Attilio Meucci

Black-Litterman and Beyond

from Normal Markets to Fully Flexible Views

ESTIMATION RISK

SCENARIO ANALYSIS

THE BLACK-LITTERMAN APPROACH

ENTROPY POOLING

CASE STUDIES

REFERENCES AND CONCLUSIONS

BL and beyond - estimation risk

$$\boldsymbol{m} \equiv \mathrm{E}\left\{\boldsymbol{R}_{T+\tau}\right\}$$: **expected returns**

$$\boldsymbol{S} \equiv \mathrm{Cov}\left\{\boldsymbol{R}_{T+\tau}\right\}$$: **covariance**

$$\boldsymbol{w}_v \equiv \underset{w \in \mathcal{C}}{\operatorname{argmax}}\left\{\boldsymbol{w}'\boldsymbol{m}\right\}$$

subject to $\boldsymbol{w}'\boldsymbol{S}\boldsymbol{w} \leq v$

$\boldsymbol{w}$ **: portfolio weights**

v **: grid of target variances**

$\mathcal{C}$ **: investment constraints, e.g.** $\boldsymbol{w}'\boldsymbol{1} = 1,\ \boldsymbol{w} \geq \boldsymbol{0}$

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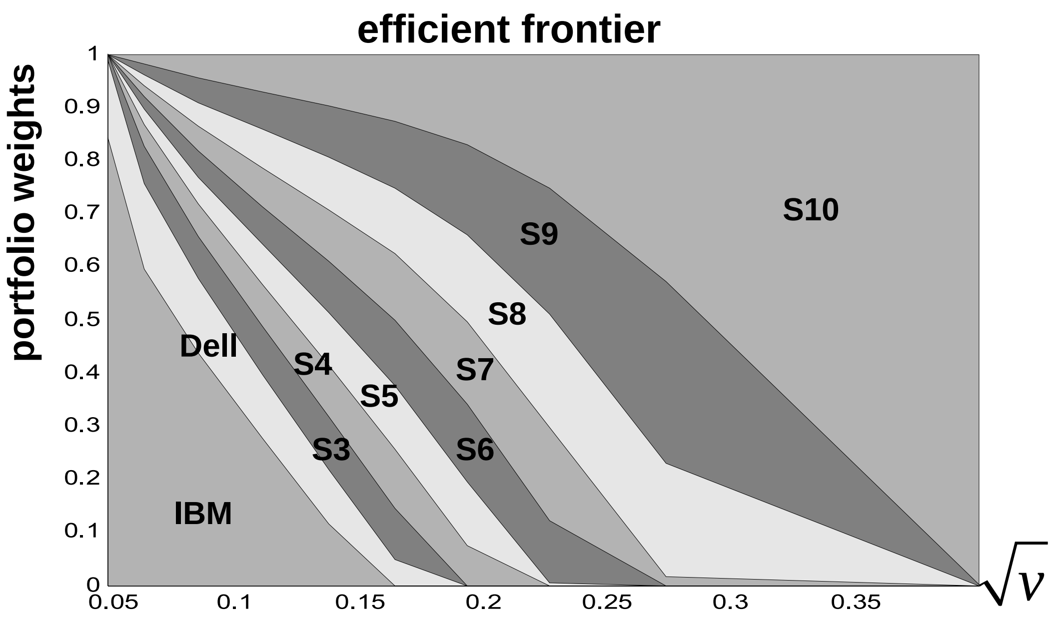

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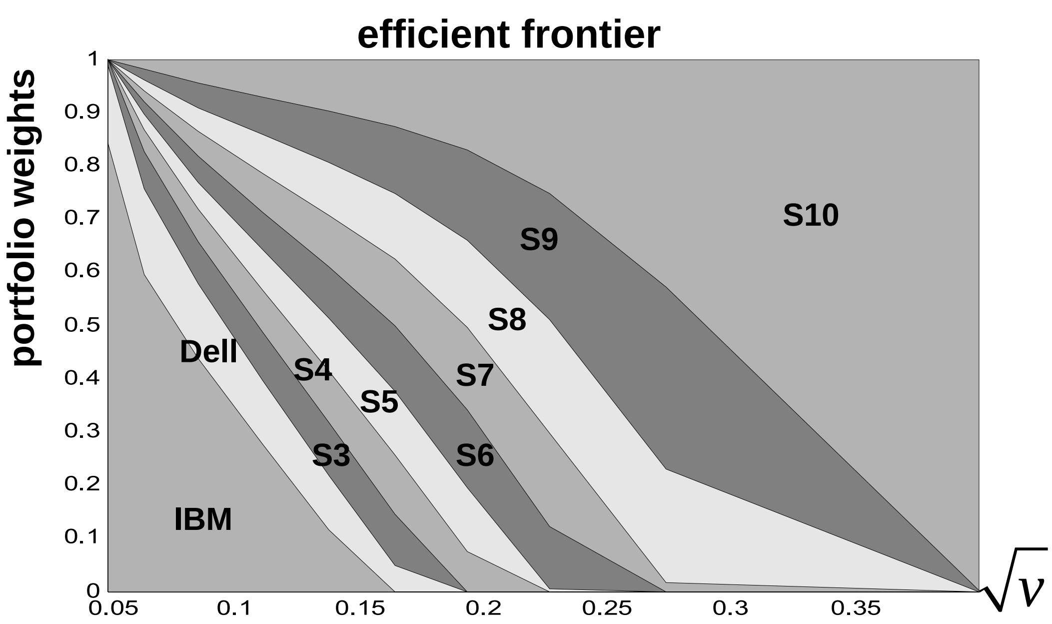

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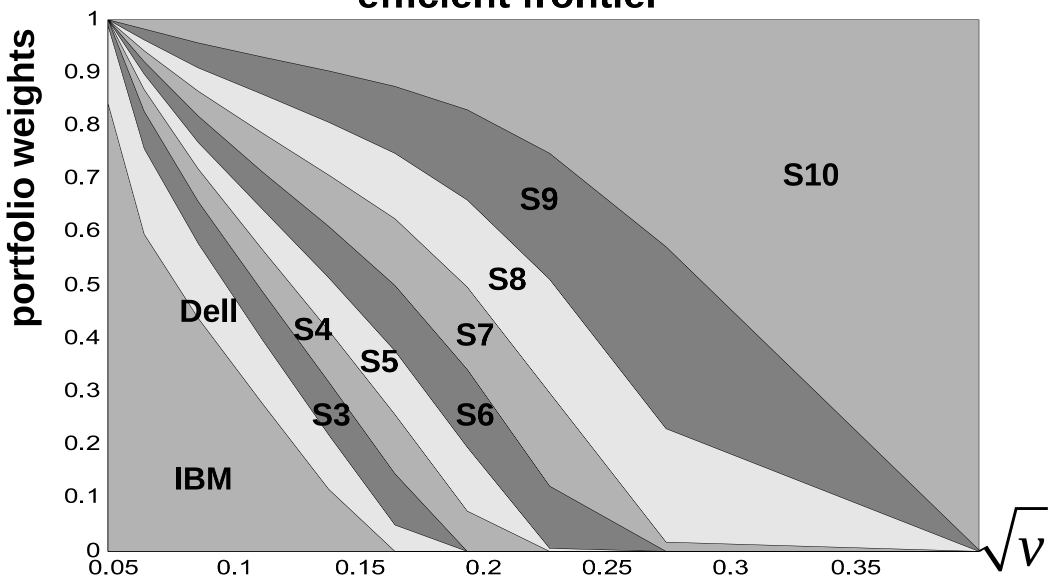
efficient frontier



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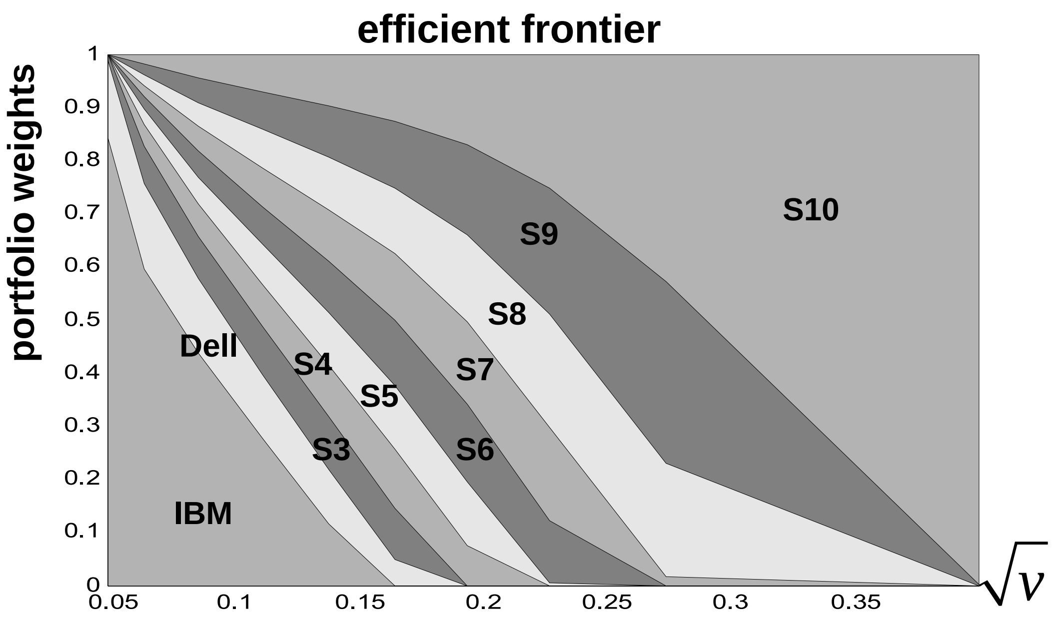

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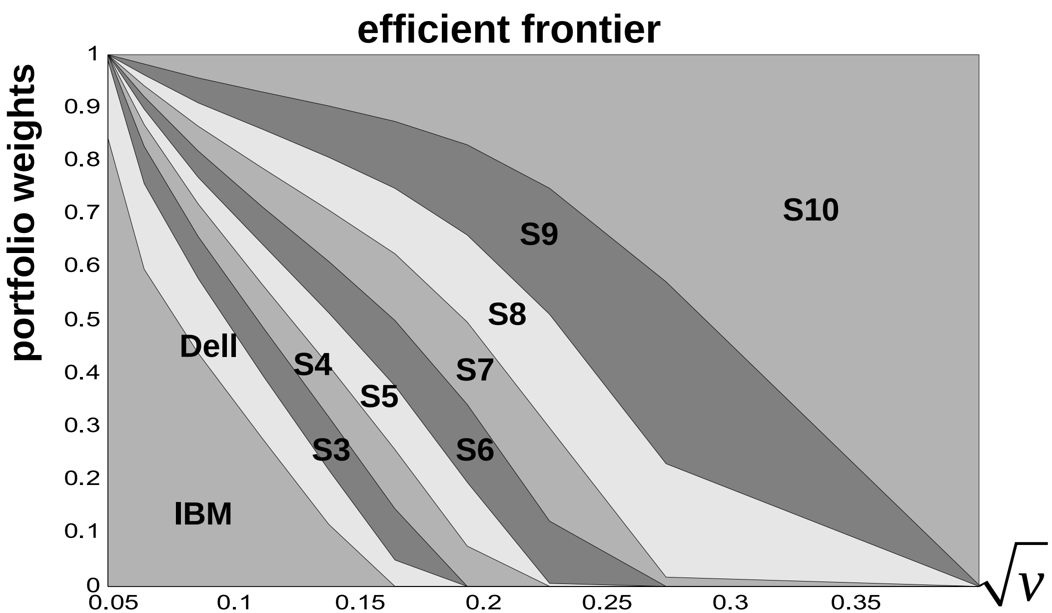

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LIVE

BL and beyond - estimation risk

1

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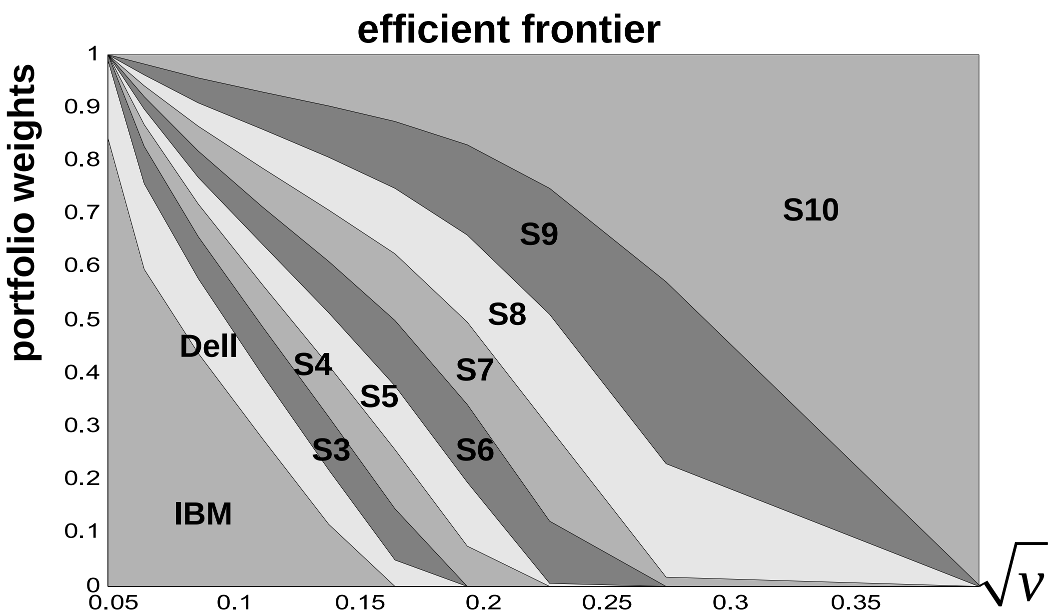

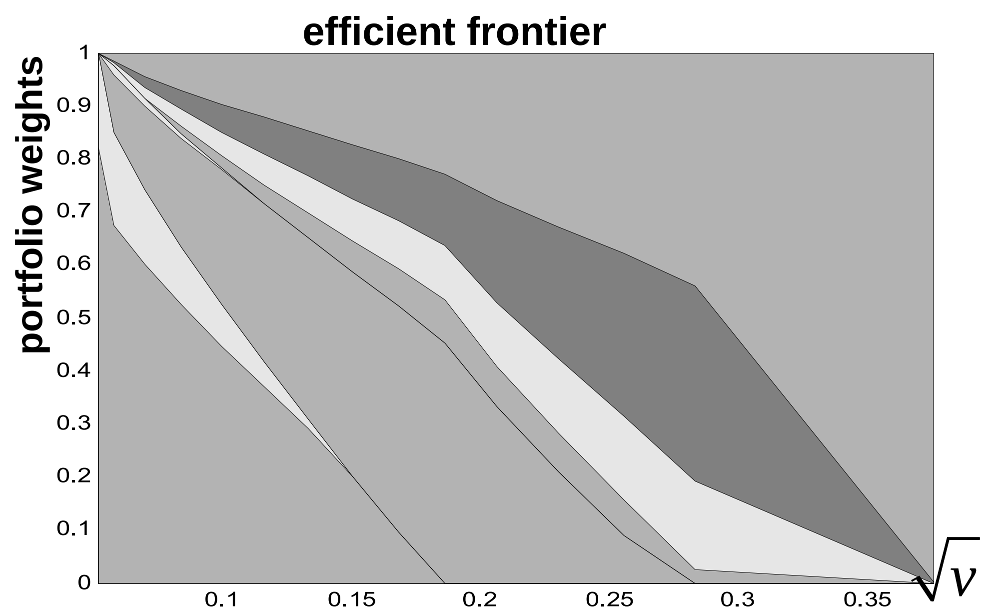

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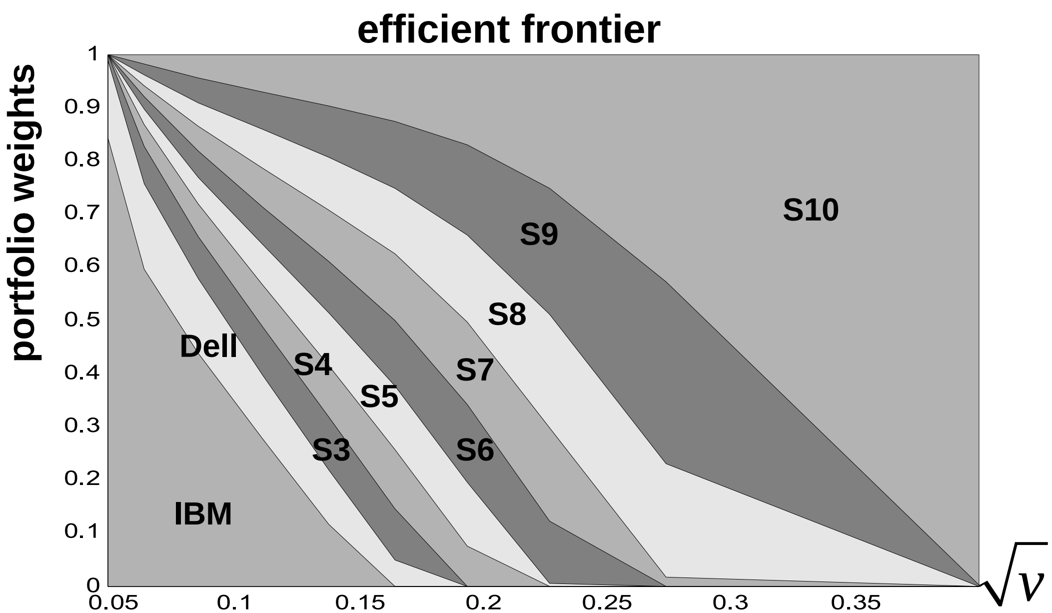

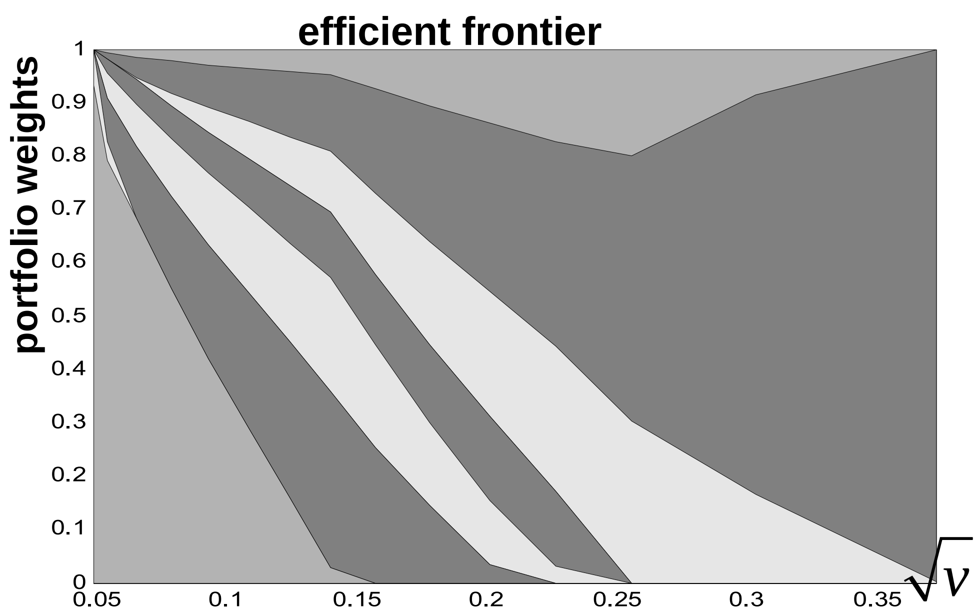

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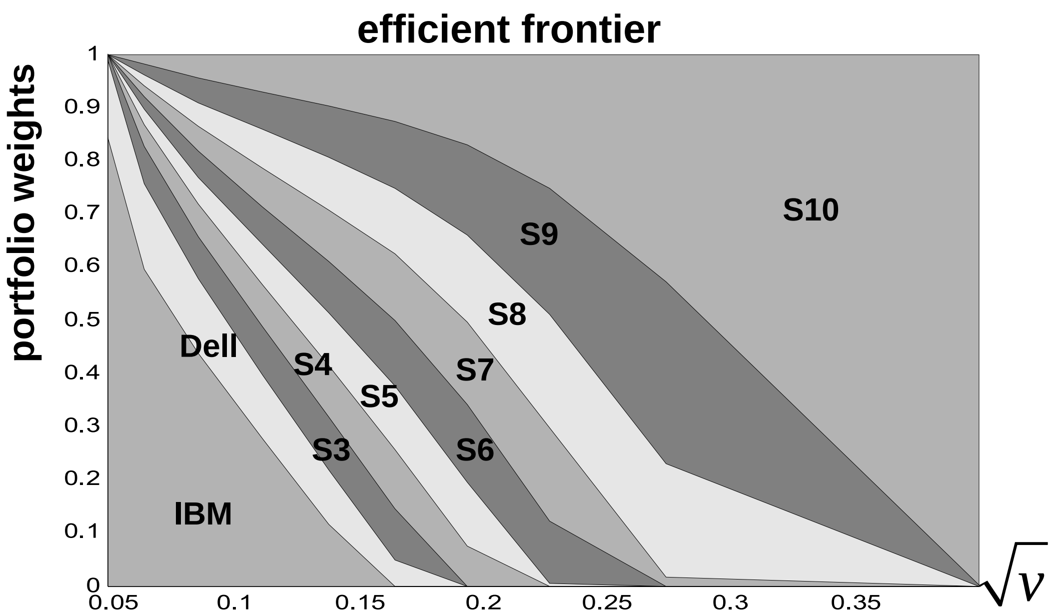
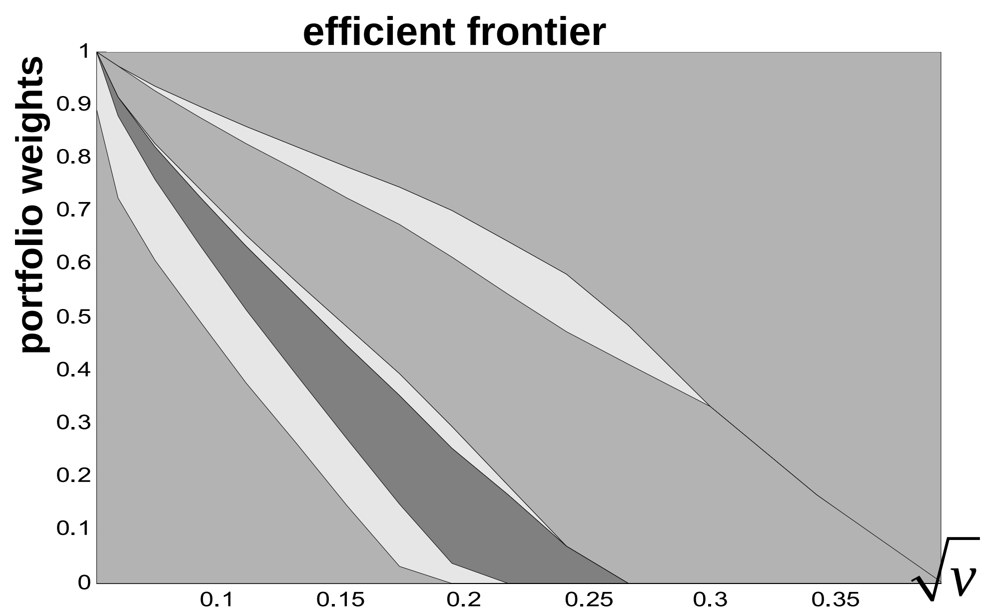
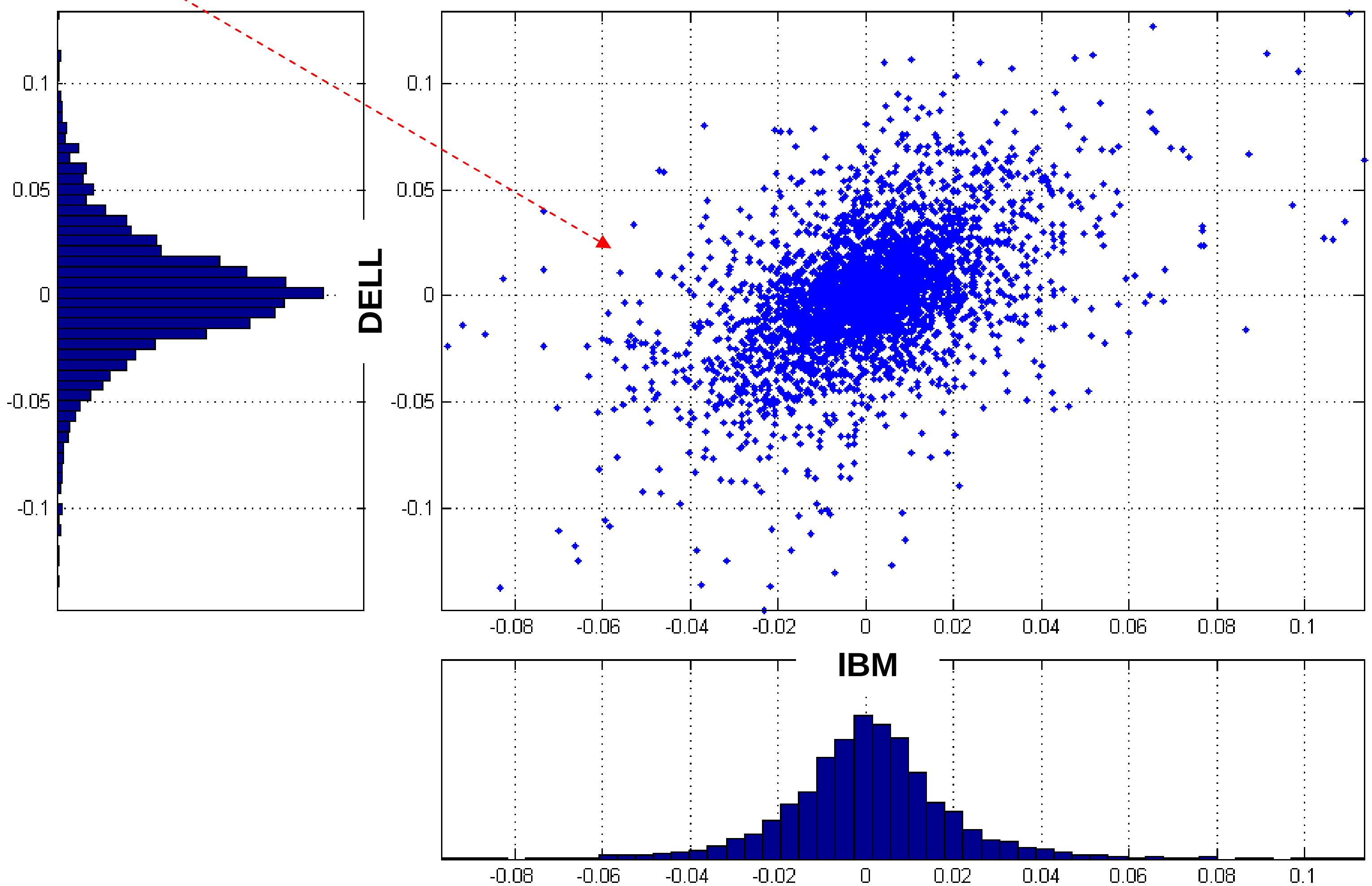




BL and beyond - scenario analysis

Market distribution

$\mathbf{X} \sim \mathrm{N}(\mu, \Sigma)$ **returns on asset classes/funds**



BL and beyond - scenario analysis

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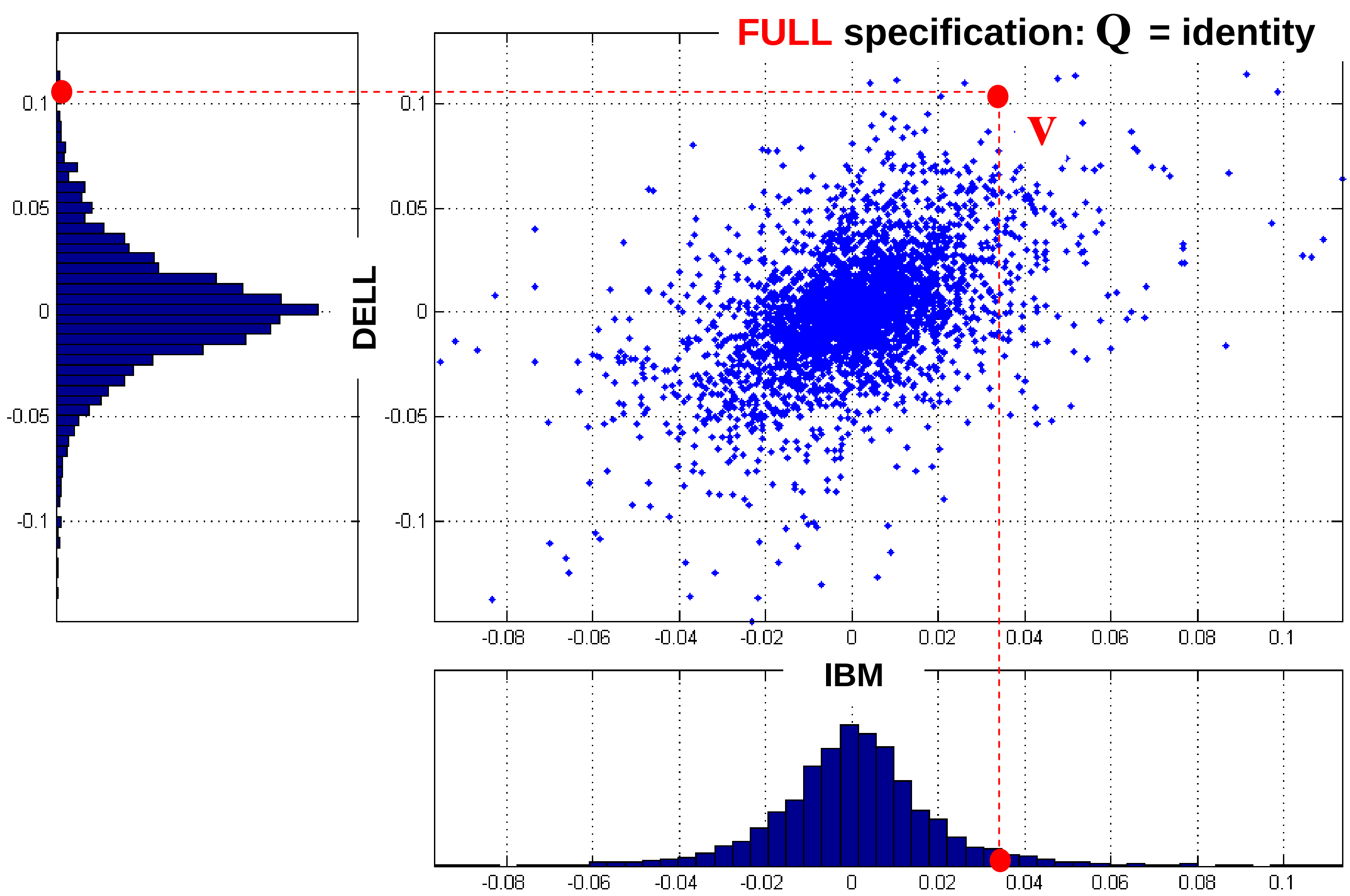

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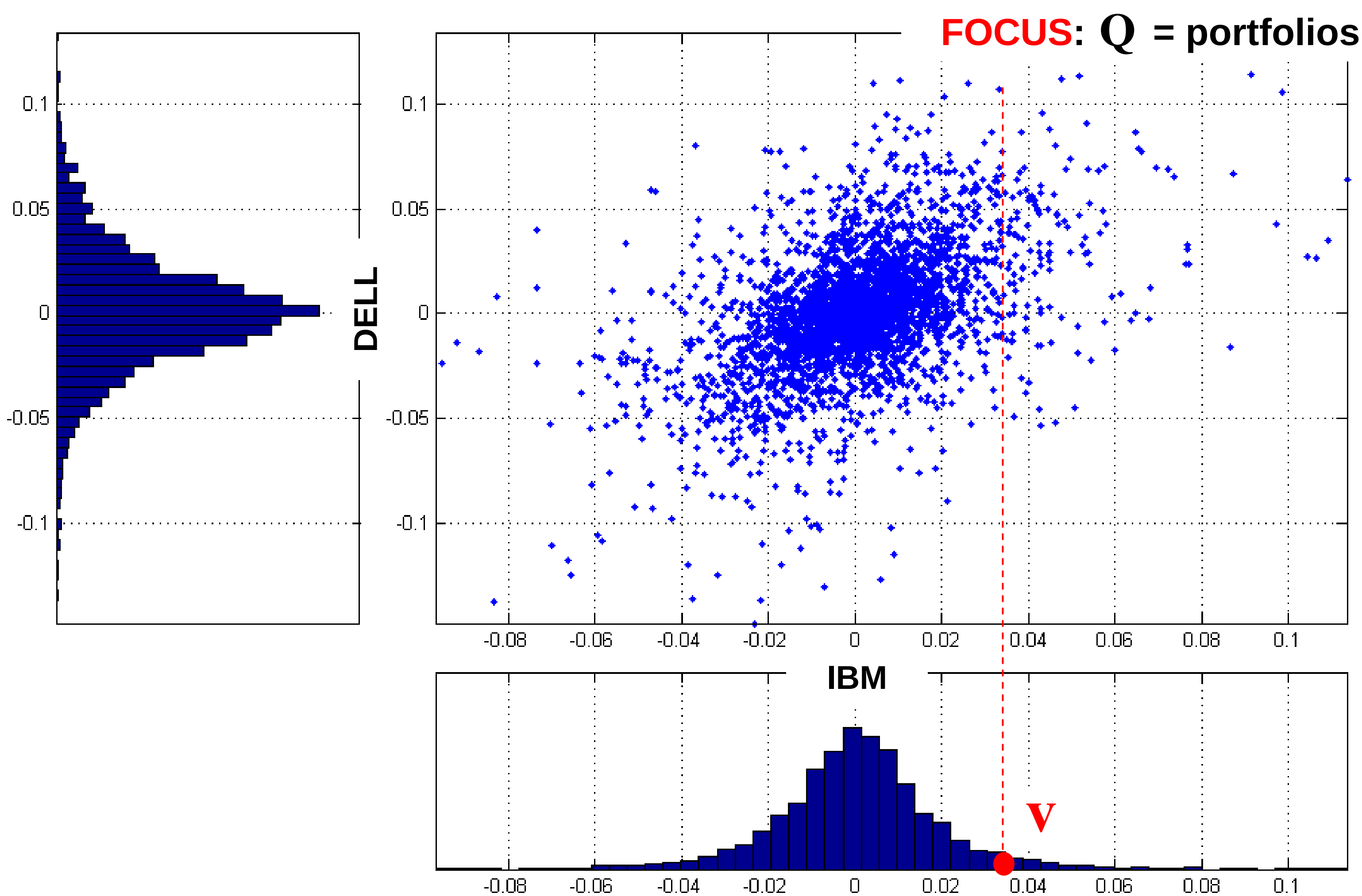

BL and beyond - scenario analysis

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Scenario analysis

$\mathbf{QX} \equiv \mathbf{v}$

Conditional formula

$$\mathbf{X}|\mathbf{v} \sim \mathrm{N}\left(\boldsymbol{\mu}_{\mathbf{x}|\mathbf{v}}, \boldsymbol{\Sigma}_{\mathbf{x}|\mathbf{v}}\right)$$

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{x}|\mathbf{v}} &\equiv \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{Q}'\left(\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}'\right)^{-1}(\mathbf{v} - \mathbf{Q}\boldsymbol{\mu}) \\ \boldsymbol{\Sigma}_{\mathbf{x}|\mathbf{v}} &\equiv \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{Q}'\left(\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}'\right)^{-1}\mathbf{Q}\boldsymbol{\Sigma}. \end{aligned}$$

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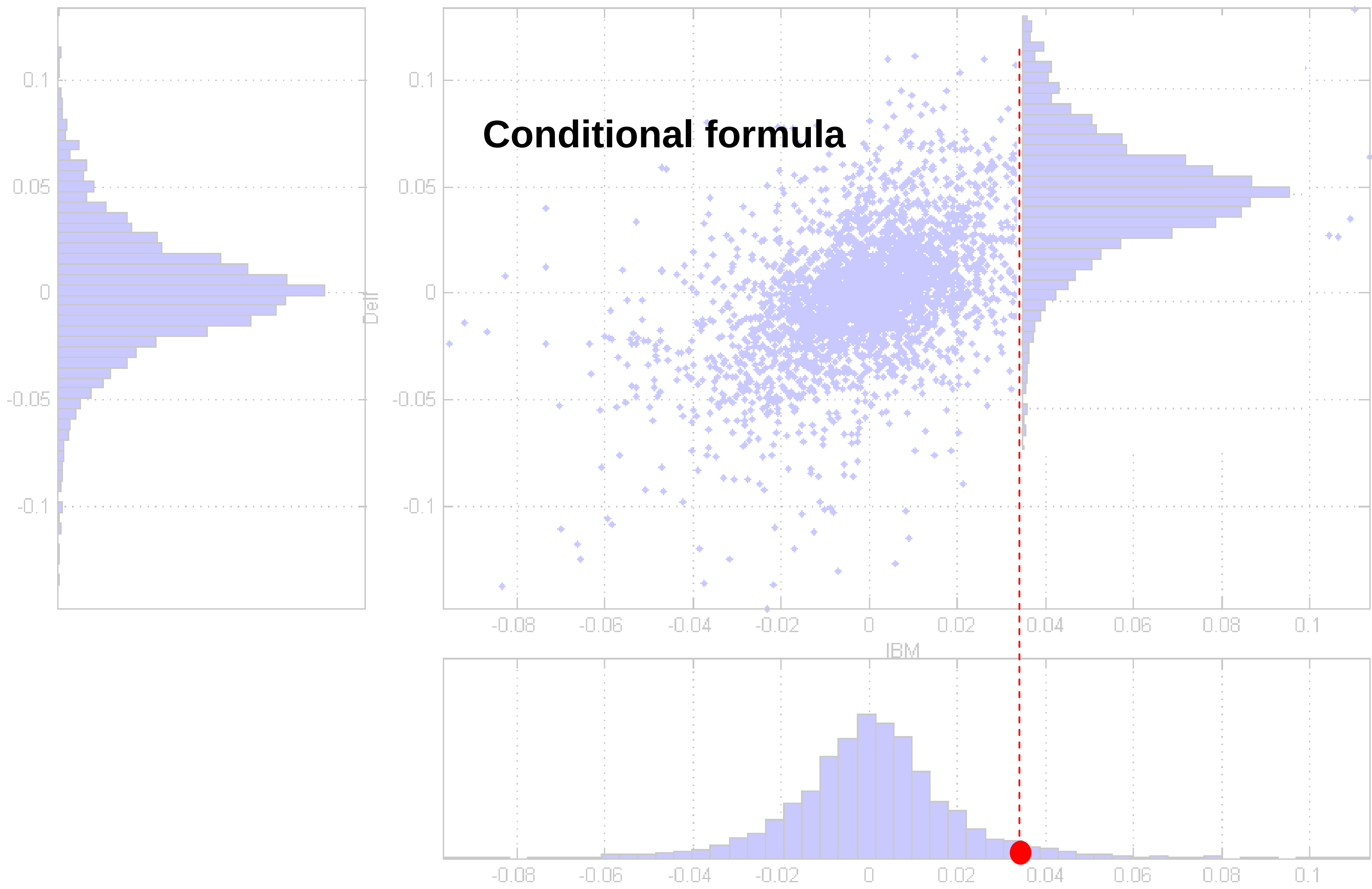

BL and beyond - scenario analysis

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Optimization

$$\mathbf{w}_{\lambda} \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}}\left\{\mathbf{w}'\boldsymbol{\mu}_{\mathbf{x}|\mathbf{v}} - \lambda\mathbf{w}'\boldsymbol{\Sigma}_{\mathbf{x}|\mathbf{v}}\mathbf{w}\right\}$$

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- **Estimation risk**
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- **Discussion**

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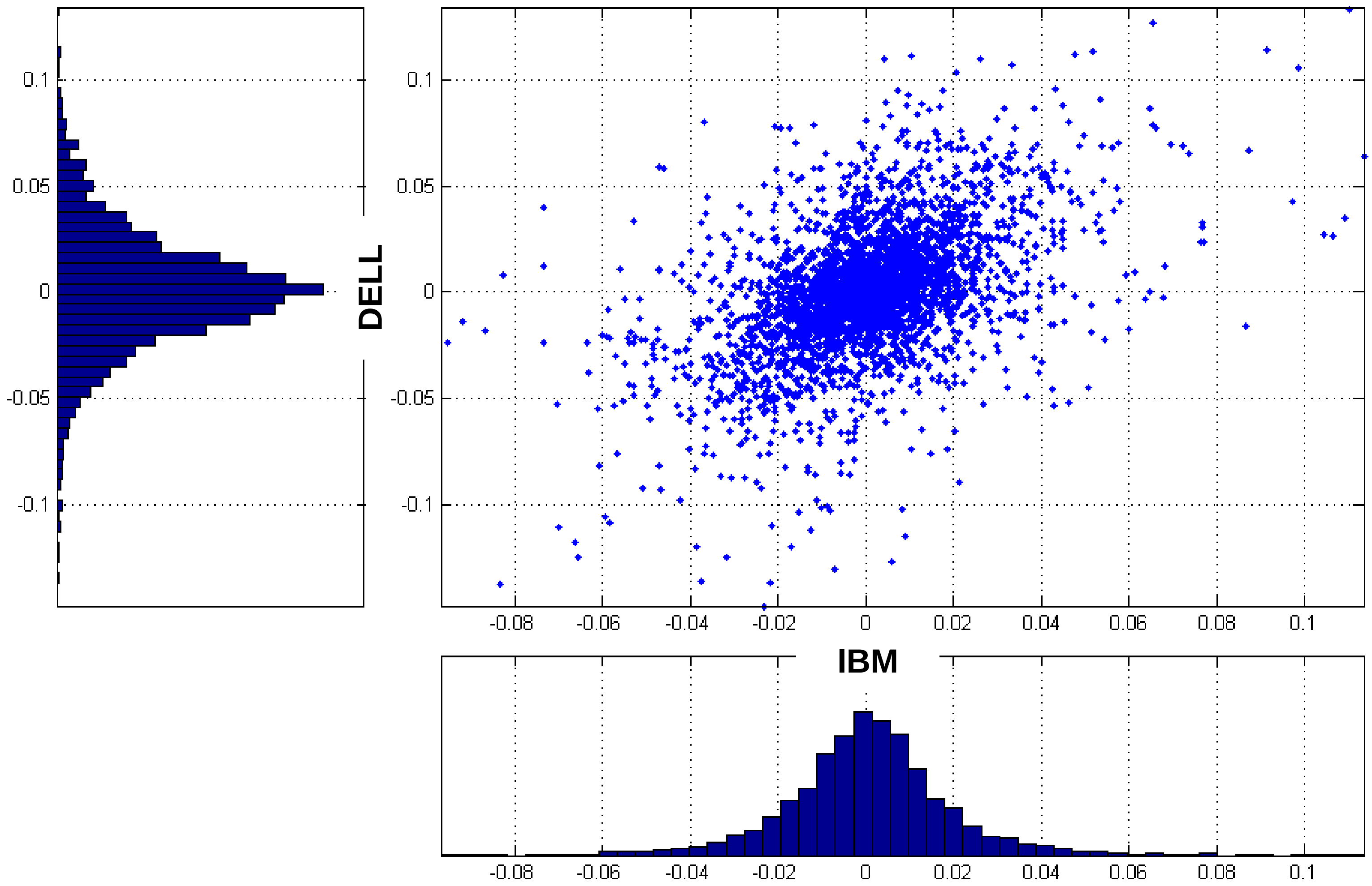

Market distribution

$$\mathbf{X} \sim \mathrm{N}(\boxed{\mu}, \boxed{\Sigma})$$

returns on asset classes/funds

?

estimation risk

Market distribution

$\mathbf{X} \sim \mathrm{N}(\mu, \Sigma)$ **returns on asset classes/funds**

Σ **estimated by exponential smoothing**

Market distribution

$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ **returns on asset classes/funds**

$\boldsymbol{\Sigma}$ **estimated by exponential smoothing**

$$\boldsymbol{\mu} \sim \mathrm{N}(\boldsymbol{\pi}, \tau\boldsymbol{\Sigma})$$

Market distribution

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$\boldsymbol{\Sigma}$ **estimated by exponential smoothing**

$\boldsymbol{\mu} \sim \mathrm{N}(\boldsymbol{\pi}, \boxed{\tau}\boldsymbol{\Sigma})$

?

$$\widehat{\boldsymbol{\mu}} \equiv \frac{1}{T}\sum_{t=1}^{T}\mathbf{X}_t \sim \mathrm{N}\left(\boldsymbol{\pi}, \frac{\boldsymbol{\Sigma}}{T}\right)$$

BL and beyond - Black-Litterman model: estimation risk

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$$\tau \approx \frac{1}{T}.$$

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equilibrium portfolio

equilibrium returns

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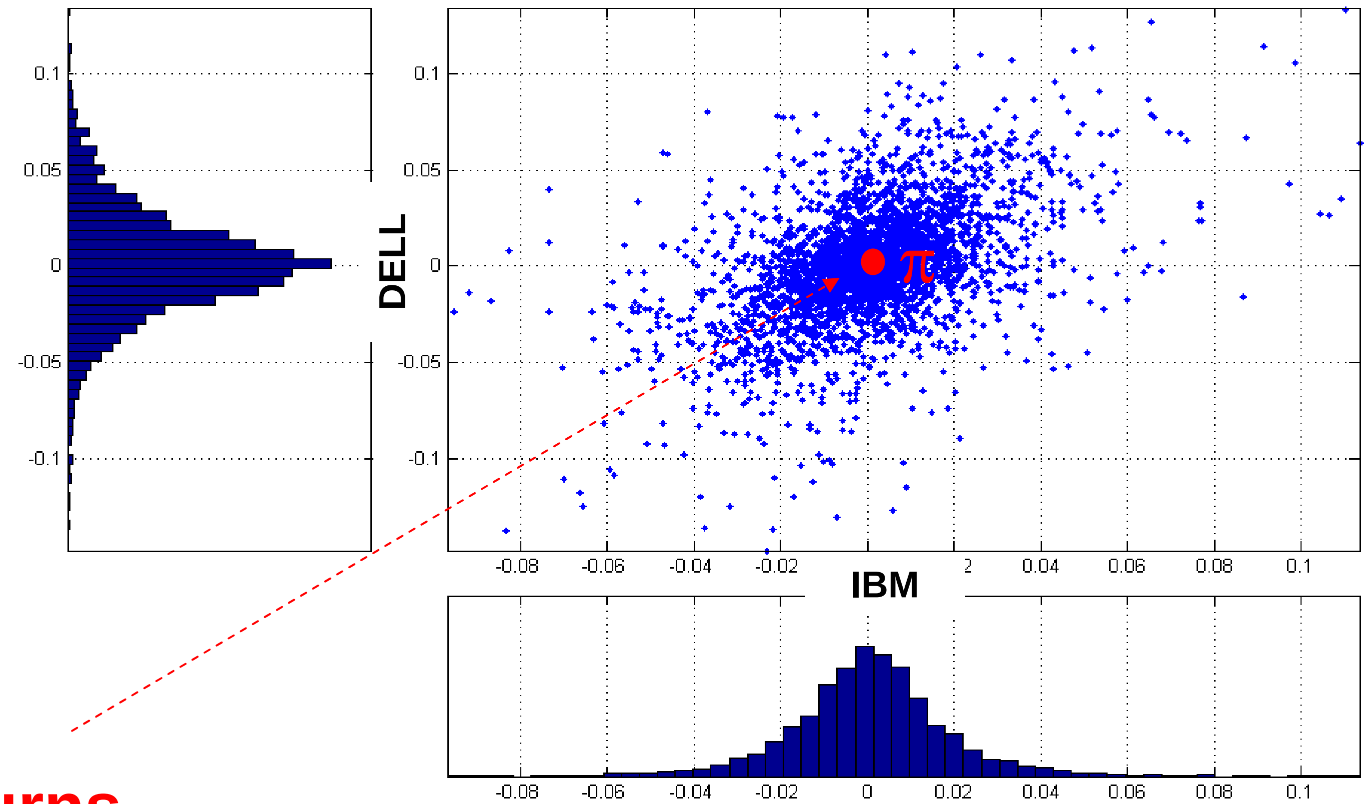

equilibrium returns

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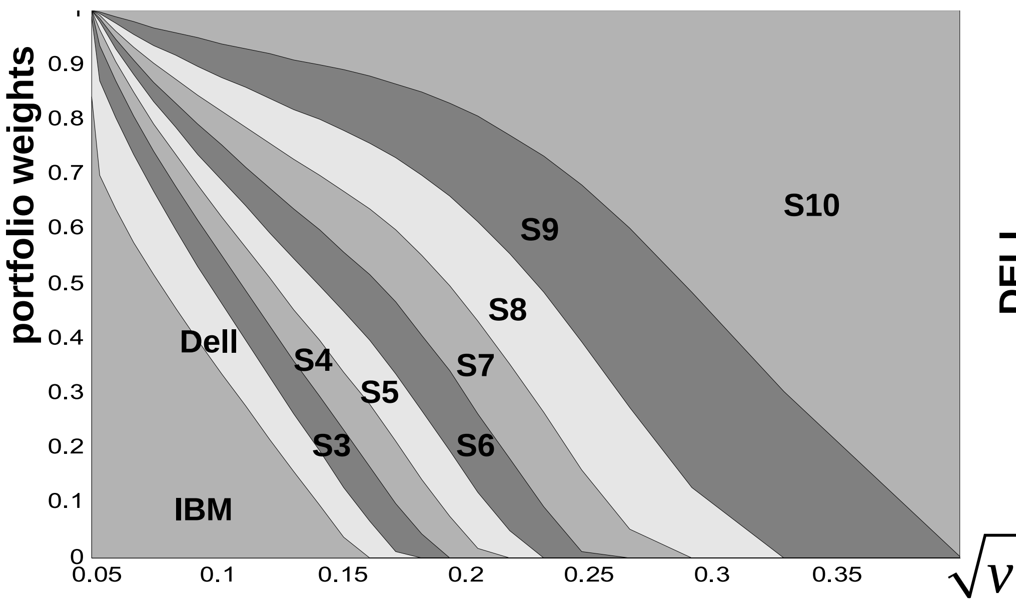
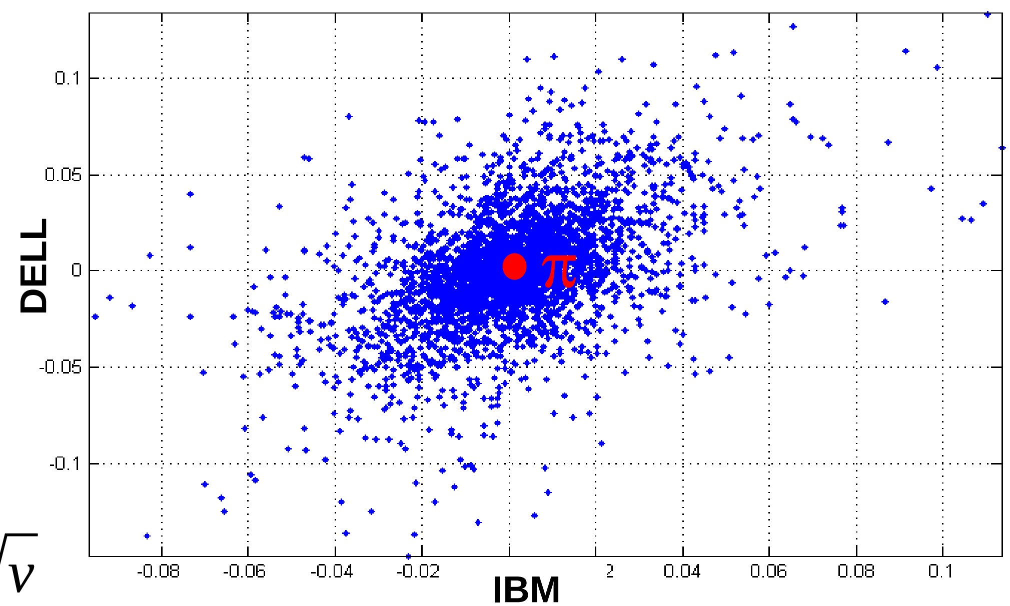
efficient frontier





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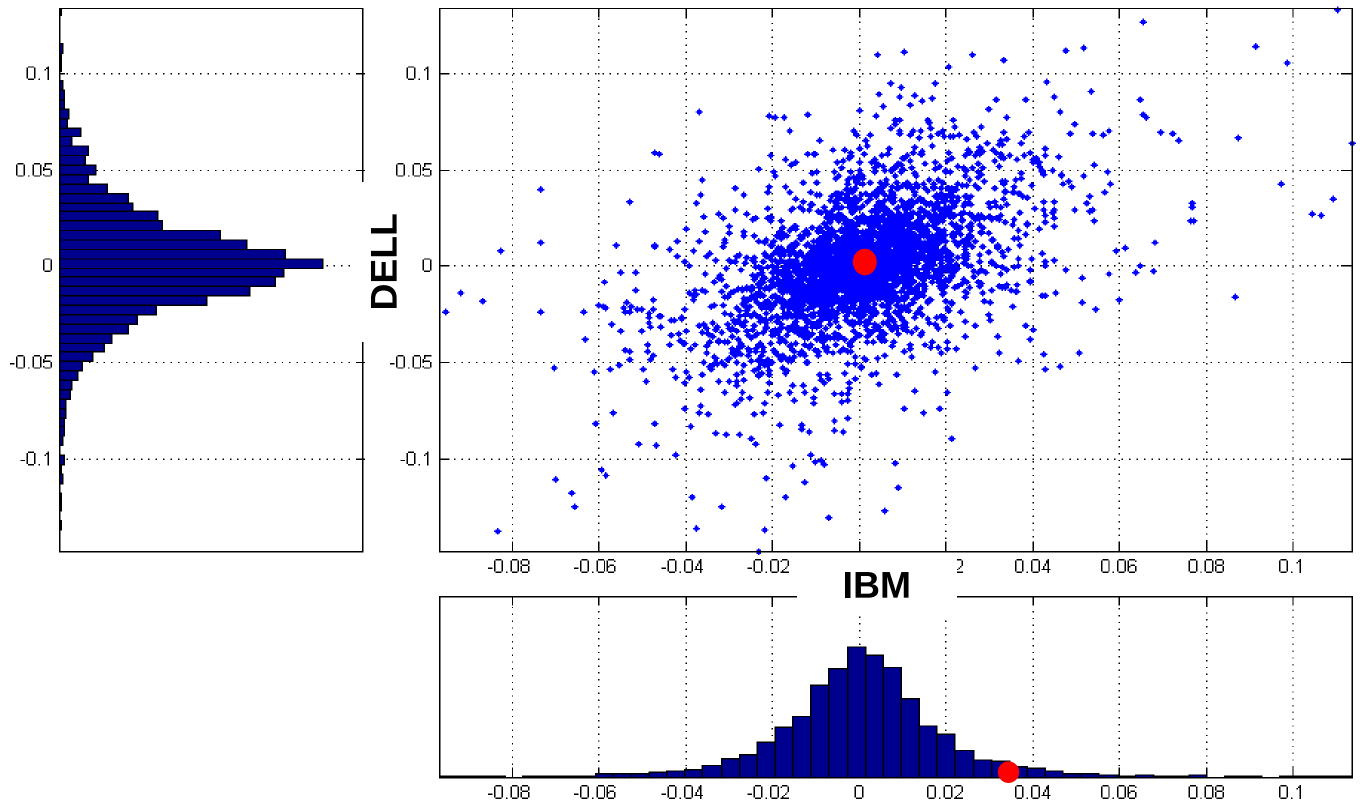
Views

scenario analysis with uncertainty



BL and beyond - Black-Litterman model: views

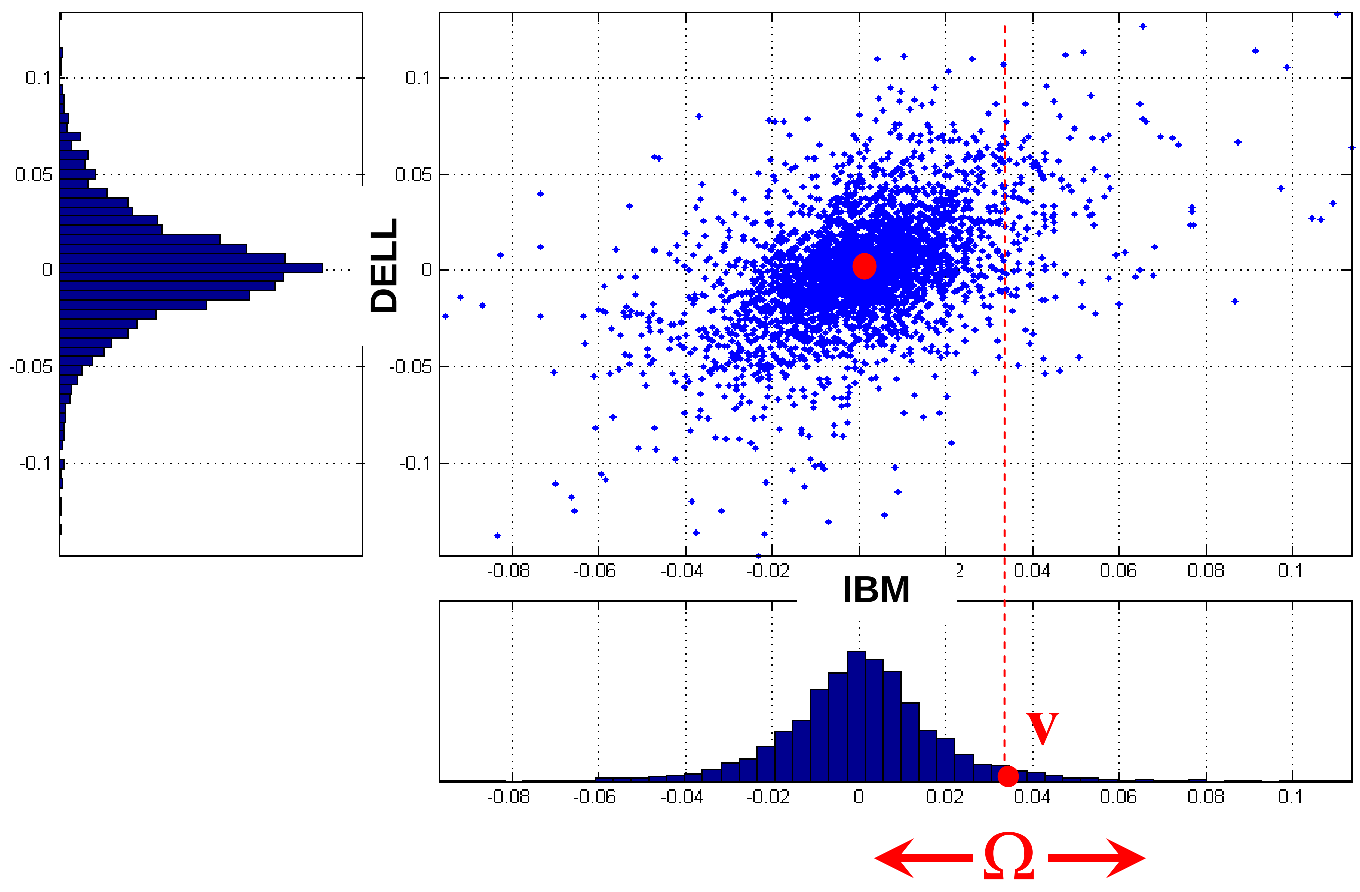
Market distribution

$\mathbf{X} \sim \mathrm{N}(\mu, \Sigma)$ **returns on asset classes/funds**

Views

$\mathbf{Q}\mu \sim \mathrm{N}(\mathbf{v}, \Omega)$

FOCUS: $\mathbf{Q}$ = portfolio



BL and beyond - Black-Litterman model: views

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Views

$\mathbf{Q}\boldsymbol{\mu} \sim \mathrm{N}(\mathbf{v}, \boldsymbol{\Omega})$

Bayes' formula

$$\mathbf{X} | \mathbf{v}; \boldsymbol{\Omega} \sim \mathrm{N}(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL})$$

$$\begin{aligned} \boldsymbol{\mu}_{BL} &= \boldsymbol{\pi} + \tau \boldsymbol{\Sigma} \mathbf{Q}' \left(\tau \mathbf{Q} \boldsymbol{\Sigma} \mathbf{Q}' + \boldsymbol{\Omega} \right)^{-1} (\mathbf{v} - \mathbf{Q}\boldsymbol{\pi}) \\ \boldsymbol{\Sigma}_{BL} &= (1 + \tau) \boldsymbol{\Sigma} - \tau^2 \boldsymbol{\Sigma} \mathbf{Q}' \left(\tau \mathbf{Q} \boldsymbol{\Sigma} \mathbf{Q}' + \boldsymbol{\Omega} \right)^{-1} \mathbf{Q} \boldsymbol{\Sigma}. \end{aligned}$$

Market distribution

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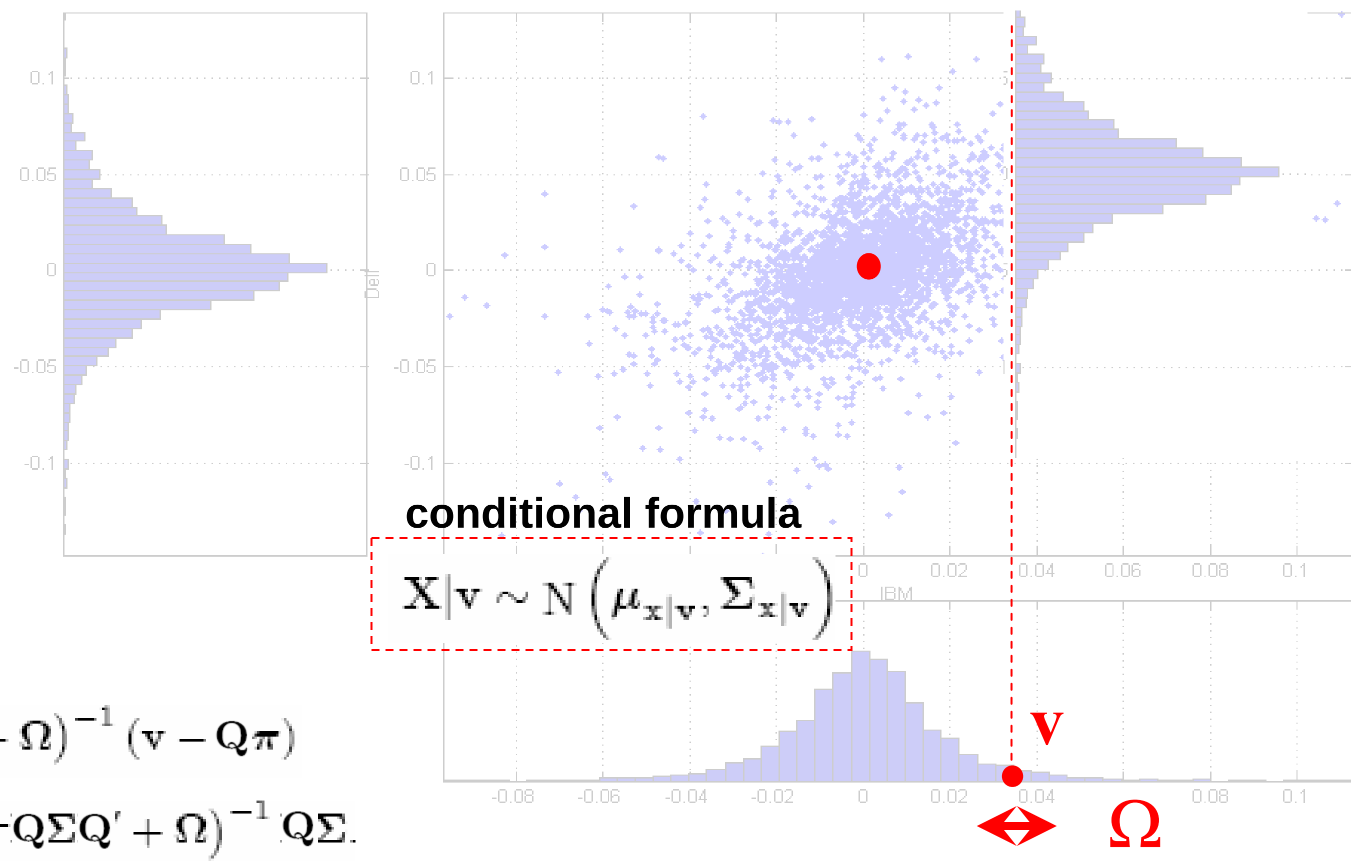
Views

$\mathbf{Q}\boldsymbol{\mu} \sim \mathrm{N}(\mathbf{v}, \boldsymbol{\Omega})$

$\boldsymbol{\Omega} \to \mathbf{0}$

small uncertainty



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conditional formula

$\mathbf{X}|\mathbf{v} \sim \mathrm{N}\left(\boldsymbol{\mu}_{\mathbf{x}|\mathbf{v}}, \boldsymbol{\Sigma}_{\mathbf{x}|\mathbf{v}}\right)$

BL and beyond - Black-Litterman model: views

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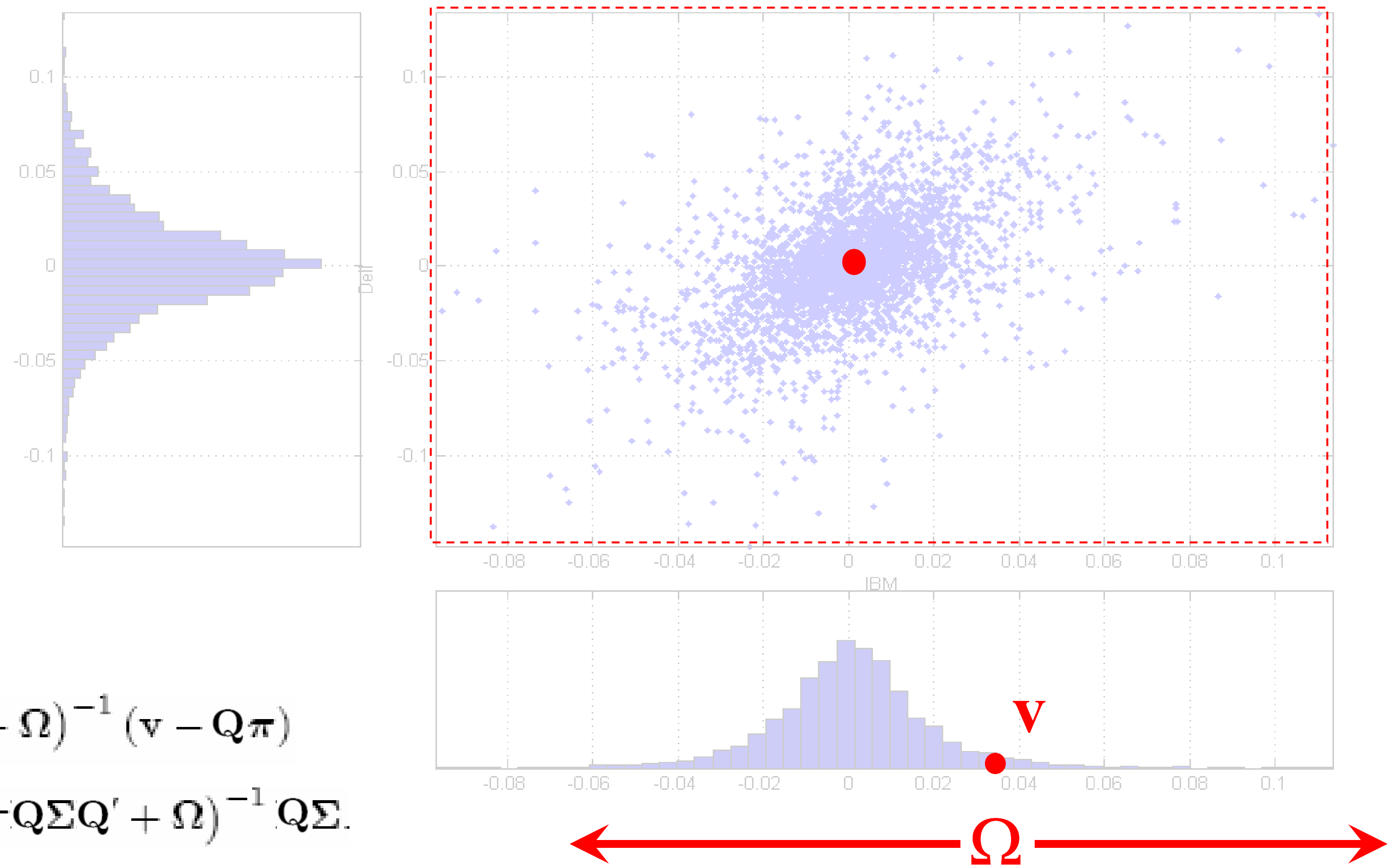
Views

$$\mathbf{Q}\boldsymbol{\mu} \sim \mathrm{N}(\mathbf{v}, \boldsymbol{\Omega})$$

$\boldsymbol{\Omega} \rightarrow \infty$

large uncertainty



Bayes' formula

$$\mathbf{X}|\mathbf{v};\boldsymbol{\Omega} \sim \mathrm{N}(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL})$$

$$\begin{aligned} \boldsymbol{\mu}_{BL} &= \boldsymbol{\pi} + \tau\boldsymbol{\Sigma}\mathbf{Q}'\left(\tau\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}' + \boldsymbol{\Omega}\right)^{-1}(\mathbf{v} - \mathbf{Q}\boldsymbol{\pi}) \\ \boldsymbol{\Sigma}_{BL} &= (1+\tau)\boldsymbol{\Sigma} - \tau^2\boldsymbol{\Sigma}\mathbf{Q}'\left(\tau\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{Q}\boldsymbol{\Sigma}. \end{aligned}$$

Market distribution

$\mathbf{X} \sim \mathrm{N}(\mu, \Sigma)$ **returns on asset classes/funds**

Views

$\mathbf{Q}\mu \sim \mathrm{N}(\mathbf{v}, \Omega)$

Bayes' formula

$\mathbf{X}|\mathbf{v};\Omega \sim \mathrm{N}(\mu_{BL}, \Sigma_{BL})$

BL and beyond - Black-Litterman model: views

Market distribution

$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ **returns on asset classes/funds**

Views

$\mathbf{Q}\boldsymbol{\mu} \sim \mathrm{N}(\mathbf{v}, \boldsymbol{\Omega})$

Bayes' formula

$\mathbf{X}|\mathbf{v};\boldsymbol{\Omega} \sim \mathrm{N}(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL})$

Optimization

$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}} \{\mathbf{w}'\boldsymbol{\mu} - \lambda\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\}$$

Optimization

$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}} \{\mathbf{w}'\boldsymbol{\mu}_{BL} - \lambda\mathbf{w}'\boldsymbol{\Sigma}_{BL}\mathbf{w}\}$$

BL and beyond - Black-Litterman model: views

Market distribution

$$\mathbf{X} \sim \mathrm{N}(\mu, \Sigma)$$

returns on asset classes/funds

Views

$$\mathbf{Q}\mu \sim \mathrm{N}(\mathbf{v}, \Omega)$$

LIVE

Optimization

$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{\mathbf{w}'\mu - \lambda \mathbf{w}'\Sigma\mathbf{w}\right\}$$

Optimization

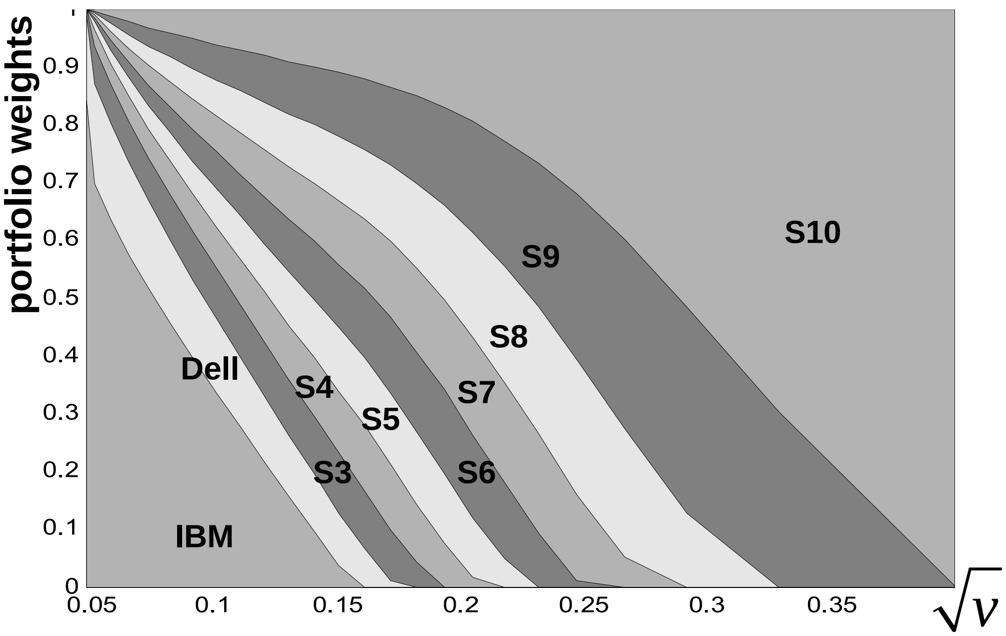
$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{\mathbf{w}'\mu_{BL} - \lambda \mathbf{w}'\Sigma_{BL}\,\mathbf{w}\right\}$$

BL and beyond - Black-Litterman model: views

Market distribution

$$\mathrm{X} \sim \mathrm{N}(\mu, \Sigma)$$ **returns on asset classes/funds**

efficient frontier



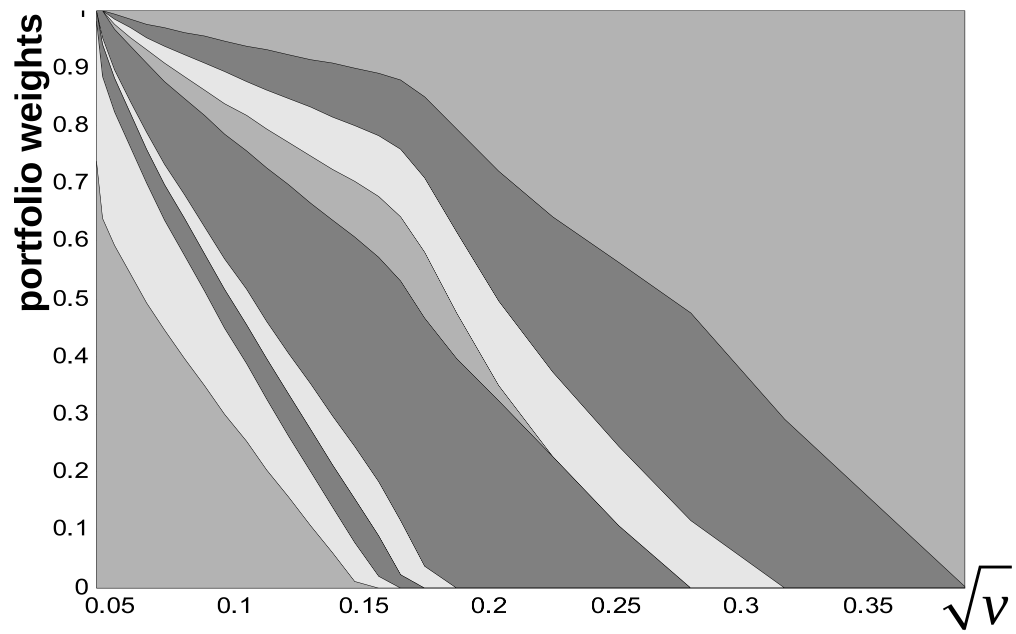
Optimization

$$\mathrm{w}_\lambda \equiv \underset{\mathrm{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{ \mathrm{w}'\mu - \lambda \mathrm{w}'\Sigma\mathrm{w} \right\}$$

Views

$$\mathrm{Q}\mu \sim \mathrm{N}(\mathrm{v}, \Omega)$$

efficient frontier



Optimization

$$\mathrm{w}_\lambda \equiv \underset{\mathrm{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{ \mathrm{w}'\mu_{BL} - \lambda \mathrm{w}'\Sigma_{BL}\, \mathrm{w} \right\}$$

ESTIMATION RISK

SCENARIO ANALYSIS

THE BLACK-LITTERMAN APPROACH

- **Estimation risk**
- **Views**
- **Discussion**

ENTROPY POOLING

CASE STUDIES

REFERENCES AND CONCLUSIONS

Market distribution

$\mathbf{X} \sim \mathrm{N}(\mu, \Sigma)$ **returns on asset classes/funds**

Views/Scenarios

$\mathbf{Q} \quad \mu \quad \equiv \mathbf{v}$ **+ uncertainty**

Optimization

$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{ \mathbf{w}'\mu - \lambda \mathbf{w}'\Sigma\mathbf{w} \right\}$$

BL and beyond - Black-Litterman model: …con's…

Market distribution

$\mathrm{X} \sim \mathrm{N}(\mu, \Sigma)$ **returns on asset classes/funds**

Views/Scenarios

$\mathbf{Q} \quad \mu \quad \equiv \mathrm{v}$ **+ uncertainty**

Market is not only returns: implied volatilities (derivatives)
rates paths (mortgages)
implied correlations (CDO's)
….

Optimization

$$\mathrm{w}_\lambda \equiv \underset{\mathrm{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{ \mathrm{w}' \mu - \lambda \mathrm{w}' \Sigma \mathrm{w} \right\}$$

BL and beyond - Black-Litterman model: …con's…

Market distribution

$\cancel{X \sim N(\mu, \Sigma)}$ **~~returns~~ on asset classes/funds**

Views/Scenarios

$Q \quad \mu \quad \equiv v$ **+ uncertainty**

Market is not only returns

Market is not only normal: fat tails, skewness, tail-risk codependence,…

Optimization

$$w_\lambda \equiv \underset{w \in \mathcal{C}}{\operatorname{argmax}} \{ w'\mu - \lambda w'\Sigma w \}$$

BL and beyond - Black-Litterman model: …con’s…

Market distribution

$\mathbf{X} \sim \cancel{\mathrm{N}}(\cancel{\mu}, \cancel{\Sigma})$ **~~returns~~ on asset classes/funds**

Views/Scenarios

$\mathbf{Q} \quad \mu \quad \equiv \mathbf{v}$ **+ uncertainty**

Market is not only returns

Market is not only normal

Market is not only equilibrium: historical estimates, implied values, …

Optimization

$$\mathbf{w}_{\lambda} \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{ \mathbf{w}' \mu - \lambda \mathbf{w}' \Sigma \mathbf{w} \right\}$$

Market distribution

$\cancel{\mathbf{X} \sim \mathrm{N}(\mu, \Sigma)}$ ~~returns~~ **returns on asset classes/funds**

Views/Scenarios

$\cancel{\mathbf{Q}}$ μ $\equiv \mathbf{v}$ **+ uncertainty**

Market is not only returns

Market is not only normal

Market is not only equilibrium

Views are not only portfolios: generic non-linear functions

Optimization

$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathbf{w}'\mu - \lambda \mathbf{w}'\Sigma \mathbf{w}\}$$

BL and beyond - Black-Litterman model: …con's…

Market distribution

$\cancel{X \sim N(\mu, \Sigma)}$ **~~returns~~ on asset classes/funds**

Views/Scenarios

$\cancel{Q}\ \cancel{\mu} \equiv v$ **+ uncertainty**

Market is not only returns

Market is not only normal

Market is not only equilibrium

Views are not only portfolios

Views are not only on expectations: correlations,
volatilities,
tail behavior,
copulas,
…

Optimization

$$w_\lambda \equiv \underset{w \in C}{\operatorname{argmax}} \left\{ w'\mu - \lambda w' \Sigma w \right\}$$

Market distribution

$\cancel{\mathbf{X} \sim \mathrm{N}(\mu, \Sigma)}$ **~~returns~~ on asset classes/funds**

Views/Scenarios

$\cancel{\mathbf{Q}} \; \cancel{\mu} \; \cancel{\equiv \mathbf{v}}$ **+ uncertainty**

Market is not only returns

Market is not only normal

Market is not only equilibrium

Views are not only portfolios

Views are not only on expectations

Views are not only equalities: stock ranking, qualitative views

Optimization

$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathbf{w}'\mu - \lambda \mathbf{w}'\Sigma\mathbf{w}\}$$

BL and beyond - Black-Litterman model: …con’s

Market distribution

$\cancel{X \sim N(\mu, \Sigma)}$ **~~returns~~ on asset classes/funds**

Views/Scenarios

$\cancel{Q}\,\cancel{\mu}\,\cancel{\equiv v}$ **+ uncertainty**

Market is not only returns

Market is not only normal

Market is not only equilibrium

Views are not only portfolios

Views are not only on expectations

Views are not only equalities

Optimization is not only mean variance: mean-CVaR, mean-VaR, …

Optimization

$$w_\lambda \equiv \cancel{\underset{w \in \mathcal{C}}{\operatorname{argmax}}} \{w'\mu - \lambda w'\Sigma w\}$$

ESTIMATION RISK

SCENARIO ANALYSIS

THE BLACK-LITTERMAN APPROACH

ENTROPY POOLING

- **Theory**
- **Analytical solution**
- **General implementation**

CASE STUDIES

REFERENCES AND CONCLUSIONS

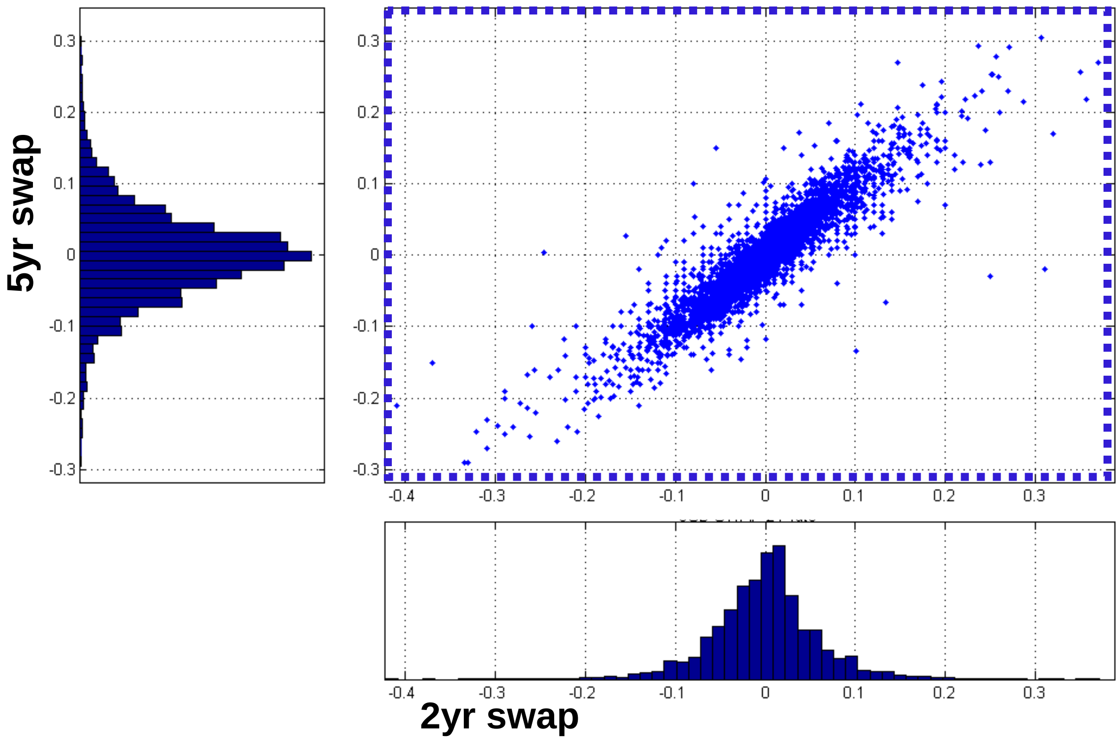
BL and beyond - entropy pooling: theory

Market distr. $X \sim \boxed{f_X}$ **not returns, not normal, not equilibrium**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**



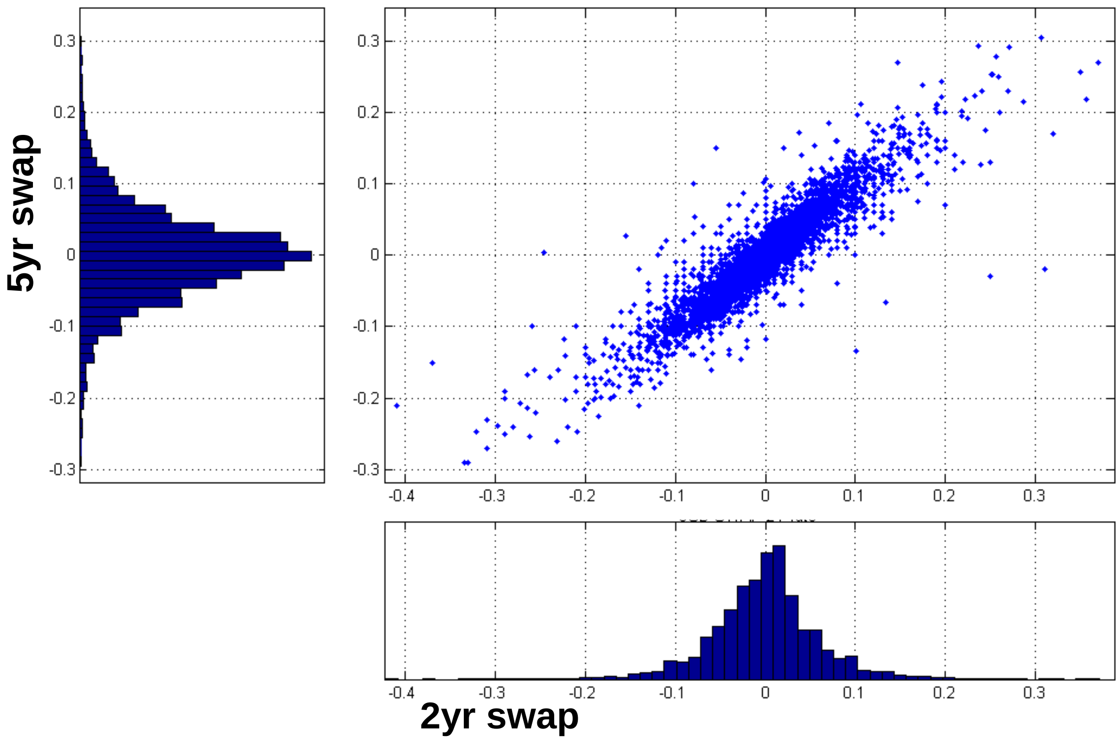
BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ **not returns, not normal, not equilibrium**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**



Pricing $P_{t+\tau} \equiv P\left(\mathbf{X}, \mathcal{I}_t\right)$ **delta/gamma/vega, full pricing, …**

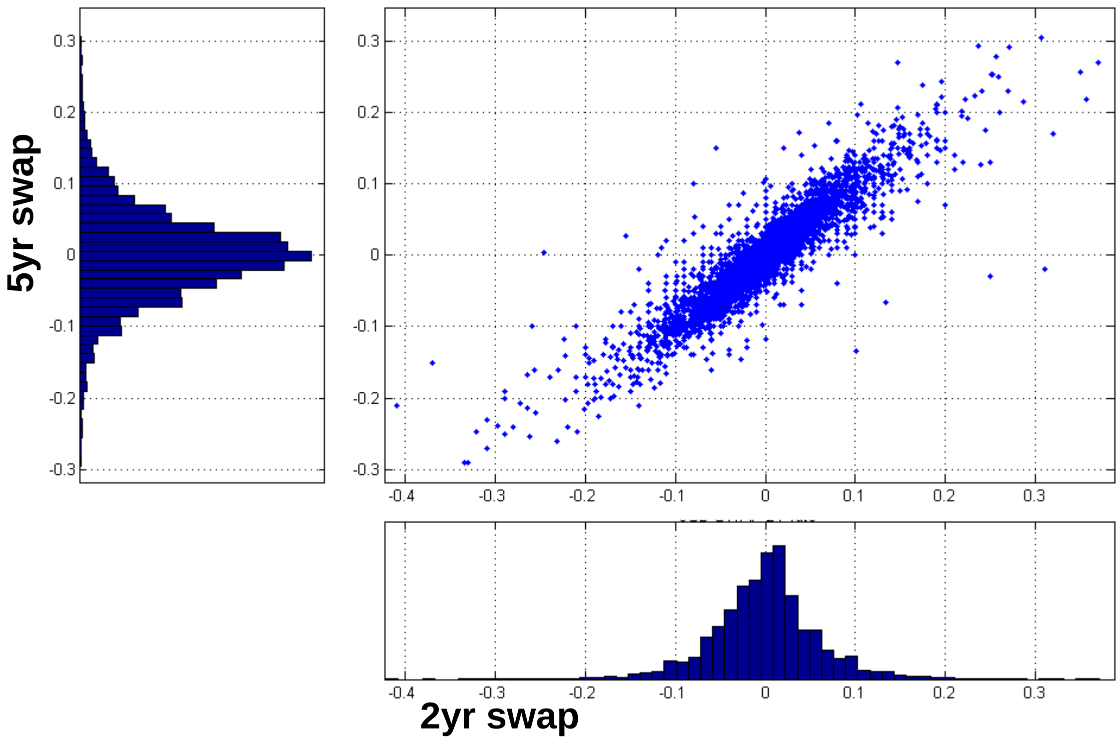
e.g.

duration + convexity

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ **not returns, not normal, not equilibrium**

e.g.
X_1 **2-yr swap rate**
X_2 **5-yr swap rate**



Pricing $P_{t+\tau} \equiv P(\mathbf{X}, \mathcal{I}_t)$ **delta/gamma/vega, full pricing, …**

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; f_{\mathbf{X}})\}$ **utility, mean-CVaR, …**

e.g.
duration + convexity
mean-variance

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1^{\,2} + X_2^{\,2}$

convexity factor

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ **not returns, not normal, not equilibrium**

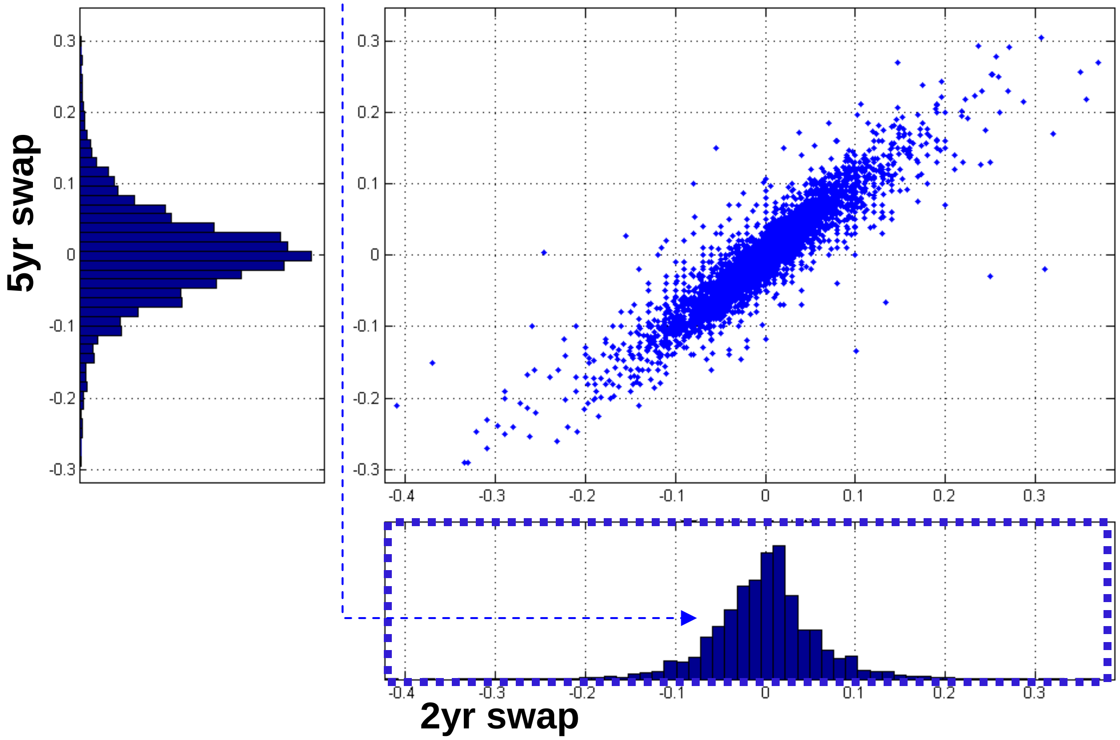
Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim \boxed{f_{\mathbf{V}}}$ **non-linear functions and external factors**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$



BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim \boxed{f_{\mathbf{V}}}$ **non-linear functions and external factors**

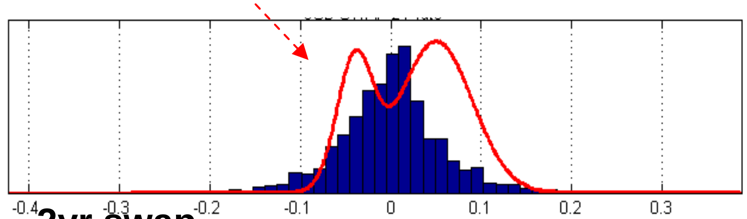
Views ✓ $\mathbf{V} \sim \boxed{\tilde{f}_{\mathbf{V}}} \neq f_{\mathbf{V}}$. **full distribution specification**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$



2yr swap

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

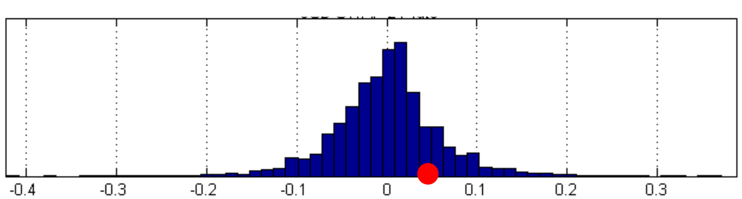
✓ $\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k};$ **view on expectations (BL), medians**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$



2yr swap

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}:$ **view on expectations (BL), medians**

$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$ **ranking**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$. **full distribution specification**

$\tilde{m}\{V_k\} \gtreqless \tilde{\mu}_{\mathbf{v},k}$: **view on expectations (BL), medians**

$\tilde{m}\{V_1\} \geq \tilde{m}\{V_2\} \geq \cdots \geq \tilde{m}\{V_K\}$ **ranking**

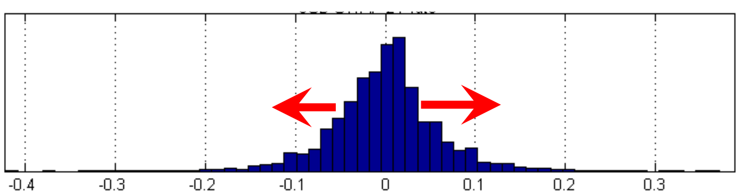
✓ $\tilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$ **views on volatilities**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$



2yr swap

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$\tilde{m}\{V_k\} \gtreqless \tilde{\mu}_{\mathbf{v},k}$: **view on expectations (BL), medians**

$\tilde{m}\{V_1\} \geq \tilde{m}\{V_2\} \geq \cdots \geq \tilde{m}\{V_K\}$ **ranking**

$\tilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$ **views on volatilities**

$\tilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$ **correlation stress-testing**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$\widetilde{m}\{V_k\} \gtreqqless \widetilde{\mu}_{\mathbf{v},k}:$ **view on expectations (BL), medians**

$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$ **ranking**

$\widetilde{\sigma}\{V_k\} \gtreqqless \varkappa\sigma\{V_k\}$ **views on volatilities**

$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$ **correlation stress-testing**

$\widetilde{Q}_V(u) \gtreqqless Q_V(u)$ **view on tail behavior**

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$$\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k};$$

$$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$$

$$\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$$

$$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$$

$$\widetilde{Q}_V(u) \gtreqless Q_V(u)$$

partial distribution specification

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$$\tilde{m}\{V_k\} \overset{\geq}{\underset{\leq}{=}} \tilde{\mu}_{\mathbf{v},k}:$$

$$\tilde{m}\{V_1\} \geq \tilde{m}\{V_2\} \geq \cdots \geq \tilde{m}\{V_K\}$$

$$\tilde{\sigma}\{V_k\} \overset{\geq}{\underset{\leq}{=}} \varkappa\sigma\{V_k\}$$

$$\tilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$$

$$\tilde{Q}_V(u) \overset{\geq}{\underset{\leq}{=}} Q_V(u)$$

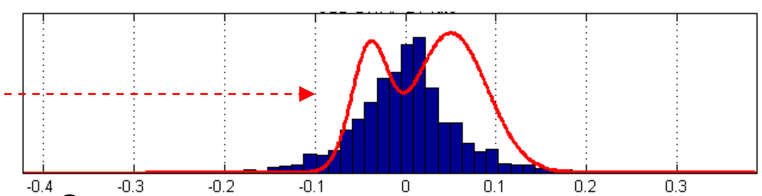
partial distribution specification

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$



2yr swap

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

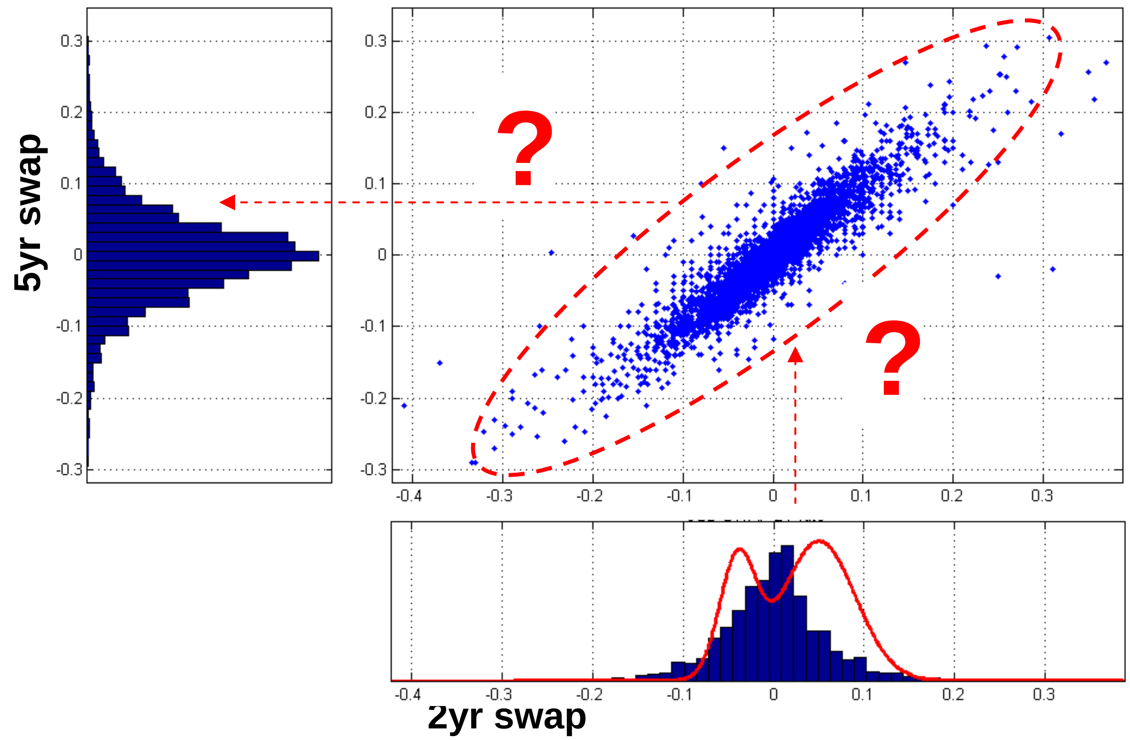
Views

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$



BL and beyond - entropy pooling: theory

e.g.

X_1 **2-yr swap rate**

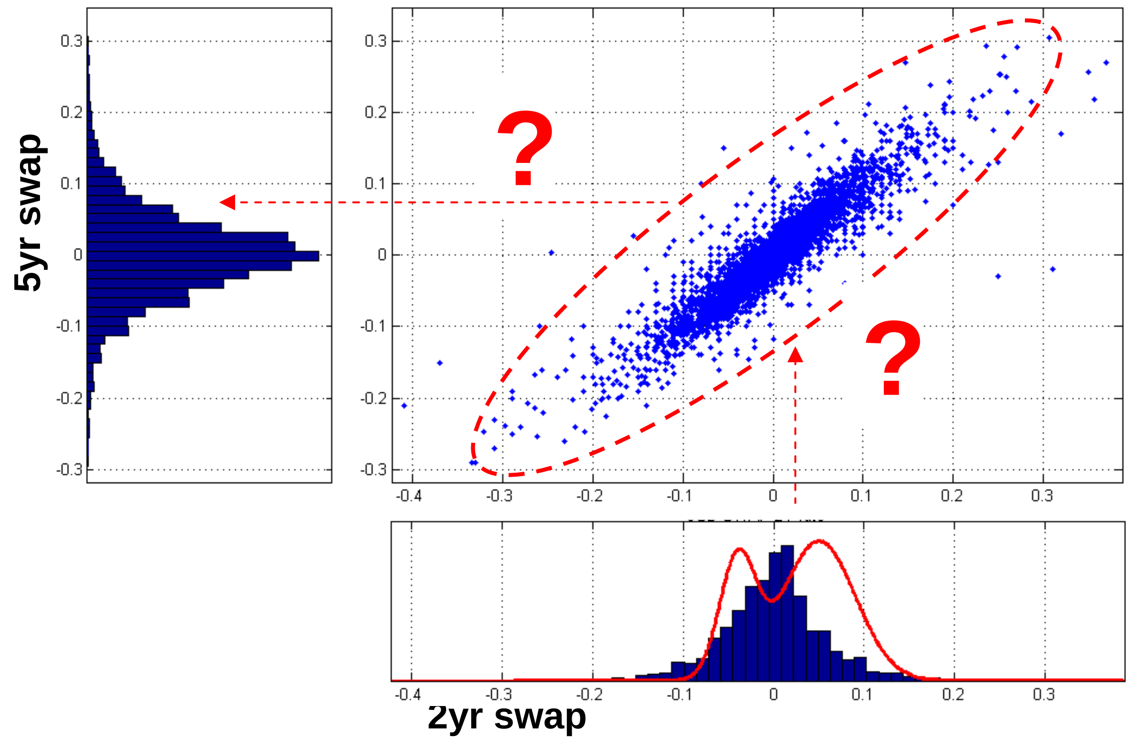
X_2 **5-yr swap rate**

$V \equiv X_1$

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views



Posterior $\tilde{f}_{\mathbf{X}}$ **?**

BL and beyond - entropy pooling: theory

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$$\left.\begin{array}{l} \tilde{m}\{V_k\} \gtreqqless \tilde{\mu}_{\mathbf{v},k}. \\ \tilde{m}\{V_1\} \geq \tilde{m}\{V_2\} \geq \cdots \geq \tilde{m}\{V_K\} \\ \tilde{\sigma}\{V_k\} \gtreqqless \varkappa\sigma\{V_k\} \\ \tilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\{\mathbf{V}\} + \rho_3 \mathbf{1}\mathbf{1}', \\ \tilde{Q}_V(u) \gtreqqless Q_V(u) \end{array}\right\}$$

partial distribution specification

Posterior $\tilde{f}_{\mathbf{X}}$ **?**

relative entropy
"distance" btw. distributions $\mathcal{E}\left(\tilde{f}_{\mathbf{X}}, f_{\mathbf{X}}\right) \equiv \int \tilde{f}_{\mathbf{X}}(\mathbf{x}) \left[\ln \tilde{f}_{\mathbf{X}}(\mathbf{x}) - \ln f_{\mathbf{X}}(\mathbf{x})\right] d\mathbf{x}.$

BL and beyond - entropy pooling: theory

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$V \equiv X_1$

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$$\left.\begin{aligned}
&\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}:\\
&\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}\\
&\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}\\
&\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',\\
&\widetilde{Q}_V(u) \gtreqless Q_V(u)
\end{aligned}\right\}$$

partial distribution specification

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \operatorname{argmin}\{\mathcal{E}(f, f_{\mathbf{X}})\}$ **least distance from prior**

relative entropy
"distance" btw. distributions $\mathcal{E}\left(\widetilde{f}_{\mathbf{X}}, f_{\mathbf{X}}\right) \equiv \int \widetilde{f}_{\mathbf{X}}(\mathbf{x}) \left[\ln \widetilde{f}_{\mathbf{X}}(\mathbf{x}) - \ln f_{\mathbf{X}}(\mathbf{x})\right] d\mathbf{x}.$

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$.

full distribution specification

$$\widetilde{m}\{V_k\} \gtreqqless \widetilde{\mu}_{\mathbf{v},k}:$$

$$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$$

$$\widetilde{\sigma}\{V_k\} \gtreqqless \varkappa\sigma\{V_k\}$$

$$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$$

$$\widetilde{Q}_V(u) \gtreqqless Q_V(u)$$

partial distribution specification

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{\boxed{f \in \mathbb{V}}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$ **least distance from prior, views satisfied**

relative entropy
"distance" btw. distributions $\mathcal{E}\left(\widetilde{f}_{\mathbf{X}}, f_{\mathbf{X}}\right) \equiv \int \widetilde{f}_{\mathbf{X}}(\mathrm{x})\left[\ln \widetilde{f}_{\mathbf{X}}(\mathrm{x}) - \ln f_{\mathbf{X}}(\mathrm{x})\right] d\mathrm{x}.$

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$$\left.\begin{array}{l} \widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}; \\ \widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\} \\ \widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\} \\ \widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}', \\ \widetilde{Q}_V(u) \gtreqless Q_V(u) \end{array}\right\}$$

partial distribution specification

Posterior $\boxed{\widetilde{f}_{\mathbf{X}} \equiv \underset{f\in\mathbb{V}}{\operatorname{argmin}}\{\mathcal{E}(f, f_{\mathbf{X}})\}}$ **least distance from prior, views satisfied**

BL and beyond - entropy pooling: theory

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$$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$$

$$\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$$

$$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$$

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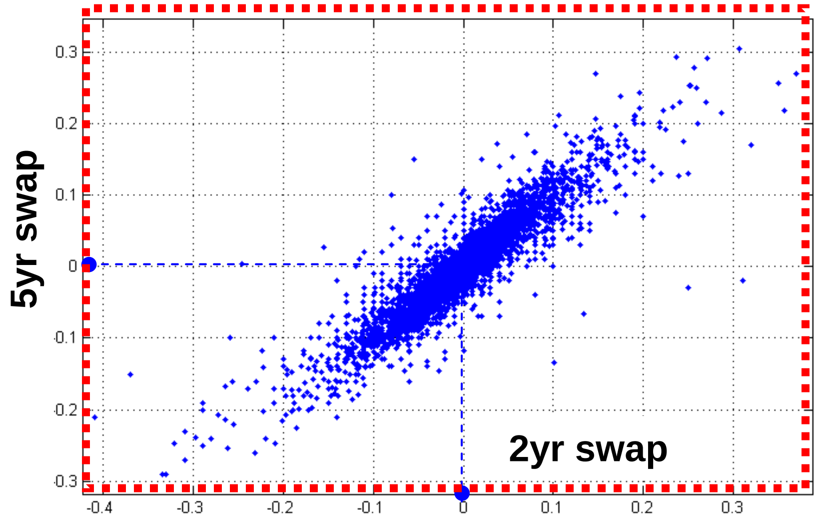
partial distribution specification

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}}\{\mathcal{E}(f, f_{\mathbf{X}})\}$ **least distance from prior, views satisfied**

LIVE

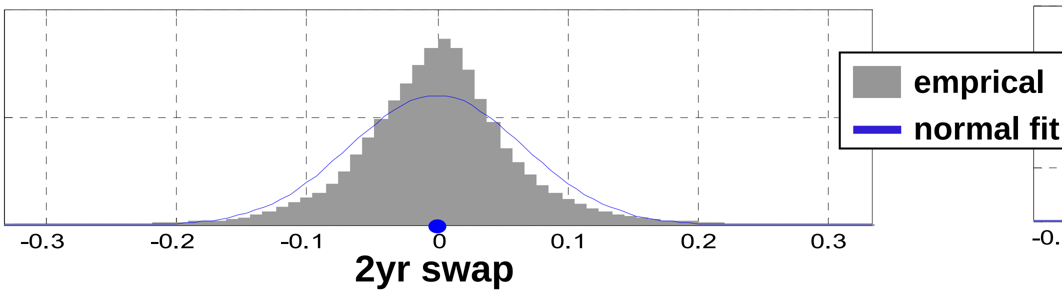
BL and beyond - entropy pooling: theory

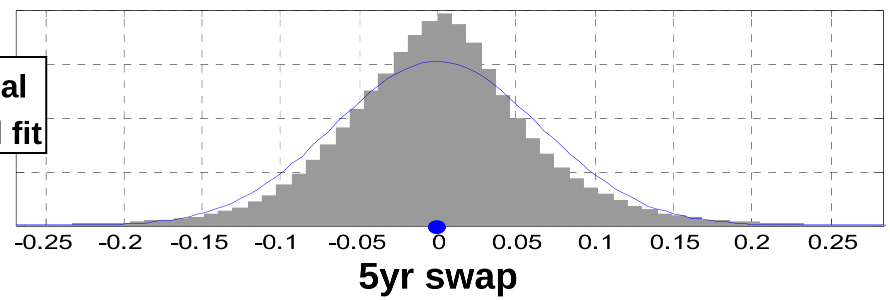
Market distr. $\mathbf{X} \sim \boxed{f_{\mathbf{X}}}$



X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

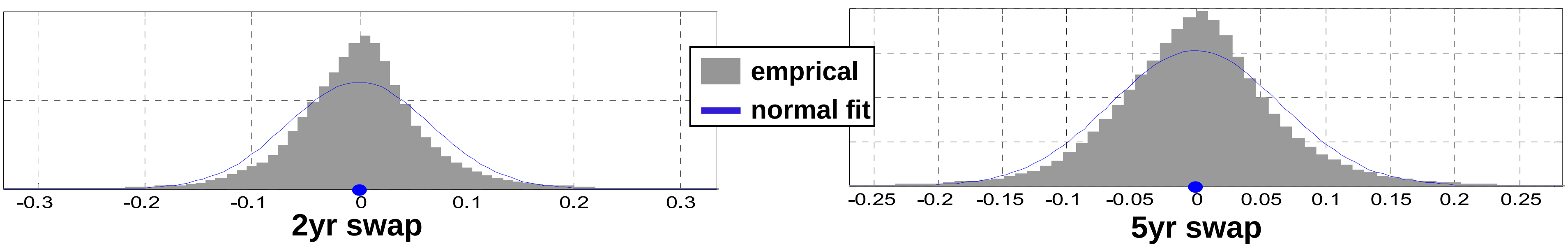




BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim \boxed{f_{\mathbf{X}}}$.

X_1 **2-yr swap rate**
X_2 **5-yr swap rate**



BL and beyond - entropy pooling: theory

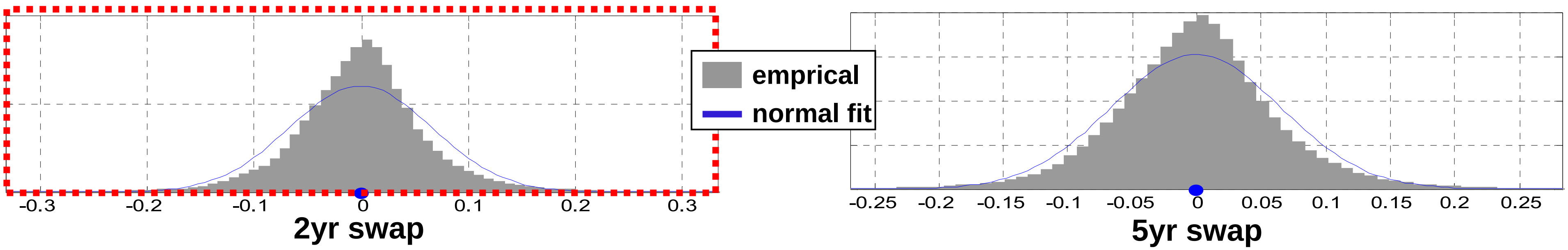
Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$.

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim \boxed{f_{\mathbf{V}}}$

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$$\boxed{V \equiv X_1}$$



BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$.

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$

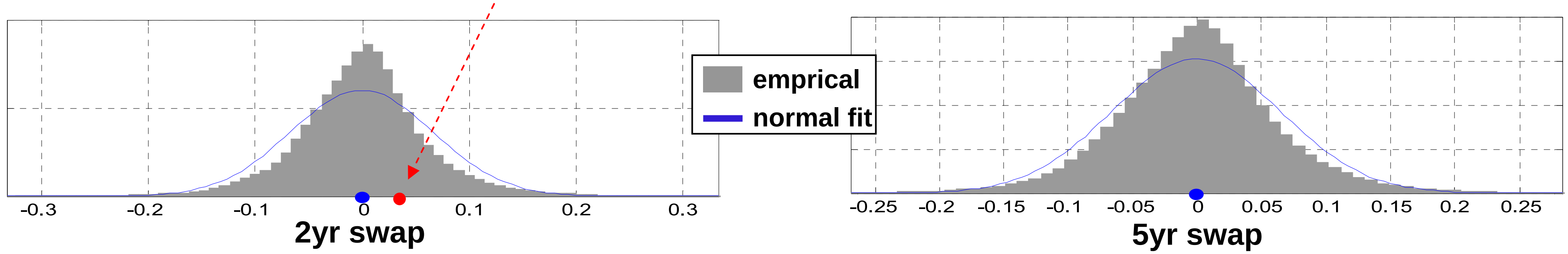
Views

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$$V \equiv X_1$$

$$\tilde{m}\{V\} \equiv m\{V\} + \frac{\sigma\{V\}}{2} \approx 3.26\ bp$$



BL and beyond - entropy pooling: theory

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$$V \equiv X_1$$

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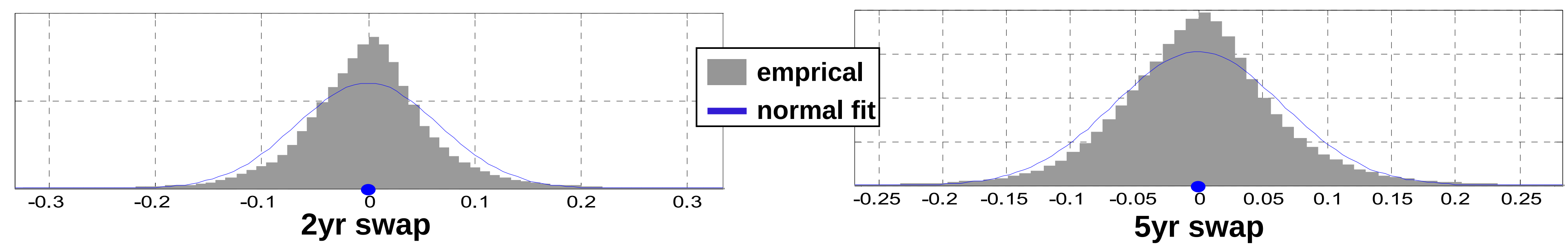
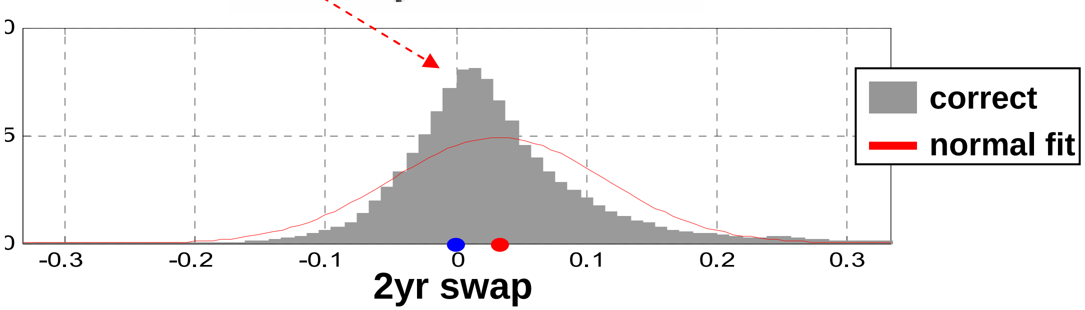


Posterior $\tilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathcal{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$



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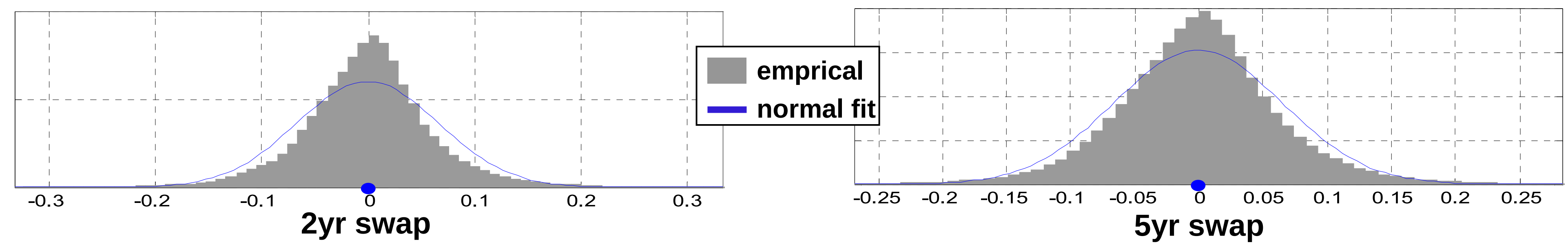

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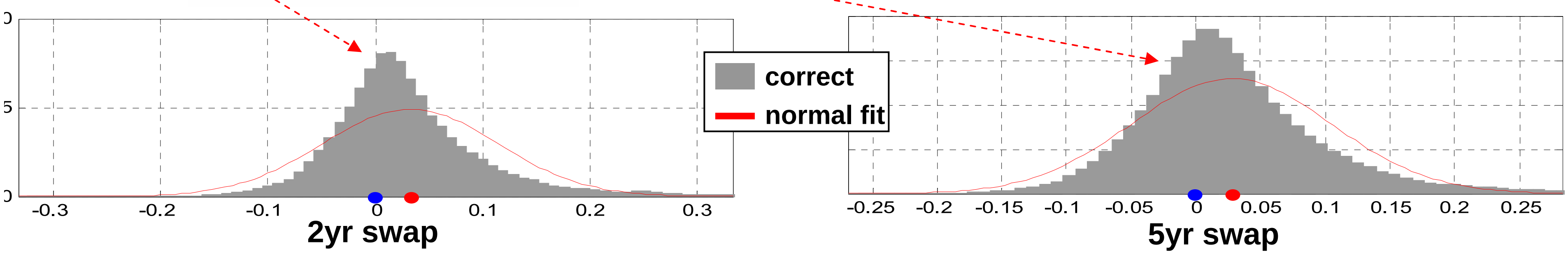

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

$$\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}:$$

$$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$$

$$\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$$

$$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\{\mathbf{V}\} + \rho_3 \mathbf{1}\mathbf{1}',$$

$$\widetilde{Q}_V(u) \gtreqless Q_V(u)$$

partial distribution specification

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$ **least distance from prior, views satisfied**

BL and beyond - entropy pooling: theory

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$$\widetilde{Q}_V(u) \gtreqless Q_V(u)$$

partial distribution specification

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$ **least distance from prior, views satisfied**

Confidence $\widetilde{f}^{c}_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$. **multi-user, multi-confidence**

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partial distribution specification

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Confidence $\widetilde{f}^c_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}.$ **multi-user, multi-confidence**

100(1-c) % of times:
PRIOR

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

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\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\} \\
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\widetilde{Q}_V(u) \gtreqless Q_V(u)
\end{array}\right\}$$

partial distribution specification

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}}\{\mathcal{E}(f, f_{\mathbf{X}})\}$ **least distance from prior, views satisfied**

Confidence $\widetilde{f}_{\mathbf{X}}^c \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}.$ **multi-user, multi-confidence**

100c % of times:
POSTERIOR

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ **not returns, not normal, not equilibrium**

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ **non-linear functions and external factors**

Views $\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ **full distribution specification**

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partial distribution specification

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Confidence $\tilde{f}^c_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\tilde{f}_{\mathbf{X}}.$ **multi-user, multi-confidence**

Pricing $P_{t+\tau} \equiv P(\tilde{\mathbf{X}}, \mathcal{I}_t)$ **delta/gamma/vega, full pricing, …**

BL and beyond - entropy pooling: theory

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. **not returns, not normal, not equilibrium**

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Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}}\{\mathcal{S}(\mathbf{w}; \widetilde{f}^{c}_{\mathbf{X}})\}$ **mean-variance, mean-CVaR, …**

ESTIMATION RISK

SCENARIO ANALYSIS

THE BLACK-LITTERMAN APPROACH

ENTROPY POOLING

- **Theory**
- **Analytical solution**
- **General implementation**

CASE STUDIES

REFERENCES AND CONCLUSIONS

BL and beyond - entropy pooling: analytical solution

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. $\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$.

$$\widetilde{m}\{V_k\} \gtreqqless \widetilde{\mu}_{\mathbf{v},k};$$

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$$\widetilde{\sigma}\{V_k\} \gtreqqless \varkappa\sigma\{V_k\}$$

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$$\widetilde{Q}_V(u) \gtreqqless Q_V(u)$$

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$

Confidence $\widetilde{f}^{c}_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$.

Pricing $P_{t+\tau} \equiv P(\widetilde{\mathbf{X}}, \mathcal{I}_t)$

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BL and beyond - entropy pooling: analytical solution

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. $\qquad \mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\qquad \mathbf{QX} \quad \mathbf{GX}$

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$.

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Confidence $\widetilde{f}^{c}_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$.

Pricing $P_{t+\tau} \equiv P(\widetilde{\mathbf{X}}, \mathcal{I}_t)$

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; \widetilde{f}^{c}_{\mathbf{X}})\}$

BL and beyond - entropy pooling: analytical solution

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$$\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}:$$

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$$\mathbb{E}\{\mathbf{QX}\} \equiv \widetilde{\boldsymbol{\mu}}_{\mathbf{Q}}$$

$$\mathbb{C}ov\{\mathbf{GX}\} \equiv \widetilde{\boldsymbol{\Sigma}}_{\mathbf{G}}$$

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Optimization $$\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; \widetilde{f}_{\mathbf{X}}^{c})\}$$

BL and beyond - entropy pooling: analytical solution

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$$\begin{cases} \widetilde{\boldsymbol{\mu}} \equiv \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{Q}'\left(\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}'\right)^{-1}\left(\widetilde{\boldsymbol{\mu}}_{\mathbf{Q}} - \mathbf{Q}\boldsymbol{\mu}\right) \\ \widetilde{\boldsymbol{\Sigma}} \equiv \boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{G}'\left(\left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{G}}\left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1} - \left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1}\right)\mathbf{G}\boldsymbol{\Sigma}. \end{cases}$$

Confidence $\widetilde{f}_{\mathbf{X}}^{c} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$

Pricing $P_{t+\tau} \equiv P\left(\widetilde{\mathbf{X}}, \mathcal{I}_t\right)$

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BL and beyond - entropy pooling: analytical solution

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$$\begin{cases} \widetilde{\boldsymbol{\mu}} & \equiv \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{Q}'\left(\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}'\right)^{-1}\left(\widetilde{\boldsymbol{\mu}}_{\mathbf{Q}} - \mathbf{Q}\boldsymbol{\mu}\right) \\ \widetilde{\boldsymbol{\Sigma}} & \equiv \boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{G}'\left(\left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{G}}\left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1} \right. \\ & \quad \left. - \left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1}\right)\mathbf{G}\boldsymbol{\Sigma}. \end{cases}$$

Confidence $\widetilde{f}^{c}_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$

$$\mathbf{X} \sim \begin{cases} \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) & (\text{probability: } 1-c) \\ \mathrm{N}\left(\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\Sigma}}\right) & (\text{probability: } c) \end{cases}$$

Pricing $P_{t+\tau} \equiv P\left(\widetilde{\mathbf{X}}, \mathcal{I}_t\right)$

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{\mathcal{S}\left(\mathbf{w}; \widetilde{f}^{c}_{\mathbf{X}}\right)\right\}$

BL and beyond - entropy pooling: analytical solution

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ $\qquad$ $\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\qquad$ $\mathbf{QX}$ $\quad$ $\mathbf{GX}$

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$

$\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}$

$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$

$\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$

$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\{\mathbf{V}\} + \rho_3 \mathbf{1}\mathbf{1}'$,

$\widetilde{Q}_V(u) \gtreqless Q_V(u)$

$\mathbb{E}\{\mathbf{QX}\} \equiv \widetilde{\boldsymbol{\mu}}_{\mathbf{Q}}$

$\mathbb{C}ov\{\mathbf{GX}\} \equiv \widetilde{\boldsymbol{\Sigma}}_{\mathbf{G}}$

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$ $\qquad$ $\mathbf{X} \sim \mathrm{N}\left(\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\Sigma}}\right)$

$$\left\{\begin{array}{lcl} \widetilde{\boldsymbol{\mu}} & \equiv & \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{Q}'\left(\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}'\right)^{-1}\left(\widetilde{\boldsymbol{\mu}}_{\mathbf{Q}} - \mathbf{Q}\boldsymbol{\mu}\right) \\ \widetilde{\boldsymbol{\Sigma}} & \equiv & \boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{G}'\left(\left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{G}}\left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1} \right. \\ & & \left. - \left(\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}'\right)^{-1}\right)\mathbf{G}\boldsymbol{\Sigma}. \end{array}\right.$$

Confidence $\widetilde{f}^{c}_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$

$$\mathbf{X} \sim \begin{cases} \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) & (\text{probability: } 1-c) \\ \mathrm{N}\left(\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\Sigma}}\right) & (\text{probability: } c) \end{cases}$$

Pricing $P_{t+\tau} \equiv P\left(\widetilde{\mathbf{X}}, \mathcal{I}_t\right)$ $\qquad$ ...

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; \widetilde{f}^{c}_{\mathbf{X}})\}$ $\qquad$ ...

ESTIMATION RISK

SCENARIO ANALYSIS

THE BLACK-LITTERMAN APPROACH

ENTROPY POOLING

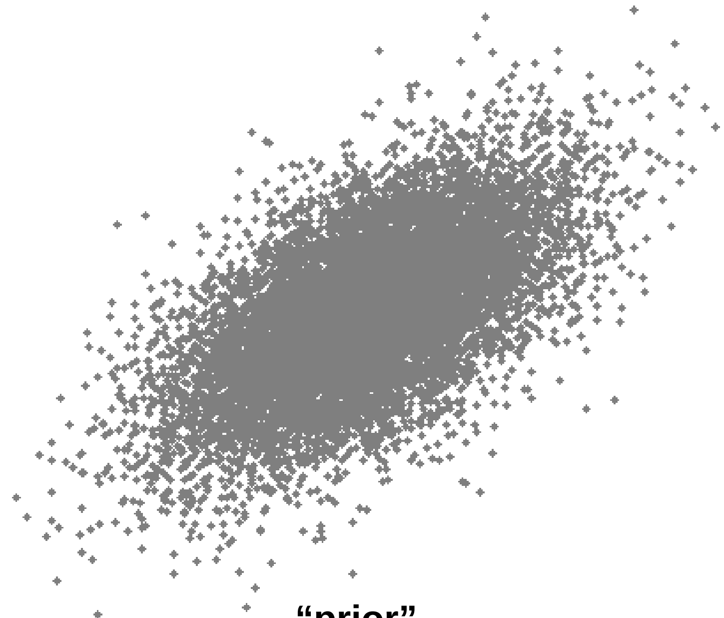
- **Theory**
- **Analytical solution**
- **General implementation**

CASE STUDIES

REFERENCES AND CONCLUSIONS

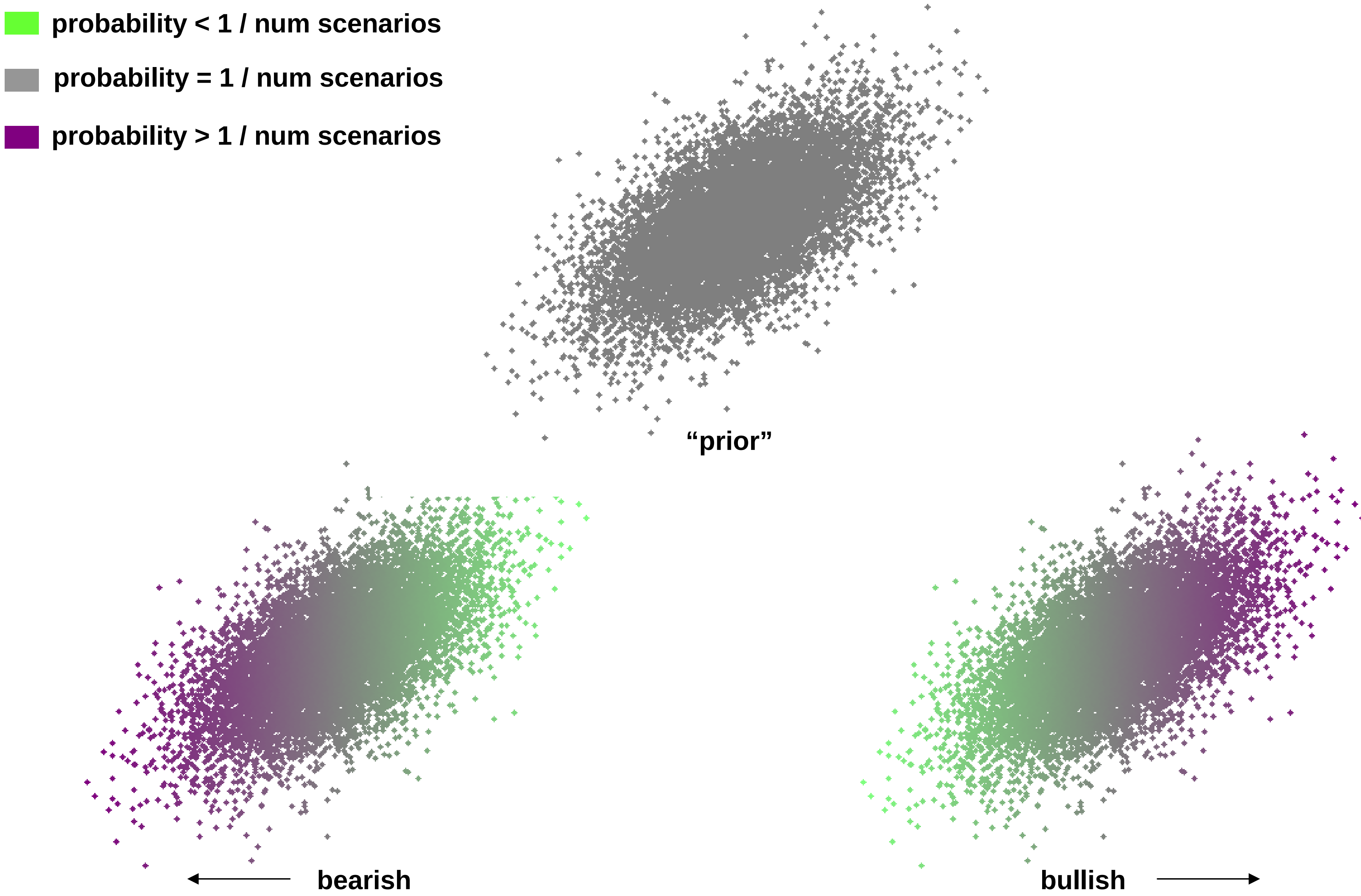
BL and beyond - entropy pooling implementation: Black-Litterman

probability = 1 / num scenarios

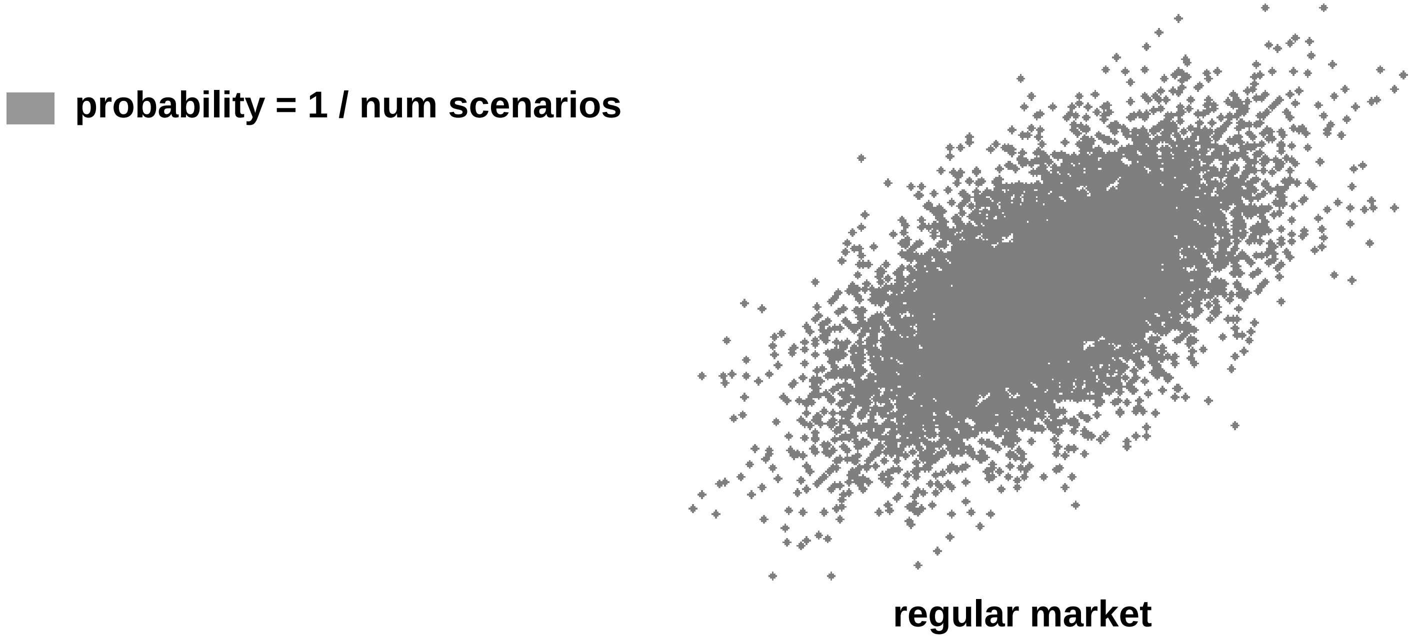


“prior”

BL and beyond - entropy pooling implementation: Black-Litterman



BL and beyond - entropy pooling implementation: stress-testing



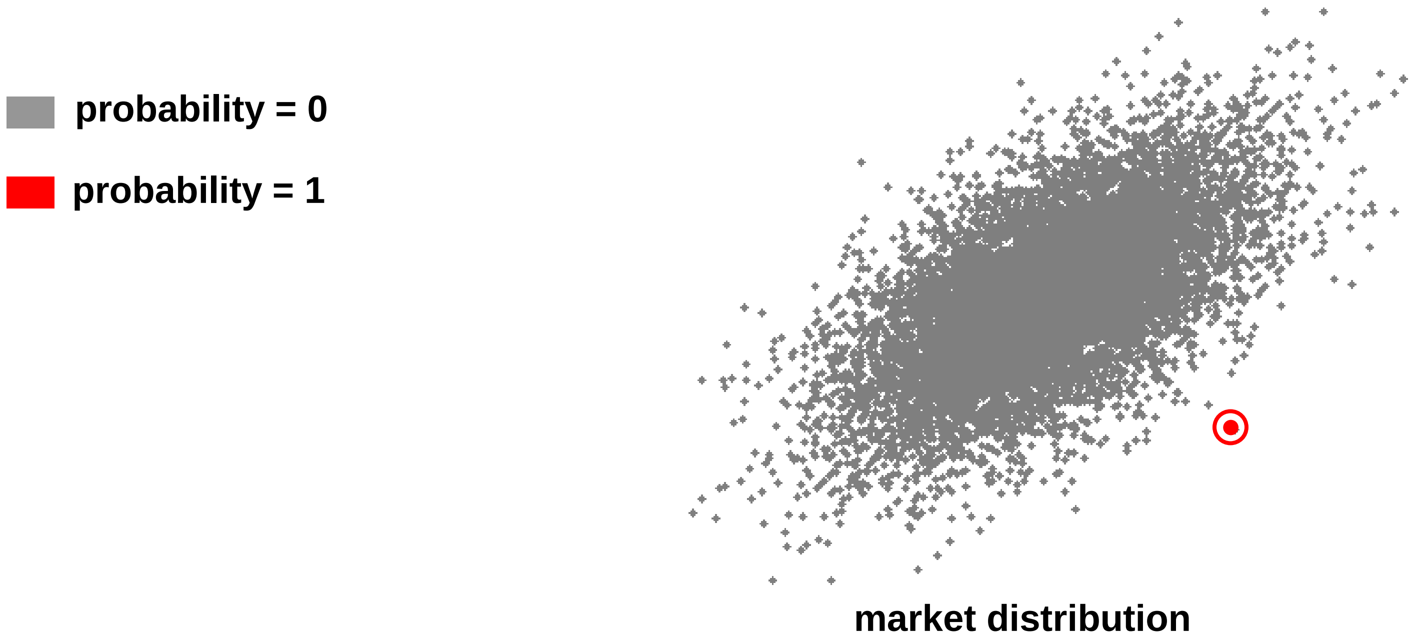
BL and beyond - entropy pooling implementation: stress-testing

BL and beyond - entropy pooling implementation: scenario analysis

probability = 1 / num scenarios

market distribution

BL and beyond - entropy pooling implementation: scenario analysis



BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ $\Leftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\mathbf{p}$ probabilities $1/J$

BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ $\Leftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\mathbf{p}$ probabilities $1/J$

Pricing $P_{t+\tau} \equiv P\left(\mathbf{X}, \mathcal{I}_t\right)$ $\Leftrightarrow$ $\mathcal{P}$ $\mathbf{p}$

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{\mathcal{S}\left(\mathbf{w}; f_{\mathbf{X}}\right)\right\}$ $\Leftrightarrow$ ...

BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. $\Leftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\mathbf{p}$ probabilities $1/J$.

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$.

$$\widetilde{m}\{V_k\} \gtreqqless \widetilde{\mu}_{\mathbf{v},k};$$

$$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$$

$$\widetilde{\sigma}\{V_k\} \gtreqqless \varkappa\sigma\{V_k\}$$

$$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$$

$$\widetilde{Q}_V(u) \gtreqqless Q_V(u)$$

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}}\{\mathcal{E}(f, f_{\mathbf{X}})\}$

Confidence $\widetilde{f}^c_{\mathbf{X}} \equiv (1-c)\, f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$.

Pricing $P_{t+\tau} \equiv P(\widetilde{\mathbf{X}}, \mathcal{I}_t)$

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}}\{\mathcal{S}(\mathbf{w}; \widetilde{f}^c_{\mathbf{X}})\}$

BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ $\Longleftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\quad \mathbf{p}$ probabilities $1/J.$

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\Longleftrightarrow$ $\mathcal{V}_{j,k} \equiv g_k\left(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N}\right)$

scenario index

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$

$$\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k};$$

$$\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$$

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$$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\{\mathbf{V}\} + \rho_3 \mathbf{1}\mathbf{1}',$$

$$\widetilde{Q}_V(u) \gtreqless Q_V(u)$$

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}}\left\{\mathcal{E}\left(f, f_{\mathbf{X}}\right)\right\}$

Confidence $\widetilde{f}_{\mathbf{X}}^{c} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}.$

Pricing $P_{t+\tau} \equiv P\left(\widetilde{\mathbf{X}}, \mathcal{I}_t\right)$

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}}\left\{\mathcal{S}\left(\mathbf{w}; \widetilde{f}_{\mathbf{X}}^{c}\right)\right\}$

BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ $\Longleftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\quad \mathbf{p}$ probabilities $1/J$

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\Longleftrightarrow$ $\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$

Views

$$\left.\begin{array}{l}
\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}} \\
\tilde{m}\{V_k\} \gtreqless \tilde{\mu}_{\mathbf{v},k} \\
\tilde{m}\{V_1\} \geq \tilde{m}\{V_2\} \geq \cdots \geq \tilde{m}\{V_K\} \\
\tilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\} \\
\tilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\{\mathbf{V}\} + \rho_3 \mathbf{1}\mathbf{1}', \\
\tilde{Q}_V(u) \gtreqless Q_V(u)
\end{array}\right\} \Longleftrightarrow \underline{\mathbf{a}} \leq \mathbf{A}\tilde{\mathbf{p}} \leq \overline{\mathbf{a}}$$

e.g.

X_1 **2-yr swap rate**

X_2 **5-yr swap rate**

$$V \equiv X_1$$

$$\tilde{m}\{V\} \equiv \tilde{\mu}$$

$$\Updownarrow$$

$$\tilde{\mu} \leq \sum_{j=1}^{J} \mathcal{V}_j \tilde{p}_j \leq \tilde{\mu}$$

Posterior $\tilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$

Confidence $\tilde{f}^{c}_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\tilde{f}_{\mathbf{X}}$

Pricing $P_{t+\tau} \equiv P(\tilde{\mathbf{X}}, \mathcal{I}_t)$

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; \tilde{f}^{c}_{\mathbf{X}})\}$

BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ $\Longleftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\mathbf{p}$ probabilities $1/J.$

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\Longleftrightarrow$ $\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$

Views

$$\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$$

$$\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}.$$

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$$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$$

$$\widetilde{Q}_V(u) \gtreqless Q_V(u)$$

$\Longleftrightarrow$ $\underline{\mathbf{a}} \leq \mathbf{A}\widetilde{\mathbf{p}} \leq \overline{\mathbf{a}}.$

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}}\{\mathcal{E}(f, f_{\mathbf{X}})\}$

$$\mathcal{E}\left(\widetilde{f}_{\mathbf{X}}, f_{\mathbf{X}}\right) \equiv \int \widetilde{f}_{\mathbf{X}}(\mathrm{x})\left[\ln \widetilde{f}_{\mathbf{X}}(\mathrm{x}) - \ln f_{\mathbf{X}}(\mathrm{x})\right] d\mathrm{x}.$$

BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. $\iff$ $\mathcal{X}$ $J \times N$ panel $\mathbf{p}$ probabilities $1/J$.

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\iff$ $\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$

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$$\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k};$$

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$$\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\{\mathbf{V}\} + \rho_3 \mathbf{1}\mathbf{1}',$$

$$\widetilde{Q}_V(u) \gtreqless Q_V(u)$$

$\iff$ $\underline{\mathbf{a}} \leq \mathbf{A}\widetilde{\mathbf{p}} \leq \overline{\mathbf{a}}$

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$

$$\mathcal{E}\left(\widetilde{f}_{\mathbf{X}}, f_{\mathbf{X}}\right) \equiv \int \widetilde{f}_{\mathbf{X}}(\mathrm{x}) \left[\ln \widetilde{f}_{\mathbf{X}}(\mathrm{x}) - \ln f_{\mathbf{X}}(\mathrm{x})\right] d\mathrm{x}. \iff \mathcal{E}(\widetilde{\mathbf{p}}, \mathbf{p}) \equiv \sum_{j=1}^{J} \widetilde{p}_j \left[\ln(\widetilde{p}_j) - \ln(p_j)\right]$$

BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. $\Leftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\quad \mathbf{p}$ probabilities $1/J$.

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\Leftrightarrow$ $\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$

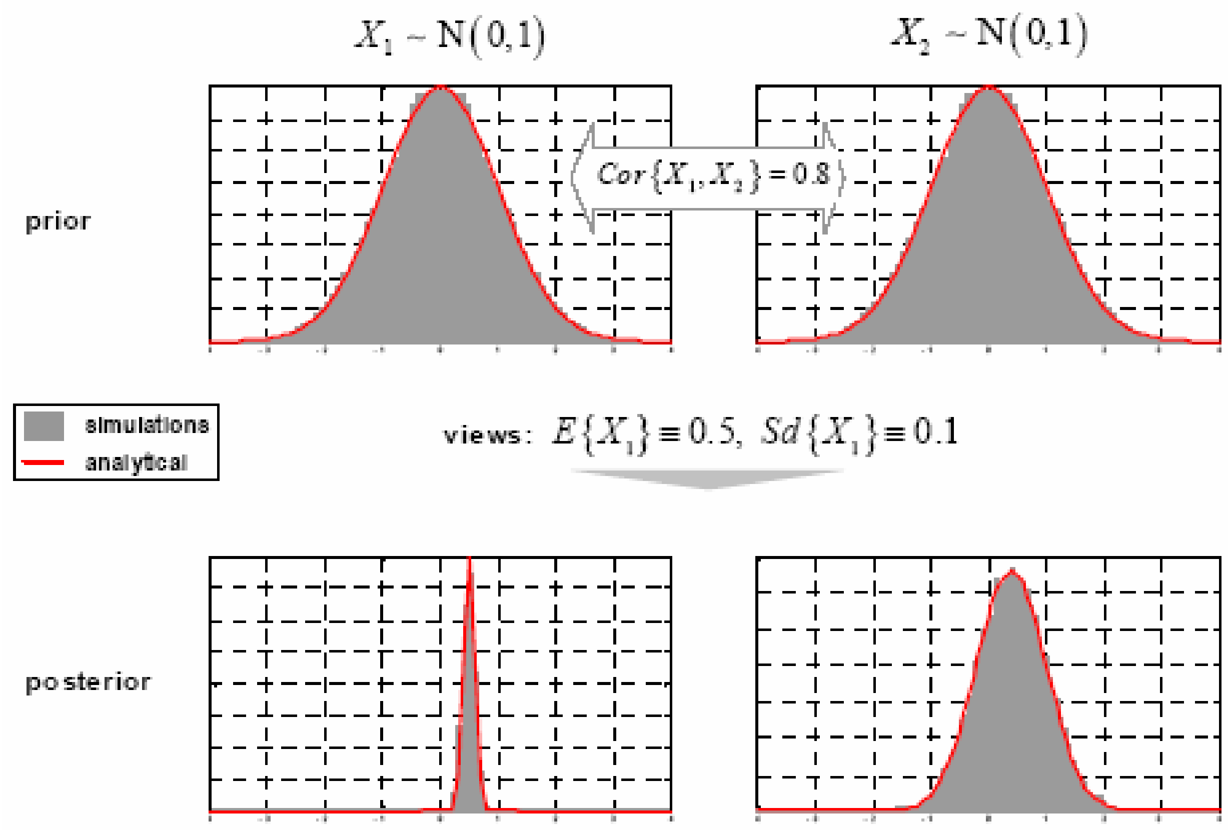
Views $\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$.

$$\left.\begin{array}{l} \tilde{m}\{V_k\} \gtreqless \tilde{\mu}_{\mathbf{v},k}; \\ \tilde{m}\{V_1\} \geq \tilde{m}\{V_2\} \geq \cdots \geq \tilde{m}\{V_K\} \\ \tilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\} \\ \tilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\{\mathbf{V}\} + \rho_3 \mathbf{1}\mathbf{1}', \\ \tilde{Q}_V(u) \gtreqless Q_V(u) \end{array}\right\} \Leftrightarrow \quad \underline{\mathbf{a}} \leq \mathbf{A}\tilde{\mathbf{p}} \leq \overline{\mathbf{a}}$$

Posterior $\tilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$ $\Leftrightarrow$ $\mathcal{X}$ $\quad \tilde{\mathbf{p}} \equiv \underset{\underline{\mathbf{a}} \leq \mathbf{A}\mathbf{f} \leq \overline{\mathbf{a}}}{\operatorname{argmin}} \{\mathcal{E}(\mathbf{f}, \mathbf{p})\}$

$$\mathcal{E}\left(\tilde{f}_{\mathbf{X}}, f_{\mathbf{X}}\right) \equiv \int \tilde{f}_{\mathbf{X}}(\mathrm{x}) \left[\ln \tilde{f}_{\mathbf{X}}(\mathrm{x}) - \ln f_{\mathbf{X}}(\mathrm{x})\right] d\mathrm{x}. \quad \Leftrightarrow \quad \mathcal{E}(\tilde{\mathbf{p}}, \mathbf{p}) \equiv \sum_{j=1}^{J} \tilde{p}_j \left[\ln(\tilde{p}_j) - \ln(p_j)\right]$$

BL and beyond - entropy pooling implementation



Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \left\{ \mathcal{E}\left(f, f_{\mathbf{X}}\right) \right\}$ $\Longleftrightarrow$ $\mathcal{X}$ $\boxed{\widetilde{\mathbf{p}} \equiv \underset{\underline{\mathbf{a}} \leq \mathbf{A}\mathbf{f} \leq \overline{\mathbf{a}}}{\operatorname{argmin}} \left\{ \mathcal{E}\left(\mathbf{f}, \mathbf{p}\right) \right\}}$

Dual formulation: linearly constrained convex optimization in # variables = # views

BL and beyond - entropy pooling implementation

Market distr.	$\mathbf{X} \sim f_{\mathbf{X}}$	$\Leftrightarrow$	$\mathcal{X}$ $J \times N$ panel $\quad \mathbf{p}$ probabilities $1/J$
Focus	$\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$	$\Leftrightarrow$	$\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$
Views	$\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$ $\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}$; $\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$ $\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$ $\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}'$, $\widetilde{Q}_V(u) \gtreqless Q_V(u)$	$\Leftrightarrow$	$\underline{\mathbf{a}} \leq \mathbf{A}\widetilde{\mathbf{p}} \leq \overline{\mathbf{a}}$
Posterior	$\widetilde{f}_{\mathbf{X}} \equiv \underset{f\in\mathbb{V}}{\operatorname{argmin}}\{\mathcal{E}(f, f_{\mathbf{X}})\}$	$\Leftrightarrow$	$\mathcal{X} \quad \widetilde{\mathbf{p}} \equiv \underset{\underline{\mathbf{a}}\leq\mathbf{A}\mathbf{f}\leq\overline{\mathbf{a}}}{\operatorname{argmin}}\{\mathcal{E}(\mathbf{f}, \mathbf{p})\}$
Confidence	$\widetilde{f}^{c}_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$		
Pricing	$P_{t+\tau} \equiv P(\widetilde{\mathbf{X}}, \mathcal{I}_t)$		
Optimization	$\mathbf{w}^* \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}}\{\mathcal{S}(\mathbf{w}; \widetilde{f}^{c}_{\mathbf{X}})\}$		

BL and beyond - entropy pooling implementation

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$ $\Leftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\quad \mathbf{p}$ probabilities $1/J$

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\Leftrightarrow$ $\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$

Views $\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$

$$\left.\begin{array}{l}
\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k} \\
\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\} \\
\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\} \\
\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\{\mathbf{V}\} + \rho_3 \mathbf{1}\mathbf{1}', \\
\widetilde{Q}_V(u) \gtreqless Q_V(u)
\end{array}\right\} \Leftrightarrow \quad \underline{\mathbf{a}} \leq \mathbf{A}\widetilde{\mathbf{p}} \leq \overline{\mathbf{a}}$$

Posterior $\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$ $\Leftrightarrow$ $\mathcal{X}$ $\quad \widetilde{\mathbf{p}} \equiv \underset{\underline{\mathbf{a}} \leq \mathbf{Af} \leq \overline{\mathbf{a}}}{\operatorname{argmin}} \{\mathcal{E}(\mathbf{f}, \mathbf{p})\}$

Confidence $\widetilde{f}^{c}_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$ $\Leftrightarrow$ $\mathcal{X}$ $\quad \mathbf{p}_c \equiv (1-c)\mathbf{p} + c\widetilde{\mathbf{p}}$

Pricing $P_{t+\tau} \equiv P(\widetilde{\mathbf{X}}, \mathcal{I}_t)$

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; \widetilde{f}^{c}_{\mathbf{X}})\}$

BL and beyond - entropy pooling implementation

Market distr.	$\mathbf{X} \sim f_{\mathbf{X}}$.	$\Longleftrightarrow$	$\mathcal{X}$ $J \times N$ panel $\mathbf{p}$ probabilities $1/J$.
Focus	$\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$	$\Longleftrightarrow$	$\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$
Views	$\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$. $\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}$; $\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$ $\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$ $\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}'$, $\widetilde{Q}_V(u) \gtreqless Q_V(u)$	$\Longleftrightarrow$	$\underline{\mathbf{a}} \leq \mathbf{A}\widetilde{\mathbf{p}} \leq \overline{\mathbf{a}}$
Posterior	$\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathcal{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$	$\Longleftrightarrow$	$\mathcal{X}$ $\widetilde{\mathbf{p}} \equiv \underset{\underline{\mathbf{a}} \leq \mathbf{A}\mathbf{f} \leq \overline{\mathbf{a}}}{\operatorname{argmin}} \{\mathcal{E}(\mathbf{f}, \mathbf{p})\}$
Confidence	$\widetilde{f}_{\mathbf{X}}^{c} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}$.	$\Longleftrightarrow$	$\mathcal{X}$ $\mathbf{p}_c \equiv (1-c)\mathbf{p} + c\widetilde{\mathbf{p}}$.
~~**Pricing**~~	$P_{t+\tau} \equiv P(\widetilde{\mathbf{X}}, \mathcal{I}_t)$	$\Longleftrightarrow$	$\boxed{\mathcal{P}}$ $\mathbf{p}_c$
Optimization	$\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; \widetilde{f}_{\mathbf{X}}^{c})\}$		

BL and beyond - entropy pooling implementation

Market distr.	$\mathbf{X} \sim f_{\mathbf{X}}.$	$\Longleftrightarrow$	$\mathcal{X}$ $J \times N$ panel $\quad \mathbf{p}$ probabilities $1/J$.
Focus	$\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$	$\Longleftrightarrow$	$\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$
Views	$\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}.$ $\widetilde{m}\{V_k\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}.$ $\widetilde{m}\{V_1\} \geq \widetilde{m}\{V_2\} \geq \cdots \geq \widetilde{m}\{V_K\}$ $\widetilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$ $\widetilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}',$ $\widetilde{Q}_V(u) \gtreqless Q_V(u)$	$\Longleftrightarrow$	$\underline{\mathbf{a}} \leq \mathbf{A}\widetilde{\mathbf{p}} \leq \overline{\mathbf{a}}$
Posterior	$\widetilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}} \{\mathcal{E}(f, f_{\mathbf{X}})\}$	$\Longleftrightarrow$	$\mathcal{X} \quad \widetilde{\mathbf{p}} \equiv \underset{\underline{\mathbf{a}} \leq \mathbf{A}\mathbf{f} \leq \overline{\mathbf{a}}}{\operatorname{argmin}} \{\mathcal{E}(\mathbf{f}, \mathbf{p})\}$
Confidence	$\widetilde{f}_{\mathbf{X}}^{c} \equiv (1-c) f_{\mathbf{X}} + c\widetilde{f}_{\mathbf{X}}.$	$\Longleftrightarrow$	$\mathcal{X} \quad \mathbf{p}_c \equiv (1-c)\mathbf{p} + c\widetilde{\mathbf{p}}.$
Pricing	$P_{t+\tau} \equiv P(\widetilde{\mathbf{X}}, \mathcal{I}_t)$	$\Longleftrightarrow$	$\mathcal{P} \quad \mathbf{p}_c$
Optimization	$\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; \widetilde{f}_{\mathbf{X}}^{c})\}$	$\Longleftrightarrow$	…

ESTIMATION RISK

SCENARIO ANALYSIS

THE BLACK-LITTERMAN APPROACH

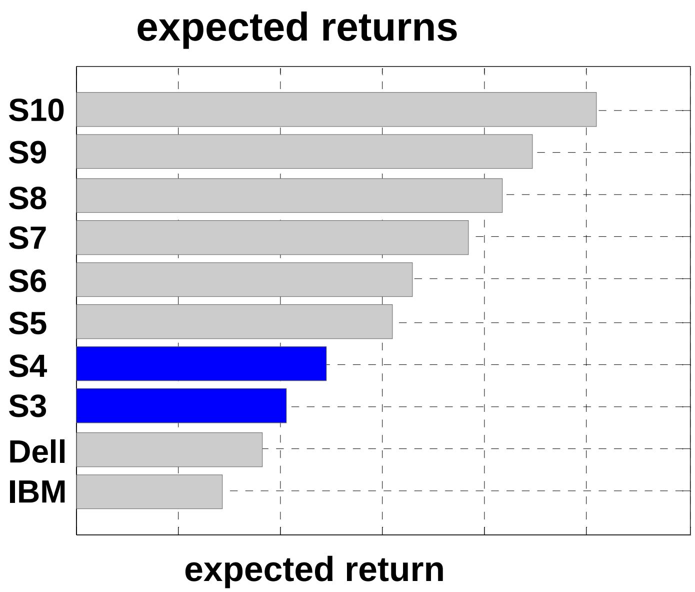
ENTROPY POOLING

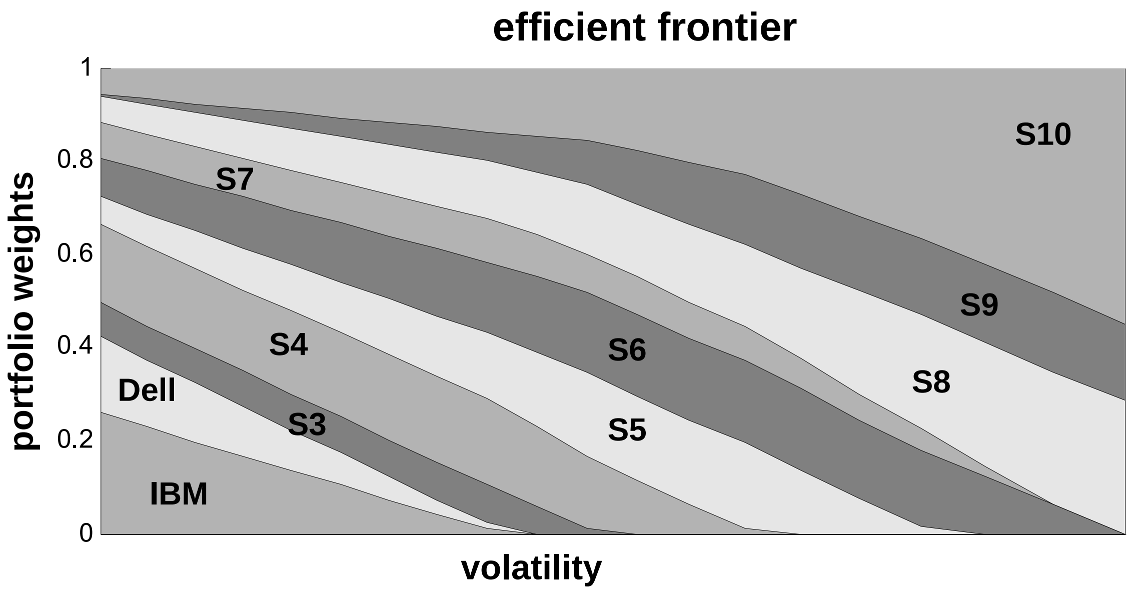
CASE STUDIES

- **Ranking allocation**
- **Option trading**

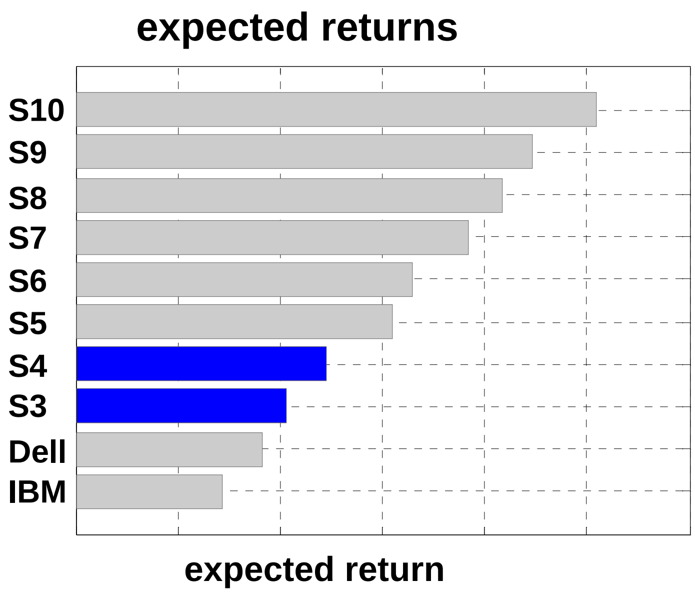
REFERENCES AND CONCLUSIONS

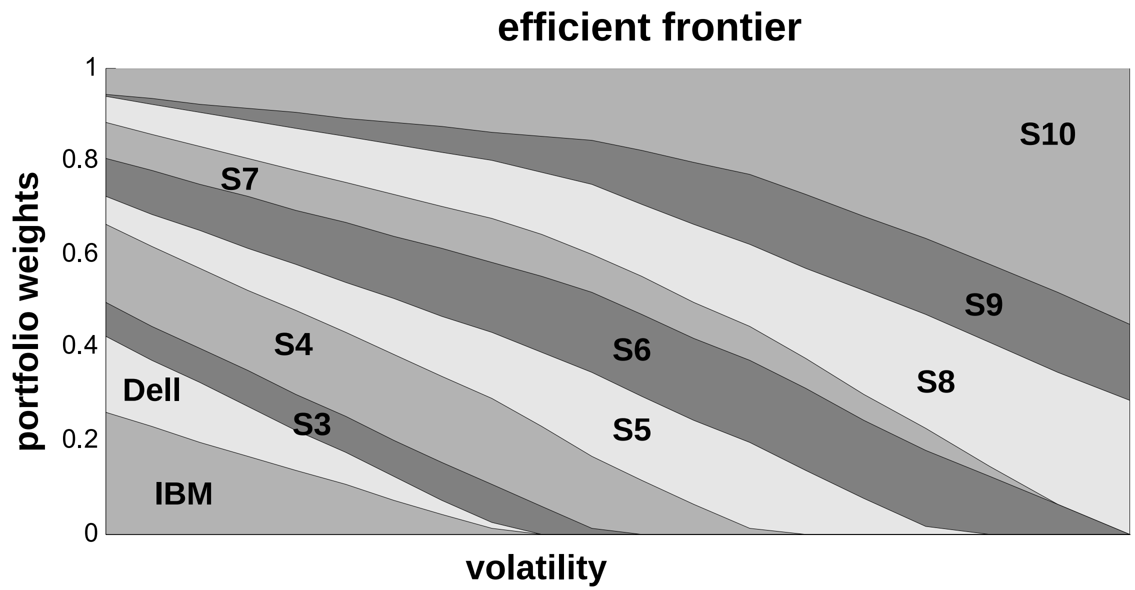
BL and beyond - EP case study: ranking allocation





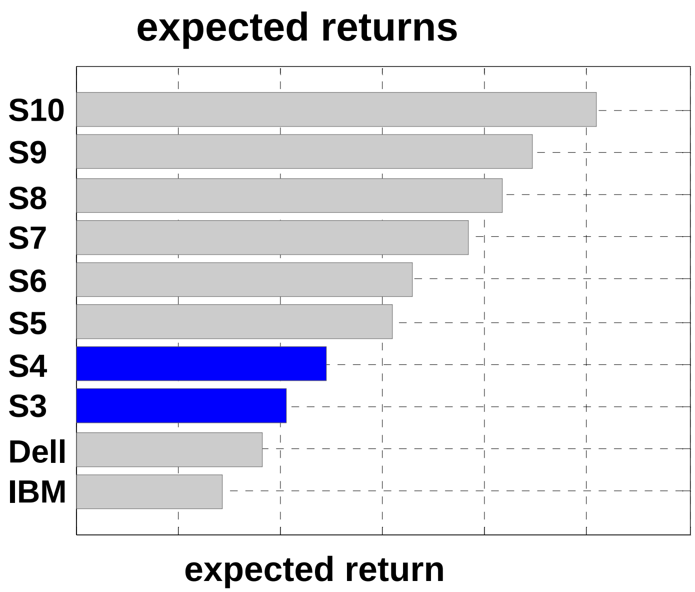
BL and beyond - EP case study: ranking allocation

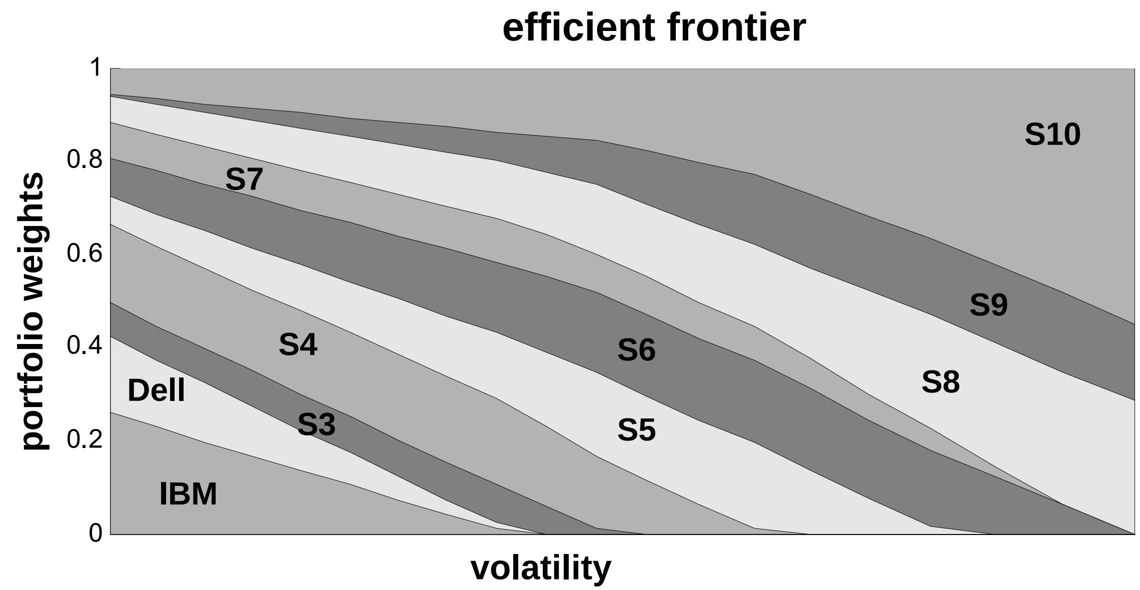




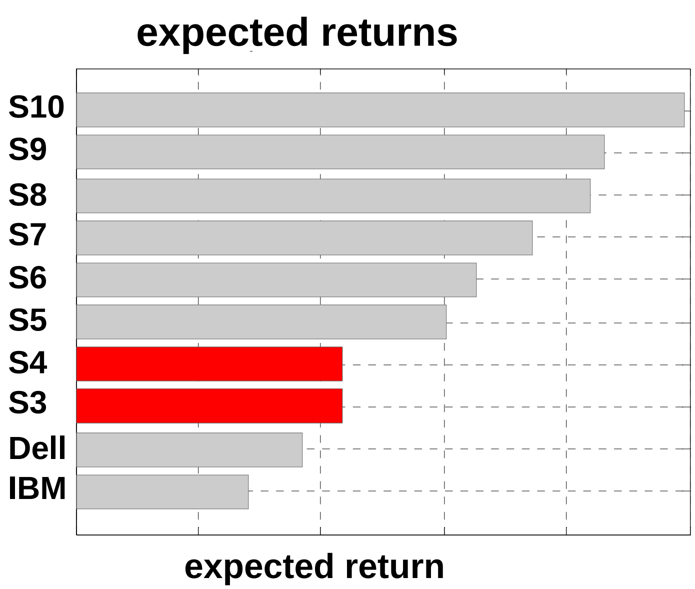
view: $E\left\{R_3\right\} \geq E\left\{R_4\right\}$

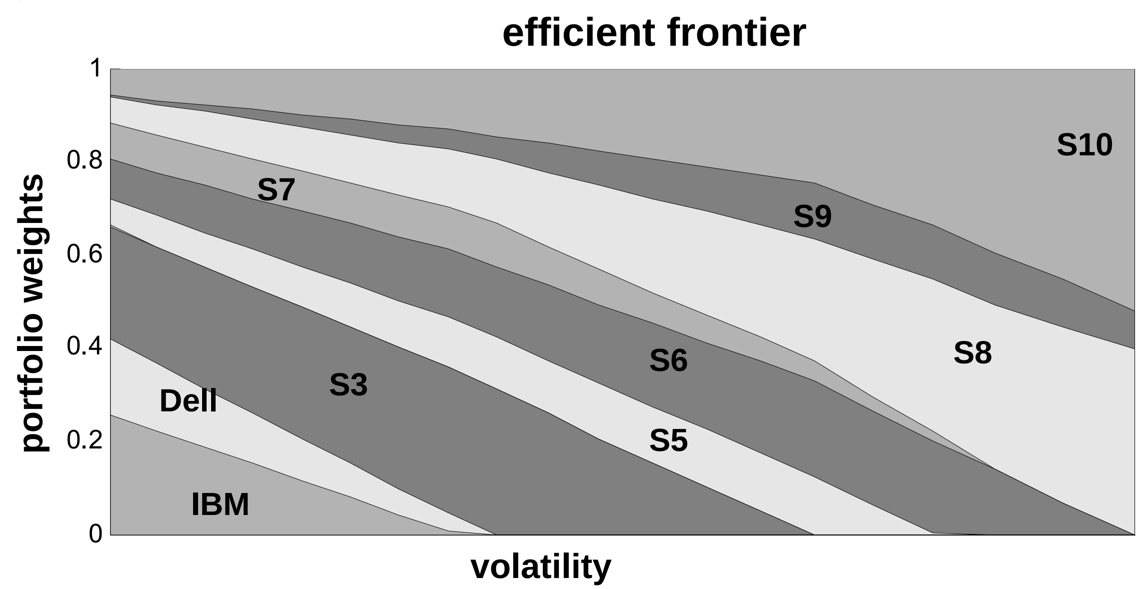
BL and beyond - EP case study: ranking allocation





view: $E\left\{R_3\right\} \geq E\left\{R_4\right\}$





ESTIMATION RISK

SCENARIO ANALYSIS

THE BLACK-LITTERMAN APPROACH

ENTROPY POOLING

CASE STUDIES

- **Ranking allocation**
- **Option trading**

REFERENCES AND CONCLUSIONS

Black-Scholes formula:
deterministic function of risk into price

$$C_{BS}\left(y,\sigma;\kappa,T,r\right)\equiv yF\left(d_1\right)-\kappa e^{-rT}F\left(d_2\right)$$

$$d_1\equiv\left(\ln\left(y/\kappa\right)+\left(r+\sigma^2/2\right)T\right)/\sigma\sqrt{T},\qquad d_2\equiv d_1-\sigma\sqrt{T};$$

Black-Scholes formula:
deterministic function of risk into price

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BL and beyond - EP case study: option trading

$$C_{BS}(y, \sigma; \kappa, T, r) \equiv yF(d_1) - \kappa e^{-rT} F(d_2)$$

$$d_1 \equiv \left(\ln(y/\kappa) + \left(r + \sigma^2/2\right)T\right)/\sigma\sqrt{T}, \qquad d_2 \equiv d_1 - \sigma\sqrt{T};$$

$$h(y, \sigma; \kappa, T) \equiv \sigma + a\frac{\ln(y/\kappa)}{\sqrt{T}} + b\left(\frac{\ln(y/\kappa)}{\sqrt{T}}\right)^2$$

empirical smirk and smile

BL and beyond - EP case study: option trading

call option price at horizon $P_{t+\tau} = C_{BS}\left(y_t e^{X_y}, h\left(y_t e^{X_y}, \sigma_t + X_\sigma, \kappa, T-\tau\right); \kappa, T-\tau, r\right)$

$$X_y \equiv \ln\left(y_{t+\tau}/y_t\right)$$

$$X_\sigma \equiv \sigma_{t+\tau} - \sigma_t$$

$$C_{BS}\left(y, \sigma; \kappa, T, r\right) \equiv yF\left(d_1\right) - \kappa e^{-rT} F\left(d_2\right)$$

$$d_1 \equiv \left(\ln\left(y/\kappa\right) + \left(r + \sigma^2/2\right)T\right)/\sigma\sqrt{T}, \qquad d_2 \equiv d_1 - \sigma\sqrt{T};$$

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BL and beyond - EP case study: option trading

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Portfolio: Microsoft 1 month
Microsoft 2 months
Microsoft 6 months
Yahoo 1 month
Yahoo 2 months
Yahoo 6 months
Google 1 month
Google 2 months
Google 6 months

$$\mathbf{X} \equiv \left(X^M, X^M_{1m}, X^M_{2m}, X^M_{6m}, \ldots, X^G_{6m}, \underbrace{X_{2y}, X_{10y}}\right)'$$

curve change (growth/inflation)
not directly in pricing

BL and beyond - EP case study: option trading

call option price at horizon $P_{t+\tau} = C_{BS}\left(y_t e^{X_y}, h\left(y_t e^{X_y}, \sigma_t + X_\sigma, \kappa, T-\tau\right); \kappa, T-\tau, r\right)$

$$X_y \equiv \ln\left(y_{t+\tau}/y_t\right)$$

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$$C_{BS}\left(y, \sigma; \kappa, T, r\right) \equiv yF\left(d_1\right) - \kappa e^{-rT} F\left(d_2\right)$$

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$$\mathbf{X} \equiv \left(X^M, X^M_{1m}, X^M_{2m}, X^M_{6m}, \ldots, X^G_{6m}, X_{2y}, X_{10y}\right)' \cancel{\sim \mathrm{N}}\left(\boldsymbol{\pi}, \boldsymbol{\Sigma}\right)$$

BL and beyond - EP case study: option trading

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$$\mathbf{X} \equiv \left(X^M, X^M_{1m}, X^M_{2m}, X^M_{6m}, \ldots, X^G_{6m}, X_{2y}, X_{10y}\right)' \not\sim \mathrm{N}\left(\boldsymbol{\pi}, \boldsymbol{\Sigma}\right)$$

$$\Pi_{\mathbf{w}} \equiv \sum_{i=1}^{I} w_i \left(C_{BS,i}\left(\mathbf{X}, \mathcal{I}_t\right) - C_{i,t}\right)$$

? **profit and loss is highly <u>non-linear</u>, highly <u>non-normal</u>**

BL and beyond - EP case study: option trading

call option price at horizon $P_{t+\tau} = C_{BS}\left(y_t e^{X_y}, h\left(y_t e^{X_y}, \sigma_t + X_\sigma, \kappa, T-\tau\right); \kappa, T-\tau, r\right)$

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$$\Pi_{\mathbf{w}} \equiv \sum_{i=1}^{I} w_i\left(C_{BS,i}\left(\mathbf{X}, \mathcal{I}_t\right) - C_{i,t}\right)$$ **?**

Mean-CVaR optimization **?**

$$\mathbf{w}_\lambda \equiv \underset{\underline{\mathbf{b}} \leq \mathbf{B}\mathbf{w} \leq \overline{\mathbf{b}}}{\operatorname{argmax}} \left\{\mathbb{E}\left\{\Pi_{\mathbf{w}}\right\} - \lambda\,\mathrm{CVaR}_\gamma\left\{\Pi_{\mathbf{w}}\right\}\right\}$$

- long-short delta-neutral
- no cash upfront
- limit on leverage

BL and beyond - EP case study: option trading

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$.

Pricing $P_{t+\tau} \equiv P(\mathbf{X}, \mathcal{I}_t)$ **?**

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; f_{\mathbf{X}})\}$ **?**

BL and beyond - EP case study: option trading

call option price at horizon $P_{t+\tau} = C_{BS}\left(y_t e^{X_y}, h\left(y_t e^{X_y}, \sigma_t + X_\sigma, \kappa, T-\tau\right); \kappa, T-\tau, r\right)$

$$X_y \equiv \ln\left(y_{t+\tau}/y_t\right)$$

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$$\mathbf{X} \equiv \left(X^M, X^M_{1m}, X^M_{2m}, X^M_{6m}, \ldots, X^G_{6m}, X_{2y}, X_{10y}\right)'$$

$$\Pi_{\mathbf{w}} \equiv \sum_{i=1}^{I} w_i\left(C_{BS,i}\left(\mathbf{X}, \mathcal{I}_t\right) - C_{i,t}\right) \quad \Leftrightarrow \quad \mathcal{P}_{j,i} \equiv C_{BS,i}\left(\underset{\downarrow}{\overset{\text{simulations}}{\mathcal{X}_{j,\cdot}}}, \mathcal{I}_t\right) - C_{i,t},$$

$$\mathbf{w}_\lambda \equiv \underset{\underline{\mathbf{b}} \leq \mathbf{B}\mathbf{w} \leq \overline{\mathbf{b}}}{\operatorname{argmax}} \left\{\mathbb{E}\left\{\Pi_{\mathbf{w}}\right\} - \lambda\,\mathrm{CVaR}_\gamma\left\{\Pi_{\mathbf{w}}\right\}\right\}$$

BL and beyond - EP case study: option trading

call option price at horizon $P_{t+\tau} = C_{BS}\left(y_t e^{X_y}, h\left(y_t e^{X_y}, \sigma_t + X_\sigma, \kappa, T-\tau\right); \kappa, T-\tau, r\right)$

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simulations

$$\Pi_{\mathbf{w}} \equiv \sum_{i=1}^{I} w_i\left(C_{BS,i}\left(\mathbf{X}, \mathcal{I}_t\right) - C_{i,t}\right) \quad \Leftrightarrow \quad \mathcal{P}_{j,i} \equiv C_{BS,i}\left(\mathcal{X}_{j,\cdot}, \mathcal{I}_t\right) - C_{i,t},$$

$$\mathbf{w}_\lambda \equiv \underset{\underline{\mathbf{b}} \leq \mathbf{B}\mathbf{w} \leq \overline{\mathbf{b}}}{\operatorname{argmax}}\left\{\mathbb{E}\left\{\Pi_{\mathbf{w}}\right\} - \lambda\,\mathrm{CVaR}_\gamma\left\{\Pi_{\mathbf{w}}\right\}\right\} \quad \Leftrightarrow \quad \textbf{linear programming}$$

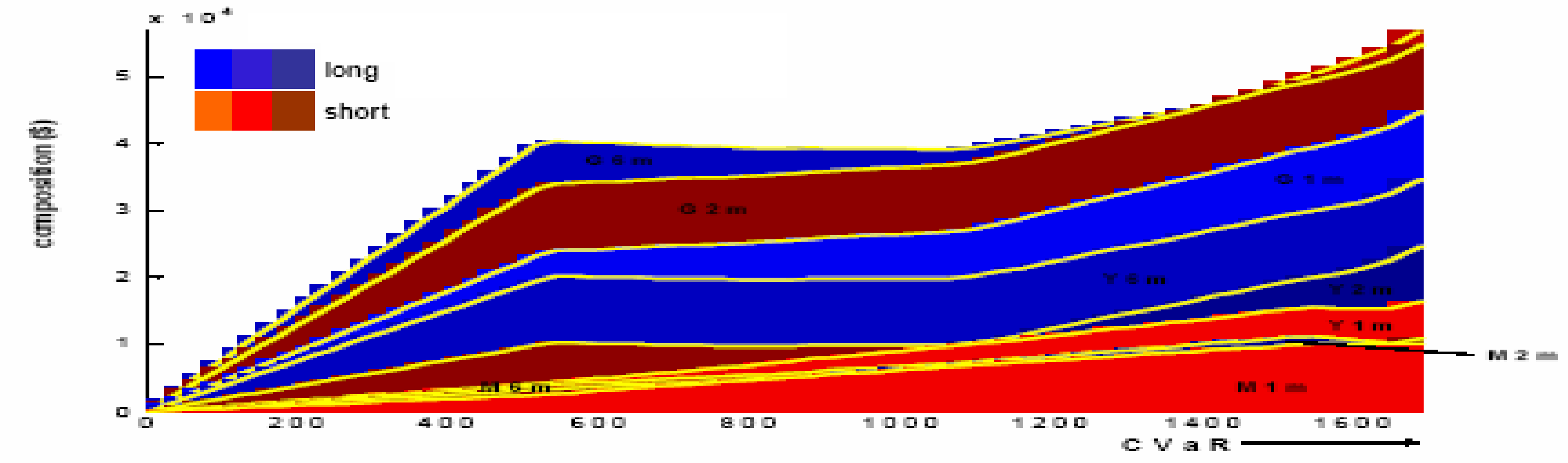
BL and beyond - EP case study: option trading

Market distr. $\mathbf{X} \sim f_{\mathbf{X}}$. $\Leftrightarrow$ $\mathcal{X}$ $\mathbf{p}$

Pricing $P_{t+\tau} \equiv P(\mathbf{X}, \mathcal{I}_t)$ $\Leftrightarrow$ $\mathcal{P}$, $\mathbf{p}$

Optimization $\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \{\mathcal{S}(\mathbf{w}; f_{\mathbf{X}})\}$ $\Leftrightarrow$ **linear programming**

BL and beyond - EP case study: option trading



BL and beyond - EP case study: option trading

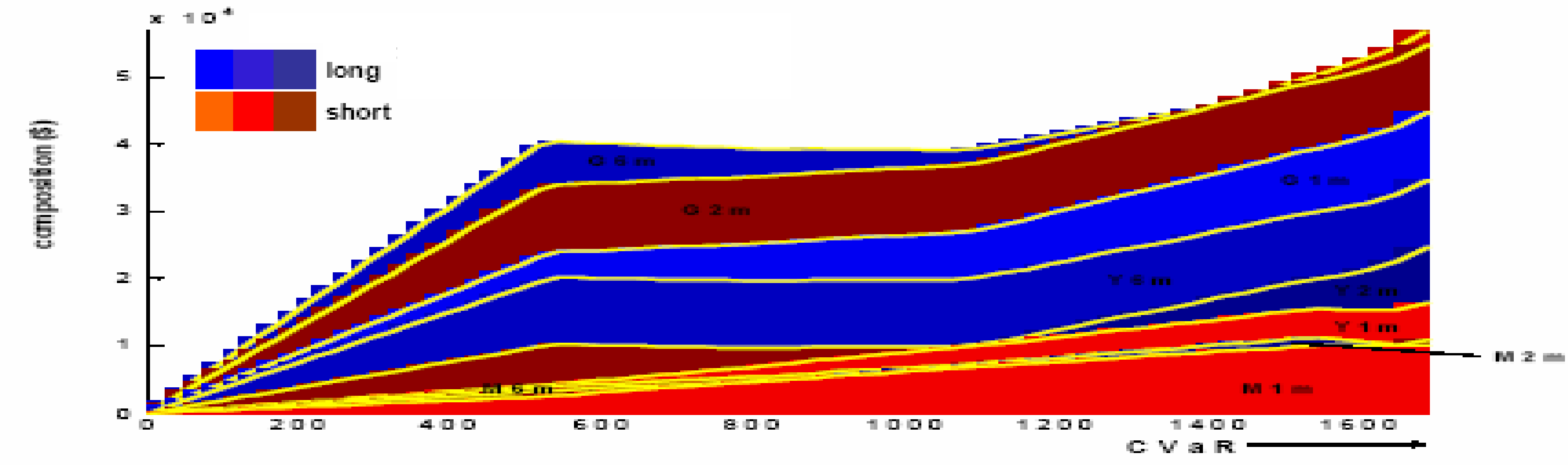
Market distr. $\mathbf{X} \sim f_{\mathbf{X}}.$ $\Leftrightarrow$ $\mathcal{X}$ $J \times N$ panel $\quad \mathbf{p}$ probabilities $1/J.$

Focus $\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$ $\Leftrightarrow$ $\mathcal{V}_{j,k} \equiv g_k\left(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N}\right)$

Views

$$\left.\begin{array}{l}
\mathbf{V} \sim \widetilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}. \\
\widetilde{m}\left\{V_k\right\} \gtreqless \widetilde{\mu}_{\mathbf{v},k}; \\
\widetilde{m}\left\{V_1\right\} \geq \widetilde{m}\left\{V_2\right\} \geq \cdots \geq \widetilde{m}\left\{V_K\right\} \\
\widetilde{\sigma}\left\{V_k\right\} \gtreqless \varkappa\sigma\left\{V_k\right\} \\
\widetilde{\mathbb{C}}\left\{\mathbf{V}\right\} \equiv \rho_1 \mathbf{I} + \rho_2 \mathbb{C}\left\{\mathbf{V}\right\} + \rho_3 \mathbf{1}\mathbf{1}', \\
\widetilde{Q}_V(u) \gtreqless Q_V(u)
\end{array}\right\} \Leftrightarrow \quad \underline{\mathbf{a}} \leq \mathbf{A}\widetilde{\mathbf{p}} \leq \overline{\mathbf{a}}.$$

BL and beyond - EP case study: option trading

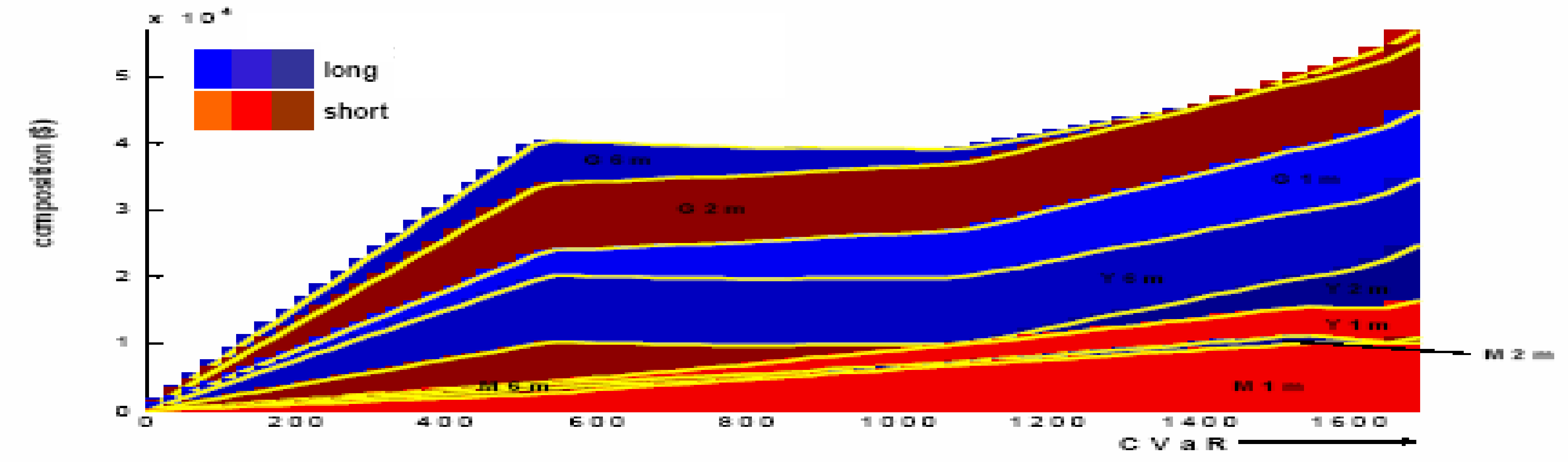


view: G 6m $\leq$ G 2m

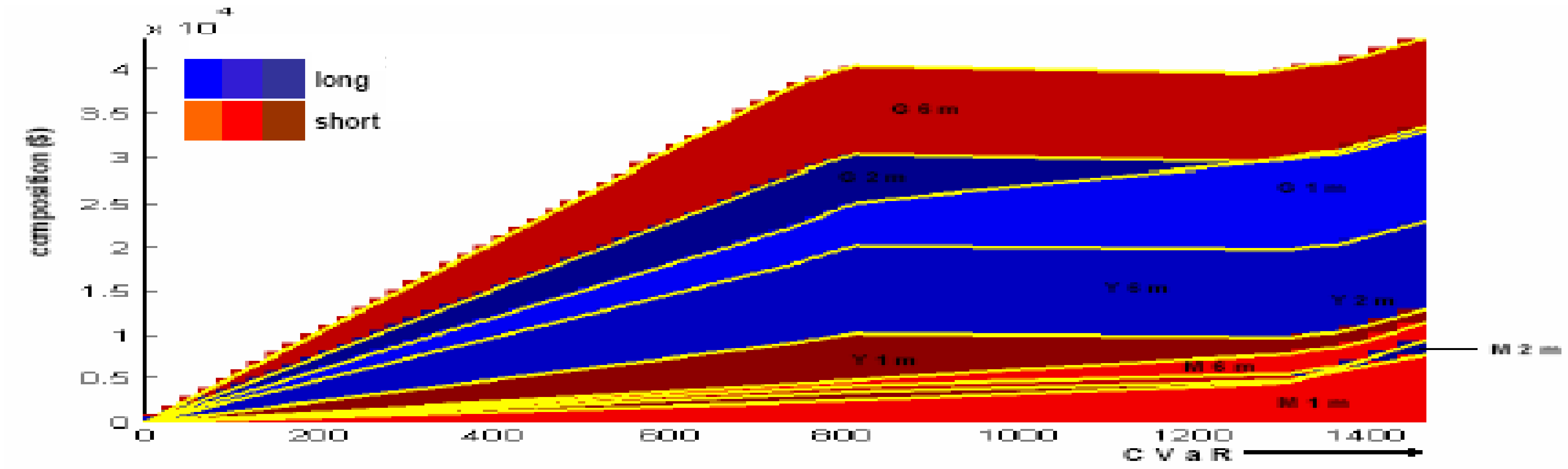
BL and beyond - EP case study: option trading

Market distr.	$\mathbf{X} \sim f_{\mathbf{X}}$.	$\Leftrightarrow$	$\mathcal{X}$ $J \times N$ panel $\quad \mathbf{p}$ probabilities $1/J$.
Focus	$\mathbf{V} \equiv \mathbf{g}(\mathbf{X}) \sim f_{\mathbf{V}}$	$\Leftrightarrow$	$\mathcal{V}_{j,k} \equiv g_k(\mathcal{X}_{j,1}, \ldots, \mathcal{X}_{j,N})$
Views	$\mathbf{V} \sim \tilde{f}_{\mathbf{V}} \neq f_{\mathbf{V}}$. $\tilde{m}\{V_k\} \gtreqless \tilde{\mu}_{\mathbf{v},k}$; $\tilde{m}\{V_1\} \geq \tilde{m}\{V_2\} \geq \cdots \geq \tilde{m}\{V_K\}$ $\tilde{\sigma}\{V_k\} \gtreqless \varkappa\sigma\{V_k\}$ $\tilde{\mathbb{C}}\{\mathbf{V}\} \equiv \rho_1\mathbf{I} + \rho_2\mathbb{C}\{\mathbf{V}\} + \rho_3\mathbf{1}\mathbf{1}'$, $\tilde{Q}_V(u) \gtreqless Q_V(u)$	$\Leftrightarrow$	$\underline{\mathbf{a}} \leq \mathbf{A}\tilde{\mathbf{p}} \leq \overline{\mathbf{a}}$
Posterior	$\tilde{f}_{\mathbf{X}} \equiv \underset{f \in \mathbb{V}}{\operatorname{argmin}}\{\mathcal{E}(f, f_{\mathbf{X}})\}$	$\Leftrightarrow$	$\mathcal{X}$ $\quad \tilde{\mathbf{p}} \equiv \underset{\underline{\mathbf{a}} \leq \mathbf{A}\mathbf{f} \leq \overline{\mathbf{a}}}{\operatorname{argmin}}\{\mathcal{E}(\mathbf{f}, \mathbf{p})\}$
Confidence	$\tilde{f}^c_{\mathbf{X}} \equiv (1-c) f_{\mathbf{X}} + c\tilde{f}_{\mathbf{X}}$.	$\Leftrightarrow$	$\mathcal{X}$ $\quad \mathbf{p}_c \equiv (1-c)\mathbf{p} + c\tilde{\mathbf{p}}$.
~~**Pricing**~~	$P_{t+\tau} \equiv P(\tilde{\mathbf{X}}, \mathcal{I}_t)$	$\Leftrightarrow$	$\boxed{\mathcal{P}}$ $\quad \mathbf{p}_c$
Optimization	$\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}}\{\mathcal{S}(\mathbf{w}; \tilde{f}^c_{\mathbf{X}})\}$	$\Leftrightarrow$	...

BL and beyond - EP case study: option trading



view: G 6m ≤ G 2m



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CASE STUDIES

REFERENCES AND CONCLUSIONS

BL and beyond - references

Black, F., and R. Litterman, 1990, Asset allocation: combining investor views with market equilibrium, *Goldman Sachs Fixed Income Research*.

	BL
normal market & linear views	✓
scenario analysis	·
correlation stress-test	·
trading desk: non-linear pricing	·
external factors: macro, etc.	·
partial specifications	·
non-normal market	·
multiple users	·
non-linear views	·
trading desk: costly pricing	·
lax constraints: ranking	·

Almgren, R., and N. Chriss, 2006, Optimal portfolios from ordering information, *Journal of Risk* 9, 1–47.

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	BL	AC
normal market & linear views	✓	·
scenario analysis	·	·
correlation stress-test	·	·
trading desk: non-linear pricing	·	·
external factors: macro, etc.	·	·
partial specifications	·	·
non-normal market	·	·
multiple users	·	·
non-linear views	·	·
trading desk: costly pricing	·	·
lax constraints: ranking	·	✓

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Qian, E., and S. Gorman, 2001, Conditional distribution in portfolio theory, *Financial Analyst Journal* 57, 44–51.

	BL	AC	QG
normal market & linear views	✓	·	✓
scenario analysis	·	·	✓
correlation stress-test	·	·	✓
trading desk: non-linear pricing	·	·	·
external factors: macro, etc.	·	·	·
partial specifications	·	·	·
non-normal market	·	·	·
multiple users	·	·	·
non-linear views	·	·	·
trading desk: costly pricing	·	·	·
lax constraints: ranking	·	✓	·

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	BL	AC	QG	P
normal market & linear views	✓	·	✓	✓
scenario analysis	·	·	✓	✓
correlation stress-test	·	·	✓	✓
trading desk: non-linear pricing	·	·	·	·
external factors: macro, etc.	·	·	·	·
partial specifications	·	·	·	✓
non-normal market	·	·	·	·
multiple users	·	·	·	·
non-linear views	·	·	·	·
trading desk: costly pricing	·	·	·	·
lax constraints: ranking	·	✓	·	·

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	BL	AC	QG	P	M
normal market & linear views	✓	·	✓	✓	✓
scenario analysis	·	·	✓	✓	✓
correlation stress-test	·	·	✓	✓	✓
trading desk: non-linear pricing	·	·	·	·	✓
external factors: macro, etc.	·	·	·	·	✓
partial specifications	·	·	·	✓	·
non-normal market	·	·	·	·	·
multiple users	·	·	·	·	·
non-linear views	·	·	·	·	·
trading desk: costly pricing	·	·	·	·	·
lax constraints: ranking	·	✓	·	·	·

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	BL	AC	QG	P	M	COP
normal market & linear views	✓	·	✓	✓	✓	✓
scenario analysis	·	·	✓	✓	✓	✓
correlation stress-test	·	·	✓	✓	✓	·
trading desk: non-linear pricing	·	·	·	·	✓	✓
external factors: macro, etc.	·	·	·	·	✓	✓
partial specifications	·	·	·	✓	·	·
non-normal market	·	·	·	·	·	✓
multiple users	·	·	·	·	·	✓
non-linear views	·	·	·	·	·	·
trading desk: costly pricing	·	·	·	·	·	·
lax constraints: ranking	·	✓	·	·	·	·

BL and beyond - references

- **Article:**

  Attilio Meucci, "Fully Flexible Views: Theory and Practice"
  ***The Risk Magazine* - October 2008, p 97-102**
  extended version available at
  www.symmys.com > Research > Working Papers

- **MATLAB examples:**

  www.symmys.com > Teaching > MATLAB

- **This presentation:**

  www.symmys.com > Teaching > Talks

	BL	AC	QG	P	M	COP	EP
normal market & linear views	✓	·	✓	✓	✓	✓	✓
scenario analysis	·	·	✓	✓	✓	✓	✓
correlation stress-test	·	·	✓	✓	✓	·	✓
trading desk: non-linear pricing	·	·	·	·	✓	✓	✓
external factors: macro, etc.	·	·	·	·	✓	✓	✓
partial specifications	·	·	·	✓	·	·	✓
non-normal market	·	·	·	·	·	✓	✓
multiple users	·	·	·	·	·	✓	✓
non-linear views	·	·	·	·	·	·	✓
trading desk: costly pricing	·	·	·	·	·	·	✓
lax constraints: ranking	·	✓	·	·	·	·	✓

Black-Litterman:

Pathbreaking approach to handle estimation risk and input views on the market

Beyond Black-Litterman:

- ✓ **Market represented by generic non-linear risk factors, not only returns**
- ✓ **Market distribution fully general, not only normal**
- ✓ **Market reference model fully general, not only based on equilibrium assumptions**
- ✓ **Views/stress-testing on any function of the market, not only linear portfolios**
- ✓ **Views on any feature, not only on expectations: median, volatility, correlations, tails**
- ✓ **Views are equalities and inequalities: ranking is possible**
- ✓ **Optimization is fully general, not only mean variance: mean-CVaR, mean-VaR, …**
- ✓ **Repricing is not necessary: complex derivatives handled**