The University of New South Wales

Department of Statistics

Session 2, 2017

MATH5905 - Statistical Inference

Assignment 1

This assignment must be submitted no later than at the beginning of the lecture at 5pm on Tuesday, 22nd August 2017. Please, declare on the first page that the assignment is your own work, except where acknowledged. State also that you have read and understood the University Rules in respect to Student Academic Misconduct.

1. Consider a decision problem with parameter space $\Theta = \{\theta_1, \theta_2\}$ and a set of non randomized decisions $D = \{d_i, 1 \leq i \leq 6\}$ with risk points $\{R(\theta_1, d_i), R(\theta_2, d_i)\}$ as follows:

i	1	2	3	4	5	6
$R(\theta_1, d_i)$	0	1	2	4	6	3
$R(\theta_2, d_i)$	6	3	1	2	5	3

a) Find the minimax rule(s) amongst the **nonrandomized** rules in D;

b) Plot the risk set of all **randomized** rules $\mathcal{D}$ generated by the set of rules in D.

c) Find the risk point of the minimax rule in $\mathcal{D}$ and determine its minimax risk.

d) Define the minimax rule in the set $\mathcal{D}$ in terms of rules in D.

e) For which prior on $\{\theta_1, \theta_2\}$ is the minimax rule in the set $\mathcal{D}$ also a Bayes rule?

f) Determine the Bayes rule and the Bayes risk for the prior $\left(\frac{1}{3}, \frac{2}{3}\right)$ on $\{\theta_1, \theta_2\}$.

g) For a small positive $\epsilon < \frac{1}{2}$, illustrate on the risk set the risk points of all rules which are ϵ-Bayes with respect to the prior $(\frac{1}{2}, \frac{1}{2})$ on $\{\theta_1, \theta_2\}$.

2. In a sequence of consecutive years $1, 2, \ldots, T$, an annual number of high-risk events is recorded by a bank. The random number N_t of high-risk events in a given year is modelled via Poisson(λ) distribution. This gives a sequence of independent counts $n_1, n_2, \ldots, n_T$. The prior on λ is Gamma(a, b) with known $a > 0, b > 0$:

$$\tau(\lambda) = \frac{\lambda^{a-1} e^{-\lambda/b}}{\Gamma(a) b^a}, \lambda > 0.$$

a) Determine the Bayesian estimator of the intensity λ with respect to quadratic loss.

b) Assume $a = 2, b = 2$. The bank claims that the yearly intensity λ is less than 2. Within the last seven years counts were $0, 2, 3, 3, 2, 2, 4$. Test the bank's claim via Bayesian testing with a zero-one loss. (You may need the `integrate` function in `R` or another numerical integration routine from your favourite programming package to answer the question.)

3. In a medical experiment, a sequence of n Bernoulli trials with outcomes $X_i =$ 1 (success) or 0 (failure) is conducted whereby the probability of success $\theta_i, i = 1, \ldots, n$

randomly varies from trial to trial. In other words, the conditional distribution $X_i|\theta_i \sim$ $Bernoulli(\theta_i)$. It is assumed that the trials are independent. The prior on each θ_i is Beta(α, β). Let $T = \sum_{i=1}^{n} X_i$ denote the total number of successful trials.

i) Calculate $E(T)$ as a function of the prior and the sample size.

ii) Calculate $Var(T)$ as a function of the prior and the sample size.

Justify all of your steps.

Hint: You can use the fact that a Beta(α, β) distributed random variable has an expected value $\frac{\alpha}{\alpha+\beta}$.

4. At a critical stage in the development of a new bullet train in Japan, a decision must be taken to continue or to abandon the project. The prime minister, Yukio Hatoyama, has a good background in Statistics, is a Member of the Statistical Society of Japan and wants to utilise his statistical expertise. His team has managed to express the financial viability of the project by a parameter $\theta \in (0, 1)$, the project being profitable if $\theta > \frac{1}{2}$. Data x provide information about θ. If $\theta < 1/2$, the cost to the taxpayer of continuing the project is $(\frac{1}{2} - \theta)$(in units of \$ billion) whereas if $\theta > 1/2$ it is zero (since the project will be privatised if profitable). If $\theta > \frac{1}{2}$ the cost of abandoning the project is $(\theta - \frac{1}{2})$ (due to contractual arrangements for purchasing the train from the Chinese), whereas if $\theta < \frac{1}{2}$ it is zero. Two actions are on the table: a_0 : continue the project and a_1 : abandon it.

i) Derive the Bayesian decision rule in terms of the posterior mean of θ given x. In particular, show that the rule reduces to comparing the posterior mean of θ given x with a threshold constant.

ii) The Transport Minister has prior density $6\theta(1 - \theta)$ for θ. Hatoyama's own prior density is $3\theta^2$. The prototype bullet train is subjected to trials, each independently having probability θ of success, and the data x consists of the total number of trials required for the first successful result to be obtained. For which value of x will there be the most serious ministerial disagreement?