INVESTOR’S OBJECTIVES EVALUATION: COHERENT MEASURES

***Risk and Asset Allocation* - Springer – symmys.com**

Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

INVESTOR'S OBJECTIVES EVALUATION: COHERENT MEASURES

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}} \quad (5.10)$$

$$\mathrm{Coh}\,(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \gamma \left\| \min\left(0, \Psi_{\boldsymbol{\alpha}} - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}\right) \right\|_{\mathbf{M};p} . \quad (5.198)$$

- **Sensibility**

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios} \Rightarrow \mathrm{Coh}\,(\boldsymbol{\alpha}) \geq \mathrm{Coh}\,(\boldsymbol{\beta}) \quad (5.193)$$

- **Positive homogeneity**

$$\mathrm{Coh}\,(\lambda\boldsymbol{\alpha}) = \lambda\,\mathrm{Coh}\,(\boldsymbol{\alpha}) \quad (5.194)$$

- **Translation invariance**

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \mathrm{Coh}\,(\boldsymbol{\alpha} + \lambda\mathbf{b}) = \mathrm{Coh}\,(\boldsymbol{\alpha}) + \lambda. \quad (5.195)$$

- **super- additivity**

$$\mathrm{Coh}\,(\boldsymbol{\alpha} + \boldsymbol{\beta}) \geq \mathrm{Coh}\,(\boldsymbol{\alpha}) + \mathrm{Coh}\,(\boldsymbol{\beta}) \quad (5.197)$$

INVESTOR'S OBJECTIVES EVALUATION: COHERENT MEASURES

Risk and Asset Allocation **- Springer –** `symmys.com`

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}} \quad (5.10)$$

$$\operatorname{Coh}(\boldsymbol{\alpha}) \equiv \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\} - \gamma \left\| \min\left(0, \Psi_{\boldsymbol{\alpha}} - \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}\right) \right\|_{\mathbf{M};p} . \quad (5.198)$$

- Money-equivalence

- **Sensibility**

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios} \Rightarrow \operatorname{Coh}(\boldsymbol{\alpha}) \geq \operatorname{Coh}(\boldsymbol{\beta}) \quad (5.193)$$

- **Positive homogeneity**

$$\operatorname{Coh}(\lambda\boldsymbol{\alpha}) = \lambda \operatorname{Coh}(\boldsymbol{\alpha}) \quad (5.194)$$

- **Translation invariance**

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \operatorname{Coh}(\boldsymbol{\alpha} + \lambda\mathbf{b}) = \operatorname{Coh}(\boldsymbol{\alpha}) + \lambda. \quad (5.195)$$

- **super- additivity**

$$\operatorname{Coh}(\boldsymbol{\alpha} + \boldsymbol{\beta}) \geq \operatorname{Coh}(\boldsymbol{\alpha}) + \operatorname{Coh}(\boldsymbol{\beta}) \quad (5.197)$$

- Concavity

$$\operatorname{Coh}(\lambda\boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta}) \geq \lambda \operatorname{Coh}(\boldsymbol{\alpha}) + (1-\lambda)\operatorname{Coh}(\boldsymbol{\beta}) \quad (5.200)$$

INVESTOR'S OBJECTIVES EVALUATION: SPECTRAL MEASURES

Risk and Asset Allocation - Springer – `symmys.com`

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}} \quad (5.10)$$

$$\mathrm{Spc}(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} \quad (5.203)$$

- Money-equivalence

- Estimability

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto \left(f_{\Psi_{\boldsymbol{\alpha}}}, F_{\Psi_{\boldsymbol{\alpha}}}, \phi_{\Psi_{\boldsymbol{\alpha}}}\right) \mapsto \mathrm{Spc}(\boldsymbol{\alpha}) \quad (5.201)$$

- Sensibility

$$\begin{aligned} &\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios} \\ &\quad \Rightarrow \mathrm{Coh}(\boldsymbol{\alpha}) \geq \mathrm{Coh}(\boldsymbol{\beta}) \end{aligned} \quad (5.193)$$

- Positive homogeneity

$$\mathrm{Coh}(\lambda\boldsymbol{\alpha}) = \lambda\,\mathrm{Coh}(\boldsymbol{\alpha}) \quad (5.194)$$

- Translation invariance

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \mathrm{Coh}(\boldsymbol{\alpha} + \lambda\mathbf{b}) = \mathrm{Coh}(\boldsymbol{\alpha}) + \lambda. \quad (5.195)$$

- super- additivity

$$\mathrm{Coh}(\boldsymbol{\alpha} + \boldsymbol{\beta}) \geq \mathrm{Coh}(\boldsymbol{\alpha}) + \mathrm{Coh}(\boldsymbol{\beta}) \quad (5.197)$$

- Co-monotonic additivity

$$\begin{aligned} &(\boldsymbol{\alpha}, \boldsymbol{\delta}) \text{ co-monotonic} \\ &\quad \Rightarrow \mathrm{Spc}(\boldsymbol{\alpha} + \boldsymbol{\delta}) = \mathrm{Spc}(\boldsymbol{\alpha}) + \mathrm{Spc}(\boldsymbol{\delta}) \end{aligned} \quad (5.202)$$

- Concavity

$$\begin{aligned} &\mathrm{Coh}(\lambda\boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta}) \geq \\ &\qquad \lambda\,\mathrm{Coh}(\boldsymbol{\alpha}) + (1-\lambda)\,\mathrm{Coh}(\boldsymbol{\beta}) \end{aligned} \quad (5.200)$$

INVESTOR'S OBJECTIVES EVALUATION: SPECTRAL MEASURES

Risk and Asset Allocation - Springer – `symmys.com`

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}} \quad (5.10)$$

$$\mathrm{Spc}(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} \quad (5.203)$$

- Money-equivalence

- Estimability

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto \left(f_{\Psi_{\boldsymbol{\alpha}}}, F_{\Psi_{\boldsymbol{\alpha}}}, \phi_{\Psi_{\boldsymbol{\alpha}}}\right) \mapsto \mathrm{Spc}(\boldsymbol{\alpha}) \quad (5.201)$$

- Sensibility

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios} \Rightarrow \mathrm{Coh}(\boldsymbol{\alpha}) \geq \mathrm{Coh}(\boldsymbol{\beta}) \quad (5.193)$$

- Consistence with weak stochastic dominance

$$Q_{\Psi_{\boldsymbol{\alpha}}}(p) \geq Q_{\Psi_{\boldsymbol{\beta}}}(p) \text{ for all } p \in (0,1) \Rightarrow \mathrm{Spc}(\boldsymbol{\alpha}) \geq \mathrm{Spc}(\boldsymbol{\beta}) \quad (5.204)$$

- Constancy

$$\Psi_{\mathbf{b}} \equiv \psi_{\mathbf{b}} \Rightarrow \mathrm{Spc}(\mathbf{b}) = \psi_{\mathbf{b}}. \quad (5.205)$$

- Positive homogeneity

$$\mathrm{Coh}(\lambda\boldsymbol{\alpha}) = \lambda\,\mathrm{Coh}(\boldsymbol{\alpha}) \quad (5.194)$$

- Translation invariance

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \mathrm{Coh}(\boldsymbol{\alpha} + \lambda\mathbf{b}) = \mathrm{Coh}(\boldsymbol{\alpha}) + \lambda. \quad (5.195)$$

- super- additivity

$$\mathrm{Coh}(\boldsymbol{\alpha} + \boldsymbol{\beta}) \geq \mathrm{Coh}(\boldsymbol{\alpha}) + \mathrm{Coh}(\boldsymbol{\beta}) \quad (5.197)$$

- Co-monotonic additivity

$$(\boldsymbol{\alpha}, \boldsymbol{\delta}) \text{ co-monotonic} \Rightarrow \mathrm{Spc}(\boldsymbol{\alpha} + \boldsymbol{\delta}) = \mathrm{Spc}(\boldsymbol{\alpha}) + \mathrm{Spc}(\boldsymbol{\delta}) \quad (5.202)$$

- Concavity

$$\mathrm{Coh}(\lambda\boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta}) \geq \lambda\,\mathrm{Coh}(\boldsymbol{\alpha}) + (1-\lambda)\,\mathrm{Coh}(\boldsymbol{\beta}) \quad (5.200)$$

- Risk aversion,

$$\mathrm{RP}(\boldsymbol{\alpha}) \geq 0.$$

INVESTOR'S OBJECTIVES EVALUATION: EXPECTED SHORTFALL

Risk and Asset Allocation - Springer – `symmys.com`

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}} \quad (5.10)$$

$$\boxed{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} = \int_{\mathbb{R}} \psi f_{\Psi_{\boldsymbol{\alpha}}}(\psi)\, d\psi = \int_0^1 Q_{\Psi_{\boldsymbol{\alpha}}}(s)\, ds} \quad (5.206)$$

INVESTOR'S OBJECTIVES EVALUATION: EXPECTED SHORTFALL

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}} \quad (5.10)$$

$$\boxed{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} = \int_{\mathbb{R}} \psi f_{\Psi_{\boldsymbol{\alpha}}}(\psi)\, d\psi = \int_0^1 Q_{\Psi_{\boldsymbol{\alpha}}}(s)\, ds} \quad (5.206)$$

$$\boxed{\mathrm{ES}_c(\boldsymbol{\alpha}) \equiv \frac{1}{1-c}\int_0^{1-c} Q_{\Psi_{\boldsymbol{\alpha}}}(s)\, ds.} \quad (5.207)$$

$$\Updownarrow$$

$$\mathrm{ES}_c(\boldsymbol{\alpha}) = \mathrm{TCE}_c(\boldsymbol{\alpha}) = \mathrm{CVaR}_c \equiv \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}} \middle| \Psi_{\boldsymbol{\alpha}} \leq \mathrm{Q}_c(\boldsymbol{\alpha})\right\} \quad (5.208)$$

INVESTOR'S OBJECTIVES EVALUATION: EXPECTED SHORTFALL

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\boxed{\Psi_{\alpha} = \alpha' \mathrm{M}} \quad (5.10)$$

$$\mathrm{ES}_c(\alpha) \equiv \frac{1}{1-c} \int_0^{1-c} Q_{\Psi_{\alpha}}(s)\, ds. \quad (5.207)$$

- Money-equivalence
- Estimability
- Sensibility
- Consistence with weak stochastic dominance
- Constancy
- Positive homogeneity
- Translation invariance
- super- additivity
- Co-monotonic additivity
- Concavity
- Risk aversion

INVESTOR'S OBJECTIVES EVALUATION: EXPECTED SHORTFALL

Risk and Asset Allocation **- Springer – `symmys.com`**

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}} \quad (5.10)$$

$$\mathrm{ES}_c(\boldsymbol{\alpha}) \equiv \frac{1}{1-c}\int_0^{1-c} Q_{\Psi_{\boldsymbol{\alpha}}}(s)\, ds. \quad (5.207)$$

- **Positive homogeneity**

$$\mathrm{ES}_c(\lambda\boldsymbol{\alpha}) = \lambda\,\mathrm{ES}_c(\boldsymbol{\alpha}) \quad (5.210)$$

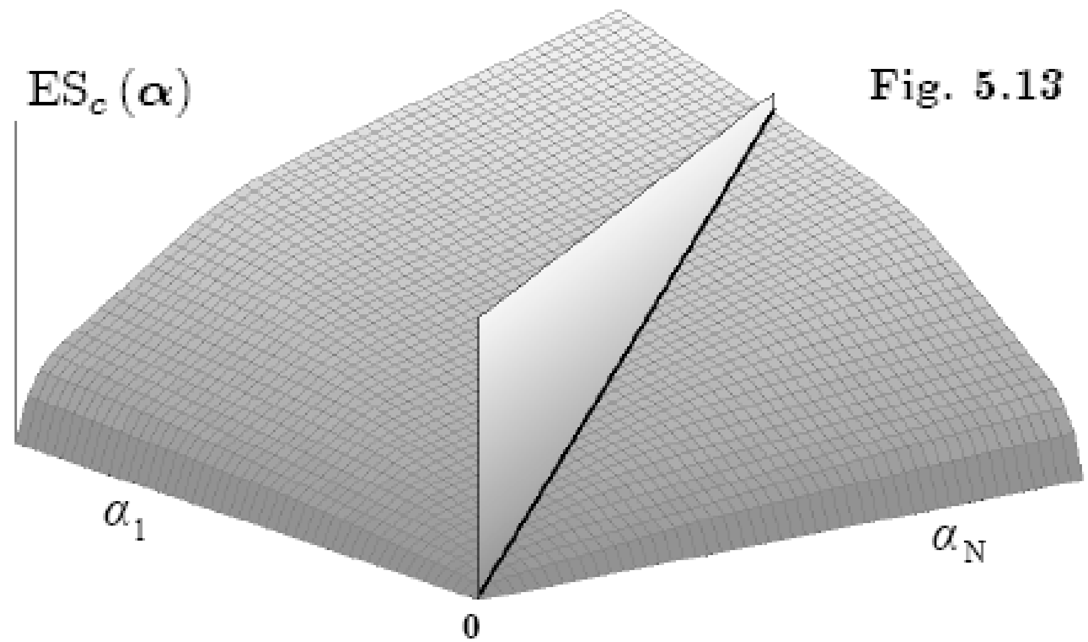

Fig. 5.13

INVESTOR'S OBJECTIVES EVALUATION: EXPECTED SHORTFALL

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}} \quad (5.10)$$

$$\mathrm{ES}_c(\boldsymbol{\alpha}) \equiv \frac{1}{1-c}\int_0^{1-c} Q_{\Psi_{\boldsymbol{\alpha}}}(s)\,ds. \quad (5.207)$$

- **Positive homogeneity**

$$\mathrm{ES}_c(\lambda\boldsymbol{\alpha}) = \lambda\,\mathrm{ES}_c(\boldsymbol{\alpha}) \quad (5.210)$$

$$\Downarrow$$

Euler:

$$\mathrm{ES}_c(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'\,\frac{\partial\,\mathrm{ES}_c}{\partial\boldsymbol{\alpha}} \quad (5.239)$$

$$\boxed{\frac{\partial\,\mathrm{ES}_c}{\partial\boldsymbol{\alpha}} = \mathrm{E}\left\{\mathbf{M}\,|\,\boldsymbol{\alpha}'\mathbf{M} \leq \mathrm{Q}_c(\boldsymbol{\alpha})\right\} \quad (5.238)}$$

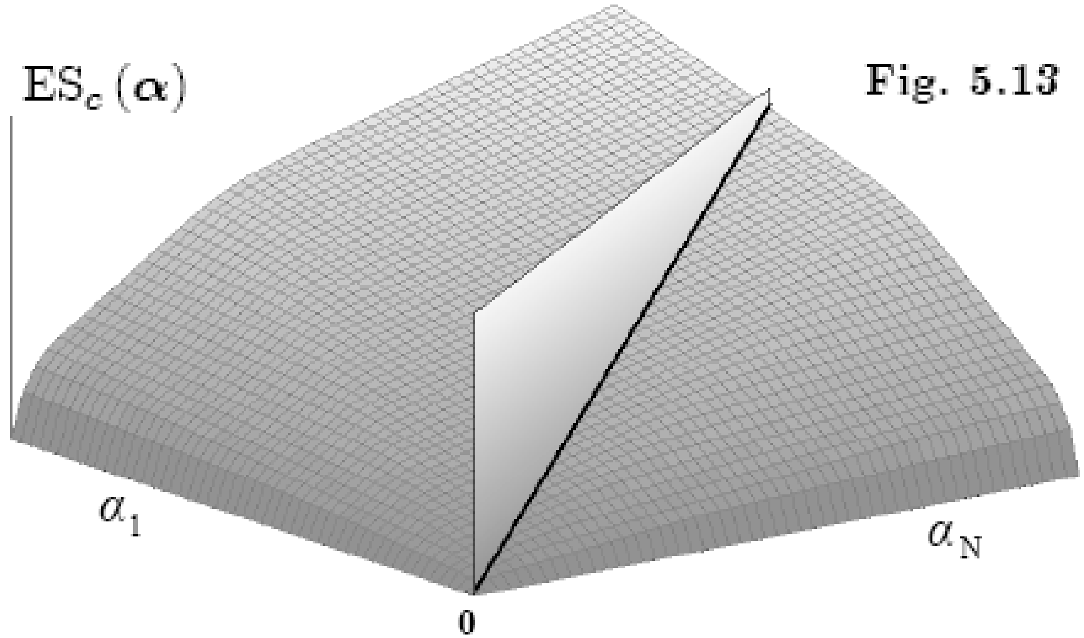

Fig. 5.13

INVESTOR'S OBJECTIVES EVALUATION: ES & SPECTRAL MEASURES

Risk and Asset Allocation **- Springer –** `symmys.com`

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}' \mathbf{M}} \quad (5.10)$$

$$\boxed{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} = \int_{\mathbb{R}} \psi f_{\Psi_{\boldsymbol{\alpha}}}(\psi)\, d\psi = \int_0^1 Q_{\Psi_{\boldsymbol{\alpha}}}(s)\, ds} \quad (5.206)$$

$$\boxed{\mathrm{ES}_c(\boldsymbol{\alpha}) \equiv \frac{1}{1-c} \int_0^{1-c} Q_{\Psi_{\boldsymbol{\alpha}}}(s)\, ds,} \quad (5.207)$$

$$\Updownarrow$$

$$\mathrm{ES}_c(\boldsymbol{\alpha}) = \mathrm{TCE}_c(\boldsymbol{\alpha}) = \mathrm{CVaR}_c \equiv \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}} \middle| \Psi_{\boldsymbol{\alpha}} \leq \mathrm{Q}_c(\boldsymbol{\alpha})\right\} \quad (5.208)$$

$$\boxed{\mathrm{Spc}_{\varphi}(\boldsymbol{\alpha}) \equiv \int_0^1 \varphi(p)\, Q_{\Psi_{\boldsymbol{\alpha}}}(p)\, dp,} \quad (5.216)$$

$$\begin{gathered} \varphi \text{ decreasing;} \\ \varphi(1) \equiv 0; \\ \int_0^1 \varphi(p)\, dp \equiv 1. \end{gathered} \quad (5.217)$$

INVESTOR'S OBJECTIVES EVALUATION: ES & SPECTRAL MEASURES

Risk and Asset Allocation **- Springer –** `symmys.com`

$$\boxed{\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}.} \quad (5.10)$$

$$\boxed{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} = \int_{\mathbb{R}} \psi f_{\Psi_{\boldsymbol{\alpha}}}(\psi)\, d\psi = \int_0^1 Q_{\Psi_{\boldsymbol{\alpha}}}(s)\, ds} \quad (5.206)$$

$$\boxed{\mathrm{ES}_c(\boldsymbol{\alpha}) \equiv \frac{1}{1-c}\int_0^{1-c} Q_{\Psi_{\boldsymbol{\alpha}}}(s)\, ds.} \quad (5.207)$$

$$\Updownarrow$$

$$\mathrm{ES}_c(\boldsymbol{\alpha}) = \mathrm{TCE}_c(\boldsymbol{\alpha}) = \mathrm{CVaR}_c \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}} | \Psi_{\boldsymbol{\alpha}} \leq \mathrm{Q}_c(\boldsymbol{\alpha})\} \quad (5.208)$$

$$\boxed{\mathrm{Spc}_{\varphi}(\boldsymbol{\alpha}) \equiv \int_0^1 \varphi(p)\, Q_{\Psi_{\boldsymbol{\alpha}}}(p)\, dp,} \quad (5.216)$$

$$\varphi \text{ decreasing.}$$
$$\varphi(1) \equiv 0. \quad (5.217)$$
$$\int_0^1 \varphi(p)\, dp \equiv 1.$$

$$\varphi_{\mathrm{ES}_c}(p) \equiv \frac{H^{(c-1)}(-p)}{1-c} \quad (5.218)$$

INVESTOR'S OBJECTIVES EVALUATION: VaR & SPECTRAL MEASURES

Risk and Asset Allocation* - Springer – *symmys.com

$$\boxed{\Psi_{\alpha} = \alpha' \mathbf{M}} \quad (5.10)$$

$$\boxed{\mathrm{E}\{\Psi_{\alpha}\} = \int_{\mathbb{R}} \psi f_{\Psi_{\alpha}}(\psi)\, d\psi = \int_0^1 Q_{\Psi_{\alpha}}(s)\, ds} \quad (5.206)$$

$$\boxed{\mathrm{ES}_c(\alpha) \equiv \frac{1}{1-c} \int_0^{1-c} Q_{\Psi_{\alpha}}(s)\, ds.} \quad (5.207)$$

$$\Updownarrow$$

$$\mathrm{ES}_c(\alpha) = \mathrm{TCE}_c(\alpha) = \mathrm{CVaR}_c \equiv \mathrm{E}\{\Psi_{\alpha} | \Psi_{\alpha} \leq \mathrm{Q}_c(\alpha)\} \quad (5.208)$$

$$\boxed{\mathrm{Spc}_{\varphi}(\alpha) \equiv \int_0^1 \varphi(p)\, Q_{\Psi_{\alpha}}(p)\, dp,} \quad (5.216)$$

$$\boxed{\varphi \text{ decreasing.}}$$

$$\varphi(1) \equiv 0. \quad (5.217)$$

$$\int_0^1 \varphi(p)\, dp \equiv 1.$$

$$\varphi_{\mathrm{ES}_c}(p) \equiv \frac{H^{(c-1)}(-p)}{1-c} \quad (5.218)$$

$$\boxed{\varphi_{\mathrm{Q}_c} \equiv \delta^{(1-c)}} \quad (5.219)$$