Attilio Meucci

Multivariate Ornstein-Uhlenbeck, Cointegration, and Statistical Arbitrage

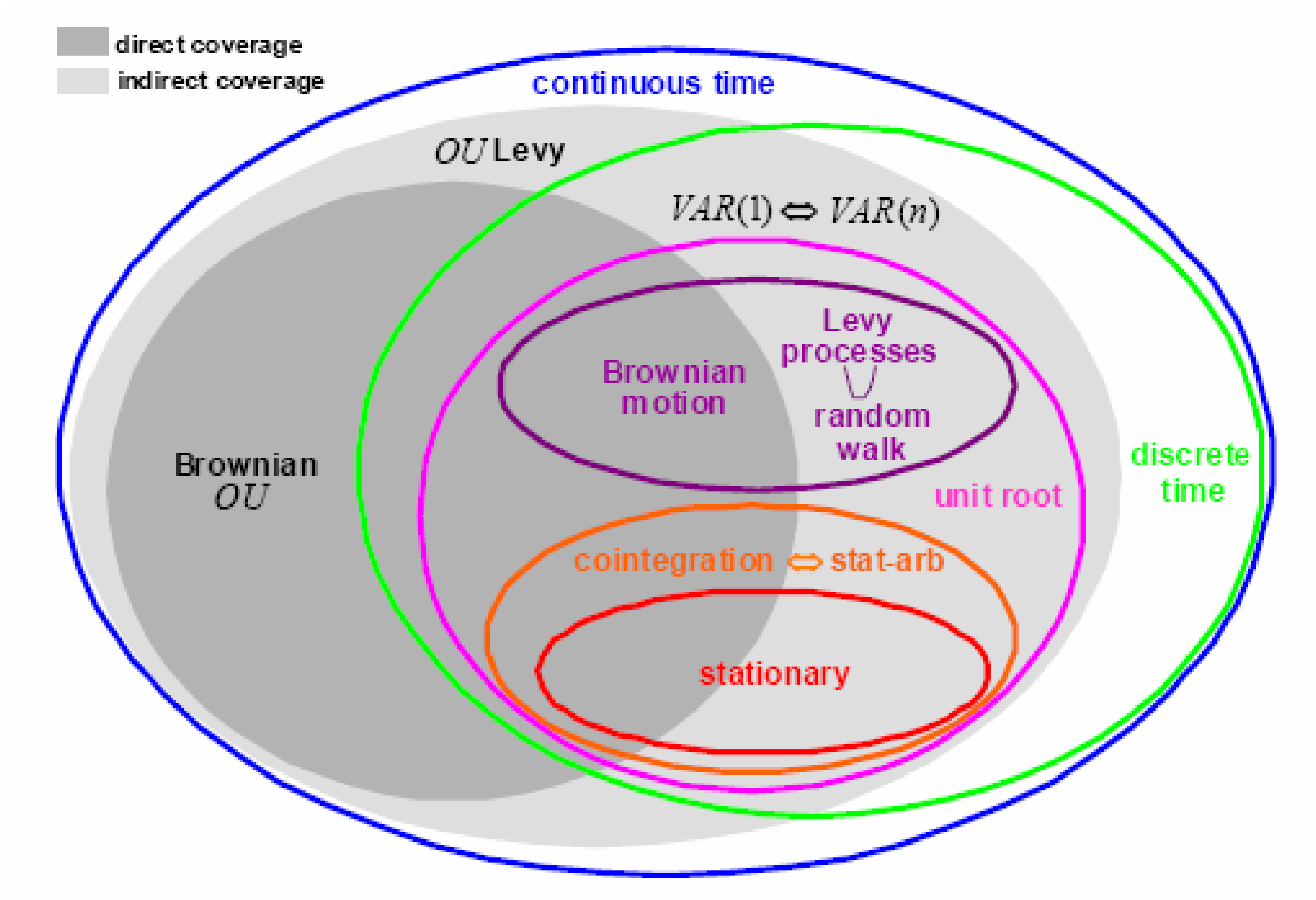
- **Slides from paper**

"Review of Statistical Arbitrage, Cointegration, and Multivariate Ornstein-Uhlenbeck"

available at www.symmys.com > Research > Working Papers

- **MATLAB code available at www.symmys.com > Teaching > MATLAB**

Multivariate Ornstein-Uhlenbeck



Multivariate Ornstein-Uhlenbeck

invariants (i.i.d.)

$$dX_t = -\theta \left(X_t - m \right) dt + \sigma \boxed{dB_t.}$$

autoregressive process

$$X_t \stackrel{d}{=} m + e^{-\theta\tau} \left(X_{t-\tau} - m \right) + \epsilon_{t,\tau},$$

$$\epsilon_{t,\tau} \sim \mathrm{N}\left(0, \frac{\sigma^2}{2\theta} \left(1 - e^{-2\theta\tau} \right) \right)$$

Multivariate Ornstein-Uhlenbeck

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Multivariate Ornstein-Uhlenbeck

$$d\mathbf{X}_t = -\Theta\left(\mathbf{X}_t - \mu\right)dt + \mathrm{S}d\mathbf{B}_t$$

autoregressive process

$$\mathbf{X}_{t+\tau} \sim \mathrm{N}\left(\mathbf{x}_{t+\tau}, \Sigma_\tau\right)$$

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autoregressive process

$$\mathbf{X}_{t+\tau} \sim \mathrm{N}\left(\mathbf{x}_{t+\tau}, \boldsymbol{\Sigma}_\tau\right)$$

$$\begin{cases} \mathbf{x}_{t+\tau} \equiv \left(\mathbf{I} - e^{-\boldsymbol{\Theta}\tau}\right)\boldsymbol{\mu} + e^{-\boldsymbol{\Theta}\tau}\mathbf{x}_t \\ \operatorname{vec}\left(\boldsymbol{\Sigma}_\tau\right) \equiv \left(\boldsymbol{\Theta} \oplus \boldsymbol{\Theta}\right)^{-1}\left(\mathbf{I} - e^{-(\boldsymbol{\Theta}\oplus\boldsymbol{\Theta})\tau}\right)\operatorname{vec}\left(\boldsymbol{\Sigma}\right) \end{cases}$$

$$\boldsymbol{\Sigma} \equiv \mathbf{S}\mathbf{S}'$$

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$\tau \to \infty$

$\tau \to 0$

$$\begin{cases} \mathrm{x}_\infty = \mu \\ \operatorname{vec}\left(\mathbf{\Sigma}_\infty\right) = \left(\Theta \oplus \Theta\right)^{-1}\operatorname{vec}\left(\mathbf{\Sigma}\right) \end{cases}$$

$$\begin{cases} \mathbf{X}_{t+\tau} \approx \mathbf{X}_t + \epsilon_{t,\tau} \\ \epsilon_{t,\tau} \sim \mathrm{N}\left(\tau\mathbf{\Theta}\mu, \tau\mathbf{\Sigma}\right) \end{cases}$$

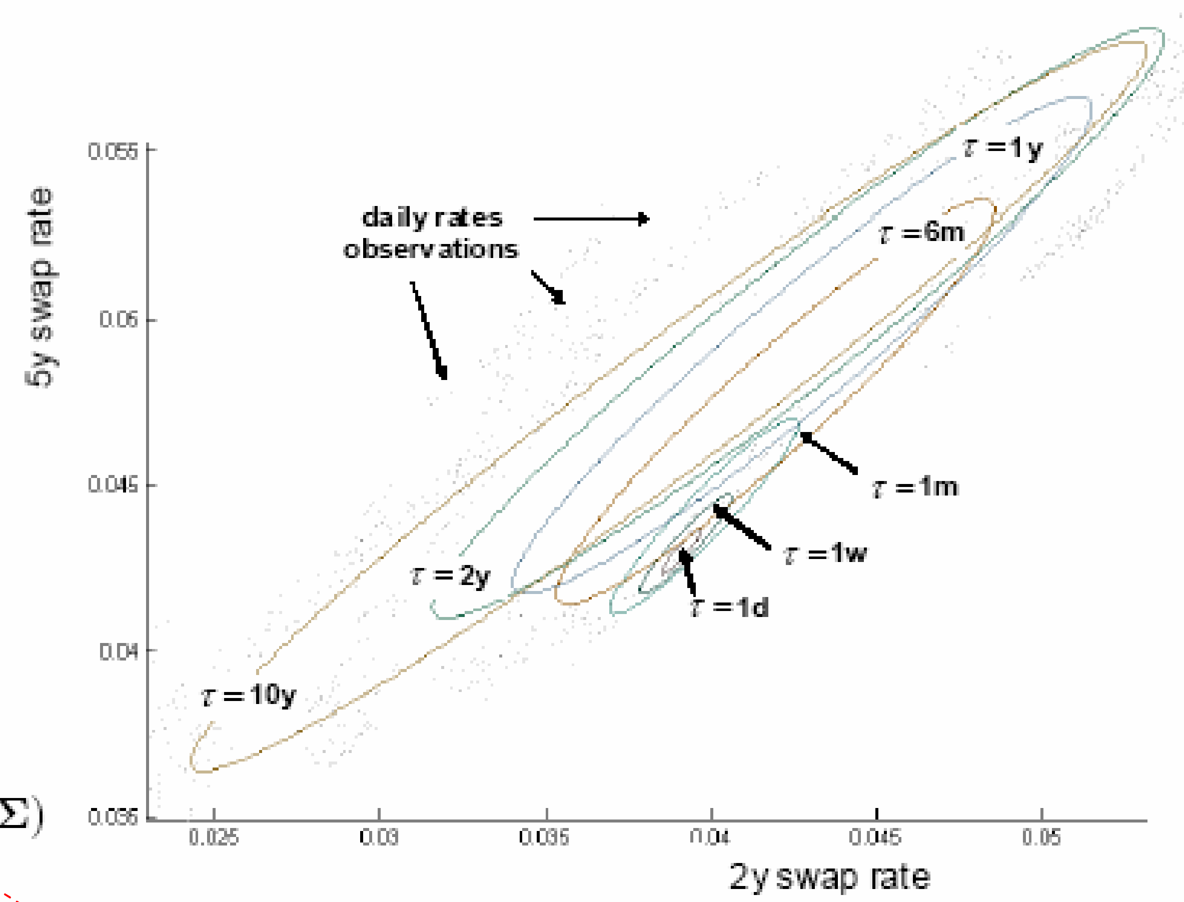
Multivariate Ornstein-Uhlenbeck

$$dX_t = -\Theta \left(X_t - \mu \right) dt + S dB_t$$

autoregressive process

$$X_{t+\tau} \sim N \left(x_{t+\tau}, \Sigma_\tau \right)$$

$$\begin{cases} x_{t+\tau} \equiv \left(I - e^{-\Theta \tau} \right) \mu + e^{-\Theta \tau} x_t \\ \text{vec}\left(\Sigma_\tau \right) \equiv \left(\Theta \oplus \Theta \right)^{-1} \left(I - e^{-(\Theta \oplus \Theta)\tau} \right) \text{vec}\left(\Sigma \right) \end{cases}$$



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$$dX_t = -\Theta \left(X_t - \mu\right) dt + S dB_t$$

Θ

B **eigenvectors (columns)**

$(\lambda_1, \ldots, \lambda_K)$ **real eigenvalues**

$(\gamma_1 \pm i\omega_1), \ldots, (\gamma_J \pm i\omega_J)$ **complex eigenvalues**

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$$\mathbf{A} \equiv \mathrm{Re}\left(\mathbf{B}\right) - \mathrm{Im}\left(\mathbf{B}\right)$$

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Multivariate Ornstein-Uhlenbeck

$$d\mathbf{X}_t = -\boldsymbol{\Theta}\left(\mathbf{X}_t - \boldsymbol{\mu}\right)dt + \mathbf{S}\,d\mathbf{B}_t$$

$\boldsymbol{\Theta}$

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$\boldsymbol{\Theta} \rightarrow \mathbf{B}$ **eigenvectors (columns)**

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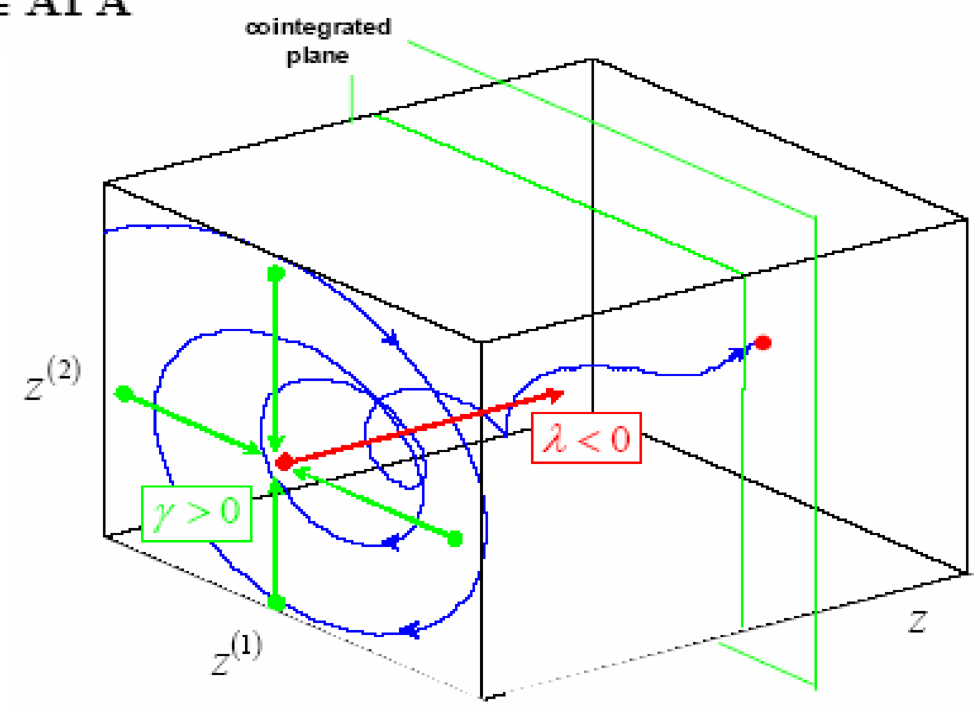

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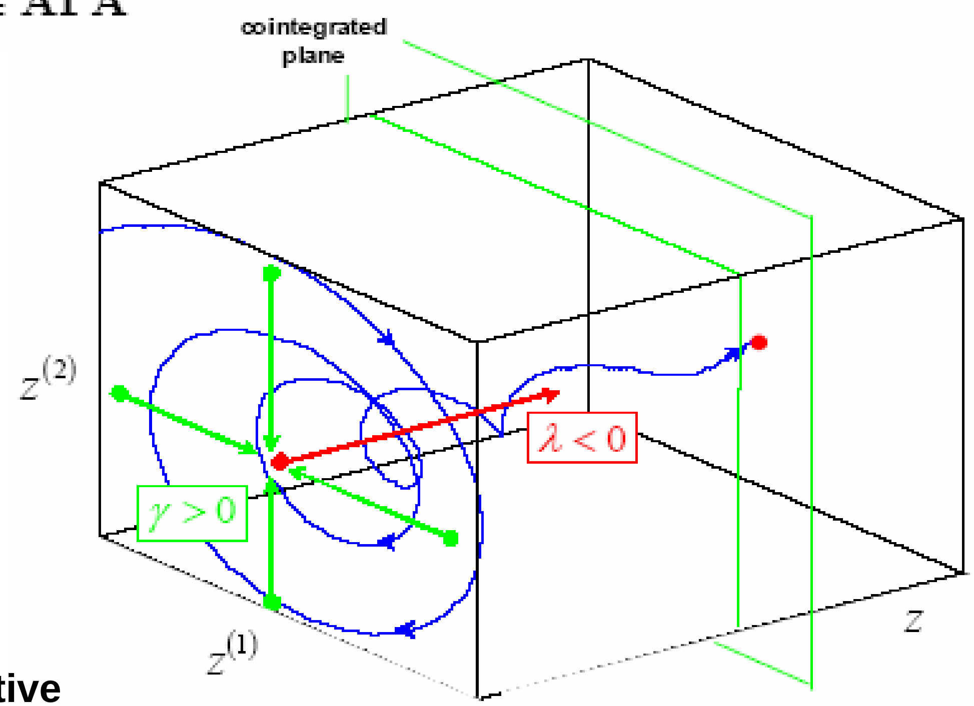
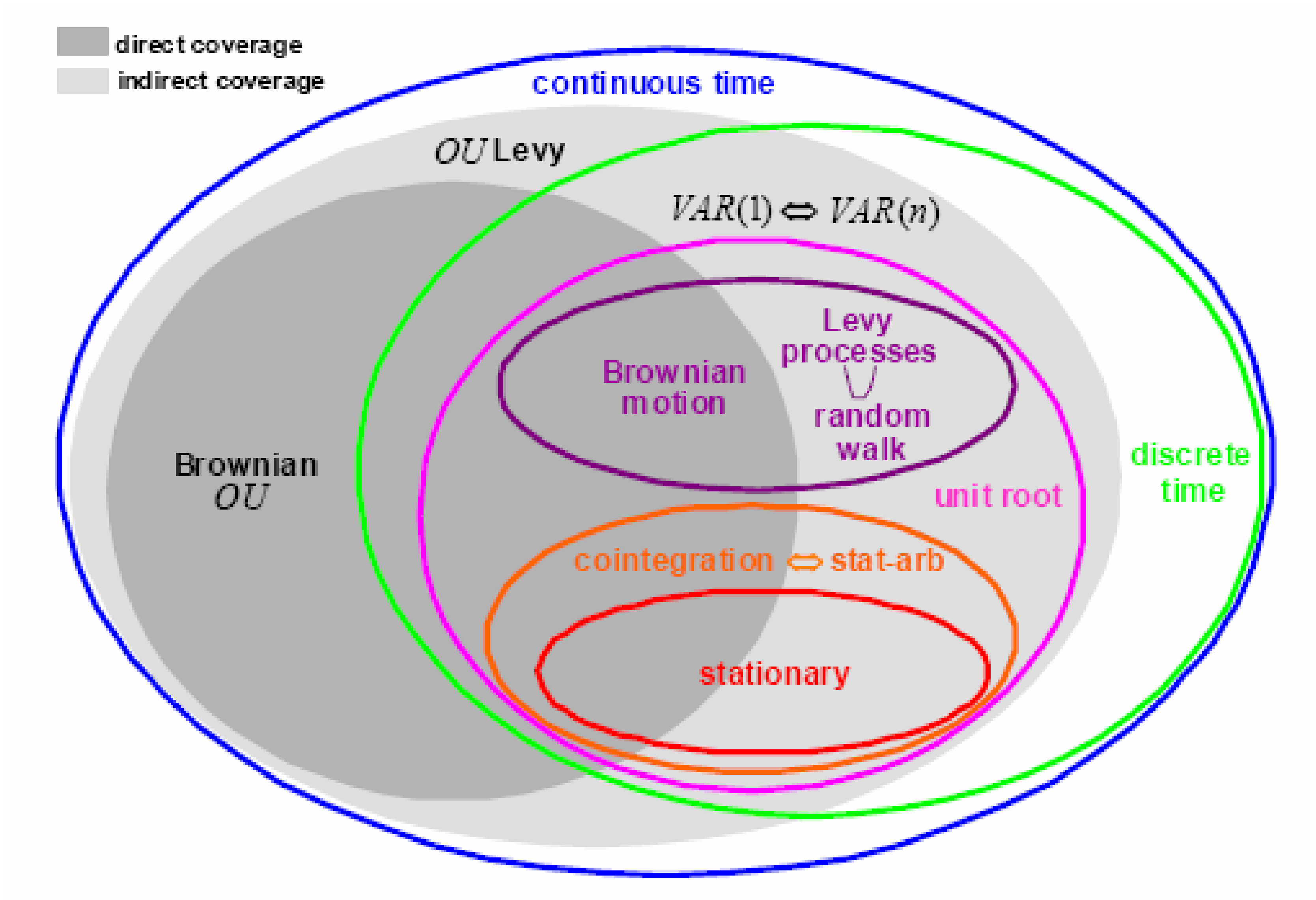


stationary: all (real part of) eigenvalues positive

random walk: null eigenvalues

cointegrated directions: (real part of) eigenvalues positive

Multivariate Ornstein-Uhlenbeck



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$$\tilde{\mathrm{w}} \equiv \underset{\|\mathbf{w}\|=1}{\operatorname{argmin}} \left[\operatorname{Var}\left\{Y_\infty^{\mathbf{w}} | \mathrm{x}_0\right\}\right]$$

$$Y_t^{\tilde{\mathrm{w}}}$$ best candidate

Cointegration

$$Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}$$

cointegrated?

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$$Y_t^{\widetilde{\mathbf{w}}}$$

best candidate

$$\boldsymbol{\Sigma}_\infty \equiv \operatorname{Cov} \left\{ \mathbf{X}_\infty | \mathbf{x}_0 \right\}, \equiv \mathbf{E} \boldsymbol{\Lambda} \mathbf{E}$$

$$\mathbf{E} \equiv \left(\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(N)} \right)$$

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Cointegration

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$$Y_t^{\widetilde{\mathbf{w}}}$$ **best candidate** $$\equiv \quad Y_t^{\mathbf{e}^{(N)}}$$

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Cointegration

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$Y_t^{\widetilde{\mathbf{w}}}$ **best candidate** $\equiv Y_t^{\mathbf{e}^{(N)}}$ $\quad Y_t^{\mathbf{e}^{(N-1)}}$ $\quad \ldots \quad$ $Y_t^{\mathbf{e}^{(1)}}$

Statistical arbitrage

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$$y_{t+\tau} \equiv \left(1 - e^{-\theta\tau}\right)\mu + e^{-\theta\tau} y_t + \epsilon_{t,\tau}$$

empirical fit

Statistical arbitrage

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empirical fit

$$\alpha \equiv \left| y_t - \operatorname{E} \left\{ y_\infty \right\} \right| = \left| y_t - \mu \right|$$

expected gain / “alpha”

$$Z_t \equiv \frac{\left| y_t - \operatorname{E} \left\{ y_\infty \right\} \right|}{\operatorname{Sd} \left\{ y_\infty \right\}} = \frac{\left| y_t - \mu \right|}{\sqrt{\sigma^2 / 2\theta}}$$

Z-score / Sharpe ratio > signal

$$\widetilde{\tau} \propto \frac{1}{\theta}.$$

half-life / expected wait to profits

Statistical arbitrage