The statistics of the maximum drawdown in financial time series

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Despite considered as a valuable risk measure of practical interest, relatively few results are currently available on the detailed dynamics of the maximum drawdown (MDD) in financial time series. The purpose of this work is to study the statistical properties of the MDD for stochastic processes characterized by the stylized facts of real financial time series. The numerical results obtained using a Monte Carlo code are firstly validated against the analytic al predictions available within the academic framework of the Brownian motion. The statistics of the maximum drawdown is then analyzed in term of both its expectation value and distribution function for processes whose increments are not independent and present non vanishing excess kurtosis and skewness. The expectations for the maximum drawdown are finally compared, along with their confidence intervals, to the events observed in the historical financial time series of different asset classes.

I. INTRODUCTION

The maximum drawdowns (MDD) in financial time series can be qualified as extreme events. The evaluation of both their expectation value and probability density function (pdf) is of big importance for practical applications, especially when building a robust framework for the risk management. This work is motivated by the need of gaining information on the statistical behavior of the MDD for stochastic processes that set aside from the academic example of the Brownian motion and are possibly closer to the stylized facts that characterize the real financial time series [14], [4], [2].

To our knowledge, analytical expressions of the distribution function of the MDD and of its expectation value have been derived in the limit of a Brownian motion with drift in Refs. [7, 8]. However these formulas involve an infinite sum of integrals without an explicit analytical solution. More recently, a method based on the numerical solution of a PDE for computing the expected MDD in the Black-Scholes framework has been proposed in Ref. [12]. Here a Monte Carlo code is proposed as a versatile tool to explore the higher order statistics available through the pdf of the MDD in cases that are hardly analytically treatable. In particular some relevant deviations with respect to the Brownian motion model are analyzed in order to quantify their impact on the statistics of the maximum drawdown: the increments of the underlying stochastic process (returns/log-returns) are not independent and present excess kurtosis (heavy tailed) and/or non zero skewness.

The work is organized as follows. Section II examines arithmetic processes driven by independent and identically distributed (iid) random variables. The case of Brownian motion with drift provides the framework where the Monte Carlo simulations are presented and validated against the analytical predictions. Section III analyzes the MDD statistics for stochastic processes obeying to a geometric model with iid increments. Section IV removes the iid ansatz: a general autoregressive model is introduced for the conditional mean and volatility of the innovations. The statistics of the maximum drawdown is numerically evaluated when varying the parameters that control the correlation, the symmetry and the tails of the increments distribution. In section V, a parametric study is performed on the expectation value of the MDD as a function of a number of parameters of practical interest. Section VI presents a comparison between the predictions of the MDD statistics and the historical time series for different asset classes. Finally sectionVII reports the preliminary conclusions of this analysis.

II. BROWNIAN MOTION WITH DRIFT

The value of a portfolio S_t is here assumed to follow the SDE

$$dS_t = \mu dt + \sigma dW_t \tag{1}$$

where W_t is a Wiener process, $0 \leq t \leq T$ and both the mean μ and the volatility σ are finite and constant. The maximum drawdown is defined as

$$\mathrm{MDD}_T = \max_{0\leq t\leq T} D_t \qquad \text{with} \qquad D_t = \max_{0\leq x\leq t} S_x - S_t \tag{2}$$

D_t is the drawdown from the previous maximum value at time t. An example is illustrated in Fig. 1. The behavior of the statistics of the MDD in this framework has been studied in [7]. An infinite series representation of its

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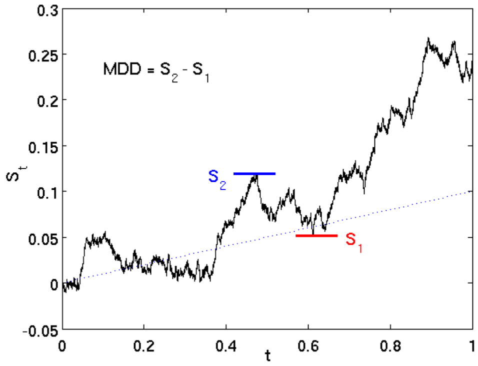

FIG. 1: MDD for a stochastic process obeying to the Eq. (1) with $\mu = 0.1$ and $\sigma = 0.1$.

distribution function is derived in the case of zero, negative and positive drift, and formulas for the expectation values for the MDD are given. In the limit of $T \to \infty$, it is found that the T dependence of $E\,(\mathrm{MDD})$ is logaritmic for $\mu > 0$, square root for $\mu = 0$ and linear for $\mu < 0$. Nevertheless, explicit analytical relations can not be immediately derived: it is required to numerically evaluate special integral functions in the case of $E\,(\mathrm{MDD})$, while the infinite series representation of the MDD distribution function involves the solution of particular eigenvalue conditions.
Here we prefer to employ a numerical simulation in order to compute the pdf of the MDD distribution function using a Monte Carlo scheme. One of the main advantages of this approach is the possibility to extend this method to different models for the underlying stochastic process.

The Monte Carlo code simulates the discretized version of Eq. (1) using the Euler scheme

$$S_{i+1} = S_i + \mu \Delta t + \sigma \sqrt{\Delta t} X_i \tag{3}$$

X_i are iid pseudo-random variables with normal distribution $\mathcal{N}\,(0,1)$, i.e. satisfying $E\,(X_i) = 0$ and $Var\,(X_i) = 1$; $\Delta t = T/N$, and N is the number of time steps in the series. The code is employed to compute the MDD defined

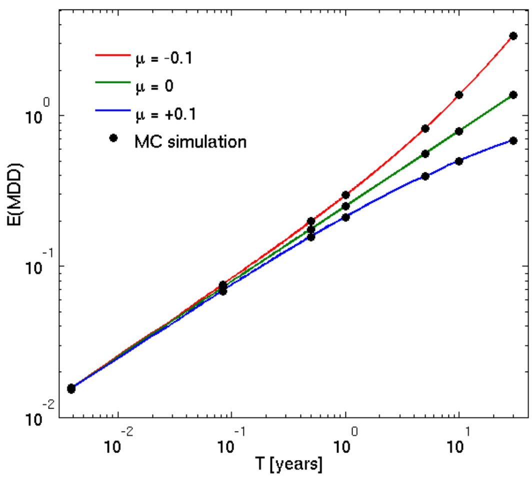

FIG. 2: $E\,(\mathrm{MDD})$ vs T, analytical expectations compared with the numerical results from the Monte Carlo simulations.

by the Eq. (2). In order to verify the goodness of this numerical approach, the code is tested against the analytical predictions of the expectation value of the MDD derived in Ref. [7]. One of the goals of this verification is to reproduce the transition from the logaritmic to the square root and the linear scaling for $T \to \infty$ when changing the drift from $\mu > 0$ to $\mu = 0$ and $\mu < 0$ respectively.
The parameters of this test case are: $\mu = -0.1,\ 0, +0.1$, $\sigma = 0.2$ and $T = 1$ d, 1 m, 6 m, 1 y, 5 y, 10 y, 30 y (1 year is

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considered as composed of 260 working days). The series that are simulated contain $N = 2 \cdot 10^4$ time steps and each simulation considers the statistics of $M = 4 \cdot 10^4$ samples. The estimator of the expectation value of the maximum drawdown is the mean of the MDD samples generated by the code. Figure 2 summarizes the numerical estimates in comparison with the analytical predictions derived in Ref. [7]. The simulations are able to reproduce very well the analytical expectations for $\mu < 0$, $\mu = 0$ and $\mu > 0$, with a relative error on $E\,(\mathrm{MDD})$ which stays always below 1%.

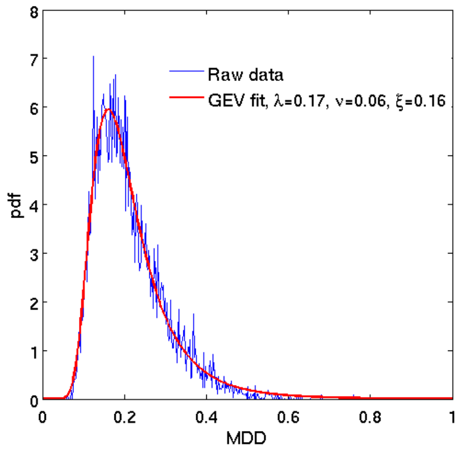

FIG. 3: $\mu = 0.1$, $\sigma = 0.2$, $T = 1$ y: pdf of the MDD obtained by the Monte Carlo simulation. The raw numerical data are fitted to a GEV distribution.

Recalling the definition of Eq. (2), the MDD is derived from the drawdown process D_t. The latter one is itself a Brownian motion reflected in its maximum; a rigorous justification of this statement can be found in Refs. [9] (p. 210) and [6]. Hence, if one is interested in the distribution function of the maxima of D_t (i.e. the maximum drawdown), the extreme value theory [3, 13] provides some powerful insight. Thanks to the Fisher-Tippett theorem, it is expected that the distribution function of the MDD converges to a generalized extreme value distribution (GEV), whose density is

$$H_\xi\,(x) = \begin{cases} \dfrac{1}{\nu}\left(1+\xi\dfrac{x-\lambda}{\nu}\right)^{-\frac{1+\xi}{\xi}} \exp\left[-\left(1+\xi\dfrac{x-\lambda}{\nu}\right)^{-\frac{1}{\xi}}\right] & \xi \neq 0 \\ \dfrac{1}{\nu}e^{-\frac{x-\lambda}{\nu}} \exp\left[-\exp\left(-\dfrac{x-\lambda}{\nu}\right)\right] & \xi = 0 \end{cases} \tag{4}$$

where $1+\xi\frac{x-\lambda}{\nu} > 0$; ξ is the shape parameter that controls the tail behavior of the distribution. For completeness, the first two moments of the GEV density are:

$$E\,(x) = \lambda + \frac{\nu}{\xi}\,(g_1 - 1) \qquad Var\,(x) = \frac{\nu^2}{\xi^2}\left(g_2 - g_1^2\right) \tag{5}$$

where $g_k = \Gamma\,(1-k\xi)$, $\Gamma\,(x)$ being the Gamma function.

The expectation of recovering the GEV density is well verified by the Monte Carlo simulations when computing the pdf of the MDD samples; the raw numerical results can in fact be fitted to the analytical expression (4). A nonlinear least squares fitting method is applied: λ, ν, ξ are the free parameters that are computed by the regression, obtaining a typical coefficient of determination $R^2 > 0.96$ for all the cases presented in this work. An example of the application of this procedure is shown in Fig. 3.

A good test for the code is also to sample the iid random variables (rvs) X_i appearing in Eq. (3) from a non normal distribution. We still require that $E\,(X_i) = 0$ and $Var\,(X_i) = 1$, but their skewness and kurtosis can take arbitrary values. In the continuous limit of $N \to \infty$ ($t = n\Delta t$) and in the case of zero drift $\mu = 0$, the P. Lévy's martingale characterization of Brownian motion theorem ([9] p. 156) establishes that S_t will still be a Brownian motion. In the more general case of $\mu \neq 0$, as suggested by the central limit theorem, the process S_t will be a semimartingale, i.e. again a Brownian motion with drift with marginal distribution $\mathcal{N}\left(\mu t, \sigma^2 t\right)$.

On the other hand, the drawdown process $D_n = \max_{0\leq j\leq n} S_j - S_n$ is a reflected random walk whose marginal distribution is the same as that of $-\min_{0\leq j\leq n} S_j$. This result as well other theorems for this kind of reflected processes are formally treated in Ref. [5]; Ref. [9] p. 210, 418 discusses in detail the reflected Brownian motion. As S_j is a discrete Brownian motion with drift, its marginal distribution is $\mathcal{N}\left(\mu j\Delta t, \sigma^2 j\Delta t\right)$. Therefore, provided that X_i are iid rvs with finite mean and variance and regardless of their moments higher than the second, the MDD for the arithmetic process described by the Eq. (3) on the time horizon T converges in the continuous limit of $N\to\infty$ to the statistics obtained in the case of the Brownian motion with drift, in terms of both density and expectation value.

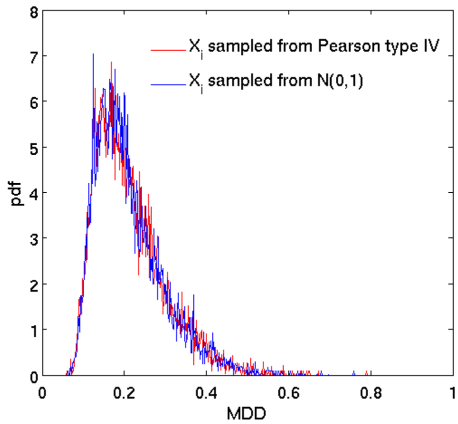

FIG. 4: $\mu = 0.1$, $\sigma = 0.2$, $T = 1$ y: pdfs of the MDD obtained by the Monte Carlo simulations sampling the iid random variables X_i from $\mathcal{N}(0,1)$ and from a Pearson type IV distribution with $s = -1.5$ and $k = 7$.

Within the Monte Carlo approach already introduced, the iid pseudo-random variables X_i of Eq. (3) can be sampled from a distribution whose third and fourth moments are non vanishing; the coefficients μ and σ and the first two moments are instead not modified. The code generates the iid sequence X_i from a Pearson type IV distribution satisfying the prescriptions $E(X_i) = 0$, $Var(X_i) = 1$, $\text{Skewness}(X_i) = s$ and $\text{Kurtosis}(X_i) = k$, where s and k can be arbitrarily prescribed. Some brief details around the Pearson family of distributions are reported in Appendix A. As expected, despite varying the skewness and the kurtosis of the underlying iid random variables, the statistics of the MDD computed by the Monte Carlo simulations systematically recovers the case of the Brownian motion with drift for a given set of μ, σ and T, as shown in Fig. 4.

III. GEOMETRIC PROCESS DRIVEN BY IID INCREMENTS

It is well known that, instead of the arithmetic model treated in the previous section, geometric processes are more appropriate to describe the financial time series. In terms of SDE, Eq. (1) will be then modified in the following way:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{6}$$

The latter equation describes a geometric Brownian motion. The natural extension of the definition of the maximum drawdown for a geometric model takes the form:

$$\text{MDD}_T = \max_{0\leq t\leq T} D_t \qquad \text{with} \qquad D_t = \frac{1}{\max_{0\leq x\leq t} S_x}\left(\max_{0\leq x\leq t} S_x - S_t\right) \tag{7}$$

The maximum drawdown is then the maximum loss relative to the previous peak; consequently, both the relations $0 \leq D_t < 1$ and $0 \leq \text{MDD}_T < 1$ hold.

Itô's calculus is a powerful instrument to handle the dynamics of Eq. (6); it is in fact possible to write $S_t = \exp(\tilde{X}_t)$, along with the SDE:

$$d\tilde{X}_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \tag{8}$$

Thanks to this result, the geometric Brownian motion (6) can be effectively treated through an arithmetic model for the process $\tilde{X}_t$ with an adjusted drift $\tilde{\mu} = \mu - \frac{1}{2}\sigma^2$. For the same reason, the analytical results of Ref. [7] concerning the expectation value and the distribution function of the MDD, can be immediately extended to the case of a geometric Brownian motion according to the new definition of Eq. (7).

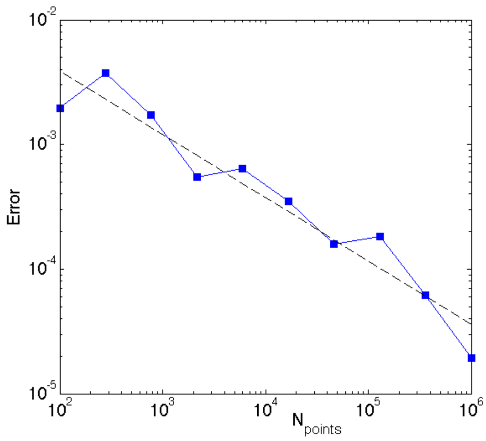

FIG. 5: Numerical simulations of a geometric process have been performed in 2 different ways: S_i^{direct} through the direct discretization of (9) and S_i^{Ito} obtained by discretizing Eq. (10) using the Itô's formula, where $i = 1...N_{points}$; in both cases the iid random variables are sampled from a Pearson type IV distribution with $s = -1.5$ and $k = 7$, moreover $\mu = 0.1$ and $\sigma = 0.2$. The plot shows the normalized relative error defined as Error $= \sqrt{\sum_{i=1}^{N}\left(S_i^{\text{Ito}} - S_i^{\text{direct}}\right)^2} / \sqrt{\sum_{i=1}^{N}\left(S_i^{\text{direct}}\right)^2}$ as a function of the number of samples. A power law fit of the type Error $\propto N^{-\alpha}$ is also displayed with $\alpha = -0.5$.

Here we are interested in considering

$$\frac{dS_t}{S_t} = \mu dt + \sigma dP_t \tag{9}$$

where P_t is a process formed by the sum of iid random variables sampled from a non normal distribution, with the assumptions $E\left(P_t\right) = 0$ and $Var\left(P_t\right) = 1$, but with arbitrary skewness and kurtosis. In the light of the previously cited Lévy's martingale characterization theorem, P_t is a Brownian motion. The Itô's formula applies if we can still write $S_t = f(\tilde{X}_t)$, where f is a C^2 function and the process $\tilde{X}_t$ is a semimartingale ([9] p. 149). It is easy to prove that the choice $f(x) = \exp(x)$ and

$$d\tilde{X}_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dP_t \tag{10}$$

fulfills these requirements; $\exp(x)$ is in fact a C^2 function and the process $\tilde{X}_t$ is a semimartingale, since it can be decomposed into a local martingale and a drift term of finite variation. Therefore the Itô's calculus prescribes that S_t is a semimartingale too and that the statistical dynamics of Eq. (9) is equivalent to the SDE (10), with $S_t = \exp(\tilde{X}_t)$. Consequently if the process P_t is driven by iid increments sampled from a non normal distribution satisfying the previous requirements, it is expected that the Euler discrete version of Eq. (9) is equivalent to the discretization of Eq. (10) with $S_t = \exp(\tilde{X}_t)$ when using a big enough number of discretization points N. The latter statement has been verified with a numerical simulation, evaluating the relative error between the two different approaches as a function of N: the results are reported in Fig. 5. It is however important to note that it is not possible to derive a direct generalization of the Itô's formula for any arbitrary distribution of the log-increments. This point is treated in detail in Ref. [10], where it is shown that only a weaker formula for the expectation value of such a process can be obtained.

As there is a formal proof that, under the previous hypotheses in the continuous limit, the geometric process (9) is equivalent to the arithmetic one (10), the same argument of the previous section can be applied for inferring the statistics of the MDD. In particular, the moments higher than the second of the iid rvs that compose the Brownian motion P_t do not affect the statistics of the MDD defined in (7) for the geometric process (9).

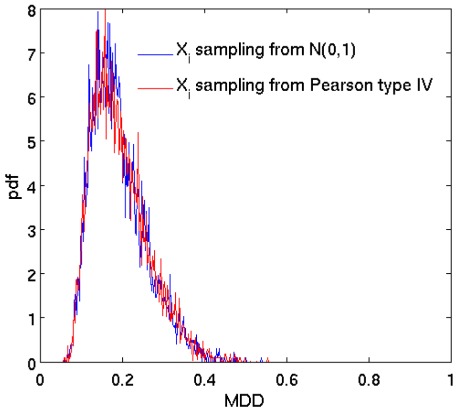

FIG. 6: $\mu = 0.1$, $\sigma = 0.2$, $T = 1$ y: pdfs of the MDD obtained by the Monte Carlo simulations for the geometric process (11) sampling the iid random variables X_i from $\mathcal{N}(0,1)$ and from a Pearson type IV distribution with $s = -1.5$ and $k = 7$.

This statement can still be recovered by the Monte Carlo simulations. The discrete version of the Eq. (9) here used is the Euler scheme

$$S_{i+1} = S_i \left(1 + \mu\Delta t + \sigma\sqrt{\Delta t} X_i\right) \tag{11}$$

As before, X_i is a sequence of iid pseudo-random variables from a Pearson type IV distribution whose first 2 moments satisfy $E(X_i) = 0$ and $Var(X_i) = 1$ ($\Delta t = T/N$). The case of sampling from the standard normal distribution $\mathcal{N}(0,1)$ is recovered when all the higher moments are vanishing.

The pdf of the MDD for the geometric model (11) has been computed exploring the following parametric space: $\mu = [-0.1, 0, +0.1]$, $\sigma = [0.1, 0.2]$, $s = [-1.3, 0]$, $k = [0, 5]$, $T = [1\text{ d}, 1\text{ y}, 10\text{ y}]$. As expected, for any set of μ, σ and T, no difference has been found when changing the skewness and/or the kurtosis of the underlying X_i, as shown in Fig 6. Obviously, as the whole pdf of the MDD does not vary, $E(\text{MDD})$ remains the same.

IV. AUTOREGRESSIVE PROCESS

In order to get a deviation from the MDD statistics of the Brownian motion with drift it is necessary to relax the strong hypothesis of iid rvs for the increments (log-returns). In this section this is done using an autoregressive model [1] for the time series. In particular we consider the following AR(1)-GARCH(1,1) model:

$$Y_i = \mu_i + \epsilon_i \tag{12}$$
$$\epsilon_i = \sigma_i Z_i \tag{13}$$
$$Z_i \equiv \text{iid rvs} \qquad \text{with} \qquad E(Z_i) = 0 \qquad Var(Z_i) = 1 \tag{14}$$

$i = 0, 1, 2...N$ ($\Delta t = T/N$), together with the prescriptions for the conditional mean and volatility

$$\mu_i = c + \phi(Y_{i-1} - c) \tag{15}$$
$$\sigma_i^2 = \alpha_0 + \alpha_1\epsilon_{i-1}^2 + \beta_1\sigma_{i-1}^2 \tag{16}$$

with $\alpha_0, \alpha_1, \beta_1 > 0$, $\alpha_1 + \beta_1 < 1$ and $|\phi| < 1$. The moments of Z_i higher than the second are explicitly not fixed here, since the skewness and the kurtosis will be varied in the following. The model is defined by the five parameters $c, \phi, \alpha_0, \alpha_1, \beta_1$.

The AR(1)-GARCH(1,1) series Y_i is considered as a sequence of log-returns; therefore, to reconstruct the portfolio value trajectory we use the discrete geometric model

$$S_{i+1} = S_i(1 + Y_i) \tag{17}$$

There is a clear analogy between the latter formulation and the previous geometric model (11) driven by iid innovations. To make a direct parallelism between them it is useful to preserve the unconditional mean μ and volatility σ over the time interval $T = N\Delta t$; this is done fixing the parameters α_0 and c of the AR(1)-GARCH(1,1) series according to

$$\alpha_0 = \sigma^2 \Delta t \left(1 - \alpha_1 - \beta_1\right) \tag{18}$$
$$c = \mu \Delta t \tag{19}$$

Therefore, for a given set of unconditional μ and σ on the time horizon T, the prescriptions (18) - (19) guarantee that the autoregressive model Eqs.(12) - (17) is completely specified by the three coefficients α_1, β_1, ϕ and comparable with the iid innovations driven model (11). The goal is in fact to evaluate how the MDD distribution is affected passing from one model to the other one. Ultimately, if Z_i has the same distribution of X_i appearing in Eq. (11) and $\alpha_1 = \beta_1 = \phi = 0$, the two models describe exactly the same statistical dynamics.

This autocorrelated geometric process is used to study the statistics of the MDD when changing the coefficient ϕ

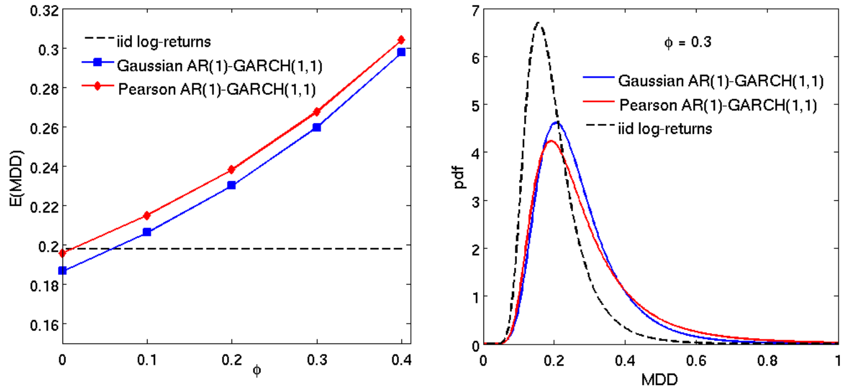

FIG. 7: Unconditional $\mu = 0.1$, $\sigma = 0.2$, $T = 1$ y. Left plot: $E\,(\mathrm{MDD})$ vs ϕ computed by the Monte Carlo code considering the AR(1)-GARCH(1,1) model and sampling the iid Z_i from $\mathcal{N}\,(0,1)$ or from a Pearson type IV distribution with $s = -1$ and $k = 0$. Right plot: comparison of the MDD pdf fitted with a GEV at fixed $\phi = 0.3$.

that controls the serial correlation of the log-returns. In this case we have fixed the following parameters: $\alpha_1 = 0.0897$ and $\beta_1 = 0.9061$ (these values come from the best fit of the daily log-returns of the S&P 500 between May 16, 1995 to April 29, 2003 to a GARCH(1,1) model [15]), the unconditional mean $\mu = 0.1$, the unconditional standard deviation $\sigma = 0.2$ and the time horizon $T = 1$ y. The Monte Carlo simulations are run sampling the iid Z_i either from $\mathcal{N}\,(0,1)$ or from a Pearson type IV distribution with $s = -1$ and $k = 0$. The kurtosis of the Pearson distribution has not been increased since the previous parameters of the AR(1)-GARCH(1,1) model already imply a high kurtosis; in particular the skewness implied by the model for the series of log-returns is 0 in the first case and ≈ -1.7 in the second one, while the kurtosis is ≈ 12 in both cases. These results are summarized in Fig. 7. The serial correlation has a quite strong impact on the expected maximum drawdown, while it appears that the skewness added in the series of log-returns increases the $E\,(\mathrm{MDD})$ at maximum by 5% with respect to the Gaussian AR(1)-GARCH(1,1) expectation. The analysis of this kind of autoregressive models still deserves further attention.

V. PARAMETRIC STUDIES

Another possible application of the Monte Carlo simulations presented in the previous sections is to study how the statistics of the MDD is related to a number of parameters of interest. Here the following dependences are explored when dealing with iid or autocorrelated log-returns:

1. $\frac{\mathrm{E(MDD)}}{\sigma}$ vs T.
2. $\frac{\mathrm{E(MDD)}}{\sigma}$ vs $\frac{\mu}{\sigma}$, μ/σ being the Sharpe ratio.
3. $\frac{\mu T}{\mathrm{E(MDD)}}$ vs $\frac{\mu}{\sigma}$, i.e. the Calmar ratio vs the Sharpe ratio.

The first case is summarized in Fig. 8, considering the time horizons $T = 1$ d, 1 w, 1 m, 6 m, 1 y. We use the unconditional mean and volatility $\mu = 0.1$ and $\sigma = 0.2$. The AR(1)-GARCH(1,1) model of the previous section is applied to generate skewed log-returns (Z_i are sampled from a Pearson distribution with $s = -1$); $\alpha_1 = 0.0897$ and

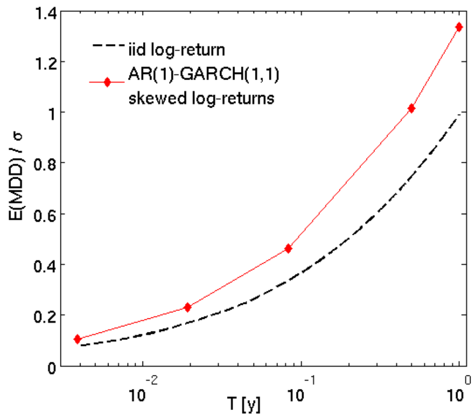

FIG. 8: Unconditional $\mu = 0.1$, $\sigma = 0.2$. $E\,(\mathrm{MDD})\,/\sigma$ vs T considering the log-returns as iid and as a AR(1)-GARCH(1,1) series with skewness ($\alpha_1 = 0.0897$, $\beta_1 = 0.9061$, $\phi = 0.3$).

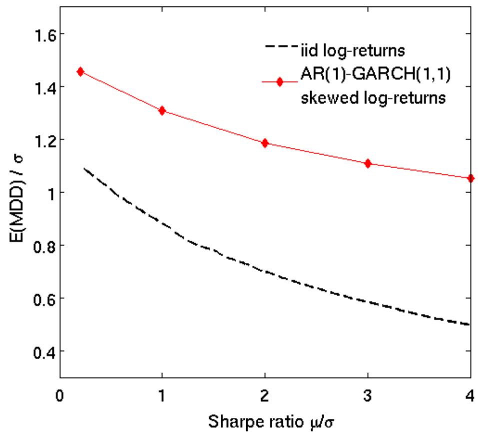

FIG. 9: $T = 1$ y. $E\,(\mathrm{MDD})\,/\sigma$ vs the Sharpe ratio μ/σ considering the log-returns as iid and as a AR(1)-GARCH(1,1) series with skewness ($\alpha_1 = 0.0897$, $\beta_1 = 0.9061$, $\phi = 0.3$).

$\beta_1 = 0.9061$ as before, and $\phi = 0.3$.

The results of the dependence of $E\,(\mathrm{MDD})\,/\sigma$ on the Sharpe ratio are reported in Fig. 9. The time horizon is $T = 1$ y and the change in the Sharpe ratio is obtained fixing the unconditional $\sigma = 0.15$ while varying the unconditional mean of the log-returns. The parameters for the AR(1)-GARCH(1,1) are the same as before. As it is clear from the figure, the Monte Carlo simulations find that the expected MDD for the autocorrelated process decays more slowly with μ/σ with respect to the case of iid innovations.

The last plot, Fig. 10, refers to the dependence of the Calmar ratio on the Sharpe ratio for the same parameters of the previous figure. The picture highlights how the risk adjusted returns are lower when the underlying series of log-returns presents positive autocorrelation and negative skewness.

Coming back to the basic hypothesis of iid log-returns with finite and constant mean and volatility described in Sec. III, despite the simplicity of such a framework it is still of practical interest to derive an heuristic approximation for the behavior of $E\,(\mathrm{MDD})$ as a function of μ, σ and T. A simple approach could be to assume that this function can be written in terms of a power law, i.e.

$$E\,(\mathrm{MDD}) = f\,(\mu, \sigma, T) \approx C_0 \mu^{c_1} \sigma^{c_2} T^{c_3} \tag{20}$$

In this framework the function $f\,(\mu, \sigma, T)$ has an analytical expression [7], but the actual value is anyway obtained through numerically computed functions. On the other hand it is possible to apply a multivariate linear regression to

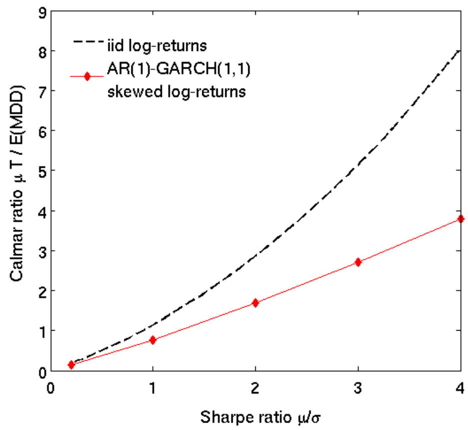

FIG. 10: $T = 1$ y. The Calmar ratio $\mu T/E\,(\mathrm{MDD})$ vs the Sharpe ratio μ/σ considering the log-returns as iid and as a AR(1)-GARCH(1,1) series with skewness ($\alpha_1 = 0.0897$, $\beta_1 = 0.9061$, $\phi = 0.3$).

fit the expected MDD to the power law Eq. (20).
Here we focus the attention on the regime of $\mu > 0$. Using the parametric space $[0.02 < \mu < 0.1]$, $[0.05 < \sigma < 0.2]$, $[1 \text{ d} < T < 6 \text{ m}]$ a linear regression has been performed on the analytical predictions, obtaining the following estimates for the coefficients of Eq. (20):

$$C_0 = 1.0305 \qquad c_1 = -0.0431 \qquad c_2 = 1.0416 \qquad c_3 = 0.4704 \tag{21}$$

The mean of the error between the power law with coefficients (21) and the analytical results is $3.75 \cdot 10^{-4}$ while the error standard deviation is 0.028. In principle a similar approach can be applied for the case of autoregressive processes, but this would require the creation of a large database of results computed by the Monte Carlo simulations.

VI. COMPARISON AGAINST HISTORICAL FINANCIAL TIME SERIES

For the purposes of the comparison against the historical financial data, the model for the underlying time series S_t is the following GARCH(1,1) one:

$$S_{t+1} = S_t\,(1 + Y_t) \tag{22}$$
$$Y_t = k + \epsilon_t \tag{23}$$
$$\epsilon_t = \sigma_t Z_t \tag{24}$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{25}$$

where Z_t are iid random variables sampled from a Student's T distribution with ν degrees of freedom and unitary variance; the drift k is a constant, $\alpha_0, \alpha_1, \beta_1 > 0$ and $\alpha_1 + \beta_1 < 1$. With respect to the model previously described in Sec. IV by the Eqs. (12) - (17), here we consider a vanishing autocorrelation of the increments and a less generic distribution for the innovations (Student's T instead of Pearson type IV): despite this simplification the latter model is able fit reasonably well the historical series of daily returns here analyzed, as it will be shown in the following.
The Monte Carlo code is used to evaluate the distribution function of the maximum drawdown $\mathrm{MDD_T}$ on the time horizon T. The simulations presented in this section consider the statistics of $M = 1 \cdot 10^6$ MDD samples, each one obtained from a time series composed by $N = 50 \cdot 10^3$ steps. The numerical predictions are compared against 6 historical financial time series, namely S&P500, CAC40, Credit Suisse, IBM, US Dollar/Japanese Yen, US treasury 30-years yield. The analysis is composed by the following steps:

1. The historical data at daily frequency have been considered on a 8 years period, namely from 1 January 2002 to 31 December 2009 (nearly 2000 points). The log-returns of the closing prices are used to perform a maximum likelihood fit to a GARCH(1,1) model with Student's T distributed innovations (see the above equations).

2. The parameters computed by the best fit are used to run the Monte Carlo code. The simulation evaluates the statistics of the $\mathrm{MDD_T}$, providing in particular the expectation value, 95% and 99% confidence intervals for the maximum drawdown.

3. *In-sample test:* The code results are compared to the historical $\mathrm{MDD_T}$ (the latter one is calculated using high and low intra-day prices) observed within the time interval used for the fit of point 1. 4 reference time horizons are considered: 10 days, 40 days (2 months), 250 days (1 year) and 2000 days (8 years).

4. *Out-of-sample test:* Using the same fit parameters obtained at point 1, the code is tested against the historical MDD in the period between 1 January 2010 and 15 March 2010. The time horizons here considered are: 10, 20 and 40 days.

The following table summarizes the number of points for each series in the 8 years period and the annualized mean and volatility of the log-returns.

Index	No. points	Mean [%]	Volatility [%]
S&P500	2015	-0.43	22.24
CAC40	2047	-1.85	25.01
Credit Suisse	2014	+1.66	48.44
IBM	2015	+0.93	26.31
USD/JPY	2057	-4.21	10.48
US 30y yield	2011	-2.25	20.35

The next table summarizes the results from the maximum likelihood fit to a GARCH(1,1) model with Student's T innovations on the previous time series (LLK in the table is the log-likelihood value obtained from the fit).

Index	k	α_0	α_1	β_1	ν	LLK
S&P500	4.89E-4	5.46E-7	0.0676	0.9298	9.90	6365.83
CAC40	6.49E-4	1.33E-6	0.0858	0.9091	11.97	6110.21
CS	6.95E-4	3.17E-6	0.0833	0.9151	6.85	4896.05
IBM	2.76E-4	1.65E-6	0.0591	0.9341	5.96	5828.22
USD/JPY	-1.29E-5	3.83E-7	0.0389	0.9534	6.14	7543.11
US 30y	-1.86E-4	5.83E-7	0.0404	0.9561	12.79	6204.93

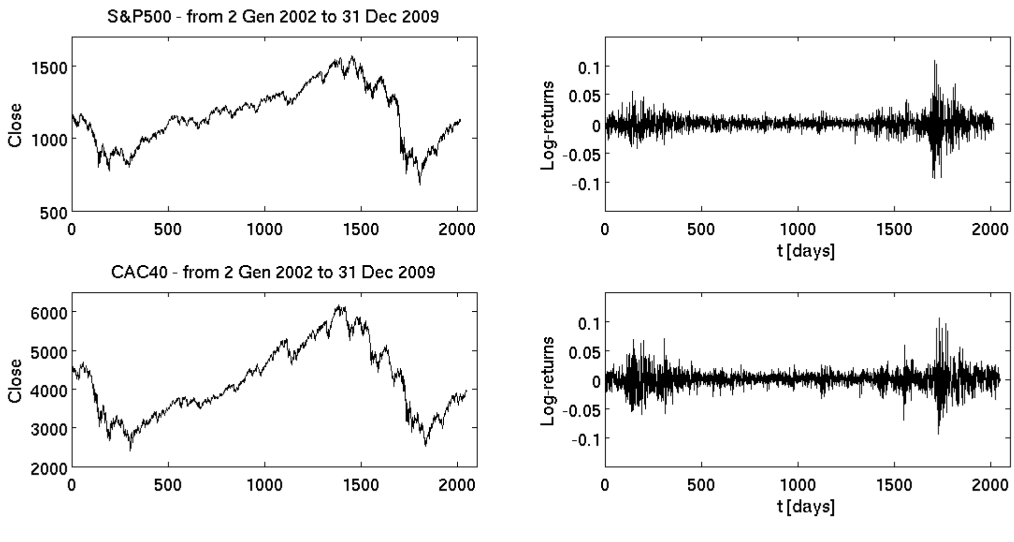

FIG. 11: S&P500 and CAC40 close prices in the time interval of interest.

The plots of Figs. 14, 15, 16, 17, 18, 19 report the in-sample comparison between the expectations of the Monte Carlo code and the maximum drawdown observed in the real financial time series.

Using the parameters obtained by the maximum likelihood fit on the data between 1 Jan 2002 and 31 Dec 2009, the code is now employed to give an out-of-sample prediction on the historical MDD values in the period between 1

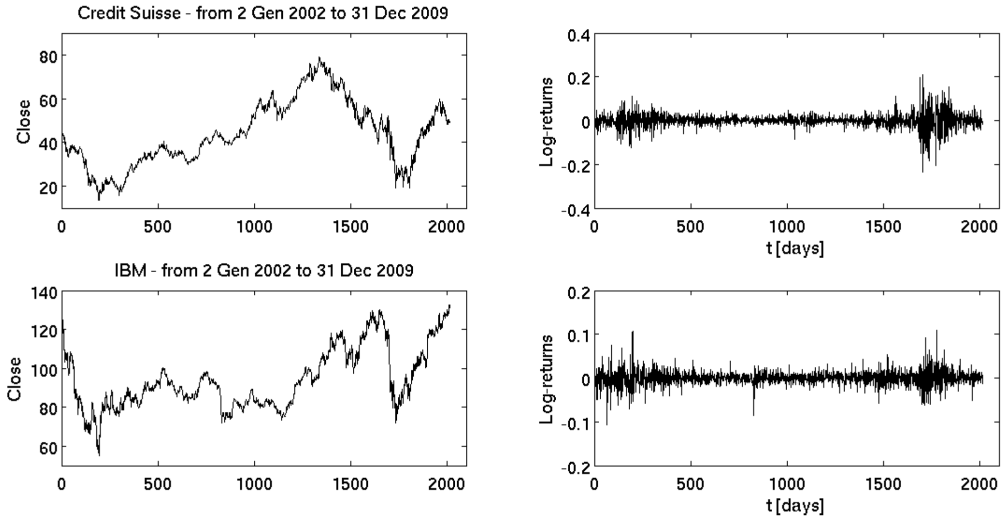

FIG. 12: Credit Suisse and IBM close prices in the time interval of interest.

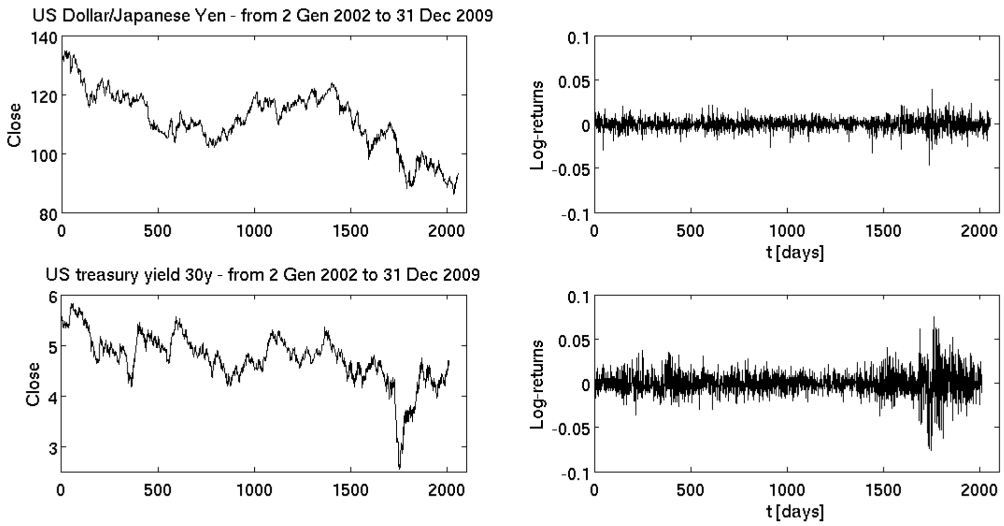

FIG. 13: US Dollar/Japanese Yen and US Treasury 30-years yields close values in the time interval of interest.

Jan 2010 and 15 Mar 2010. The results are shown in Figs. 20, 21, 22, 23, 24, 25.

In conclusion, the Monte Carlo code has been used to evaluate a total number of 42 MDD observations (24 in the 1 Jan 2002 to 31 Dec 2009 period and 18 in the 1 Jan 2010 to 15 Mar 2010 period). Hence the expected number of violations from the model is 2.1 when working with 95% CI, and 0.42 when working with 99% CI. The present analysis reported 3 violations from the 95% CI and 0 violations from the 99% CI.

VII. CONCLUSIONS

The preliminary conclusions on the present work can be summarized as it follows.
The analytical predictions for the expectation value of the maximum drawdown in the case of a Brownian motion with drift are very well recovered by the Monte Carlo code here presented. The numerical simulations allow also to consistently recover the MDD distribution function in terms of a generalized extreme value distribution.

In the continuous limit of the discretized version of a geometric stochastic process, the MDD statistics is shown to be unaffected by the moments higher than the second that characterize the distribution of the underlying increments provided that they are iid. This result is coherent with the expectations from the central limit theorem and the Itô's formula and it is recovered by the Monte Carlo code.

When relaxing the iid constraint for the increments of the process and considering an autoregressive AR(1)-GARCH(1,1) model, the MDD statistics is strongly affected; for example, in the cases here examined, the expectation value of the MDD increases almost linearly with the serial correlation coefficient.

In both cases of geometric Brownian motion and autoregressive stochastic process, some parametric studies have been performed in order to derive the heuristic behavior of the MDD expected value as a function of time, mean return, volatility and Sharpe ratio.

Finally, under the hypothesis of a GARCH(1,1) process driven by Student's T distributed innovations, the MDD statistics predicted by the Monte Carlo code has been compared to the events observed in different historical financial time series. The relevance of this comparison relies not only on the MDD expectation value but most of all on its confidence interval (at 95% and 99%), available through the MDD distribution function derived from the simulation. Both the in-sample and the out-of-sample tests show a good match between the numerical expectations and the historical data.

Appendix A: Some details on the Pearson distributions

The Pearson family of distributions [11] is particularly well suited to generate random variables with arbitrary skewness and kurtosis. Seven distribution types are in fact encompassed in the Pearson system; particular members of that family are normal, beta, gamma, Student's t, F, inverse Gaussian and Pareto distributions. The general probability density function $p(x)$ of a Pearson distribution is defined through the differential equation

$$\frac{d \log p(x)}{dx} = \frac{x - \alpha}{a_0 + a_1 x + a_2 x^2} \tag{A1}$$

The normal distribution is for example recovered for $a_1 = a_2 = 0$.

To our purposes, the Pearson type IV distribution is of particular importance and some its properties will be here briefly recalled. When $a_1 \neq 0$ and $a_2 \neq 0$, the roots of the denominator of Eq. (A1) are in general the complex quantities $b \pm ia$. The density of the Pearson type IV distribution can be written as

$$p(x) \propto \left(1 + \frac{x^2}{a^2}\right)^{-m} \exp\left[\delta \arctan\left(\frac{x}{a}\right)\right] \tag{A2}$$

where $m = -\frac{1}{2} a_2$ and $\delta = (b - a)/a a_2$. Eq. (A2) is an asymmetric leptokurtic density function. Denoting the skewness by s and the kurtosis by k (only in this context $k = 3$ for the normal distribution), the following relations between the parameters of the Pearson type IV distribution (A2) and s and k can be derived:

$$r = \left[\frac{6\left(k - s^2 - 1\right)}{2k - 3s^2 - 6}\right] \qquad \text{with} \qquad r = 2m - 2 \tag{A3}$$

$$\delta = \frac{r(r-2)s}{\sqrt{16(r-1) - s^2(r-2)^2}} \tag{A4}$$

$$a = \sqrt{\frac{\mu_2}{16}\left[16(r-1) - s^2(r-2)^2\right]} \tag{A5}$$

where μ_2 is the second central moment. The parameters α, a_0, a_1, a_2 appearing in the differential equation (A1) can then be determined by fixing the first 4 moments of the distribution.

The Pearson family of distributions is hence used in this work in order to generate a sequence of iid random variables given a set of mean μ, standard deviation σ, skewness s and kurtosis k for the underlying density.

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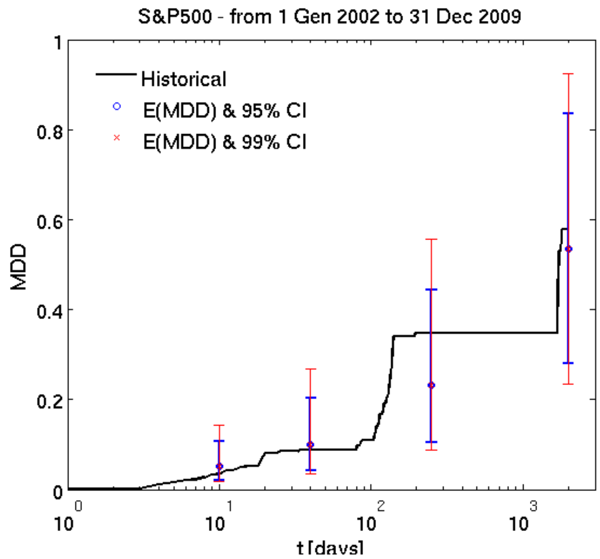

FIG. 14: The time evolution of the MDD of the historical S&P500 series between 1 Jan 2002 and 31 Dec 2009. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

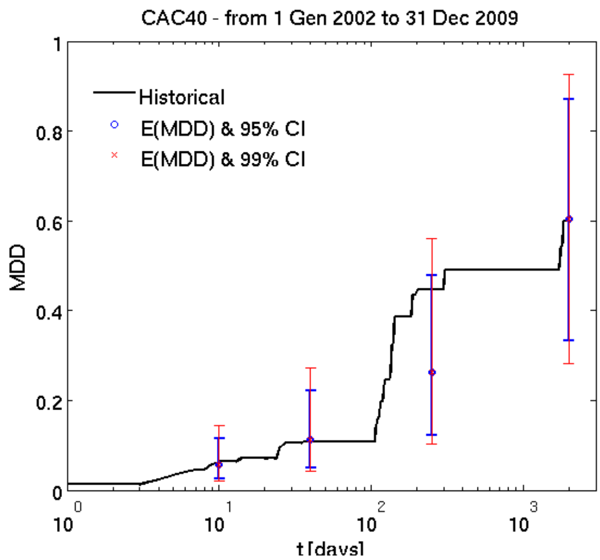

FIG. 15: The time evolution of the MDD of the historical CAC40 series between 1 Jan 2002 and 31 Dec 2009. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

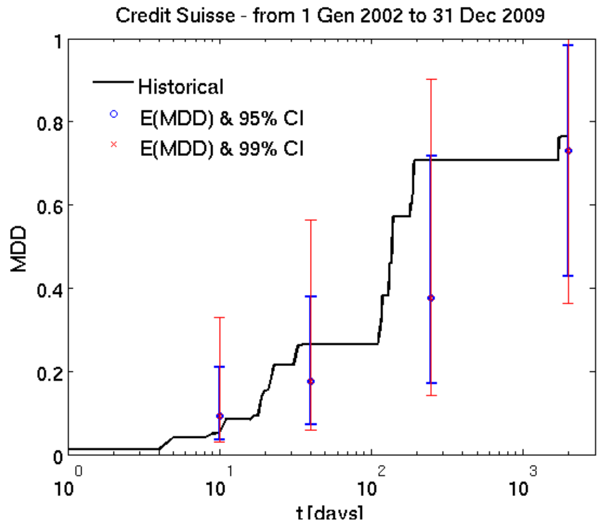

FIG. 16: The time evolution of the MDD of the historical Credit Suisse series between 1 Jan 2002 and 31 Dec 2009. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

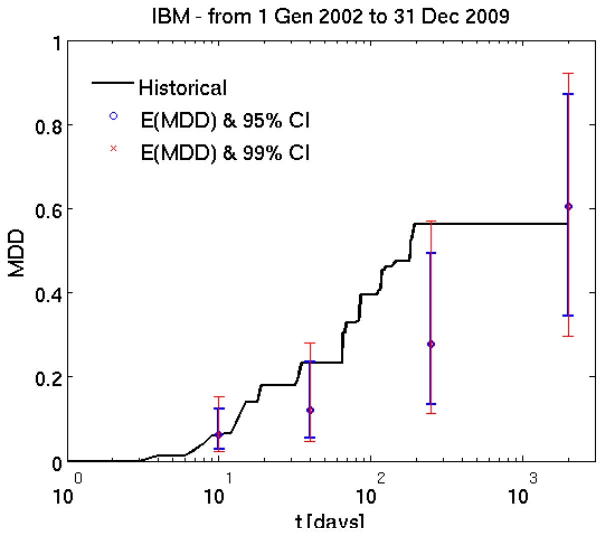

FIG. 17: The time evolution of the MDD of the historical IBM series between 1 Jan 2002 and 31 Dec 2009. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

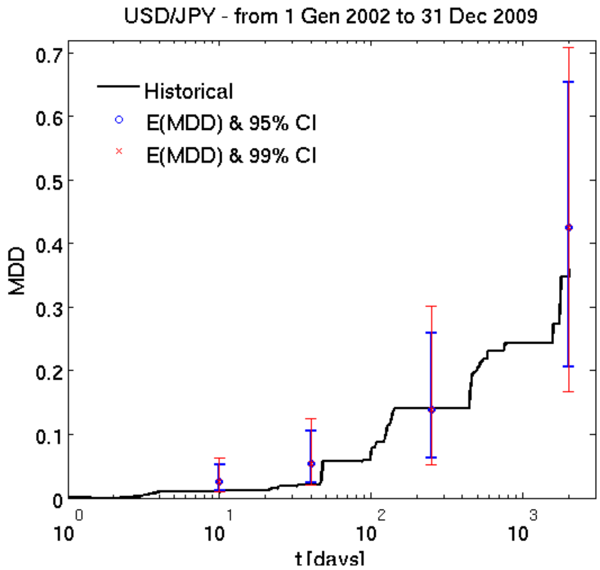

FIG. 18: The time evolution of the MDD of the historical USD/JPY series between 1 Jan 2002 and 31 Dec 2009. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

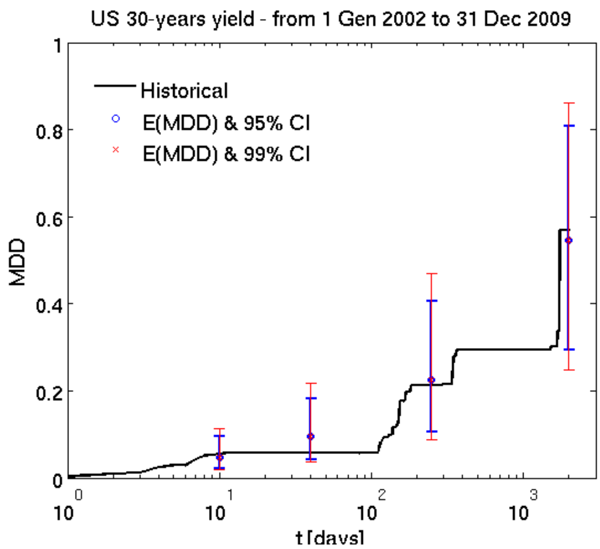

FIG. 19: The time evolution of the MDD of the historical US 30-years yield series between 1 Jan 2002 and 31 Dec 2009. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

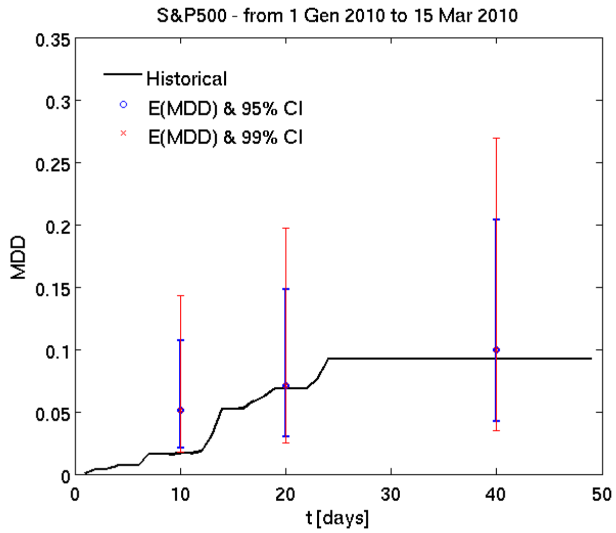

FIG. 20: The time evolution of the MDD of the historical S&P500 series between 1 Jan 2010 and 15 Mar 2010. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

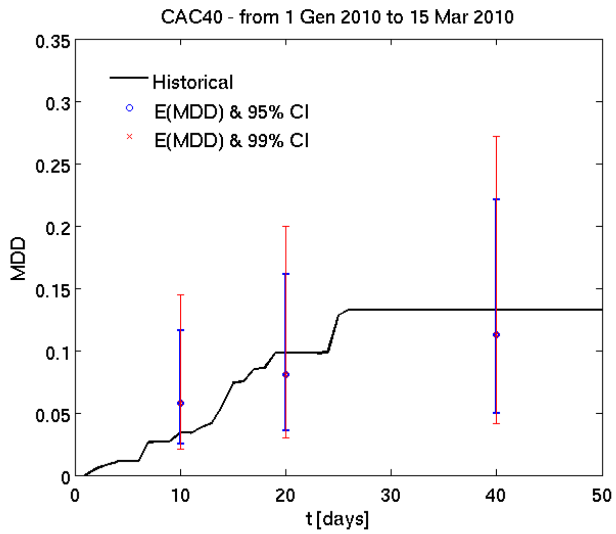

FIG. 21: The time evolution of the MDD of the historical CAC40 series between 1 Jan 2010 and 15 Mar 2010. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

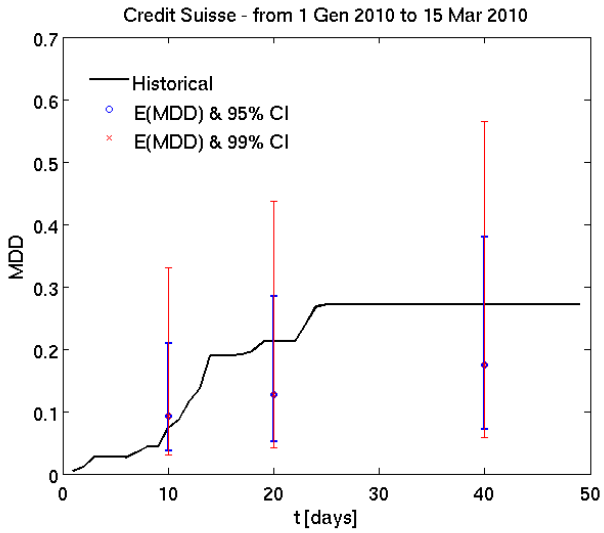

FIG. 22: The time evolution of the MDD of the historical Credit Suisse series between 1 Jan 2010 and 15 Mar 2010. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

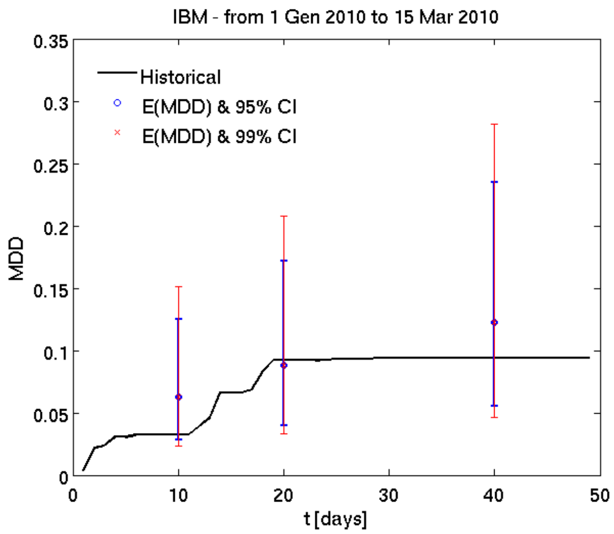

FIG. 23: The time evolution of the MDD of the historical IBM series between 1 Jan 2010 and 15 Mar 2010. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

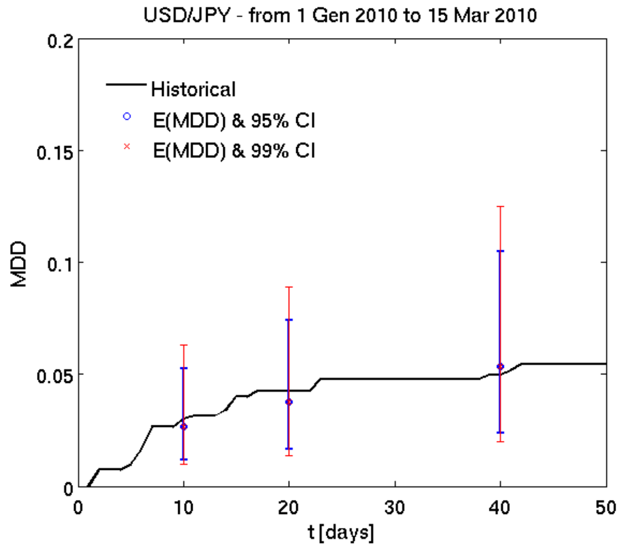

FIG. 24: The time evolution of the MDD of the historical USD/JPY series between 1 Jan 2010 and 15 Mar 2010. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.

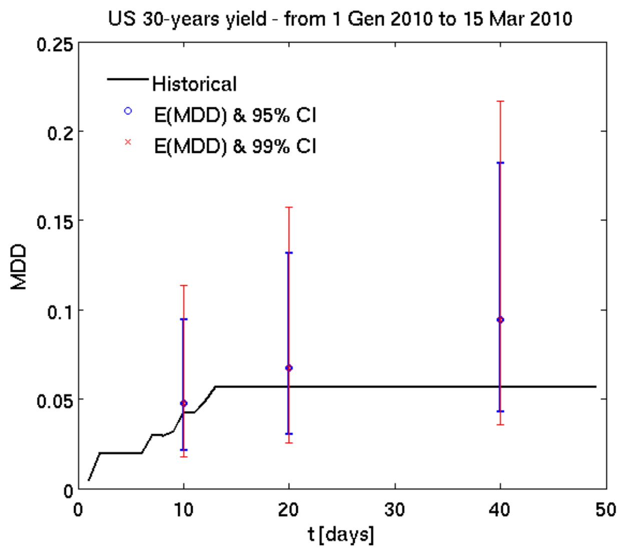

FIG. 25: The time evolution of the MDD of the historical US 30-years yield series between 1 Jan 2010 and 15 Mar 2010. The real data are compared with the expectation value and the confidence intervals computed by the Monte Carlo code.