

Lecture 20 Volatility Modeling and Calibration

This lecture is adapted from a survey paper I wrote entitled “**Inference and Computation for Stochastic Volatility Models Related to Option Pricing**”, which reviews the Black-Scholes (BS) theory and its extensions in several aspects:

- logarithmic stochastic volatility (SV) models and square-root SV models with jumps;
- statistical inference using Markov chain Monte Carlo (MCMC) methods;
- Generalized Black-Scholes (GBS) pricing formulas and their role in calibration of SV models using both returns and option data;
- Certain probability approximation schemes proposed recently which reduce the dimensionality in Monte Carlo computation of some stochastic integrals involved in the GBS formulas.

Note: With the presence of high-dimensional latent volatility processes, numerical integration for computing option prices is required at every time point and every iteration of MCMC. There is an urgent need for developing approximation schemes that reduce numerical integration from a high-dimensional space (of diffusion sample paths) to a low-dimensional space (of 2D or 3D random vectors). We propose using bivariate Gaussian or gamma mixtures of Gaussian to approximate joint distributions of certain integrated volatilities and additive functionals in the lifetime of relevant options or other derivatives. When implementing those schemes to computation of various derivatives prices represented by GBS formulas, they significantly improve the efficiency (speed/accuracy) of related MCMC algorithms. This will have an impact on a wide range of problems in financial econometrics.

20.1 Introduction

The celebrated BS theory [2] in financial economics lays a foundation for three important aspects: a probability model, an option pricing formula and a statistical inference procedure for volatilities. Although the past few decades have seen a great deal of progress in various extensions of the BS theory, a brief recap still provides a clear theme for our review and ongoing research.

- *The geometric Brownian motion [BS Stochastic Differential Equation (SDE)]:*

$$dS_t/S_t = \mu dt + \sigma dW_t, \tag{20.1}$$

where S_t represents a stock price at time $t \in [0, T]$, the constants μ and σ represent the expected rate of return and volatility respectively, and $\{W_t\}$ is a standard Wiener process.

The BS model consists of only two primary assets, i.e. the stock $\{S_t\}$ and a bank account $\{B_t\}$ with constant interest rate r :

$$B_t = B_0 e^{rt}, \quad t \in [0, T]. \quad (20.2)$$

- *Pricing formulas for derivative securities*

To simplify the notation, we denote the maturity of a derivative also by T , same as the finite horizon. Later, we will make a necessary distinction that the maturity time is denoted by $T^* \leq T$. A contingent claim $g(S_T)$ written on the stock price S_T can be given a time t “fair” price $P(t, S_t)$ in two equivalent ways. The analytical approach yields $P(t, x)$, $t \in [0, T]$, $x \geq 0$ as the solution to the boundary value problem

$$\begin{aligned} \frac{\partial P(t, x)}{\partial t} + rx \frac{\partial P(t, x)}{\partial x} + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 P(t, x)}{\partial x^2} - r P(t, x) &= 0 \\ \text{with } P(T, x) &= g(x). \end{aligned} \quad (20.3)$$

The probabilistic approach expresses $P(t, S_t)$ as a conditional expectation given \mathcal{F}_t under a risk-neutral probability measure Q :

$$P(t, S_t) = E^Q \left[e^{-r(T-t)} g(S_T) \mid \mathcal{F}_t \right], \quad (20.4)$$

where Q is defined through its Radon-Nikodým derivative with respect to the real-world probability measure P (also referred to as the physical measure or objective measure) that governs the BS SDE (20.1)

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \exp \left[-\frac{\mu - r}{\sigma} W_t - \frac{t}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \right]; \quad (20.5)$$

and $\{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by the Wiener process $\{W_t\}$. Note in this case, the information up to time t , i.e. \mathcal{F}_t , is summarized by the current state S_t . In the special case of an European call option, we have $g(S_T) = \max\{S_T - K, 0\}$, where the positive constant K is called the strike price at the maturity time T . The time t price $P(t, S_t)$, denoted by a function $C^{BS}(S_t, \sigma^2, r, K, T - t)$, enjoys a closed form solution by using either approach:

$$C^{BS}(S_t, \sigma^2, r, K, T - t) = S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2), \quad (20.6)$$

where $\Phi(\cdot)$ is the cdf of $N(0, 1)$ distribution, and

$$d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[\log \frac{S_t}{K} + (r + \sigma^2/2)(T-t) \right], \quad (20.7)$$

$$d_2 = d_1 - \sigma \sqrt{T-t}. \quad (20.8)$$

Remarks:

(i) The equivalence between (20.3) and (20.4) was established via the Feynman-Kac formula (cf. [46], [56]). In what follows, we will follow the probabilistic approach and refer to [23] for the analytical approach.

(ii) The fundamental risk-neutral valuation principle for pricing derivatives originated from the seminal papers of [2], [53]. A rigorous probability framework was due to [35] and [36]. Several excellent books were written in mathematical finance at different levels, such as [38], [1], [13], [15], to name a few. For a systematic treatment of stochastic calculus, including the Girsanov theorem, see [46] and [56].

- *Estimation of volatilities*

To put the pricing formula (20.6) in action, numerical values of t, T, S_t, K, r, σ need to be specified. Among them, t, T, S_t, K are known or observable, while r and σ require more attention. In a simple case, it may not be hard to estimate r . However, some other situations go beyond the scope of this work, in which r is not only changing over time, but also considered to be stochastic. This is a rich research area of interest rate modeling. We refer to [44] and [58] for more detailed accounts.

Focusing on estimation of volatility σ , there are two substantially different methods:

- (i) *Historical volatility*

This is just to estimate a population variance by a sample variance. Suppose we observe $S_{t_0}, S_{t_1}, \dots, S_{t_n}$ at $t_0 < t_1 < \dots < t_n$. Let

$$R_i = \log S_{t_i} - \log S_{t_{i-1}} \quad (20.9)$$

be the return incurred in $(t_{i-1}, t_i]$. Assume $t_i - t_{i-1} = \Delta$, $\forall i = 1, \dots, n$. Then R_1, \dots, R_n are iid normal random variables with mean $(\mu - \sigma^2/2)\Delta$ and variance $\sigma^2\Delta$. Naturally, a historical volatility estimate for σ is given by

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^n (R_i - \bar{R})^2}{(n-1)\Delta}} \quad \text{with} \quad \bar{R} = \frac{1}{n} \sum_{i=1}^n R_i.$$

Note that a common sense expression of return R_i , given by $(S_{t_i} - S_{t_{i-1}})/S_{t_{i-1}}$, is a first-order approximation of the definition in (20.9) with small Δ .

- (ii) *Implied volatility*

An argument against the use of historical volatility is based on (a) volatility is not constant, but changes over time; (b) historical volatility only gives an estimate for the volatility over a past time period, but what is more relevant concerns the *market expectation* of the volatility

in the near future. More specifically, if we want to determine the time t price of an option with maturity time T , then the market expectation of the volatility in the life time $(t, T]$ of the option should be incorporated consistently in the volatility estimation. One way to realize the market expectation is the following: Note that $C^{BS}(\cdot)$ in (20.6) is an invertible map of σ with other variables held fixed. Get a market price c_t from a “benchmark” call option written on the same stock $\{S_t\}$, then find σ by inverting $C^{BS}(\cdot)$ from c_t . Such a value of σ is called the implied volatility.

Mathematically, implied volatility does not follow the traditional statistical inference guideline as historical volatility does. It is similar to a dynamical programming strategy where at each step a conditional expectation of future reward given the current state is compared to the instantaneous reward, in order to determine an optimal strategy. The idea is far reaching. We will revisit it in later sections when studying calibration of SV models using both returns and option data.

20.2 SV Models and MCMC Calibration

20.2.1 ARCH/GARCH models

Several drawbacks of the BS model were widely recognized, particularly the constant volatility σ . Since then, many efforts have been made to model volatility as a time-varying stochastic quantity. Along this line, the ARCH/GARCH (*autoregressive conditional heteroscedasticity/ generalized ~*) family is an important development in modeling volatility.

ARCH(p):

$$X_t = \sigma_t \epsilon_t, \quad (20.10)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i X_{t-i}^2, \quad (20.11)$$

where $\epsilon_1, \epsilon_2, \dots$ are iid $N(0, 1)$ random variables, and $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \dots, p$ are parameters. Note that X_t is considered to be the return in $(t-1, t]$, thus $\sigma_t^2 = E(X_t^2 | \mathcal{F}_{t-1}) = \text{Var}(X_t | \mathcal{F}_{t-1})$ is the conditional variance or second moment of X_t . In particular, α_0 represents the long-run volatility.

GARCH(p, q):

$$X_t = \sigma_t \epsilon_t, \quad (20.12)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i X_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2 \quad (20.13)$$

with the additional parameters $\beta_j \geq 0$, $j = 1, \dots, q$. Apparently, $\text{ARCH}(p)$ is a special case of $\text{GARCH}(p, q)$ with $q = 0$.

Since the original works of [18] and [3], ARCH/GARCH models have enjoyed great popularity. The likelihood functions have simple forms which make inference and prediction very convenient. There are many extended versions in the ARCH/GARCH family, among which [55] is noteworthy in relating GARCH and SV models. The books [22] and [65] also devote a couple of chapters to ARCH/GARCH models.

In ARCH/GARCH models, the sequence $\{\sigma_t^2\}$ represents time-varying volatilities, and they are stochastic. However, the term “stochastic volatility” usually is not meant to include ARCH/GARCH models, but reserved for a different model class we are to discuss. The main distinction is that there is a single source of uncertainty from asset returns, and each σ_t^2 is defined as a deterministic function of returns. In contrast, volatility in a SV model is defined as a stochastic process generated from an uncertainty source (often termed “factor”) in addition to that for asset returns. Sometimes more than one factors are introduced. The necessity for SV models lies in the modeling flexibility. Certain interesting behaviors in financial markets, such as the famous “volatility smile”, can be modeled by SV models, but not by ARCH/GARCH models. Here is a brief account: A call option is said to be “in the money” (ITM) at time t if $S_t > K$ and “out of the money” (OTM) if $S_t < K$. For put option with the contingent claim $g(S_T) = \max\{K - S_T, 0\}$, the inequalities are reversed. If $S_t = K$, the option is said to be “at the money” (ATM). Having observed the market prices of a number of European call options with the same maturity date T on the same underlying stock, the plot of implied volatility as a function of strike price K should be a horizontal straight line, if the assumption of constant volatility σ in the BS model is correct. Contrary to this, empirical evidences show that options far OTM or deep ITM are traded at higher implied volatilities than ATM options. The graph of the observed implied volatility function often looks like a smile shape, which gives rise to the term “volatility smile”.

20.2.2 SV models

Let $S = \{S_t\}$ denote a continuous-time process for an asset price under a physical measure P . The following SV model was considered in [6]:

$$y_t = \log S_t, \tag{20.14}$$

$$dy_t = \mu dt + e^{h_t/2} \left(\sqrt{1 - \rho^2} dW_{1t} + \rho dW_{2t} \right), \tag{20.15}$$

$$dh_t = (\alpha + \beta h_t) dt + \sigma dW_{2t}, \tag{20.16}$$

where $W = \{W_t\} = \{(W_{1t}, W_{2t})\}$ is a standard 2D Wiener process defined on a probability space (Ω, \mathcal{F}, P) . Let $\{\mathcal{F}_t : 0 \leq t \leq T\}$ denote the filtration generated by W . The parameter ρ is the correlation between the asset return and the volatility factor.

This model consists of an observation equation (20.15) and a state evolution equation (20.16). It can be viewed as an example in state-space models or in filtering problems. The logarithmic volatility formulation guarantees that the resulting volatility $e^{h_t/2}$ is always positive. Although other forms of volatility evolution are available, log-SV models were introduced earlier and considered in [10], [64], [62], [51], [25], [41], [14], [31], [60], [61], [47], [23], [30], [9], [42], and [8].

To price an option on S , a standard approach is to assume no arbitrage and specify two risk premia processes. See [57] and [20]. In doing so, the SDEs describing the returns process have the same functional form under the physical measure P and the risk neutral measure Q . These two risk premia processes are

$$\tilde{\nu}_t = \nu_1 + \nu_2 h_t, \quad (20.17)$$

$$\lambda_t = \frac{1}{\sqrt{1-\rho^2}} \left\{ e^{-h_t/2} \left[(\mu - r) + e^{h_t/2} \right] - \rho \tilde{\nu}_t \right\}, \quad (20.18)$$

where r is the short rate, presumed constant. Assume the L_2 norms of the risk premia are bounded and $E \left[\exp \left(\int_0^T (\lambda_u^2 + \tilde{\nu}_u^2) du \right) \right] < \infty$. As shown in [35] and [36], there exists an equivalent martingale measure Q such that the time t price of a contingent claim $g(S_{t'})$ (with $t' > t$) can be expressed as the conditional expectation $E^Q[e^{-r(t'-t)}g(S_{t'}) | \mathcal{F}_t]$. For the purpose of computing the expectation, the measure Q can be expressed either as a Radon-Nikodým derivative with respect to the measure P ,

$$\begin{aligned} & \left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} \\ &= \exp \left(- \int_0^t \lambda_u dW_{1u} - \int_0^t \tilde{\nu}_u dW_{2u} - \frac{1}{2} \int_0^t \lambda_u^2 du - \frac{1}{2} \int_0^t \tilde{\nu}_u^2 du \right), \end{aligned} \quad (20.19)$$

or as a system of SDEs in terms of the Wiener processes

$$W_{1t}^Q = W_{1t} + \int_0^t \lambda_u du, \quad (20.20)$$

$$W_{2t}^Q = W_{2t} + \int_0^t \tilde{\nu}_u du, \quad (20.21)$$

which is

$$dS_t/S_t = r dt + e^{h_t/2} [\sqrt{1-\rho^2} dW_{1t}^Q + \rho dW_{2t}^Q], \quad (20.22)$$

$$dh_t = [\alpha - \nu_1 \sigma + (\beta - \nu_2 \sigma) h_t] dt + \sigma dW_{2t}^Q. \quad (20.23)$$

SV models enjoy tremendous modeling flexibility. They can model various important market behaviors, such as the volatility smile. A key idea, termed GBS pricing formulas, was hinted in [39] but advocated explicitly in [60], [61], [59], [27] and [29]. Here is the GBS formula for the log-SV

model (20.15)-(20.16) and (20.22)-(20.23):

$$P(S_t, h_t, \theta, \nu) = E^Q \left[C^{BS} \left(S_t e^{Z_{t,T^*}}, \bar{V}_{t,T^*}, r, K, T^* - t \right) | \mathcal{F}_t \right], \quad (20.24)$$

$$Z_{t,T^*} = \rho \int_t^{T^*} e^{h_u/2} dW_{2u}^Q - \frac{\rho^2}{2} \int_t^{T^*} e^{h_u} du, \quad (20.25)$$

$$\bar{V}_{t,T^*} = \frac{1 - \rho^2}{T^* - t} \int_t^{T^*} e^{h_u} du, \quad (20.26)$$

where $\theta = \{\mu, \alpha, \beta, \sigma, \rho\}$ and $\nu = \{\nu_1, \nu_2\}$ denote the parameters in (20.15)–(20.16) and (20.22)–(20.23); Q is a risk-neutral measure [Note: Q is not unique, which is specified by the choice of risk premia (20.17) and (20.18)]; $C^{BS}(\cdot)$ is the BS call option pricing formula given by (20.6) with life span $(t, T^*]$. Hence the GBS pricing formula is interpreted as a conditional expectation of the original BS option price with certain adjusted parameters, taken over the unobservable log-volatilities h_u , $t < u \leq T^*$ in the life time of the option. The special case with constant volatility and $\rho = 0$ will turn (20.24) to the original BS call option price. A put option can be priced using the parity relation between a put (P_t) and a call (C_t), which is $P_t = C_t - S_t + K \exp[-r(T^* - t)]$. Other derivatives can also be priced in a similar way.

Later we will elaborate on how to compute GBS formulas efficiently by using certain probability approximations.

20.2.3 MCMC calibration

(i) *Estimation based on return data only*

Statistical inference for the SV model (20.15)–(20.16) concerns the model parameter θ (parameter estimation) and latent volatilities $h = \{h_{t+i\Delta}, i = 0, 1, \dots, n_y - 1\}$ (state estimation), based on a time series of return data $Y = \{y_{t+i\Delta}, i = 0, 1, \dots, n_y - 1\}$ generated from a physical measure P . Note that Δ is the sampling window of observations; the initial time t is arbitrary to maintain flexibility; the risk-neutral measure Q , risk premia parameter ν are irrelevant because no option data are used for the time being (inference with option data to be studied later).

Discretization of the diffusion model (20.15)–(20.16) via an Euler scheme (see [48]) yields a likelihood. The presence of latent volatility sequence h poses an indirect inference problem, which can be tackled by several methods: EM algorithms, generalized method of moments (GMM), efficient method of moments (EMM), MCMC, etc. As for references, see [52] for EM algorithms, [33] and [5] for GMM, [24] for EMM, and [50] for MCMC. When applying to calibration of SV models, each method has its own pros and cons. No attempts are made in this paper for comparisons among different methods. Here we only mention [26] and [8] as examples of using GMM or EMM, and [41] as the earliest contribution to MCMC calibration of SV models. More references will be given when discussing inference on SV models using both returns and option data in case (ii).

In the current context, MCMC methods generate a discrete-time Markov chain whose stationary distribution is the joint posterior distribution of model parameter θ and latent variables h , hence samples from the posterior distribution of any marginal are easy to obtain by selecting elements from the joint series.

Discretization can introduce bias. This bias may be eliminated by using any one of the fill-in methods proposed by [19] or [17]. But as verified by [20], for the type of data and class of models considered here, accuracy is adequate for a Δ of one day so there is no need to use fill-in methods if data sampled at a daily frequency are available.

The MCMC method we adopt is a type of “Metropolis within Gibbs”. In the spirit of Gibbs, the unknowns θ and h are classified in several blocks. Each block is sampled conditioning on the rest and one cycles through different blocks. Because the normalizing factors for the conditional distributions involved are not known, a Metropolis sampler is used to generate runs within each block. Implementation of a Metropolis step consists of generating a candidate move from a proposal density and deciding whether to accept the proposed move based on a Metropolis ratio. The art is that the proposal density is usually specified in a problem-dependent manner. In other words, a few guidelines are available, but no rigid rules.

Denote the posterior distribution for parameters and latent variables by $P(h, \theta | Y)$. Holding Y fixed, this density is proportional to the likelihood times the prior

$$P(h, \theta | Y) \propto P(Y, h | \theta) P(\theta) = P(Y | h, \theta) P(h | \theta) P(\theta) \quad (20.27)$$

with

$$\begin{aligned} & P(Y|h, \theta) P(h|\theta) \\ &= \prod_{i=0}^{n-1} p(y_{t+(i+1)\Delta} | y_{t+i\Delta}, h_{t+i\Delta}, \theta) p(h_{t+(i+1)\Delta} | h_{t+i\Delta}, \theta), \end{aligned} \quad (20.28)$$

where $p(y_{t+(i+1)\Delta} | y_{t+i\Delta}, h_{t+i\Delta}, \theta)$ is the normal density of $y_{t+(i+1)\Delta}$ with the conditional mean $\mu\Delta + y_{t+i\Delta}$ and conditional variance $e^{h_{t+i\Delta}}\Delta$, and $p(h_{t+(i+1)\Delta} | h_{t+i\Delta}, \theta)$ is the normal density of $h_{t+(i+1)\Delta}$ with the conditional mean $\alpha\Delta + (1 + \beta\Delta)h_{t+i\Delta}$ and conditional variance $\sigma^2\Delta$, following the discretization of (20.15)–(20.16).

The Gibbs blocking is as follows. The elements within the block of h are moved one at a time. The coordinate variable $h_{t+i\Delta}$ at each i is generated from the conditional density

$$p(h_{t+i\Delta} | h_{t+(i-1)\Delta}, h_{t+(i+1)\Delta}; y_{t+(i-1)\Delta}, y_{t+i\Delta}, y_{t+(i+1)\Delta}; \theta)$$

due to the Markovian structure in the dynamics of (20.15)–(20.16). The drift parameters μ and $\{\alpha, \beta\}$ are moved as two separate blocks and the diffusion parameters $\{\sigma, \rho\}$ as another. The priors are uniform on their supports with the exception for μ which has a conjugate normal prior, allowing μ to be updated directly without Metropolis-Hastings acceptance/rejection intermediation. All priors are proper with boundary parameters set in the experiments to be effectively noninformative.

The single-move Gibbs sampler is not as efficient as some other Bayesian MCMC inference methods for calibrating SV models when using only returns data Y . In particular, the multi-move Gibbs sampler with Kalman filters introduced in [63], [47] and [9] is more efficient computationally. Note in case (ii) we still use the single-move Gibbs sampler for calibrating SV models based on both return and option data because, to the best of our knowledge, no multi-move MCMC methods are available for that task. Given that we must use it for that purpose, we also use it for returns alone to reduce our coding and debugging burden.

(ii) *Estimation based on both return and option data*

The MCMC (Bayesian) inference on SV models is essentially an extension of the (frequentist) BS historic volatility approach in a broad sense that it only uses return data. Many financial econometricians argue that option data are at least as relevant as return data, if not more, in volatility calibration. Technically, the idea of BS implied volatility cannot be extended to SV models naively because the pricing formula, as a map from latent volatilities to an option price, is not invertible, i.e. it is an ill-posed inverse problem due to the high dimensionality of h . Nevertheless, we could and should still incorporate observed option data in volatility estimation by checking the discrepancy (called a pricing error) between an observed option price and one calculated via a GBS formula.

Consider the additional information provided by options written on Y . The option prices C we consider are on homogeneous instruments sampled at the same time as $y_{t+i\Delta}$ so that C has elements $c_{t+j\Delta}$ for $j = 0, 1, \dots, n_c - 1$, where $c_{t+j\Delta}$ is the logarithm of the observed option price. (Note: Usually $n_c < n_y$ and the time t where C begins is larger than that of Y . But we do not consider those complications here.)

As is common in derivative pricing applications (cf. [20]), the distribution of C is determined by both an option pricing formula, which is (20.24) in our case, and an assumption of pricing errors. Pricing errors are not only required to permit application of Bayesian and similar likelihood based methods but are also plausible reflections of bid-ask bounce and similar micro structure considerations (cf. [59], [43]). We assume that errors are additive in log-scale,

$$c_{t+j\Delta} = f_{t+j\Delta} + \delta\tilde{\epsilon}_j, \quad j = 0, 1, \dots, n_c - 1, \quad (20.29)$$

where, from (20.24),

$$f_{t+j\Delta} = \log [P(S_{t+j\Delta}, h_{t+j\Delta}, \theta, \nu)], \quad (20.30)$$

and the errors $\tilde{\epsilon}_j$, $j = 0, 1, \dots, n_c - 1$ are iid $N(0, 1)$ random variables, independent of the processes S and h . Note that (20.29) treats observed option data as a noisy version of the option prices predicted based on a GBS pricing formula. In [45], a different but somewhat equivalent approach

is taken by using the implied volatility data (VIX) to be a noisy version of the latent volatility variables.

There are several unknowns now, including $\theta = \{\mu, \alpha, \beta, \rho, \sigma\}$ and h that appear in the returns dynamics (20.15)–(20.16), the risk premia parameters $\nu = \{\nu_1, \nu_2\}$ that appear in the risk-neutral dynamics (20.22)–(20.23), and the pricing error standard deviation δ from (20.29).

We assume the three blocks $\{Y, h, \theta\}$, ν , and δ are mutually independent under the prior distribution. Under this assumption, the joint posterior distribution of interest becomes

$$P(h, \theta, \nu, \delta | C, Y) \propto P(C | Y, h, \theta, \nu, \delta) P(Y | h, \theta) P(h | \theta) P(\theta) P(\nu) P(\delta). \quad (20.31)$$

We have expressions for $P(Y | h, \theta)$ and $P(h | \theta)$ from case (i). Following (20.29), the likelihood of the option prices is

$$P(C | Y, h, \theta, \nu, \delta) \propto \prod_{j=0}^{n_c-1} \delta^{-1} \exp\left(-\frac{(c_{t+j\Delta} - f_{t+j\Delta})^2}{2\delta^2}\right). \quad (20.32)$$

The Gibbs blocking is as follows. As in case (i) of this subsection, the elements $h_{t+i\Delta}$ of h are moved one at a time, the drift parameters μ and $\{\alpha, \beta\}$ are moved as two separate blocks, and the diffusion parameters $\{\sigma, \rho\}$ as another. The risk premium parameters ν and the pricing error parameter δ are moved as two separate blocks. The prior on ν is uniform. The prior for δ is the conjugate, so that δ can be directly sampled from a product of inverse Gamma distributions without Metropolis-Hastings acceptance/rejection intermediation. All priors are proper with parameters set to be effectively noninformative.

20.3 GBS Formulas and Probability Approximations for Numerical Integration

A price that comes with the modeling capability by using SV models is the computational intensity required for calibrating SV models using financial data from two sources: underlying asset returns and option prices. Adding option data to this task creates a significant challenge for the following reason. In a SV model, a volatility time series consists of latent variables h_t , $t = 0, 1, \dots, T$. Calibration of parameter θ (including the risk-neutral parameter ν as some components) and volatilities $\{h_t\}$ is an indirect inference problem, inviting certain Monte Carlo based algorithms, such as EMM or MCMC. There is an extensive literature in calibration of SV models using both return and option data. Here we only mention a few. See [7], [57] for EMM, and [30], [45], [20], [6] for MCMC. Those algorithms yield updated values $\theta^{(m)}$ and $h_t^{(m)}$, $t = 0, 1, \dots, T$ in the m th iteration, $m = 1, \dots, M$. In particular, as an important part of the m th iteration in fitting a SV model, an option price $f_t^{(m)}$ is calculated by using a pricing formula based on the SV model with the current parameter value $\theta^{(m-1)}$ and volatility value $h_t^{(m-1)}$, and compared to the observed option price C_t . The comparison

will lead to an adjustment of $h_t^{(m-1)}$ to $h_t^{(m)}$, and also updating of $\theta^{(m-1)}$ to $\theta^{(m)}$ based on $h_t^{(m)}$, $t = 0, 1, \dots, T$. Notice that a general method of computing $f_t^{(m)}$ is to perform high-dimensional numerical integration — treated as a conditional expectation over the space of sample paths of future volatilities, and this has to be done for *every* t and *every* m in the algorithm. The required computational time quickly adds up with large values of T (several hundred days) and M (typically more than 100,000 iterations). So far, there have been basically two ways to handle the computation in practice. A basic strategy is to use “brute force” simulation, i.e. to generate a large number of “additional” sample paths of future volatilities based on which the Monte Carlo integration yields an approximation to $f_t^{(m)}$. An alternative approach relies on the availability of closed-form option pricing formulas which can avoid the aforementioned brute force numerical integration. However, Heston’s model (cf. [37]) appears to be the only case beyond the original BS setting that enjoys a closed-form solution (still, certain low-dimensional numerical integration needed for a Fourier inversion strategy). The intractability of the brute force calculation and the limitation with closed-form pricing formulas present an urgent need for developing new efficient computational methodology.

First, we present a Gaussian approximation scheme for numerical computation of the option price and summarize the related work. The proposed Gaussian approximation scheme promotes a significant dimension reduction for numerical integration, from the space of volatility sample paths (with dimensionality of several hundreds) to the sample space of bivariate Gaussian vectors. See [6] (<http://www.stat.unc.edu/faculty/cji/research.html>) for detailed analytical arguments and numerical results. Next, we will also include some ongoing work in [66] for Gamma-Gaussian approximations applied to a SV model with jumps.

20.3.1 Gaussian approximations for logarithmic SV models

The obvious (and usual) approach to computing the integrals in (20.24)–(20.26) is to discretize (20.22)–(20.23), generate sample paths according to the recursions implied by the discretization, evaluate (20.24)–(20.26) over each sample path, and take an average. Indeed, this is how we check the accuracy of our proposed Gaussian approximations by creating a benchmark value of the option price. A number of useful tips with solid theoretical justifications are given in [48], [16], [32] regarding Monte Carlo simulation of SDEs. For the purpose of calibrating SV models via MCMC considered in this work, the brute force simulation is too computationally intensive to be practicable. Our proposal is as follows.

With a small time increment Δ , the Euler discretization of the volatility process under the risk-neutral measure Q , i.e. (20.23) yields the recursion

$$h_{t+\Delta} = a + b h_t + c \epsilon_{t+\Delta}, \quad (20.33)$$

where $\epsilon_{t+\Delta}$ is a $N(0, 1)$ random variable, which is independent of the past, i.e. independent of the

spot volatility h_t , of the drift terms $a = (\alpha - \nu_1\sigma)\Delta$ and $b = 1 + (\beta - \nu_2\sigma)\Delta$, and of the diffusion term $c = \sqrt{\Delta}\sigma$. Notice that our notation departs from convention because we have incorporated Δ into the expressions for a , b , and c in order to simplify later formulas. Let $n = (T^* - t)/\Delta$. Our task is to derive the asymptotic distribution of the two sums

$$U_n = \sum_{j=0}^{n-1} e^{h_{t+j\Delta}}, \quad (20.34)$$

$$V_n = \sum_{j=0}^{n-1} e^{h_{t+j\Delta}/2} \epsilon_{t+(j+1)\Delta}, \quad (20.35)$$

which appear in the discretization of (20.25)–(20.26) using (20.33), i.e.

$$U_n \Delta \approx \int_t^{T^*} e^{h_u} du \quad (20.36)$$

$$V_n \sqrt{\Delta} \approx \int_t^{T^*} e^{h_u/2} dW_{2u}^Q. \quad (20.37)$$

For fixed t , let $E_t(\cdot)$, $Var_t(\cdot)$ and $Cov_t(\cdot)$ denote the conditional expectation, variance and covariance operators respectively, given h_t .

Theorem 20.1 *Assume $|b| < 1$. Fix t , Δ and an initial state h_t . As $n \rightarrow \infty$, the limiting distribution of $n^{-1/2}(U_n - E_t U_n, V_n - E_t V_n)$ conditioning on h_t is a bivariate normal distribution with mean $(0, 0)$ and covariance matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ provided $a_{11}a_{22} > a_{12}^2$, with*

$$\begin{aligned} a_{11} &= \lim_{n \rightarrow \infty} n^{-1} Var_t(U_n), \\ a_{12} &= a_{21} = \lim_{n \rightarrow \infty} n^{-1} Cov_t(U_n, V_n), \\ a_{22} &= \lim_{n \rightarrow \infty} n^{-1} Var_t(V_n); \end{aligned}$$

where

$$E_t U_n = \sum_{i=0}^{n-1} \exp \left[\frac{a(1-b^i)}{1-b} + b^i h_t + \frac{c^2(1-b^{2i})}{2(1-b^2)} \right]; \quad (20.38)$$

$$E_t V_n = 0; \quad (20.39)$$

$$Var_t(U_n) = \sum_{i=0}^{n-1} Var_t(e^{h_{t+i\Delta}}) + 2 \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} Cov_t(e^{h_{t+i\Delta}}, e^{h_{t+j\Delta}}), \quad (20.40)$$

$$\begin{aligned} Cov_t(U_n, V_n) &= \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} c b^{i-j-1} \exp \left[\left(b^i + b^j/2 \right) h_t + \frac{a(3/2 - b^i - b^j/2)}{1-b} \right. \\ &\quad \left. + \frac{c^2(5/4 - b^{2i} - b^{2j}/4 + b^{i-j} - b^{i+j})}{2(1-b^2)} \right] \\ Var_t(V_n) &= E_t U_n; \end{aligned} \quad (20.41)$$

with

$$\begin{aligned}
& \text{Var}_t(e^{h_{t+i\Delta}}) \\
&= \exp\left(2b^i h_t + \frac{2a(1-b^i)}{1-b} + \frac{c^2(1-b^{2i})}{1-b^2}\right) \left[\exp\left(\frac{c^2(1-b^{2i})}{1-b^2}\right) - 1\right], \\
& \text{Cov}_t(e^{h_{t+i\Delta}}, e^{h_{t+j\Delta}}) \\
&= \exp\left((b^i b^j) h_t + \frac{a(2-b^i-b^j)}{1-b} + \frac{c^2(2-b^{2i}-b^{2j})}{2(1-b^2)}\right) \cdot \left[\exp\left(\frac{c^2(b^{j-i}-b^{j+i})}{1-b^2}\right) - 1\right].
\end{aligned}$$

Theorem 20.1 follows from a CLT for mixing sequences (cf. [40], [4]). The detailed proof, implementation of MCMC, and numerical results in simulation studies and an application to real foreign exchange data can be found in [6]. In particular, the assessment of computational efficiency (speed/accuracy) shows great promises of the proposed Gaussian approximations.

20.3.2 Gamma-Gaussian approximations for SV models with jumps

Among several important recent contributions to calibration of SV models in recent years, SV models with jumps (simply denoted by SVJ in what follows) have received much attention. See [57], [21], [20], [8]. Jumps and SV are two ways to incorporate additional sources for risk and uncertainty into a model (SDE) for asset returns. Each of them has its own risk premium. Merton was the first one to introduce a jump-diffusion model for stock returns in [54]. An encyclopedic treatment of financial models with jumps is given in the excellent book [12]. Here we present a SVJ model, a GBS pricing formula, and related Gaussian approximations. The notation in this subsection, following [66], is kept “local” and need not be consistent with those used in previous sections.

(i) SVJ models

Consider the SVJ model

$$d(\log S_t) = (r_t + \eta_s V_t - V_t/2) dt + \sqrt{V_t} dW_t^s + U_{N_t} dN_t - \mu_1 \lambda V_t dt, \quad (20.42)$$

$$dV_t = \kappa(\theta - V_t) dt + \sigma_v \sqrt{V_t} dW_t^v, \quad (20.43)$$

where for $t \geq 0$, $S = \{S_t\}$ and $V = \{V_t\}$ represent continuous-time asset price and SV processes respectively. r_t is the interest rate at time t (assumed to be known). Both $\{W_t^s\}$ and $\{W_t^v\}$ are standard 1D Wiener processes, which satisfy $\text{Cov}(W_t^s, W_t^v) = \rho t$ with the correlation coefficient ρ

to incorporate the leverage effect. The SV dynamic (20.43) corresponds to a square-root diffusion with a mean-reversion level θ and a reversion rate κ .

Price jumps are captured via a pure jump process that contains two components: a jump time process and a jump size process. Successive jumps at $0 < \tau_1 < \tau_2 < \dots$ form a point process with a state-dependent intensity function $\{\lambda V_t : t \geq 0\}$ for some constant $\lambda > 0$, which allows jumps to occur more frequently in high volatility regimes. Given the arrival of the i th jump, the price jumps from S_{τ_i-} to $S_{\tau_i-} \exp(U_i^s)$, where $U_i^s, i = 1, 2, \dots$ are iid normal random variables with mean μ_J and variance σ_J^2 . Assume $\{U_i^s\}$ is independent of processes $\{\tau_i\}$, $\{W_t^s\}$ and $\{W_t^v\}$. Denote the mean relative jump size, conditioning on a jump event, by $\mu_1 = E(\exp(U_1^s) - 1) = \exp(\mu_J + \sigma_J^2/2) - 1$. It follows that the last term in (20.42) compensates the instantaneous price increment introduced by the pure jump process. The model parameters are $\eta_s, \kappa, \theta, \sigma_v, \lambda, \mu_J, \sigma_J$.

(ii) Candidate pricing kernels

A pricing kernel, introduced in [34], is basically the Radon-Nikodým derivative $\frac{dP}{dQ}$ of a risk-neutral measure Q with respect to a physical measure P .

The unhedgeable risk factors introduced by the SV and jump processes make an incomplete market in our study. Consequently, a pricing kernel is not unique. To price those risk factors (the diffusive price shocks, the price jumps, and the diffusive volatility shocks), we define a candidate pricing kernel π_t by

$$d(\log \pi_t) = h(V_t) dt + a(V_t) dW_t^v + b(V_t) dW_t^{v\perp} + (\log \pi_t - \log \pi_{t-}) dN_t, \quad (20.44)$$

where $W = (W^v, W^{v\perp})$ is a standard 2D Wiener process. N_t represents the number of price jumps up to time t , i.e. we assume that $\{\pi_t\}$ and $\{S_t\}$ share the same jump time process. Let $\log \pi_t - \log \pi_{t-} = U_i^\pi$ if $\tau_i = t$. Assume $U_i^\pi, i = 1, 2, \dots$ are iid normal random variables with mean μ_π and variance σ_π^2 , and $\{U_i^\pi\}$ is independent of W and $\{N_t\}$. However, we allow the correlation $\text{Corr}(U_1^\pi, U_1^s) = \rho_\pi$.

The assumption of no arbitrage requires that the pricing kernel defined in (20.44) satisfies the following constraints:

$$r_t + h(V_t) + \frac{1}{2}a^2(V_t) + \frac{1}{2}b^2(V_t) + \mu_2\lambda V_t = 0, \quad (20.45)$$

$$\eta_s V_t - V_t/2 + \sqrt{V_t} [\rho a(V_t) + \sqrt{1 - \rho^2} b(V_t)] = -(\mu_3 - \mu_2 - \mu_1)\lambda V_t, \quad (20.46)$$

where

$$\mu_2 = E[\exp(U_1^\pi) - 1] = \exp(\mu_\pi + \sigma_\pi^2/2) - 1,$$

$$\mu_3 = E[\exp(U_1^\pi + U_1^s) - 1] = \exp(\mu_\pi + \sigma_\pi^2/2 + \rho_\pi \sigma_\pi \sigma_J + \mu_J + \sigma_J^2/2) - 1.$$

(iii) *GBS pricing formulas*

Having introduced a pricing kernel $\{\pi_t\}$, all conditional expectations in what follows are taken under the physical measure P , and no risk-neutral measure Q is necessary. Let F_{t,T^*} denote the σ -field generated by $\{V_u, N_u : t < u \leq T^*\}$. Define

$$B^*(t, T^*) = E_t \left(\frac{\pi_{T^*}}{\pi_t} \middle| F_{t,T^*} \right), \quad (20.47)$$

$$\xi_{t,T^*} = E_t \left(\frac{\pi_{T^*}}{\pi_t} \frac{S_{T^*}}{S_t} \middle| F_{t,T^*} \right). \quad (20.48)$$

The time t price $P(S_t, V_t, t, T^*)$ of an European call option with maturity time T^* follows the GBS pricing formula:

$$\begin{aligned} P(S_t, V_t, t, T^*) &= E_t \left[\frac{\pi_{T^*}}{\pi_t} (S_{T^*} - K)^+ \right] \\ &= E_t \left[C^{BS} \left(S_t \xi_{t,T^*}, (\overline{\sigma_{t,T^*}})^2, \{r_u, t \leq u \leq T^*\}, K, T^* - t \right) \right] \\ &= E_t [S_t \xi_{t,T^*} \Phi(d_1) - K B^*(t, T^*) \Phi(d_2)], \end{aligned} \quad (20.49)$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sqrt{T^* - t} \overline{\sigma_{t,T^*}}} \left[\log \left(\frac{S_t \xi_{t,T^*}}{K B^*(t, T^*)} \right) + \frac{1}{2} (\overline{\sigma_{t,T^*}})^2 (T^* - t) \right], \\ d_2 &= d_1 - \overline{\sigma_{t,T^*}} \sqrt{T^* - t}, \\ (\overline{\sigma_{t,T^*}})^2 &= \frac{(1 - \rho^2) \int_t^{T^*} V_u du + \sigma_J^2 (N_{T^*} - N_t)}{T^* - t}, \end{aligned} \quad (20.50)$$

$$\begin{aligned} B^*(t, T^*) &= \left\{ \exp \left[\int_t^{T^*} \left(-r_u - \frac{1}{2} a^2(V_u) - \mu_2 \lambda V_u \right) du \right. \right. \\ &\quad \left. \left. + \int_t^{T^*} a(V_u) dW_u^v \right] \right\} (\mu_2 + 1)^{N_{T^*} - N_t}, \end{aligned} \quad (20.51)$$

$$\xi_{t,T^*} \quad (20.52)$$

$$= \left[\exp \left(- \int_t^{T^*} \mu_3 \lambda V_u du \right) \right] \left\{ \exp \left[\int_t^{T^*} \left(a(V_u) + \rho \sqrt{V_u} \right) dW_u^v \right. \right. \\ \left. \left. - \frac{1}{2} \int_t^{T^*} \left(a(V_u) + \rho \sqrt{V_u} \right)^2 du \right] \right\} (\mu_3 + 1)^{N_{T^*} - N_t}.$$

(20.49) clearly demonstrates the idea of GBS. The BS option price in this case is a function measurable on F_{t,T^*} , i.e. a function of unobserved latent variables $\{V_u, N_u, t < u \leq T^*\}$. Taking another conditional expectation over $\{V_u, N_u, t < u \leq T^*\}$ then yields the option price $P(S_t, V_t, t, T^*)$ for the SVJ model. Note that in this GBS formula, $B^*(t, T^*)$, ξ_{t,T^*} and $\overline{\sigma_{t,T^*}}$ are interpreted as the bond price (or the discount factor), the stock price and the (integrated) volatility respectively in a *virtual* world where the information about future volatilities and jumps in F_{t,T^*} were known at time t .

(iv) *Gamma-Gaussian approximations in computation of $P(S_t, V_t, t, T^*)$*

To compute $P(S_t, V_t, t, T^*)$ numerically, we propose a Gamma-Gaussian approximation scheme. There are a number of integrals in (20.50)-(20.52): $\int_t^{T^*} V_u du$, $\int_t^{T^*} a^2(V_u) du$, $\int_t^{T^*} \sqrt{V_u} a(V_u) du$; $\int_t^{T^*} \sqrt{V_u} dW_u^v$, $\int_t^{T^*} a(V_u) dW_u^v$. The specification $a(V_u) = \text{constant } \sqrt{V_u}$ will enable us to focus on the joint distribution of two integrals: $\int_t^{T^*} V_u du$ and $\int_t^{T^*} \sqrt{V_u} dW_u^v$. Once again, let $n = (T^* - t)/\Delta$ and define

$$U_{1n} = \sum_{j=0}^{n-1} V_{t+j\Delta} \approx \Delta^{-1} \int_t^{T^*} V_u du, \\ U_{2n} = \sum_{j=0}^{n-1} \sqrt{V_{t+j\Delta}} \epsilon_{t+(j+1)\Delta} \approx \Delta^{-1/2} \int_t^{T^*} \sqrt{V_u} dW_u^v,$$

where $\epsilon_{t+\Delta}$, $\epsilon_{t+2\Delta}$, ... are iid $N(0, 1)$ random variables. For fixed Δ and large n , a Gamma approximation scheme is proposed for the distribution of U_{1n} ; then the conditional distribution of U_{2n} given U_{1n} is approximated by a normal distribution with mean 0 and variance U_{1n} . Here we present the first and second moments of (U_{1n}, U_{2n}) , which will be used in determination of those parameters in the proposed Gamma and Gaussian distributions.

An Euler discretization scheme yields the SV dynamics

$$V_{t+\Delta} = a + b V_t + c \sqrt{V_t} \epsilon_{t+\Delta}, \quad (20.53)$$

where

$$\begin{aligned} a &= \kappa \theta \Delta \\ b &= 1 - \kappa \Delta \\ c &= \sigma_v \sqrt{\Delta} \end{aligned}$$

are transformations of the original parameters κ , θ and σ_v .

For fixed t , let $E_t(\cdot)$, $Var_t(\cdot)$ and $Cov_t(\cdot)$ denote the conditional expectation, variance and covariance operators respectively, given V_t .

Theorem 20.2

$$E_t U_{1n} = \frac{a}{1-b} \left(n - \frac{1-b^n}{1-b} \right) + \frac{1-b^n}{1-b} V_t \quad (20.54)$$

$$E_t U_{2n} = 0 \quad (20.55)$$

$$Var_t U_{1n} = B_0 + B_1 V_t + B_2 V_t^2 \quad (20.56)$$

$$Var_t U_{2n} = E_t U_{1n} \quad (20.57)$$

$$\begin{aligned} Cov_t(U_{1n}, U_{2n}) &= \frac{ac}{(1-b)^3} [n-2-nb+nb^{n-1}-(n-2)b^n] \\ &\quad + \frac{c[1-nb^{n-1}+(n-1)b^n]}{(1-b)^2} V_t \end{aligned} \quad (20.58)$$

where the coefficients B_0 , B_1 and B_2 in (20.56) enjoy the closed-form expressions

$$B_0 = \frac{ac^2(n-2)}{(1-b)^2(1+b)} - \frac{abc^2(1-b^{n-2})(1+2b-b^n)}{(1-b)^3(1+b)^2} \quad (20.59)$$

$$\begin{aligned} &+ \frac{2abc^2}{(1-b)^3(1+b)} \left[(n-3)(1+b^{n-2}+b^{n-1}) \right. \\ &\quad \left. - \frac{(1-b^{n-3})(2b+2b^2+b^{n+1})}{(1-b)(1+b)} \right] + \frac{2a^2b^n(1+b)}{(1-b)^2} \left(n-1 - \frac{1-b^{n-1}}{1-b} \right) \end{aligned}$$

$$B_1 = \frac{1-b^{n-1}}{(1-b)^3} [c^2(1+b^n) + 2ab^n(1-b^2)] - \frac{2(n-1)b^{n-1}c^2}{(1-b)^2} \quad (20.60)$$

$$B_2 = 0 \quad (20.61)$$

20.4 Conclusion

Following the spirit of the Black-Scholes theory, several SV models are studied along with the related GBS pricing formulas and MCMC inference methods. As a new contribution, certain proposed probability approximation schemes significantly reduce the dimensionality in Monte Carlo computation of some stochastic integrals involved in the GBS formulas, which makes the Monte Carlo based inference applicable more efficiently.

There are several related areas not presented in this paper, including the PDE approach for option pricing (cf. [23]), realized volatilities and high frequency data (cf. [12]), and the utility-based approach for option pricing with equilibrium models (cf. [49], [11], [28]). The cited references will provide more detailed information and treatments.

Bibliography

- [1] Björk, T. (2004). *Arbitrage Theory in Continuous Time* (2nd edition). Oxford Univ. Press.
- [2] Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81**, 637–659.
- [3] Bollerslev, T. (1986). Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics* **31**, 307–327.
- [4] Bradley, R.C. (1986). Basic properties of strong mixing conditions. *Dependence in Probability and Statistics* (edited by E. Eberlein and M.S. Taqqu), 165–192. Birkhäuser.
- [5] Campbell, J.Y., Lo, A.W. and MacKinlay, A.C. (1997). *The Econometrics of Financial Markets*. Princeton Univ. Press.
- [6] Cheng, A., Gallant, A.R., Ji, C., and Lee, B. (2005). A Gaussian approximation scheme for computation of option prices for stochastic volatility models. Revised for *Journal of Econometrics*.
- [7] Chernov, M. and Ghysels, E. (2000). A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation. *Journal of Financial Economics* **56**, 407–458.
- [8] Chernov, M., Gallant, A.R., Ghysels, E., and Tauchen, G. (2003). Alternative models for stock price dynamics. *Journal of Econometrics* **116**, 225–257.
- [9] Chib, S., Nardari, F. and Shephard, N. (2002). Markov chain Monte Carlo methods for stochastic volatility models. *Journal of Econometrics* **108**, 281–316.
- [10] Clark, P.K. (1973). A subordinated stochastic process model with finite variances for speculative prices. *Econometrica* **41**, 135–156.
- [11] Cochrane, J.H. (2005). *Asset Pricing* (revised edition). Princeton Univ. Press.
- [12] Cont, R. and Tankov, P. (2004). *Financial Modelling with Jump Processes*. Chapman & Hall.
- [13] Cvitanović, J. and Zapatero, F. (2004). *Introduction to the Economics and Mathematics of Financial Markets*. The MIT Press.

- [14] Danielson, J. (1994). Stochastic volatility in asset prices: estimation with simulated maximum likelihood. *Journal of Econometrics* **63**, 375–400.
- [15] Duffie, D. (1996). *Dynamic Asset Pricing Theory* (2nd edition). Princeton Univ. Press.
- [16] Duffie, D. and Glynn, P. (1995). Efficient Monte Carlo simulation of security prices. *Annals of Applied Probability* **5**, 897–905.
- [17] Elarian, O., Chib, S. and Shephard, N. (2001). Likelihood inference for discretely observed non-linear diffusions. *Econometrica* **69**, 959–994.
- [18] Engle, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of the United Kingdom inflation. *Econometrica* **50**, 987–1007.
- [19] Eraker, B. (2001). MCMC analysis of diffusion models with applications to finance. *Journal of Business and Economic Statistics* **19**, 177–191.
- [20] Eraker, B. (2004). Do stock prices and volatility jump? Reconciling evidence from spot and option prices. *Journal of Finance* **54**, 1367–1404.
- [21] Eraker, E., Johannes, M., and Polson, N. (2003). The impact of jumps in returns and volatility. *Journal of Finance* **53**, 1269–1300.
- [22] Fan, J. and Yao, Q. (2003). *Nonlinear Time Series*. Springer-Verlag.
- [23] Fouque, J.P., Papanicolaou, G. and Sircar, R. (2000). *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge Univ. Press.
- [24] Gallant, A.R. and Tauchen, G.E. (1996). Which moments to match? *Econometric Theory* **12**, 657–681.
- [25] Gallant, A.R., Hsieh, D.A., and Tauchen, G.E. (1991). On fitting a recalcitrant series: the Pound/Dollar exchange rate, 1974–1983, nonparametric and semiparametric methods in econometrics and statistics. *Proceedings of the Fifth International Symposium in Economic Theory and Econometrics* (edited by Barnett, W.A., Powell, J., and Tauchen, G.E.), 199–240. Cambridge Univ. Press.
- [26] Gallant, A.R., Hsu, C.-T. and Tauchen, G. (1999). Using daily range data to calibrate volatility diffusions and extract the forward integrated variance. *The Review of Economics and Statistics* **81**, 617–631.
- [27] Garcia, R., Ghysels, E. and Renault, E. (2003). The econometrics of option pricing. Preprint.
- [28] Garcia, R., Luger, R. and Renault, E. (2003). Empirical assessment of an intertemporal option pricing model with latent variables. *Journal of Econometrics* **116**, 49–83.

- [29] Garcia, R., Luger, R. and Renault, E. (2005). Option prices, preferences, and state variables. Preprint.
- [30] Ge, X. (2000). Bayesian calibration of stochastic volatility models. Ph.D. dissertation. Univ. of North Carolina at Chapel Hill.
- [31] Ghysels, E., A. Harvey and E. Renault (1996): Stochastic volatility. *Handbook of Statistics* **14** (edited by Maddala, G.S. and Rao, C.R.). Elsevier Science B.V.
- [32] Glasserman, P. (2000). *Monte Carlo Methods in Financial Engineering*. Springer-Verlag.
- [33] Hansen, L. (1982). Large sample properties of generalized method of moments estimators. *Econometrica* **50**, 1029-1054.
- [34] Hansen, L. and Richard, S. (1987). The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models. *Econometrica* **55**, 587-613.
- [35] Harrison, M. and Kreps, D. (1979). Martingales and arbitrage in multiperiod security markets. *Journal of Economic Theory* **20**, 381-408.
- [36] Harrison, M. and Pliska, S. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process and Their Application*. **11**, 215-260.
- [37] Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with application to bond and currency options. *Review of Financial Studies* **6**, 327-343.
- [38] Hull, J.C. (2000). *Options, Futures, and Other Derivatives* (4th edition). Prentice Hall.
- [39] Hull, J. and White, A. (1987). The pricing of options on assets with stochastic volatility. *Journal of Finance* **42**, 281-300.
- [40] Ibragimov, I.A. and Linnik, Yu.V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff Publishing, Groningen.
- [41] Jacquier, E., Polson, N., and Rossi, P. (1994). Bayesian analysis of stochastic volatility models (with discussion). *Journal of Business and Economic Statistics* **12**, 371-417.
- [42] Jacquier, E., Polson, N., and Rossi, P. (2004). A Bayesian analysis fat-tailed stochastic volatility models with correlated errors. *Journal of Econometrics* **122**, 185-212.
- [43] Jacquier, E. and Jarrow, R. (2000). Bayesian analysis of contingent claim model error. *Journal of Econometrics* **94**, 145-180.
- [44] James, J. and Webber, N. (2000). *Interest Rate Modelling*. John Wiley & Sons.
- [45] Jones, C.S. (2003). The dynamics of stochastic volatility: evidence from underlying and option markets. *Journal of Econometrics*. **116**, 181-224.

- [46] Karatzas, I. and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus* (2nd edition). Springer-Verlag.
- [47] Kim, S., Shephard, N., and Chib, S. (1998). Stochastic volatility: likelihood inference and comparison with ARCH models. *Review of Economic Studies* **65**, 361–393.
- [48] Kloeden, P. E., and Platen, E. (1992). *Numerical Solution of Stochastic Differential Equations*. Springer-Verlag.
- [49] Kou, S.G. (2002). A jump-diffusion model for option pricing. *Management Science* **48**, 1086–1101.
- [50] Liu, J. (2001). *Monte Carlo Strategies in Scientific Computing*. Springer-Verlag.
- [51] Melino, A. and Turnbull, S. (1990). Pricing foreign currency options with stochastic volatility. *Journal of Econometrics* **45**, 239–265.
- [52] Meng, X.-L. and van Dyk, D.A. (1997). The EM algorithm — an old folk song sung to a fast new tune. *Journal of Royal Statistical Society, Series B* **59**, 511–567.
- [53] Merton, R. (1973). The theory of rational option pricing. *Bell Journal of Economics and Management* **7**, 141–183.
- [54] Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* **3**, 125–144.
- [55] Nelson, D.B. (1991). Conditional heteroscedasticity in asset returns: a new approach. *Econometrica* **59**, 347–370.
- [56] Øksendal, B. (1995). *Stochastic Differential Equations* (5th edition). Springer-Verlag.
- [57] Pan, J. (2002). The jump-risk premia implicit in options: evidence from an integrated time-series study. *Journal of Financial Economics* **63**, 3–50.
- [58] Rebonato, R. (1996). *Interest-rate Option Models* (2nd edition). John Wiley & Sons.
- [59] Renault, E. (1997). Econometric models of option pricing errors. *Advances in Economics and Econometrics* (edited by D. Kreps and K. Wallis), 223–278. Cambridge Univ. Press.
- [60] Renault, E. and Touzi, N. (1996). Option hedging and implied volatilities in a stochastic volatility model. *Mathematical Finance* **6**, 279–302.
- [61] Romano, M. and Touzi, N. (1997). Contingent claims and market completeness in a stochastic volatility model. *Mathematical Finance* **7**, 399–412.
- [62] Scott, L. (1987). Option pricing when the variance changes randomly: theory, estimation and an application. *Journal of Financial and Quantitative Analysis* **22**, 419–438.

- [63] Shephard, N. and Pitt, M. (1997). Likelihood analysis of non-Gaussian measurement time series. *Biometrika* **84**, 653–667.
- [64] Taylor, S.J. (1986). *Modelling Financial Time Series*. John Wiley & Sons.
- [65] Tsay, R.S. (2002). *Analysis of Financial Time Series*. John Wiley & Sons.
- [66] Yoon, J., Renault, E. and Ji, C. (2007). Calibration of stochastic volatility models with jumps and computation using generalized Black-Scholes pricing formulas. Work in preparation.