0 Day 2 of Week 3

Recall

1. $L_p(\Omega)(1 \le p \le \infty)$ is complete. The outline of proof is here:

   Step 1. Show that if $(f_n)_{n\in\mathbb{N}}$ is Cauchy (in norm), then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure.

   Step 2. Show that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, then $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ that converges to a measurable function f μ-a.e..

   Step 3. Use Fatou's lemma to show that $(f_{n_j})_{j\in\mathbb{N}} \xrightarrow{\|\ \|_p} f$.

   Step 4. Show that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\|\ \|_p} f$ and $f \in L_p$

2. About quotient space. Given a normed space $(X, \|\ \|)$ and a closed subspace $X_0 \hookrightarrow X$. We can define the quotient space

   $$X/X_0 := \{[x] = x + X_0 : x \in X\},$$

   whose norm is

   $$\|[x]\| = \inf_{y\in X_0} \|x - y\| = \inf_{y\in[x]} \|y(-0)\|.$$

   The second equality can be verified by change $y \in [x] \iff y = x + x_0, x_0 \in X_0$.

3. Norm and semi-norm (p, $p(x) = 0 \not\Longrightarrow x = 0$). Let X be a linear semi-normed space, with the semi-norm p. A familiar linear semi-normed is $\mathcal{L}_p(1 \le p \le \infty)$. Let $X_0 := \{x \in X : p(x) = 0\} \hookrightarrow X$.

   Claim. X_0 is closed subspace of X (so, X/X_0 is allowed, see (**??**)).

   Proof. X_0 is a linear subspace, since p is a semi-norm.

   p is a continuous map, since the triangle inequality holds. Then $N = p^{-1}(0)$ must be closed. □

Now, (**??**) ensures that $\|\ \|: X/X_0, [x] \mapsto p(x)$ is a norm on X/X_0.

Proof. It should be verified that p is well-defined (though this should have been proved in (**??**)). Suppose $[x] = [y]$, that is $[x - y] = [y - x] = [0]$. Since p is a semi-norm, we have the triangle inequality

$$p(x) + p(y - x) \ge p(y), p(y) + p(x - y) \ge p(x),$$

and $[x - y] = [y - x] = 0 \implies p(x - y) = p(y - x) = 0$, that is $p(x) = p(y)$. Thus, $[x] \mapsto p(x)$ is well-defined. And

(1) $\|[x]\| = 0 \iff p(x) = 0 \iff x \in X_0 = [0] \iff [x] = [0]\left(\in {}^{X}\!/_{X_0}\right)$.

(2) $\|k[x]\| = \|[kx]\| = p(kx) = |k|p(x) = |k|\|x\|$.

(3) $\|[x] + [y]\| = \|[x + y]\| = p(x + y) \leq p(x) + p(y) = \|[x]\| + \|[y]\|$.

Above all, $\| \ \|$ is a norm on $[X]$. □

0.1 Completion

In this class, X is a linear noremd space, unless otherwise specified.

Definition (Isometry)**.** Suppose X, Y are two linear normed spaces. We say X is isomtetric with Y, if there is a linear surjection $T\colon X \to Y$ such that

$$\|Tx\| = \|x\|(\forall x \in X),$$

or equivalently $\| \ \|_Y \circ T = \| \ \|_X$.

Remark. Isometry is automatically injective, since $Tx = 0 \iff \|Tx\| = \|x\| = 0 \iff x = 0$. That is $\ker T = \{0\}$. Therefore, T is automatically injective and hence bijective as we want.

Definition (Density)**.** Let $(X, \| \ \|)$ be a liner normed space and $X_0 \hookrightarrow X$. X_0 is said to be dense in X, if $\overline{X_0} = X$.

Question. How to verify $\overline{X_0} = X$?

$\overline{X_0} = X$, if

$$\forall x \in X \forall \varepsilon > 0 \exists x_\varepsilon \in X_0 (\|x_\varepsilon - x\| < \varepsilon.)$$

And equivalently

$$\forall x \in X \forall n \in \mathbb{N} \exists x_n \in X_0 (\|x_\varepsilon - x\| < 1/n.)$$

That is, $\exists (x_n)_{n\in\mathbb{N}} \subseteq X_0$ that converges to x.

Theorem 0.1 (Completion Theorem)**.** Let $(X, \| \ \|)$ be a linear normed space. There is a Banach space $(\widehat{X}, \| \ \|)$ such that X is isometric to a dense subspace of $\widehat{X}$.

Remark. in fact, the completion $\widehat{X}$ is unique up to an isometry.

Definition. $\widehat{X}$ is called the completion of X.

Proof. We will construct a complection of X. Let

$$\mathcal{E} := \{(x_n)_{n\in\mathbb{N}} \subseteq X : (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence}\},$$

and define $p\colon \mathcal{E} \to \mathbb{R}, x(= (x_n)_{n\in\mathbb{N}}) \mapsto \lim_n \|x_n\|$. Here $\lim_n \|x_n\|$ exists in $\mathbb{R}$, because $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence implies that $\|x\|(= (\|x_n\|)_{n\in\mathbb{N}})$ is a Cauchy sequence in $\mathbb{R}$, and $\mathbb{R}$ is complete. Moreover, p is a semi-morn on $\mathcal{E}$. Now define $N := p^{-1}(0)$. Then $N \hookrightarrow \mathcal{E}$ and N is closed (by the continuity of p). Therefore we can consider $\widehat{X} := {}^{\mathcal{E}}\!/_{N}$, with the norm $\| \ \|\colon \widehat{X} \to \mathbb{R}, x + N \mapsto p(x)$.

Now, we prove this theorem in 3 steps.

Step 1. X is isometric to a subspace of $\widehat{X}$. Let $X_0 := \{[(x)_{n\in\mathbb{N}}] : x \in X\}$ and

$$T\colon X \to X_0, x \mapsto [(x)_{n\in\mathbb{N}}] = (x)_{n\in\mathbb{N}} + N,$$

where $(x)_{n\in\mathbb{N}}$ means the constant sequence $(x, \ldots, x, \ldots)$. That is, $T(x) = (x, \ldots, x, \ldots) + N$. Clearly T is a linear surjection. We want to show T is isometric, that is $\forall x \in X, \|T(x)\| = \|x\|$. By definiton

$$\begin{aligned} \|T(x)\| &= \|[(x)_{n\in\mathbb{N}}]\| && \text{(def of } T\text{)} \\ &= p\big((x)_{n\in\mathbb{N}}\big) && \text{(def of } \|\ \|_{\widehat{X}}\text{)} \\ &= \lim_n \|x\| && \text{(def of } p\text{)} \\ &= \|x\|. \end{aligned}$$

To sum up, T is a linear isometric surjection as we want.

Step 2. $X_0 \hookrightarrow \widehat{X}$ is dense. As discussed above, it suffices to show that $\forall [x] = (x_1, \ldots, x_n, \ldots) + N \in \widehat{X}$, there is a sequence in X_0 converge to X. Let

$$[x]^{(m)}\colon \mathbb{N} \to [(x_m)_{n\in\mathbb{N}}] = (x_m, \ldots, x_m, \ldots) + N,$$

and we prove that the sequence $([x^{(m)}])_{m\in\mathbb{N}}$ is convergent to $[x]$.

$$\begin{aligned} \lim_m \left\| [x]^{(m)} - [x] \right\| &= \lim_m \|(x_m - x_1, \ldots, x_m - x_n, \ldots) + N\| && \text{(def of } \pm\text{)} \\ &= \lim_m p\big((x_m - x_n)_{n\in\mathbb{N}}\big) && \text{(def of } \|\ \|\text{)} \\ &= \lim_m \lim_n \|x_m - x_n\| && \text{(def of } p\text{)} \\ &= 0. && \text{(see remark)} \end{aligned}$$

Step 3. $\widehat{X}$ is a Banach space. That is $\widehat{X}$ is complete. Let $([x]^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widehat{X}$. By the density of $X_0 = TX$, we have a sequence $(y_n)_{n\in\mathbb{N}} \subseteq X$ such that

$$\forall n \in \mathbb{N} \left\| T(y_n) - [x]^{(n)} \right\| \leq 1/n.$$

Claim. $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned} \|y_m - y_n\| &= \|T(y_m) - T(y_n)\| \\ &\leq \left\| T(y_m) - [x]^{(m)} \right\| + \left\| [x]^{(m)} - [x]^{(n)} \right\| + \left\| T(y_n) - [x]^{(n)} \right\| \\ &\leq 1/m + \left\| [x]^{(m)} - [x]^{(n)} \right\| + 1/n. \end{aligned}$$

Apply $\limsup_{m,n}$ on both sides and we have

$$\limsup_{m,n} \|y_m - y_n\| \leq 0.$$

Therefore, $(y_n)_{n\in\mathbb{N}}$ is Cauchy, and $(y_n)_{n\in\mathbb{N}} \in \mathcal{E}$. Now we show that $([x]^{(n)})_{n\in\mathbb{N}} \to [y] = (y_1, \ldots, y_n, \ldots) + N$. By definition of $\| \, \|_{\hat{X}}$

$$\begin{aligned}
\|[x]^m - [y]\| &\leq \|[x]^m - T(y_m) + T(y_m) - [y]\| \\
&\leq \|[x]^m - T(y_m)\| + \|T(y_m) - [y]\| \\
&\leq 1/m + p\big((y_n - y_m)_{n\in\mathbb{N}}\big) \\
&= 1/m + \lim_n \|y_n - y_m\|,
\end{aligned}$$

and let $m \to \infty$, we have

$$\limsup_m \|[x]^m - [y]\| \leq \limsup_m 1/m + \limsup_m \lim_n \|y_n - y_m\|.$$

The second limit must be 0, since $\lim_m \lim_n \|y_n - y_m\| = 0$ (see remark) .

□

Remark. Here we explain why $\lim_m \lim_n \|x_m - x_n\| = 0$. We may wan to write: suppose $\lim_n x_n = x$, then

$$\lim_m \lim_n \|x_m - x_n\| = \lim_m \|x_m - x\| = 0,$$

where the first equality is using the continuity of $\| \, \|$ and the second equality follows from the definition of $\lim_n x_n = x$. Everything makes sense, except $\lim_n x_n = x$. Notice that is a sequence in X and none said that X is complete.

So, why $\lim_m \lim_n \|x_m - x_n\| = 0$ holds? Search in what we know and there is something like this, that is $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. For convenience, let $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}, (m,n) \mapsto f_{m,n} := \|x_m - x_n\|$. Therefore, it suffices to show that we have

$$\lim_m \lim_n f_{m,n} = \lim_{m,n} f_{m,n}$$

whenever $\lim_{m,n\to\infty} f_{m,n}$ exists.

Proof. Let $x = \lim_{m,n} f_{m,n}$. $\lim_{m,n} f_{m,n} = x$, we have

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \forall m > N \;\; f_{m,n} \in (x - \varepsilon, x + \varepsilon).$$

This implies that

$$\forall m > N (x - \varepsilon \leq \liminf_n f_{m,n} \leq \limsup_n f_{m,n} \leq x + \varepsilon).$$

Let $m \to \infty$ and we have

$$a - \varepsilon \leq \liminf_m \liminf_n f_{m,n} \leq \limsup_m \limsup_n f_{m,n} \leq a + \varepsilon.$$

Clearly

$$\liminf_m \liminf_n f_{m,n} \leq \liminf_m \limsup_n f_{m,n} \leq \limsup_m \limsup_n f_{m,n}.$$

Therefore,

$$a - \varepsilon \leq \liminf_{m} \liminf_{n} f_{m,n} \leq \liminf_{m} \limsup_{n} f_{m,n} \leq a + \varepsilon.$$

ε is arbitrary, so $\liminf_m \liminf_n f_{m,n} = \liminf_m \limsup_n f_{m,n} = a$. That is $\liminf_m \lim_n f_{m,n} = a$. Similarly, $\limsup_m \lim_n f_{m,n} = a$. Above all,

$$\liminf_{m} \lim_{n} f_{m,n} = \limsup_{m} \lim_{n} f_{m,n} = a. \qquad \square$$