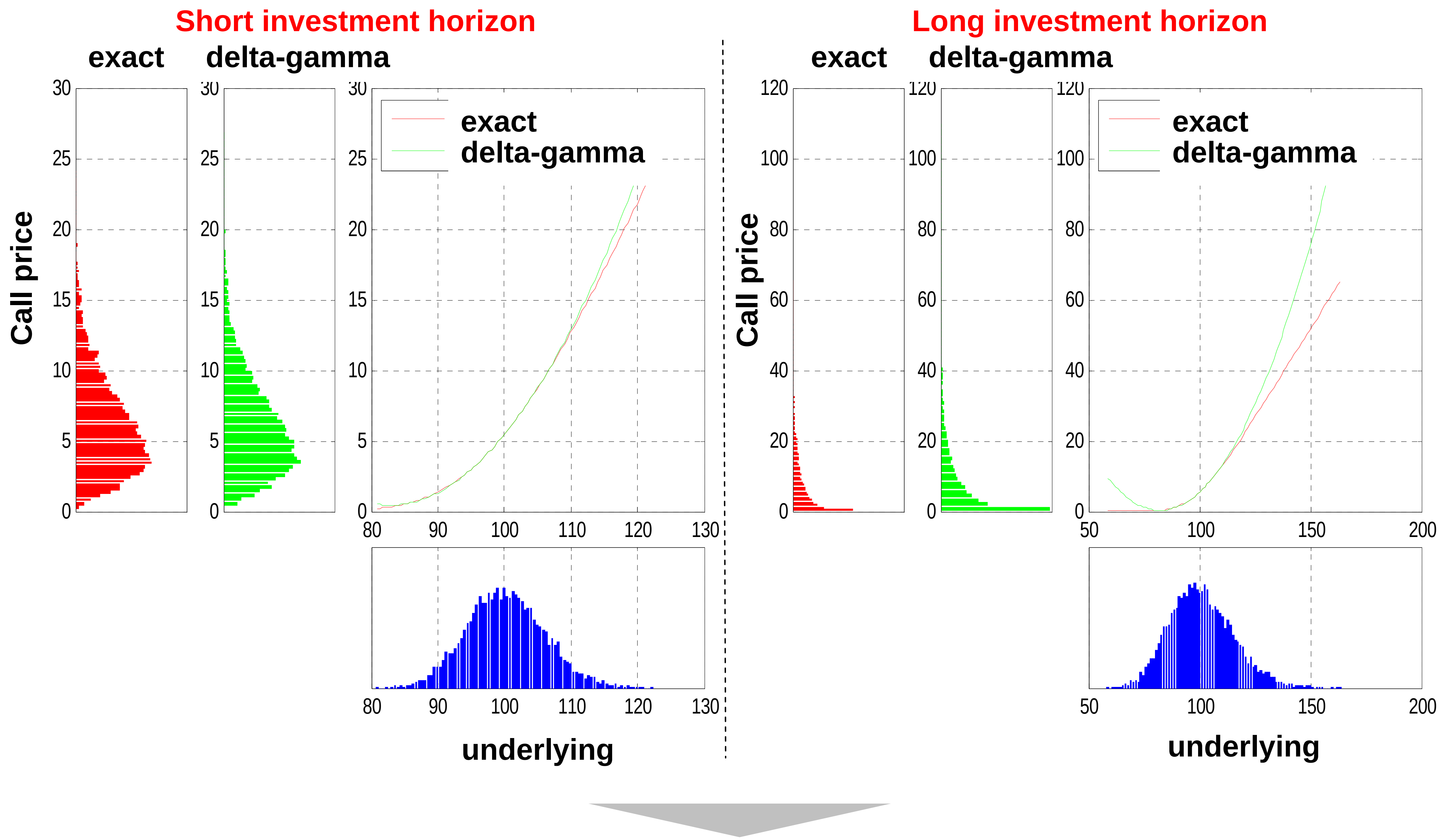
Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Risk attribution to basis of portfolios

Applications – No-Greek Hedging



Greeks approximation becomes inadequate for long investment horizons

Risk drivers

$$\mathbf{X} \equiv \mathbf{BF} + \mathbf{U}$$

Arbitrary estimation criterion

Attribution factors

$$\mathbf{Z}$$

Returns of hedging instruments

Pricing

$$R_n = g_n \left(X_1, \ldots, X_S \right)$$

Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \hat{\mathrm{E}} \left[\left(\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d} \right)^2 \right] \right\}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

Optimal no-Greek hedges

Risk drivers

$$\mathbf{X} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}$$

Arbitrary estimation criterion

Attribution factors

$$\mathbf{Z}$$

Returns of hedging instruments

Pricing

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{CVaR}}\left[\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d}\right] \right\}$$

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\mathbf{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{E}}\left[\left(\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d}\right)^2\right] \right\}$$

Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Optimal no-Greek hedges that promote upside

Risk drivers

$$\mathbf{X} \equiv \mathbf{BF} + \mathbf{U}$$

Arbitrary estimation criterion

e.g.

- compounded return of one underlying
- compounded returns of vol. surf

Attribution factors

$$\mathbf{Z}$$

Returns of hedging instruments

e.g.

- linear return of one underlying

Units of underlying to hedge one call option

	100 days	150 days	200 days	250 days	300 days
FOD	5.8	5.3	5.0	4.9	4.8
BS	5.7	5.4	5.2	5.1	5.0

Applications – Best Pool on Demand

Risk drivers

$$\mathbf{X} \equiv \mathbf{BF} + \mathbf{U}$$

Arbitrary estimation criterion

Attribution factors

$$\mathbf{Z}$$

Returns of hedging instruments

Pricing

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

Attribution

$$\mathrm{d}_{\mathbf{w}} \equiv \underset{\boxed{\mathrm{d} \in \mathcal{C}}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{CVaR}}\left[\mathcal{R}\mathrm{w} - \mathcal{Z}\mathrm{d}\right] \right\}$$

$$\mathrm{d}_{\mathbf{w}} \equiv \underset{\mathrm{d} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \widehat{\mathrm{E}}\left[\left(\mathcal{R}\mathrm{w} - \mathcal{Z}\mathrm{d}\right)^2\right] \right\}$$

Includes cardinality constraint

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Aggregation

$$R_{\mathbf{w}} = \mathrm{w}'\mathbf{R}$$

Best pool of hedges that promote upside

Applications – Best Pool on Demand

Risk drivers

$$\mathbf{X} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}$$

Arbitrary estimation criterion

Attribution factors

$$\mathbf{Z}$$

GICS Industry index returns

Pricing

$$R_n = g_n\left(X_1, \ldots, X_S\right)$$

Attribution

$$\mathbf{d}_{\mathbf{w}} \equiv \underset{\boxed{\mathbf{d} \in \mathcal{C}}}{\operatorname{argmin}} \left\{ \hat{\mathrm{E}}\left[\left(\mathcal{R}\mathbf{w} - \mathcal{Z}\mathbf{d} \right)^2 \right] \right\}$$

Includes cardinality constraint

Aggregation

$$R_{\mathbf{w}} = \mathbf{w}'\mathbf{R}$$

$$R_{\mathbf{w}} \equiv \sum_{k=1}^{K} d_{\mathbf{w},k} Z_k + \eta_{\mathbf{w}}$$

Best portfolio-specific factor model

ESTIMATION VERSUS ATTRIBUTION

STANDARD APPROACH TO FACTOR MODELING

RATIONALE OF FACTORS ON DEMAND

IMPLEMENTATION OF FACTORS ON DEMAND

APPLICATIONS OF FACTORS ON DEMAND

REFERENCES

Appendix: factor models pitfalls

- **Article:**

  Attilio Meucci - "Factors on Demand"
  ***Risk*, July 2010, p 84-89**
  available at http://ssrn.com/abstract=1565134

- **MATLAB examples:**

  MATLAB Central Files Exchange (see above article)

- **This presentation:**

  www.symmys.com > Teaching > Talks

APPENDIX: FACTOR MODELS PITFALLS

FINANCIAL THEORY

QUEST FOR INVARIANCE

NATURE OF RESIDUAL

Factor Models Pitfalls - Financial Theory

$$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n .$$

$\mathbf{R}$	$N \times 1$	**Returns of securities**
$\mathbf{D}$	$N \times K$	**Exposures of returns to factors**
$\mathbf{Z}$	$K \times 1$	**Systematic factors**
$\boldsymbol{\eta}$	$N \times 1$	**Idiosyncratic shocks**

Independent

Supported by Arbitrage Pricing Theory

APT: if $\mathbf{R} = \mathbf{DZ} + \boldsymbol{\eta} \quad \Rightarrow \quad \mathrm{E}\{\mathbf{R}\} = \xi_0 \mathbf{1} + \mathbf{D}\boldsymbol{\xi}$

APPENDIX: FACTOR MODELS PITFALLS

FINANCIAL THEORY

QUEST FOR INVARIANCE

NATURE OF RESIDUAL

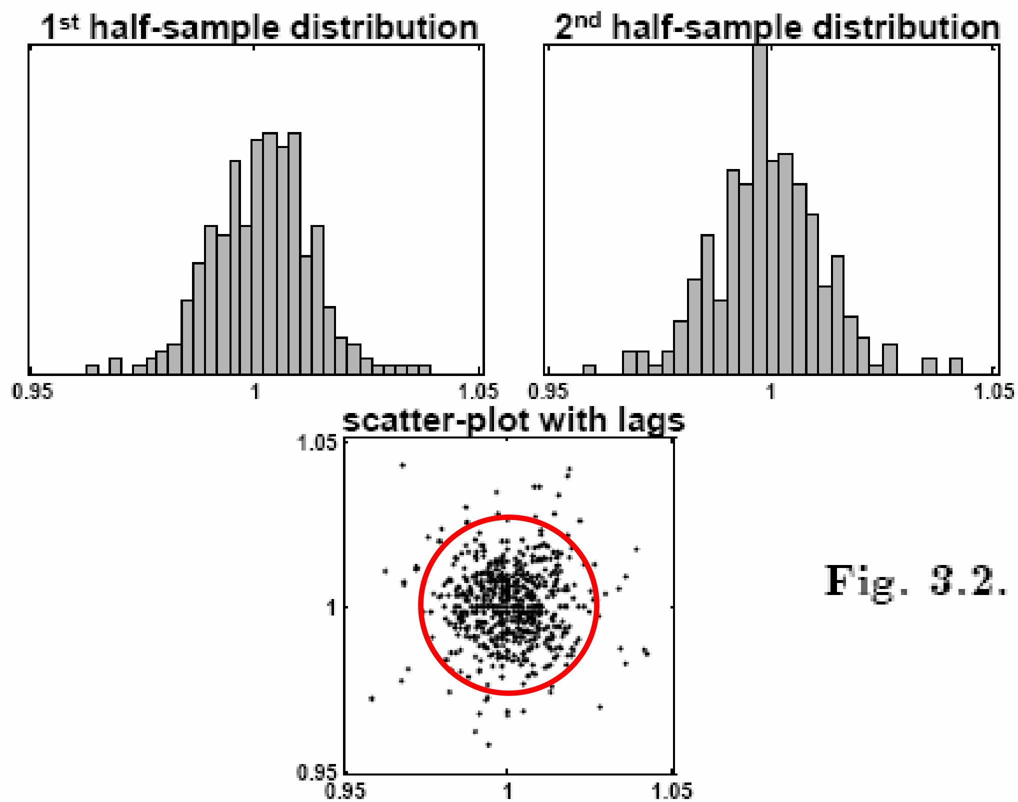


Fig. 3.2.

total returns

$$H_t \equiv \frac{P_t}{P_{t-1}} \quad (3.9)$$

total returns $$H_t \equiv \frac{P_t}{P_{t-1}} \qquad (3.9)$$

linear returns

$$R_t \equiv \frac{P_t}{P_{t-1}} - 1 \qquad (3.10)$$

$$\Longleftrightarrow$$

compounded returns

$$C_t \equiv \ln\left(\frac{P_t}{P_{t-1}}\right) \qquad (3.11)$$

$C_t(K,E)$

price at time t of call with strike K expiring at time E

Factor Models Pitfalls - Quest for Invariance - Derivatives

$C_t(K,E)$

price at time t of call with strike K expiring at time E

$$R_t \equiv \frac{C_t(K,E)}{C_{t-1}(K,E)} - 1$$

return

Factor Models Pitfalls - Quest for Invariance - Derivatives

$C_t(K,E)$

price at time t of call with strike K expiring at time E

$$R_t \equiv \frac{C_t(K,E)}{C_{t-1}(K,E)} - 1$$

returns are **NOT** invariants

Invariants: compounded returns

$$\epsilon_t \equiv \ln S_t - \ln S_{t-1}$$

theory > price:

$$C_t(K,E) = C_{BS}(t, S_t, \sigma; K, E)$$

price at time t of call with strike K expiring at time E

$$\begin{cases} C_{BS}(t,S,\sigma;K,E) \equiv S\Phi(d_1) - Ke^{-r(E-t)}\Phi(d_2) \\ d_1 \equiv \dfrac{-\ln\left(\frac{K}{S}\right) + (E-t)\left(r+\frac{\sigma^2}{2}\right)}{\sqrt{\sigma^2(E-t)}} \\ d_2 \equiv \dfrac{-\ln\left(\frac{K}{S}\right) + (E-t)\left(r-\frac{\sigma^2}{2}\right)}{\sqrt{\sigma^2(E-t)}} \end{cases}$$

Invariants: compounded returns

$$\epsilon_t \equiv \ln S_t - \ln S_{t-1}$$

theory > price:

$$C_t(K,E) = C_{BS}(t, S_t, \sigma; K, E)$$

implied volatility surface

$$(t, K, E) \mapsto \sigma_t(K, E)$$

Factor Models Pitfalls - Quest for Invariance - Derivatives

Invariants: compounded returns

$$\epsilon_t \equiv \ln S_t - \ln S_{t-1}$$

theory > price:

$$C_t(K, E) = C_{BS}(t, S_t, \sigma; K, E)$$

implied volatility surface

$$(t, K, E) \mapsto \sigma_t(K, E)$$

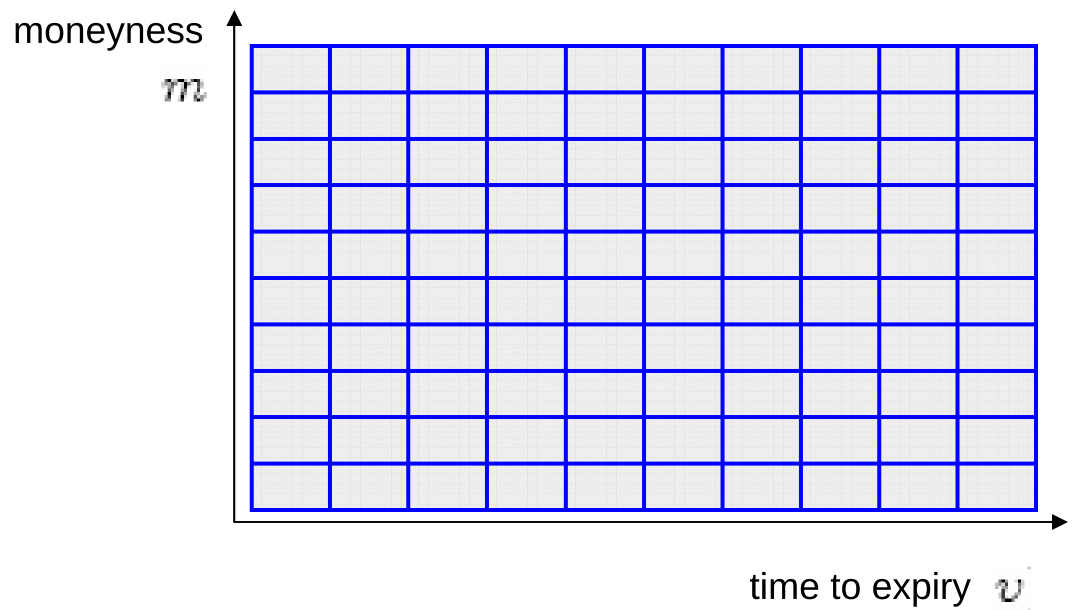
invariant coordinates

$$(t, m, \upsilon) \mapsto \sigma_t(m, \upsilon)$$

moneyness

time to expiry

volatility surface $\sigma_t(m, \upsilon)$



Invariants: compounded returns

$$\epsilon_t \equiv \ln S_t - \ln S_{t-1}$$

theory > price:

$$C_t(K,E) = C_{BS}(t, S_t, \sigma; K, E)$$

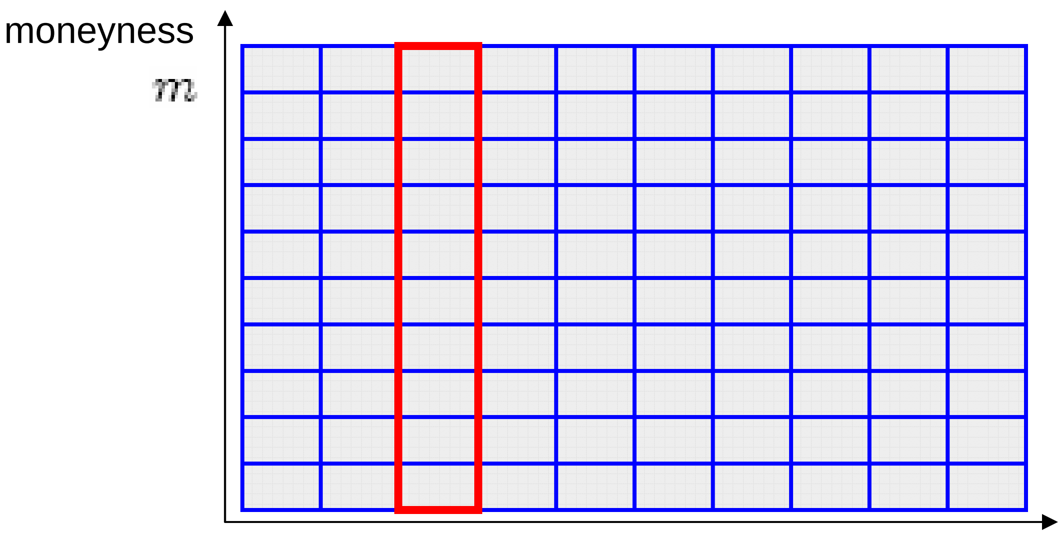
implied volatility surface

$$(t, K, E) \mapsto \sigma_t(K, E)$$

invariant coordinates **volatility slice**

$$(t, m, v) \mapsto \sigma_t(m, v) \quad \Leftrightarrow \quad \boldsymbol{\sigma}_t$$

volatility surface $\sigma_t(m, v)$



volatility slice $\boldsymbol{\sigma}_t$

Invariants: compounded returns

$$\epsilon_t \equiv \ln S_t - \ln S_{t-1}$$

theory > price:

$$C_t(K, E) = C_{BS}(t, S_t, \sigma; K, E)$$

implied volatility surface

$$(t, K, E) \mapsto \sigma_t(K, E)$$

invariant coordinates **volatility slice**

$$(t, m, v) \mapsto \sigma_t(m, v) \quad \Leftrightarrow \quad \boldsymbol{\sigma}_t$$

Invariants: compounded returns of volatility slice

$$\mathbf{X}_t \equiv \ln \boldsymbol{\sigma}_t - \ln \boldsymbol{\sigma}_{t-1}$$

Factor Models Pitfalls - Quest for Invariance - Derivatives

Invariants: compounded returns

$$\epsilon_t \equiv \ln S_t - \ln S_{t-1}$$

theory > price:

$$C_t(K,E) = C_{BS}(t, S_t, \sigma; K, E)$$

implied volatility surface

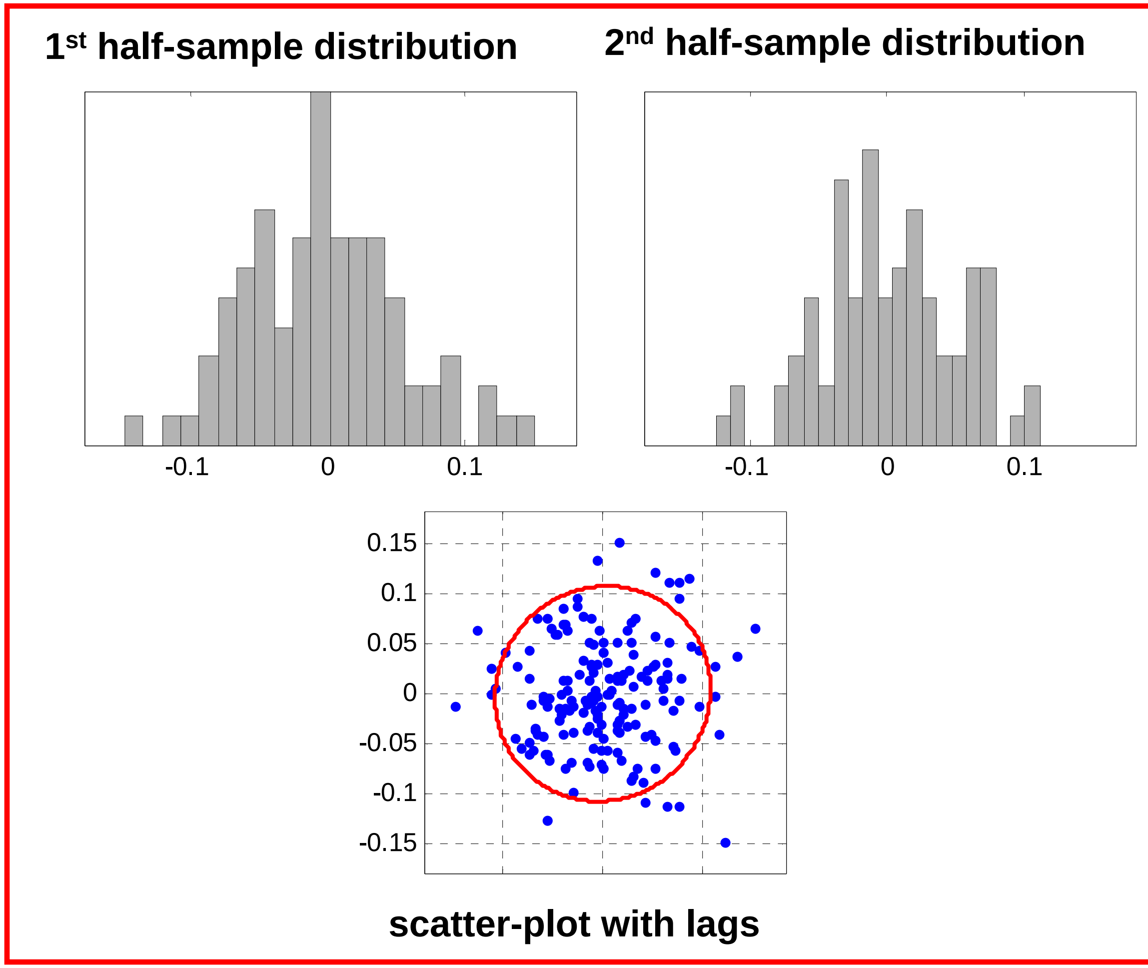
$$(t,K,E) \mapsto \sigma_t(K,E)$$

invariant coordinates

$$(t,m,v) \mapsto \sigma_t(m,v)$$

Invariants: compounded returns of volatility slice

$$\boxed{X_t} \equiv \ln \sigma_t - \ln \sigma_{t-1}$$



Equities: $R = e^{\overset{\text{log-return}}{X}} - 1$

Derivatives: $R = \dfrac{BS\left(X_1, X_2; \theta\right)}{P_0} - 1$

log-return of underlying (X_1)
log-return of implied volatility (X_2)

Bonds: $R = \dfrac{P\left(X_1, X_2; \theta\right)}{P_0} - 1$

government yield changes (X_1)
spread changes (X_2)

Returns R are fully determined by risk drivers / invariants X

Estimation must be performed on risk-drivers/invariants, not on returns

APPENDIX: FACTOR MODELS PITFALLS

FINANCIAL THEORY

QUEST FOR INVARIANCE

NATURE OF RESIDUAL

Factor Models Pitfalls - Nature of Residual

$$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n.$$

$\mathbf{R}$	$N \times 1$	**Returns of securities**
$\mathbf{D}$	$N \times K$	**Exposures of returns to factors**
$\mathbf{Z}$	$K \times 1$	**Systematic factors**
$\boldsymbol{\eta}$	$N \times 1$	**Idiosyncratic shocks**

Independent

Factor Models Pitfalls - Nature of Residual

$$R_n = \sum_{k=1}^{K} d_{n,k} Z_k + \eta_n.$$

$\mathbf{R}$	$N \times 1$	**Returns of securities**
$\mathbf{D}$	$N \times K$	**Exposures of returns to factors**
$\mathbf{Z}$	$K \times 1$	**Systematic factors**
$\boldsymbol{\eta}$	$N \times 1$	**Idiosyncratic shocks**

Independent

…more in general …

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

$\mathbf{X}$	$N \times 1$	**Risk drivers**
$\mathbf{B}$	$N \times K$	**Loadings**
$\mathbf{F}$	$K \times 1$	**Risk factors**
$\mathbf{U}$	$N \times 1$	**Residuals idiosyncratic**

Independent

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n \qquad \begin{cases} \mathbf{X} & N \times 1 & \textbf{Risk drivers} \\ \mathbf{B} & N \times K & \textbf{Loadings} \\ \mathbf{F} & K \times 1 & \textbf{Risk factors} \\ \mathbf{U} & N \times 1 & \textbf{Residuals} \end{cases}$$

$$\underbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_n \\ \vdots \\ X_N \end{pmatrix}}_{\text{risk drivers}} = \underbrace{\begin{pmatrix} b_{1,1} & \cdots & b_{1,k} & \cdots & b_{n,K} \\ \vdots & & \vdots & & \vdots \\ b_{n,1} & \cdots & b_{n,k} & \cdots & b_{n,K} \\ \vdots & & \vdots & & \vdots \\ b_{N,1} & \cdots & b_{N,k} & \cdots & b_{N,K} \end{pmatrix}}_{\text{exposure of drivers to factors}} \underbrace{\begin{pmatrix} F_1 \\ \vdots \\ F_k \\ \vdots \\ F_K \end{pmatrix}}_{\text{factors}} + \underbrace{\begin{pmatrix} U_1 \\ \vdots \\ U_n \\ \vdots \\ U_N \end{pmatrix}}_{\text{shocks for risk driver}}$$

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n \qquad \left\{ \begin{array}{lll} \mathbf{X} & N \times 1 & \textbf{Risk drivers} \\ \mathbf{B} & N \times K & \textbf{Loadings} \\ \mathbf{F} & K \times 1 & \textbf{Risk factors} \\ \mathbf{U} & N \times 1 & \textbf{Residuals} \end{array} \right.$$

Independent

correlated

$$\overbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_n \\ \vdots \\ X_N \end{pmatrix}}^{\text{risk drivers}} = \overbrace{\begin{pmatrix} b_{1,1} & \cdots & b_{1,k} & \cdots & b_{n,K} \\ \vdots & & \vdots & & \vdots \\ b_{n,1} & \cdots & b_{n,k} & \cdots & b_{n,K} \\ \vdots & & \vdots & & \vdots \\ b_{N,1} & \cdots & b_{N,k} & \cdots & b_{N,K} \end{pmatrix}}^{\text{exposure of drivers to factors}} \overbrace{\begin{pmatrix} \boxed{\begin{matrix} F_1 \\ \vdots \\ F_k \\ \vdots \\ F_K \end{matrix}} \end{pmatrix}}^{\text{factors}} + \overbrace{\begin{pmatrix} \boxed{\begin{matrix} U_1 \\ \vdots \\ U_n \\ \vdots \\ U_N \end{matrix}} \end{pmatrix}}^{\text{shocks for risk driver}}$$

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

$\mathbf{X}$	$N \times 1$	**Risk drivers**
$\mathbf{B}$	$N \times K$	**Loadings**
$\mathbf{F}$	$K \times 1$	**Risk factors**
$\mathbf{U}$	$N \times 1$	**Residuals ~~idiosyncratic~~**

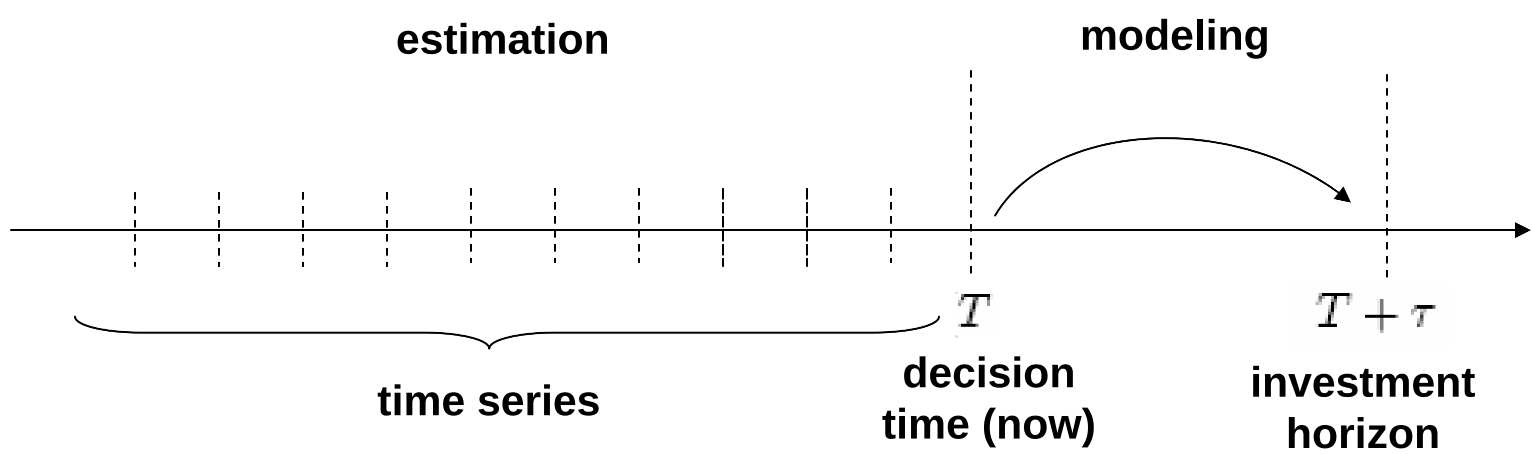
$$\overbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_n \\ \vdots \\ X_N \end{pmatrix}}^{\text{risk drivers}} = \overbrace{\begin{pmatrix} b_{1,1} & \cdots & b_{1,k} & \cdots & b_{n,K} \\ \vdots & & \vdots & & \vdots \\ b_{n,1} & \cdots & b_{n,k} & \cdots & b_{n,K} \\ \vdots & & \vdots & & \vdots \\ b_{N,1} & \cdots & b_{N,k} & \cdots & b_{N,K} \end{pmatrix}}^{\text{exposure of drivers to factors}} \overbrace{\begin{pmatrix} F_1 \\ \vdots \\ F_k \\ \vdots \\ F_K \end{pmatrix}}^{\text{factors}} + \overbrace{\begin{pmatrix} \boxed{U_1} \\ \vdots \\ \boxed{U_n} \\ \vdots \\ U_N \end{pmatrix}}^{\text{shocks for risk driver}}$$

correlated

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

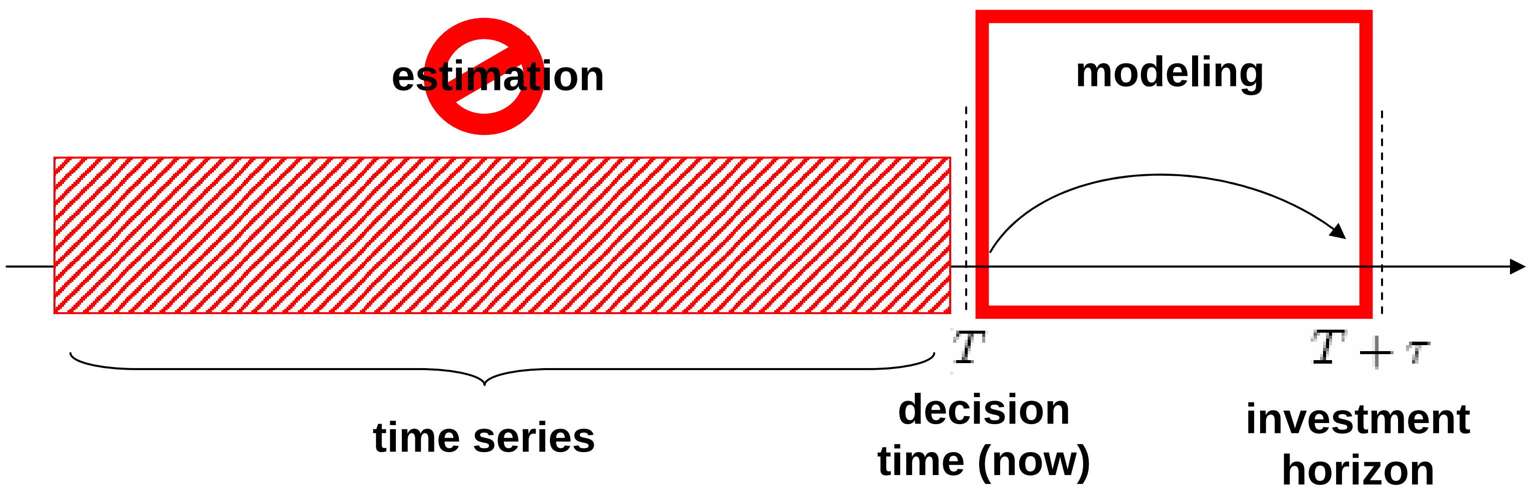
$$\left\{ \begin{array}{lll} \mathbf{X} & N \times 1 & \text{Risk drivers} \\ \mathbf{B} & N \times K & \text{Loadings} \\ \mathbf{F} & K \times 1 & \text{Risk factors} \\ \mathbf{U} & N \times 1 & \text{Residuals} \end{array} \right.$$



Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ **Risk drivers with known distribution** $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ **Loadings**
- $\mathbf{F}$ $K \times 1$ **Risk factors**
- $\mathbf{U}$ $N \times 1$ **Residuals**



Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

$\mathbf{X}$	$N \times 1$	**Risk drivers with known distribution** $f_{\mathbf{X}}$
$\mathbf{B}$	$N \times K$	**Loadings**
$\mathbf{F}$	$K \times 1$	**Risk factors**
$\mathbf{U}$	$N \times 1$	**Residuals**

Optimality Criteria

$$K \ll N$$

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

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Optimality Criteria

$$K \ll N$$

$$\mathrm{Cor}\{\mathbf{F}, \mathbf{U}\} = \mathbf{0}_{K \times N}$$

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

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- $\mathbf{B}$ $N \times K$ **Loadings**
- $\mathbf{F}$ $K \times 1$ **Risk factors**
- $\mathbf{U}$ $N \times 1$ **Residuals**

Optimality Criteria

$K \ll N$

$\mathrm{Cor}\{\mathbf{F}, \mathbf{U}\} = \mathbf{0}_{K \times N}$

$\mathbf{U}$ **"small"** $\Leftrightarrow$ $R^2\{\mathbf{X}, \mathbf{BF}\}$ **large**

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

$\mathbf{X}$	$N \times 1$	**Risk drivers with known distribution $f_{\mathbf{X}}$**
$\mathbf{B}$	$N \times K$	**Loadings**
$\mathbf{F}$	$K \times 1$	**Risk factors**
$\mathbf{U}$	$N \times 1$	**Residuals**

Optimality Criteria

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Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ Risk drivers with known distribution $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ Loadings
- $\mathbf{F}$ $K \times 1$ Risk factors
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F},\mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F},\mathbf{B}$ Optimized

Optimality Criteria

$K \ll N$

$\mathrm{Cor}\{\mathbf{F},\mathbf{U}\} = \mathbf{0}_{K \times N}$

$\mathbf{U}$ "small" $\Leftrightarrow$ $R^2\{\mathbf{X},\mathbf{BF}\}$ large

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Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ Risk drivers with **known** distribution $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ Loadings, **known**
- $\mathbf{F}$ $K \times 1$ Risk factors, **known** distributions $f_{\mathbf{F}}$, $f_{\mathbf{X},\mathbf{F}}$
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F},\mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F},\mathbf{B}$ Optimized

"Residual" approach

e.g.

- $\mathbf{X}$ Bond returns
- $\mathbf{B}$ Key rate durations
- $\mathbf{F}$ Changes in key rates

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ Risk drivers with known distribution $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ Loadings, known
- $\mathbf{F}$ $K \times 1$ Risk factors, known distributions $f_{\mathbf{F}}$, $f_{\mathbf{X},\mathbf{F}}$
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

"Residual" approach

e.g.

- $\mathbf{X}$ Bond returns
- $\mathbf{B}$ Key rate durations
- $\mathbf{F}$ Changes in key rates

Optimality Criteria

- ✓ $K \ll N$
- ✗ $\mathrm{Cor}\{\mathbf{F}, \mathbf{U}\} = \mathbf{0}_{K \times N}$
- ✗ $\mathbf{U}$ "small" $\Leftrightarrow R^2\{\mathbf{X}, \mathbf{BF}\}$ large
- ✗ $\mathbf{U}$ idiosyncratic

 Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n \qquad \begin{cases} \mathbf{X} \quad N \times 1 & \text{Risk drivers with known distribution } f_{\mathbf{X}} \\ \mathbf{B} \quad N \times K & \text{Loadings} \\ \mathbf{F} \quad K \times 1 & \text{Risk factors, known distributions } f_{\mathbf{F}},\ f_{\mathbf{X},\mathbf{F}} \\ \mathbf{U} \quad N \times 1 & \text{Residuals} \end{cases}$$

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

"Time series" approach (misnomer)

e.g. $\mathbf{X}$ **stock returns**

$\mathbf{B}$ **"betas"**

$\mathbf{F}$ **- S&P index return, - industry indices, …**

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ Risk drivers with known distribution $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ Loadings
- $\mathbf{F}$ $K \times 1$ Risk factors, known distributions $f_{\mathbf{F}}$, $f_{\mathbf{X},\mathbf{F}}$
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

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$$\mathbf{B}_r \equiv \underset{\mathbf{B}}{\operatorname{argmax}}\, R^2 \{\mathbf{X}, \mathbf{BF}\}$$
$$= \mathrm{E}\{\mathbf{XF}'\}\,\mathrm{E}\{\mathbf{FF}'\}^{-1}$$

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ Risk drivers with known distribution $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ Loadings
- $\mathbf{F}$ $K \times 1$ Risk factors, known distributions $f_{\mathbf{F}}$, $f_{\mathbf{X},\mathbf{F}}$
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

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$$\mathbf{B}_r \equiv \underset{\mathbf{B}}{\operatorname{argmax}} R^2 \{\mathbf{X}, \mathbf{BF}\} = \mathrm{E}\{\mathbf{XF}'\} \mathrm{E}\{\mathbf{FF}'\}^{-1}$$

Optimality Criteria

✓ $K \ll N$

✗ $\mathrm{Cor}\{\mathbf{F}, \mathbf{U}\} = \mathbf{0}_{K \times N}$

~ $\mathbf{U}$ "small" $\Leftrightarrow$ $R^2 \{\mathbf{X}, \mathbf{BF}\}$ large

✗ $\mathbf{U}$ idiosyncratic

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ Risk drivers with known distribution $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ Loadings
- $\mathbf{F} \mapsto \begin{pmatrix} 1 \\ \mathbf{F} \end{pmatrix}$ Risk factors, known distributions $f_{\mathbf{F}}$, $f_{\mathbf{X},\mathbf{F}}$
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

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✓ $K \ll N$

✓ $\mathrm{Cor}\{\mathbf{F}, \mathbf{U}\} = \mathbf{0}_{K \times N}$

~ $\mathbf{U}$ "small" ⇔ $R^2\{\mathbf{X}, \mathbf{BF}\}$ large

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Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

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$\mathbf{B}$	$N \times K$	Loadings, known
$\mathbf{F}$	$K \times 1$	Risk factors
$\mathbf{U}$	$N \times 1$	Residuals

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

"Cross section" approach

e.g. $\mathbf{X}$ stock returns

$\mathbf{B}$ GICS 1/0 industry partition

$\mathbf{F}$ industry factors

 Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ **Risk drivers with known distribution** $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ **Loadings, known**
- $\mathbf{F}$ $K \times 1$ **Risk factors**
- $\mathbf{U}$ $N \times 1$ **Residuals**

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

"Cross section" approach

e.g. $\mathbf{X}$ stock returns

$\mathbf{B}$ GICS 1/0 industry partition

$\mathbf{F}$ industry factors

$$\mathbf{F}_c \equiv \underset{\mathbf{F} \equiv \mathbf{A}'\mathbf{X}}{\operatorname{argmax}} R^2 \{\mathbf{X}, \mathbf{BF}\}$$

$$= (\mathbf{B}'\mathbf{B})^{-1} \mathbf{B}'\mathbf{X}$$

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

- $\mathbf{X}$ $N \times 1$ Risk drivers with known distribution $f_{\mathbf{X}}$
- $\mathbf{B}$ $N \times K$ Loadings, known
- $\mathbf{F}$ $K \times 1$ Risk factors
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\boxed{\mathbf{B} \text{ Exogenous, } \mathbf{F} \text{ Optimized}}$	$\mathbf{F}, \mathbf{B}$ Optimized

"Cross section" approach

e.g.
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$$= (\mathbf{B}'\mathbf{B})^{-1} \mathbf{B}'\mathbf{X}$$

Optimality Criteria

- ✓ $K \ll N$
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- ~ $\mathbf{U}$ "small" $\Leftrightarrow$ $R^2\{\mathbf{X}, \mathbf{BF}\}$ large
- ✗ $\mathbf{U}$ idiosyncratic

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

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- $\mathbf{B}$ $N \times K$ Loadings
- $\mathbf{F}$ $K \times 1$ Risk factors
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\boxed{\mathbf{F}, \mathbf{B} \text{ Optimized}}$

Principal component analysis

e.g. $\mathbf{X}$ yield curve changes

$\mathbf{B}$ market / slope / butterfly

$\mathbf{F}$ parallel shift / tilt / twist

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

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- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

Principal component analysis

e.g. $\mathbf{X}$ yield curve changes

$\mathbf{B}$ market / slope / butterfly

$\mathbf{F}$ parallel shift / tilt / twist

$$(\mathbf{B}_p, \mathbf{A}_p) \equiv \underset{\mathbf{B},\mathbf{A}}{\operatorname{argmax}} R^2 \{\mathbf{X}, \mathbf{B}\mathbf{A}'\mathbf{X}\}$$

$$\mathbf{A} = \mathbf{B} = \mathbf{E}_K$$

$$\operatorname{Cov}\{\mathbf{X}\} \equiv \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}'$$

$$\mathbf{E}_K \equiv \left(\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(K)}\right)$$

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

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$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

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Optimality Criteria

✓ $K \ll N$

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- $\mathbf{F}$ $K \times 1$ **Risk factors**
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$\mathbf{F}, \mathbf{B}$ **Exogenous**	$\mathbf{F}$ **Exogenous,** $\mathbf{B}$ **Optimized**	$\mathbf{B}$ **Exogenous,** $\mathbf{F}$ **Optimized**	$\boxed{\mathbf{F}, \mathbf{B} \text{ Optimized}}$

Factor analysis

e.g. $\mathbf{X}$ **stock returns**

$\mathbf{B}$ **statistical loadings**

$\mathbf{F}$ **hidden factors**

Factor Models Pitfalls - Nature of Residual

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- $\mathbf{F}$ $K \times 1$ Risk factors
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$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\boxed{\mathbf{F}, \mathbf{B} \text{ Optimized}}$

Factor analysis

e.g. $\mathbf{X}$ stock returns

$\mathbf{B}$ statistical loadings

$\mathbf{F}$ hidden factors

$$\text{Cov}\{\mathbf{X}\} \approx \mathbf{B}\mathbf{B}' + \mathbf{\Delta}$$

diagonal

Factor Models Pitfalls - Nature of Residual

$$X_n = \sum_{k=1}^{K} b_{n,k} F_k + U_n$$

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- $\mathbf{B}$ $N \times K$ Loadings
- $\mathbf{F}$ $K \times 1$ Risk factors
- $\mathbf{U}$ $N \times 1$ Residuals

$\mathbf{F},\mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F},\mathbf{B}$ Optimized

Factor analysis

e.g. $\mathbf{X}$ stock returns

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Optimality Criteria

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$\mathbf{F}, \mathbf{B}$ Exogenous	$\mathbf{F}$ Exogenous, $\mathbf{B}$ Optimized	$\mathbf{B}$ Exogenous, $\mathbf{F}$ Optimized	$\mathbf{F}, \mathbf{B}$ Optimized

Factor analysis

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✗ $\mathbf{U}$ idiosyncratic **!**