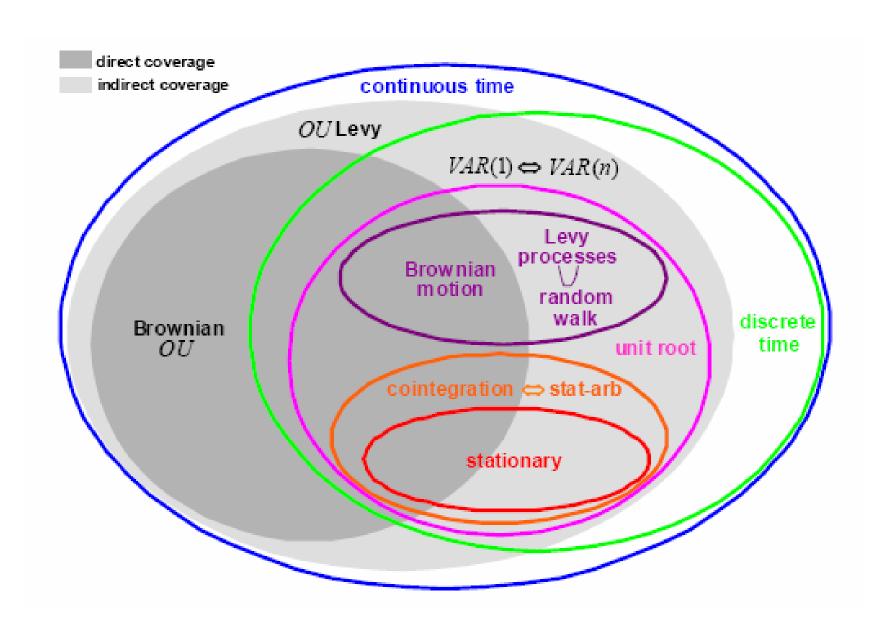
Attilio Meucci

Multivariate Ornstein-Uhlenbeck, Cointegration, and Statistical Arbitrage

- > Slides from paper
- "Review of Statistical Arbitrage, Cointegration, and Multivariate Ornstein-Uhlenbeck" available at www.symmys.com > Research > Working Papers
- ➤ MATLAB code available at www.symmys.com > Teaching > MATLAB



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invariants (i.i.d.)

$$dX_{t} = -\theta \left(X_{t} - m \right) dt + \sigma dB_{t}.$$

$$\begin{split} X_t &\stackrel{d}{=} m + e^{-\theta \tau} \left(X_{t-\tau} - m \right) + \epsilon_{t,\tau}, \\ \\ \epsilon_{t,\tau} &\sim \mathrm{N} \left(0, \frac{\sigma^2}{2\theta} \left(1 - e^{-2\theta \tau} \right) \right) \end{split}$$

invariants (i.i.d.)

$$d\mathbf{X}_{t} = -\Theta\left(\mathbf{X}_{t} - \mu\right)dt + \mathbf{S}d\mathbf{B}_{t}$$

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$$d\mathbf{X}_{t} = -\Theta\left(\mathbf{X}_{t} - \mu\right)dt + \mathbf{S}d\mathbf{B}_{t}$$



$$X_{t+\tau} \sim N(x_{t+\tau}, \Sigma_{\tau})$$

$$dX_{t}=-\theta \left(X_{t}-m\right) dt+\sigma dB_{t}.$$

$$\begin{split} X_t &\stackrel{d}{=} m + e^{-\theta \tau} \left(X_{t-\tau} - m \right) + \epsilon_{t,\tau}, \\ \\ \epsilon_{t,\tau} &\sim \mathrm{N} \left(0, \frac{\sigma^2}{2\theta} \left(1 - e^{-2\theta \tau} \right) \right) \end{split}$$

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$$d\mathbf{X}_{t} = -\Theta\left(\mathbf{X}_{t} - \mu\right)dt + \mathbf{S}d\mathbf{B}_{t}$$

autoregressive process

$$\mathbf{X}_{t+\tau} \sim \mathbf{N}\left(\mathbf{x}_{t+\tau}, \mathbf{\Sigma}_{\tau}\right)$$

$$\begin{cases} \mathbf{x}_{t+\tau} \equiv \left(\mathbf{I} - e^{-\mathbf{\Theta}\tau}\right) \mu + e^{-\mathbf{\Theta}\tau} \mathbf{x}_t \\\\ \operatorname{vec}\left(\Sigma_{\tau}\right) \equiv \left(\Theta \oplus \Theta\right)^{-1} \left(\mathbf{I} - e^{-(\mathbf{\Theta} \oplus \mathbf{\Theta})\tau}\right) \operatorname{vec}\left(\Sigma\right) \\\\ \Sigma \equiv \mathbf{SS}' \end{cases}$$

$$dX_{t} = -\theta \left(X_{t} - m \right) dt + \sigma dB_{t}.$$

$$\begin{split} X_t &\stackrel{d}{=} m + e^{-\theta \tau} \left(X_{t-\tau} - m \right) + \epsilon_{t,\tau}, \\ \\ \epsilon_{t,\tau} &\sim \mathcal{N} \left(0, \frac{\sigma^2}{2\theta} \left(1 - e^{-2\theta \tau} \right) \right) \end{split}$$

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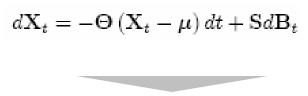
$$d\mathbf{X}_{t} = -\Theta\left(\mathbf{X}_{t} - \mu\right)dt + \mathbf{S}d\mathbf{B}_{t}$$

$$X_{t+\tau} \sim N(x_{t+\tau}, \Sigma_{\tau})$$

$$\begin{cases} \mathbf{x}_{t+\tau} \equiv \left(\mathbf{I} - e^{-\mathbf{\Theta}\tau}\right) \mu + e^{-\mathbf{\Theta}\tau} \mathbf{x}_{t} \\ \operatorname{vec}\left(\Sigma_{\tau}\right) \equiv \left(\Theta \oplus \Theta\right)^{-1} \left(\mathbf{I} - e^{-(\mathbf{\Theta} \oplus \mathbf{\Theta})\tau}\right) \operatorname{vec}\left(\Sigma\right) \\ \tau \rightarrow \infty & \tau \rightarrow 0 \\ \\ \mathbf{x}_{\infty} = \mu \\ \operatorname{vec}\left(\Sigma_{\infty}\right) = \left(\Theta \oplus \Theta\right)^{-1} \operatorname{vec}\left(\Sigma\right) \end{cases}$$

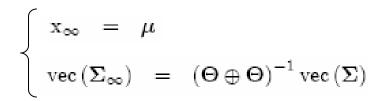
$$\left\{ \begin{array}{l} \mathbf{X}_{t+\tau} \approx \mathbf{X}_t + \boldsymbol{\epsilon}_{t,\tau} \\ \\ \boldsymbol{\epsilon}_{t,\tau} \sim \mathrm{N}\left(\tau \boldsymbol{\Theta} \boldsymbol{\mu}, \tau \boldsymbol{\Sigma}\right) \end{array} \right.$$

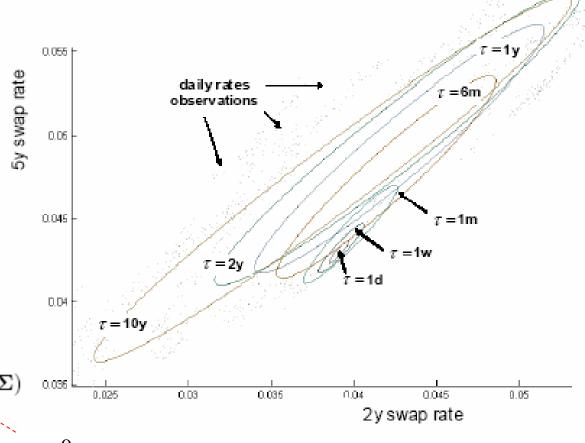
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$$\mathbf{X}_{t+\tau} \sim \mathbf{N}\left(\mathbf{x}_{t+\tau}, \boldsymbol{\Sigma}_{\tau}\right)$$

$$\begin{cases} \mathbf{x}_{t+\tau} \equiv \left(\mathbf{I} - e^{-\mathbf{\Theta}\tau}\right) \mu + e^{-\mathbf{\Theta}\tau} \mathbf{x}_{t} \\\\ \operatorname{vec}\left(\mathbf{\Sigma}_{\tau}\right) \equiv \left(\mathbf{\Theta} \oplus \mathbf{\Theta}\right)^{-1} \left(\mathbf{I} - e^{-(\mathbf{\Theta} \oplus \mathbf{\Theta})\tau}\right) \operatorname{vec}\left(\mathbf{\Sigma}\right) \end{cases}$$





$$\left\{egin{aligned} \mathbf{X}_{t+ au} pprox \mathbf{X}_t + oldsymbol{\epsilon}_{t, au} \ & oldsymbol{\epsilon}_{t, au} \sim \mathrm{N}\left(au\mathbf{\Theta}oldsymbol{\mu}, au\mathbf{\Sigma}
ight) \end{aligned}
ight.$$

$$d\mathbf{X}_t = -\Theta\left(\mathbf{X}_t - \mu\right)dt + \mathbf{S}d\mathbf{B}_t$$

$$\Theta \underbrace{\begin{array}{c} \mathbf{B} \ \ \text{eigenvectors (columns)} \\ (\lambda_1, \dots, \lambda_K) \ \ \text{real eigenvalues} \\ (\gamma_1 \pm i\omega_1), \dots, (\gamma_J \pm i\omega_J) \ \ \text{complex eigenvalues} \end{array}}$$

$$d\mathbf{X}_{t}=-\Theta\left(\mathbf{X}_{t}-\boldsymbol{\mu}\right)dt+\mathbf{S}d\mathbf{B}_{t}$$

$$\Theta \xrightarrow{\quad \mathbf{B} \quad \text{eigenvectors (columns)}} \\ (\lambda_1, \dots, \lambda_K) \quad \text{real eigenvalues} \\ (\gamma_1 \pm i\omega_1), \dots, (\gamma_J \pm i\omega_J) \quad \text{complex eigenvalues}}$$

$$\begin{split} \mathbf{A} &\equiv \operatorname{Re}\left(\mathbf{B}\right) - \operatorname{Im}\left(\mathbf{B}\right) \\ \mathbf{\Gamma} &\equiv \operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{K}, \mathbf{\Gamma}_{1}, \ldots, \mathbf{\Gamma}_{J}\right) \qquad \mathbf{\Gamma}_{j} \equiv \left(\begin{array}{cc} \gamma_{j} & \omega_{j} \\ -\omega_{j} & \gamma_{j} \end{array}\right) \\ \mathbf{\Theta} &\equiv \mathbf{A} \mathbf{\Gamma} \mathbf{A}^{-1} \end{split}$$

$$d\mathbf{X}_{t}=-\Theta\left(\mathbf{X}_{t}-\boldsymbol{\mu}\right)dt+\mathbf{S}d\mathbf{B}_{t}$$

$$\mathbf{z} \equiv \mathbf{A}^{-1} (\mathbf{x} - \mathbf{m})$$

$$\mathbf{V} \equiv \mathbf{A}^{-1} \mathbf{S}$$

$$d\mathbf{Z}_t = -\Gamma \mathbf{Z}_t dt + \mathbf{V} d\mathbf{B}_t$$

$$\Theta \xrightarrow{\quad \mathbf{B} \quad \text{eigenvectors (columns)}} \\ (\lambda_1, \dots, \lambda_K) \quad \text{real eigenvalues} \\ (\gamma_1 \pm i\omega_1), \dots, (\gamma_J \pm i\omega_J) \quad \text{complex eigenvalues}$$

$$\begin{split} \mathbf{A} &\equiv \operatorname{Re}\left(\mathbf{B}\right) - \operatorname{Im}\left(\mathbf{B}\right) \\ \Gamma &\equiv \operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{K}, \Gamma_{1}, \ldots, \Gamma_{J}\right) \qquad \Gamma_{j} \equiv \left(\begin{array}{cc} \gamma_{j} & \omega_{j} \\ -\omega_{j} & \gamma_{j} \end{array}\right) \\ \Theta &\equiv \mathbf{A}\Gamma\mathbf{A}^{-1} \end{split}$$

$$d\mathbf{X}_{t}=-\Theta\left(\mathbf{X}_{t}-\boldsymbol{\mu}\right)dt+\mathbf{S}d\mathbf{B}_{t}$$

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$$\mathbf{A} \equiv \operatorname{Re}\left(\mathbf{B}\right) - \operatorname{Im}\left(\mathbf{B}\right)$$

$$\mathbf{\Gamma} \equiv \operatorname{diag}\left(\lambda_{1}, \dots, \lambda_{K}, \mathbf{\Gamma}_{1}, \dots, \mathbf{\Gamma}_{J}\right)$$
 $\mathbf{\Gamma}_{j} \equiv \left(\mathbf{G}_{j}, \dots, \mathbf{\Gamma}_{J}, \dots, \mathbf{\Gamma}_{J}\right)$

$$\Gamma_{j} \equiv \left(\begin{array}{cc} \gamma_{j} & \omega_{j} \\ -\omega_{j} & \gamma_{j} \end{array} \right)$$

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$$d\mathbf{Z}_t = -\mathbf{\Gamma} \mathbf{Z}_t dt + \mathbf{V} d\mathbf{B}_t$$

$$dz_{k,t} = -\lambda_k z_{k,t} dt$$
.

$$d\mathbf{z}_{j,t} = -\Gamma_j \mathbf{z}_{j,t} dt.$$

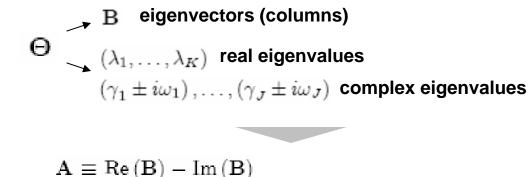
$$\Theta \begin{tabular}{ll} \hline & B & eigenvectors (columns) \\ \hline & (\lambda_1,\dots,\lambda_K) & real eigenvalues \\ & (\gamma_1\pm i\omega_1)\,,\dots,(\gamma_J\pm i\omega_J) & complex eigenvalues \\ \hline \end{tabular}$$

$$\mathbf{A} \equiv \operatorname{Re}(\mathbf{B}) - \operatorname{Im}(\mathbf{B})$$

$$\mathbf{\Gamma} \equiv \operatorname{diag}\left(\lambda_{1}, \dots, \lambda_{K} \middle[\mathbf{\Gamma}_{1}, \dots, \mathbf{\Gamma}_{J} \middle] \quad \mathbf{\Gamma}_{j} \equiv \begin{pmatrix} \gamma_{j} & \omega_{j} \\ -\omega_{j} & \gamma_{j} \end{pmatrix}$$

$$\mathbf{C} = \mathbf{A} \mathbf{\Gamma} \mathbf{A}^{-1}$$

$$d\mathbf{X}_{t} = -\Theta\left(\mathbf{X}_{t} - \mu\right)dt + \mathbf{S}d\mathbf{B}_{t}$$



$$\mathbf{z} \equiv \mathbf{A}^{-1} (\mathbf{x} - \mathbf{m})$$

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$$d\mathbf{Z}_t = -\Gamma \mathbf{Z}_t dt + \mathbf{V} d\mathbf{B}_t$$

$$\mathbf{A} \equiv \operatorname{Re}(\mathbf{B}) - \operatorname{Im}(\mathbf{B})$$

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$$dz_{k,t} = -\lambda_k z_{k,t} dt$$
. $z_{k,t} \equiv e^{-\lambda_k t} z_{k,0}$

$$d\mathbf{z}_{j,t} = -\Gamma_j \mathbf{z}_{j,t} dt$$
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$$d\mathbf{Z}_t = -\mathbf{\Gamma} \mathbf{Z}_t dt + \mathbf{V} d\mathbf{B}_t$$

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$$\mathbf{O} = \mathbf{A} \Gamma \mathbf{A}^{-1}$$

$$dz_{k,t} = -\lambda_k z_{k,t} dt$$
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$$d\mathbf{z}_{j,t} = -\Gamma_j \mathbf{z}_{j,t} dt$$
, $\mathbf{z}_{j,t} \equiv e^{-\Gamma_j t} \mathbf{z}_{j,0}$

$$\begin{array}{ccc} z_{j;t}^{(1)} & \equiv & e^{-\gamma_{j}t} \left(z_{j,0}^{(1)} \cos \omega_{j} t - z_{j,0}^{(2)} \sin \omega_{j} t \right) \\ \\ z_{j;t}^{(2)} & \equiv & e^{-\gamma_{j}t} \left(z_{j,0}^{(1)} \sin \omega_{j} t + z_{j,0}^{(2)} \cos \omega_{j} t \right) \end{array}$$

$$d\mathbf{X}_{t} = -\Theta\left(\mathbf{X}_{t} - \mu\right)dt + \mathbf{S}d\mathbf{B}_{t}$$

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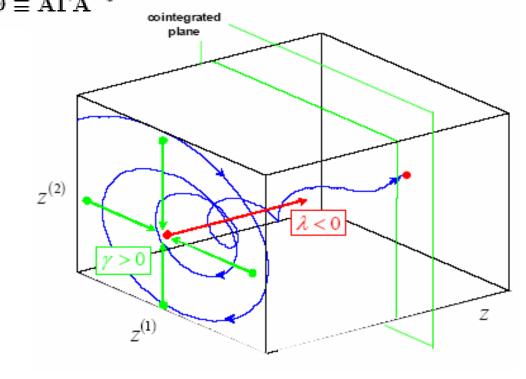
$$d\mathbf{z}_{j,t} = -\Gamma_j \mathbf{z}_{j,t} dt, \quad \mathbf{z}_{j,t} \equiv e^{-\Gamma_j t} \mathbf{z}_{j,0}$$

$$z_{j;t}^{(1)} \equiv e^{-\gamma_j t} \left(z_{j,0}^{(1)} \cos \omega_j t - z_{j,0}^{(2)} \sin \omega_j t \right)$$

$$z_{j;t}^{(2)} \equiv e^{-\gamma_j t} \left(z_{j,0}^{(1)} \sin \omega_j t + z_{j,0}^{(2)} \cos \omega_j t \right)$$

$$\Theta \xrightarrow{\quad \mathbf{B} \quad \text{eigenvectors (columns)}} \\ (\lambda_1, \dots, \lambda_K) \quad \text{real eigenvalues} \\ (\gamma_1 \pm i\omega_1), \dots, (\gamma_J \pm i\omega_J) \quad \text{complex eigenvalues}$$

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$$d\mathbf{X}_{t}=-\Theta\left(\mathbf{X}_{t}-\boldsymbol{\mu}\right)dt+\mathbf{S}d\mathbf{B}_{t}$$

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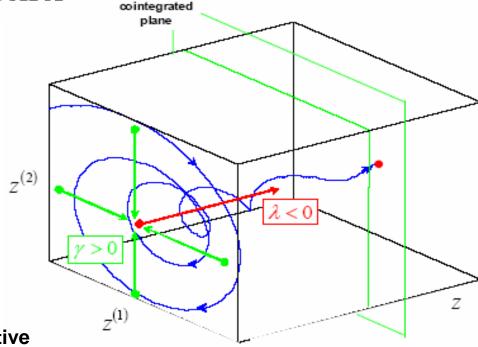
<u>stationary</u>: all (real part of) eigenvalues positive
<u>random walk</u>: null eigenvalues

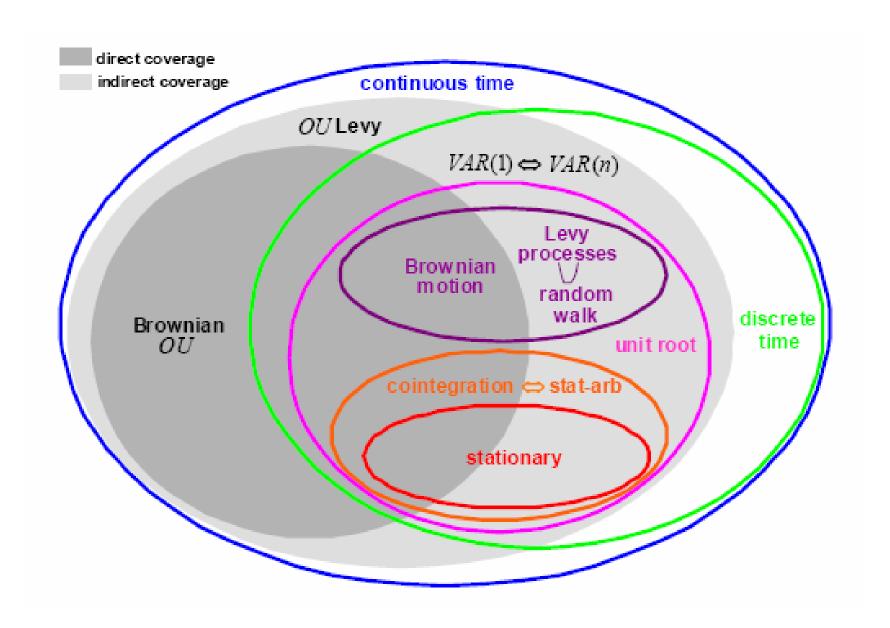
cointegrated directions: (real part of) eigenvalues positive

B eigenvectors (columns)
$$(\lambda_1,\dots,\lambda_K) \text{ real eigenvalues}$$

$$(\gamma_1\pm i\omega_1),\dots,(\gamma_J\pm i\omega_J) \text{ complex eigenvalues}$$

$$\begin{split} \mathbf{A} &\equiv \operatorname{Re}\left(\mathbf{B}\right) - \operatorname{Im}\left(\mathbf{B}\right) \\ \mathbf{\Gamma} &\equiv \operatorname{diag}\left(\lambda_{1}, \dots, \lambda_{K} \middle| \mathbf{\Gamma}_{1}, \dots, \mathbf{\Gamma}_{J} \middle| \mathbf{\Gamma}_{j} \equiv \left(\begin{array}{cc} \gamma_{j} & \omega_{j} \\ -\omega_{j} & \gamma_{j} \end{array} \right) \\ \Theta &\equiv \mathbf{A} \mathbf{\Gamma} \mathbf{A}^{-1} \end{split}$$





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 $Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}$ cointegrated?

$$Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}$$
 cointegrated?

$$\widetilde{\mathbf{w}} \equiv \underset{\|\mathbf{w}\|=1}{\operatorname{argmin}} \left[\operatorname{Var} \left\{ Y_{\infty}^{\mathbf{w}} | \mathbf{x_0} \right\} \right]$$

$$Y_t^{\widetilde{\mathbf{W}}^{\scriptscriptstyle{\parallel}}}$$
 best candidate

$$Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}$$
 cointegrated?

$$\widetilde{\mathbf{w}} \equiv \underset{\|\mathbf{w}\|=1}{\operatorname{argmin}} \left[\operatorname{Var} \left\{ Y_{\infty}^{\mathbf{w}} | \mathbf{x}_{0} \right\} \right]$$

$$Y_t^{\widetilde{\mathbf{W}}^{\scriptscriptstyle{\dagger}}}$$
 best candidate

$$\begin{split} \boldsymbol{\Sigma}_{\infty} &\equiv \operatorname{Cov} \big\{ \mathbf{X}_{\infty} \big| \mathbf{x}_{0} \big\}, \equiv \mathbf{E} \boldsymbol{\Lambda} \mathbf{E} \\ &\mathbf{E} \equiv \left(\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(N)} \right) \\ &\boldsymbol{\Lambda} \equiv \operatorname{diag} \left(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(N)} \right) \end{split}$$

$Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}$ cointegrated?

$$\widetilde{\mathbf{w}} \equiv \underset{\|\mathbf{w}\|=1}{\operatorname{argmin}} \left[\operatorname{Var} \left\{ Y_{\infty}^{\mathbf{w}} | \mathbf{x}_{0} \right\} \right]$$

$$Y_t^{\widetilde{\mathbf{w}}_t}$$
 best candidate $= Y_t^{\mathbf{e}^{(N)}}$

$$\begin{split} \boldsymbol{\Sigma}_{\infty} &\equiv \operatorname{Cov} \, \{ \mathbf{X}_{\infty} | \mathbf{x}_{0} \}, \equiv \mathbf{E} \boldsymbol{\Lambda} \mathbf{E} \\ &\mathbf{E} \equiv \left(\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(N)} \right) \\ &\boldsymbol{\Lambda} \equiv \operatorname{diag} \left(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(N)} \right) \end{split}$$

$$Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}$$
 cointegrated?

$$\widetilde{\mathbf{w}} \equiv \underset{\|\mathbf{w}\|=1}{\operatorname{argmin}} \left[\operatorname{Var} \left\{ Y_{\infty}^{\mathbf{w}} | \mathbf{x}_{0} \right\} \right]$$

$$Y_t^{\widetilde{\mathbf{w}}}$$
 best candidate $\equiv Y_t^{\mathbf{e}^{(N)}}$

$$egin{aligned} oldsymbol{\Sigma}_{\infty} &\equiv \operatorname{Cov}\left\{\mathbf{X}_{\infty} \middle| \mathbf{x}_{0}
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ight) \\ &\mathbf{\Lambda} \equiv \operatorname{diag}\left(\lambda^{(1)}, \ldots, \lambda^{(N)}
ight) \end{aligned}$$

$$Y_t^{\mathbf{e}^{(N-1)}} \dots Y_t^{\mathbf{e}^{(1)}}$$

Statistical arbitrage

$$Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}$$
 cointegrated?

$$\widetilde{\mathbf{w}} \equiv \underset{\|\mathbf{w}\|=1}{\operatorname{argmin}} \left[\operatorname{Var} \left\{ Y_{\infty}^{\mathbf{w}} | \mathbf{x}_{0} \right\} \right]$$

$$egin{aligned} \Sigma_{\infty} &\equiv \mathrm{Cov}\,\{\mathbf{X}_{\infty}|\mathbf{x}_{0}\}, \equiv \mathbf{E}\Lambda\mathbf{E} \ \\ &\mathbf{E} \equiv \left(\mathbf{e}^{(1)},\ldots,\mathbf{e}^{(N)}\right) \ \\ &\Lambda \equiv \mathrm{diag}\left(\lambda^{(1)},\ldots,\lambda^{(N)}\right) \end{aligned}$$

$$Y_t^{\widetilde{\mathbf{W}}}$$
 best candidate $\equiv Y_t^{\mathbf{e}^{(N)}}$ $Y_t^{\mathbf{e}^{(N-1)}}$... $Y_t^{\mathbf{e}^{(1)}}$

$$y_{t+ au} \equiv \left(1-e^{- heta au}
ight)\mu + e^{- heta au}y_t + \epsilon_{t, au}$$
 empirical fit

Statistical arbitrage

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$$Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}$$
 cointegrated?

$$\widetilde{\mathbf{w}} \equiv \underset{\|\mathbf{w}\|=1}{\operatorname{argmin}} \left[\operatorname{Var} \left\{ Y_{\infty}^{\mathbf{w}} | \mathbf{x}_{0} \right\} \right]$$

$$\Sigma_{\infty} \equiv \operatorname{Cov} \{ \mathbf{X}_{\infty} | \mathbf{x}_{0} \}, \equiv \mathbf{E} \Lambda \mathbf{E}$$

$$\mathbf{E} \equiv \left(\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(N)} \right)$$

$$\boldsymbol{\Lambda} \equiv \operatorname{diag}\left(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(N)}\right)$$

$$Y_t^{\widetilde{\mathbf{w}}}$$
 best candidate $\equiv Y_t^{\mathbf{e}^{(N)}}$

$$= Y_t^{\mathbf{e}^{(N)}}$$

$$Y_t^{e^{(N-1)}}$$

$$Y_{t}^{e^{(1)}}$$

$$y_{t+\tau} \equiv \left(1 - e^{-\theta\tau}\right)\mu + e^{-\theta\tau}y_t + \epsilon_{t,\tau}$$

empirical fit

$$\alpha \equiv |y_t - \mathbb{E}\{y_\infty\}| = |y_t - \mu|$$

expected gain / "alpha"

$$Z_t \equiv \frac{|y_t - \operatorname{E} \left\{ y_\infty \right\}|}{\operatorname{Sd} \left\{ y_\infty \right\}} = \frac{|y_t - \mu|}{\sqrt{\sigma^2/2\theta}}$$

Z-score / Sharpe ratio > signal

$$\tilde{\tau} \propto \frac{1}{\theta}$$
.

half-life / expected wait to profits

