Functional Analysis Class Note

Spammer

2022 Autumn Semester, 1-16 Weeks

This note is taken for the Functional Analysiscourse, lectured by Professor Jiao Yong. In fact, this note contain full of my fragmentary thoughts, so all errors in this notes should be mine.

Here are some conventions:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are fields as you should learn in Mathematical Analysis. $\mathbb{K}$ is one of $\mathbb{R}$ and $\mathbb{C}$, usually used to state different cases conveniently. $\mathbb{N}$ is the set of **positive** integers.
- $\forall, \exists$ and $\exists!$ means "for all, there is and there is unique" respectively.
- Formula $A := B$ means A is defined as B. For example, $\mathbb{C} := \mathbb{R}[x]/(1+x^2)$ means $\mathbb{C}$ is defined as the quotient ring $\mathbb{R}[x]/(1+x^2)$.
- For each set A, the identity map is $\mathrm{id}_A \colon A \to A, a \mapsto a$.
- For $a, b \in \mathbb{R}$, define minimum function

  $$\wedge \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto \frac{a+b-|a-b|}{2},$$

  and maximum function

  $$\vee \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto \frac{a+b+|a-b|}{2}.$$

- Semminus of sets A, B is $A \setminus B := \{x \in A : x \notin B\}$.
- For a sets A, $\mathcal{P}(A)$ means the power set of A.
- For proposition p, q, we use $p \wedge q$ to mean the proposition "p and q", $\wedge$ has the truth table as follow:

  Similarly, we define $p \vee q$.

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

- Addition of real/complex-valued functions is defined pointwisely. That is: let $f, g\colon X \to \mathbb{K}$, we define a funtion $f + g\colon X \to \mathbb{K}$ by $x \mapsto f(x) + g(x)$.
- For $f\colon X \to \mathbb{K}$ and $k \in \mathbb{K}$, we define that function $f + k$ by $x \mapsto f(x) + k$. That is, respect k as a constant function $x \mapsto k$.
- Somewhere you can see color different, that is reminding you to think about what here should be. (Just like $1 + 1 = 2$.)
- $\lim_n$ means $\lim_{n\to\infty}$ for short.
- We say a diagram commutes, if all the morphisms (and their possible compositions) with the same domain and same codomain coincide.
- The Kronecker symbol on a set is defined as $\delta\colon X \times X \to \{0, 1\}, (x, y) \mapsto \delta_y^x := \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$.
- Let $(\Omega, \mathcal{F})$ be a measurable space. We say a function $f\colon \Omega \to \mathbb{R}$ is measurable, if the preimage of Borel subsets of $\mathbb{K}$ under f is $\mathcal{F}$-measurable. That is, assume $\mathbb{K}$ is equipped with Borel σ-algebra.

Contents

0 Introduction

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Syllabus

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1 Week 1, Lecture 1

We begin from **Banach Spaces and Metric**. Before the definition of Banach Space, we should recall the definition of Vector spaces(or Linear Spaces). Given a set X, a vector space is a triple $(X, +, \cdot)$ where $+\colon X \times X \to X$ is called the addition on X, and $\cdot\colon \mathbb{K} \times X \to X$ is called scalar-multiplication on X, satisfying 8 axioms.

Recall:An isomorphism between vector space means a bijection that keeps the linear structure, that is $\varphi\colon X \to Y$ satisfies: $\forall k, l \in \mathbb{K}, \forall x, x' \in X$ we have $\varphi(kx + lx') = k\varphi(x) + l\varphi(x')$. Isomorphisms in categoryshould be in mind.

1.1 Linear Normed space

Definition (Linear Normed space)**.** Let X be a linear space. Define a map $\|\ \|\colon X \to \mathbb{R}_{\geq 0}$ satisfying:

(i) $\|x\| = 0(\in \mathbb{K}) \iff x = 0(\in X)$;

(ii) $\|kx\| = |k| \cdot \|x\|(\forall k \in \mathbb{K}, x \in X)$;

(iii) $\|x + y\| \leq \|x\| + \|y\|(\forall x, y \in X)$.

Then $\|\ \|$ is called a **norm** over X, and $(X, \|\ \|$ is called a **linear normed space**.

Remark. There is some similar weaker definitions:

- If(only)**(i)** is not satisfied, we call $\|\ \|$ a semi-norm.
- If(only)**(iii)** becomes $\|x + y\| \leq C(\|x\| + \|y\|)$ for some $C \in \mathbb{R}_{>1}$, we call $\|\ \|$ a quasi-norm.

Example (Euclidean Spaces)**.** $(\mathbb{R}^n, \|\ \|)$ is a linear normed space, whose norm is defined as follow:

$$\|\ \|\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}, x = (x_1, \ldots, x_n) \mapsto \left(\sum_{j=1}^{n} x_j^2\right)^{1/2} (= d(x, 0)).$$

Triangle inequality for this norm comes to be the particular triangle inequality for the metric, which can be shown by Cauchy-Schwarz inequality for real numbers.

Example (Continuous Functions Spaces)**.** $\left(C([a, b], \mathbb{K}), \max_{[a,b]}|\ |\right)$ is a linear normed space. Recall the definition of $C([a, b], \mathbb{K})$ the family of continuous function from $[a, b]$ to $\mathbb{K}$. whose norm is defined as follow:

$$\max_{[a,b]}|\ |\colon (C([a, b], \mathbb{K}) \to [0, \infty), f \mapsto \max_{x \in [a,b]} |f(x)|.$$

Recall why $C([a,b],\mathbb{K})$ is a vector space. What is needed to show is just "addition of continuous functions is continuous", and there is lots of ways to do this, see remark. Notice that $[a,b]$ is compact and so is $f([a,b])$, guaranteeing the existence of $\max_{x\in[a,b]}|f(x)|$. Compatibility with multiplication and triangle inequality is trivial.

Remark. We have follow methods for proving "addition of continuous functions is continuous". They give the same result with different standpoints. Suppose $f,g \in C([a,b],\mathbb{K})$

1. By definition of continuity. We prove pointwisely: Fix $x \in [a,b]$. $\forall \varepsilon > 0$,we can find $\delta_1, \delta_2 > 0$ such that $\forall y : 0 < |y-x| < \delta_1, |f(y)-f(x)| < \varepsilon/2$ and $\forall y : 0 < |y-x| < \delta_2, |g(y)-g(x)| < \varepsilon/2$. Therefore, let $\delta := \delta_1 \wedge \delta_2$ and we have $\forall y : 0 < |y-x| < \delta$,

$$\begin{aligned} |(f+g)(y)-(f+g)(x)| =& |f(y)+g(y)-f(x)-g(x)| \\ \le& |f(y)-f(x)|+|g(y)-g(x)| \\ <& \varepsilon/2+\varepsilon/2 \\ =& \varepsilon. \end{aligned}$$

   Therefore, $f+g$ is continuous at x.

2. By Heine's Theorem: We prove pointwisely: Fix $x \in [a,b]$. Suppose sequence $(x_n)_{n\in\mathbb{N}} \subseteq [a,b]$ converges to x, then:

$$\begin{aligned} \lim_{n\to\infty}(f+g)(x_n) =& \lim_{n\to\infty}\big(f(x_n)+g(x_n)\big) \\ =& \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) \\ =& f(x)+g(x) \\ =& (f+g)(x). \end{aligned}$$

   Therefore, $f+g$ is continuous at x.

3. By topological definition($\mathbb{K}=\mathbb{R}$ case): an observation :

$$(f+g)^{-1}(t,\infty) = \bigcup_{r\in\mathbb{R}} \big(f^{-1}(t-r,\infty)\cap g^{-1}(r,\infty)\big),$$

   which should be prove by $A \subseteq B \wedge B \subseteq A \implies A = B$. Right hand side is union of intersection of two open sets, and similarly for $(f+g)^{-1}(-\infty,t)$.We're done.

4. Addition is continuous($\mathbb{K}=\mathbb{R}$ case). We decompose $f+g$ as

following communicative diagrams

$$\begin{tikzcd}
{[a,b]} \arrow[r, "f\times g"] \arrow[dr, "f+g"'] & \mathbb{R}\times\mathbb{R} \arrow[d, "+"] \\
& \mathbb{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dr, mapsto] & (f(x),g(x)) \arrow[d, mapsto] \\
& f(x)+g(x)
\end{tikzcd}$$

The right diagram explains what the functions in the left diagram mean. By the property of product topology and continuity of f and g, we know $f \times g$ is continuous. Continuity of $+\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is trivial. Therefore $f + g = + \circ (f \times g)$ is continuous.

To get rid of the assumption $\mathbb{K} = \mathbb{R}$, use the fact that $f\colon X \to \mathbb{C}$ is continuous if and only if both $\mathrm{Re}(f), \mathrm{Im}(f)$ are continuous.

Example (p-summable sequence spaces)**.** Given $p \in [1, \infty]$ we define $(\ell_p, \| \ \|_p)$, where

$$\begin{aligned}
\ell_p &:= \Big\{(a_n)_{n\in\mathbb{N}} : \sum_{n\geq 1} |a_n|^p < \infty\Big\}, \text{(for } p < \infty\text{)} \\
\ell_\infty &:= \Big\{(a_n)_{n\in\mathbb{N}} : \sup_{n\in\mathbb{N}} |a_n| < \infty\Big\}.
\end{aligned}$$

And norms are

$$\begin{aligned}
\|a\|_p &:= \Big(\sum_{n\geq 1} |a_n|^p\Big)^{1/p}, && \text{(for } p < \infty\text{)} \\
\|a\|_\infty &:= \sup_{n\in\mathbb{N}} |a_n|. && \text{(Here } a \text{ means } (a_n)_{n\in\mathbb{N}}\text{)}
\end{aligned}$$

Proposition 1.1. $(\ell_\infty, \| \ \|_\infty)$ is a normed space.

Proof. Clearly ℓ_∞ is a vector space. Now we prove $\| \ \|_\infty$ is a norm.

1. $\|a\|_\infty \geq 0$ and $\|a\|_\infty = 0 \iff a = 0$: $\|a\|_\infty \geq 0$ is trivial. Suppose $\|a\|_\infty = 0$, that is $\sup_{n\in\mathbb{N}}|a_n| = 0$. By definition of supermum, $|a_n| \leq 0 (\forall n \in \mathbb{N})$. Therefore, $a = 0$.

2. $\forall k \in \mathbb{K}$, by property of absolute value we know $\|ka\|_\infty = |k|\|a\|_\infty$.

3. Let $a, b \in \ell_\infty$ and $M_a = \|a\|_\infty, M_b = \|b\|_\infty$. Now from definition of supermum

$$\forall n \in \mathbb{N} : |a_n + b_n| \leq |a_n| + |b_n| \leq M_a + M_b$$

Again using definition of supermum, we get $\|a + b\|_\infty \leq M_a + M_b$, which was what we wanted.

$\square$

Remark. In general, for a measure space $(\Omega, \mathcal{F}, \mu)$ the inequality $\|f+g\|_p \leq \|f\|_p + \|g\|_p (p \geq 1)$ is called the Minkowski inequality.

Example. ℓ_∞ has linear subspaces: $c_0 \subseteq c \subseteq \ell_\infty$, where

$$
\begin{aligned}
c :=& \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is convergent sequence}\}, \\
c_0 :=& \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is convergent sequence, with limit } 0\}.
\end{aligned}
$$

2 Week 1, Lecture 2

2.1 p -integrable function spaces

Recall the left problem: Minkowski inequality, which makes $(\ell_p, \| \, \|_p)$ a normed space. Now, we need a lemma.

Lemma (Hölder's Inequality)**.** Let $a \in \ell^p, b \in \ell^q$ for $p \in (1, \infty)$ and $q \in (1, \infty)$ satisfy $1/p + 1/q = 1$, we have:

$$\|ab\|_1 \leq \|a\|_p \|b\|_q, \tag{1}$$

Remark. $q = p/(p-1)$ is also called the dual index of p, usually denoted by p'.

Remark. Before start of the proof, we have a look at (1). Recall what we have learned in mathematical analysis, and have a problem in mind: is there anything similar? That is Cauchy-Schwarz Inequality, since they coincide when $p = q = 2$. Now we have a direct goal.

Aim. Prove (1) by imitating the proof of Cauchy-Schwarz Inequality.

Now, recall all the proofs of Cauchy-Schwarz Inequality you know and think: Which would be useful in this case? [4] Lagerange's Idendity, Schwarz's argument(inner product $\langle x + ty, x + ty \rangle \geq 0$), or just $2xy \leq x^2 + y^2$? When $p \neq 2$, Schwarz's argument is a nonstarter since there is no quadratic polynomial in sight. Similarly, the absence of a quadratic form means that one is unlikely to find an effective analog of Lagrange's identity.

This brings us to our most robust proof of Cauchy-Schwarz Inequality, the one that starts with the so-called "humble bound,"

$$xy \leq \frac{x^2}{2} + \frac{y^2}{2}, \forall x, y \in \mathbb{R}. \tag{2}$$

(2) proves Cauchy's inequality as follows.

Proof of Cauchy's inequality from (2). Without lost of generality, suppose that $\sum_{n\geq 1} a_n^2 \neq 0$ and $\sum_{n\geq 1} b_n^2 \neq 0$. Let

$$a_j' = a_j / \Big(\sum_{n\geq 1} a_n^2 \Big)^{1/2}, b_j' = b_j / \Big(\sum_{n\geq 1} b_n^2 \Big)^{1/2}, \forall j \in \mathbb{N}.$$

Notice that $\sum_{n\geq 1} a_n' = \sum_{n\geq 1} b_n' = 1$. Now, since (2) holds, we obtain

$$\sum_{n\geq 1} a_n b_n \leq \sum_{n\geq 1} (a_n^2 + b_n^2)/2 = \sum_{n\geq 1} a_n^2/2 + \sum_{n\geq 1} b_n^2/2 = 1.$$

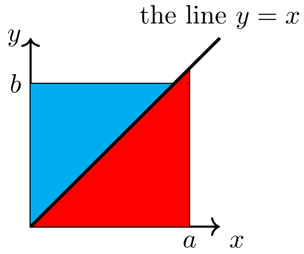

Figure 1: Area meaning of (2)

And, in terms of $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and multiply $(\sum_{n\geq 1} a_n^2)^{1/2}(\sum_{n\geq 1} a_n^2)^{1/2}$ on both sides, we have

$$\sum_{n\geq 1} a_n b_n \leq \Big(\sum_{n\geq 1} a_n^2\Big)^{1/2}\Big(\sum_{n\geq 1} b_n^2\Big)^{1/2}. \qquad \square$$

This bound may now remind us that the general AM-GM inequality

$$x^p y^q \leq \frac{x}{p} + \frac{y}{q} \quad \text{for all } x, y \geq 0 \text{ and } q = p'(p, q > 1). \tag{3}$$

(3) is the perfect analog of the "humble boun"(2).

Proof of (2). There is many ways to to this, see[4]. We choose the way by area of regions. Consider the region under the function $x \mapsto x$:

$$A := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq a\}, B := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq b\}.$$

Then (1)[Figure 1] shows that $m(A) + m(B) \geq m([0, a] \times [0, b])$, where m denotes the Lebesgue measure on $\mathbb{R}^2$. $\square$

Now, by imitating the proof of (2), we need to get the x^p/p as area of some region under a function, so consider the function $x \mapsto x^{p-1}$.

Proof of (3). It's easy to verify that

$$m(A) = \int_{[0,a]} f \,\mathrm{d}m, m(B) = b^{\frac{p}{p-1}} - \int_{[0,b^{p/(p-1)}]} f \,\mathrm{d}m,$$

where m is the Lebesgue measure on $\mathbb{R}$. By simple calculation, we have $m(A) = \frac{1}{p}a^p, m(B) = \frac{1}{q}b^q$. Notice that $A \cup B$ contains $[0, a] \times [0, b]$, we're done. $\square$

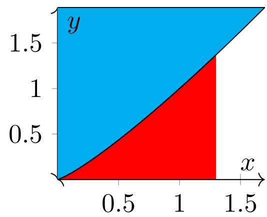

Figure 2: Area meaning of (3)

Proof of (1). Without loss of generality, suppose $\|a+b\|_p \neq 0$. And suppose $a \neq 0 (\in \ell_p), b \neq 0 (\in \ell_q)$. As what we do in the proof of Cauchy's inequality, let

$$a'_j = a_j / \|a\|_p, b'_j = b_j / \|b\|_p, \forall j \in \mathbb{N}.$$

Notice that $\|a'\|_p = \|b'\|_q = 1$. Now, apply (3) to $|a_j b_j|$, we have

$$\sum_{n \geq 1} |a'_n b'_n| \leq \sum_{n \geq 1} |a'_n|^p / p + \sum_{n \geq 1} |b'_n|^q / q = 1/p + 1/q = 1,$$

which implies

$$\|ab\|_1 \leq \|a\|_p \|b\|_p. \qquad \square$$

Proof of Minkowski inequality.

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{n \geq 1} |(x+y)_n|^p \\
&= \sum_{n \geq 1} |x_n + y_n|^{p-1} |x_n + y_n| \\
&\leq \sum_{n \geq 1} |x_n + y_n|^{p-1} (|x_n| + |y_n|) (\text{Triangle inequality on } \mathbb{R}) \\
&= \sum_{n \geq 1} |x_n + y_n|^{p-1} |x_n| + \sum_{n \geq 1} |x_n + y_n|^{p-1} |y_n| \\
&= \left\| (x+y)^{p-1} x \right\|_1 + \left\| (x+y)^{p-1} y \right\|_1 (\text{def of norm}) \\
&\leq \left\| (x+y)^{p-1} \right\|_q \|x\|_p + \left\| (x+y)^{p-1} \right\|_q \|y\|_p (\text{see } (1)) \ (*) \\
&= \|(x+y)\|_p^{p/q} (\|x\|_p + \|y\|_p) ((p-1)q = p),
\end{aligned}$$

and divide $\|x+y\|_p^{p/q} (\neq 0)$ from both sides, getting

$$\|x+y\|_p^{p-p/q} \leq \|x\|_p + \|y\|_p.$$

We're done, since $p - p/q = 1$. $\square$

To summarize what we have done, we need the language of measure.

Definition (σ-algebra)**.** A **σ-algrbra** on a set Ω is a subset Ω, satisfying:

1. $\Omega, \varnothing \in \mathcal{F}$;
2. $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$;
3. $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F} \implies \bigcup_{n\geq 1} A_n \in \mathcal{F}$.

Definition (Measurable Spaces, Measurable sets)**.** A **measurable space** is a double $(\Omega, \mathcal{F})$ where Ω is an aritrary set and $\mathcal{F}$ is a σ-algebra over Ω. Elements of $\mathcal{F}$ is called **measurable sets** of $(\Omega, \mathcal{F})$.

Definition (Measure,Measure spaces)**.** A **measure** is a σ-additive function from $\mathcal{F}$ to $[0, \infty]$. A triple $(\Omega, \mathcal{F}, \mu)$ is called a **measure space**, if $(\Omega, \mathcal{F})$ is a measurable space and μ is a measure.

Definition (Integral with respect to measure)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We have a glance at "how to define integral with respect to measure". For the detail, see [2].

Step 1: Define **integral** $\int$ for measurable simple nonnegative function:

$$\sum_{k=1}^{n} a_k \chi_{A_k} \longmapsto \sum_{k=1}^{n} a_k \mu(A_k).$$

Step 2: Define **integral** $\int$ for measurable nonnegative function:

$$f \longmapsto \sup\Big\{ \int \varphi : \varphi \leq f, \varphi \text{ is nonnegative simple function}\Big\}.$$

Step 3: Define **integral** $\int$ for measurable function:

$$f \longmapsto \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu,$$

where $f^+ = f\chi_{f^{-1}[0,\infty)}, f^- = -f\chi_{f^{-1}(-\infty,0]}$.

Definition (p-integrable space)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, then the p-integrable space over $\mathcal{L}_p(\Omega, \mathcal{F}, \mu)$ is defined as

$$\mathcal{L}_p(\Omega, \mathcal{F}, \mu) := \Big\{ f \in \mathbb{K}^\Omega : f \text{ is measurable and } \int f \,\mathrm{d}\mu < \infty \Big\}.$$

Fact. The proof of Minkowski inequalityover ℓ_p actually proved the Minkowski inequalityof **every** p-integrable space $\mathcal{L}_p(\Omega, \mathcal{F}, \mu)$.

To understand this fact, we should have another way to illustrate $\sum$. That is, $\sum$ is a kind of integral.

Definition (Counting Measure)**.** Given a measurable space $(\Omega, \mathcal{F})$. Define $\mu\colon \mathcal{F} \to [0, \infty], A \mapsto \sharp A$. Where $\sharp A = \infty$ if A is an infinite set, and $\sharp A = n$ if A has exactly n elements. μ is called the **counting measure** over $(\Omega, \mathcal{F})$.

Remark. It can be shown that,[1] for real sequence $(a_n)_{n\in\mathbb{N}}$ (equivalent to a function $a\colon \mathbb{N} \to \mathbb{R}$), we have

$$\sum_{n\geq 1} a_n = \int a \,\mathrm{d}\mu.$$

That's why we can respect $\sum$ as $\int$. And hence, the fact above is just regard $\sum$ as integral with respect to coungting measure, and the proof works for arbitrary measure space.

Remark. We can also prove Minkowski inequalityof L^p by using the $L^{p'}$. Since

$$\|f\|_p = \sup\left\{\left|\int fg \,\mathrm{d}\mu\right| : g \in L^{p'}(\Omega, \mathcal{F}, \mu), \|g\|_{p'} \leq 1\right\}.$$

3 Week 2, Lecture 1

3.1 Quotient Spaces

Let X be a vector space with a linear subspace X_0, denoted as $X_0 \hookrightarrow X$.

Definition (Cosets). $\forall x \in X$, the coset of x (with respect to X_0), denoted as $[x]$ or $x + X_0$ is defined as

$$[x] = x + X_0 := \{x + y : y \in X_0\}.$$

Definition (Quotient Spaces). ${}^{X}\!/\!{}_{X_0} := \{[x] : x \in X\}$, called the quotient space of X (with respect to X_0).

We want ${}^{X}\!/\!{}_{X_0}$ to be a vector space, so we define operations as follows:

$$\oplus : {}^{X}\!/\!{}_{X_0} \times {}^{X}\!/\!{}_{X_0} \to {}^{X}\!/\!{}_{X_0}, ([x],[y]) \mapsto [x+y];$$
$$\odot : \mathbb{K} \times {}^{X}\!/\!{}_{X_0} \to {}^{X}\!/\!{}_{X_0}, ([x],k) \mapsto [kx].$$

Where $[x+y]$ means the addtion (and take the coset), and the $[kx]$ means the scalar multiplication of X (and take the coset). You should verify that the operations are well defined. For simplicity, we write $+, \cdot$ instead of $\oplus, \odot$.

Claim. $({}^{X}\!/\!{}_{X_0}, +, \cdot)$ is a vector space.

Question. Think this questions:

1. Clearly, the zero element in ${}^{X}\!/\!{}_{X_0}$ is $[0]$. But, $[0] = ?$;
2. If $[x] \neq [y]$, what is $[x] \cap [y]$?
3. Show that $x \in [y] \iff x - y \in X_0$.

Answers are as follows:

1. $[0] = X_0$, from definition of coset.
2. $\varnothing$. Since (3) implies $[x] \cap [y] \neq \varnothing$ means $\exists z : z - x, z - y \in X_0$, therefore $x - y = (z - y) - (z - x) \in X_0$ since X_0 is a linear subspace. Now, $\forall a \in [x]$, from $a = x + w(w \in X_0)$, we have $a = y + \big(w + (x - y)\big)$ and $\big(w + (x - y)\big) \in X_0$ so $a \in [y]$. Above all, $[x] \subseteq [y]$. It is the same to know $[y] \subseteq [x]$.

3. Since

$$\begin{aligned} x \in [y] &\iff x = y + z \text{ for some } z \in X_0 \\ &\iff x - y = z (= 0 + z) \text{ for some } z \in X_0 \\ &\iff x - y \in [0] = X_0. \end{aligned}$$

Let's see a simple example:

Example. From previous example, $c_0 \hookrightarrow c \hookrightarrow \ell_\infty$. And we introduce a new notion:

Definition (Codimension)**.** Suppose X a vector space and $X_0 \hookrightarrow X$. Then the codimension of X_0, is $\operatorname{codim}_X X_0 := \dim {}^{X}\!/_{X_0}$. Also denoted by just $\operatorname{codim}(X_0)$ if there is no confusion.

Claim. $\operatorname{codim}_c c_0 = 1$.

Proof. Let $(1_n)_{n\in\mathbb{N}}$ be the sequence with all elements 1. We want to show that $\{(1_n)_{n\in\mathbb{N}}\}$ is a basis of ${}^{c}\!/_{c_0}$. Let $(x_n)_{n\in\mathbb{N}} \in c$, and suppose $\lim_n x_n = x \in \mathbb{K}$. We have $[(x_n)_{n\in\mathbb{N}}] = [x(1_n)_{n\in\mathbb{N}}]$, since $x(1_n)_{n\in\mathbb{N}}$ is just the sequence with all elements x, and clearly $\lim_n(x_n - x) = 0 \implies (x_n)_{n\in\mathbb{N}} - x(1_n)_{n\in\mathbb{N}} \in c_0$. That is, $[(x_n)_{n\in\mathbb{N}}] = [x(1_n)_{n\in\mathbb{N}}] = x[(1_n)_{n\in\mathbb{N}}]$. We're done. $\square$

Remark. There is an isomorphism from ${}^{c}\!/_{c_0}$ to $\mathbb{K}$, that is $[(x_n)_{n\in\mathbb{N}}] \mapsto \lim_n x_n$.

Example. Consider $X = \mathbb{R}^2$, $X_0 \hookrightarrow X$ with $\dim X_0 = 1$. It is easy to see that $\forall x \in \mathbb{R}$, the coset containing x is just translating X_0 such that $0 (\in X_0)$ is translated to x. And

$${}^{X}\!/_{X_0} = \{X_0\} \cup \{\text{all lines that are parallel to } X\}.$$

Now we want to define a norm on ${}^{X}\!/_{X_0}$. An intuitive norm is the distance between X_0 and the coset.

Definition (Norm on ${}^{X}\!/_{X_0}$)**.** Define

$$\| \ \| \colon {}^{X}\!/_{X_0} \to \mathbb{R}_{\geq 0}, [x] \mapsto \inf_{y \in X_0} \|x - y\|.$$

The norm in green color is the usual norm in $\mathbb{R}^2$, see previous example.

We should verify that $\| \ \|$ is actually a norm. That is

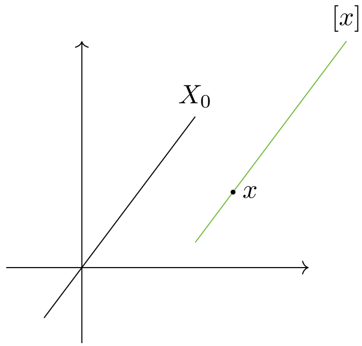

Figure 3: X, X_0 and points of ${}^{X}\!/_{X_0}$

Question. Verify that :

1. $\forall [x] \in {}^{X}\!/_{X_0} : \|[x]\| \geq 0$ and $\|x\| = 0 \iff x = X_0$;
2. $\forall [x] \in {}^{X}\!/_{X_0} : \|k[x]\| = |k| \cdot \|x\|$;
3. $\|[x] + [y]\| \leq \|[x]\| + \|[y]\|$.

Proof. For (1): Only needed is to show that $\|x\| = 0 \iff x = X_0$. Here we use a Theorem (in the below remark) and a trivial fact:

Fact. X_0 is a closed subset of X.

Now suppose $[x] \in {}^{X}\!/_{X_0}$ satisfying $\|[x]\| = 0$. By definition, we have $\inf_{y \in X_0} \|x - y\| = 0$. From the definition of infimum : $\forall n \in \mathbb{N} \exists y_n \in X_0$ such that $\|x - y_n\| < 1/n$, therefore we have a sequence $(y_n)_{n \in \mathbb{N}} \subseteq X_0$ converging to x. From the theorem below, we know $x \in X_0$, so $[x] = X_0$ as we wanted.

(2): It holds naturally when $k = 0$. If $k \neq 0$, it just follows from property of norm and $k^{-1} X_0 = X_0$.

(3): From intuition, we have

$$
\begin{aligned}
\|[x] + [y]\| &= \|[x + y]\| \\
&= \inf_{z \in X_0} \|x + y - 2z\| \\
&\leq \inf_{z \in X_0} (\|x - z\| + \|y - z\|) \text{(triangle inequality of norm on } X\text{)} \\
&\leq \inf_{z \in X_0} \|x - z\| + \inf_{z \in X_0} \|y - z\| \\
&= \|[x]\| + \|[y]\|.
\end{aligned} \tag{4}
$$

So easy, isn't it? However, look at $\leq$, this inequality is non-trivial and we should prove. By simple application of definition of infimum, we find: the inequality is **reversed**! But (4) can be corrected: $\forall \varepsilon > 0$, $\exists z_\varepsilon \in X_0, w_\varepsilon \in X_0$ such that

$$\|x - z_\varepsilon\| < \inf_{z \in X_0} \|x - z\| + \varepsilon/2 = \|x\| + \varepsilon/2,$$
$$\|y - w_\varepsilon\| < \inf_{z \in X_0} \|y - z\| + \varepsilon/2 = \|y\| + \varepsilon/2.$$

Therefore we have

$$\inf_{z \in X_0} (\|x - z\| + \|y - z\|) \leq \|x\| + \|y\| + \varepsilon.$$

Since ε is arbitrary, we know $\inf_{z \in X_0}(\|x - z\| + \|y - z\|) \leq \|x\| + \|y\|$ and then $\|x + y\| \leq \|x\| + \|y\|$.

However, this is wrong again. Since z_ε may not coincide with w_ε. To fix this, write

$$\|[x + y]\| = \inf_{z,w \in X_0} \|x + y - (z + w)\|. \tag{5}$$

By (5), and $\|x + y - (z + w)\| \leq \|x - z\| + \|y - w\|$, we use the definition of inf for $\inf_{z \in X_0} \|x - z\|, \inf_{w \in X_0} \|y - w\|$. We can find $z_\varepsilon, w_\varepsilon$ as above and get $\|[x + y]\| \leq \|[x]\| + \|[y]\| + \varepsilon$, we're done.

Above all, $\| \ \|$ is actually a norm. □

Remark. We define the topology of linear normed space as follows:

Definition (Topology of linear normed space)**.** Let $(X, \| \ \|)$ be a linear normed space. Then there is a natural metric on X, that is $d \colon X \times X \to \mathbb{R}_{\geq 0}, (x, y) \mapsto \|x - y\|$. The topology induced by this metric is called the (usual) topology of $(X, \| \ \|)$.

Now we have a topology of X, and we have a result characterizing the closed subsets of X.

Theorem. Given a linear normed space X with $X_0 \hookrightarrow X$. Then, X is closed **if and only if** for all $(x_n)_{n \in \mathbb{N}} \subseteq X_0$ such that $\lim_n x_n = x \in X$, we have $x \in X_0$.

Remark. A quotient semi-norm in X/X_0 is a norm if and only if X_0 is closed.

4 Week 2, Lecture 2

Substitute teacher: Wu Lian.

4.1 Metric Spaces

Definition (Metric, Metric Spaces)**.** Let X be a set. $d\colon X \times X \to \mathbb{R}$ is called a metric, if d satisfies:

1. $\forall x, y \in X\colon d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$.
2. $\forall x, y \in X\colon d(x,y) = d(y,x)$.
3. $\forall x, y, z \in X\colon d(x,y) + d(y,z) \geq d(x,z)$.

The ordered pair (X, d) is called a metric space.

Remark. Every metric space has a topology, we will discuss this later.

Remark. Let's compare normed spaces and metric spaces: normed space need linear structures but metric spaces don't need. A normed space $(X, \|\ \|)$ is naturally a metric space by the metric induced by norm $d\colon X \times X \to \mathbb{R}, (x,y) \mapsto \|x - y\|$.

Remark. Let X be an arbitrary set, we can define a metric on X by the Kronecker symbol δ.

Example. $(\mathbb{R}^n, d)$ is a metric space, where

$$d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) \mapsto \Big(\sum_{j=1}^{n} (x_j - y_j)^2\Big)^{1/2}.$$

Example. $(\mathbb{R}^{\mathbb{N}}, d)$ is a metric space, where

$$d\colon \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}, \big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\big) \mapsto \sum_{j\geq 1} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}.$$

d is well-defined, since the series can be dominated by $\sum_{j=1}^{\infty} 1/2^j$. To verify the triangle inequality, we use the monotone function $f\colon [0,\infty) \to [0,1), x \mapsto x/(1+x)$. So, $|x_j - y_j| + |y_j - z_j| \geq |x_j - z_j|$ implies

$$\frac{|x_j - y_j| + |y_j - z_j|}{1 + |x_j - y_j| + |y_j - z_j|} \geq \frac{|x_j - z_j|}{1 + |x_j - z_j|},$$

and clearly the left-hand side is no more than $f(|x_j - y_j|) + f(|y_j - z_j|)$. Sum for $j \in \mathbb{N}$ and we're done.

Example. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let $\mathcal{L}_0(\Omega)$ be the space of all $\mathcal{F}$-measurable functions from Ω to $\mathbb{K}$, written $\mathcal{L}_0$ for short. Define $\mathcal{Z} := \{f \in \mathcal{L}_0(\Omega) : f(x) = 0$ for μ-almost every $x \in \Omega\}$, the (linear) subspace containing all functions equal 0 μ-almost everywhere. Now consider the quotient space ${\mathcal{L}_0}/_{\mathcal{Z}}$. We define

$$\begin{aligned} d\colon {\mathcal{L}_0}/_{\mathcal{Z}} \times {\mathcal{L}_0}/_{\mathcal{Z}} &\longrightarrow \mathbb{R} \\ (f + \mathcal{Z}, g + \mathcal{Z}) &\longmapsto \int_\Omega \frac{|f-g|}{1+|f-g|}\,\mathrm{d}\mu. \end{aligned} \tag{6}$$

Integrand on the right-hand side can be dominated by $1_\Omega (= 1)$, hence the integral is finite. The definition of d involves the selection of representative element, so we should verify that d is well-defined. Suppose $f + \mathcal{Z} = f' + \mathcal{Z}, g = g' + \mathcal{Z}$, and suppose f, g is finite everywhere, then

$$\begin{aligned} &\exists A_1 : \mu(A_1) = 0\ \forall x \in A_1^c\ f(x) = f'(x); \\ &\exists A_2 : \mu(A_2) = 0\ \forall x \in A_1^c\ g(x) = g'(x). \end{aligned} \tag{7}$$

Then $f(x) - g(x) = f'(x) - g'(x)$ for all $x \in (A_1 \cup A_2)^c$ and $\mu(A_1 \cup A_2) = 0$. Therefore $f - g = f' - g'$ μ-almost everywhere, and hence $\frac{|f-g|}{1+|f-g|} = \frac{|f'-g'|}{1+|f'-g'|}$ μ-almost everywhere, implying that their integration coincide. Above all, $d(f + \mathcal{Z}, g + \mathcal{Z}) = d(f' + \mathcal{Z}, g' + \mathcal{Z})$ whenever $f - f' \in \mathcal{Z}, g - g' \in \mathcal{Z}$.

Proof of "d is a metric" is the same as the previous example.

Example. These are all metric spaces, since they are linear normed spaces: $\ell_p, c_0, c, C([a,b], \mathbb{K}), L_p, \mathbb{R}^n$.

Definition (Convergence in metric space)**.** Let (X, d) be a metric space. A sequence in X, say $(x_n)_{n\in\mathbb{N}} \subseteq X$. We say $(x_n)_{n\in\mathbb{N}}$ is convergent to $x \in X$, if $\lim_n d(x_n, x) = 0$(limit of real sequence). $(x_n)_{n\in\mathbb{N}}$ is convergent to x is usually denoted by $(x_n)_{n\in\mathbb{N}} \xrightarrow{d} x$(or $(x_n)_{n\in\mathbb{N}} \to x$ if there is no aml

Example. Suppose X is an arbitrary set. (X, δ) is a metric space, where δ means the Kronecker symbol. Then

$$(x_n)_{n\in\mathbb{N}} \to x \iff \exists N \in \mathbb{N}\ \forall n \geq N\ x_n = x.$$

Example. Consider $\big((C[a,b], \mathbb{K}), d\big)$, where

$$d\colon (C[a,b], \mathbb{K}) \times (C[a,b], \mathbb{K}) \to \mathbb{R}, (f, g) \mapsto \max_{[a,b]} |f - g|.$$

Then $(f_n)_{n\in\mathbb{N}} \xrightarrow{d} f \iff (f_n)_{n\in\mathbb{N}}$ converge to f uniformly, as we learned in Mathematical Analysis.

Example. Recall $({}^{L_0}\!/\!_{\mathcal{Z}}, d)$, $(f+\mathcal{Z}_n)_{n\in\mathbb{N}} \xrightarrow{d} f+\mathcal{Z} \iff (f_n)_{n\in\mathbb{N}} \xrightarrow{\mu} f$.

Proof. Necessity:$(f+\mathcal{Z}_n)_{n\in\mathbb{N}} \xrightarrow{d} f+\mathcal{Z}$ means

$$\lim_n \int_\Omega \frac{|f_n - f|}{1+|f_n-f|}\,\mathrm{d}\mu = 0.$$

Given $\sigma > 0$. Define a set $E_n^\sigma := \{x \in \Omega : |f_n(x) - f(x)| > \sigma\}$, we need to show $\lim_n \mu(E_n^\sigma) = 0$. By Chebyshev's inequality:

$$\begin{aligned}
\mu(E_n^\sigma) =& \mu\{x \in \Omega : |f_n(x) - f(x)| > \sigma\} \\
=& \mu\Big\{x \in \Omega : \frac{|f_n(x) - f(x)|}{1+|f_n(x)-f(x)|} > \frac{\sigma}{1+\sigma}\Big\} \\
\leq& \frac{1+\sigma}{\sigma}\int_{E_\sigma^n} \frac{|f_n(x) - f(x)|}{1+|f_n(x)-f(x)|}\,\mathrm{d}\mu \\
\leq& \frac{1+\sigma}{\sigma}\int_{\Omega} \frac{|f_n(x) - f(x)|}{1+|f_n(x)-f(x)|}\,\mathrm{d}\mu \\
=& \frac{1+\sigma}{\sigma} d(f_n+\mathcal{Z}, f+\mathcal{Z}).
\end{aligned}$$

$\lim_n d(f_n+\mathcal{Z}, f+\mathcal{Z}) = 0$ implies $\lim_n \mu(E_n^\sigma) = 0$, that is $f_n \xrightarrow{\mu} f$.

Sufficiency: Given $\sigma \in (0,1)$, we know:

$$\Big\{x \in \Omega : \frac{|f_n - f|}{1+|f_n-f|} > \sigma\Big\} = \{x \in \Omega : |f_n - f| > \frac{\sigma}{1-\sigma}\}.$$

This implies that $\frac{|f_n-f|}{1+|f_n-f|} \xrightarrow{\mu} 0$.

Now, from the dominated convergence theorem (1_Ω being the dominated function), we have:

$$\lim_n d(f_n+\mathcal{Z}, f+\mathcal{Z}) = \lim_n \int_\Omega \frac{|f_n - f|}{1+|f_n-f|}\,\mathrm{d}\mu = \int_\Omega \lim_n \frac{|f_n - f|}{1+|f_n-f|}\,\mathrm{d}\mu = 0.$$

□

Remark. I didn't get this solution for sufficiency at the class. So, it is meaningful to have a look after class.

Topology of metric spaces

Definition (Topology of metric space)**.** The topology of a metric space (X,d) is generated by the base

$$\mathcal{B} = \{B(x,r) \colon x \in X, r \in (0,\infty)\},$$

where $B(x,r) := \{y \in X \colon d(y,x) < r\}$.

Remark. Now we can define these things for metric spaces:

- Interior points of a set.
- Interior of sets.
- Limit points of a set.
- Derived sets.
- Closure.
- Isolated point.
- Boundary.

Fact. For a metric space (X, d):

1. A set G is open $\iff \forall x \in G\ \exists r > 0\ B(x, r) \subseteq G$.

   Proof. Sufficiency is trivial. For necessity, since each open set is union of bases, then $x \in G$ must lie in a open ball contained in G, and we can find some $r > 0$ such that $B(x, r)$ is contained in the open ball. $\square$

2. Intersection of open sets may not be open. For example,

$$\bigcap_{n\in\mathbb{N}} (-1/n, 1/n) = \{0\}.$$

Definition (Continuity for Met)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces. We say $f \colon X \to Y$ is continuous at $x \in X$, if $\forall \varepsilon > 0\ \exists r > 0$ such that $f\big(B(x, r)\big) \subseteq B\big(f(x), \varepsilon\big)$(two balls are in X and Y respectively). f is continuous if f is continuous at every $x \in X$.

Theorem (Continuity's equivalent conditions)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces. A map $f \colon X \to Y$ is continuous at x if and only if $\forall (x_n)_{n\in\mathbb{N}} \subseteq X(\lim_n x_n = x \implies \lim_n f(x_n) = f(x))$.

Proof. Suppose f is continuous at x and $(x_n)_{n\in\mathbb{N}} \to x$. $\forall \varepsilon > 0$, by continuity of f at x, $\exists r > 0$ such that $f\big(B(x, r)\big) \subseteq B\big(f(x), \varepsilon\big)$. For this $r > 0$, by convergence of $(x_n)_{n\in\mathbb{N}}$, $\exists N \in \mathbb{N}$ such that $\forall n > N\ x_n \in B(x, r)$ and hence $\forall n > N\ f(x_n) \in B\big(f(x), \varepsilon\big)$. Therefore, $\lim_n f(x_n) = f(x)$.

Suppose $\forall (x_n)_{n\in\mathbb{N}} \subseteq X(\lim_n x_n = x \implies \lim_n f(x_n) = f(x))$. If f is not continuous at x, by definition of continuity,

$$\exists \varepsilon_0 > 0 \forall \delta > 0 \exists y \in B(x, \delta) f(y) \notin B\big(f(x), \varepsilon_0\big).$$

In particular, take $\delta_n = 1/n$. Then there is $y_n \in B(x, 1/n)$ and $f(y_n) \notin B\big(f(x), \varepsilon_0\big)$. Now we have a sequence $(y_n)_{n\in\mathbb{N}}$ converge to x but $\lim_n f(y_n) \neq x$, contradiction. Therefore, f must be continuous at x. $\square$

Definition (Continuity for Top)**.** Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two topological spaces. We say $f\colon X \to Y$ is continuous if $\forall O \in \mathcal{U}\ f^{-1}(O) \in \mathcal{T}$.

Theorem (Equivalence of definitions of continuity)**.** $f\colon (X, d) \to (Y, d)$ is continuous if and only if $f\colon (X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$ is continuous.

Remark. Here we mean $f\colon (X, d) \to (Y, d)$ is continuous, if it satisfies the definition of continuous maps between metric spaces. And "$f\colon (X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$ is continuous" means it satisfies the definition of continuous maps between topological spaces.

Proof. Suppose $f\colon (X, d) \to (Y, d)$ is continuous. Since $(Y, \mathcal{T}_{d_Y})$ has the topology base

$$\mathcal{B}_Y = \{B(y, r) : y \in Y, r \in (0, \infty)\},$$

it suffices to show that $\forall B(y, r) \in \mathcal{B}_Y$ we have $f^{-1}\big(B(y, r)\big) \in \mathcal{T}_{d_X}$. Suppose $f^{-1}\big(B(y, r)\big) \neq \varnothing$, else it's automatically open. Since $f(x_1) \in B(y, r)$, $\exists r_1 > 0$ such that $B(f(x_1), r_1) \subseteq B(y, r)$. Using the continuity of f at x_1, $\exists \delta > 0$ such that $f\big(B(x_1, \delta)\big) \subseteq B\big(f(x_1), r_1\big) \subseteq B(y, r)$. Therefore $B(x_1, \delta) \subseteq f^{-1}\big(B(y, r)\big)$. This means $f^{-1}\big(B(y, r)\big)$ contains a neighbourhood for each point of itself, and hence $f^{-1}\big(B(y, r)\big)$ is open.

Suppose $f\colon (X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$ is continuous. $\forall x \in X$, $f^{-1}\big(B(f(x), r)\big)$ is open for all $r > 0$. $x \in f^{-1}\big(f(x), r\big)$ and $f^{-1}\big(B(f(x), r)\big)$ is union of sets like $B(x_0, \delta_0)$, so we can suppose $x \in B(x_0, \delta_0)$ for some $x_0 \in X, \delta_0 > 0$. Now choose $\delta > 0$ such that $B(x, \delta) \subseteq B(x_0, \delta_0)$ and we have $f\big(B(x, \delta)\big) \subseteq f\big(B(x_0, \delta_0)\big) \subseteq f\Big(f^{-1}\big(B(f(x), r)\big)\Big) \subseteq B(f(x), r)$. We're done. $\square$

5 Week 3, Lecture 1

Recall

Every linear normed space $(X, \| \ \|)$ has a metric (induced by its norm) $d\colon X \times X \to \mathbb{R}, (x, y) \mapsto \|x - y\|$. This is surely a metric, ensured by the properties of norm. However, a metric space (X, d) need not to be a linear normed space, since it is possible that X has no linear structure.

Now, suppose (X, d) a metric space, where X is a linear space. We have a question: is there some norm $\| \ \|$ such that d is induced from $\| \ \|$? If there is a norm that we want, it is clear that $\| \ \|\colon X \to \mathbb{R}, x \mapsto \|x\| := d(x, 0)$. We want $\| \ \|$ is a norm, so it should satisfy:

1. $\| \ \| \geq 0$ and $\|x\| = 0 \iff x = 0$. This holds, since d is a metric.

2. $\forall k \in \mathbb{K}, x \in X, d(kx, 0) = |k|d(x, 0)$. This should be satisfied.

3. $d(x, 0) + d(y, 0) \geq d(x + y, 0)$ as the triangle inequality.

Moreover, d should satisfy $d(x+z, y+z) = d(x, y)$, since $(x+z)-(y+z) = x - y$. In fact, the following conditions ensure that d is induced by a norm:

Condition 1. $d(kx, 0) = |k|d(x, 0)$.

Condition 2. d is translation-invariant, that is $d(x+z, y+z) = d(x, y)$.

Suppose d satisfies condition 1 and condition 2, then it is enough to show that $\| \ \|$ satisfies the triangle inequality.

Proof.

$$\begin{aligned}
\|x + y\| &= d(x + y, 0) \\
&= d(x + y, -y + y) \\
&= d(x, -y) && \text{(condition 2)} \\
&\leq d(x, 0) + d(0, -y) && \text{(triangle inequality of } d\text{)} \\
&= d(x, 0) + d(-y, 0) && (d \text{ is symmetric}) \\
&= d(x, 0) + d(y, 0) && \text{(condition 1)} \\
&= \|x\| + \|y\|.
\end{aligned}$$

We're done. □

Here comes an important notion of functional analysis.

5.1 Banach Space

Definition (Banach Space)**.** A **complete** linear normed space $(X, \| \ \|)$ is called a **Banach Space**.

Here the word "complete" should be defined.

Definition (Completeness)**.** A metric space (X, d) is complete if every Cauchy sequence in X converges.

Definition (Cauchy sequence)**.** Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be a Cauchy sequence, if

$$\lim_{m,n} \|x_m - x_n\| = 0.$$

Remark. Here $\{\|x_m - x_n\|\}_{m,n\in\mathbb{N}}$ is a double index real sequence, and "the double index limit is 0 " should be interpreted as

$$\forall \varepsilon > 0 \exists M \in \mathbb{N} \exists N \in \mathbb{N} (\forall m > M \forall n > N \ |\|x_m - x_n\| - 0| < \varepsilon).$$

Warning. Convergent sequence must be Cauchy sequence (from definition), while Cauchy sequence may not converge (as the following examples).

Example. Let $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto |x - y|$ be the normal metric on $\mathbb{R}$. Consider $(\mathbb{Q}, d|_{\mathbb{Q}\times\mathbb{Q}})$. This is not a complete metric space, since $\mathbb{Q}$ is dense in $\mathbb{R}$ and for arbitrary $x \in \mathbb{R}$ we can find a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x in $\mathbb{R}$. Consider $x \in \mathbb{R} \setminus \mathbb{Q}$ and we get a sequence in $\mathbb{Q}$, that is Cauchy in $(\mathbb{Q}, d|_{\mathbb{Q}\times\mathbb{Q}})$ and doesn't converge to any $r \in \mathbb{Q}$.

Example. Consider $\big(C[0, 1], \| \ \|_{L_1}\big)$, where $C[0, 1]$ means $C([0, 1], \mathbb{K})$ for short and $\| \ \|_{L_1}$ means the norm

$$\| \ \|_{L_1} : C[0, 1] \to \mathbb{R}, f \mapsto \int_{[0,1]} |f| \, \mathrm{d}m.$$

This is a norm, since $\|f\|_{L_1} = 0 \iff |f| = 0$ m-a.e, and continuity of f ensures $f = 0$. Other conditions for norm is trivial. And this is a incomplete normed vector space, since $C[0, 1]$ is dense (with respect to the norm $\| \ \|_{L_1}$) in L_1.

From now on, $C_p[a, b]$ means $(C[0, 1], \| \ \|_{L_p})$.

Remark. The completion (which will be defined the next class) of $C_p[a, b], 1 \leq p < \infty$ is $L_p[a, b]$, since $C[a, b]$ is dense in $L_1[a, b]$

Example. Let $P[a,b] := \{\text{Polynomial functions defined on}[a,b]\}$, then the linear normed space $(P[a,b], \max_{[a,b]}|\ |)$ is incomplete. Since $\exists f \in C[a,b]$ such that f is not a polynomial, such as $f = \exp|_{[a,b]}$. Suppose $\exp\colon \mathbb{R} \to \mathbb{R}$ is defined as the power series for convenience. Then by **Weierstrass Approximation Theorem**, for each fixed $\varepsilon > 0$, we can find a $p \in P[a,b]$ such that $\|p - f\| < \varepsilon$.

In fact, for $f = \exp|_{[a,b]}$, it is enough to take

$$p_n\colon [a,b] \to \mathbb{R}, x \mapsto \sum_{j=1}^{n} \frac{x^j}{j!}.$$

By the result in power series theory, we know $p_n \xrightarrow{\max_{[a,b]}|\ |} f$.

Now we compare two normed spaces sharing the underlying set $C[a,b]$. $C[a,b]$ means the normed space $(C[a,b], \max_{[a,b]}|\ |)$ somewhere. And we will prove the completeness of $C[a,b]$.

Normed space	$C[a,b]$	$C_p[a,b]$
Underlying set	$C[a,b]$	$C[a,b]$
Norm	$\max_{[a,b]}\|\ \|$	$\\|\ \\|_p$
Completeness	complete	incomplete

Proof of completeness. Let $(f_n)_{n\in\mathbb{N}} \subseteq C[a,b]$ be a Cauchy sequence. That is

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m,n \geq N \max_{[a,b]} |f_m - f_n| < \varepsilon.$$

Therefore, given any $x \in [a,b]$ we have

$$|f_m(x) - f_n(x)| \leq \max_{[a,b]} |f_m - f_n| < \varepsilon.$$

That is the sequence $\big(f_n(x)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of $\mathbb{R}$, $\big(f_n(x)\big)_{n\in\mathbb{N}}$ converge. Then we can define a function

$$f\colon [a,b] \to \mathbb{R}, x \mapsto \lim_n f_n(x).$$

$\lim_n f_n(x)$ is surely a real number, as explained above. And we have two claims.

Claim. $f_n \xrightarrow{\max_{[a,b]}|\ |} f$.

$\forall n > N$, we have
$$\max_{\in[a,b]}|f_m - f_n| < \varepsilon.$$
It's equivalent to
$$|f_m(x) - f_n(x)| < \varepsilon(\forall x \in [a,b]),$$
and let $m \to \infty$, using the continuity of $|\ |$ (to change the order of $\lim_m$ and $|\ |$)
$$|f(x) - f_n(x)| < \varepsilon(\forall x \in [a,b]),$$
which is equivalent to
$$\max_{\in[a,b]}|f - f_n| < \varepsilon.$$
Therefore, $f_n \xrightarrow{\max_{[a,b]}|\ |} f$.

Claim. $f \in C[a,b]$.

It suffices to show that f is uniformly continuous. Given arbitrary $\varepsilon > 0$, by the convergence of $(f_n)_{n\in\mathbb{N}}$
$$\exists N \forall n \geq N \max_{[a,b]}|f_n - f| < \varepsilon/3.$$
Fix this N, and the continuity (equivalent to uniform continuity for functions on $[a,b]$) of f_N ensures that $\exists\delta > 0$ such that
$$\forall x \forall y (|x-y| < \delta \implies |f_N(x) - f_N(y)| < \varepsilon/3).$$
And $\forall x \forall y$ such that $|x-y| < \delta$, we have
$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &\leq \max_{[a,b]}|f_N - f| + \varepsilon/3 + \max_{[a,b]}|f_N - f| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon.\end{aligned}$$
Thus f is uniformly continuous. □

Example. Suppose $1 \leq p \leq \infty$. then $L_p(\Omega, \mathcal{F}, \mu)$ is a Banach space.

Proof. First, suppose $1 \leq p < \infty$. Here is a proof different from our textbook. Suppose $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure (by Chebyshev's Inequality). By the lemma,

$\exists$ a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ such that $f_{n_j} \to f$ μ-a.e.. Therefore, by **Fatou's Lemma**:1

$$\begin{aligned}\lim_j \left\|f_{n_j} - f\right\|_p^p &= \lim_j \int_\Omega \left|f_{n_j} - f\right|^p \mathrm{d}\mu \\ &\leq \int_\Omega \liminf_j \left|f_{n_j} - f\right|^p \mathrm{d}\mu && \text{(Fatou's Lemma)} \\ &= 0. && (f_{n_j} \to f \ \mu\text{-a.e.})\end{aligned}$$

While the inequality should be reversed. This can be corrected:

$$\begin{aligned}\left\|f_{n_j} - f\right\|_p^p &= \int_\Omega \lim_n \left|f_{n_j} - f_n\right|^p \mathrm{d}\mu \\ &\leq \liminf_j \int_\Omega \left|f_{n_j} - f\right|^p \mathrm{d}\mu, && \text{(Fatou's Lemma)}\end{aligned}$$

and

$$\begin{aligned}\lim_{n_j} \left\|f_{n_j} - f\right\|_p^p &= \lim_{n_j} \int_\Omega \lim_n \left|f_{n_j} - f_n\right|^p \mathrm{d}\mu \\ &\leq \lim_{n_j} \liminf_n \int_\Omega \left|f_{n_j} - f_n\right|^p \mathrm{d}\mu && \text{(Fatou's Lemma)} \\ &= 0. && \text{(Cauchy sequence)}\end{aligned}$$

So $f_{n_j} \xrightarrow{\| \|_{L_p}} f$. Minkowski's inequality shows

$$\|f_n - f\| \leq \left\|f_n - f_{n_j}\right\| + \left\|f - f_{n_j}\right\|.$$

Let $n_j, n \to \infty$ and use the fact that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in norm, we have $f_n \xrightarrow{\| \|_{L_p}} f$.

If $f \in L_p$, we are done. f is a μ- a.e. limit of $(f_{n_j})_{j\in\mathbb{N}}$ and hence is measurable. Minkowski's inequality shows

$$\|f\|_p \leq \left\|f - f_{n_j}\right\|_p + \left\|f_{n_j}\right\|_p.$$

The first term is bounded (since the real sequence has limit 0), and the second term is finite since $f_{n_j} \in L_p$.

Then, suppose $p = \infty$. There is $(A_{m,n})_{m,n\in\mathbb{N}} \in \mathcal{F}$ such that $\mu(A_{m,n}) = 0 \forall m, n \in \mathbb{N}$ and

$$\forall \omega \in A_{m,n}^c \ |f_m(\omega) - f_n(\omega)| \leq \|f_n - f_m\|_\infty.$$

Clearly for $A := \cup_{m,n\geq 1} A_{m,n}$, we have$\mu() = 0$. And we have

$$\forall \omega \in A^c |f_n(\omega) - f_m(\omega)| \leq \|f_n - f_m\|_\infty.$$

Let $m \to \infty$

$$\forall \omega \in A^c |f_n(\omega) - f(\omega)| \leq \lim_m \|f_n - f_m\|_\infty,$$

and hence

$$\|f_n - f\|_\infty \leq \lim_m \|f_n - f_m\|_\infty.$$

Let $n \to \infty$ and use

$$\lim_n \lim_m \|f_n - f_m\|_\infty = 0.$$

We're done. $\square$

Lemma. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}_0(\Omega)$ is Cauchy in measure, where

$$\mathcal{L}_0(\Omega) := \{f \colon \Omega \to (\mathbb{K}, \mathcal{B}(\mathbb{K})) \text{ that is measurable}\}.$$

Then there is a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $g_n \to f$ $\mu-$ a.e.. Here $f \in \mathcal{L}_0(\Omega)$.

Proof. We can choose a subsequence $(g_n)_{n\in\mathbb{N}} = \left(f_{n_j}\right)_{j\in\mathbb{N}}$ such that if $E_j := |g_j - g_{j+1}|^{-1}[2^{-j}, \infty)$ then $\mu(E_j) \leq 2^{-j}$. Because

$$\forall j \in \mathbb{N} \quad \lim_{m,n\to\infty} |f_m - f_n|_* \mu[2^{-j}, \infty) = 0.$$

And pick n_j inductively, such that $n_{j+1} > n_j$ and

$$\mu_* |f_m - f_n| [2^{-j}, \infty) < 2^{-j} \ \forall m, n \geq n_j.$$

Set $F_k := \bigcup_{j\geq k} E_j$ then $\mu(F_k) \leq \sum_{j\geq k} 2^{-j} = 2^{1-k}$. Continuity from above is allowed! If $x \notin F_k$, for $i \geq j \geq k$ we have

$$|g_i(x) - g_j(x)| \leq \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq \sum_{l=j}^{i-1} 2^{-l} \leq 2^{1-j},$$

which ensures that $\forall x \in F_k^c$, $\left(g_j(x)\right)_{j\in\mathbb{N}}$ is a Cauchy sequence. Let

$$F = \bigcap_{j\geq 1} F_j = \limsup_j E_j,$$

we have $\mu(F) = \mu(\lim_j F_j) = \lim_j \mu(E_j) = 0$. $\square$

Exercise. Prove that ℓ_p is complete when $1 \leq p < \infty$.

Suppose $(X, \| \ \|)$ is a linear normed space. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ satisfies $\sum_{n\geq 1}\|x_n\| < \infty$, and we can define the infinite sum for this sequence as

$$\sum_{n\geq 1} x_n := \lim_{n\to\infty} S_n, \text{ where } S\colon \mathbb{N} \to X, j \mapsto \sum_{j=1}^{N} x_j.$$

Theorem 5.1. $(X, \| \ \|)$ is a Banach space if and only if $\forall (x_n)_{n\in\mathbb{N}} \subseteq X$,

$$\sum_{n\geq 1}\|x_n\| < \infty \implies \sum_{n\geq 1} x_n < \infty.$$

Here $\sum_{n\geq 1} x_n < \infty$ means $\sum_{n\geq 1} x_n$ exists for short.

Proof. Necessity: suppose X is a Banach space, then $\sum_{n\geq 1}\|x_n\| < \infty$ implies

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \Big(\sum_{j=1}^{p} \|x_{n+j}\| < \varepsilon (\forall p \in \mathbb{N}) \Big),$$

and therefore $\forall n > N \|S_{n+p} - S_n\| \leq \sum_{j=1}^{p}\|x_{n+j}\| < \varepsilon$, this means that $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. X is complete, so $(S_n)_{n\in\mathbb{N}}$ converges. That is $\sum_{n\geq 1} x_n < \infty$.

Sufficiency: suppose X satisfies the condition above. If X is not complete, then $\exists (x_n)_{n\in\mathbb{N}} \subseteq X$ that is Cauchy but has no limit in X. Now, select a subsequence of $(x_n)_{n\in\mathbb{N}}$, say $\left(x_{n_j}\right)_{j\in\mathbb{N}}$ such that

$$\forall j \in \mathbb{N} \left\|x_{n_{j+1}} - x_{n_j}\right\| < 2^{-j}.$$

Define $y\colon \mathbb{N} \to X, j \mapsto x_{n_{j+1}} - x_{n_j}$, then $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, satisfying

$$\forall j \in \mathbb{N} \ \|y_j\| < 2^{-j}.$$

Therefore, $\sim_{n\geq 1} \|y_j\| < \infty$. Then X satisfies the condition, which implies that $\sum_{n\geq 1} y_n < \infty$. Equivalently, $\lim_j x_{n_j}$ exists in X. While $(x_n)_{n\in\mathbb{N}}$ is Cauchy, so $\lim_n x_n = \lim_j x_{n_j}$ exists, that's a contradiction (see how we selected $(x_n)_{n\in\mathbb{N}}$). $\square$

6 Week 3, Lecture 2

Recall

1. $L_p(\Omega)(1 \leq p \leq \infty)$ is complete. The outline of proof for $p < \infty$ is here:

   Step 1. Show that if $(f_n)_{n\in\mathbb{N}}$ is Cauchy (in norm), then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure.

   Step 2. Show that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, then $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ that converges to a measurable function f μ-a.e..

   Step 3. Use Fatou's lemma to show that $(f_{n_j})_{j\in\mathbb{N}} \xrightarrow{\|\ \|_p} f$.

   Step 4. Show that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\|\ \|_p} f$ and $f \in L_p$

2. About quotient space. Given a normed space $(X, \|\ \|)$ and a closed subspace $X_0 \hookrightarrow X$. We can define the quotient space

$$X/_{X_0} := \{[x] = x + X_0 : x \in X\},$$

   whose norm is

$$\|[x]\| = \inf_{y\in X_0} \|x - y\| = \inf_{y\in[x]} \|y(-0)\|.$$

   The second equality can be verified by change $y \in [x] \iff y = x + x_0, x_0 \in X_0$.

3. Norm and semi-norm $(p, p(x) = 0 \not\Longrightarrow x = 0)$. Let X be a linear semi-normed space, with the semi-norm p. A familiar linear semi-normed is $\mathcal{L}_p(1 \leq p \leq \infty)$. Let $X_0 := \{x \in X : p(x) = 0\} \hookrightarrow X$.

   Claim. X_0 is closed subspace of X (so, $X/_{X_0}$ is allowed, see this remark.

   Proof. X_0 is a linear subspace, since p is a semi-norm.

   p is a continuous map, since the triangle inequality holds. Then $N = p^{-1}(0)$ must be closed. □

Now, the remark ensures that $\|\ \|: X/_{X_0}, [x] \mapsto p(x)$ is a norm on $X/_{X_0}$.

Proof. It should be verified that p is well-defined (though this should have been proved in the remark). Suppose $[x] = [y]$, that is $[x-y] = [y-x] = [0]$. Since p is a semi-norm, we have the triangle inequality

$$p(x) + p(y-x) \geq p(y), p(y) + p(x-y) \geq p(x),$$

and $[x-y] = [y-x] = 0 \implies p(x-y) = p(y-x) = 0$, that is $p(x) = p(y)$. Thus, $[x] \mapsto p(x)$ is well-defined. And

(1) $\|[x]\| = 0 \iff p(x) = 0 \iff x \in X_0 = [0] \iff [x] = [0] \left(\in {}^{X}\!/\!_{X_0}\right)$.

(2) $\|k[x]\| = \|[kx]\| = p(kx) = |k|p(x) = |k|\|x\|$.

(3) $\|[x] + [y]\| = \|[x+y]\| = p(x+y) \leq p(x) + p(y) = \|[x]\| + \|[y]\|$.

Above all, $\| \; \|$ is a norm on $[X]$. $\square$

6.1 Completion

In this class, X is a linear noremd space, unless otherwise specified.

Definition (Isometry)**.** Suppose X, Y are two linear normed spaces. We say X is isomtetric with Y, if there is a linear surjection $T\colon X \to Y$ such that

$$\|Tx\| = \|x\| (\forall x \in X),$$

or equivalently $\| \; \|_Y \circ T = \| \; \|_X$.

Remark. Isometry is automatically injective, since $Tx = 0 \iff \|Tx\| = \|x\| = 0 \iff x = 0$. That is $\ker T = \{0\}$. Therefore, T is automatically injective and hence bijective as we want.

Definition (Density)**.** Let $(X, \| \; \|)$ be a liner normed space and $X_0 \hookrightarrow X$. X_0 is said to be dense in X, if $\overline{X_0} = X$.

Question. How to verify $\overline{X_0} = X$?

$\overline{X_0} = X$, if

$$\forall x \in X \forall \varepsilon > 0 \exists x_\varepsilon \in X_0 (\|x_\varepsilon - x\| < \varepsilon.)$$

And equivalently

$$\forall x \in X \forall n \in \mathbb{N} \exists x_n \in X_0 (\|x_\varepsilon - x\| < 1/n.)$$

That is, $\exists (x_n)_{n \in \mathbb{N}} \subseteq X_0$ that converges to x.

Theorem 6.1 (Completion thm)**.** Let $(X, \| \|)$ be a linear normed space. There is a Banach space $(\widehat{X}, \| \|)$ such that X is isometric to a dense subspace of $\widehat{X}$.

Remark. in fact, the completion $\widehat{X}$ is unique up to an isometry.

Definition (Completion)**.** $\widehat{X}$ (together with the isometric inclusive mapping) is called the completion of X.

Proof. We will construct a complection of X. Let

$$\mathcal{E} := \{(x_n)_{n\in\mathbb{N}} \subseteq X : (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence}\},$$

and define $p\colon \mathcal{E} \to \mathbb{R}, x(= (x_n)_{n\in\mathbb{N}}) \mapsto \lim_n \|x_n\|$. Here $\lim_n \|x_n\|$ exists in $\mathbb{R}$, because $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence implies that $\|x\|(= (\|x_n\|)_{n\in\mathbb{N}})$ is a Cauchy sequence in $\mathbb{R}$, and $\mathbb{R}$ is complete. Moreover, p is a semi-morn on $\mathcal{E}$. Now define $N := p^{-1}(0)$. Then $N \hookrightarrow \mathcal{E}$ and N is closed (by the continuity of p). Therefore we can consider $\widehat{X} := \mathcal{E}/N$, with the norm $\| \|\colon \widehat{X} \to \mathbb{R}, x + N \mapsto p(x)$.

Now, we prove this thm in 3 steps.

Step 1. X is isometric to a subspace of $\widehat{X}$. Let $X_0 := \{[(x)_{n\in\mathbb{N}}] : x \in X\}$ and

$$T\colon X \to X_0, x \mapsto [(x)_{n\in\mathbb{N}}] = (x)_{n\in\mathbb{N}} + N,$$

where $(x)_{n\in\mathbb{N}}$ means the constant sequence $(x, \ldots, x, \ldots)$. That is, $T(x) = (x, \ldots, x, \ldots) + N$. Clearly T is a linear surjection. We want to show T is isometric, that is $\forall x \in X, \|T(x)\| = \|x\|$. By definiton

$$\begin{aligned}
\|T(x)\| &= \|[(x)_{n\in\mathbb{N}}]\| && \text{(def of } T) \\
&= p\big((x)_{n\in\mathbb{N}}\big) && \text{(def of } \| \|_{\widehat{X}}) \\
&= \lim_n \|x\| && \text{(def of } p) \\
&= \|x\|.
\end{aligned}$$

To sum up, T is a linear isometric surjection as we want.

Step 2. $X_0 \hookrightarrow \widehat{X}$ is dense. As discussed above, it suffices to show that $\forall [x] = (x_1, \ldots, x_n, \ldots) + N \in \widehat{X}$, there is a sequence in X_0 converge to X. Let

$$[x]^{(m)}\colon \mathbb{N} \to [(x_m)_{n\in\mathbb{N}}] = (x_m, \ldots, x_m, \ldots) + N,$$

and we prove that the sequence $([x^{(m)}])_{m\in\mathbb{N}}$ is convergent to $[x]$.

$$\begin{aligned}
\lim_m \left\| [x]^{(m)} - [x] \right\| &= \lim_m \|(x_m - x_1, \dots, x_m - x_n, \dots) + N\| && \text{(def of } \pm\text{)} \\
&= \lim_m p\big((x_m - x_n)_{n\in\mathbb{N}}\big) && \text{(def of } \|\ \|\text{)} \\
&= \lim_m \lim_n \|x_m - x_n\| && \text{(def of } p\text{)} \\
&= 0. && \text{(see remark)}
\end{aligned}$$

Step 3. $\widehat{X}$ is a Banach space. That is $\widehat{X}$ is complete. Let $([x]^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widehat{X}$. By the density of $X_0 = TX$, we have a sequence $(y_n)_{n\in\mathbb{N}} \subseteq X$ such that

$$\forall n \in \mathbb{N} \left\| T(y_n) - [x]^{(n)} \right\| \le 1/n.$$

Claim. $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned}
\|y_m - y_n\| &= \|T(y_m) - T(y_n)\| \\
&\le \left\| T(y_m) - [x]^{(m)} \right\| + \left\| [x]^{(m)} - [x]^{(n)} \right\| + \left\| T(y_n) - [x]^{(n)} \right\| \\
&\le 1/m + \left\| [x]^{(m)} - [x]^{(n)} \right\| + 1/n.
\end{aligned}$$

Apply $\limsup_{m,n}$ on both sides and we have

$$\limsup_{m,n} \|y_m - y_n\| \le 0.$$

Therefore, $(y_n)_{n\in\mathbb{N}}$ is Cauchy, and $(y_n)_{n\in\mathbb{N}} \in \mathcal{E}$. Now we show that $([x]^{(n)})_{n\in\mathbb{N}} \to [y] = (y_1, \dots, y_n, \dots) + N$. By definition of $\|\ \|_{\widehat{X}}$

$$\begin{aligned}
\|[x]^m - [y]\| &\le \|[x]^m - T(y_m) + T(y_m) - [y]\| \\
&\le \|[x]^m - T(y_m)\| + \|T(y_m) - [y]\| \\
&\le 1/m + p\big((y_n - y_m)_{n\in\mathbb{N}}\big) \\
&= 1/m + \lim_n \|y_n - y_m\|,
\end{aligned}$$

and let $m \to \infty$, we have

$$\limsup_m \|[x]^m - [y]\| \le \limsup_m 1/m + \limsup_m \lim_n \|y_n - y_m\|.$$

The second limit must be 0, since $\lim_m \lim_n \|y_n - y_m\| = 0$ (see remark) .

$\square$

Remark. Here we explain why $\lim_m \lim_n \|x_m - x_n\| = 0$. We may wan to write: suppose $\lim_n x_n = x$, then

$$\lim_m \lim_n \|x_m - x_n\| = \lim_m \|x_m - x\| = 0,$$

where the first equality is using the continuity of $\| \ \|$ and the second equality follows from the definition of $\lim_n x_n = x$. Everything makes sense, except $\lim_n x_n = x$. Notice that is a sequence in X and none said that X is complete.

So, why $\lim_m \lim_n \|x_m - x_n\| = 0$ holds? Search in what we know and there is something like this, that is $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. For convenience, let $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}, (m, n) \mapsto f_{m,n} := \|x_m - x_n\|$. Therefore, it suffices to show that we have

$$\lim_m \lim_n f_{m,n} = \lim_{m,n} f_{m,n}.$$

whenever $\lim_{m,n\to\infty} f_{m,n}$ exists.

Proof. This proof is wrong, consider $f_{m,n} = (-1)^m/n + (-1)^n/m$. For correct proof, see MSE. Let $a = \lim_{m,n} f_{m,n}$, we have

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \forall m > N \ f_{m,n} \in (a - \varepsilon, a + \varepsilon).$$

This implies that

$$\forall m > N (a - \varepsilon \le \liminf_n f_{m,n} \le \limsup_n f_{m,n} \le a + \varepsilon).$$

Let $m \to \infty$ and we have

$$a - \varepsilon \le \liminf_m \liminf_n f_{m,n} \le \limsup_m \limsup_n f_{m,n} \le a + \varepsilon.$$

Clearly

$$\liminf_m \liminf_n f_{m,n} \le \liminf_m \limsup_n f_{m,n} \le \limsup_m \limsup_n f_{m,n}.$$

Therefore,

$$a - \varepsilon \le \liminf_m \liminf_n f_{m,n} \le \liminf_m \limsup_n f_{m,n} \le a + \varepsilon.$$

ε is arbitrary, so $\liminf_m \liminf_n f_{m,n} = \liminf_m \limsup_n f_{m,n} = a$. That is $\liminf_m \lim_n f_{m,n} = a$. Similarly, $\limsup_m \lim_n f_{m,n} = a$. Above all,

$$\liminf_m \lim_n f_{m,n} = \limsup_m \lim_n f_{m,n} = a. \qquad \square$$

Theorem (Uniqueness of completion)**.** The completion of a linear normed space X is unique up to an unique isometry (that conincides with the two inclusions). That is, if $\widehat{X}, Y$ with isometric inclusion map ι, ι' respectively are completions of X, then the following diagram commutes

$$\begin{tikzcd} & X \arrow[dl, hook', "\iota"'] \arrow[dr, hook, "\iota'"] & \\ \widehat{X} \arrow[rr, dashed, "\exists\varphi"'] & & Y \end{tikzcd}$$

Proof. Consider the corestriction of ι, that is $\iota_0 := \iota|^{\iota(X)}$. Clearly ι_0 is an isometry from X to $\iota(X)\Big($which is dense in $\widehat{X}\Big)$. Now we define a map φ_0 by the following diagram (i.e. $\varphi_0 := \iota' \circ (\iota_0)^{-1}$)

$$\begin{tikzcd} & X \arrow[dr, "\iota'"] & \\ \iota(X) \arrow[ur, "(\iota_0)^{-1}"] \arrow[rr, "\varphi_0"] & & Y \end{tikzcd}$$

Now φ_0 is linear and keeps norm. Since $\iota(X)$ is dense in $\widehat{X}$, Y is complete and φ_0 is uniformly continuous (φ_0 keeps norm and hence is uniformly continuous), we can extend φ_0 to a uniformly continuous map $\varphi\colon \widehat{X} \to Y$ (see the textbook, Thm 2.3.4).

To show that φ is an isometry, we should show that:

1. φ is linear;
2. φ keeps norm.
3. φ is surjective;

First, we prove that φ is linear. Since $\varphi|_{\iota(X)} = \varphi_0$ and $\iota(X)$ is dense in $\widehat{X}$, $\forall x, y \in \widehat{X}$ $\exists (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq \iota(X)$ such that $(x_n)_{n\in\mathbb{N}} \to$

$x, (y_n)_{n\in\mathbb{N}} \to y$. Then $\forall k \in \mathbb{K}$, we have

$$\begin{aligned}
\varphi(kx+y) &= \varphi\big(\lim_n (kx_n + y_n)\big) \\
&= \lim_n \varphi(kx_n + y_n) && \text{(continuity of } \varphi) \\
&= \lim_n \varphi_0(kx_n + y_n) && \Big(\varphi|_{\iota(X)} = \varphi_0\Big) \\
&= \lim_n \big(k\varphi_0(x_n) + \varphi_0(y_n)\big) && (\varphi_0 \text{ is linear}) \\
&= k\lim_n \varphi_0(x_n) + \lim_n \varphi_0(y_n) && \Big(\lim_n \text{ is linear}\Big) \\
&= k\lim_n \varphi(x_n) + \lim_n \varphi(y_n) && \Big(\varphi|_{\iota(X)} = \varphi_0\Big) \\
&= k\varphi(x) + \varphi(y). && \text{(continuity of } \varphi)
\end{aligned}$$

Therefore, φ is linear.

Second, φ keeps norm. $\forall x \in \widehat{X}$, $\exists (x_n)_{n\in\mathbb{N}} \subseteq \iota(X)$ such that $(x_n)_{n\in\mathbb{N}} \to x$.

$$\begin{aligned}
\|x\| &= \Big\|\lim_n x_n\Big\| \\
&= \lim_n \|x_n\| && \text{(continuity of } \|\ \|) \\
&= \lim_n \|\varphi_0(x_n)\| && (\varphi_0 \text{ is an isometry}) \\
&= \lim_n \|\varphi(x_n)\| && \Big(\varphi|_{\iota(X)} = \varphi_0\Big) \\
&= \Big\|\lim_n \varphi(x_n)\Big\| && \text{(continuity of } \|\ \|) \\
&= \|\varphi(x)\| && \text{(continuity of } \varphi).
\end{aligned}$$

Thirdly, φ is surjective. $\forall y \in Y$, by the density of $\iota'(X)$, $\exists (y_n)_{n\in\mathbb{N}} \subseteq \iota'(X)$ such that $(y_n)_{n\in\mathbb{N}} \to y$. And $\forall n \in \mathbb{N}$, let $x_n := \varphi_0^{-1}(y_n)$ then $(x_n)_{n\in\mathbb{N}} \subseteq \iota(X) \subseteq \widehat{X}$ is well-defined and Cauchy (since $(y_n)_{n\in\mathbb{N}}$ is Cauchy and φ keeps norm). Now

$$y = \lim_n y_n = \lim_n \varphi(x_n) = \varphi(\lim_n x_n) = \varphi(x).$$

The last equality used the completeness of $\widehat{X}$. Therefore, φ is surjective.

Above all, φ is an isometry. If there is another isometry $\phi\colon \widehat{X} \to Y$ such that the diagram commutes, then $\varphi|_{\iota(X)} = \phi|_{\iota(X)} = \varphi_0$. φ and ϕ conincide on a dense subset of $\widehat{X}$ and hence $\varphi = \phi$. $\square$

7 Week 4, Lecture 1

Recall

No recall today.

7.1 Exercise course

We have only 3 exercises this course.

Question. Let $(X, \| \|)$ be a linear normed space, $X_0 \hookrightarrow X$. If X is complete and X_0 is closed then X_0 is complete.

Question. Let (X, d) be a metric space. $T \colon X \to X$ such that $\exists \lambda \in (0, 1)$

$$d(T(x), T(y)) \leq \lambda d(x, y), \forall x, y \in X.$$

Prove that $\exists! x_0 \in X$ such that $Tx_0 = x_0$.

Remark. This result doesn't hold when $\lambda = 1$. To see this, consider

$$(X, d) = ([0, \infty), d), T \colon X \to X, x \mapsto \sqrt{1 + x^2}.$$

And completeness is necessary too, consider $(X, d) = ((0, \infty), d)$ and $T \colon X \to X$, $x \mapsto x/2$. Other examples can be found.

Question. Let $(X, \| \|)$ be a linear normed space. Then X is a Banch space if and only if for each closed decreasing non-empty subsets sequence $(A_n)_{n \in \mathbb{N}}$, $\bigcap_{n \geq 1} A_n$ is a singleton set whenever $\lim_n \operatorname{diam}(A_n) = 0$.

There are answers in the next section.

8 Week 4, Lecture 2

Recall

For all l.n.s $(X, \| \|)$, there is a Banach space $\widehat{X}$ such that $X \cong X_0 (\hookrightarrow) \widehat{X}$, where X_0 is a dense subspace of $\widehat{X}$. It's ok to say $X = X_0 \hookrightarrow \widehat{X}$, and hence $\overline{X} = \widehat{X}$. The proof has 3 steps: construction of $\widehat{X}$, embedding X to $\widehat{X}$ and showing the completeness.

Remark. In the final exam and Phd qualifying exam, stating this theorem and its proof is common.

8.1 Review of exercise class

Here are the proofs of the questions of the exercise class.

Proof of the first. Suppose $(x_n)_{n\in\mathbb{N}} \subseteq X_0$ is a Cauchy sequence in X_0, then $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X. X is complete so $\exists x \in X$ such that $(x_n)_{n\in\mathbb{N}} \to x$. Now, X_0 is closed and hence $x \in X_0$. Thus, $(x_n)_{n\in\mathbb{N}} \to x \in X_0$. That is every Cauchy sequence in X_0 is convergent to some point $x \in X_0$, which is equivalent to X_0's completeness. □

Proof of the second. Let a be an arbitrary point in X. Define a sequence inductively:

$$(x_n)_{n\in\mathbb{N}} \colon \mathbb{N} \mapsto X, n \mapsto x_n := \begin{cases} a, & n = 1; \\ T(x_{n-1}), & n \geq 2. \end{cases}$$

Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy, because for all $n \geq 2$

$$d(x_{n+1}, x_n = d(T(x_n), T(x_{n-1}) \leq \lambda(x_n, x_{n-1}).$$

By induction, we have $d(x_{n+1}, x_n) \leq \lambda^{n-1} d(x_2, x_1)$, and hence the series

$$\sum_{n\geq 1} d(x_{n+1}, x_n) \leq \sum_{n\geq 1} \lambda^{n-1} d(x_2, x_1) = \frac{1}{1-\lambda} d(x_2, x_1) < \infty.$$

Therefore, the sequence $(S_n)_{n\in\mathbb{N}}$ is Cauchy, where $S_n := \sum_{j=1}^{n} d(x_j, x_{j+1})$. The triangle inequality implies that

$$S_{m\vee n} - S_{m\wedge n-1} \geq d(x_m, x_n),$$

which ensures that $(x_n)_{n\in\mathbb{N}}$ is Cauchy (let $S_0 = 0$ and then the inequality above always holds). By the completeness of X, $\exists! x_0 \in X$ such that $(x_n)_{n\in\mathbb{N}} \to x$. Now, the continuity (from $d(T(x), T(y)) \le \lambda d(x, y)$) of T implies

$$T(x_0) = \lim_n T(x_n) = \lim_n x_{n+1} = x_0.$$

This proves the existence. Supppose there is $y \in X$ such that $T(y) = y$, then

$$d(y, x_0) = d(T(y), T(x_0)) \le \lambda d(y, x_0).$$

$\lambda < 1$ implies that $d(y, x_0) = 0$. Equivalently, $x_0 = y$. This proves the uniqueness. □

Proof of the third. I think this proof is similar to the proof of [3, Chapter 5, Thm 2].

Necessity: suppose X is a Banach space. Given a closed decreasing non-empty subsets sequence $(A_n)_{n\in\mathbb{N}}$, choose $x_n \in A_n$ for each $n \in \mathbb{N}$. This is possible since $\forall n \in \mathbb{N}\ A_n \ne \varnothing$. Since $(A_n)_{n\in\mathbb{N}}$ is decreasing, we have

$$\forall m, n \in \mathbb{N}(x_m \in A_{m\wedge n}, x_n \in A_{m\wedge n}),$$

and hence

$$d(x_m, x_n) \le \operatorname{diam} A_{m\wedge n} \to 0(m, n \to \infty).$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then the completeness of X ensures that $\exists a \in X$ such that $(x_n)_{n\in\mathbb{N}} \to a$. $\forall n \in \mathbb{N}$, since A_n is closed and $x_j \in A_n$ for all except for finite $j \in \mathbb{N}$, we have $a \in A_n$. Therefore, $a \in \bigcap_{n\ge 1} A_n$. Clearly $\bigcap_{n\ge 1} A_n$ cann't have more than 1 elements. If so, $\exists y \in A_n \forall n \in \mathbb{N}$ and hence $\operatorname{diam}(A_n) \ge d(x, y) \ge 0$. That's a contradiction.

Sufficiency: suppose X satisfies the condition above. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X, define $(A_n)_{n\in\mathbb{N}}$ as follows

$$\forall n \in \mathbb{N}, A_n := \{x_m \in X : m \ge n\}.$$

Then $(\overline{A}_n)_{n\in\mathbb{N}}$ satisfies the condition for set sequence: clearly $(\overline{A}_n)_{n\in\mathbb{N}}$ is decreasing, and $\operatorname{diam}(\overline{A_n}) = \operatorname{diam}(A) \to 0$ since $(x_n)_{n\in\mathbb{N}}$ is Cauchy. The reason of $\operatorname{diam}(\overline{A_n}) = \operatorname{diam}(A_n)$ is written in remark. Therefore, $\exists! a \in \bigcap_{n\ge 1} A_n$. Now, it suffices to show that $(x_n)_{n\in\mathbb{N}} \to a$. This follows from

$$d(x_n, a) \le \operatorname{diam}(\overline{A_n}) \to 0(n \to \infty).$$ □

Remark. $\forall n \in \mathbb{N}$, we want to show that $\operatorname{diam}(\overline{A_n}) = \operatorname{diam}(A_n)$. Since n is fixed, we can omit the index. Given $A \subseteq X$ and $\varepsilon > 0$, $\forall x, y \in \overline{A}$, there is $x_\varepsilon, y_\varepsilon \in A$ such that

$$\|x - x_\varepsilon\| < \varepsilon/2, \|y - y_\varepsilon\| < \varepsilon/2.$$

Therefore

$$\|x - y\| \le \|x - x_\varepsilon\| + \|x_\varepsilon - y_\varepsilon\| + \|y_\varepsilon - y\| \le \|x_\varepsilon - y_\varepsilon\| + \varepsilon,$$

and use $\|x_\varepsilon - y_\varepsilon\| \le \operatorname{diam}(A)$,

$$\|x - y\| \le \operatorname{diam}(A) + \varepsilon.$$

Since $x, y \in \overline{A}$ are arbitrary, we have

$$\operatorname{diam}(\overline{A}) \le \operatorname{diam}(A) + \varepsilon.$$

And ε is arbitrary, so

$$\operatorname{diam}(\overline{A}) \le \operatorname{diam}(A).$$

The reversed inequality is trivial.

8.2 Banach fixed-point theorem

Here we introduce a classical result about Banach spaces.

Definition (Contractive mapping)**.** Given a metric space (X, d). Then a mapping $T\colon X \to X$ is called a contraction if $\exists \lambda \in (0,1)$ such that $d(T(x), T(y)) \le d(x, y)$.

Remark. Every linear normed space $(X, \| \, \|)$ has the natural metric $d(x, y) = \|x - y\|$ and hence a contraction on $(X, \| \, \|)$ means $T\colon X \to X$, such that $\exists \lambda \in (0,1), \forall x, y \in X$

$$\|T(x) - T(y)(\ne T(x - y))\| \le \lambda \|x - y\|.$$

The $\ne$ above means that T may not be a linear map.

It is easy to verify that each contraction is continuous.

Theorem 8.1 (Banach fixed-point theorem)**.** Suppose (X, d) is a complete metric space and T is a contraction on X. Then $\exists! x_0 \in X$ such that $Tx_0 = x_0$.

Proof. See the second proof in 8.1. □

Let's have some applications. Suppose X is a Banach space and $U\colon X \to X$. We want to solve the equation $U(x) = y$.

Proof. To use 8.2, we should rewrite the equation $U(x) = y$ as $T(x) = x$ for some T.

$$U(x) = y \iff U(x) - y = 0 \iff U(x) + x - y = x,$$

thus consider $T\colon X \to X, x \mapsto U(x) + x - y$. And

$$\|T(u) - T(u)\| = \|U(u) + u - y - U(v) - v + y\|.$$

If it's verified that T is a contraction, then 8.2 (Banach fixed-point theorem) implies that T has a unique fixed-point, i.e. $U(x) = y$ has a unique solution. □

Example. X is a Banach space, on which U is a contraction. Prove that $U(x) = x + y$ has a unique solution.

Proof. We want solve $U(x) - x = y$, i.e. $(U - \mathrm{id})(x) - y = 0$. So the discussion above tells us that we should consider $T = U - \mathrm{id} + \mathrm{id} - y = U - y$. Let $x_1 \in X$ be an arbitrary point. Define $x_{n+1} = T(x_n) = U(x_n) - y$ for all $n \in \mathbb{N}$. Then T is a contraction since

$$\|T(a) - T(b)\| = \|U(a) - U(b)\|,$$

and U is a contraction. Then use Theorem 8.2 (Banach fixed-point theorem) and we're done. □

9 Week 5, Lecture 1

Recall

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