Incorporating Trading Strategies in the Black-Litterman Framework

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The simplicity and the intuitive appeal of portfolio construction using modern portfolio theory have attracted significant attention both in academia and in practice. Yet, despite considerable effort it took many years until portfolio managers started using modern portfolio theory for managing real money. Unfortunately, in real world applications there are many problems associated with it, and portfolio optimization is still considered by many practitioners to be difficult to apply.

The outline of this paper is as follows. We start by providing a general overview of some of the common practical problems encountered in mean-variance optimization. Then, we turn our attention to the Black-Litterman model. As the model is often misunderstood and thought to be complicated, we provide a straightforward and step-by-step derivation using classical regression analysis. Introducing a simple cross-sectional momentum strategy, we then show how we can combine this strategy with market equilibrium using the Black-Litterman model in the mean-variance framework to rebalance the portfolio on a monthly basis.

PROBLEMS ENCOUNTERED IN MEAN-VARIANCE OPTIMIZATION IN PRACTICE

Optimized portfolios do normally not perform as well in practice as one would expect from theory. For example, they are often outperformed by simple allocation strategies such as the equally weighted portfolio (Jobson and Korkie [1981]) or the global minimum variance portfolio (Jorion, [1991]).[1] Simply put, the "optimized" portfolio is not optimal at all.

Portfolio weights are often not stable over time but change significantly each time the portfolio is reoptimized, leading to unnecessary turnover and increased transaction costs. Moreover, these portfolios typically present extreme holdings ("corner solutions") in a few securities while other securities have close to zero weight. Consequently, these "optimized" portfolios are not necessarily well diversified and exposed to unnecessary *ex post* risk (Michaud [1989] and Green and Hollifield [1992]). The reason for these phenomena is not a sign that mean-variance optimization does not work, but rather that the modern portfolio theory framework is very sensitive to small changes in the inputs.

In a portfolio optimization context, securities with large expected returns and low standard deviations will be overweighted and conversely, securities with low expected returns and high standard deviations will be underweighted. Therefore, large estimation errors in expected returns and/or variances/covariances will introduce errors in the optimized portfolio weights. For this reason, people often cynically refer to optimizers as "error maximizers." Uncertainty from estimation error in expected returns tends to have more influence than in the covariance matrix in a mean-variance optimization (Best and Grauer [1991], [1992]). The relative importance depends on the investor's risk aversion, but as a general rule of thumb, errors in the expected returns are about 10 times more important than errors in the covariance matrix, and errors in the variances are about twice as important as errors in the covariances.[2]

In classical mean-variance optimization we need to provide estimates of the expected returns and covariances of all the securities in the investment universe considered. This is of course a humongous task, given the number of securities available today. Portfolio managers are very unlikely to have a detailed understanding of all the securities, companies, industries, and sectors that they have at their disposal. Typically, however, portfolio managers have reliable return forecasts for only a small subset of these assets. This is probably one of the major reasons why the mean-variance framework has not been adopted by practitioners in general. It is simply unrealistic for the portfolio manager to produce good estimates of all the inputs required for classical portfolio theory.

Many trading strategies used today cannot easily be turned into forecasts of expected returns and covariances. In particular, not all trading strategies produce views on *absolute* return, but rather just provide *relative* rankings of securities that are predicted to outperform/underperform other securities. For example, consider two stocks, A and B. An *absolute view* is of the form "the one-month expected returns on A and B are 1.2% and 1.7% with a standard deviation of 5% and 5.5%, respectively," while a *relative view* may be expressed as "B will outperform A by 0.5% over the next month" or simply "B will outperform A over the next month." Clearly, it is not an easy task to translate any of these relative views into the inputs required for portfolio optimization.

In practice, the Black-Litterman model has proven successful in mitigating estimation errors in the mean-variance framework.[3] The Black-Litterman model does not require that the portfolio manager provide forecasts on all the securities in the universe managed. Rather, it "blends" any views (this could be a forecast on just one or a few securities, or all them) the investor might have with market equilibrium. The investor's views can be formulated in different ways. They can be the result of a quantitative model or come from a more traditional process. They can be formulated as absolute or as relative. When no views are present, the resulting Black-Litterman expected returns are just the expected returns consistent with market equilibrium. Conversely, when the investor has views on some of the assets, the resulting expected returns deviate from market equilibrium.

Recently, robust optimization techniques, originally developed in the area of operations research and optimization theory, have received significant interest by the investment management community. These techniques allow the portfolio manager to incorporate estimation errors directly into the portfolio optimization. In a nutshell, standard optimizers take inputs as given with complete certainty, where as robust optimizers incorporate the uncertainty introduced by estimation error directly into the optimization process.[4] Improved estimation techniques, such as the Black-Litterman model, can be effectively applied in combination with robust portfolio optimization.

THE BLACK-LITTERMAN MODEL

In this section we provide a simple derivation of the Black-Litterman model. We have divided the exposition into three steps: basic assumptions and starting point, expressing an investor's views, and combining an investor's views with market equilibrium.

Step 1: Basic Assumptions and Starting Point

One of the basic assumptions underlying the Black-Litterman model is that the expected return of a security should be consistent with market equilibrium *unless* the investor has a specific view on the security. In other words, an investor who does not have any views on the market should hold the market portfolio.

Our starting point is the CAPM

$$E(R_i) - R_f = \beta_i(E(R_M) - R_f)$$

where $E(R_i)$, $E(R_M)$, and R_f are the expected return on security i, the expected return on the market portfolio, and the risk-free rate, respectively. Furthermore,

$$\beta_i = \frac{\mathrm{cov}(R_i, R_M)}{\sigma_M^2}$$

where σ_M^2 is the variance of the market portfolio. Let us denote by $\mathbf{w}_b = (w_{b1}, \ldots, w_{bN})'$ the market capitalization or benchmark weights, so that with an asset universe of N securities the return on the market can be written as

$$R_M = \sum_{j=1}^{N} w_{bj} R_j$$

Then by the CAPM, the expected excess return on asset i, $\Pi_i = E(R_i) - R_f$, becomes

$$\begin{aligned}\Pi_i &= \beta_i(E(R_M) - R_f) \\ &= \frac{\text{cov}(R_i, R_M)}{\sigma_M^2}(E(R_M) - R_f) \\ &= \frac{E(R_M) - R_f}{\sigma_M^2} \sum_{i=1}^{N} \text{cov}(R_i, R_j) w_{bj}\end{aligned}$$

We can also express this in matrix-vector form as

$$\boldsymbol{\Pi} = \boldsymbol{\delta \Sigma w}$$

where we define the market price of risk as

$$\delta = \frac{E(R_M) - R_f}{\sigma_M^2},$$

the expected excess return vector

$$\boldsymbol{\Pi} = \begin{bmatrix} \Pi_1 \\ \vdots \\ \Pi_N \end{bmatrix},$$

and the covariance matrix of returns

$$\boldsymbol{\Sigma} = \begin{bmatrix} \text{cov}(R_1, R_1) & \cdots & \text{cov}(R_1, R_N) \\ \vdots & & \vdots \\ \text{cov}(R_N, R_1) & \cdots & \text{cov}(R_N, R_N) \end{bmatrix}$$

The true expected returns $\boldsymbol{\mu}$ of the securities are unknown. However, we assume that our equilibrium model above serves as a reasonable estimate of the true expected returns in the sense that

$$\boldsymbol{\Pi} = \boldsymbol{\mu} + \boldsymbol{\varepsilon_\Pi}, \boldsymbol{\varepsilon_\Pi} \sim N(0, \boldsymbol{\tau\Sigma})$$

for some small parameter $\tau << 1$. We can think about $\tau\boldsymbol{\Sigma}$ as our confidence in how well we can estimate the equilibrium expected returns. In other words, a small τ implies a high confidence in our equilibrium estimates and vice versa.

Because the market portfolio is on the efficient frontier, as a consequence of the CAPM an investor will hold a portfolio consisting of the market portfolio and a risk-free instrument earning the risk-free rate. But let us now see what happens if an investor has a particular view on some of the securities.

Step 2: Expressing an Investor's Views

Formally, K views in the Black-Litterman model are expressed as a K-dimensional vector $\mathbf{q}$ with

$$\mathbf{q} = \mathbf{P}\boldsymbol{\mu} + \boldsymbol{\varepsilon}_\mathbf{q}, \boldsymbol{\varepsilon}_\mathbf{q} \sim N(0, \boldsymbol{\Omega})$$

where $\mathbf{P}$ is a $K \times N$ matrix (explained in the following example) and $\boldsymbol{\Omega}$ is a $K \times K$ matrix expressing the confidence in the views. In order to understand this mathematical specification better, let us take a look at an example.

Let us assume that the asset universe that we consider has five stocks ($N = 5$) and that an investor has the following two views:

1. Stock 1 will have a return of 1.5%.
2. Stock 3 will outperform Stock 2 by 4%.

We recognize that the first view is an absolute view where as the second one is a relative view. Mathematically, we express the two views together as

$$\begin{bmatrix} 1.5\% \\ 4\% \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

The first row of the $\mathbf{P}$ matrix represents the first view, and similarly, the second row describes the second view. In this example, we chose the weights of the second view such that they add up to zero, but other weighting schemes are also possible.

We also remark at this point that the error terms ε_1, ε_2 do not explicitly enter into the Black-Litterman model—but their variances do. Quite simply, these are just the variances of the different views. For example,

$$\boldsymbol{\Omega} = \begin{bmatrix} 1\%^2 & 0 \\ 0 & 1\%^2 \end{bmatrix}$$

corresponds to a higher confidence in the views, and conversely,

$$\boldsymbol{\Omega} = \begin{bmatrix} 5\%^2 & 0 \\ 0 & 7\%^2 \end{bmatrix}$$

represents a much lower confidence in the views. The off diagonal elements of $\boldsymbol{\Omega}$ are typically set to zero. The reason for this is that the error terms of the individual views are most often assumed to be independent of one another.

Step 3: Combining an Investor's Views with Market Equilibrium

Having specified the market equilibrium and an investor's views separately, we are now ready to combine the two together. There are two different, but equivalent, approaches that can be used to derive the Black-Litterman model. We will describe a derivation that relies upon standard econometric techniques, in particular, the so-called *mixed estimation technique* described by Theil [1971]. The approach based on Bayesian statistics has been explained in some detail by Satchel and Scowcroft [2000].

First, we combine the specifications of market equilibrium from Step 1 and the investor's views from Step 2 in the form,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\mu} + \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \sim N(0, \mathbf{V})$$

where

$$\mathbf{y} = \begin{bmatrix} \boldsymbol{\Pi} \\ \mathbf{q} \end{bmatrix}, \mathbf{X} = \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \tau\boldsymbol{\Sigma} & \\ & \boldsymbol{\Omega} \end{bmatrix}$$

with $\boldsymbol{I}$ denoting the $N \times N$ identity matrix. We observe that this is just a linear model for the expected returns $\boldsymbol{\mu}$. Calculating the Generalized Least Squares (GLS) estimator for $\boldsymbol{\mu}$, we obtain

$$\begin{aligned}
\hat{\boldsymbol{\mu}}_{BL} &= (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \\
&= \left(\begin{bmatrix} \mathbf{I} & \mathbf{P}' \end{bmatrix} \begin{bmatrix} (\tau\boldsymbol{\Sigma})^{-1} & \\ & \boldsymbol{\Omega}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{P}' \end{bmatrix} \begin{bmatrix} (\tau\boldsymbol{\Sigma})^{-1} & \\ & \boldsymbol{\Omega}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Pi} \\ \mathbf{q} \end{bmatrix} \\
&= \left(\begin{bmatrix} \mathbf{I} & \mathbf{P}' \end{bmatrix} \begin{bmatrix} (\tau\boldsymbol{\Sigma})^{-1} \\ \boldsymbol{\Omega}^{-1}\mathbf{P} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{P}' \end{bmatrix} \begin{bmatrix} (\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} \\ \boldsymbol{\Omega}^{-1}\mathbf{q} \end{bmatrix} \\
&= [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{q}]
\end{aligned}$$

The last line in the above formula is the Black-Litterman expected returns that "blend" the market equilibrium with the investor's views.

SOME PROPERTIES OF THE BLACK-LITTERMAN MODEL

To provide a better intuitive understanding of the formula, next we discuss some of its important properties.

Absence of Views

If the investor has no views (that is, $\mathbf{q} = \boldsymbol{\Omega} = 0$) or the confidence in the views is zero, then the Black-Litterman expected return becomes $\hat{\boldsymbol{\mu}}_{BL} = \boldsymbol{\Pi}$. Consequently, the investor will end up holding the market portfolio as predicted by the CAPM. In other words, the optimal portfolio in the absence of views is the defined market.

If we were to plug return targets of zero or use the available cash rates, for example, into an optimizer to represent the absence of views, the result would be an optimal portfolio that looks very much different from the market. The equilibrium returns are those forecasts that in the absence of any other views will produce an optimal portfolio equal to the market portfolio. Intuitively speaking, the equilibrium returns in the Black-Litterman model are used to "center" the optimal portfolio around the market portfolio.

A Confidence Weighted Expected Return

By using $\mathbf{q} = \mathbf{P}\boldsymbol{\mu} + \boldsymbol{\varepsilon}_{\mathbf{q}}$, we have that the investor's views alone imply the estimate of expected returns $\hat{\boldsymbol{\mu}} = (\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'\mathbf{q}$. Since where $\mathbf{P}(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}' = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix, we can rewrite the Black-Litterman expected returns in the form

$$\hat{\boldsymbol{\mu}}_{BL} = [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\hat{\boldsymbol{\mu}}]$$

Now we see that the Black-Litterman expected return is a "confidence weighted" linear combination of market equilibrium $\boldsymbol{\Pi}$ and the expected return $\hat{\boldsymbol{\mu}}$ implied by the investor's views. The two weighting matrices are given by

$$\begin{aligned}
\mathbf{w}_{\boldsymbol{\Pi}} &= [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}]^{-1}(\tau\boldsymbol{\Sigma})^{-1} \\
\mathbf{w}_{\mathbf{q}} &= [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}]^{-1}\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}
\end{aligned}$$

where

$$\mathbf{w}_{\boldsymbol{\Pi}} + \mathbf{w}_{\mathbf{q}} = \mathbf{I}$$

In particular, $(\tau\boldsymbol{\Sigma})^{-1}$ and $\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}$ represent the confidence we have in our estimates of the market equilibrium and the views, respectively. Therefore, if we have low confidence in the views, the resulting expected returns will be close to the ones implied by market equilibrium. Conversely, with higher confidence in the views, the resulting expected returns will deviate from the market equilibrium implied expected returns. We say that we "tilt" away from market equilibrium.

It is straightforward to show that the Black-Litterman expected returns can also be written in the form

$$\hat{\boldsymbol{\mu}}_{BL} = \boldsymbol{\Pi} + \tau\boldsymbol{\Sigma}\mathbf{P}'(\boldsymbol{\Omega} + \tau\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi})$$

where we now immediately see that we tilt away from the equilibrium return $\boldsymbol{\Pi}$ by a vector proportional to $\boldsymbol{\Sigma}\mathbf{P}'(\boldsymbol{\Omega} + \tau\ \mathbf{P}\ \boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi})$.

The Black-Litterman Expected Returns as the Solution of an Optimization Problem

The Black-Litterman expected returns can be shown to be the solution to the optimization problem

$$\begin{aligned}\hat{\boldsymbol{\mu}}_{BL} = \arg\min_{\mu}\{&(\boldsymbol{\Pi} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Pi} - \boldsymbol{\mu})\\ &+ \tau(\mathbf{q} - \mathbf{P}\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\mu})\}\end{aligned}$$

From this formulation we see that $\hat{\boldsymbol{\mu}}_{BL}$ is chosen such that it is *simultaneously* as close to $\boldsymbol{\Pi}$, and $\mathbf{P}\boldsymbol{\mu}$ is as close to $\mathbf{q}$ as possible. The distances are determined by $\boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{\Omega}^{-1}$. Furthermore, the relative importance of the equilibrium versus the views is determined by τ. For example, for τ large the weight of the views is increased, whereas for τ small the weight of the equilibrium is higher. We also see that τ is a "redundant" parameter as it can be absorbed into $\boldsymbol{\Omega}$.

The Covariance Matrix of the Black-Litterman Returns

It is straightforward to calculate the variance of the Black-Litterman expected return. By the standard "sandwich formula," we obtain

$$\begin{aligned}\mathrm{var}(\hat{\boldsymbol{\mu}}_{BL}) &= (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\\ &= [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}]^{-1}\end{aligned}$$

How the Influence of Estimation Errors Is Mitigated

The most important feature of the Black-Litterman model is that it uses the mixed estimation procedure to adjust the *entire* market equilibrium implied expected return vector with an investor's views. Because security returns are correlated, views on just a few assets will, due to these correlations, imply changes to the expected returns on *all* assets. Mathematically speaking, this follows from the fact that although the vector $\mathbf{q}$ can have dimension $K << N$, $\mathbf{P}'\boldsymbol{\Omega}^{-1}$ is an $N \times K$ matrix that "propagates" the K views into N components, $\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{q}$. This effect is stronger the more correlated the different securities are. In the absence of this adjustment of the expected return vector, the differences between the equilibrium expected return and an investor's forecasts will be interpreted as an arbitrage opportunity by a mean-variance optimizer and result in portfolios concentrated in just a few assets ("corner solutions"). Intuitively, any estimation errors are spread out over all assets, making the Black-Litterman expected return vector less sensitive to errors in individual views. This effect contributes to the mitigation of both estimation risk and error maximization in the optimization process.

INCORPORATING TRADING STRATEGIES IN THE BLACK-LITTERMAN MODEL

In this subsection we discuss how to incorporate factor models and cross-sectional rankings in this framework. Furthermore, we describe some simple approaches to estimate the confidence for a view in cases where it is not directly available.

Factor Models

Let's assume we have a factor model of the form

$$R_i = \alpha_i + \mathbf{F}\beta_i + \varepsilon_i,\ \ i \in I$$

where $I \subset \{1, 2, \ldots, N\}$, $\mathbf{F} = (F_1, \ldots, F_K)$, $\boldsymbol{\beta}_i = (\beta_{i1}, \ldots, \beta_{iK})$, and $K < N$. Typically, from a factor model it is easy to obtain an estimate of the residual variance, $\mathrm{var}(\varepsilon_i)$. In this case, we set

$$q_i = \begin{cases}\alpha + \mathbf{F}\beta_i, & i \in I\\ 0, & \text{otherwise}\end{cases}$$

and the corresponding confidence

$$\omega_{ii}^2 = \begin{cases}\mathrm{var}(\varepsilon_i), & i \in I\\ 0, & \text{otherwise}\end{cases}$$

The $\mathbf{P}$ matrix is defined by

$$p_{ii} = \begin{cases}1, & i \in I\\ 0, & \text{otherwise}\end{cases}$$

$$P_{ij} = 0,\ \ i \neq j$$

Of course in a practical implementation we would omit rows with only zeros.

Cross-Sectional Rankings

Many quantitative investment strategies do not *a priori* produce expected returns, but rather just a simple ranking of the securities. Let us consider a ranking of securities from "best to worst" (from an outperforming to an underperforming perspective, etc.). For example, a value manager might consider ranking securities in terms of increasing book-to-price ratio (B/P), where a low B/P would indicate an undervalued stock (potential to increase in value) and high B/P an overvalued stock (potential to decrease in value). From this ranking we form a long-short portfolio where we purchase the top half of the stocks (the group that is expected to outperform) and we sell short the second half of stocks (the group that is expected to underperform). The view $\mathbf{q}$ in this case becomes a scalar, equal to the expected return on the long-short portfolio. The confidence of the view can be decided from backtests, as we describe below. Further, here the $\mathbf{P}$ matrix is a $1 \times N$ matrix of ones and minus ones. The corresponding column component is set to one if the security belongs to the outperforming group, or minus one if it belongs to the underperforming group.

Determining the Confidence Level

In many cases we may not have a direct estimate of the confidence (variance) of a view. We describe a few different approaches here that can be used to determine the confidence.

One of the advantages of a quantitative strategy is that it can be backtested. In the case of the long-short portfolio strategy discussed above, we could estimate its historical variance through simulation with historical data. Of course, we cannot completely judge the performance of a strategy going forward from our backtests. Nevertheless, the back-test methodology allows us to obtain an estimate of the Black-Litterman view and confidence for a particular view/strategy.

Another approach of deriving estimates of the confidence of the view is through simple statistical assumptions. To illustrate, let us consider the second view in the example above: "Stock 3 will outperform Stock 2 by 4%." If we don't know its confidence, we can come up with an estimate for it from the answers to a few simple questions. We start by asking ourselves with what certainty we believe the strategy will deliver a return between 3% and 5% ($4\% \pm \alpha$ where α is some constant, in this case $\alpha = 1\%$). Let us say that we believe there is a two out of three chance that this will happen, $2/3 \approx 67\%$. If we assume normality, we can interpret this as a 67% confidence interval for the future return to be in the interval [3%, 5%]. From this confidence interval we calculate that the implied standard deviation is equal to about 0.66%. Therefore, we would set the Black-Litterman confidence equal to $(0.66\%)^2 = 0.43\%$.

THE BLACK-LITTERMAN MODEL: AN EXAMPLE

In this section we provide an illustration of the Black-Litterman model by combining a cross-sectional momentum strategy with market equilibrium. The resulting Black-Litterman expected returns are subsequently fed into a mean-variance optimizer. We start by describing the momentum strategy before we turn to the optimized strategy.

A Cross-Sectional Momentum Strategy

Practitioners and researchers alike have identified several ways to successfully predict future security returns based on historical returns. Among these findings, perhaps the most popular ones are those of momentum strategies.

The basic idea of a momentum strategy is to buy stocks that have performed well and to sell the stocks that have performed poorly with the hope that the same trend will continue in the near future. This effect was first documented in the academic literature by Jegadeesh and Titman [1993] for the U.S. stock market and has thereafter been shown to be present in many other international equity markets by Rouwenhorst [1998]. The empirical findings show that stocks that outperformed (underperformed) over a horizon of 6 to 12 months will continue to perform well (poorly) on a horizon of 3 to 12 months to follow. Typical backtests of these strategies have historically earned about 1% per month over the following 12 months.

In this example, we developed a cross-sectional momentum strategy using daily returns of the individual constituents (developed market country indices) of the MSCI World Index over the period 1/1/1980 through 5/31/2004. The MSCI World Index is a free float-adjusted market capitalization index that is designed to measure global developed market equity performance. As of

December 2004, the MSCI World Index consisted of the following 23 developed market country indices: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Hong Kong, Ireland, Italy, Japan, the Netherlands, New Zealand, Norway, Portugal, Singapore, Spain, Sweden, Switzerland, the United Kingdom, and the United States. Other constituents that were part of the index at some point throughout the sample period were Malaysia, Mexico, and South African Gold Mines.[5]

We construct a cross-sectional momentum portfolio using all the country indices from the MSCI World Index at a point in time t ("today") and hold the portfolio for one month. We sort the countries based on their "one-day lagged" past nine-month return normalized by their individual volatilities. In other words, the ranking is based on the quantity

$$z_{t,i} = \frac{P_{t-1\,\text{day},i} - P_{t-1\,\text{day}-9\,\text{months},i}}{P_{t-1\,\text{day}-9\,\text{months},i} \cdot \sigma_i}$$

where $P_{t-1\text{ day},\,i}$, $P_{t-1\text{ day—9 months},\,i}$ and σ_i denote the prices of security i at one day before t, one day and nine months before t, and the volatility of security i, respectively. After the ranking, the securities in the top half are assigned a weight of

$$w_i = \frac{1}{\sigma_i \cdot \kappa}$$

where κ is a scaling factor chosen such that the resulting annual portfolio volatility is equal to 20%.[6] Similarly, the securities in the bottom half are assigned a weight of

EXHIBIT 1
Summary Statistics of the Momentum Strategy

	Start Date	End Date	Mean	Volatility	Sharpe Ratio	Skew	Kurtosis	Min	Max	Alpha	Beta
1st Qtr	Jan-81	Dec-85	23.0%	19.4%	1.18	0.12	2.82	–10.4%	17.1%	11.7%	0.25
2nd Qtr	Jan-86	Dec-91	22.1%	21.7%	1.02	0.50	4.90	–14.9%	21.8%	14.3%	0.06
3rd Qtr	Jan-92	Dec-97	26.9%	20.9%	1.29	–0.09	4.87	–18.8%	20.2%	22.3%	–0.02
4th Qtr	Jan-98	May-04	3.7%	20.8%	0.18	0.54	3.33	–13.1%	16.9%	–0.1%	–0.05
1st Half	Jan-81	Dec-91	22.5%	20.6%	1.09	0.36	4.23	–14.9%	21.8%	12.9%	0.12
2nd Half	Jan-92	May-04	14.8%	21.1%	0.70	0.23	3.82	–18.8%	20.2%	10.7%	–0.03
Full	Jan-81	May-04	18.4%	20.9%	0.88	0.29	4.01	–18.8%	21.8%	11.7%	0.05

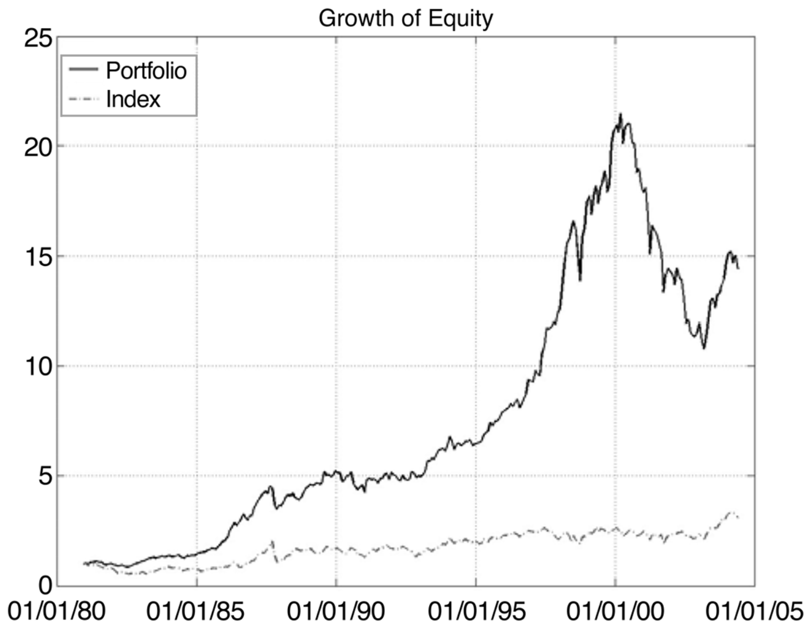
Notes: The columns Mean, Volatility, Sharpe Ratio, and Alpha are the annualized mean returns, volatilities, Sharpe ratios, and alphas of the portfolio over the different periods. Min and Max are the daily minimum and maximum portfolio returns respectively. Skew and Kurtosis are calculated as the third and fourth normalized centered moments. Alphas and beats are calculated using 1-month LIBIOR.

EXHIBIT 2
Summary Statistics of the MSCI World Index

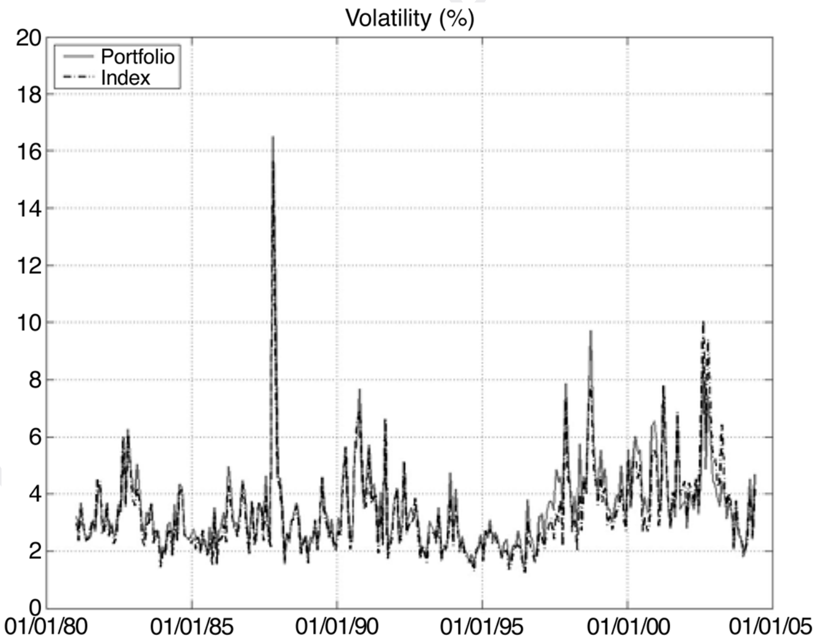
	Start Date	End Date	Mean	Volatility	Sharpe Ratio	Skew	Kurtosis	Min	Max
1st Qtr	Jan-81	Dec-85	10.2%	11.5%	0.88	–0.29	2.70	–7.6%	7.7%
2nd Qtr	Jan-86	Dec-91	13.2%	16.4%	0.81	–0.21	4.05	–14.6%	12.8%
3rd Qtr	Jan-92	Dec-97	9.6%	9.8%	0.98	0.62	3.28	–3.9%	9.5%
4th Qtr	Jan-98	May-04	2.9%	17.2%	0.17	0.17	3.49	–12.3%	16.0%
1st Half	Jan-81	Dec-91	11.8%	14.3%	0.83	–0.21	4.22	–14.6%	12.8%
2nd Half	Jan-92	May-04	6.1%	14.1%	0.43	0.15	4.32	–12.3%	16.0%
Full	Jan-81	May-04	8.8%	14.2%	0.62	–0.02	4.22	–14.6%	16.0%

Notes: The columns Mean, Volatility, and Sharpe Ratio are the annualized mean returns, volatilities, and Sharpe ratios of the index over the different periods. Min and Max are the daily minimum and maximum Index returns, respectively. Skew and Kurtosis are calculated as the third and fourth normalized centered moments.

E X H I B I T 3
Growth of Equity of the Optimized Strategy and the MSCI World Index



E X H I B I T 4
Monthly Portfolio Volatility of the Optimized Strategy Compared to Monthly Volatility of the MSCI World Index



$$w_i = -\frac{1}{\sigma_i \cdot \kappa}$$

We make the portfolio weights a function of the individual volatilities in order not to overweight the most volatile assets. This is not a zero cost long-short portfolio as the portfolio weights do not sum up to zero. It is straightforward to modify the weighting scheme to achieve a zero cost portfolio, but for our purposes this does not matter and will not significantly change the results.

The momentum strategy outperforms the MSCI World Index on both an "alpha" and a Sharpe ratio basis. The Sharpe ratio of the strategy over the full period is 0.88 versus 0.62 for the index. The full period-annualized alpha is 11.7%, consistent with the standard results in the momentum literature. The beta of the strategy is very low, only 0.05 for the full sample. The realized correlation between the momentum strategy and the index is 3.5%. Summary statistics of the MSCI World Index and the momentum strategy are provided in Exhibits 1 and 2.

This simple momentum strategy has an average monthly portfolio turnover of 23.7% with a cross-sectional standard deviation of 9.3%. The United Kingdom has the highest average turnover (40.6%) and New Zealand has the lowest (10.8%).

Incorporating the Momentum Strategy in the Black-Litterman Model

In the previous section we introduced a simple cross-sectional momentum strategy. In this section we demonstrate how it can be combined with market equilibrium in a portfolio optimization framework by using the Black-Litterman model.

In this case, we only have one view—the momentum strategy. The covariance matrices needed for the portfolio optimization are calculated from daily historical data with geometric weighting (monthly decay

EXHIBIT 5

Portfolio Summary Statistics of the Optimized Strategy Rebalanced Monthly

	Start Date	End Date	Mean	Volatility	Sharpe Ratio	Skew	Kurtosis	Min	Max	Alpha	Beta
1st Qtr	Jan-81	Dec-89	18.9%	15.0%	1.26	–0.33	4.39	–15.1%	13.2%	9.2%	0.28
2nd Qtr	Jan-90	Dec-94	13.8%	13.7%	1.01	0.35	3.92	–9.4%	12.8%	11.23%	0.40
3rd Qtr	Jan-95	Dec-99	23.8%	14.0%	1.70	0.19	4.12	–9.1%	14.4%	18.1%	0.39
4th Qtr	Jan-00	May-04	–2.9%	15.3%	–0.19	–0.28	2.82	–11.9%	8.5%	–5.0%	0.65
1st Half	Jan-81	Dec-94	16.5%	14.6%	1.13	–0.09	4.08	–15.1%	13.2%	11.4%	0.31
2nd Half	Jan-95	May-04	11.4%	15.2%	0.75	–0.13	3.60	–11.9%	14.4%	6.3%	0.37
Full	Jan-81	May-04	13.6%	14.9%	0.92	–0.11	3.88	–15.1%	14.4%	8.3%	0.36

Notes: The columns Mean, Volatility, Sharpe Ratio, and Alpha are the annualized mean returns, volatilities, Sharpe ratios, and alphas of the portfolio over the different periods. Min and Max are the daily minimum and maximum portfolio returns, respectively. Skew and Kurtosis are calculated as the third and fourth normalized centered moments. Alphas and betas are calculated using 1-month LIBOR.

of 0.95) and with the standard "Newey-West" correction for autocorrelation (2 lags).[7] We choose $\tau = 0.1$ for the Black-Litterman model.

After computing the Black-Litterman expected returns, we use the risk aversion formulation of the mean-variance optimization problem

$$\max_{w}(\mathbf{w}'\hat{\boldsymbol{\mu}}_{BL} - \boldsymbol{\lambda}\mathbf{w}'\Sigma\mathbf{w})$$

subject to

$$\mathbf{w}'\boldsymbol{\iota} = 1, \boldsymbol{\iota}' = [1,1,\ldots,1]$$

We choose a risk aversion coefficient of $\lambda = 2$ (calibrated to achieve about the same volatility as the index). We calculate optimal portfolio weights and rebalance the portfolio monthly. Before rebalancing at the end of each month, we calculate the realized portfolio return and its volatility. Results and summary statistics are presented in Exhibits 3, 4, and 5.

The optimized strategy has a full sample Sharpe ratio of 0.92 versus 0.62 for the index and an "alpha" of 8.3%. We observe that in the last quarter the Sharpe ratio and the "alpha" of the strategy were negative, largely due to the general downturn in the market during that period. In contrast to the standalone momentum strategy that we discussed in the previous section, since the optimized strategy is a blend of momentum and market equilibrium, its resulting correlation with the index is significantly different from zero. For example, the full sample correlation with the index in this case is 0.36.[8]

CONCLUSIONS

The Black-Litterman model is a simple and effective way to mitigate estimation errors and to incorporate different trading strategies into the same portfolio optimization framework.

ENDNOTES

[1]For an illustration showing that a global minimum-variance portfolio and the equally weighted portfolio can significantly outperform the mean-variance portfolio, see Chapter 4 in Fabozzi, Focardi, and Kolm [2006].

[2]For a further discussion, see Chapter 8 in Fabozzi, Focardi, and Kolm [2006] and the references therein.

[3]See, Black and Litterman [1990], Bevan and Winkelmann (1998), and Litterman [2003].

[4]For more details on robust portfolio optimization, see Goldfarb and Garud (2003) and Fabozzi, Focardi, and Kolm [2006].

[5]We would like to thank Morgan Stanley Capital International, Inc., http://www.msci.com, for providing us with the data set.

[6]κ can be estimated from past portfolio returns at each time of rebalancing. Typically its value does not change significantly from period to period.

[7]This particular covariance matrix estimator is described in Newey and West [1987].

[8]One possibility to decrease the correlation of the strategy with the index is to impose zero β constraints in the optimization.

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