On the Ω-Ratio

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ABSTRACT

Despite the fact that the Ω-ratio captures the complete shape of the underlying return distribution, selecting a portfolio by maximizing the value of the Ω-ratio at a given return threshold does not necessarily produce a portfolio that can be considered optimal over a reasonable range of investor preferences. Specifically, this selection criterion tends to select a feasible portfolio that maximizes available leverage (whether explicitly applied by the investor or realized internally by the capital structure of the investment itself) and, therefore, does not trade off risk and reward in a manner that most investors would find acceptable.

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1. The Ω-Ratio

In Keating and Shadwick (2002a) and (2002b) a new performance measure, the Omega ratio (Ω-ratio), is introduced "which captures the effects of all higher moments fully and which may be used to rank and evaluate manager performance" (Keating and Shadwick, 2002a, p. 4). Keating's and Shadwick's basic criticisms of many current performance measures are difficult to argue with, and in particular, "that mean and variance cannot capture all of the risk and reward features in a financial return distribution" (Keating and Shadwick, 2002b, p. 2).

The standard application of the Ω -ratio is to select a return threshold and then rank investments according to the value of their Ω-ratios at that threshold, with higher ratios preferred to lower. They state, "No assumptions about risk preferences or utility are necessary" (Keating and Shadwick, 2002b, p. 4). This methodology is extended to portfolio optimization in a straightforward way: Select a return threshold and then construct the portfolio with the highest Ω–ratio at that threshold. Their claim is that the Ω-ratio is a universal performance measure because it represents a complete description of the return distribution.

Keating and Shadwick (2002b, p. 10) note that any invertible transformation applied to the return distribution will yield another valid distribution from which an Ω-ratio can be constructed. Such transformations can be used as alternatives to utility functions. It is difficult to see, however, how such a transformation ends up being different or superior to a direct application of utility theory. This idea is not developed further nor does there appear to be any further development of it in the available literature.

A number of papers have explored the application of the Ω-ratio to selecting from among or creating portfolios of investments whose return characteristics are not adequately captured within the tradition mean-variance framework. See, *e.g.*, Barreto (2006), Betrand (2005), Favre-Bulle and Pache (2003), Gilli *et al.* (2008) and Togher and Barsbay (2007).

These studies, however, focus on examples that best illustrate the Ω-ratio's strengths. True understanding of any measure demands that we also understand how it breaks down. The notion that the Ω-ratio "avoids the need for utility functions," as stated in Keating and Shadwick (2002a p.4), is far too strong a statement. As will be shown in what follows, the Ω-ratio has a serious bias that can lead to excessive risk taking.

1.1 Background

The Ω-ratio is the ratio of probability weighted gains and losses about a specified return threshold. Let the random variable X represent the return of a particular investment, μ_X its mean and F_X its CDF. For the sake of expositional simplicity in what follows, assume that F_X is continuous and strictly increasing over the range $-\infty > x > \infty$. Keating and Shadwick (2002a, p.8 and 2002b, p.3) define the Ω-ratio of X at threshold τ as shown in Equation 1.

$$\Omega_X[\tau] = \frac{\int_{\tau}^{\infty} \left(1 - F_X(x)\right) dx}{\int_{-\infty}^{\tau} F_X(x) dx}$$

Equation 1 - The Omega Ratio

And a graphical representation of the Ω-ratio follows below in Figure 1.

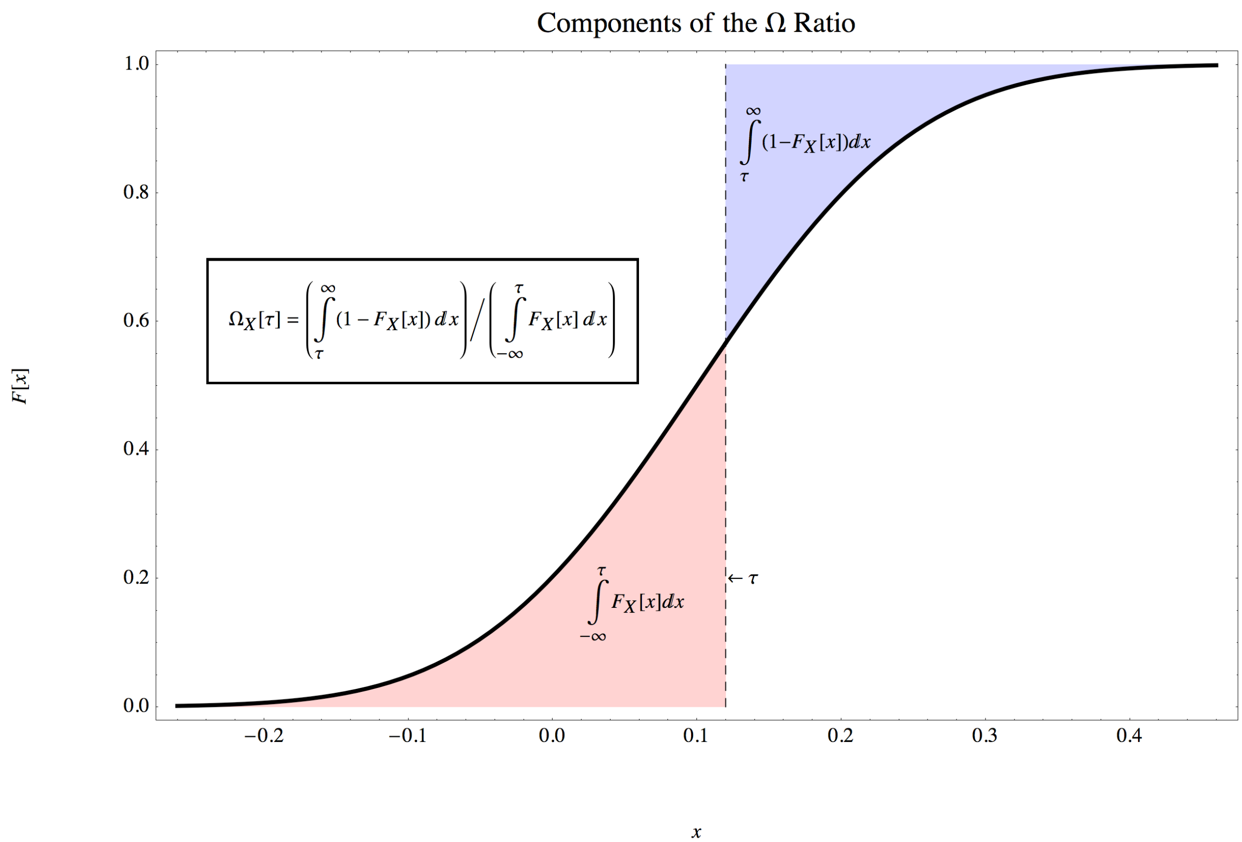

Figure 1 - Graphical Representation of the Omega Ratio

For all practical purposes, the Ω-ratio is an invertible transform of the CDF. Keating and Shadwick (2002a, pp. 10-12) derive some of its important properties. In particular, that

- $\Omega_X[\mu_X] = 1$;
- $\Omega_X[x]$ is a monotone decreasing function from $(-\infty, \infty)$ to $(\infty, 0)$;
- $\Omega_X[x]$ is an affine invariant of the returns distribution, *i.e*, if $Y = a + b\,X$, then $\Omega_Y[a + b\,\tau] = \Omega_X[\tau]$.

A plot of the Ω-ratio for a Standard Normal distribution appears below in Figure 2.

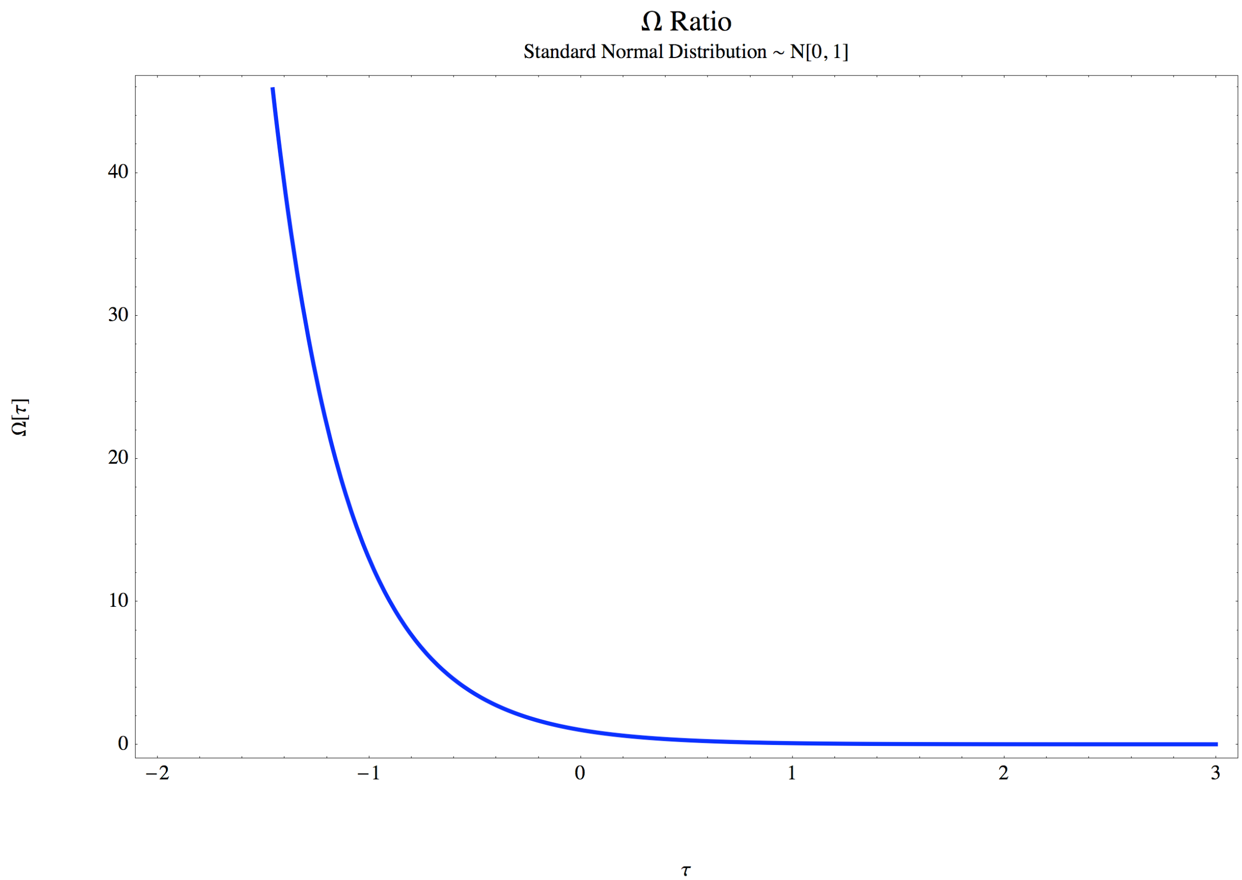

Figure 2 - Omega Ratio of the Standard Normal Distribution

Several researchers have investigated its properties. Kazemi, Schneeweis and Gupta (2003) show that the Ω-ratio can be written as the ratio of a call option to a put option with strikes at the threshold. Kaplan and Knowles (2004) develop a more generalized measure, the Kappa ratio (K-ratio), which subsumes the Ω and Sortino ratios as special cases.

1.2 The Ω Leverage Bias

Let the return of an asset be a random variable X with CDF F_X, mean μ_X and standard deviation $\infty > \sigma_X > 0$. Assume that there exists a risk-free asset with return r_f, that an investor can lend or borrow at that rate and that $0 < r_f, < \mu_X$.

Denote the leverage λ as the amount invested in the risky asset with $(1-\lambda)$ invested or borrowed at r_f and designate the random variable of the return of the leveraged position as Y with CDF F_Y. Note that the term leverage is used in a general way that encompasses placing some capital in cash, $\lambda < 1$, being fully invested, $\lambda = 1$, or borrowing to finance a position greater than one's capital, $\lambda > 1$.

The mean and standard deviation of Y are $\lambda\, \mu_X + (1-\lambda)\, r_f$ and $\lambda\, \sigma_X$, respectively. Of course, both have the same Sharpe ratio. The CDFs of X and Y are related; *viz.*, $F_Y[\lambda\, x + (1-\lambda)\, r_f] = F_X[x]$. Consequently, we can assert that $\Omega_Y[\lambda\, \tau + (1-\lambda)\, r_f] = \Omega_X[\tau]$, and, hence, the single crossing point is $\Omega_Y[r_f] = \Omega_X[r_f]$.

Consider the case for $\lambda > 1$ and $\tau > r_f$. Then $F_X[\tau] > F_Y[\tau]$. This in turn implies that the numerator of $\Omega_X[\tau]$ is less than the numerator of $\Omega_Y[\tau]$ and the denominator of $\Omega_X[\tau]$ is greater than the denominator $\Omega_Y[\tau]$. Thus, $\Omega_X[\tau] < \Omega_Y[\tau]$. Analogous results hold for alternate restrictions on λ and τ.

Thus, for a given investment X, an investor who uses the Ω-ratio with a threshold higher than the risk-free rate as a selection criterion *always* prefers more leverage on X to less, independent of the distribution of X. This is called the *Ω leverage bias (ΩLB)* in what follows.

1.3. A Simple Example of the ΩLB

Let the risk-free return $r_f = 0.03$ and consider an investment with a Normally distributed return with mean $\mu = 0.10$ and standard deviation $\sigma = 0.12$. Assume we leverage 3:2, *i.e.*, $\lambda = 1.5$. A plot of the respective Ω-ratios of the unleveraged and leveraged investments appears below in Figure 3. As expected, the crossover occurs at r_f after which the leveraged investment dominates the unleveraged one. Most investors would select thresholds greater than the risk-free rate; hence, maximizing the Ω-ratio would drive most investors to maximize the leverage of their positions.

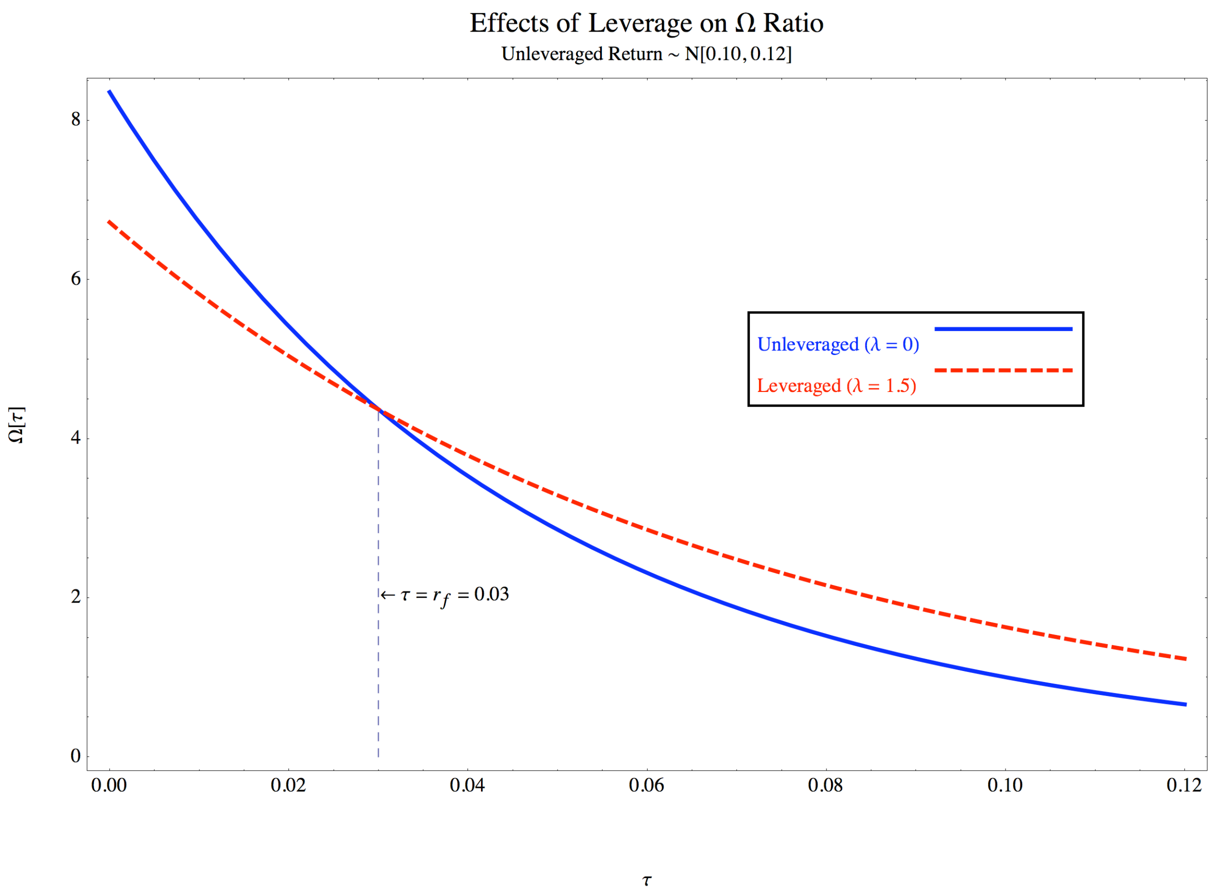

Figure 3 - Effects of Leverage on a Normally Distributed Investment

2. Implications of the ΩLB

One may well counter that the whole point of the Ω-ratio is to select "better" return distributions. In the example above the leveraged and unleveraged investments are both Normally distributed. In a portfolio optimization in which leverage is not a factor might not the Ω-ratio still select a portfolio whose return distribution is the best trade-off of up- and downside performance *vis-à-vis* the investor's return threshold?

The answer is no, or at least not necessarily. Given multiple investment strategies, those with otherwise less desirable distributions can improve their Ω-ratio by taking on more leverage. Unfortunately, this increase in leverage can also mean greater exposure to those aspects of the investment's return that made it less desirable.

Leverage may occur internally within a fund or company; therefore, an investor who is not explicitly allowing leverage at the portfolio level may nevertheless find it implicitly represented in the candidates available. Thus, the ΩLB can drive portfolio selection towards choices that have high levels of leverage, whether internal or external.

Finally, if we consider a factor model as a suitable representation of the returns across a set of investments, then higher factor exposures, which closely resemble the effects of leverage, also tend to result in higher Ω-ratios because of the ΩLB. These issues will be explored in the following sections.

2.1. Example: Normal *vs*. Leptokurtotic and Negatively Skewed

We will follow the approach followed in several introductory papers on the Ω-ratio and use it to compare different return distributions. Consider three potential investments: A, B and C. A plot of their returns' PDFs is below in Figure 4.

The returns of A are Normally distributed with mean $\mu_A = 0.1175$ and standard deviation $\sigma_A = 0.1047$. The returns of B are represented by a finite normal mixture with component weights $\mathbf{w}_B = (0.95, 0.05)^T$, component means $\boldsymbol{\mu}_B = (0.13, -0.12)^T$, and component standard deviations $\boldsymbol{\sigma}_B = (0.085, 0.15)^T$. The mean and standard deviation of B

are the same as that of A. Investment C is a leveraged version of B with $\lambda = 1.5$, achieved by borrowing at the risk-free rate $r_f = 0.03$. This situation can arise if, *e.g.*, B and C represent different share classes of the same underlying hedge fund, or if B and C are separate investments that follow similar strategies with C running at a higher risk-level.

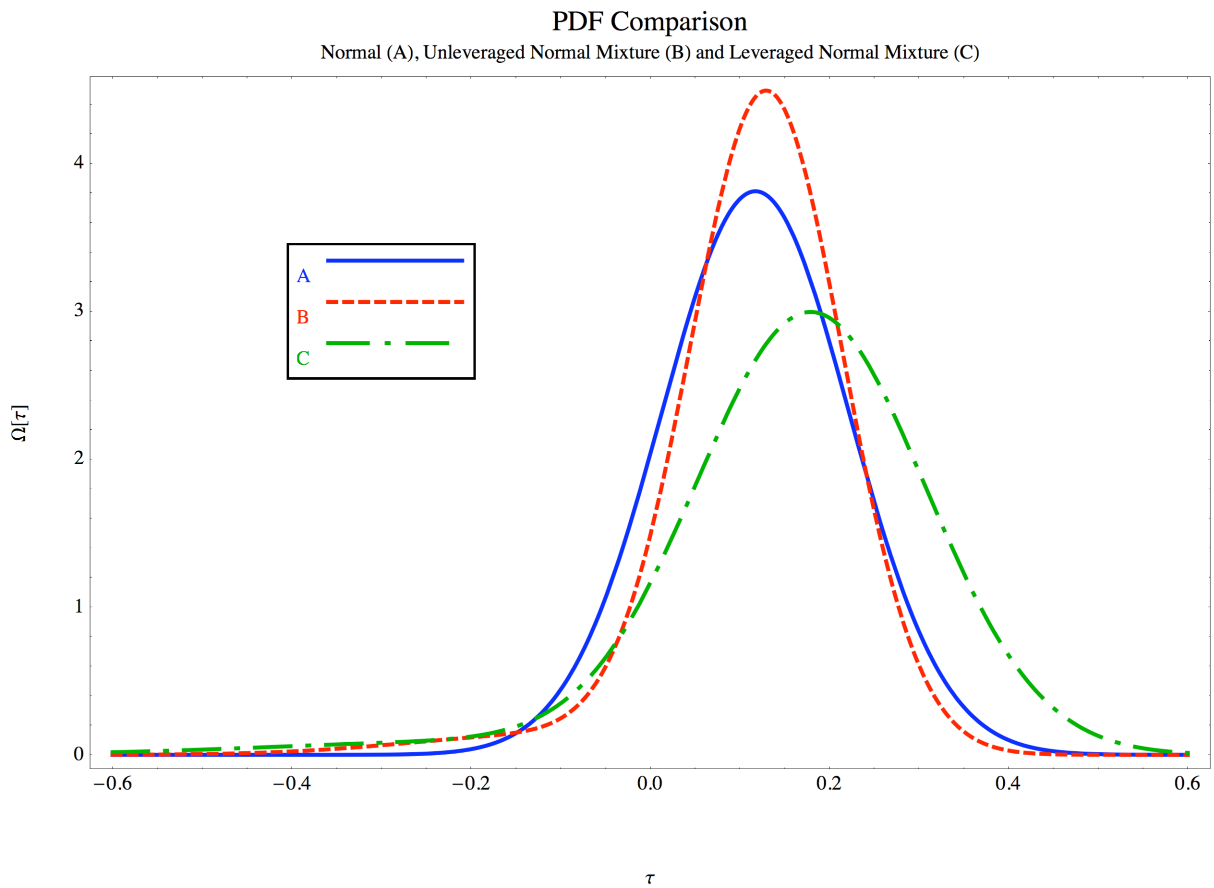

Figure 4 - PDFs of Investments *A*, *B* and *C*

The mean and standard deviation of B are the same as that of A. Investments B and C are negatively skewed and leptokurtotic. All three investments have the same Sharpe ratio. A plot of the lower tails of the CDFs below in Figure 5 clearly illustrates dramatic differences in downside risk. The probability of a loss that is greater than 40% is about 1% for C but it is negligible for A and B.

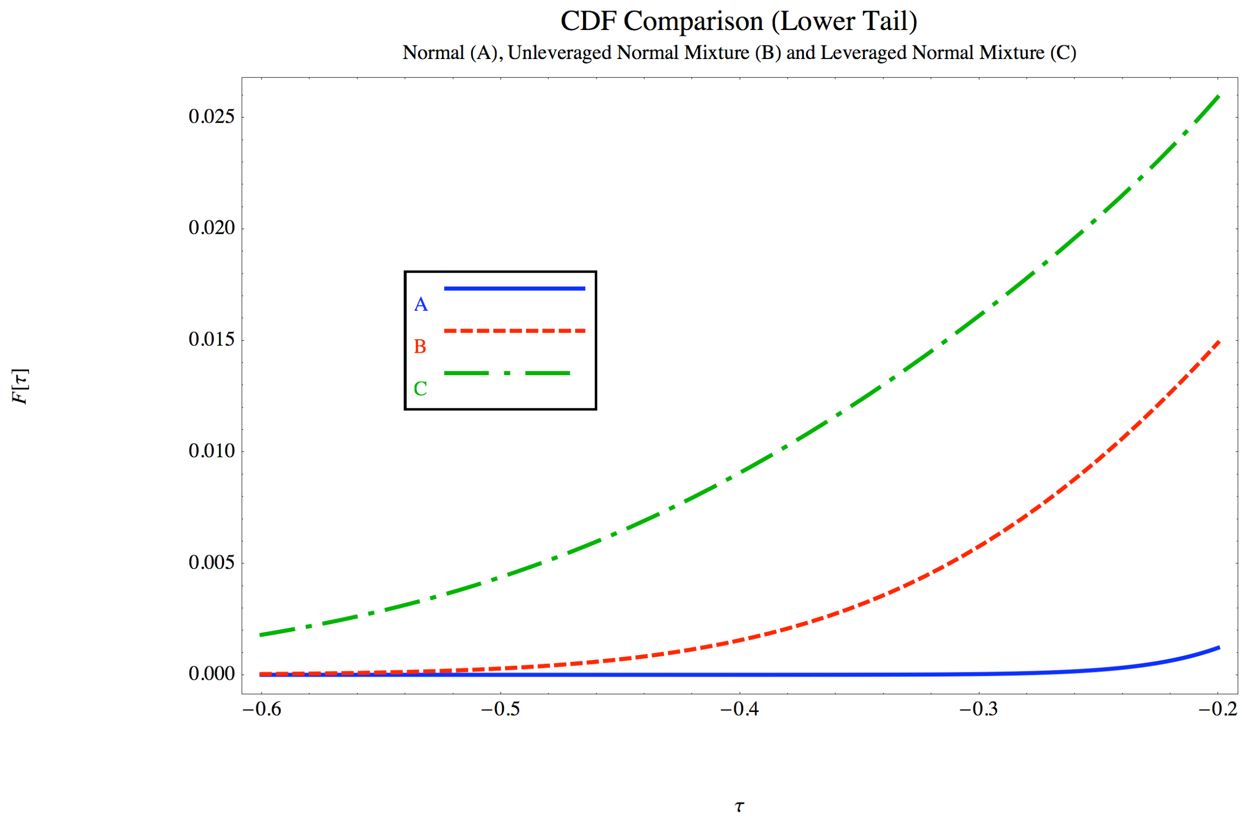

Figure 5 - CDFs of Loss Tails of Investments *A*, *B* and *C*

For thresholds greater than roughly 0.05, a comparison of the Ω-ratios of *A* and *B*, shown below in Figure 6, does not indicate a marked preference for *A* or *B*. For thresholds $\tau \geq 0.037$, *C* dominates both *A* and *B*. Now it is certainly possible that some investors with modest return thresholds may prefer to leverage up *B* to *C*, despite the fact that its lower tail is materially fatter than that of *A*'s; however, it is not tenable to assert that *all* would. In fact, many would be terrified of doing so.

Investments with less desirable distributions can improve their Ω-ratio by taking on more leverage. Thus, the ΩLB can drive portfolio selection towards choices that have undesirable distributions but high levels—whether external or internal—of leverage.

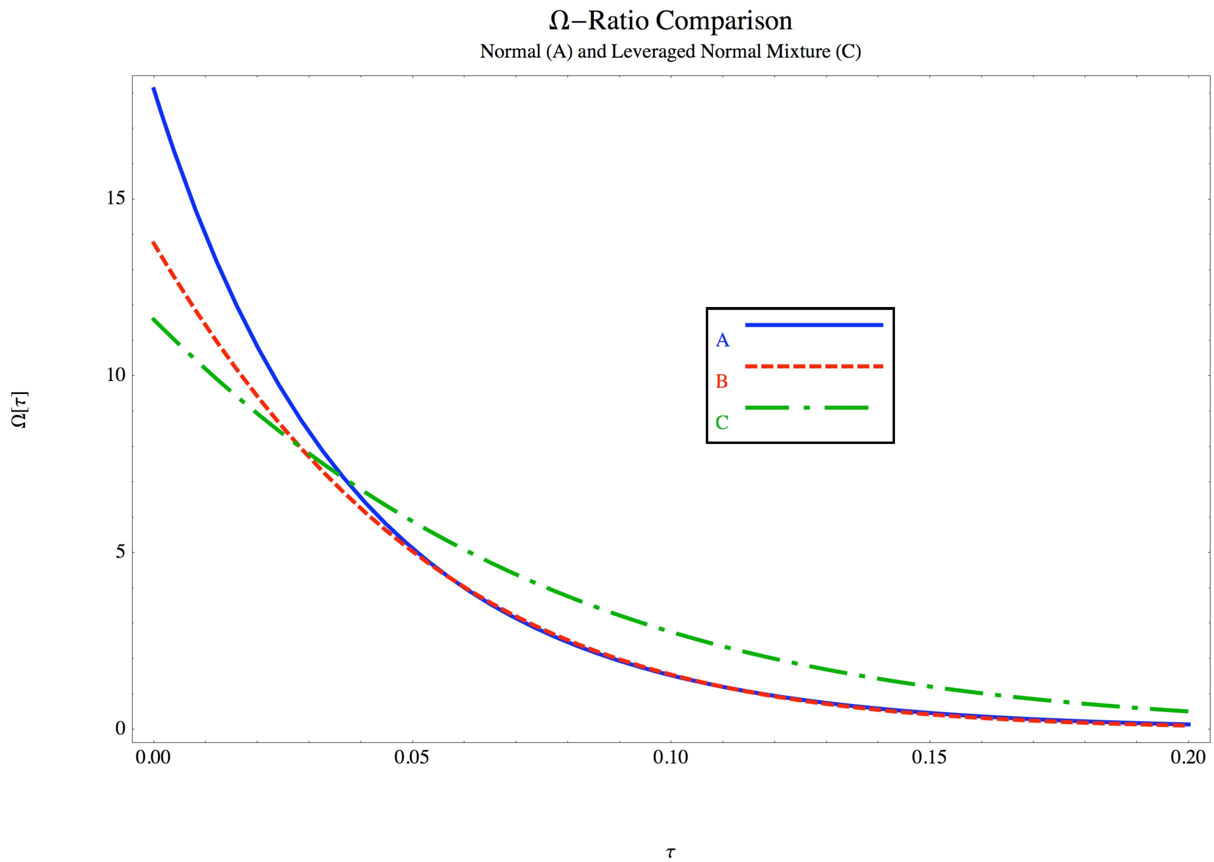

Figure 6 - Omega Ratios of Investments *A*, *B* and *C*

2.2 Example: Normal *vs.* Bimodal

A second, more extreme example explores this further. Consider three potential investments: *D*, *E* and *F*. The returns of *D* are Normally distributed with mean $\mu_D = 0.1$ and standard deviation $\sigma_D = 0.155$. The returns of *E* can be modeled as a finite normal mixture with component weights $\mathbf{w}_E = (0.5, 0.5)^T$, component means $\boldsymbol{\mu}_E = (0.25, -0.05)^T$, and component standard deviations $\boldsymbol{\sigma}_E = (0.04, 0.04)^T$. Investment *F* is a leveraged version of *E* with $\lambda = 1.5$, achieved by borrowing at the risk-free rate $r_f = 0.03$. All three investments have the same Sharpe ratio. A plot comparing the distributions of *D*, *E* and *F* is below.

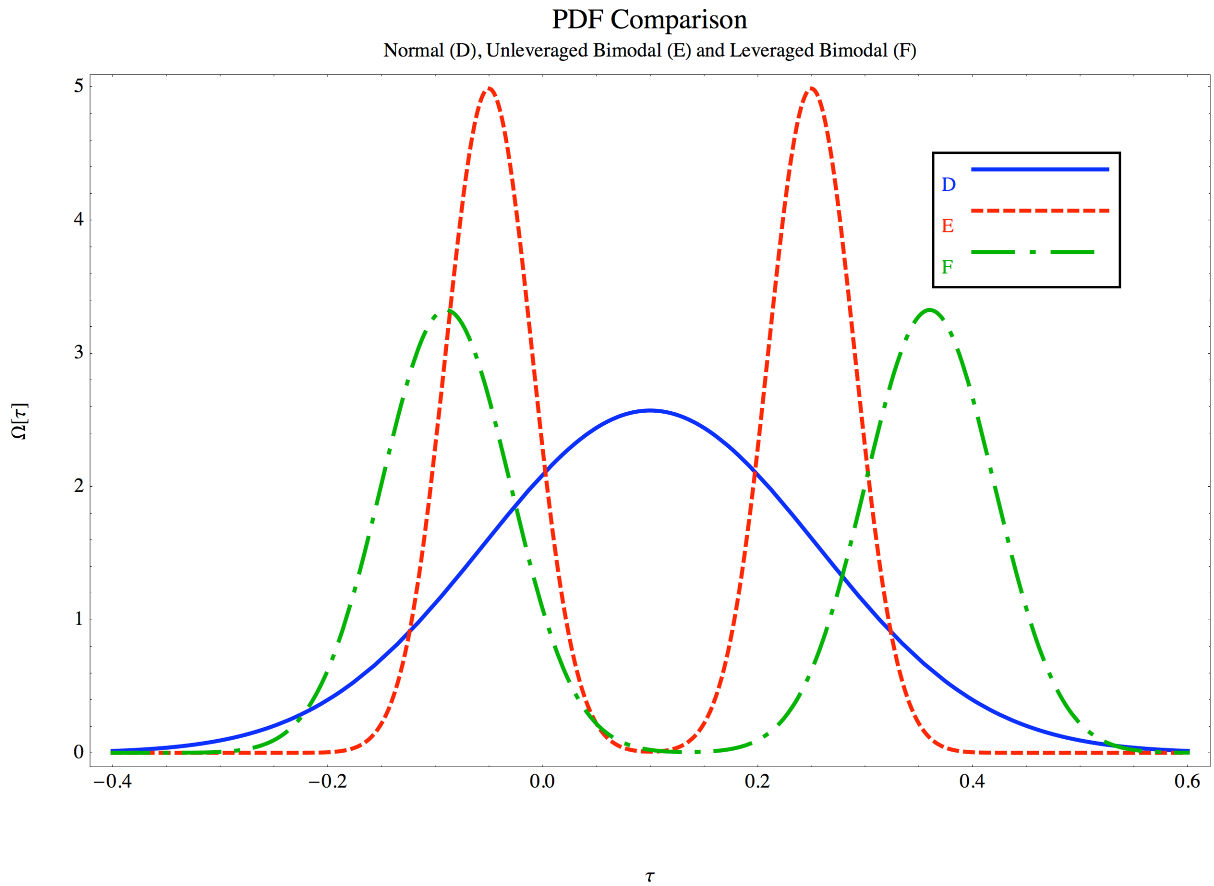

Figure 7 - PDFs of Investments *D*, *E* and *F*

The plot below compares the Ω-ratios of these three alternatives. It shows that *D* and *E* interweave with one another, although for thresholds $\tau < 0.1$ *D* is the clear preference. As expected, the leveraged *F* dominates its unleveraged counterpart *E* for thresholds $\tau > r_f$. However, it also dominates the Normally distributed *D* for thresholds $\tau >$ 0.054.

The effect of leveraging up *E* to *F* takes a distribution with modes at –0.05 and 0.25 and produces one with modes –0.09 and 0.36. Using the Ω-ratio as a selection criterion would lead investors with modest thresholds to select a highly leveraged investment with considerable downside risk. For many investors a preference for F over D and E may make sense, but, as is the case for the example in the previous section, it is not likely to make sense for all of them. The single parameter available to investors, the threshold, isn't much use in articulating different investors' preferences in this regard.

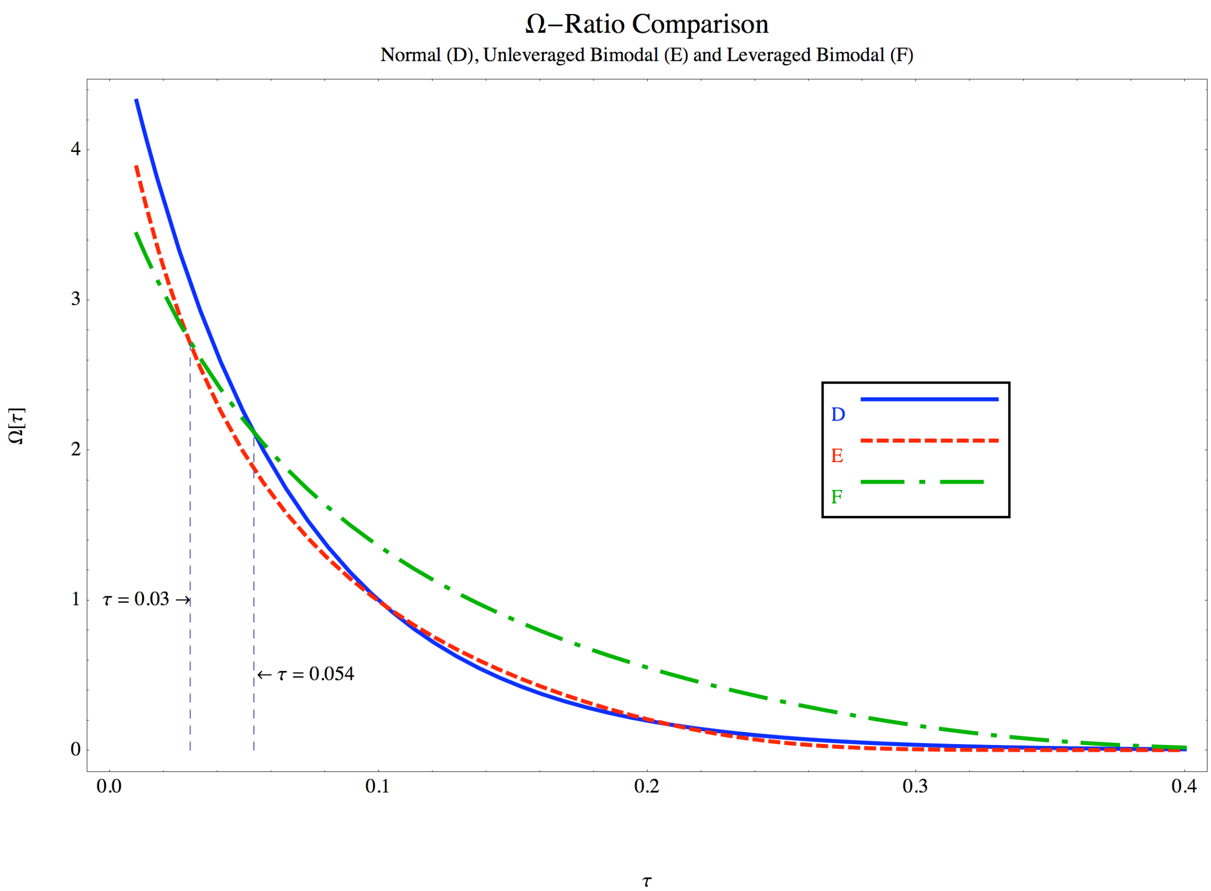

Figure 8 - Omega Ratios of Investments *D*, *E* and *F*

2.3 The ΩLB for Factor Returns and Other Replicating Strategies

Investment returns can typically be replicated by their exposures to a limited number of factors plus and uncorrelated idiosyncratic component. In cases in which traditional factor models have not proven adequate, researchers have found forms of so called alternative or exotic beta—such as hedge fund or commodity indices (Litterman

2005)—or dynamic trading strategies—which use traditional investments to replicate the properties of alternative investments (Kat and Palaro, 2005, 2006a and 2006b). Thus, even when non-linearities or multiple regimes are involved, as with alternative investments or direct investment in derivatives, the returns can often be modeled by a small number of basis functions, perhaps including indices or trading strategies, *qua* factor returns. Heightened factor exposures can have an effect on the Ω-ratio similar to that of increased leverage.

Consider, for example, a single factor models such as the Capital Asset Pricing Model (CAPM). Under the CAPM, investment returns can be modeled by their exposure to a single market factor. Let $x_i(t)$ be the return of investment i at time t, β_i the exposure of i to the systematic market return $m(t)$ and $\varepsilon_i(t)$ an idiosyncratic random error that is uncorrelated with $m(t)$ and other investments' errors (*i.e.*, $\mathrm{Cov}[m(t), \varepsilon_j(t)] = 0$ and $\mathrm{Cov}[\varepsilon_i(t), \varepsilon_j(t)] = 0$ for $j \neq i$). The CAPM asserts that $x_i(t) - r_f = \beta_i(m(t) - r_f) + \varepsilon_i(t)$. Rearranging terms yields $x_i(t) = \beta_i m(t) + (1 - \beta_i) r_f + \varepsilon_i(t)$. Thus, except for the idiosyncratic component, the relationship between x_i and m is the same as that of the return of m leveraged by a factor $\lambda = \beta_i$.

Once a moderate level of diversification has been achieved the impact of the idiosyncratic risks is diversified away. Under these conditions and for thresholds $\tau > r_f$, one would expect that the ΩLB would drive portfolio construction relentlessly towards a portfolio with high β exposure. In more complex multi-factor models the effect would be the same and would occur across the multiple factors present.

3. Conclusions

The work of Kaplan and Knolwes (2003) mentioned earlier may help to provide additional insight into those conditions in which the Ω-ratio may break down. They show that the Ω-ratio is related to K_1-ratio (the ratio of excess expected return to threshold divided by the first lower partial moment with regard to threshold) as shown in Equation 2.

$$\Omega_X[\tau] = \mathrm{K}_1[\tau] + 1 = \frac{\mu_X - \tau}{\int_{-\infty}^{\tau} (x - \tau) dF[x]} + 1$$

Equation 2 - Relationship Between the Omega and Kappa Ratios

At a given threshold τ, the Ω-ratio obscures much of the downside tail structure. It is difficult to see how picking τ establishes an investor's risk-reward preferences compared with, for example, defining a utility function. While is it true that by varying τ one can gain more insight, one has to wonder if a more natural and intuitive insight could not be gained by a direct comparison of the return distributions themselves.

The Ω-ratio is not a "universal performance measure" as claimed in Keating and Shadwick (2002a). To be sure, other measures, such as the Sharpe ratio, have their deficiencies. The Sharpe ratio, however, is neutral towards leverage; it does not actively seek it out.

Unfortunately, for modest return thresholds, the ΩLB can drive portfolios towards investments with more highly leveraged capital structures even when those choices involve significant downside risk. This may be what *some* investors *sometimes* want, but it is difficult to argue that it is what *all* investors *always* want.

A *Mathematica*® (Version 7) notebook containing all of the code, computations and graphics produced in this paper can be found at http://www.ams.sunysb.edu/~frey/Research/. Included are some useful tools for computing and working with Ω-ratios. The notebook also contains additional examples not used in this paper.

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