SHRINKAGE ESTIMATORS – LOCATION PARAMETER

Risk and Asset Allocation **- Springer –** `symmys.com`

Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

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$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \qquad (4.132)$$

$$\widehat{\boldsymbol{\mu}}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t . \qquad (4.135)$$

$$\widehat{\boldsymbol{\mu}}[I_T] \sim \mathrm{N}\left(\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{T}\right) \qquad (4.102)$$

$$\mathrm{Err}^2(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) = \frac{1}{T}\mathrm{tr}(\boldsymbol{\Sigma}) \qquad (4.136)$$

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$$\boxed{\widehat{\boldsymbol{\mu}}^S \equiv (1-\alpha)\,\widehat{\boldsymbol{\mu}} + \alpha\mathbf{b}.} \qquad (4.138)$$

$$\alpha \equiv \frac{1}{T}\frac{N\overline{\lambda} - 2\lambda_1}{(\widehat{\boldsymbol{\mu}} - \mathbf{b})'(\widehat{\boldsymbol{\mu}} - \mathbf{b})} \qquad (4.139)$$

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$$\boldsymbol{\Sigma} \mapsto \widehat{\boldsymbol{\Sigma}}.$$

$$\mathbf{b} \mapsto \frac{\mathbf{1}'\widehat{\boldsymbol{\mu}}}{N}\mathbf{1} \qquad (4.141)$$

$$\mathbf{b} \mapsto \frac{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\widehat{\boldsymbol{\mu}}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\mathbf{1}. \qquad (4.142)$$

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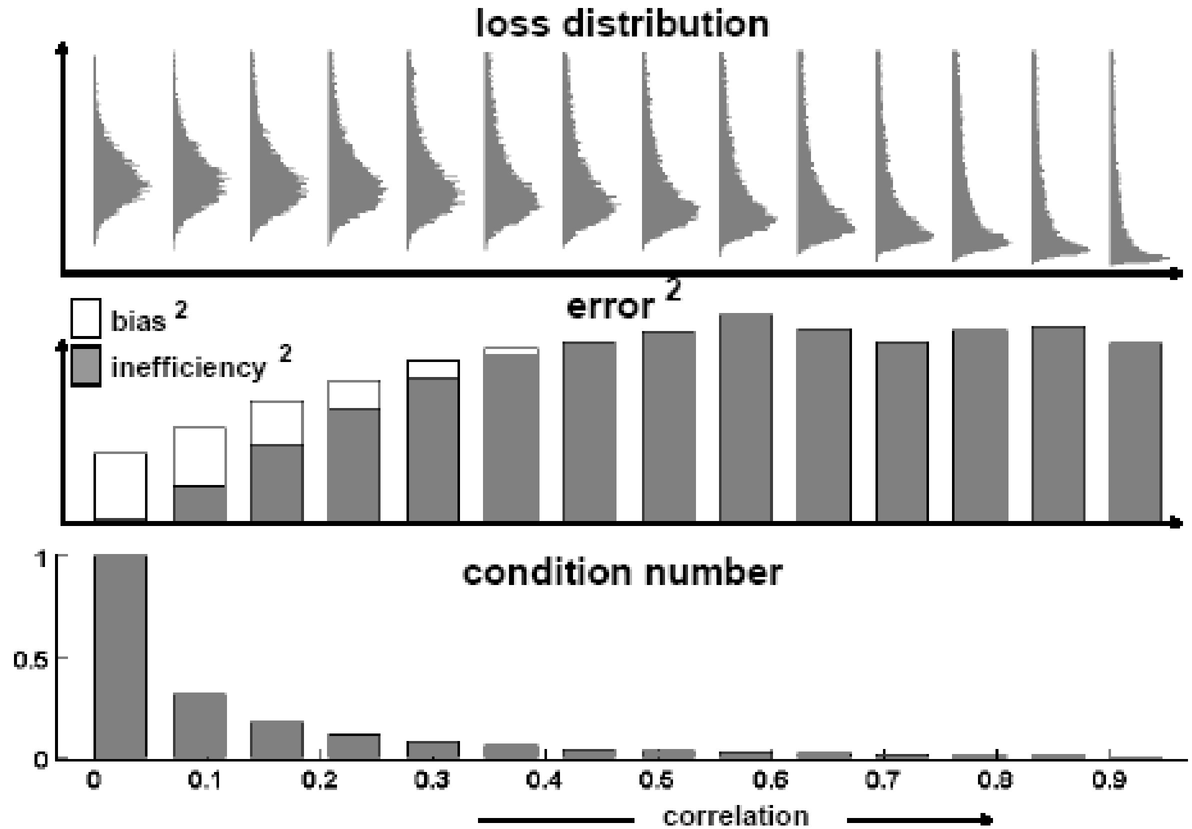

Fig. 4.13. Shrinkage estimator of mean: evaluation

$$\boldsymbol{\Sigma} \mapsto \widehat{\boldsymbol{\Sigma}}.$$

$$\mathbf{b} \mapsto \frac{\mathbf{1}'\widehat{\boldsymbol{\mu}}}{N}\mathbf{1} \qquad (4.141)$$

$$\mathbf{b} \mapsto \frac{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\widehat{\boldsymbol{\mu}}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\mathbf{1}. \qquad (4.142)$$

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$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.143)$$

$$\widehat{\boldsymbol{\Sigma}}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T}[\mathrm{x}_t - \widehat{\boldsymbol{\mu}}[i_T]][\mathrm{x}_t - \widehat{\boldsymbol{\mu}}[i_T]]' . \quad (4.146)$$

$$T\widehat{\boldsymbol{\Sigma}}[I_T] \sim \mathrm{W}(T-1, \boldsymbol{\Sigma}) \quad (4.103)$$

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$$\widehat{\mathrm{CN}}\{\mathbf{X}\} \equiv \frac{\widehat{\lambda}_N}{\widehat{\lambda}_1} < \frac{\lambda_N}{\lambda_1} \equiv \mathrm{CN}\{\mathbf{X}\} \quad (4.156)$$

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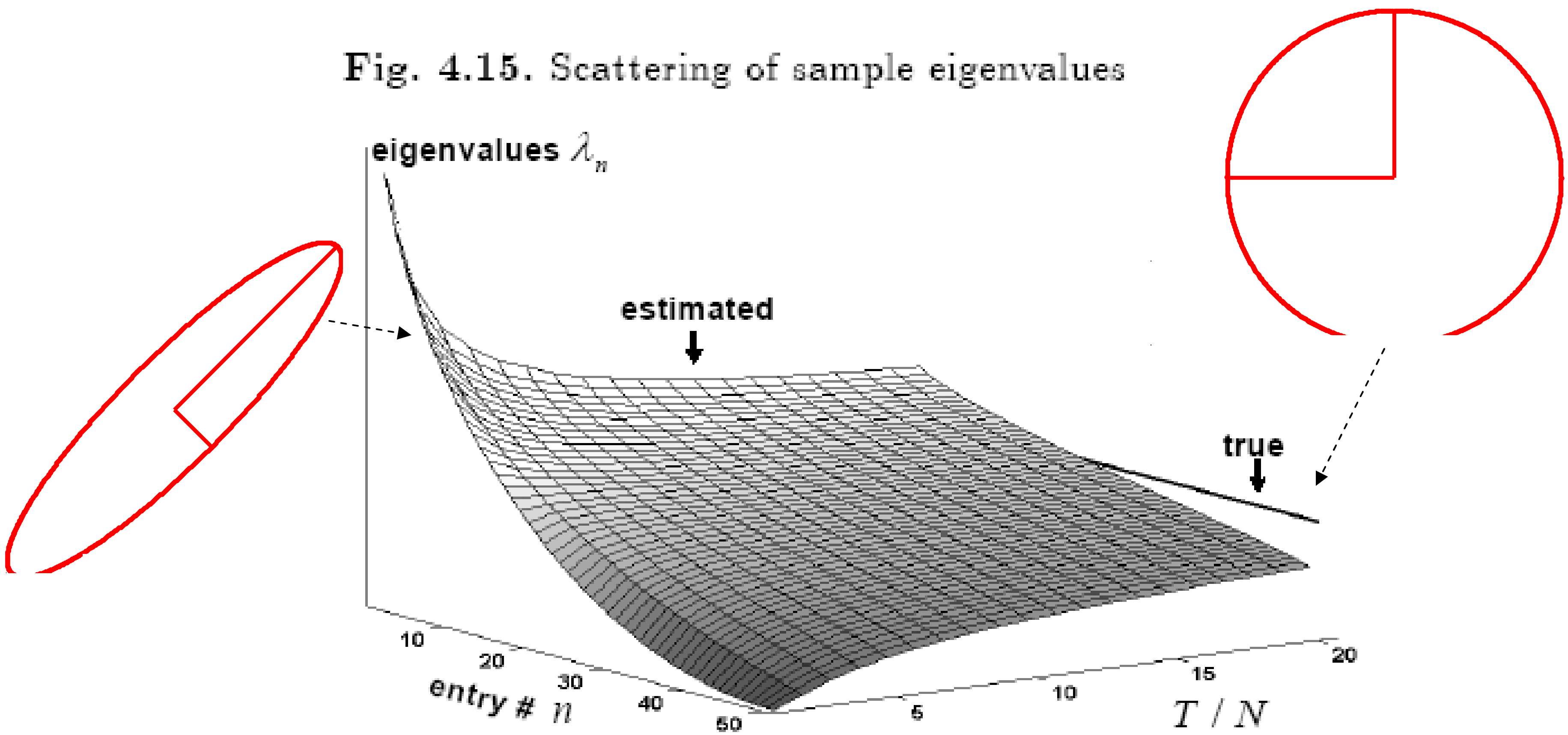
Fig. 4.15. Scattering of sample eigenvalues



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$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.143)$$

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$$\boxed{\widehat{\boldsymbol{\Sigma}}^S \equiv (1-\alpha)\widehat{\boldsymbol{\Sigma}} + \alpha\widehat{\mathbf{C}}.} \quad (4.160)$$

$$\widehat{\mathbf{C}} \equiv \frac{\sum_{n=1}^{N}\widehat{\lambda}_n}{N}\mathbf{I}. \quad (4.159)$$

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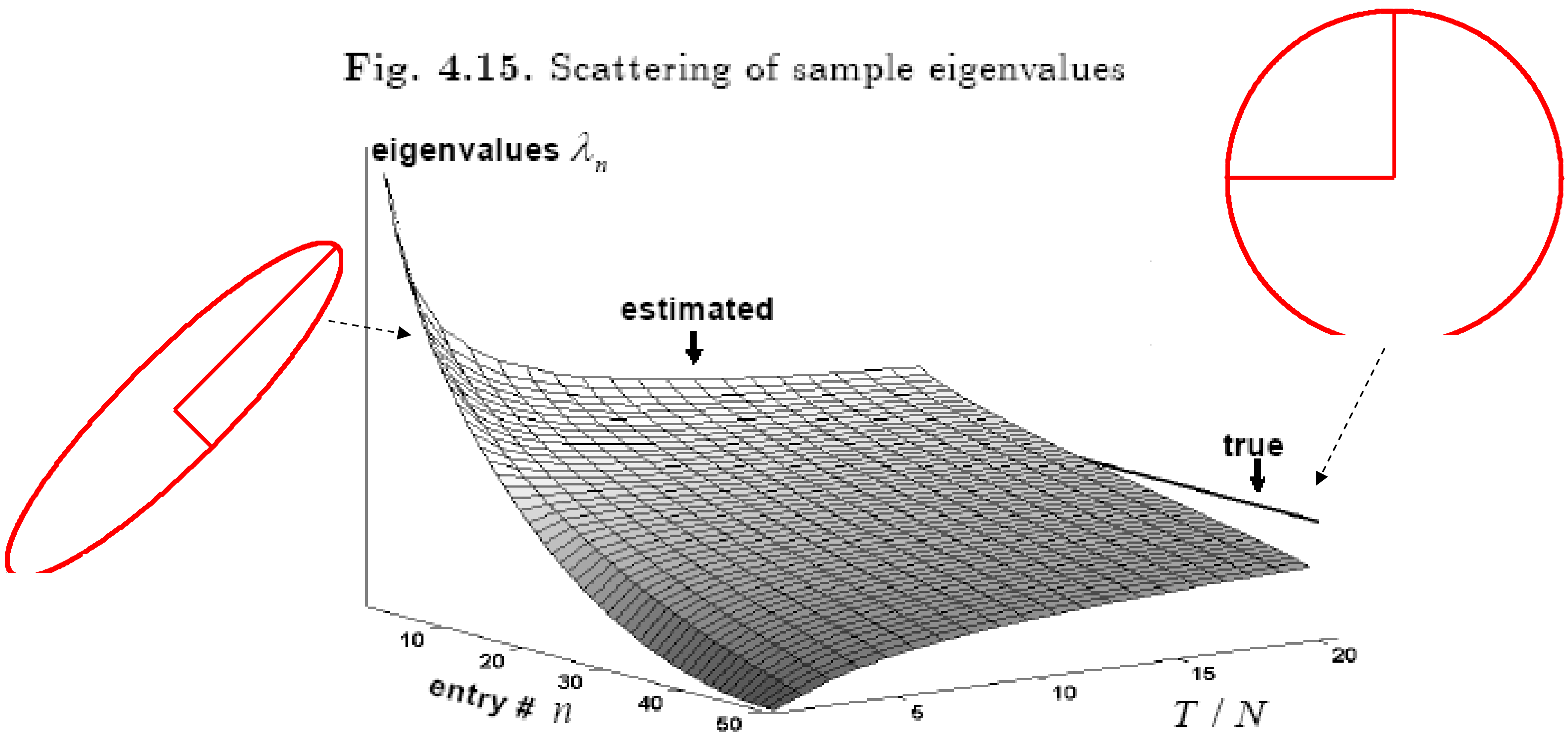

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$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.143)$$

$$\boxed{\widehat{\boldsymbol{\Sigma}}^S \equiv (1-\alpha)\,\widehat{\boldsymbol{\Sigma}} + \alpha \widehat{\mathbf{C}}.} \quad (4.160)$$

$$\alpha \equiv \frac{1}{T} \frac{\frac{1}{T}\sum_{t=1}^{T} \operatorname{tr}\left\{\left(\mathbf{x}_t \mathbf{x}_t' - \widehat{\boldsymbol{\Sigma}}\right)^2\right\}}{\operatorname{tr}\left\{\left(\widehat{\boldsymbol{\Sigma}} - \widehat{\mathbf{C}}\right)^2\right\}} \quad (4.161)$$

$$\widehat{\boldsymbol{\Sigma}}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T} [\mathbf{x}_t - \widehat{\boldsymbol{\mu}}[i_T]]\,[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}[i_T]]'. \quad (4.146)$$

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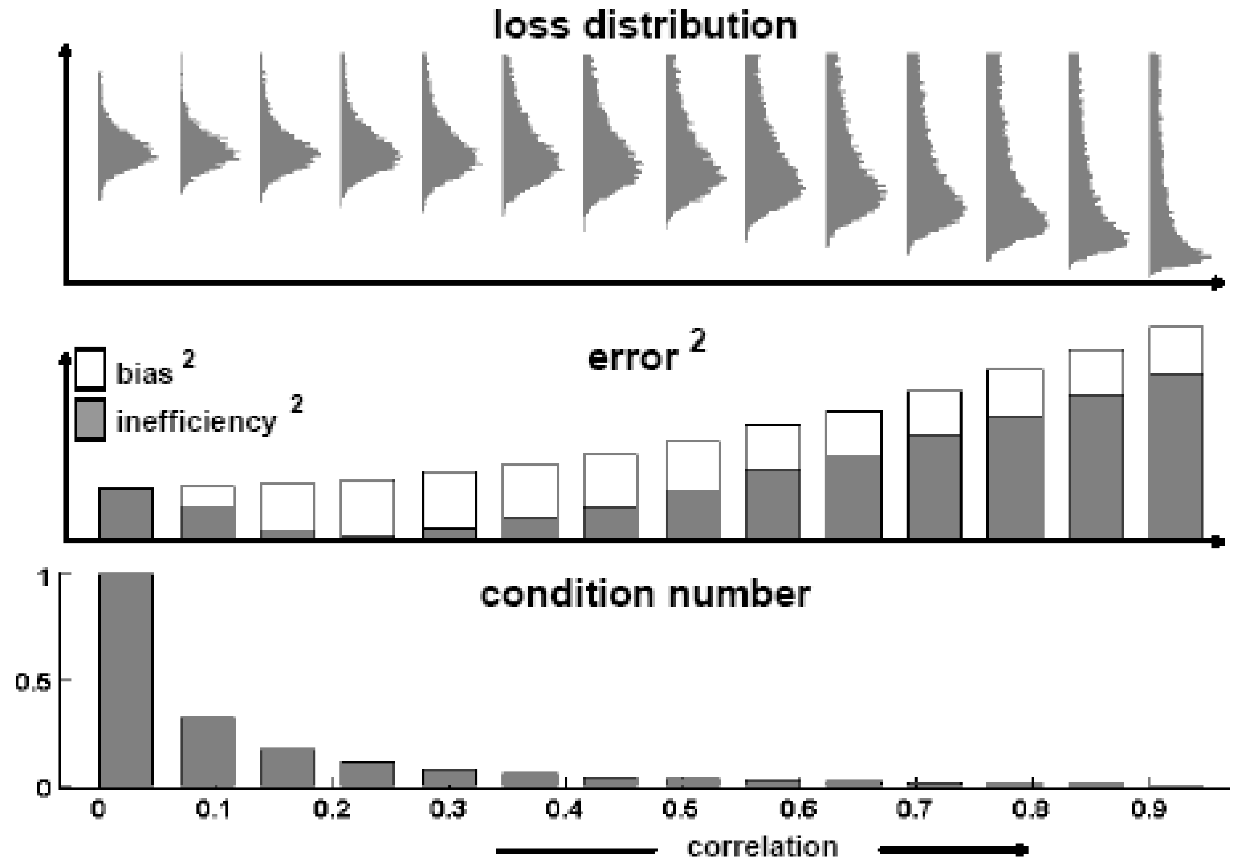

Fig. 4.16. Shrinkage estimator of covariance: evaluation