Functional Analysis Class Note

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This note is taken for the Functional Analysiscourse, lectured by Professor Jiao Yong. In fact, this note contain full of my fragmentary thoughts, so all errors in this notes should be mine.

Here are some conventions:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are fields as you should learn in Caculus. $\mathbb{K}$ is one of $\mathbb{R}$ and $\mathbb{C}$, usually used to state different cases conveniently. $\mathbb{N}$ is the set of **positive** integers.
- Formula $A := B$ means A is defined as B. For example, $\mathbb{C} := \mathbb{R}[x]/(1+x^2)$ means $\mathbb{C}$ is defined as the quotient ring $\mathbb{R}[x]/(1+x^2)$.
- For $a, b \in \mathbb{R}$, define minimum function

$$\wedge\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto \frac{a+b-|a-b|}{2},$$

  and maximum function

$$\vee\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto \frac{a+b+|a-b|}{2}.$$

- Semminus of sets A, B is $A \setminus B := \{x \in A : x \notin B\}$.
- For a sets A, $\mathcal{P}(A)$ means the power set of A.
- For proposition p, q, we use $p \wedge q$ to mean the proposition "p and q", $\wedge$ has truth table as follow:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

  Similarly, we define $p \vee q$.

- Addition of real/complex-valued functions is defined pointwisely. That is: let $f, g\colon X \to \mathbb{K}$, we define a funtion $f + g\colon X \to \mathbb{K}$ by $x \mapsto f(x) + g(x)$.

- Somewhere you can see color different, that is reminding you to think about what here should be. (Just like $1+1=$ 2 .)
- We say a diagram commutes, if all the morphisms (and their possible compositions) with the same domain and same codomain coincide.

Contents

0 Introduction

Here is left to write after the final...

1 Day 2 of Week 1

L^p spaces

Recall the left problem: Minkowski inequality, which makes $(\ell_p, \| \, \|_p)$ a normed space. Now, we need a lemma.

Lemma (Hölder's Inequality)**.** Let $a \in \ell^p, b \in \ell^q$ for $p \in (1, \infty)$ and $q \in (1, \infty)$ staisfying $1/p + 1/q = 1$, we have:

$$\|ab\|_1 \leq \|a\|_p \|b\|_q, \tag{1}$$

Remark. $q = p/(p-1)$ is also called the dual index of p, usually denoted by p'.

Remark. Before start of the proof, we have a look at (1). Recall what we have learned in mathematical analysis, and have a problem in mind: is there anything similar? That is Cauchy-Schwarz Inequality, since they coincide when $p = q = 2$. Now we have a direct goal.

Aim. Prove (1) by imitating the proof of Cauchy-Schwarz Inequality.

Now, recall all the proofs of Cauchy-Schwarz Inequality you know and think: Which would be useful in this case? [3] Lagerange's Idendity, Schwarz's argument(inner product $\langle x + ty, x + ty \rangle \geq 0$), or just $2xy \leq x^2 + y^2$? When $p \neq 2$, Schwarz's argument is a nonstarter since there is no quadratic polynomial in sight. Similarly, the absence of a quadratic form means that one is unlikely to find an effective analog of Lagrange's identity.

This brings us to our most robust proof of Cauchy-Schwarz Inequality, the one that starts with the so-called "humble bound,"

$$xy \leq \frac{x^2}{2} + \frac{y^2}{2}, \forall x, y \in \mathbb{R}. \tag{2}$$

(2) proves Cauchy's inequality as follows.

Proof of Cauchy's inequality from (2)*.* Without lost of generality, suppose that $\sum_{n\geq 1} a_n^2 \neq 0$ and $\sum_{n\geq 1} b_n^2 \neq 0$. Let

$$a'_j = a_j / \Big(\sum_{n\geq 1} a_n^2\Big)^{1/2}, b'_j = b_j / \Big(\sum_{n\geq 1} b_n^2\Big)^{1/2}, \forall j \in \mathbb{N}.$$

Notice that $\sum_{n\geq 1} a'_n = \sum_{n\geq 1} b'_n = 1$. Now, since (2) holds, we obtain

$$\sum_{n\geq 1} a_n b_n \leq \sum_{n\geq 1} (a_n^2 + b_n^2)/2 = \sum_{n\geq 1} a_n^2/2 + \sum_{n\geq 1} b_n^2/2 = 1.$$

And, in terms of $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$, and multiply $(\sum_{n\geq 1} a_n^2)^{1/2} (\sum_{n\geq 1} a_n^2)^{1/2}$ on both sides, we have

$$\sum_{n\geq 1} a_n b_n \leq \Big(\sum_{n\geq 1} a_n^2\Big)^{1/2} \Big(\sum_{n\geq 1} b_n^2\Big)^{1/2}. \qquad \square$$

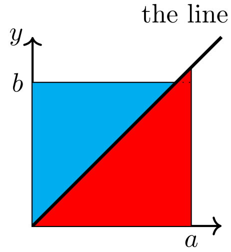

Figure 1: Area meaning of (2)

This bound may now remind us that the general AM-GM inequality

$$x^p y^q \le \frac{x}{p} + \frac{y}{q} \quad \text{for all } x, y \ge 0 \text{ and } q = p'(p, q > 1). \tag{3}$$

(3) is the perfect analog of the “humble boun”(2).

Proof of (2). There is many ways to to this, see[3]. We choose the way by area of regions. Consider the region under the function $x \mapsto x$:

$$\color{red}{A} := \{(x, y) \in \mathbb{R}^2 : 0 \le y \le x \le a\}, \color{cyan}{B} := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le y \le b\}.$$

Then (1)[Figure 1] shows that $m(A)+m(B) \ge m([0, a]\times[0, b])$, where m denotes the Lebesgue measure on $\mathbb{R}^2$. $\square$

Now, by imitating the proof of (2), we need to get the x^p/p as area of some region under a function, so consider the function $x \mapsto x^{p-1}$.

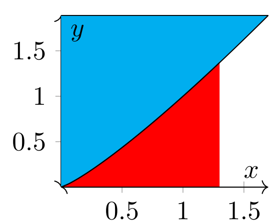

Figure 2: Area meaning of (3)

Proof of (3). It's easy to verify that

$$m(A) = \int_{[0,a]} f \, \mathrm{d}\mu, m(B) = b^{\frac{p}{p-1}} - \int_{[0,b^{p/(p-1)}]} f \, \mathrm{d}\mu,$$

where μ is the Lebesgue measure on $\mathbb{R}$. By simple calculation, we have $m(A) = \frac{1}{p}a^p, m(B) = \frac{1}{q}b^q$. Notice that $A \cup B$ contains $[0, a] \times [0, b]$, we're done. $\square$

Proof of (1). Without loss of generality, suppose $\|a+b\|_p \neq 0$. And suppose $a \neq 0(\in \ell_p), b \neq 0(\in \ell_q)$. As what we do in the proof of Cauchy's inequality, let

$$a'_j = a_j/\|a\|_p, b'_j = b_j/\|b\|_p, \forall j \in \mathbb{N}.$$

Notice that $\|a'\|_p = \|b'\|_q = 1$. Now, apply (3) to $|a_j b_j|$, we have

$$\sum_{n\geq 1}|a'_n b'_n| \leq \sum_{n\geq 1}|a'_n|^p/p + \sum_{n\geq 1}|b'_n|^q/q = 1/p + 1/q = 1,$$

which implies

$$\|ab\|_1 \leq \|a\|_p \|b\|_p.$$

∎

Proof of Minkowski inequality.

$$\begin{aligned}
\|x+y\|_p^p = &\sum_{n\geq 1}|(x+y)_n|^p \\
= &\sum_{n\geq 1}|x_n+y_n|^{p-1}|x_n+y_n| \\
\leq &\sum_{n\geq 1}|x_n+y_n|^{p-1}(|x_n|+|y_n|)\text{(Triangle inequality on }\mathbb{R}) \\
= &\sum_{n\geq 1}|x_n+y_n|^{p-1}|x_n| + \sum_{n\geq 1}|x_n+y_n|^{p-1}|y_n| \\
= &\left\|(x+y)^{p-1}x\right\|_1 + \left\|(x+y)^{p-1}y\right\|_1\text{(def of norm)} \\
\leq &\left\|(x+y)^{p-1}\right\|_q\|x\|_p + \left\|(x+y)^{p-1}\right\|_q\|y\|_p\text{(see (1)) } (*) \\
= &\|(x+y)\|_p^{p/q}(\|x\|_p + \|y\|_p)((p-1)q = p),
\end{aligned}$$

and divide $\|x+y\|_p^{p/q}(\neq 0)$ from both sides, getting

$$\|x+y\|_p^{p-p/q} \leq \|x\|_p + \|y\|_p.$$

We're done, since $p - p/q = 1$. ∎

To summarize what we have done, we need the language of measure.

Definition (σ-algebra)**.** A **σ-algrbra** on a set Ω is a subset Ω, satisfying:

1. $\Omega, \varnothing \in \mathcal{F}$;
2. $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$;
3. $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F} \implies \bigcup_{n\geq 1} A_n \in \mathcal{F}$.

Definition (Measurable Spaces, Measurable sets)**.** A **measurable space** is a double $(\Omega, \mathcal{F})$ where Ω is an aritrary set and $\mathcal{F}$ is a σ-algebra over Ω. Elements of $\mathcal{F}$ is called **measurable sets** of $(\Omega, \mathcal{F})$.

Definition (Measure,Measure spaces)**.** A **measure** is a σ-additive function from $\mathcal{F}$ to $[0, \infty]$. A triple $(\Omega, \mathcal{F}, \mu)$ is called a **measure space**, if $(\Omega, \mathcal{F})$ is a measurable space and μ is a measure.

Definition (Integral with respect to measure)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We have a glance at "how to define integral with respect to measure". For the detail, see [2].

Step 1: Define **integral** $\int$ for measurable simple nonnegative function:

$$\sum_{k=1}^{n} a_k \chi_{A_k} \longmapsto \sum_{k=1}^{n} a_k \mu(A_k).$$

Step 2: Define **integral** $\int$ for measurable nonnegative function:

$$f \longmapsto \sup \left\{ \int \varphi : \varphi \leq f, \varphi \text{ is nonnegative simple function} \right\}.$$

Step 3: Define **integral** $\int$ for measurable function:

$$f \longmapsto \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu,$$

where $f^+ = f\chi_{f^{-1}[0,\infty)}, f^- = -f\chi_{f^{-1}(-\infty,0]}$.

Definition (p-integrable space)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, then the p-integrable space over $\mathcal{L}_p(\Omega, \mathcal{F}, \mu)$ is defined as

$$\mathcal{L}_p(\Omega, \mathcal{F}, \mu) := \left\{ f \in \mathbb{K}^\Omega : f \text{ is measurable and } \int f \, \mathrm{d}\mu < \infty \right\}.$$

Fact. The proof of Minkowski inequalityover ℓ_p actually proved the Minkowski inequalityfor **every** p-integrable space $\mathcal{L}_p(\Omega, \mathcal{F}, \mu)$.

To understand this fact, we should have another way to illustrate $\sum$. That is, $\sum$ is a kind of integral.

Definition (Counting Measure)**.** Given a measurable space $(\Omega, \mathcal{F})$. Define $\mu \colon \mathcal{F} \to [0, \infty], A \mapsto \sharp A$. Where $\sharp A = \infty$ if A is an infinite set, and $\sharp A = n$ if A has exactly n elements. μ is called the **counting measure** over $(\Omega, \mathcal{F})$.

Remark. It can be shown that,[1] for real sequence $(a_n)_{n \in \mathbb{N}}$ (equivalent to a function $a \colon \mathbb{N} \to \mathbb{R}$), we have

$$\sum_{n \geq 1} a_n = \int a \, \mathrm{d}\mu.$$

That's why we can respect $\sum$ as $\int$. And hence, the fact above is just regard $\sum$ as integral with respect to coungting measure, and the proof works for arbitrary measure space.

Example.

In the next class, we will talk about the completeness of $\mathcal{L}_p(\Omega, \mathcal{F}, \mu)$. For this goal, we need to have some knowledge about the quotient spaces first.

Definition (Quotient space)**.** Let V be a vector space over $\mathbb{K}$ with a subspace U. For every $v \in V$, we define $v + U = \{v + u : u \in U\}$ called the affine subset of V parallel to U. The **quotient space** V/U is the set of all affine subsets of V parallel to U. In other words,

$$V/U := \{v + U : v \in V\}.$$

Remark. To make V/U a vector space, we define **addition** and **scalar multiplication** on V/U by

$$\begin{aligned}(v + U) + (w + U) :=& (v + w) + U,\\ k(v + U) :=& (kv) + U.\end{aligned}$$

for all $v, w \in V$ and $k \in \mathbb{K}$. It is easy to show that both operations is well-defined(irrelevant to the selection of representative element). Verification of V/U is a linear space is omitted here.

Definition (Quotient map)**.** Suppose U is a subspace of V. The **quotient map** π (with respect to U) is the linear map

$$\pi\colon V \to V/U, v \mapsto v + U.$$

π is also called the canonical projection

Remark. Linearity should be verified by using the definition of addition over U/V.

Theorem (Universal property)**.** Property (a) as follows, uniquely determined the quotient space up to linear isomorphism. That is

(a) Let V be a vector space and U a subspace of X. Let $\pi\colon V \to V/U$ be the canonical projection. For all linear maps $f : V \to W$ such that $(U =) \ker \pi \subseteq \ker f$, there exists a unique linear map $\tilde{f} : V/U \to W$ such that $f = \tilde{f} \circ \pi$. In other words, the following diagram commutes

(b) If vector space V_1 satisfies: $\exists$ a linear map $\pi_1\colon V \to V_1$, and for all linear maps $g : V \to W$ satisfying $\ker \pi_1 \subseteq \ker g$, there exists a unique linear map $\tilde{g} : V_1 \to W$ such that $g = \tilde{g} \circ \pi_1$, then $V_1 \cong V/U$.

Property (a) is called the universal property of quotient space.

Proof. (a) holds: given V, U, π and f satisfy the conditions above, we can directly define

$$\tilde{f}\colon V/U \to W, v + U \mapsto f(v).$$

Let's check : $\tilde{f}$ makes the diagram commutes and $\tilde{f}$ is unique. $\tilde{f}$ is well-defined, since $\ker f \supseteq U$ ensures that $u - v \in U \implies f(u-v) = 0$. Linearity of f is trivial. To prove $f = \tilde{f} \circ \pi$, given $v \in V$:

$$\tilde{f} \circ \pi(v) = \tilde{f}\big(\pi(v)\big) = \tilde{f}(v + U) = f(v),$$

we're done. Uniqueness of $\tilde{f}$ can be ensured by $f = \tilde{f} \circ \pi$. Suppose there is another linear map $\tilde{f}'$ satisfyig $f = \tilde{f}' \circ \pi$, then $\forall v + U \in V/U$

$$\tilde{f}'(v + U) = \tilde{f}'\big(\pi(v)\big) = \tilde{f}' \circ (\pi)(v) = f(v).$$

From this equatlity and definition of $\tilde{f}$, we get $\tilde{f} = \tilde{f}'$. Therefore $\tilde{f}$ is unique.

(b) holds: given V_1, π_1 as stated in (b), we have two communicative diagrams Hence, this diagram commutes While, another communicative diagram as follows Now, uniqueness stated in (b) ensures that $\tilde{\pi_1} \circ \tilde{\pi} = 1_{V_1}$. Similarly, we have $\tilde{\pi} \circ \tilde{\pi_1} = 1_{V/U}$. Therefore $\tilde{\pi}$ is an isomorphism form V_1 to V/U. $\square$

Remark. This theorem contains results as follows:

1. Quotient spaces have property (a).

2. spaces "with" property (a) must be isomorphic to the quotient sapce. The meaning of "with" is explained in (b).