Functional Analysis Class Note

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2022 Autumn Semester, 1-16 Weeks

This note is taken for the Functional Analysiscourse, lectured by Professor Jiao Yong. In fact, this note contain full of my fragmentary thoughts, so all errors in this notes should be mine.

Here are some conventions:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are fields as you should learn in Caculus. $\mathbb{K}$ is one of $\mathbb{R}$ and $\mathbb{C}$, usually used to state different cases conveniently. $\mathbb{N}$ is the set of **positive** integers.
- Formula $A := B$ means A is defined as B. For example, $\mathbb{C} := \mathbb{R}[x]/(1+x^2)$ means $\mathbb{C}$ is defined as the quotient ring $\mathbb{R}[x]/(1+x^2)$.
- For $a, b \in \mathbb{R}$, define minimum function

$$\wedge\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto \frac{a+b-|a-b|}{2},$$

  and maximum function

$$\vee\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto \frac{a+b+|a-b|}{2}.$$

- Semminus of sets A, B is $A \setminus B := \{x \in A : x \notin B\}$.
- For a sets A, $\mathcal{P}(A)$ means the power set of A.
- For proposition p, q, we use $p \wedge q$ to mean the proposition "p and q", $\wedge$ has truth table as follow:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

  Similarly, we define $p \vee q$.

- Addition of real/complex-valued functions is defined pointwisely. That is: let $f, g\colon X \to \mathbb{K}$, we define a funtion $f + g\colon X \to \mathbb{K}$ by $x \mapsto f(x) + g(x)$.

- For $f\colon X \to \mathbb{K}$ and $k \in \mathbb{K}$, we define that function $f + k$ by $x \mapsto f(x) + k$. That is, respect k as a constant function $x \mapsto k$.
- Somewhere you can see color different, that is reminding you to think about what here should be. (Just like $1 + 1 = 2$.)
- $\lim_n$ means $\lim_{n\to\infty}$ for short.
- We say a diagram commutes, if all the morphisms (and their possible compositions) with the same domain and same codomain coincide.
- The Kronecker symbol on a set is defined as $\delta\colon X \times X \to \{0, 1\}, (x, y) \mapsto \delta^x_y := \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$.
- Let $(\Omega, \mathcal{F})$ be a measurable space. We say a function $f\colon \Omega \to \mathbb{R}$ is measurable, if the preimage of Borel subsets of $\mathbb{K}$ under f is $\mathcal{F}$-measurable. That is, assume $\mathbb{K}$ is equipped with Borel σ-algebra.

Contents

0 Introduction

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1 Day 1 of Week 1

We begin from **Banach Spaces and Metric**. Before the definition of Banach Space, we should recall the definition of Vector spaces(or Linear Spaces). Given a set X, a vector space is a triple $(X, +, \cdot)$ where $+\colon X \times X \to X$ is called the addition on X, and $\cdot\colon \mathbb{K} \times X \to X$ is called scalar-multiplication on X, making diagrams commute:

Recall:An isomorphism between vector space means a bijection that keeps the linear structure, that is $\varphi\colon X \to Y$ satisfies: $\forall k, l \in \mathbb{K}, \forall x, x' \in X$ we have $\varphi(kx + lx') = k\varphi(x) + l\varphi(x')$. Isomorphisms in categoryshould be in mind,

Linear Normed space

Definition (Linear Normed space)**.** Let X be a linear space. Define a map $\|\ \|\colon X \to \mathbb{R}_{\geq 0}$ satisfying:

(i) $\|x\| = 0 (\in \mathbb{K}) \iff x = 0 (\in X)$;

(ii) $\|kx\| = |k| \cdot \|x\| (\forall k \in \mathbb{K}, x \in X)$;

(iii) $\|x + y\| \leq \|x\| + \|y\| (\forall x, y \in X)$.

Then $\|\ \|$ is called a **norm** over X, and $(X, \|\ \|$ is called a **linear normed space**.

Remark. There is some similar weaker definitions:

- If(only)**(i)** is not satisfied, we call $\|\ \|$ a semi-norm.
- If(only)**(iii)** becomes $\|x + y\| \leq C(\|x\| + \|y\|)$ for some $C \in \mathbb{R}_{>1}$, we call $\|\ \|$ a quasi-norm.

Example (Euclidean Spaces)**.** $(\mathbb{R}^n, \|\ \|)$ is a linear normed space, whose norm is defined as follow:

$$\|\ \|\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}, x = (x_1, \ldots, x_n) \mapsto \Big(\sum_{j=1}^{n} x_j^2\Big)^{1/2} (= d(x, 0)).$$

Triangle inequality for this norm comes to be the particular triangle inequality for the metric, which can be shown by Cauchy-Schwarz inequality for real numbers.

Example (Continuous Functions Spaces)**.** $\big(C([a, b], \mathbb{K}), \max_{[a,b]}|\ |\big)$ is a linear normed space. Recall the definition of $C([a, b], \mathbb{K})$ the family of continuous function from $[a, b]$ to $\mathbb{K}$. whose norm is defined as follow:

$$\max_{[a,b]}|\ |\colon (C([a, b], \mathbb{K}) \to [0, \infty), f \mapsto \max_{x \in [a,b]} |f(x)|.$$

Recall why $C([a, b], \mathbb{K})$ is a vector space. What is needed to show is just "addition of continuous functions is continuous", and there is lots of ways to do this, see remark. Notice that $[a,b]$ is compact and so is $f([a,b])$, guaranteeing the existence of $\max_{x\in[a,b]}|f(x)|$. Compatibility with multiplication and triangle inequality is trivial.

Remark. We have follow methods for proving "addition of continuous functions is continuous". They give the same result with different standpoint. Suppose $f,g \in C([a,b],\mathbb{K})$

1. By definition of continuity. We prove pointwisely: Fix $x \in [a,b]$. $\forall\varepsilon > 0$,we can find $\delta_1, \delta_2 > 0$ such that $\forall y : 0 < |y-x| < \delta_1, |f(y)-f(x)| < \varepsilon/2$ and $\forall y : 0 < |y-x| < \delta_2, |g(y)-g(x)| < \varepsilon/2$. Therefore, let $\delta := \delta_1 \wedge \delta_2$ we have $\forall y : 0 < |y-x| < \delta$,

$$\begin{aligned}|(f+g)(y)-(f+g)(x)| =&|f(y)+g(y)-f(x)-g(x)|\\ \leq&|f(y)-f(x)|+|g(y)-g(x)|\\ <&\varepsilon/2+\varepsilon/2\\ =&\varepsilon.\end{aligned}$$

   Therefore, $f+g$ is continuous at x.

2. By Heine's Theorem: We prove pointwisely: Fix $x \in [a,b]$. Suppose sequence $(x_n)_{n\in\mathbb{N}} \subseteq [a,b]$ converges to x, then:

$$\begin{aligned}\lim_{n\to\infty}(f+g)(x_n) =& \lim_{n\to\infty}\big(f(x_n)+g(x_n)\big)\\ =& \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n)\\ =&f(x)+g(x)\\ =&(f+g)(x).\end{aligned}$$

   Therefore, $f+g$ is continuous at x.

3. By topological definition($\mathbb{K}=\mathbb{R}$ case): an observation :

$$(f+g)^{-1}(t,\infty) = \bigcup_{r\in\mathbb{R}} \big(f^{-1}(t-r,\infty)\cap g^{-1}(r,\infty)\big),$$

   which should be prove by $A \subseteq B \wedge B \subseteq A \implies A = B$. Right hand side is union of intersection of two open sets, and similarly for $(f+g)^{-1}(-\infty,t)$.We're done.

4. Addition is continuous($\mathbb{K}=\mathbb{R}$ case). We decompose $f+g$ as following communicative diagrams

$$\begin{tikzcd}
{[a,b]} \arrow[r, "f\times g"] \arrow[dr, "f+g"'] & \mathbb{R}\times\mathbb{R} \arrow[d, "+"] & x \arrow[r, mapsto] \arrow[dr, mapsto] & (f(x),g(x)) \arrow[d, mapsto] \\
& \mathbb{R} & & f(x)+g(x)
\end{tikzcd}$$

The right diagram explains what the functions in the left diagram mean. By the property of product topology and continuity of f and g, we know $f \times g$ is continuous. Continuity of $+\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is trivial. Therefore $f + g = + \circ (f \times g)$ is continuous.

To get rid of the assumption $\mathbb{K} = \mathbb{R}$, use the fact that $f\colon X \to \mathbb{C}$ is continuous if and only if both $\mathrm{Re}(f), \mathrm{Im}(f)$ are continuous.

Example (p-summable sequence spaces)**.** Given $p \in [1, \infty]$ we define $(\ell_p, \| \ \|_p)$, where

$$\begin{aligned} \ell_p :=& \Big\{ (a_n)_{n\in\mathbb{N}} : \sum_{n\geq 1} |a_n|^p < \infty \Big\}, \text{(for } p < \infty\text{)} \\ \ell_\infty :=& \Big\{ (a_n)_{n\in\mathbb{N}} : \sup_{n\in\mathbb{N}} |a_n| < \infty \Big\}. \end{aligned}$$

And norms are

$$\begin{aligned} \|a\|_p :=& \Big(\sum_{n\geq 1} |a_n| \Big)^{1/p}, && \text{(for } p < \infty\text{)} \\ \|a\|_\infty :=& \sup_{n\in\mathbb{N}} |a_n|. && \text{(Here } a \text{ means } (a_n)_{n\in\mathbb{N}}\text{)} \end{aligned}$$

Proposition 1.1. $(\ell_\infty, \| \ \|_\infty)$ is a normed space.

Proof. Clearly ℓ_∞ is a vector space. Now we prove $\| \ \|_\infty$ is a norm.

1. $\|a\|_\infty \geq 0$ and $\|a\|_\infty = 0 \iff a = 0$: $\|a\|_\infty \geq 0$ is trivial. Suppose $\|a\|_\infty = 0$, that is $\sup_{n\in\mathbb{N}} |a_n| = 0$. By definition of supermum, $|a_n| \leq 0 (\forall n \in \mathbb{N})$. Therefore, $a = 0$.

2. $\forall k \in \mathbb{K}$, by property of absolute value we know $\|ka\|_\infty = |k| \|a\|_\infty$.

3. Let $a, b \in \ell_\infty$ and $M_a = \|a\|_\infty, M_b = \|b\|_\infty$. Now from definition of supermum
$$\forall n \in \mathbb{N} : |a_n + b_n| \leq |a_n| + |b_n| \leq M_a + M_b$$
Again using definition of supermum, we get $\|a + b\|_\infty \leq M_a + M_b$, which was what we wanted.

$\square$

Remark. In general, for a measure space $(\Omega, \mathcal{F}, \mu)$ the inequality $\|f + g\|_p \leq \|f\|_p + \|g\| (p \geq 1)$ is called the Minkowski inequality.

Example. [Subspaces of ℓ_∞] ℓ_∞ has linear subspaces: $c_0 \subseteq c \subseteq \ell_\infty$, where

$$\begin{aligned} c :=& \{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is convergent sequence} \}, \\ c_0 :=& \{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is convergent sequence, with limit } 0 \}. \end{aligned}$$

2 Day 2 of Week 1

L^p spaces

Recall the left problem: Minkowski inequality, which makes $(\ell_p, \| \ \|_p)$ a normed space. Now, we need a lemma.

Lemma (Hölder's Inequality)**.** Let $a \in \ell^p, b \in \ell^q$ for $p \in (1, \infty)$ and $q \in (1, \infty)$ staisfying $1/p + 1/q = 1$, we have:

$$\|ab\|_1 \leq \|a\|_p \|b\|_q, \tag{1}$$

Remark. $q = p/(p-1)$ is also called the dual index of p, usually denoted by p'.

Remark. Before start of the proof, we have a look at (1). Recall what we have learned in mathematical analysis, and have a problem in mind: is there anything similar? That is Cauchy-Schwarz Inequality, since they coincide when $p = q = 2$. Now we have a direct goal.

Aim. Prove (1) by imitating the proof of Cauchy-Schwarz Inequality.

Now, recall all the proofs of Cauchy-Schwarz Inequality you know and think: Which would be useful in this case? [3] Lagerange's Idendity, Schwarz's argument(inner product $\langle x + ty, x + ty \rangle \geq 0$), or just $2xy \leq x^2 + y^2$? When $p \neq 2$, Schwarz's argument is a nonstarter since there is no quadratic polynomial in sight. Similarly, the absence of a quadratic form means that one is unlikely to find an effective analog of Lagrange's identity.

This brings us to our most robust proof of Cauchy-Schwarz Inequality, the one that starts with the so-called "humble bound,"

$$xy \leq \frac{x^2}{2} + \frac{y^2}{2}, \forall x, y \in \mathbb{R}. \tag{2}$$

(2) proves Cauchy's inequality as follows.

Proof of Cauchy's inequality from (2). Without lost of generality, suppose that $\sum_{n\geq 1} a_n^2 \neq 0$ and $\sum_{n\geq 1} b_n^2 \neq 0$. Let

$$a'_j = a_j / \Big(\sum_{n\geq 1} a_n^2 \Big)^{1/2}, b'_j = b_j / \Big(\sum_{n\geq 1} b_n^2 \Big)^{1/2}, \forall j \in \mathbb{N}.$$

Notice that $\sum_{n\geq 1} a'_n = \sum_{n\geq 1} b'_n = 1$. Now, since (2) holds, we obtain

$$\sum_{n\geq 1} a_n b_n \leq \sum_{n\geq 1} (a_n^2 + b_n^2)/2 = \sum_{n\geq 1} a_n^2/2 + \sum_{n\geq 1} b_n^2/2 = 1.$$

And, in terms of $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$, and multiply $(\sum_{n\geq 1} a_n^2)^{1/2} (\sum_{n\geq 1} a_n^2)^{1/2}$ on both sides, we have

$$\sum_{n\geq 1} a_n b_n \leq \Big(\sum_{n\geq 1} a_n^2 \Big)^{1/2} \Big(\sum_{n\geq 1} b_n^2 \Big)^{1/2}. \qquad \square$$

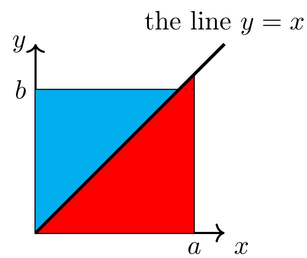

Figure 1: Area meaning of (2)

This bound may now remind us that the general AM-GM inequality

$$x^p y^q \leq \frac{x}{p} + \frac{y}{q} \quad \text{for all } x, y \geq 0 \text{ and } q = p'(p, q > 1). \tag{3}$$

(3) is the perfect analog of the "humble boun"(2).

Proof of (2). There is many ways to to this, see[3]. We choose the way by area of regions. Consider the region under the function $x \mapsto x$:

$$A := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq a\}, B := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq b\}.$$

Then (1)[Figure 1] shows that $m(A)+m(B) \geq m([0, a]\times[0, b])$, where m denotes the Lebesgue measure on $\mathbb{R}^2$. □

Now, by imitating the proof of (2), we need to get the x^p/p as area of some region under a function, so consider the function $x \mapsto x^{p-1}$.

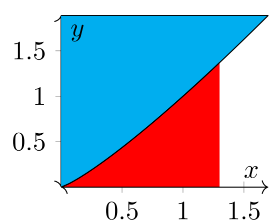

Figure 2: Area meaning of (3)

Proof of (3). It's easy to verify that

$$m(A) = \int_{[0,a]} f \, \mathrm{d}\mu, m(B) = b^{\frac{p}{p-1}} - \int_{[0,b^{p/(p-1)}]} f \, \mathrm{d}\mu,$$

where μ is the Lebesgue measure on $\mathbb{R}$. By simple calculation, we have $m(A) = \frac{1}{p}a^p, m(B) = \frac{1}{q}b^q$. Notice that $A \cup B$ contains $[0, a] \times [0, b]$, we're done. □

Proof of (1). Without loss of generality, suppose $\|a+b\|_p \neq 0$. And suppose $a \neq 0 (\in \ell_p), b \neq 0 (\in \ell_q)$. As what we do in the proof of Cauchy's inequality, let

$$a'_j = a_j/\|a\|_p, b'_j = b_j/\|b\|_p, \forall j \in \mathbb{N}.$$

Notice that $\|a'\|_p = \|b'\|_q = 1$. Now, apply (3) to $|a_j b_j|$, we have

$$\sum_{n\geq 1} |a'_n b'_n| \leq \sum_{n\geq 1} |a'_n|^p/p + \sum_{n\geq 1} |b'_n|^q/q = 1/p + 1/q = 1,$$

which implies

$$\|ab\|_1 \leq \|a\|_p \|b\|_p.$$

□

Proof of Minkowski inequality.

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{n\geq 1} |(x+y)_n|^p \\
&= \sum_{n\geq 1} |x_n + y_n|^{p-1} |x_n + y_n| \\
&\leq \sum_{n\geq 1} |x_n + y_n|^{p-1} (|x_n| + |y_n|) \text{(Triangle inequality on } \mathbb{R}\text{)} \\
&= \sum_{n\geq 1} |x_n + y_n|^{p-1} |x_n| + \sum_{n\geq 1} |x_n + y_n|^{p-1} |y_n| \\
&= \left\|(x+y)^{p-1} x\right\|_1 + \left\|(x+y)^{p-1} y\right\|_1 \text{(def of norm)} \\
&\leq \left\|(x+y)^{p-1}\right\|_q \|x\|_p + \left\|(x+y)^{p-1}\right\|_q \|y\|_p \text{(see (1)) } (*) \\
&= \|(x+y)\|_p^{p/q} (\|x\|_p + \|y\|_p) ((p-1)q = p),
\end{aligned}$$

and divide $\|x+y\|_p^{p/q} (\neq 0)$ from both sides, getting

$$\|x+y\|_p^{p-p/q} \leq \|x\|_p + \|y\|_p.$$

We're done, since $p - p/q = 1$. □

To summarize what we have done, we need the language of measure.

Definition (σ-algebra)**.** A **σ-algrbra** on a set Ω is a subset Ω, satisfying:

1. $\Omega, \varnothing \in \mathcal{F}$;
2. $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$;
3. $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F} \implies \bigcup_{n\geq 1} A_n \in \mathcal{F}$.

Definition (Measurable Spaces, Measurable sets)**.** A **measurable space** is a double $(\Omega, \mathcal{F})$ where Ω is an aritrary set and $\mathcal{F}$ is a σ-algebra over Ω. Elements of $\mathcal{F}$ is called **measurable sets** of $(\Omega, \mathcal{F})$.

Definition (Measure,Measure spaces)**.** A **measure** is a σ-additive function from $\mathcal{F}$ to $[0,\infty]$. A triple $(\Omega,\mathcal{F},\mu)$ is called a **measure space**, if $(\Omega,\mathcal{F})$ is a measurable space and μ is a measure.

Definition (Integral with respect to measure)**.** Let $(\Omega,\mathcal{F},\mu)$ be a measure space. We have a glance at "how to define integral with respect to measure". For the detail, see [2].

Step 1: Define **integral** $\int$ for measurable simple nonnegative function:

$$\sum_{k=1}^{n} a_k\chi_{A_k} \longmapsto \sum_{k=1}^{n} a_k\mu(A_k).$$

Step 2: Define **integral** $\int$ for measurable nonnegative function:

$$f \longmapsto \sup\Big\{\int \varphi : \varphi \le f, \varphi \text{ is nonnegative simple function}\Big\}.$$

Step 3: Define **integral** $\int$ for measurable function:

$$f \longmapsto \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu,$$

where $f^+ = f\chi_{f^{-1}[0,\infty)}, f^- = -f\chi_{f^{-1}(-\infty,0]}$.

Definition (p-integrable space)**.** Let $(\Omega,\mathcal{F},\mu)$ be a measure space, then the p-integrable space over $\mathcal{L}_p(\Omega,\mathcal{F},\mu)$ is defined as

$$\mathcal{L}_p(\Omega,\mathcal{F},\mu) := \Big\{f \in \mathbb{K}^\Omega : f \text{ is measurable and } \int f \,\mathrm{d}\mu < \infty\Big\}.$$

Fact. The proof of Minkowski inequalityover ℓ_p actually proved the Minkowski inequalityof **every** p-integrable space $\mathcal{L}_p(\Omega,\mathcal{F},\mu)$.

To understand this fact, we should have another way to illustrate $\sum$. That is, $\sum$ is a kind of integral.

Definition (Counting Measure)**.** Given a measurable space $(\Omega,\mathcal{F})$. Define $\mu\colon \mathcal{F} \to [0,\infty], A \mapsto \sharp A$. Where $\sharp A = \infty$ if A is an infinite set, and $\sharp A = n$ if A has exactly n elements. μ is called the **counting measure** over $(\Omega,\mathcal{F})$.

Remark. It can be shown that,[1] for real sequence $(a_n)_{n\in\mathbb{N}}$ (equivalent to a function $a\colon \mathbb{N} \to \mathbb{R}$), we have

$$\sum_{n\ge 1} a_n = \int a \,\mathrm{d}\mu.$$

That's why we can respect $\sum$ as $\int$. And hence, the fact above is just regard $\sum$ as integral with respect to coungting measure, and the proof works for arbitrary measure space.

Remark. We can also prove Minkowski inequalityof L^p by using the $L^{p'}$. Since

$$\|f\|_p = \sup\Big\{\Big|\int fg \,\mathrm{d}\mu\Big| : g \in L^{p'}(\Omega,\mathcal{F},\mu), \|g\|_{p'} \le 1\Big\}.$$

3 Day 1 of Week 2

Quotient Spaces

Let X be a vector space with a linear subspace X_0, denoted as $X_0 \hookrightarrow X$.

Definition (Cosets)**.** $\forall x \in X$, the coset of x (with respect to X_0), denoted as $[x]$ or $x + X_0$ is defined as

$$[x] = x + X_0 := \{x + y : y \in X_0\}.$$

Definition (Quotient Spaces)**.** ${}^{X}\!/\!_{X_0} := \{[x] : x \in X\}$, called the quotient space of X (with respect to X_0).

We want ${}^{X}\!/\!_{X_0}$ to be a vector space, so we define operations as follows:

$$\oplus : {}^{X}\!/\!_{X_0} \times {}^{X}\!/\!_{X_0} \to {}^{X}\!/\!_{X_0}, ([x],[y]) \mapsto [x+y];$$
$$\odot : \mathbb{K} \times {}^{X}\!/\!_{X_0} \to {}^{X}\!/\!_{X_0}, ([x],k) \mapsto [kx].$$

Where $[x+y]$ means the addtion (and take the coset), and the $[kx]$ means the scalar multiplication of X (and take the coset). You should verify that the operations are well defined. For simplicity, we write $+, \cdot$ instead of $\oplus, \odot$.

Claim. $({}^{X}\!/\!_{X_0}, +, \cdot)$ is a vector space.

Question. Think this questions:

1. Clearly, the zero element in ${}^{X}\!/\!_{X_0}$ is $[0]$. But, $[0] = ?$;
2. If $[x] \neq [y]$, what is $[x] \cap [y]$?
3. Show that $x \in [y] \iff x - y \in X_0$.

Answers are as follows:

1. $[0] = X_0$, from definition of coset.
2. $\varnothing$. Since (3) implies $[x] \cap [y] \neq \varnothing$ means $\exists z : z - x, z - y \in X_0$, therefore $x - y = (z - y) - (z - x) \in X_0$ since X_0 is a linear subspace. Now, $\forall a \in [x]$, from $a = x + w(w \in X_0)$, we have $a = y + \big(w + (x - y)\big)$ and $\big(w + (x - y)\big) \in X_0$ so $a \in [y]$. Above all, $[x] \subseteq [y]$. It is the same to know $[y] \subseteq [x]$.
3. Since

$$\begin{aligned} x \in [y] &\iff x = y + z \text{ for some } z \in X_0 \\ &\iff x - y = z (= 0 + z) \text{ for some } z \in X_0 \\ &\iff x - y \in [0] = X_0. \end{aligned}$$

Let's see a simple example:

Example. From example (1), $c_0 \hookrightarrow c \hookrightarrow \ell_\infty$. And we introduce a new notion:

Definition (Codimension)**.** Suppose X a vector space and $X_0 \hookrightarrow X$. Then the codimension of X_0, is $\operatorname{codim}_X X_0 := \dim {}^{X}\!/_{X_0}$. Also denoted by just $\operatorname{codim}(X_0)$ if there is no confusion.

Claim. $\operatorname{codim}_c c_0 = 1$.

Proof. Let $(1_n)_{n\in\mathbb{N}}$ be the sequence with all elements 1. We want to show that $\{(1_n)_{n\in\mathbb{N}}\}$ is a basis of ${}^{c}\!/_{c_0}$. Let $(x_n)_{n\in\mathbb{N}} \in c$, and suppose $\lim_n x_n = x \in \mathbb{K}$. We have $[(x_n)_{n\in\mathbb{N}}] = [x(1_n)_{n\in\mathbb{N}}]$, since $x(1_n)_{n\in\mathbb{N}}$ is just the sequence with all elements x, and clearly $\lim_n(x_n - x) = 0 \implies (x_n)_{n\in\mathbb{N}} - x(1_n)_{n\in\mathbb{N}} \in c_0$. That is, $[(x_n)_{n\in\mathbb{N}}] = [x(1_n)_{n\in\mathbb{N}}] = x[(1_n)_{n\in\mathbb{N}}]$. We're done. $\square$

Remark. There is an isomorphism from ${}^{c}\!/_{c_0}$ to $\mathbb{K}$, that is $[(x_n)_{n\in\mathbb{N}}] \mapsto \lim_n x_n$.

Example. Consider $X = \mathbb{R}^2$, $X_0 \hookrightarrow X$ with $\dim X_0 = 1$. It is easy to see that $\forall x \in \mathbb{R}$, the coset containing x is just translating X_0 such that $0 (\in X_0)$ is translated to x. And

$${}^{X}\!/_{X_0} = \{X_0\} \cup \{\text{all lines that are parallel to } X\}.$$

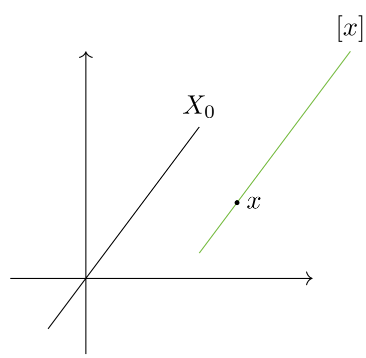

Figure 3: X, X_0 and points of ${}^{X}\!/_{X_0}$

Now we want to define a norm on ${}^{X}\!/_{X_0}$. An intuitive norm is the distance between X_0 and the coset.

Definition (Norm on ${}^{X}\!/_{X_0}$)**.** Define

$$\|\ \|\colon {}^{X}\!/_{X_0} \to \mathbb{R}_{\geq 0}, [x] \mapsto \inf_{y\in X_0} \|x - y\|.$$

The norm in green color is the usual norm in $\mathbb{R}^2$, see (1).

We should verify that $\| \;\|$ is actually a norm. That is

Question. Verify that :

1. $\forall [x] \in {}^{X}\!/_{X_0} : \|[x]\| \geq 0$ and $\|x\| = 0 \iff x = X_0$;
2. $\forall [x] \in {}^{X}\!/_{X_0} : \|k[x]\| = |k| \cdot \|x\|$;
3. $\|[x] + [y]\| \leq \|[x]\| + \|[y]\|$.

Proof. For (1): Only needed is to show that $\|x\| = 0 \iff x = X_0$. Here we use Theorem (3) (in the below remark) and a trivial fact:

Fact. X_0 is a closed subset of X.

Now suppose $[x] \in {}^{X}\!/_{X_0}$ satisfying $\|[x]\| = 0$. By definition, we have $\inf_{y \in X_0} \|x - y\| = 0$. From the definition of infimum : $\forall n \in \mathbb{N} \exists y_n \in X_0$ such that $\|x - y_n\| < 1/n$, therefore we have a sequence $(y_n)_{n \in \mathbb{N}} \subseteq X_0$ converging to x. From (3), we know $x \in X_0$, so $[x] = X_0$ as we wanted.

(2): It holds naturally when $k = 0$. If $k \neq 0$, it just follows from property of norm and $k^{-1} X_0 = X_0$.

(3): From intuition, we have

$$\begin{aligned} \|[x] + [y]\| =& \|[x + y]\| \\ =& \inf_{z \in X_0} \|x + y - 2z\| \\ \leq& \inf_{z \in X_0} (\|x - z\| + \|y - z\|) \text{(triangle inequality of norm on } X\text{)} \\ \leq& \inf_{z \in X_0} \|x - z\| + \inf_{z \in X_0} \|y - z\| \\ =& \|[x]\| + \|[y]\|. \end{aligned} \tag{4}$$

So easy, isn't it? However, look at $\leq$, this inequality is non-trivial and we should prove. By simple application of definition of infimum, we find: the inequality is **reversed**! But (4) can be corrected: $\forall \varepsilon > 0$, $\exists z_\varepsilon \in X_0, w_\varepsilon \in X_0$ such that

$$\|x - z_\varepsilon\| < \inf_{z \in X_0} \|x - z\| + \varepsilon/2 = \|x\| + \varepsilon/2,$$
$$\|y - w_\varepsilon\| < \inf_{z \in X_0} \|y - z\| + \varepsilon/2 = \|y\| + \varepsilon/2.$$

Therefore we have

$$\inf_{z \in X_0} (\|x - z\| + \|y - z\|) \leq \|x\| + \|y\| + \varepsilon.$$

Since ε is arbitrary, we know $\inf_{z \in X_0} (\|x - z\| + \|y - z\|) \leq \|x\| + \|y\|$ and then $\|x + y\| \leq \|x\| + \|y\|$.

However, this is wrong again. Since z_ε may not coincide with w_ε. To fix this, write

$$\|[x + y]\| = \inf_{z, w \in X_0} \|x + y - (z + w)\|. \tag{5}$$

And by (5), and $\|x + y - (z + w)\| \leq \|x - z\| + \|y - w\|$, we use the definition of inf for $\inf_{z \in X_0} \|x - z\|, \inf_{w \in X_0} \|y - w\|$. We can find $z_\varepsilon, w_\varepsilon$ as above and get $\|[x + y]\| \leq \|[x]\| + \|[y]\| + \varepsilon$, we're done.

Above all, $\| \ \|$ is actually a norm. □

Remark. We define the topology of linear normed space as follows:

Definition (Topology of linear normed space)**.** Let $(X, \| \ \|)$ be a linear normed space. Then there is a natural metric on X, that is $d\colon X \times X \to \mathbb{R}_{\geq 0}, (x, y) \mapsto \|x - y\|$. The topology induced by this metric is called the (usual) topology of $(X, \| \ \|)$.

Now we have topology of X, and we have a result characterizing the closed subsets of X.

Theorem. Given a linear normed space X with $X_0 \hookrightarrow X$. Then, X is closed **if and only if** for all $(x_n)_{n \in \mathbb{N}} \subseteq X_0$ such that $\lim_n x_n = x \in X$, we have $x \in X_0$.

4 Day 2 of Week 2

Substitute teacher: Wu Lian.

Metric Spaces

Definition (Metric, Metric Spaces)**.** Let X be a set. $d\colon X \times X \to \mathbb{R}$ is called a metric, if d satisfies:

1. $\forall x, y \in X\colon d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$.
2. $\forall x, y \in X\colon d(x,y) = d(y,x)$.
3. $\forall x, y, z \in X\colon d(x,y) + d(y,z) \geq d(x,z)$.

The ordered pair (X, d) is called a metric space.

Remark. Every metric space has a topology, we will discuss this later.

Remark. Let's compare normed spaces and metric spaces: normed space need linear structures but metric spaces don't need. A normed space $(X, \|\ \|)$ is naturally a metric space by the metric induced by norm $d\colon X \times X \to \mathbb{R}, (x,y) \mapsto \|x - y\|$.

Remark. Let X be an arbitrary set, we can define a metric on X by the Kronecker symbol δ.

Example. $(\mathbb{R}^n, d)$ is a metric space, where

$$d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) \mapsto \Big(\sum_{j=1}^{n} (x_j - y_j)^2\Big)^{1/2}.$$

Example. $(\mathbb{R}^{\mathbb{N}}, d)$ is a metric space, where

$$d\colon \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}, \big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\big) \mapsto \sum_{j\geq 1} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}.$$

d is well-defined, since the series can be dominated by $\sum_{j=1}^{\infty} 1/2^j$. To verify the triangle inequality, we use the monotone function $f\colon [0,\infty) \to [0,1), x \mapsto x/(1+x)$. So, $|x_j - y_j| + |y_j - z_j| \geq |x_j - z_j|$ implies

$$\frac{|x_j - y_j| + |y_j - z_j|}{1 + |x_j - y_j| + |y_j - z_j|} \geq \frac{|x_j - z_j|}{1 + |x_j - z_j|},$$

and clearly the left-hand side is no more than $f(|x_j - y_j|) + f(|y_j - z_j|)$. Sum for $j \in \mathbb{N}$ and we're done.

Example. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let $\mathcal{L}_0(\Omega)$ be the space of all $\mathcal{F}$-measurable functions from Ω to $\mathbb{K}$, written $\mathcal{L}_0$ for short. Define $\mathcal{Z} := \{f \in \mathcal{L}_0(\Omega) : f(x) = 0 \text{ for } \mu\text{-almost every } x \in \Omega\}$, the (linear) subspace containing all functions equal 0 μ-almost everywhere. Now consider the quotient space $\mathcal{L}_0/\mathcal{Z}$. We define

$$\begin{aligned} d\colon \mathcal{L}_0/\mathcal{Z} \times \mathcal{L}_0/\mathcal{Z} &\longrightarrow \mathbb{R} \\ (f+\mathcal{Z}, g+\mathcal{Z}) &\longmapsto \int_\Omega \frac{|f-g|}{1+|f-g|}\,\mathrm{d}\mu. \end{aligned} \tag{6}$$

Integrand on the right-hand side can be dominated by $1_\Omega (= 1)$, hence the integral is finite. The definition of d involves the selection of representative element, so we should verify that d is well-defined. Suppose $f + \mathcal{Z} = f' + \mathcal{Z}, g = g' + \mathcal{Z}$, and suppose f, g is finite everywhere, then

$$\begin{aligned} &\exists A_1 : \mu(A_1) = 0 \ \forall x \in A_1^c \ f(x) = f'(x); \\ &\exists A_2 : \mu(A_2) = 0 \ \forall x \in A_1^c \ g(x) = g'(x). \end{aligned} \tag{7}$$

Then $f(x) - g(x) = f'(x) - g'(x)$ for all $x \in (A_1 \cup A_2)^c$ and $\mu(A_1 \cup A_2) = 0$. Therefore $f - g = f' - g'$ μ-almost everywhere, and hence $\frac{|f-g|}{1+|f-g|} = \frac{|f'-g'|}{1+|f'-g'|}$ μ-almost everywhere, implying that their integration coincide. Above all, $d(f + \mathcal{Z}, g + \mathcal{Z}) = d(f' + \mathcal{Z}, g' + \mathcal{Z})$ whenever $f - f' \in \mathcal{Z}, g - g' \in \mathcal{Z}$.

Proof of "d is a metric" is the same as the previous example.

Example. These are all metric spaces, since they are linear normed spaces: $\ell_p, c_0, c, C([a, b], \mathbb{K}), L_p, \mathbb{R}^n$.

Definition (Convergence in metric space)**.** Let (X, d) be a metric space. A sequence in X, say $(x_n)_{n\in\mathbb{N}} \subseteq X$. We say $(x_n)_{n\in\mathbb{N}}$ is convergent to $x \in X$, if $\lim_n d(x_n, x) = 0$(limit of real sequence). $(x_n)_{n\in\mathbb{N}}$ is convergent to x is usually denoted by $(x_n)_{n\in\mathbb{N}} \xrightarrow{d} x$(or $(x_n)_{n\in\mathbb{N}} \to x$ if there is no ambiguity).

Example. Suppose X is an arbitrary set. (X, δ) is a metric space, where δ means the Kronecker symbol. Then

$$(x_n)_{n\in\mathbb{N}} \to x \iff \exists N \in \mathbb{N} \ \forall n \geq N \ x_n = x.$$

Example. Consider $\big((C[a, b], \mathbb{K}), d\big)$, where

$$d\colon (C[a, b], \mathbb{K}) \times (C[a, b], \mathbb{K}) \to \mathbb{R}, (f, g) \mapsto \max_{[a,b]} |f - g|.$$

Then $(f_n)_{n\in\mathbb{N}} \xrightarrow{d} f \iff (f_n)_{n\in\mathbb{N}}$ converge to f uniformly, as we learned in Mathematical Analysis.

Example. Recall $(L_0/\mathcal{Z}, d)$, $(f + \mathcal{Z}_n)_{n\in\mathbb{N}} \xrightarrow{d} f + \mathcal{Z} \iff (f_n)_{n\in\mathbb{N}} \xrightarrow{\mu} f$.

Proof. Necessity:$(f+\mathcal{Z}_n)_{n\in\mathbb{N}} \xrightarrow{d} f+\mathcal{Z}$ means

$$\lim_n \int_\Omega \frac{|f_n - f|}{1+|f_n-f|}\,\mathrm{d}\mu = 0.$$

Given $\sigma > 0$. Define a set $E_n^\sigma := \{x \in \Omega : |f_n(x) - f(x)| > \sigma\}$, we need to show $\lim_n \mu(E_n^\sigma) = 0$. By Chebyshev's inequality:

$$\begin{aligned}
\mu(E_n^\sigma) =& \mu\{x \in \Omega : |f_n(x) - f(x)| > \sigma\} \\
=& \mu\Big\{x \in \Omega : \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} > \frac{\sigma}{1+\sigma}\Big\} \\
\le& \frac{1+\sigma}{\sigma}\int_{E_\sigma^n} \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|}\,\mathrm{d}\mu \\
\le& \frac{1+\sigma}{\sigma}\int_{\Omega} \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|}\,\mathrm{d}\mu \\
=& \frac{1+\sigma}{\sigma} d(f_n+\mathcal{Z}, f+\mathcal{Z}).
\end{aligned}$$

$\lim_n d(f_n+\mathcal{Z}, f+\mathcal{Z}) = 0$ implies $\lim_n \mu(E_n^\sigma) = 0$, that is $f_n \xrightarrow{\mu} f$.

Sufficiency: Given $\sigma \in (0,1)$, we know:

$$\Big\{x \in \Omega : \frac{|f_n - f|}{1+|f_n-f|} > \sigma\Big\} = \{x \in \Omega : |f_n - f| > \frac{\sigma}{1-\sigma}\}.$$

This implies that $\frac{|f_n-f|}{1+|f_n-f|} \xrightarrow{\mu} 0$.

Now, from the dominated convergence theorem (1_Ω being the dominated function), we have:

$$\lim_n d(f_n+\mathcal{Z}, f+\mathcal{Z}) = \lim_n \int_\Omega \frac{|f_n-f|}{1+|f_n-f|}\,\mathrm{d}\mu = \int_\Omega \lim_n \frac{|f_n-f|}{1+|f_n-f|}\,\mathrm{d}\mu = 0. \quad \square$$

Remark. I didn't get this solution for sufficiency at the class. So, it is meaningful to have a look after class.

Topology of metric spaces

Definition (Topology of metric space)**.** The topology of a metric space (X,d) is generated by the base

$$\mathcal{B} = \{B(x,r) \colon x \in X, r \in (0,\infty)\},$$

where $B(x,r) := \{y \in X \colon d(y,x) < r\}$.

Remark. Now we can define these things for metric spaces:

- Interior points of a set.
- Interior of sets.

- Limit points of a set.
- Derived sets.
- Closure.
- Isolated point.
- Boundary.

Fact. For a metric space (X, d):

1. A set G is open $\iff \forall x \in G\ \exists r > 0\ B(x, r) \subseteq G$.

   Proof. Sufficiency is trivial. For necessity, since each open set is union of bases, then $x \in G$ must lie in a open ball contained in G, and we can find some $r > 0$ such that $B(x, r)$ is contained in the open ball. □

2. Intersection of open sets may not be open. For example,

$$\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}.$$

Definition (Continuity of maps between metric spaces)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces. We say $f\colon X \to Y$ is continuous at $x \in X$, if $\forall \varepsilon > 0\ \exists r > 0$ such that $f\big(B(x, r)\big) \subseteq B\big(f(x), \varepsilon\big)$(two balls are in X and Y respectively). f is continuous if f is continuous at every $x \in X$.

Theorem (Continuity's equivalent sequential condition)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces. A map $f\colon X \to Y$ is continuous at x if and only if $\forall (x_n)_{n\in\mathbb{N}} \subseteq X(\lim_n x_n = x \implies \lim_n f(x_n) = f(x))$.

Proof. Suppose f is continuous at x and $(x_n)_{n\in\mathbb{N}} \to x$. $\forall \varepsilon > 0$, by continuity of f at x, $\exists r > 0$ such that $f\big(B(x, r)\big) \subseteq B\big(f(x), \varepsilon\big)$. For this $r > 0$, by convergence of $(x_n)_{n\in\mathbb{N}}$, $\exists N \in \mathbb{N}$ such that $\forall n > N\ x_n \in B(x, r)$ and hence $\forall n > N\ f(x_n) \in B\big(f(x), \varepsilon\big)$. Therefore, $\lim_n f(x_n) = f(x)$.

Suppose $\forall (x_n)_{n\in\mathbb{N}} \subseteq X(\lim_n x_n = x \implies \lim_n f(x_n) = f(x))$. If f is not continuous at x, by definition of continuity,

$$\exists \varepsilon_0 > 0 \forall \delta > 0 \exists y \in B(x, \delta) f(y) \notin B\big(f(x), \varepsilon_0\big).$$

In particular, take $\delta_n = 1/n$. Then there is $y_n \in B(x, 1/n)$ and $f(y_n) \notin B\big(f(x), \varepsilon_0\big)$. Now we have a sequence $(y_n)_{n\in\mathbb{N}}$ converge to x but $\lim_n f(y_n) \neq x$, contradiction. Therefore, f must be continuous at x. □

Definition (Continuity of maps between topological spaces)**.** Let $(X, \mathcal{T}), (Y, \mathcal{U})$ be two topological spaces. We say $f\colon X \to Y$ is continuous if $\forall O \in \mathcal{U}\ f^{-1}(O) \in \mathcal{T}$.

Theorem (Equivalence of definitions of continuity)**.** $f\colon (X, d) \to (Y, d)$ is continuous if and only if $f\colon (X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$ is continuous.

Remark. Here we mean $f\colon (X,d)\to(Y,d)$ is continuous, if it satisfies the definition of continuous maps between metric spaces. And "$f\colon (X,\mathcal{T}_{d_X})\to(Y,\mathcal{T}_{d_Y})$ is continuous" means it satisfies the definition of continuous maps between topological spaces.

Proof. Suppose $f\colon (X,d)\to(Y,d)$ is continuous. Since $(Y,\mathcal{T}_{d_Y})$ has the topology base

$$\mathcal{B}_Y=\{B(y,r): y\in Y, r\in(0,\infty)\},$$

it suffices to show that $\forall B(y,r)\in\mathcal{B}_Y$ we have $f^{-1}\big(B(y,r)\big)\in\mathcal{T}_{d_X}$. Suppose $f^{-1}\big(B(y,r)\big)\neq\varnothing$, else it's automatically open. Since $f(x_1)\in B(y,r)$, $\exists r_1>0$ such that $B(f(x_1),r_1)\subseteq B(y,r)$. Using the continuity of f at x_1, $\exists\delta>0$ such that $f\big(B(x_1,\delta)\big)\subseteq B\big(f(x_1),r_1\big)\subseteq B(y,r)$. Therefore $B(x_1,\delta)\subseteq f^{-1}\big(B(y,r)\big)$. This means $f^{-1}\big(B(y,r)\big)$ contains a neighbourhood for each point of itself, and hence $f^{-1}\big(B(y,r)\big)$ is open.

Suppose $f\colon (X,\mathcal{T}_{d_X})\to(Y,\mathcal{T}_{d_Y})$ is continuous. $\forall x\in X$, $f^{-1}\big(B(f(x),r)\big)$ is open for all $r>0$. $x\in f^{-1}\big(f(x),r\big)$ and $f^{-1}\big(B(f(x),r)\big)$ is union of sets like $B(x_0,\delta_0)$, so we can suppose $x\in B(x_0,\delta_0)$ for some $x_0\in X,\delta_0>0$. Now choose $\delta>0$ such that $B(x,\delta)\subseteq B(x_0,\delta_0)$ and we have $f\big(B(x,\delta)\big)\subseteq f\big(B(x_0,\delta_0)\big)\subseteq f\Big(f^{-1}\big(B(f(x),r)\big)\Big)\subseteq B(f(x),r)$. We're done. $\square$

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