Review of Dynamic Allocation Strategies
Convex versus Concave Management

Attilio Meucci[1]
attilio_meucci@symmys.com

this version: August 8, 2010
latest version available at http://ssrn.com/abstract=1635982

Abstract

We review the main approaches to dynamically allocate capital between a risky portfolio and a safe account: horizon-driven strategies, such as the expected utility maximization and the option-based portfolio insurance (OBPI); and myopic heuristics such as the constant proportion portfolio insurance (CPPI), the constant weight/exposure portfolio, and the simple buy & hold portfolio.

We present a refresher of the theory under general assumptions. We discuss the connections among the different approaches, as well as their relationship with convex and concave strategies. We provide explicit, practicable solutions with all the computations. Fully documented code for all the strategies is also provided.

JEL Classification: C1, G11

Keywords: CPPI, OBPI, option replication, dynamic programming, Hamilton-Jacobi-Bellman equation, Pontryagin principle, geometric Brownian motion, power utility.

[1] The author is grateful to Eva Chan and Gianluca Fusai

1 Introduction

We consider investments that dynamically allocate capital between a risky portfolio and a safe account. The remainder of this article is organized as follows. In Section 1.1 we introduce the dynamics of the two investment vehicles and in Section 1.2 we discuss the rules to build strategies in full generality. In Section 2 we discuss the first of two types of horizon-driven strategies, namely strategies that maximize the expected utility of the payoff at the investment horizon. The solution of this type of problem in full generality requires advanced techniques, but a special notable case can be solved explicitly with little effort. In Section 3 we discuss the second type of horizon-driven strategy, namely dynamic replication of a given payoff at the horizon, also known under the misnomer of option-based-portfolio-insurance. In Section 4 we cover dynamic heuristics, such as the constant proportion portfolio insurance. In Section 5 we comment on the connections among the above approaches and how they relate to convex and concave strategies. Two major issues are not covered in this refresher: transaction costs and estimation risk.

1.1 The market

We denote by D_t the value of one unit of the risk-free asset at the generic time t, which evolves deterministically. The benchmark model for the evolution of the deterministic asset D_t is the exponential deterministic growth with constant risk-free rate r

$$\frac{dD_t}{D_t} = rdt. \tag{1}$$

We can think of this asset as the money market.

We denote by P_t the value of one unit of the risky asset at the generic time t, which evolves stochastically. We assume that P_t follows a general diffusive process

$$dP_t = \mu\left(t, P_t\right) dt + \sigma\left(t, P_t\right) dB_t. \tag{2}$$

In this expression the deterministic drift $\mu\left(t,p\right)$ and the volatility $\sigma\left(t,p\right)$ are smooth functions of their arguments; B_t is a Brownian motion, i.e. a stochastic process such that all non-overlapping increments are independently and normally distributed with null expectation and variance equal to the time step $B_{t+s} - B_t \sim \mathrm{N}\left(0, s\right)$; and dB_t denotes the infinitesimal increments of the Brownian motion $dB_t \equiv B_{t+dt} - B_t \sim \mathrm{N}\left(0, dt\right)$.

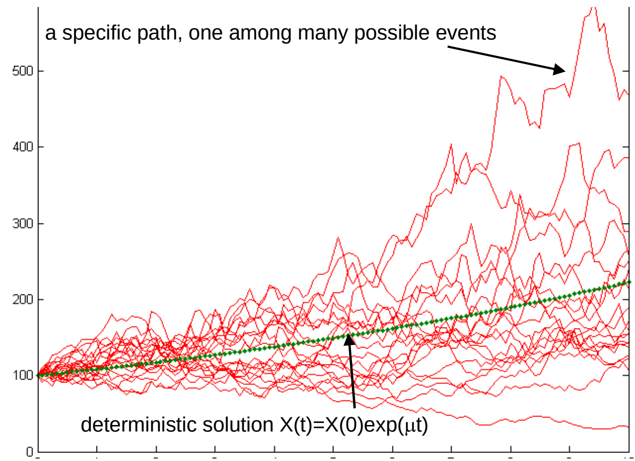

Figure 1: Geometric Brownian motion

The benchmark model for the evolution of the stochastic asset P_t is the geometric Brownian motion with constant drift and constant volatility adopted among others by Merton (1969) and Black and Scholes (1973)

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t, \tag{3}$$

which is the special case of the diffusion (2) with

$$\mu(t,p) \equiv \mu p, \quad \sigma(t,p) \equiv \sigma p. \tag{4}$$

This process integrates to a lognormal random variable at any horizon

$$P_t = P_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma(B_t - B_0)}. \tag{5}$$

In Figure 1 we plot some paths of this process.

We can think of the risky investment P_t as a stock, a bond, a commodity futures, but we emphasize that the risky investment can also be portfolio.

For instance, the geometric Brownian motion dynamics for the risky investment (3) is consistent with a portfolio of N securities where the weights $\mathbf{w}_t \equiv (w_{t,1}, \ldots, w_{t,N})'$ remain constant $\mathbf{w}_t \equiv \mathbf{w}$, and the prices $\mathbf{P}_t \equiv (P_{t,1}, \ldots, P_{t,N})'$ evolve according to a multivariate geometric Brownian motion

$$\frac{d\mathbf{P}_t}{\mathbf{P}_t} = \boldsymbol{\mu} dt + \boldsymbol{\Sigma} d\mathbf{B}_t, \tag{6}$$

where the division is meant entry-by-entry, $\boldsymbol{\mu}$ is a N-dimensional vector, $\boldsymbol{\Sigma}$ is a full-rank $N \times N$ matrix, and $\mathbf{B}_t$ are N independent Brownian motions. Indeed, as we show in Appendix A.1, in this case the portfolio follows the process (3), where

$$\mu \equiv \mathbf{w}'\boldsymbol{\mu}, \quad \sigma^2 \equiv \mathbf{w}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{w}. \tag{7}$$

1.2 Strategies

We denote by α_t the number of units of the risky asset at time t and by β_t the number of units of the risk-free asset. We denote the current time by $t \equiv 0$ and the investment horizon by $t \equiv \tau$.

A strategy is a sequence of allocations that rebalances between the two assets at the generic time t throughout the investment period based on the information available at time t

$$\{\alpha_t, \beta_t\}_{t\in[0,\tau]} \,. \tag{8}$$

The combined value S_t of a the strategy at the generic time t is

$$S_t = \alpha_t P_t + \beta_t D_t. \tag{9}$$

The strategy must be self-financing, i.e. whenever a rebalancing occurs $(\alpha_t, \beta_t) \mapsto (\alpha_{t+\delta t}, \beta_{t+\delta t})$ the following must hold true

$$S_{t+\delta t} \equiv \alpha_t P_{t+\delta t} + \beta_t D_{t+\delta t} \equiv \alpha_{t+\delta t} P_{t+\delta t} + \beta_{t+\delta t} D_{t+\delta t}. \tag{10}$$

The strategy (8) can be represented equivalently in terms of the relative weights of the risky and risk-free investments respectively

$$w_t \equiv \frac{\alpha_t P_t}{S_t}, \quad u_t \equiv \frac{\beta_t D_t}{S_t}. \tag{11}$$

The self-financing constraint (10) is equivalent to the weight of the risk-free asset being equal to $u_t \equiv 1 - w_t$ at all times.

Therefore the strategy (8) can be represented equivalently by the free evolution of the weight of the risky asset $\{w_t\}_{t\in[0,\tau]}$. When w_t is close to 1 the strategy evolves as the risky asset P_t. Conversely, if w_t is close to zero, the strategy evolves as the risk-free asset. More in general, as we show in Appendix A.1, the strategy evolves as

$$\frac{dS_t}{S_t} = (1 - w_t)\, r dt + w_t \frac{dP_t}{P_t}. \tag{12}$$

The path for the weight of the risky asset $\{w_t\}_{t\in[0,\tau]}$ and the initial budget constraint

$$S_0 \text{ given.} \tag{13}$$

determine the distribution of the final payoff of the strategy

$$S_0, \{w_t\}_{t\in[0,\tau]} \mapsto S_\tau. \tag{14}$$

As we show in Appendix A.2, when the risky investment follows a geometric Brownian motion (3), the final value (14) is lognormally distributed

$$S_{\tau} = S_0 e^{Y_{w(\cdot)}}, \tag{15}$$

where Y is normal

$$Y_{w(\cdot)} \sim \mathrm{N}\left(m_{w(\cdot)}, s^2_{w(\cdot)}\right) \tag{16}$$

with expected value

$$m_{w(\cdot)} \equiv r\tau + \int_0^{\tau} \left((\mu - r)\, w_t - \frac{\sigma^2}{2} w_t^2 \right) dt \tag{17}$$

and variance

$$s^2_{w(\cdot)} = \int_0^{\tau} \sigma^2 w_t^2 dt. \tag{18}$$

2 Expected utility maximization

Here we discuss strategies that maximize the expected utility of the payoff at the investment horizon, an approach pioneered by Merton (1969).

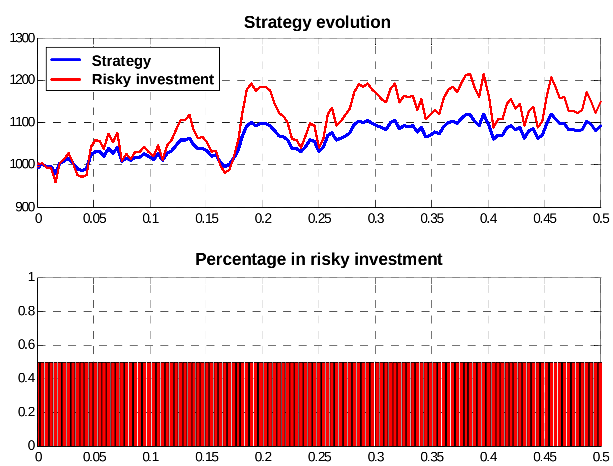

Figure 2: Constant weight dynamic strategy: one path

First of all, we introduce a utility function u that allows us to measure the satisfaction ensuing from the horizon payoff S_{τ} of a strategy in terms of the expected utility $\mathrm{E}\{u(S_{\tau})\}$. Assuming that "rich is better than poor", the function u must be increasing

$$u' > 0. \tag{19}$$

Also, assuming that "richer is better when poor, than when rich", it is common practice to assume that u is concave

$$u'' < 0. \tag{20}$$

The benchmark utility function is the power function

$$u(s) = \frac{s^{\gamma}}{\gamma}, \tag{21}$$

where $\gamma < 1$. This function satisfies (19) and (20). Indeed

$$u'(s) = s^{\gamma-1} > 0 \tag{22}$$

$$u''(s) = (\gamma - 1) s^{\gamma-2} < 0. \tag{23}$$

As in (14), the final payoff S_τ of a strategy depends on the budget constraint (13) on the initial value S_0 and the dynamic rebalancing path $w_{(\cdot)}$. The investor selects the optimal dynamic allocation path $w^*_{(\cdot)}$ in such a way to maximize the expected utility of the strategy at the horizon under the budget constraint and a potential set of additional constraints $\mathcal{C}$

$$w^*_{(\cdot)} \equiv \underset{S_0, w_{(\cdot)} \in \mathcal{C}}{\operatorname{argmax}} \left(\mathrm{E}\left\{ u\left(S_\tau\right) \right\} \right). \tag{24}$$

There exist two approaches to solve this problem in general: dynamic programming, as in Merton (1969) and Merton (1992), and martingale methods explored by Pliska (1986), Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987).

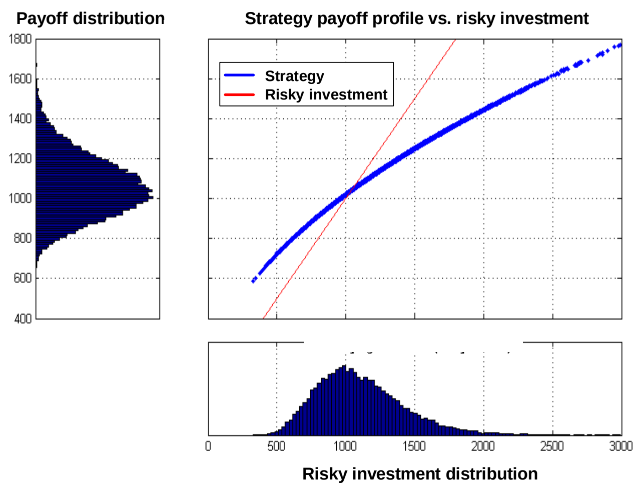

Figure 3: Constant weight dynamic strategy: final payoff in many paths

In our simplified market dynamics (12) with power utility (21) and under no other constraint than the budget, we can solve for the optimal strategy (24) by direct computation, as in Omberg (2001).

From (15) and the expression for the expectation of a lognormal variable we obtain

$$\mathrm{E}\left\{u\left(S_{\tau}\right)\right\}=\frac{S_{0}^{\gamma}}{\gamma}\,\mathrm{E}\{e^{\gamma Y_{w(\cdot)}}\}=\frac{S_{0}^{\gamma}}{\gamma}e^{\gamma\left(m_{w(\cdot)}+\frac{\gamma}{2}s_{w(\cdot)}^{2}\right)}. \tag{25}$$

Substituting (17) and (18), the optimal strategy (24) solves

$$w_{(\cdot)}^{*}\equiv\underset{w_{(\cdot)}}{\operatorname{argmax}}\left\{\int_{0}^{\tau}\left(w_{t}\left(\mu-r\right)-w_{t}^{2}\frac{\sigma^{2}}{2}\left(1-\gamma\right)\right)dt\right\}. \tag{26}$$

The solution to this problem is the value that maximizes the integrand at each time. Therefore, the solution is the constant

$$w_{(\cdot)}^{*}\equiv w\equiv\frac{1}{1-\gamma}\frac{\mu-r}{\sigma^{2}}. \tag{27}$$

As in standard mean-variance, the more promising the risky investment, i.e. large μ or small σ, the more the investor will allocated in the risky asset. Also, γ can be interpreted as a risk-propensity parameter: the closer the parameter γ to the upper boundary 1, the more the investor borrows cash to buy the risky asset.

In Figure 2 we plot the evolution of the constant weight strategy in a specific path. In Figure 3 we plot the payoff profile of this strategy as a function of the risky asset in many paths. Please refer to Meucci (2009) for more details and to download the fully commented code that generated the figures.

3 Dynamic payoff replication

Here we discuss the second of two types of horizon-driven strategies, namely option-based-portfolio-insurance, proposed by Rubinstein and Leland (1981).

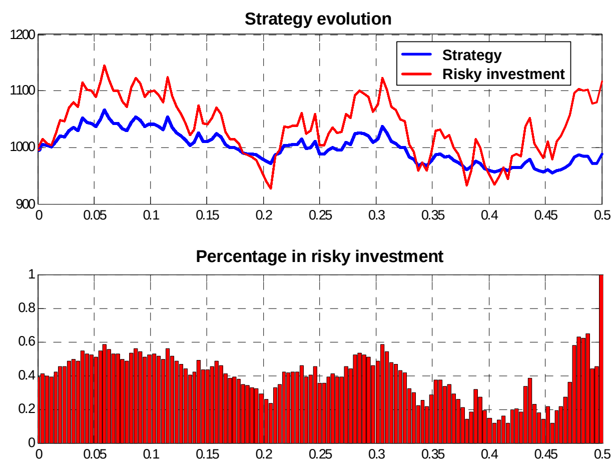

Figure 4: Dynamic replication of call option profile: one path

The final payoff S_τ of a dynamic strategy is stochastic, but all the randomness is driven by the process P_t for the risky asset. Although it is clearly not possible to specify the desired outcome of an investment strategy, quite surprisingly it is possible to specify the desired outcome as an arbitrary function of the outcome of the risky asset P_τ at the investment horizon

$$S_\tau \equiv s\left(P_\tau\right). \tag{28}$$

Any such payoff is attainable with an initial budget S_0 which depends on the profile and can be computed exactly.

Indeed, define $G\left(t, p\right)$ as the solution to the following partial differential equation

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial p} r + \frac{1}{2} \frac{\partial^2 G}{\partial p^2} \left(\sigma\left(t, p\right)\right)^2 - Gr = 0, \tag{29}$$

with boundary condition

$$G\left(\tau, p\right) \equiv s\left(p\right), \quad p > 0, \tag{30}$$

where r and σ in (29) are the risk-free rate (1) and the volatility of the risky investment (2).

Then the required budget is

$$S_0 \equiv G\left(0, P_0\right) \tag{31}$$

and the following dynamic allocation in the risky asset provides the desired payoff (28)

$$w_t \equiv \frac{P_t}{S_t} \frac{\partial G(t, P_t)}{\partial P_t}, \tag{32}$$

see the proofs in Appendix A.3.

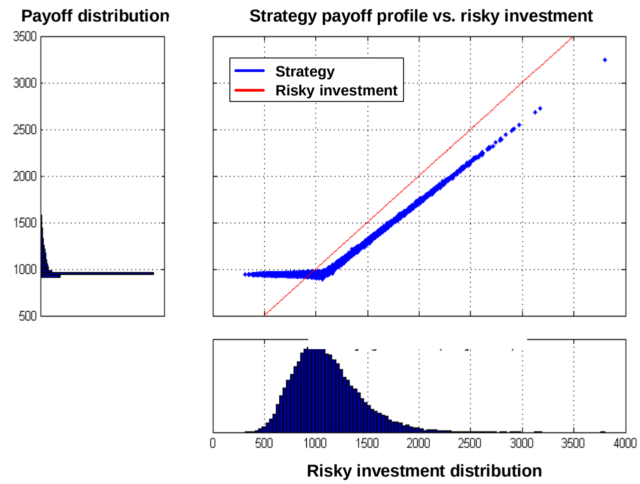

Figure 5: Dynamic replication of call option profile: final payoff in many paths

For instance, consider a call option profile with strike K

$$s(p) = \max\{0, p - K\}. \tag{33}$$

This profile is a fundamental brick that allows to generate arbitrary payoffs, see Appendix A.4.

Assume that the risky investment follows a geometric Brownian motion (3). Then

$$G(t, p) = p\Phi(d_1) - e^{-r(\tau - t)} K \Phi(d_2), \tag{34}$$

where Φ is the cumulative distribution for the standard normal distribution and

$$d_1(t, p) \equiv \frac{1}{\sigma\sqrt{\tau - t}} \left(\ln\left(\frac{p}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(\tau - t) \right) \tag{35}$$

$$d_2(t, p) \equiv d_1(t, p) - \sigma\sqrt{\tau - t}. \tag{36}$$

This is the celebrated result in Black and Scholes (1973), the proof of which can be easily found on the internet.

From the explicit analytical expression (34) we can derive the expression for the weight (32) of the risky asset

$$w_t = \frac{P_t}{S_t} \Phi(d_1(t, P_t)). \tag{37}$$

In Figure 4 we plot the evolution of this strategy along a specific path. In Figure 5 we plot the payoff profile of the strategy as a function of the risky asset in many paths. Please refer to Meucci (2009) for more details and to download the fully commented code that generated the figures.

4 Dynamic heuristics

Here we discuss the most popular myopic heuristic to dynamically rebalance between the risky investment and the risk-free asset, namely the constant proportion portfolio insurance (CPPI) by Black and Perold (1992).

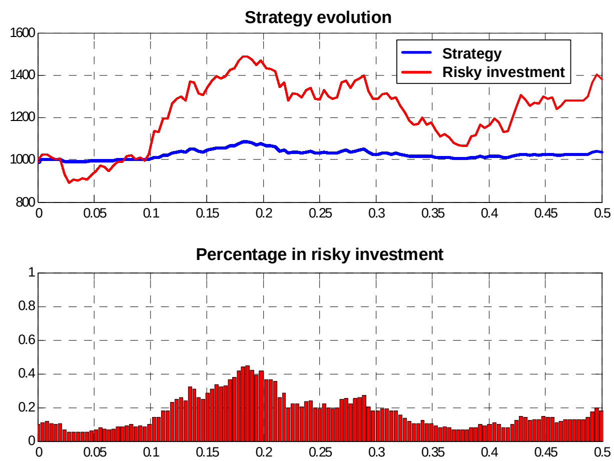

Figure 6: CPPI: one path

According to this strategy the investor first specifies a deterministically increasing floor F_t that satisfies the budget constraint

$$F_0 \leq S_0 \tag{38}$$

and grows to a guaranteed value H at the horizon

$$F_t \equiv He^{-r(\tau-t)}, \quad t \in [0, \tau]. \tag{39}$$

At all times t, for any level of the strategy S_t there is an excess cushion

$$C_t \equiv \max(0, S_t - F_t). \tag{40}$$

According to the CPPI, a constant multiple m of the cushion is invested in the risky asset, therefore obtaining the dynamic strategy's weight

$$\underline{w} \leq w_t \equiv \frac{mC_t}{S_t} \leq \overline{w}, \tag{41}$$

where $\underline{w}$ and $\overline{w}$ are lower and upper boundaries. As the strategy gains value, the investor puts a relatively larger amount of money in the risky asset. As wealth shrinks toward the floor (39) the investor turns more and more conservative

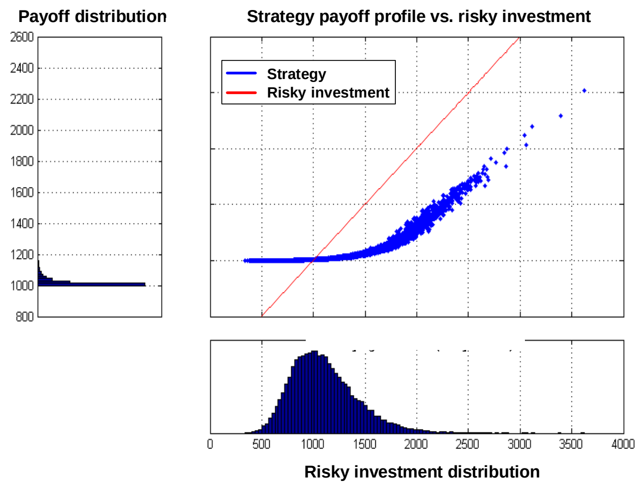

Figure 7: CPPI: final payoff in many paths

To illustrate, we can perform the CPPI under the geometric Brownian motion assumption (3) for the evolution of the risky asset. Bouye (2009) provides the analytical expression for the cushion. In Figure 6 we plot the evolution of this strategy in a specific path. In Figure 7 we plot the payoff profile of this strategy as a function of the risky asset in many paths. Please refer to Meucci (2009) for more details and to download the fully commented code that generated the figures.

5 Concave versus convex management

As we can appreciate in Figure 3, the constant-weight strategy ensuing from utility maximization is concave: concave strategies perform better when the risky asset moves sideways, but worse when the risky asset goes down.

On the other hand, the call option dynamic replication in Figure 5 and the CPPI in Figure 7 are convex strategies: convex strategies protect the investor when the risky asset underperforms, although they fail to capture all the upside when the risky asset rallies. Therefore convex strategies are suitable to provide portfolio insurance.

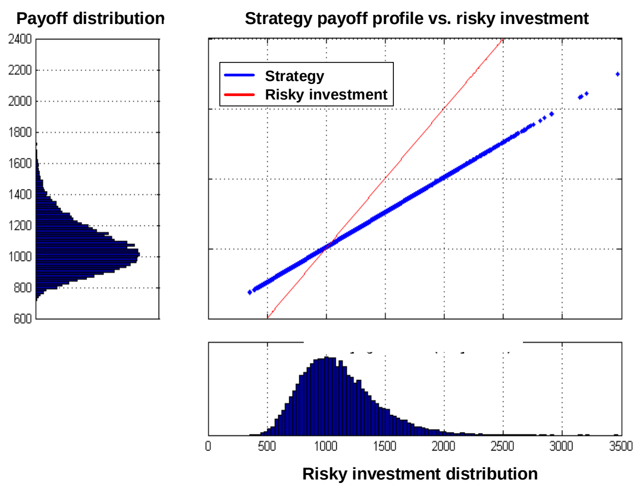

Figure 8: Buy & hold strategy: final payoff in many paths

Finally, the buy and hold strategy is linear, see Figure 8: only a percentage of the risk in the risky asset is reflected by the strategy, so that both upside and downside are limited. Notice that the buy and hold strategy is not the same as constant weight strategy, see Figure 9: indeed, in the buy & hold strategy as the risky asset rallies, the respective weight in the portfolio increases.

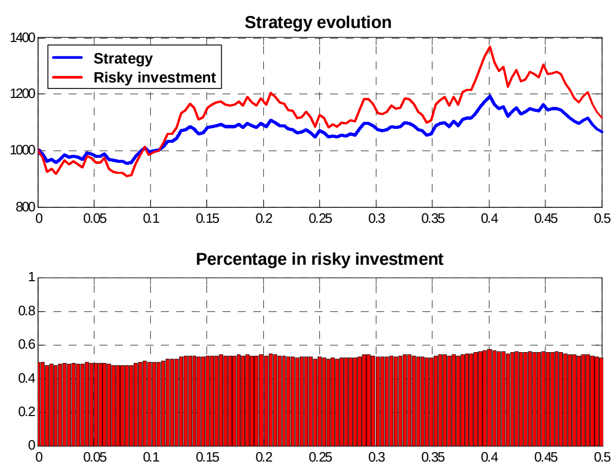

Figure 9: Buy & hold strategy: one path

More in general, all the different dynamic strategies discussed so far are interconnected and can display convex, linear, or concave profiles, see also Bertrand and Prigent (2001), Bouye (2009) and references therein.

The dynamic payoff replication discussed in Section 3 can be linear, convex, or concave by definition, as this depends on the subjective choice of the shape of the final payoff (28). Therefore, the term "option based portfolio insurance" given to this approach is a misnomer, because portfolio insurance refers to convex strategies only.

The expected utility maximization discussed in Section 2 can also be both convex and concave. This follows from a result in Brennan and Solanki (1981), that connects expected utility maximization with dynamic payoff replication. Indeed, given an arbitrary concave utility function u, the strategy that maximizes expected utility is the same as the dynamic replication of the payoff

$$s(p) \equiv u'^{-1}\left(\lambda \frac{\pi(p)}{f(p)}\right). \tag{42}$$

In this expression u' can be inverted because u is concave (20); π is the discounted risk-neutral pdf of the risky investment at the horizon P_τ, or state-price density; f is the real-measure pdf of P_τ;and λ is a constant, set in terms of the quantile function Q of f in such a way to satisfy the budget constraint:

$$\int_0^1 \frac{\pi(Q(t))}{f(Q(t))} u'^{-1}\left(\lambda \frac{\pi(Q(t))}{f(Q(t))}\right) dt \equiv S_0. \tag{43}$$

In Appendix A.5 we provide the proof in full detail.

Finally, dynamic heuristics include dynamic payoff replication and expected utility maximization as special cases, therefore dynamic heuristics include convex profiles as the CPPI, as well as concave profiles.

References

Bertrand, P., and J. L. Prigent, 2001, Portfolio insurance strategies: OBPI versus CPPI, *Working Paper* Available at http://ssrn.com/abstract=299688.

Black, F., and A. Perold, 1992, Theory of constant proportion portfolio insurance, *Journal of Economic Dynamics and Control* 16, 403–426.

Black, F., and M. S. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–654.

Bouye, E., 2009, Portfolio insurance: A short introduction, *Working Paper* Available at http://ssrn.com/abstract=1416790.

Brennan, M. J., and R. Solanki, 1981, Optimal portfolio insurance, *Journal of Financial and Quantitative Analysis* 16, 279–300.

Cox, J., and C. F. Huang, 1989, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33–83.

Cox, J. C., and H. E. Leland, 2000, On dynamic investment strategies, *Journal of Economic Dynamics and Control* 24, 1859–1880.

Dybvig, P. H., 1988, Inefficient dynamic portfolio strategies or how to throw away a million dollars in the stock market, *Review of Financial Studies* 1, 67–88.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve, 1987, Optimal portfolio and consumption decisions for a small investor on a finite horizon, *SIAM Journal of Control and Optimization* 27, 1157–1186.

Merton, R. C., 1969, Lifetime portfolio selection under uncertainty: The continuous case, *Review of Economical Statistics* 51, 247–257.

———, 1992, *Continuous-Time Finance* (Blackwell).

Meucci, A., 2005, *Risk and Asset Allocation* (Springer).

———, 2009, Exercises in advanced risk and portfolio management - with step-by-step solutions and fully documented code, *Free E-Book* available at http://ssrn.com/abstract=1447443.

Omberg, E., 2001, Non-myopic asset-allocation with stochastic interest rates, *Working Paper*.

Pliska, S. R., 1986, A stochastic calculus model of continuous trading: Optimal portfolios, *Mathematics of Operations Research* 11, 371–382.

Rubinstein, M., and H. E. Leland, 1981, Replicating options with positions in stock and cash, *Financial Analysts Journal* 37, 63–72.

A Appendix

In this appendix we discuss technical results that can be skipped at first reading.

A.1 Strategy dynamics, the general case

The self-financing condition in a market of N securities reads

$$\sum_{n=1}^{N} \alpha_n \left(P_n + dP_n\right) \equiv \sum_{n=1}^{N} \left(\alpha_n + d\alpha_n\right)\left(P_n + dP_n\right), \tag{44}$$

which simplifies to

$$\sum_{n=1}^{N} d\alpha_n \left(P_n + dP_n\right) \equiv 0 \tag{45}$$

The evolution of a strategy S_t is described by Ito's rule

$$\begin{aligned} dS &= \sum_{n=1}^{N} d\left(\alpha_n P_n\right) = \sum_{n=1}^{N} d\alpha_n P_n + \alpha_n dP_n + d\alpha_n dP_n \\ &= \sum_{n=1}^{N} \alpha_n dP_n = \sum_{n=1}^{N} \alpha_n P_n \frac{dP_n}{P_n}, \end{aligned} \tag{46}$$

where we made use of equation (45). The last expression implies

$$\frac{dS}{S} = \sum_{n=1}^{N} \frac{\alpha_n P_n}{S_t} \frac{dP_n}{P_n} = \sum_{n=1}^{N} w_n \frac{dP_n}{P_n}, \tag{47}$$

where w_n are the portfolio weights.

A.2 Strategy dynamics, gBm market

If as in (6) the prices follow a multivariate geometric Brownian motion

$$\frac{d\mathbf{P}_t}{\mathbf{P}_t} = \boldsymbol{\mu} dt + \boldsymbol{\Sigma} d\mathbf{B}_t, \tag{48}$$

then from (47) we obtain

$$\frac{dS_t}{S_t} = \mathbf{w}_t' \frac{d\mathbf{P}_t}{\mathbf{P}_t} = \mathbf{w}_t' \boldsymbol{\mu} dt + \mathbf{w}_t' \boldsymbol{\Sigma} d\mathbf{B}_t. \tag{49}$$

Since $d\mathbf{B}_t \sim \mathrm{N}\left(0, \mathbf{I}dt\right)$ it follows

$$\mathbf{w}_t' \boldsymbol{\Sigma} d\mathbf{B}_t \sim \mathrm{N}\left(0, \mathbf{w}_t' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \mathbf{w}_t dt\right). \tag{50}$$

Therefore

$$\frac{dS_t}{S_t} \overset{d}{=} \mu_{\mathbf{w}_t} dt + \sigma_{\mathbf{w}_t} dB_t, \tag{51}$$

where

$$\mu_{\mathbf{w}_t} \equiv \mathbf{w}_t' \boldsymbol{\mu}, \quad \sigma^2_{\mathbf{w}_t} \equiv \mathbf{w}_t' \boldsymbol{\Sigma\Sigma}' \mathbf{w}_t. \tag{52}$$

Then Ito's rule yields

$$\begin{aligned} d\ln S &= \frac{1}{S} dS - \frac{1}{2S^2} (dS)^2 \qquad (53)\\ &= \mu_{\mathbf{w}_t} dt + \sigma_{\mathbf{w}_t} dB - \frac{1}{2S^2} \left(S\mu_{\mathbf{w}_t} dt + S\sigma_{\mathbf{w}_t} dB_t \right)^2 \\ &= \left(\mu_{\mathbf{w}_t} - \frac{\sigma^2_{\mathbf{w}_t}}{2} \right) dt + \sigma_{\mathbf{w}_t} dB_t, \end{aligned}$$

or

$$\ln S_\tau - \ln S_0 = \int_0^\tau \left(\mu_{\mathbf{w}_t} - \frac{\sigma^2_{\mathbf{w}_t}}{2} \right) dt + \int_0^\tau \sigma_{\mathbf{w}_t} dB_t. \tag{54}$$

Using Ito's isometry we obtain

$$\ln S_\tau - \ln S_0 \overset{d}{=} \int_0^\tau \left(\mu_{\mathbf{w}_t} - \frac{\sigma^2_{\mathbf{w}_t}}{2} \right) dt + Z \int_0^\tau \sigma^2_{\mathbf{w}_t} dt, \tag{55}$$

where $Z \sim \mathrm{N}(0,1)$.

A.3 Payoff replication

In this section we consider the two-security market (1)-(2). Assume that we can compute the solution $G(t,p)$ of the following partial differential equation

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial p} pr + \frac{1}{2} \frac{\partial^2 G}{\partial p^2} (\sigma(t,p))^2 = Gr, \tag{56}$$

with boundary condition

$$G(\tau, p) \equiv s(p), \quad p > 0. \tag{57}$$

Now consider a strategy S_t that Invests the initial budget

$$S_0 \equiv G(0, P_0) \tag{58}$$

and contentiously rebalances between a risk-free asset and the risky asset, with the following weight

$$w_t \equiv \frac{P_t}{S_t} \frac{\partial G(t, P_t)}{\partial P_t}. \tag{59}$$

Then the strategy evolves in such a way that the following identity holds at all times, and in particular at $t \equiv \tau$

$$S_t \equiv G(t, P_t), \quad t \in [0, \tau]. \tag{60}$$

Indeed, using Ito's rule and denoting by $\mu_t \equiv \mu(t, P_t)$ and $\sigma_t \equiv \sigma(t, P_t)$ the functions in the diffusion process.

$$\begin{aligned} dG_t &= \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial P_t} dP_t + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} (dP_t)^2 \\ &= \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial P_t} (\mu_t dt + \sigma_t dB) + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} \sigma_t^2 dt \\ &= \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial P_t} \mu_t + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} \sigma_t^2 \right) dt + \frac{\partial G}{\partial P_t} \sigma_t dB_t. \end{aligned} \tag{61}$$

Also, from (12) and (59) we obtain

$$\begin{aligned} dS_t &= S_t r dt + S_t w_t \left(\frac{dP_t}{P_t} - r dt \right) \\ &= S_t r dt + P_t \frac{\partial G}{\partial P_t} \left(\frac{dP_t}{P_t} - r dt \right) \\ &= S_t r dt + \frac{\partial G}{\partial P_t} (\mu_t dt + \sigma_t dB_t - P_t r dt) \end{aligned} \tag{62}$$

Therefore

$$\begin{aligned} d(G_t - S_t) &= \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial P_t} \mu_t + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} \sigma_t^2 \right) dt + \frac{\partial G}{\partial P_t} \sigma_t dB_t \\ &\quad - S_t r dt - \frac{\partial G}{\partial P_t} (\mu_t dt + \sigma_t dB_t - P_t r dt) \\ &= \left(\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} \sigma_t^2 - S_t r + \frac{\partial G}{\partial P_t} P_t r \right) dt \end{aligned} \tag{63}$$

Using (56) we obtain

$$\frac{d(G_t - S_t)}{(G_t - S_t)} = r dt, \tag{64}$$

Which means

$$(G_\tau - S_\tau) = (G_0 - S_0) e^{r\tau}. \tag{65}$$

Since from (58) we have $G_0 - S_0 = 0$, it follows that (60) holds true.

A.4 General derivative payoffs in terms of calls

Define

$$\begin{aligned} g(x) &\equiv f(k) + (\partial_x f)|_k (x - k) + \int_k^{+\infty} \left(\partial_u^2 f\right)\big|_u (x - u)^+ du \\ &\quad + \int_{-\infty}^{k} \left(\partial_u^2 f\right)\big|_u (u - x)^+ du, \end{aligned} \tag{66}$$

where $(x)^+ \equiv \max(0, x)$, f is a smooth function and k is a given arbitrary fixed point. Notice that $g(x)$ is a continuum linear combination of call option profiles

(33). We want to prove that $g(x) = f(x)$. We will do this by showing that the two functions are the same in the point k: $g(k) = f(k)$.and the derivatives of any order s are the same $\left.(\partial_x^s g)\right|_x = \left.(\partial_x^s f)\right|_x$. If we prove this, then we easily obtain

$$g(x) = g(k) + \sum_{s=1}^{\infty} \left.(\partial_x^s g)\right|_k \frac{(x-k)^s}{s!} = f(k) + \sum_{s=1}^{\infty} \left.(\partial_x^s f)\right|_k \frac{(x-k)^s}{s!} = f(x) \quad (67)$$

The first point follows from:

$$\begin{aligned} g(k) &= f(k) + \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u (k-u)^+ \, du + \int_{-\infty}^{k} \left.\left(\partial_u^2 f\right)\right|_u (u-k)^+ \, du \\ &= f(k) . \end{aligned} \quad (68)$$

As for the second point, we first recall some results. Denote by Θ the Heaviside function

$$\Theta(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases} \quad (69)$$

and by δ the Dirac delta

$$\int_{-\infty}^{+\infty} \delta(x) f(x) \, dx = f(0) , \quad (70)$$

Then

$$\partial_x (x-u)^+ = \Theta(x-u) \quad (71)$$

$$\partial_x^2 (x-u)^+ = \delta(x-u) \quad (72)$$

$$\partial_x (u-x)^+ = -\Theta(u-x) \quad (73)$$

$$\partial_x^2 (u-x)^+ = \delta(x-u) . \quad (74)$$

For a proof, it suffices to smoothen the Dirac delta with a Gaussian kernel as in Meucci (2005), derive the results for the smooth version, and then consider the limit as the bandwidth of the kernel goes to zero.

First we consider the first derivative of both sides of (66). By the above differentiation rules we obtain

$$\begin{aligned} \left.(\partial_x g)\right|_x &= \left.(\partial_x f)\right|_k + \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u \left.\partial_x (x-u)^+\right|_x du + \int_{-\infty}^{k} \left.\left(\partial_u^2 f\right)\right|_u \left.\partial_x (u-x)^+\right|_x du \\ &= \left.(\partial_x f)\right|_k + \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u \Theta(x-u) \, du - \int_{-\infty}^{k} \left.\left(\partial_u^2 f\right)\right|_u \Theta(u-x) \, du \\ &= \left.(\partial_x f)\right|_k + \Theta(x-k) \int_k^{x} \left.\left(\partial_u^2 f\right)\right|_u du - \Theta(k-x) \int_x^{k} \left.\left(\partial_u^2 f\right)\right|_u du \quad (75) \\ &= \left.(\partial_x f)\right|_k + \left.(\partial_x f)\right|_x - \left.(\partial_x f)\right|_k \\ &= \left.(\partial_x f)\right|_x \end{aligned}$$

Next we consider the second derivative of both sides of (66)

$$\begin{aligned}
\left(\partial_x^2 g\right)\big|_x &= \int_k^{+\infty} \left(\partial_u^2 f\right)\big|_u \, \partial_x^2 (x-u)^+\Big|_x \, du + \int_{-\infty}^k \left(\partial_u^2 f\right)\big|_u \, \partial_x^2 (u-x)^+\Big|_x \, du \qquad (76) \\
&= \int_k^{+\infty} \left(\partial_u^2 f\right)\big|_u \, \delta(x-u)\, du + \int_{-\infty}^k \left(\partial_u^2 f\right)\big|_u \, \delta(x-u)\, du \\
&= \Theta(x-k) \int_k^{+\infty} \left(\partial_u^2 f\right)\big|_u \, \delta(x-u)\, du + \Theta(k-x) \int_{-\infty}^k \left(\partial_u^2 f\right)\big|_u \, \delta(x-u)\, du \\
&= \left(\partial_x^2 f\right)\big|_x
\end{aligned}$$

Finally we consider the higher order $s > 2$ derivatives of both sides of (66)

$$\begin{aligned}
\left(\partial_x^s g\right)|_x &= \partial_x^s \left(\int_k^{+\infty} \left(\partial_u^2 f\right)\big|_u (x-u)^+ \, du + \int_{-\infty}^k \left(\partial_u^2 f\right)\big|_u (u-x)^+ \, du \right) \qquad (77) \\
&= \partial_x^{s-2} \left(\int_k^{+\infty} \left(\partial_u^2 f\right)\big|_u \, \partial_x^2 (x-u)^+\Big|_x \, du + \int_{-\infty}^k \left(\partial_u^2 f\right)\big|_u \, \partial_x^2 (u-x)^+\Big|_x \, du \right) \\
&= \partial_x^{s-2} \left(\partial_x^2 f\right)\big|_x \\
&= \left(\partial_x^s f\right)|_x
\end{aligned}$$

Therefore the result follows.

A.5 Utility maximization versus payoff replication

First we recall a result from control theory. Consider an objective functional to be minimized

$$J[\psi, x] \equiv \int_0^T L(x_t, \psi_t)\, dt + \Gamma(x_T), \qquad (78)$$

where the control $\psi_t \in \Psi$ and where the controlled path satisfies

$$\dot{x}_t = h(x_t, \psi_t) \qquad (79)$$

$$x_0 \equiv a, \quad x_T \equiv b \qquad (80)$$

To determine the optimal path x_t^* and the optimal control ψ_t^* we introduce the Hamiltonian

$$H(x, \psi, \lambda) \equiv \lambda h(x, \psi) + L(x, \psi), \qquad (81)$$

Then Pontryagin's principle states

$$\psi_t^* = \underset{\psi \in \Psi}{\operatorname{argmin}} \left\{ H(x_t^*, \psi_t, \lambda_t^*) \right\} \qquad (82)$$

$$\dot{\lambda}_t^* = -\partial_x H(\lambda_t^*, x_t^*, \psi_t^*). \qquad (83)$$

Going back to our original problem of the equivalence between final payoff allocation and expected utility maximization, we want to prove that, given a

utility function u we can find a strategy payoff S_τ such that
a) The payoff S_τ is a function of the risky asset at the horizon P_τ as in (28)

$$S_\tau \equiv \varphi\left(P_\tau\right). \tag{84}$$

b) The payoff S_τ maximizes the expected utility as in (24). Denoting by f the pdf of P_τ and using (84) this condition becomes

$$\varphi^* \equiv \underset{\varphi}{\operatorname{argmax}} \left\{ \int_0^\infty u\left(\varphi\left(p\right)\right) f\left(p\right) dp \right\}, \tag{85}$$

where
c) The payoff S_τ satisfies the budget constraint (13). Using the fundamental theorem of asset pricing, there exists a probability measure, called risk-neutral, such that

$$\frac{S_0}{D_0} = \mathrm{E}\left\{ \frac{S_\tau}{D_\tau} \right\}. \tag{86}$$

Denoting by π the risk-neutral pdf of P_τ times the discounting $D_0/D_\tau = e^{-r\tau}$, the budget constraints reads

$$S_0 \equiv \int_0^\infty \varphi\left(p\right) \pi\left(p\right) dp \tag{87}$$

Therefore, denoting by Q quantile function of the distribution f the problem (84)-(85)-(86) becomes as in Brennan and Solanki (1981)

$$\varphi^* \equiv \underset{\varphi}{\operatorname{argmax}} \left\{ \int_0^1 u\left(\varphi\left(Q\left(t\right)\right)\right) dt \right\}, \tag{88}$$

such that

$$S_0 \equiv \int_0^1 \varphi\left(Q\left(t\right)\right) \frac{\pi\left(Q\left(t\right)\right)}{f\left(Q\left(t\right)\right)} dt \tag{89}$$

Defining

$$x_t \equiv \int_0^t \varphi\left(Q\left(z\right)\right) \frac{\pi\left(Q\left(z\right)\right)}{f\left(Q\left(z\right)\right)} dz \tag{90}$$

This means that the constraint reads

$$\dot{x}_t \equiv \varphi\left(Q\left(t\right)\right) \frac{\pi\left(Q\left(t\right)\right)}{f\left(Q\left(t\right)\right)}, \quad x_0 \equiv 0, \quad x_1 \equiv S_0. \tag{91}$$

We can rephrase (88)-(89) as (78)-(79)-(80) with the following substitutions

$$\begin{aligned}
L\left(x_t, \psi_t\right) &\equiv -u\left(\psi_t\right) && (92)\\
h\left(x_t, \psi_t\right) &\equiv \psi_t \gamma_t && (93)\\
\Gamma &\equiv 0 && (94)\\
\Psi &\equiv \mathbb{R} && (95)\\
T &\equiv 1 && (96)\\
a &\equiv 0, \quad b \equiv S_0 && (97)
\end{aligned}$$

where

$$\begin{aligned}\psi_t &\equiv \varphi\left(Q\left(t\right)\right) && (98)\\ \gamma_t &\equiv \frac{\pi\left(Q\left(t\right)\right)}{f\left(Q\left(t\right)\right)}. && (99)\end{aligned}$$

Therefore (82)-(83) become

$$\begin{aligned}\psi_t^* &= \underset{\psi}{\operatorname{argmin}}\left\{\lambda_t^*\psi\gamma_t^* - u\left(\psi\right)\right\} && (100)\\ \dot{\lambda}_t^* &= -\partial_x\left\{\lambda_t^*\psi_t^*\gamma_t - u\left(\psi_t^*\right)\right\} = 0. && (101)\end{aligned}$$

From (101) we obtain that λ_t^* must be constant

$$\lambda_t^* \equiv \lambda^*. \quad (102)$$

From (100) we obtain

$$\psi_t^* = u'^{-1}\left(\lambda^*\gamma_t\right), \quad (103)$$

or

$$\varphi^*\left(q\right) = u'^{-1}\left(\lambda^*\frac{\pi\left(q\right)}{f\left(q\right)}\right), \quad (104)$$

where λ^* is set from (89) in such a way that

$$\int_0^1 \frac{\pi\left(Q\left(t\right)\right)}{f\left(Q\left(t\right)\right)} u'^{-1}\left(\lambda^*\frac{\pi\left(Q\left(t\right)\right)}{f\left(Q\left(t\right)\right)}\right) dt \equiv S_0. \quad (105)$$