Note of Stochastic process

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Contents

Chapter 1

Poisson process

1.1 Basic definition

Definition. A counting process $\{N(t), t \geq 0\}$ is said to be a **Poisson process**, having rate $\lambda > 0$, if

1. N(0)=0
2. The process has independent increments
3. The number of events in any interval of length t is poisson distributed with mean λt. that is

$$P\{N(t+s) - N(t) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad \forall s, t \geq 0, n \in N^*$$

Remark. We said that this process has rate λ, because the number of events during the time t has mean $E[N(t)] = \lambda t$.

There is alternative definition of poisson process, and it's an easier method to check a process is poisson.

Definition. A counting process $\{N(t), t \geq 0\}$ is said to be poisson process with rate λ if

1. N(0)=0
2. The process has stationary and independent increments .
3. $P\{N(h) = 1\} = \lambda h + o(h)$
4. $P\{N(h) \geq 2\} = o(h)$

1.2 The distribution of interarrival and waiting time

1.2.1 The interarrival time

Definition. Let X_n denote the time between the (n-1)st and the nth event. $\{X_n, n \geq 1\}$ is called the sequence of interarrival time.

Theorem 1.2.1 (Distribution of X_n)**.** The interarrival time $\{X_n, n \geq 1\}$ of a poisson process with rate λ, are independent identically distributed exponential random variables having mean $\frac{1}{\lambda}$.

Proof. First note that the event $\{X_1 > t\}$ happen if and only if no event of the poisson process occur in the interval $[0, t]$, and thus

$$P\{X_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

Hence X_1 has an exponential distribution with mean $\frac{1}{\lambda}$.

Now we consider X_2. Take condition on X_1.

$$\begin{aligned} P\{X_2 > t | X_1 = s\} &= P\{0 \text{ events in (s,s+t]} | X_1 = s\} \\ &= P\{0 \text{ event in (s,s+t]}\} \textit{By independent increments} \\ &= P\{0 \text{ event in (0, t]}\} \textit{By stationary increments} \\ &= e^{-\lambda t} \end{aligned}$$

Therefore we can see X_2 is independent of X_1, and is an exponential random variable mean $\frac{1}{\lambda}$. Repeating the same argument we can get the theorem. □

1.2.2 The waiting time

Definition. Let

$$S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1$$

denote the arrival time of the nth event, also called the waiting time until the nth event.

Theorem 1.2.2 (Distribution of S_n)**.** The distribution of S_n is

$$F_{S_n}(t) = P\{S_n \leq t\} = P\{N(t) \geq n\} = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

and the density function of S_n is

$$f_{S_n} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Proof. The distribution function of S_n is easy to check. To get the density function, consider

$$\frac{dF_{S_n}}{dt} = f_{S_n}(t).$$

□

1.3 Conditional distribution of the waiting time

1.3.1 Main result

We want to know the distribution of the waiting time (or arrival time) S_n on the condition of $N(t) = n$.

Remark. The definition of order statistics and the joint density function of the order statistics corresponding to n independent random variables uniformly distributed on the interval (0,t) doesn't mention here. If you are unfamiliar to these, please read the book.

Theorem 1.3.1. Given that $N(t) = n$, the n arrival times $S_1, \cdots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval (0,t).

Proof. Let $0 < t_1 < \cdots < t_n < t_{n+1} = t$, and let h_i small enough so that $t_i + h_i < t_{i+1}, i = 1, 2, ..., n$. Consider the joint density function of $\{S_n\}$, we need to make a disturbance of the value of S_n. That is

$$
\begin{aligned}
&P\{t_i \leq S_i \leq t_i + h_i, i = 1, 2, ..., n \mid N(t) = n\} \\
&= \frac{P\{\text{exactly one event in each } [t_i, t_i + h_i], \text{ and no events elsewhere in } [0,t]\}}{P\{N(t) = n\}} \\
&= \frac{\prod_{i=1}^{n} \lambda h_i e^{-\lambda h_i} e^{-\lambda(t-h_1-\cdots-h_n)}}{e^{-\lambda t}(\lambda t)^n/n!} \\
&= \frac{n!}{t^n} h_1 h_2 \cdots h_n
\end{aligned}
$$

Hence the joint conditional density of $\{S_n\}$ is

$$
\lim_{h_i \to 0} \frac{P\{t_i \leq S_i \leq t_i + h_i, i = 1, 2, ..., n \mid N(t) = n\}}{h_1 \cdots h_n} = \frac{n!}{t^n}
$$

this is the same as the joint density function as the order statistics corresponding to n indenpendent random variables uniformly distributed on the interval (0,t). □

Here is an application of this theorem.

1.3.2 An application

Example 1. Suppose that travelers arrive at a train station in accordance with a poisson process with rate λ. If the train departs at time t, let us compute the expected sum of the waiting times of travelers arriving in (0,t). That is

$$
E\left[\sum_{i=1}^{N(t)} (t - S_i)\right]
$$

where S_i is the arrival time of the ith traveler.

Solution. Clearly, we need to take condition on $N(t) = n$. Then

$$E\left[\sum_{i=1}^{N(t)}(t - S_i)\right] = \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{n}(t - S_i) \mid N(t) = n\right] P\{N(t) = n\}$$
$$= \sum_{n=1}^{\infty}(nt - E\left[\sum_{i=1}^{n} S_i | N(t) = n\right])P\{N(t) = n\}$$

Now we counting the expectation $E\left[\sum_{i=1}^{n} S_i | N(t) = n\right]$. By upper theorem, on condition of $N(t) = n$, S_i has the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval (0,t). Then we have

$$E\left[\sum_{i=1}^{n} S_i | N(t) = n\right] = E\left[\sum_{i=1}^{n} U_{(i)}\right] = E\left[\sum_{i=1}^{n} U_i\right] = \frac{nt}{2}$$

Hence

$$E\left[\sum_{i=1}^{N(t)}(t - S_i)\right] = \sum_{n=1}^{\infty}(nt - \frac{nt}{2})P\{N(t) = n\} = \frac{t}{2}E\left[N(t)\right] = \frac{\lambda t^2}{2}$$

$\square$

1.4 Type-I,II Poisson process

1.4.1 Introduction and main result

In this section we consider an important class of poisson process. Consider a poisson process with rate λ. If every events of this process can be classified as two type, namely I-type and II-type, we said this is a two type poisson process.

Specifically, suppose that an event in poisson process occur at time s, it is classified as being a type-I event with probability $P(s)$ (means that the probability of being classified as type-I is depending on when the event occur.), and it is classified as being a type-II event with probability $Q(s)$. Since every event must be classified as type-I or type-II, we write $(1 - P(s))$ instead of $Q(s)$.

Definition. For a poisson process that has two-type, we have define $N(t)$ as the number of events during time t. Similarly we can define

1. $N_1(t)$:the number of type-I events that occur by time t
2. $N_2(t)$:the number of type-II events that occur by time t

Here we would like to know the means of N_1 and N_2.

Theorem 1.4.1. *$N_1(t)$ and $N_2(t)$ are independent poisson random variables and having means λtp and $\lambda t(1-p)$, where*

$$p = \frac{1}{t}\int_0^t P(s)ds$$

(which we can view as the average probability of begin classified as type-I during time t)

Proof. Let's compute the joint distribution of $N_i(t), i = 1, 2$ by conditioning on $N(t)$.

$$\begin{aligned}
&P\{N_1(t) = n, N_2(t) = m\} \\
&= \sum_{k=0}^{\infty} P\{N_1(t) = n, N_2(t) = m \mid N(t) = k\}P\{N(t) = k\} \\
&= P\{N_1(t) = n, N_2(t) = m \mid N(t) = m+n\}P\{N(t) = m+n\}
\end{aligned}$$

i) Since $\{N(t), t \geq 0\}$ is a poisson process, by definition we have

$$P\{N(t) = m+n\} = e^{-\lambda t}\frac{(\lambda t)^{m+n}}{n!}$$

ii) Given an arbitrary event during time t, according to theorem th, we know that this event will occur at some time uniformly distributed on (0,t).(Since the event we chosen was arbitrary.) Hence the probability of an arbitrary event that will be classified as type-I event is

$$p = \frac{1}{t}\int_0^t P(s)ds$$

Since it's independent of the other events, hence

$$P\{N_1(t) = n, N_2(t) = m \mid N(t) = m+n\} = \binom{n}{r}p^n(1-p)^m$$

Consequently,

$$\begin{aligned}P\{N_1(t)=n, N_2(t)=m\} &= \tbinom{n}{r} p^n(1-p)^m e^{-\lambda t}\frac{(\lambda t)^{m+n}}{n!} \\ &= e^{-\lambda tp}\frac{(\lambda tp)^n}{n!} e^{-\lambda t(1-p)}\frac{(\lambda t(1-p))^m}{m!}\end{aligned}$$

Which implies that $N_1(t)$ is independent of $N_2(t)$, and they are poisson random variables having respective means λtp and $\lambda t(1-p)$. □

1.4.2 An example of two type Poisson process

Know we would like to give an example of two type Poisson process.

Example 2 (The infinite server Poisson Queue)**.** A infinite server poisson queue is a service system, that satisfied

1. Suppose that customer arrive at a service station which has infinite servers with a poisson process with rate λ.
2. The service times are independent with a common distribution G.
3. the service times are independent of the arrive process.

Know fix a time t, then every customer arrived before time t can be classified as two type. That is

$$\begin{cases}\text{type-I customer;} & \text{If it does not complete service at t.} \\ \text{type-II customer;} & \text{If it complete his service before or equal to the time t.}\end{cases}$$

And let

$$\begin{cases}N_1(t); & \text{denote the number of type-I customer during the time } (0,t) \\ N_2(t); & \text{denote the number of type-II customer during the time } (0,t)\end{cases}$$

We would like to compute the distribution of $N_1(t)$ and $N_2(t)$.

Solution. Suppose a customer enter this system at time $s < t$ (Which we should view it as a poisson event occur at time s.). Then it has the probability $G(t-s)$ to be classified as type-I, and has the probability $1-G(t-s)$ to be classified as type-II.

By the theorem above. Let

$$p = \frac{1}{t}\int_0^t G(t-S)ds$$

then

$$\begin{aligned}\lambda_1 = E[N_1(t)] = \lambda pt = \lambda\int_0^t G(t-s)ds = \lambda\int_0^t G(y)dy \\ \lambda_2 = E[N_2(t)] = \lambda(1-p)t = \lambda\int_0^t (1-G(t-s))ds = \lambda\int_0^t (1-G(y))dy\end{aligned}$$

And $N_i(t), i=1,2$ is a poisson variable with means λ_i. □

1.5 M/G/1 Busy Period

1.5.1 Introduction of M/G/1

Front word: In the last section, we have learn a service system with infinite servers. Now we consider a service system has only one server. Formally

Definition. M/G/1 system is

1. A server system which has exactly one server.
2. Customer arrive this service system in accordance with a poisson process with rate λ.
3. The service time of each customer are independent and identically distributed according to a distribute function G. Also the service time independent of the arrival process.
4. If a customer arrived and the server are free, then the service will begin immediately.
5. If a customer arrived and the server are busy, then he should wait until every customer in front of him complete the service before he began to receiving service.
6. When a customer arrive we say he enter this system. When he receiving service or waiting for service, we say he is in the system. When he complete his service, we say he leave this system.

We would like to know, as the server began to serve customer, when will he take a break. That is

Definition (Busy period)**.** When an arrival finds the server free, he began to receiving service, and we say the busy period begins. And this busy period will ends until there is no customer in the service system.

1.5.2 preparation of computing the distribution

In this section, our goal is to compute the distribution of the busy period. Specifically, the

$$P\{\text{busy period of length t, and consist of } n \text{ service}\}$$

First we should find a equivalent conditions. Let

1. S_n denote the time until n additional customers have arrived.
2. $\{Y_n\}$ denote the sequence of service time.

Then the busy period will last a time t and will consist of n service if and only if

1. $S_k \leq Y_1 + \cdots + Y_k, \quad \text{forall } k = 1, \cdots, n-1$
2. $Y_1 + \cdots + Y_n = t$
3. There are $n-1$ arrivals during the time $(0, t)$.

(i) ensure the busy time won't end until the busy time containing n service.

Hence we have

$$
\begin{aligned}
&P\{\text{busy period of length t, and consist of } n \text{ service}\}\\
=&P\{Y_1+\cdots+Y_n=t,\ n-1 \text{ arrivals in (0,t)},\ S_k\le Y_1+\cdots+Y_k, k=1,2,\cdots,n-1\}\\
=&P\{S_k\le Y_1+\cdots+Y_k, k=1,\cdots,n-1 \mid n-1 \text{ arrivals in (0,t)},\ Y_1+\cdots+Y_n=t\}\\
\times&P\{n-1 \text{ arrivals in (0,t)},\ Y_1+\cdots+Y_n=t\}
\end{aligned}
$$

It's easy to calculate the probability

$$P\{n-1 \text{ arrivals in (0,t)},\ Y_1+\cdots+Y_n=t\}$$

But the probability

$$P\{S_k\le Y_1+\cdots+Y_k, k=1,\cdots,n-1 \mid n-1 \text{ arrivals in (0,t)},\ Y_1+\cdots+Y_n=t\}$$

require some lemma, without proving them, that is

Lemma 1. Let $Y_1,\cdots,Y_n$ be independent and identically distributed nonnegative random variables. Then

$$E[Y_1+\cdots+Y_k \mid Y_1+\cdots+Y_n=y]=\frac{k}{n}y$$

Lemma 2. Let $\{U_{(i)}\}$ denote the ordered values from a set of n independent uniform random variables on $(0,t)$. $Y_1,\cdots,Y_n$ be independent and identically distributed nonnegative random variables, and are also independent of $\{U_{(i)}\}$. Then when $0<y<t$

$$P\{Y_1+\cdots+Y_k\le U_{(k)}, k=1,2,\cdots,n \mid Y_1+\cdots+Y_n=y\}=1-y/t$$

Lemma 3. Let $\{U_{(i)}\}$ denote the ordered values from a set of $n-1$ independent uniform random variables on $(0,t)$. $Y_1,\cdots,Y_n$ be independent and identically distributed nonnegative random variables, and are also independent of $\{U_{(i)}\}$. Then

$$P\{Y_1+\cdots+Y_k\le U_{(k)}, k=1,2,\cdots,n-1 \mid Y_1+\cdots+Y_n=y\}=1/n$$

The proof of them is quite long, if you're interesting in it, see the textbook on page 77.

1.5.3 Distribution of busy period

In the last subsection, we have

$$
\begin{aligned}
&P\{\text{busy period of length t, and consist of } n \text{ service}\}\\
=&P\{S_k\le Y_1+\cdots+Y_k, k=1,\cdots,n-1 \mid n-1 \text{ arrivals in (0,t)},\ Y_1+\cdots+Y_n=t\}\\
\times&P\{n-1 \text{ arrivals in (0,t)},\ Y_1+\cdots+Y_n=t\}
\end{aligned}
$$

The second probability

$$
\begin{aligned}
&P\{n-1 \text{ arrivals in (0,t)},\ Y_1+\cdots+Y_n=t\}\\
=&P\{n-1 \text{ arrivals in (0,t)}\}P\{Y_1+\cdots+Y_n=t\}\heartsuit\\
=&e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}dG_n(t) \quad (\textit{where } G_n \textit{ is the } n-fold \textit{ convolution of } G)
\end{aligned}
$$

The step ♡ is because the arrival process independent of the service time.

The first probability is

$$P\{S_k \leq Y_1 + \cdots + Y_k, k = 1, \cdots, n-1 \mid n-1 \text{ arrivals in (0,t)},\ Y_1 + \cdots + Y_n = t\} = 1/n$$

by the lemma above.

(The detail proof of this probability is on the book, since I think this section is not that important in the final exam, hence in this section I don't want to spend much time in these complex proof of the lemmas.)

So if we let $B(t,n) = P\{$busy period is of length $\leq t$, n customers served in a busy period$\}$, then

$$\frac{d}{dt}B(t,n) = e^{-\lambda t}\frac{(\lambda t)^{n-1}}{n!}dG_n(t)$$

More over

$$B(t,n) = \int_0^t e^{-\lambda t}\frac{(\lambda t)^{n-1}}{n!}dG_n(t)$$

1.6 Poisson process with a variable rate

1.6.1 Definition of conditional poisson process

Front word: Recall that in the former section, the Poisson process we've mention all have a constant rate λ. In this section we will consider a special poisson process that it's rate which is a positive random variable having distribution G.

Formally, Let Λ be a positive variable having distribution G, and let $\{N(t), t \geq 0\}$ be a counting process such that, given that $\Lambda = \lambda$, $\{N_\lambda(t), t \geq 0\}$ is a poisson process having rate λ. Thus

$$P\{N(t+s) - N(s) = n\} = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dG(\lambda)$$

We call the process $\{N(t), t \geq 0\}$ is a **conditional poisson process**. (that's because if you take condition on $\Lambda = \lambda$, then the conditional process is a poisson process with rate λ.)

Property of conditional poisson process

1. It does have stationary increments.

2. It does not must have independent increments.

Proof. (i) is easy to check. We now show that it does not have independent increments. Consider the a random time t_1 and t_2. It's suffice to show that

$$P\{N(t_1) - N(0) = 0\} P\{N(t_2) - N(t_1) = 0\} \neq P\{N(t_1) - N(0) = 0, N(t_2) - N(t_1) = 0\}$$

since

$$P\{N(t_1) - N(0) = 0\} = \int_0^\infty e^{-\lambda t_1} dG(\lambda)$$

and

$$P\{N(t_2) - N(t_1) = 0\} = \int_0^\infty e^{-\lambda(t_2 - t_1)} dG(\lambda)$$

$$P\{N(t_1) - N(0) = 0, N(t_2) - N(t_1) = 0\} = P\{N(t_2) = 0\} = \int_0^\infty e^{-\lambda t_2} dG(\lambda)$$

Hence it's not independent increments. □

1.6.2 Conditional distribution of Λ on $N(t)$

We would like to discuss the distribution of the rate on the condition of $N(t) = n$. For a small $d\lambda$.

$$\begin{aligned}
&P\{\Lambda \in (\lambda, \lambda + d\lambda) \mid N(t) = n\} \\
=&\frac{P\{N(t) = n \mid \Lambda \in (\lambda, \lambda + d\lambda)\} P\{\Lambda \in (\lambda, \lambda + d\lambda)\}}{P\{N(t) = n\}} \\
=&\frac{e^{-\lambda t} \dfrac{(\lambda t)^n}{n!} dG(\lambda)}{\int_0^\infty e^{-\lambda t} \dfrac{(\lambda t)^n}{n!} dG(\lambda)}
\end{aligned}$$

and so the conditional distribution of Λ, given that $N(t) = n$, is

$$P\{\Lambda \le x \mid N(t) = n\} = \frac{\int_0^x e^{-\lambda t}\frac{(\lambda t)^n}{n!}dG(\lambda)}{\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}dG(\lambda)}$$

1.6.3 An example of conditional poisson process

Example 3. Suppose an process $\{N(t), t \ge 0\}$ is a poisson process with rate either λ_1 or λ_2 with probability p to be the rate of λ_1, and with probability $1-p$ to be the rate of λ_2. Now if there is n event arrived during time t, what's the probability of it's the rate λ_1.

Solution. Let Λ be a random variable that is either λ_1 or λ_2 with the probability p and $1-p$ respectively. What we should compute is

$$P\{\Lambda = \lambda_1 \mid N(t) = n\}$$

By the method above,

$$\begin{aligned}&P\{\Lambda = \lambda_1 \mid N(t) = n\}\\ =&\frac{P\{N(t) = n \mid \Lambda = \lambda_1\}P\{\Lambda = \lambda_1\}}{P\{N(t) = n\}}\\ =&\frac{e^{-\lambda_1 t}\frac{(\lambda_1 t)^n}{n!}p}{e^{-\lambda_1 t}\frac{(\lambda_1 t)^n}{n!}p + e^{-\lambda_2 t}\frac{(\lambda_2 t)^n}{n!}(1-p)}\end{aligned}$$

Similarly we can compute $P\{\Lambda = \lambda_2 \mid N(t) = n\}$ □

1.7 Exercise

Exercise

Proof these two definition are equivalent.
poisson process: A counting process $\{N(t), t \geq 0\}$ is said to be a poisson process, having rate $\lambda > 0$, if

1. N(0)=0
2. The process has independent increments
3. The number of events in any interval of length t is poisson distributed with mean λt. that is

$$P\{N(t+s) - N(t) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad \forall s, t \geq 0, n \in N^*$$

poisson process: A counting process $\{N(t), t \geq 0\}$ is said to be poisson process with rate λ if

1. N(0)=0
2. The process has stationary and independent increments .
3. $P\{N(h) = 1\} = \lambda h + o(h)$
4. $P\{N(h) \geq 2\} = o(h)$

Solution. Here we only prove ② → ①.
(a): We prove that $p_0(t+s) = p_0(t)p_0(s)$.

$$\begin{aligned} p_0(t+s) :=& P\{N(t+s) = 0\} \\ =& P\{N(t+s) - N(s) \mid N(s) = 0\}P\{N(s) = 0\} \\ =& P\{N(t) = 0\}P\{N(s)\} \\ =& p_0(t)p_0(s) \end{aligned}$$

(b): We prove the interarrival times X_n are independent exponential random variables with mean $\dfrac{1}{\lambda}$. Since $P\{X_1 > t\} = P\{N(t) = 0\} = p_0(t)$, and

$$\begin{aligned} & p_0(t+h) = p_0(h)p_0(t) \\ \Rightarrow & p_0(t+h) = (1 - \lambda h + o(h))p_0(t) \\ \Rightarrow & p_0(t+h) - p_0(t) = -\lambda h p_0(t) + o(h) \\ \Rightarrow & p_0'(t) = -\lambda p_0(t) \\ \Rightarrow & p_0(t) = e^{-\lambda t} \end{aligned}$$

Hence X_1 is exponential random variable with mean $\frac{1}{\lambda}$.More over,

$$\begin{aligned}P\{X_2 > t \mid X_1 = s\} &= P\{N(t+s) - N(s) = 0 \mid X_1 = s\}\\ &= P\{N(t+s) - N(s) = 0\} \ (independent\ increments)\\ &= p_0(t) \ (stationary\ increments)\\ &= e^{-\lambda t}\end{aligned}$$

Hence X_2 is independent of X_1 and is exponential random variable with mean $\frac{1}{\lambda}$. Repeat this method, we get the proposition (b).

(c): We prove $N(t)$ is a poisson random variable with mean λt.

$$\begin{aligned}&P\{N(t) \geq n\}\\ =&P\{X_1 + \cdots + X_n \leq t\}\\ =&\sum_{j=n}^{\infty} e^{-\lambda t}\frac{(\lambda t)^j}{j!}\end{aligned}$$

Hence $P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$. □

Exercise

For a poisson process, show that for $s < t$,

$$P\{N(s) = k \mid N(t) = n\} = \binom{n}{k}(\frac{s}{t})^k(1-\frac{s}{t})^{n-k}, \quad k = 0, 1, \cdots, n$$

Proof. Let $\{U_i\}$ denote n independent random variables uniformly distributed on the interval $(0, t)$. And $\{U_{(i)}\}$ are the order statistics of $\{U_i\}$. By the theorem 1.3.1,

$$\begin{aligned}P\{N(s) = k \mid N(t) = n\} &= P\{S_k \leq s, S_{k+1} > s \mid N(t) = n\}\\ &= P\{U_{(k)} \leq s, U_{(k+1)} > s\}\\ &= \binom{n}{k}(\frac{s}{t})^k(1-\frac{s}{t})^{n-k}\end{aligned}$$

And we get the result. □

Exercise

The number of trails to be performed is a poisson random variables with mean λ. Each trails has n possible outcomes, number i with probability p_i, $\sum_{i=1}^{n} = 1$. Let X_j denote the number of outcomes that occur exactly j times, compute

$$E[X_j] \quad \text{and} \quad \mathrm{Var}(X_j)$$

Solution. Let Y_i denote the number of the occurrence of the ith outcome. Form theorem 1.4.1, $\{Y_i\}$ are independent poisson variables with mean λp_i. If we let

$$I_i = \begin{cases} 1 & \text{the ith result occur exactly j times} \\ 0 & \text{else} \end{cases}$$

Then $X_j = \sum_{i=1}^{n} I_i$, and

$$E[X_j] = \sum_{i=1}^{n} E[I_i] = \sum_{i=1}^{n} P\{Y_i = j\} = \sum_{i=1}^{n} e^{-\lambda p_i} \frac{(\lambda p_i)^j}{j!}$$

And

$$\mathrm{Var}(X_j) = \sum_{i=1}^{n} \mathrm{Var}(I_i)\heartsuit = \sum_{i=1}^{n} P\{Y_i = j\}(1 - P\{Y_i = j\})\heartsuit$$

1. The first $\heartsuit$ here is because I_i and I_j are independent when $i \neq j$.
2. The second $\heartsuit$ here is because I_n is a $0-1$ distribution, and the variance of it is $p(1-p)$.

$\square$

Chapter 2

Renewal process

2.1 Basic definition

We know that the interarrival times for the Poisson process are independent and identically distribute exponential random variable. Now the renewal process generalize it to an arbitrary distribution.

Definition. Let $\{X_n : n = 1, 2, \cdots\}$ be a sequence of nonnegative independent random variable with common distribution F. If we interpret X_n as the time between the $(n-1)$st and nth even. Then $S_n = \sum_{i=1}^{n} X_i$ is the time of the nth event. The counting process

$$N(t) = \sup\{n : S_n \leq t\}$$

is called a renewal process.

Remark. To avoid the trivialities suppose, we often suppose that $F(0) = P\{X_n = 0\} < 1$.

Here is some notation we may usually use.

Definition. :

1. $\mu = E[X_n] = \int_0^\infty x dF(x)$ denote as the mean time between successive events.
2. $m(t) = E[N(t)]$ is called the renewal function
3. F_n is the distribution function of S_n, which is the n-fold convolution of F with itself.
4. $S_{N(t)}$ the time of the last renewal prior to or at the time t.
5. $S_{N(t)+1}$ the time of the first renewal after time t. (cannot be equal to.)

It's easy to check the relation of them :

Theorem 2.1.1. :

1. By the strong law of the large numbers, with probability 1, we have

$$\lim_{n\to\infty} \frac{S_n}{n} = \mu$$

2. $P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$
3. $m(t) = \sum_{i=1}^{n} F_n(t)$
4. $S_{N(t)} \leq t < S_{N(t)+1}$

Proof. we only prove (3) here. Let

$$I_n = \begin{cases} 1, & \text{if the } n\text{th renewal occurred in [0,t]} \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

then $N(t) = \sum_{n=1}^{\infty} I_n$. Hence

$$
\begin{aligned}
E[N(t)] &= E\left[\sum_{n=1}^{\infty} I_n\right] \\
&= \sum_{n=1}^{\infty} E[I_n] \quad (\heartsuit) \\
&= \sum_{n=1}^{\infty} P\{I_n = 1\} \\
&= \sum_{n=1}^{\infty} F_n(t)
\end{aligned}
$$

$\square$

Remark. Explain of the step $(\heartsuit)$. By Levi monotonic convergence theorem, we know that if $\{f_n\}$ is a monotone sequence of nonnegative measurable functions. $f_n \to f$ $a.e$, then

$$\lim_{n\to\infty} \int_\Omega f_n dx = \int_\Omega f dx$$

Hence if $\{f_n\}$ are nonnegative measurable functions, then

$$\int_\Omega \sum_{n=1}^{\infty} f_n dx = \sum_{n=1}^{\infty} \int_\Omega f_n dx$$

That why we can interchange of expectation and summation, because $\{I_n\}$ is nonnegative.

2.2 Limit Theorems

2.2.1 some limit theorems

About how many renewals can occur in finite/infinite time, we have

Theorem 2.2.1. :

1. It can't occur an infinite number of renewals in a finite time.
2. It must occur an infinite number of renewals in an infinite time. In other words $N(\infty) = \infty$ (with the probability 1).
3. The expectation of $N(t)$, $E[N(t)] = m(t) < \infty$, if $0 \leq t < \infty$.

Proof. i) By 1, we know that

$$\frac{S_n}{n} \to \mu \quad \text{as } n \to \infty$$

hence S_n must be going to infinity as n goes to infinity. Thus , for a finite time t, S_n can be less than or equal to t for at most a finite number of values of n. Since $N(t) = \sup\{n : S_n \leq t\}$, we know that $N(t)$ must finite. And we can write $N(t) = \max\{n : S_n \leq t\}$ when the time are finite.

ii) It's equal to prove that $P\{N(\infty) < \infty\} = 0$. We have

$$\begin{aligned} P\{N(\infty) < \infty\} &= P\{X_n = \infty \text{,for some n}\} \\ &= P\{\bigcup_{n=1}^{\infty}\{X_n = \infty\}\} \\ &\leq \sum_{n=1}^{\infty} P\{X_n = \infty\} \\ &= 0 \end{aligned}$$

iii) Since $F(0) = P\{X_n = 0\} < 1$, there is an $\alpha > 0$, such that $P\{X_n \geq \alpha\} > 0$. Now we can consider a related renewal process $\{\overline{X_n}, n = 1, 2, \cdots\}$ by

$$\overline{X_n} = \begin{cases} 0 & \text{if } X_n < \alpha \\ \alpha & \text{if } X_n \geq \alpha \end{cases}$$

then this process can only renewals at times $t = n\alpha, n = 1, 2, \cdots$. And the number of renewals at each of these times are independent geometric random variables with mean

$$\frac{1}{P\{X_n \geq \alpha\}}$$

Thus these renewal points of time only $\left[\frac{t}{\alpha}\right]$. Hence

$$E\left[N(t)\right] \leq E\left[\overline{N(t)}\right] \leq \left[\frac{t}{\alpha}\right] \frac{1}{P\{X_n \geq \alpha\}} < \infty$$

□

Now we consider the rate at which $N(t)$ and $E[N(t)]$ grows.

Theorem 2.2.2. :

1. with probability 1,
$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty$$
2. with probability 1,
$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty$$

Proof. (i) Since $S_{N(t)} \leq t < S_{N(t)+1}$, we see that

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

Since $N(t) \to \infty$ when $t \to \infty$, $\dfrac{S_{N(t)}}{N(t)} \to \mu$ when $N(t) \to \infty$, we can conclude that $\dfrac{S_{N(t)}}{N(t)} \to \infty$ when $t \to \infty$. More over

$$\frac{S_{N(t)}}{N(t)} = \frac{S_{N(t)}}{N(t)+1} \frac{N(t)+1}{N(t)}$$

by the same reasoning, $\dfrac{S_{N(t)+1}}{N(t)} \to \mu$ when $t \to \infty$. Hence we get that

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty.$$

$\square$

Remark. :
i) All " $\to \infty$" or " $\to \mu$" are under the meaning of with the probability 1.
ii) For this reason we call $\dfrac{1}{\mu}$ is the rate of the renewal process.

2.2.2 preparation of proving the elementary renewal theorem

Before proving $\dfrac{m(t)}{t} \to \dfrac{1}{\mu}$ as $t \to \infty$, we need some preparation.

Definition. An integer-valued random variable N is said to be a **stopping time** for the sequence $\{X_n\}$. If the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \cdots$.

Theorem 2.2.3 (Wald equation)**.** If $\{X_n\}$ are independent and identically distributed random variables having finite expectation, and if N is a stopping time for $\{X_n\}$, and $E[N] < \infty$, then

$$E\left[\sum_{n=1}^{N} X_n\right] = E[N]\,E[X]$$

Proof. Letting

$$I_n = \begin{cases} 1 & \text{if } N \geq n \\ 0 & \text{if } N < n \end{cases} \tag{2.2}$$

then $\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_n$ Hence,

$$E\left[\sum_{n=1}^{N} |X_n|\right] = E\left[\sum_{n=1}^{\infty} |X_n| I_n\right] = \sum_{n=1}^{\infty} E\left[|X_n| I_n\right]$$

Since I_n is independent of X_n we thus obtain

$$\begin{aligned} E\left[\sum_{n=1}^{N} |X_n|\right] &= \sum_{n=1}^{\infty} E\left[|X_n| I_n\right] \\ &= E\left[|X_n|\right] \sum_{n=1}^{\infty} E\left[I_n\right] (\heartsuit) \\ &= E\left[|X_n|\right] \sum_{n=1}^{\infty} P\{N \geq n\} \\ &= E\left[|X|\right] E\left[N\right] \end{aligned}$$

Since $E\left[|X|\right] E\left[N\right] < \infty (\heartsuit)$, by dominated convergence theorem, we can conclude that

$$\begin{aligned} E\left[\sum_{n=1}^{N} X_n\right] &= \sum_{n=1}^{\infty} E\left[X_n I_n\right] \\ &= E\left[X_n\right] \sum_{n=1}^{\infty} E\left[I_n\right] \\ &= E\left[X_n\right] \sum_{n=1}^{\infty} P\{N \geq n\} \\ &= E\left[X\right] E\left[N\right] \end{aligned}$$

□

Remark. The first $\heartsuit$ is because the Levi monotonic convergence theorem. And the second $\heartsuit$ is because the definition of expectation that X exist finite expectation if and only if $|X|$ exist finite expectation.

Example 4. If $X_n, n = 1, 2, \cdots$ are independent variables such that

$$P\{X_n = 1\} = P\{X_n = 0\} = \frac{1}{2}$$

then

$$N = \min\{n : X_1 + \cdots + X_n = 10\}$$

is a stopping time.

By Wald equation, $E\left[X_1 + \cdots + X_N\right] = \frac{1}{2} E\left[N\right]$. By definition, $X_1 + \cdots + X_N = 10$, so $E\left[N\right] = 20$.

Corollary 2.2.1. An important use of stopping time is that we consider $\{X_n\}$ are the interarrival times of a renewal process. Then $N = N(t) + 1$ is a stopping time for the sequence of $\{X_n\}$. By Wald equation, we have

$$E\left[X_1 + \cdots + X_{N(t)+1}\right] = E[X]E[N(t) + 1]$$

or equivalently

$$E\left[S_{N(t)+1}\right] = E[X]E[N(t) + 1] = \mu[m(t) + 1].$$

2.2.3 proof of the elementary renewal theorem

Theorem 2.2.4 (The elementary renewal theorem)**.** For an renewal process $\{N(t), t \geq 0\}$, with interarrival time sequence $\{X_n\}$, $\mu = E[X_n]$, $m(t) = E[N(t)]$

$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad as\ t \to \infty.$$

Proof. We will prove this theorem by proving that

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu} \quad and \quad \limsup_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

(a): Suppose $\mu < \infty$. Since

$$S_{N(t+1)} > t,\ E[S_{N(t)+1}] > t$$

By the corollary 2.2.1,

$$\begin{aligned}
&\mu(m(t) + 1) > t \\
\Rightarrow &\frac{(m(t) + 1}{t} > \frac{1}{\mu} \\
\Rightarrow &\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}
\end{aligned}$$

(b): Fix a constant $M > 0$, define a new renewal process with interarrival time sequence $\{\overline{X_n}\}$ as

$$\overline{X_n} = \begin{cases} X_n & \text{if } X_n \leq M \\ M & \text{if } X_n \geq M \end{cases}$$

Then $\overline{S_n} = \sum_{i=1}^{n} \overline{X_n}$, and $\overline{N(t)} = \{n : \overline{X_n} \leq t\}$ defines a new renewal process. We obtain

$$\overline{S_{N(t)+1}} \leq t + M$$

Hence by corollary 2.2.1 again

$$(\overline{m(t)} + 1)\overline{\mu} \leq t + M$$

Thus

$$\limsup_{t\to\infty} \frac{\overline{m(t)}}{t} \leq \frac{1}{\overline{\mu}}$$

It's easy to see that $N(t) \leq \overline{N(t)}$, therefore $m(t) \leq \overline{m(t)}$. Moreover

$$\lim_{M\to\infty} \overline{\mu} = \mu$$

then we get the result. $\square$

2.3 The key renewal process

2.3.1 Main result

Here are the main result in this section

Definition. A nonnegative random variable X is said to be lattice, if X can only takes on integral multiples of some nonnegative number d. Then we can also say the distribution function F of X is lattice. The largest d having this property is said to be the period of X.

Theorem 2.3.1 (Blackwell's Theorem)**.** For a renewal process, F is the distribution function of it's interarrival time X.$\mu = E[X], m(t) = E[N(t)] = \sum_{n=1}^{\infty} F_n$. Then

1. If F is not lattice, then

$$m(t+a) - m(t) \to \frac{a}{\mu} \quad \text{as } t \to \infty$$

   for all $a \geq 0$.

2. If F is lattice, with period d, then

$$E[\text{number of renewals at nd}] \to \frac{d}{\mu} \quad \text{as } t \to \infty$$

Definition. $h(t)$ be a function defined on $[0, \infty]$, for any $a > 0$, let $\underline{m_n}(a)$ be the supremum and $\overline{m_n}(a)$ be the infinum of $h(t)$ over the interval $(n-1)a \leq t \leq na$. We say that h is directly Riemann integrable if $\sum_{n=1}^{\infty} \overline{m_n}(a)$ and $\sum_{n=1}^{\infty} \underline{m_n}(a)$ are finite for all $a > 0$, and

$$\lim_{a \to 0} a \sum_{n=1}^{\infty} \overline{m_n}(a) = \lim_{a \to 0} \sum_{n=1}^{\infty} \underline{m_n}(a)$$

Theorem 2.3.2 (A sufficient condition for directly Riemann integrable)**.** h is Riemann integrable if

1. $h(t) \geq 0$ for all $t \geq 0$.

2. $h(t)$ is non increasing.

3. $\int_0^{\infty} h(t)dt < \infty$

Theorem 2.3.3 (The key renewal theorem)**.** For a renewal process, with it's interarrival time has distribution function F. If F is not lattice, and $h(t)$ is directly Riemann integrable, then

$$\lim_{t \to \infty} \int_0^t h(t-x)dm(x) = \frac{1}{\mu} \int_0^t h(t)dt$$

where $\mu = E[X], m(t) = E[N(t)]$.

Recall that $\frac{1}{\mu}$ is the rate of the renewal process.

Lemma 4. For a renewal process

$$P\{S_{N(t)} \leq s\} = \overline{F}(t) + \int_0^s \overline{F}(t-y)dm(y) \quad t \geq s \geq 0.$$

Corollary 2.3.1. :

1. $P\{S_{N(t)} = 0\} = \overline{F}(t)$
2. $dF_{S_{N(t)}}(y) = \overline{F}(t-y)dm(y)$

Explain: Since we have

$$\begin{aligned} dm(y) &= \sum_{n=1}^{\infty} f_n(y)dy \\ &= \sum_{n=1} P\{n\text{th renewal occurs in (y,y+dy)}\} \\ &= P\{\text{renewal occurs in (y,y+dy)}\} \end{aligned}$$

So when $F_{S_{N(t)}}$ is continuous, the probability density of $S_{N(t)}$ is

$$\begin{aligned} f_{S_{N(t)}}(y)dy &= P\{\text{renewal in}(y, y+dy), \text{next interarrival} > t-y\} \\ &= dm(y)\overline{F}(t-y) \end{aligned}$$

2.3.2 alternating renewal process

Consider a system that can be in one of two states: "on" or "off". Initially it is "on". Now suppose the time of it remain "on" is a random variable Z_1, then "off" for a time Y_1, then "on"... And then we get a sequence of random variables $\{Z_n\}$ representing each "on" remaining time , and a sequence of random variables $\{Y_n\}$ representing each "off" remaining time. We allow Z_n and Y_n to be dependent. Let H be the distribution function of $\{Z_n\}$, and G be the distribution function of $\{Y_n\}$, F be the distribution function of $Z_n + Y_n$.

Now we would like to now the probability of it's "on" at the time t. That is

$$P(t) = P\{\text{on at time t}\}$$

Theorem 2.3.4. If $E[Z+Y] < \infty$, and F is not lattice, then

$$\lim_{t\to\infty} P(t) = \frac{E[Z_n]}{E[Z_n] + E[Y_n]}$$

Proof. We can view $X_n = Z_n + Y_n$ be the interarrival time sequence, then it's a renewal process if we let $S_n = \sum_{i=1}^n X_i$, $N(t) = \{n : S_n \leq t\}$. We want to know whether it's "on" or "off" at the time t. We can take condition of the last renewal time before t, that is

$$\begin{aligned} P(t) =& P\{\text{on at t} \mid S_{N(t)=0}\}P\{S_{N(t)=0}\} \\ &+ \int_0^t P\{\text{on at t} \mid S_{N(t)} = y\}dF_{S_{N(t)}}(y) \end{aligned}$$

Now

$$P\{\text{on at t} \mid S_{N(t)=0}\} = \overline{H}(t)/\overline{F}(t)$$

and

$$P\{\text{on at t} \mid S_{N(t)=y}\} = \overline{H}(t-y)/\overline{F}(t-y)$$

by corollary 2.3.1

$$dF_{S_{N(t)}} = dm(y)\overline{F}(t-y)$$

Hence we have

$$P(t) = \overline{H}(t) + \int_0^t \overline{H}(t-y)dm(y)$$

By key renewal theorem 2.3.3

$$\lim_{t\to\infty} P(t) = \frac{\int_0^\infty \overline{H}(t)dt}{\mu} = \frac{E[Z]}{E[Z+Y]}$$

□

Here is an example of alternating renewal process

Example 5. Suppose that customers arrive at a store, which sells only one commodity. The arrival process is a renewal process with the distribution function F of it's interarrival time. Suppose F is not lattice. The amounts desired by the customers are assumed to be independent with a common distribution G. The store uses the following (s, S) policy: That is

1. If the inventory level after serving a customer is below s, then the server will add the commodity to the amount of S.
2. Otherwise the server will not add the commodity.

Let $X(t)$ denote the inventory level of time t, and $X(0) = S$, we want to know the probability $P\{X(t) \geq x\}$ when $t \to \infty$, .

Solution. We said this system is "on" whenever the inventory level is more than or equal to x, and is "off" otherwise. By the theorem above, we know

$$\begin{aligned}\lim_{t\to\infty} P\{X(t) \geq x\} &= \lim_{t\to\infty} P\{\text{on at time t}\} \\ &= \frac{E[\text{time of the inventory} \geq x \text{ in a cycle}]}{E[\text{time of a cycle}]}\end{aligned}$$

Let $Y_1, Y_2, \cdots$ denote the demand of the arrived customer in a cycle and

$$N_x = \min\{n : Y_1 + \cdots + Y_n > S - x\}$$

$$N_s = \min\{n : Y_1 + \cdots + Y_n > S - s\}$$

Then the

$$\text{amount of "on" time in a cycle} = \sum_{i=1}^{N_x} X_i$$

$$\text{time of a cycle} = \sum_{i=1}^{N_s} X_i$$

As we can check N_x and N_s are a stopping time of $\{X_i\}$, hence by Wald equation 2.2.3, we have

$$\lim_{t\to\infty} P\{X(t) \geq x\} = \frac{E[\sum_{i=1}^{N_x} X_i]}{E[\sum_{i=1}^{N_s} X_i]} = \frac{E[N_x]}{E[N_s]}$$

To counting the expectation $E[N_x]$, suppose that $\{Y_n\}$ is a interarrival time of a renewal process $\{N(t), t \geq 0\}$. Then

$$N(S-x) = \max\{n : Y_1 + \cdots + Y_n \leq S - x\} = N_x - 1$$

Hence

$$E[N_x - 1] = E[N(S-x)] = m_G(S-x)$$

Where $m_G(t) = \sum_{n=1}^{\infty} G_n(t)$, and G_n is n-fold of G. Hence

$$\lim_{t\to\infty} P\{X(t) \geq x\} = \frac{m_G(S-x)+1}{m_G(S-s)+1} \quad s \leq x \leq S$$

□

Example 6 (excess life and age). Suppose there is a renewal process, with the interarrival time is not lattice. Let

1. $Y(t) = S_{N(t)+1} - t$ denote the time from t to the next renewal.
2. $A(t) = t - S_{N(t)}$ denote the time from t since the last renewal.

$Y(t)$ is called the excess time and $A(t)$ is called the age. Know we want to compute the probability $P\{A(t) \leq x\}$.

To use the alternating renewal process, we say the system is "on" at time t if the age at t is less than or equal to x, and "off" otherwise. Then

$$\begin{aligned}\lim_{t\to\infty} P\{A(t) \leq x\} &= \lim_{t\to\infty} P\{\text{on at t}\} \\ &= \frac{E[\min(X,x)]}{E[X]} \quad \heartsuit \\ &= \frac{\int_0^{\infty} P\{\min(x,X) > y\}dy}{E[X]} \\ &= \frac{\int_0^{x} \overline{F}(y)dy}{\mu}\end{aligned}$$

$\heartsuit$ here is because the open time in each cycle is $\begin{cases} x & \text{if } x < X \\ X & \text{if } x \geq X \end{cases}$, where X is the time of a cycle, which equal to the interarrival time of the renewal process.

Example 7 (The distribution of $X_{N(t)+1}$). Still consider a renewal process with interarrival time sequence $\{X_n\}$.

$$S_n = \sum_{i=1}^{n} X_n$$
$$X_{N(t)+1} = S_{N(t)+1} - S_{N(t)}$$

One may thought the distribution of $X_{N(t)+1}$ is just as the distribution of X, which is F. However it might be wrong. In the exercises we will prove that $P\{X_{N(t)+1}\} \geq \overline{F}(x)$.

Now we want to get the limiting distribution of $X_{N(t)+1}$. We have

$$P\{X_{N(t)+1} > x\} = P\{\text{length of renewal interval containing } t > x\}$$

To use the alternating renewal process, again let an on-off cycle correspond to a renewal interval. That is, if the length of a renewal interval is greater than x, we say it's "on" in the whole cycle, otherwise, it's "off" in the whole cycle. Then each cycle either "on" at all or "off" at all. Now

$$\begin{aligned}
\lim_{t\to\infty} P\{X_{N(t)+1} > x\} &= \lim_{t\to\infty} P\{\text{length of renewal interval containing} t > x\} \\
&= \lim_{t\to\infty} P\{\text{on at time } t\} \\
&= \frac{\text{the expectation of the time of it's on in a cycle}}{\text{the expectation of the time of a cycle}} \\
&= \frac{E[X > x]}{\mu} \\
&= \frac{\int_x^\infty ydF(y)}{\mu}
\end{aligned}$$

Until now we still don't know the distribution of $X_{N(t)+1}$, However, if we have know the exact time of $S_{N(t)}$, then

$$\{X_{N(t)+1} > y \mid S_{N(t)} = s\}$$

has the same distribution with

$$\{X > y \mid X > t - s, \text{renewal at s}\}$$

We will use this property in the next section and exercise. See 2.6

2.4 Delayed renewal process

We often consider a counting process for which the first interarrival time has a different distribution from the remaining ones. Formally, let

1. $\{X_n, n \geq 1\}$ be a sequence of independent nonnegative random variables with X_1 having distribution G, and others having distribution F.

2. Let $S_n = \sum_{i=1}^{n} X_i$.

3. $N_D(t) = \{n : S_n \leq t\}$, then $\{N_D(t), t \geq 0\}$

To distinguish delayed renewal process and renewal process, we often add a subscript D.

Definition. :

1. The counting process $\{N_D(t), t \geq 0\}$ defined above is called a delayed renewal process.

2. $m_D(t) := E[N_D(t)]$

3. μ denote as the expectation of $X_j, J \geq 2$, that is

$$\mu = \int_0^\infty x dF(x)$$

Familiar to the result in former section, it's easy to get

Proposition 1. For a delayed renewal process $\{N_D(t), t \geq 0\}$, with the first interarrival time has distribution G, and others have distribution F, we have:

1. The distribution of S_n is $G * F_{n-1}$, which means the convolution of G and $(n-1)$ F.

2. The distribution of $N_D(t)$ is

$$P\{N_D(t) = n\} = G * F_{n-1}(t) - G * F_n(t)$$

3. $m_D(t) = \sum_{n=1}^{\infty} G * F_{n-1}$

And by the same way we can get a key renewal theorem in delayed renewal process

Theorem 2.4.1. For a delayed renewal process, with probability 1, we have

1. $\dfrac{N_D(t)}{t} \to \dfrac{1}{\mu} \quad \text{as } t \to \infty$

2. $\dfrac{m_D(t)}{t} \to \dfrac{1}{\mu} \quad \text{as } t \to \infty$

3. If F is not lattice, then

$$m_D(t+a) - m_D(t) \to \frac{a}{\mu} \quad \text{as } t \to \infty$$

4. If F is lattice with period d, then

$$E[\text{number of renewals at nd}] \to \frac{d}{n} \quad \text{as } n \to \infty$$

5. If F is not lattice, $\mu < \infty$, and h is a directly R-integrable function, then

$$\int_0^\infty h(t-x)dm_D(x) \to \int_0 \infty h(t)dt/\mu$$

2.5 Renewal Reward Process

Definition. Consider a renewal process, with the interarrival time sequence $\{X_n\}$, and it's distribution function F. Suppose that each time a renewal occur and we receive a reward, which denote by R_n the reward earned at the time of the nth renewal. Usually we allow the $R_n, n \geq 1$ are independent and identically distributed, but depend on X_n, the length of the nth renewal interval, that is (X_n, R_n) are independent and identically distributed. And such this renewal process is called the renewal reward process.

Let

$$R(t) = \sum_{n=1}^{N(t)} R_n, \ E[R] = E[R_n], \ \mu = E[X] = E[X_n]$$

which represents the total reward earned by time t. We would like to know the relation between them.

Theorem 2.5.1. If $E[R] < \infty, E[X] < \infty$, then

1. with probability 1
$$\frac{R(t)}{t} \to \frac{E[R]}{E[X]} \quad \text{as } t \to \infty$$

2. with probability 1
$$\frac{E[R(t)]}{t} \to \frac{E[R]}{E[X]} \quad \text{as } t \to \infty$$

Proof. (i) Since

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \cdot \frac{N(t)}{t}$$

We know that when $t \to \infty, N(t) \to \infty$. Hence by strong law of large numbers, we obtain that

$$\sum_{n=1}^{N(t)} R_n/N(t) \to E[R] \quad \text{as } t \to \infty$$

and by the theorem2.2.2, we have

$$\frac{N(t)}{t} \to \frac{1}{\mu} = \frac{1}{E[X]} \quad \text{as } t \to \infty$$

Thus (i) is proven.

(ii) Just as the proof of the elementary renewal theorem, we found out that the random variable $N = N(t) + 1$ is the stopping time of $\{R_n\}$. By Wald's equation we have

$$E[\sum_{n=1}^{N(t)} R_n] = E[\sum_{n=1}^{N(t)+1} R_n] - E[R_{N(t)+1}] = (m(t) + 1)E[R] - E[R_{N(t)+1}]$$

Hence

$$\frac{E[R(t)]}{t} = \frac{m(t) + 1}{t} E[R] - \frac{E[R_{N(t)+1}]}{t}$$

It's easy to see

$$\lim_{t\to\infty} \frac{m(t)+1}{t} E[R] = \frac{E[R]}{E[X]}$$

Therefore our goal is to prove that

$$E[R_{N(t)+1}]/t \to 0 \quad \text{as } t \to \infty$$

Let

$$\begin{aligned} g(t) :&= E[R_{N(t)+1}] \\ &= E[R_{N(t)+1} \mid S_{N(t)} = 0]\overline{F}(t) + \int_0^t E[R_{N(t)+1} \mid S_{N(t)} = s]\overline{F}(t-s)dm(s) \\ &= E[R_1 \mid X_1 > t]\overline{F}(t) + \int_0^t E[R \mid X > t-s]\overline{F}(t-s)dm(s) \end{aligned}$$

Let

$$h(t) := E[R_1 \mid X_1 > t]\overline{F}(t) = \int_t^\infty E[R \mid X = x]dF(x)$$

Since

$$E[|R|] = \int_0^\infty E[|R| \mid X = x]dF(x) < \infty$$

So $|h(t)| < E[|R|]$, and $\lim_{t\to\infty} h(t) = 0$, which means $\forall \varepsilon > 0, \exists T > 0$, when $t > T, |h(t)| < \varepsilon$. Hence when $t > T$

$$\begin{aligned} \frac{|g(t)|}{t} &\le \frac{|h(t)|}{t} + \int_0^t \frac{|h(t-s)|}{t} dm(s) \\ &\le \frac{\varepsilon}{t} + \int_0^{t-T} \frac{|h(t-s)|}{t} dm(s) + \int_{t-T}^t \frac{|h(t-s)|}{t} dm(s) \\ &\le \frac{\varepsilon}{t} + \frac{\varepsilon}{t} m(t-T) + E[|R|]\frac{m(t)-m(t-T)}{t} \\ &\to 0 \quad \text{as } t \to \infty \end{aligned}$$

Above we have prove this theorem. □

In the last section we have know that the distribution of $X_{N(t)}$ is not as the same with distribution of $\{X_n\}$ in general cases. Since $R_{N(t)}$ is related to $X_{N(t)}$, thus the distribution of $R_{N(t)}$ should not as the same with $\{R_n\}$.

Above we just assume that the reward is received all at once at the end of the renewal cycle. However if the reward is earned gradually during the renewal cycle. This theorem remain true. To see this, we have

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} \le \frac{R(t)}{t} \le \frac{\sum_{n=1}^{N(t)} R_n}{t} + \frac{R_{N(t)+1}}{t}$$

Where $R(t)$ represent the reward earned during the time t, but not $R(t) = \sum_{n=1}^{N(t)} R_n$.

Example 8. For a random renewal process which has the interarrival time sequence $\{X_n\}$. Let $A(t)$ be the age at t of this renewal process, which is $t - S_{N(t)}$, and let $Y(t)$ be the recess time at t, which is $S_{N(t)+1} - t$. We want to know the limit

$$\lim_{t\to\infty} \frac{\int_0^t A(s)ds}{t}$$

and

$$\lim_{t\to\infty} \frac{\int_0^t Y(s)ds}{t}$$

Solution. Assume we received a reward $A(t)$ at the time t, then $R(t) = \int_0^t A(s)ds$. By the theorem above

$$\frac{\int_0^t A(s)ds}{t} \to \frac{E[\text{reward during a renewal cycle}]}{E[\text{time of a renewal cycle}]}$$

Since

$$E[\text{reward during a renewal cycle}] = \int_0^X sds = \frac{X^2}{2}$$

Then

$$\frac{\int_0^t A(s)ds}{t} \to \frac{E[X^2]}{2E[X]}$$

Just as the same method, we can get

$$\frac{\int_0^t Y(s)ds}{t} \to \frac{E[X^2]}{2E[X]}$$

□

2.6 Exercises

Exercise

For a renewal process, Verify that is these proposition true or false

1. $N(t) < n$ if and only if $S_n > t$
2. $N(t) \leq n$ if and only if $S_n \geq t$
3. $N(t) > n$ if and only if $S_n < t$

Solution. :

1. true.
2. false. Since $S_n = t$, needn't have $N(t) \leq n$, because the X_{n+1} may equals to zero.
3. false. By the same reasoning above. Suppose $N(t) > n$, there might be $S_n = t$ and $X_{n+1} = 0$.

□

Exercise

For a renewal process, suppose the distribution function of the interarrival time F, satisfy $F(\infty) = P\{X < \infty\} < 1$, that is the probability $P\{X = \infty\} > 0$. Then after each renewal, there is a probability $1 - F(\infty)$ that makes the process will no have further renewal. Prove that in this situation the total number of renewals, call it $N(\infty)$, is such that $1 + N(t)$ has a geometric distribution with mean $1/(1 - F(\infty))$

Solution. We have

$$
\begin{aligned}
&P\{N(\infty) = k\} \\
=&P\{\text{the former k-1 interarrival time is finite, and the } k\text{th interarrival time is infinite}\} \\
=&F(\infty)^k(1 - F(\infty))
\end{aligned}
$$

Hence

$$
\begin{aligned}
P\{1 + N(\infty) = k\} &= P\{N(\infty) = k - 1\} \\
&= F(\infty)^{k-1}(1 - F(\infty))
\end{aligned}
$$

Here we have proved the proposition. □

Exercise

Prove that $P\{X_{N(t)+1} > x\} \geq \overline{F}(x)$, and calculate that when $F(x) = 1 - e^{-\lambda x}$, what's the $P\{X_{N(t)+1} > x\}$

Solution.

$$
\begin{aligned}
P\{X_{N(t)+1} > x\} &= \int_0^t P\{X_{N(t)+1} > x \mid S_{N(t)} = y\} dF_{S_{N(t)}} \\
&= \int_0^t P\{X > x \mid X > t-y, \text{renewal at } y\ \} dF_{S_{N(t)}} \\
&= \int_0^t P\{X > x, X > t-y, \text{renewal at } y\}/P\{X > t-y, \text{renewal at } y\} dF_{S_{N(t)}} \\
&= \int_0^t P\{X > x, X > t-y, \text{renewal at } y\}/P\{X > t-y, \text{renewal at } y\} dF_{S_{N(t)}} \\
&= \int_0^t P\{X > max\{x, t-y\}\}/P\{X > t-y\} dF_{S_{N(t)}} \\
&= \int_0^t [1 - F(\max\{x, t-y\})]/[1 - F(t-y)] dF_{S_{N(t)}} \\
&= \int_0^t \min\{1, [1 - F(x)]/[1 - F(t-y)]\} dF_{S_{N(t)}} \qquad \heartsuit \\
&\geq \int_0^t (1 - F(x)) dF_{S_{N(t)}} \\
&= 1 - F(x)
\end{aligned}
$$

Take $F(x) = 1 - e^{-\lambda x}$ into the step $\heartsuit$, then we get

$$
\begin{aligned}
P\{X_{N(t)+1} \geq x\} &= \int_0^\infty \min\{1, e^{-\lambda x}/e^{-\lambda(t-s)}\} dF_{S_{N(t)}} \\
&= \int_0^{t-x} dF_{S_{N(t)}}(s) + \int_{t-x}^t e^{-\lambda(x+s-t)} dF_{S_{N(t)}}(s) \\
&= \int_0^{t-x} e^{-\lambda(t-s)} dm(s) + \int_{t-x}^t e^{-\lambda x} dm(s) \\
&= e^{-\lambda t} \int_0^{t-x} e^{\lambda s} d\lambda s + e^{-\lambda x} \int_{t-x}^t d\lambda s \\
&= (1 + \lambda x) e^{-\lambda x} - e^{-\lambda t}
\end{aligned}
$$

□

Exercise

Prove the equation

$$
m(t) = F(t) + \int_0^t m(t-x) dF(x)
$$

Solution.

$$\begin{aligned} m(t) &= E[N(t)] \\ &= \int_0^t E[N(t) \mid X_1 = x]dF(x) \\ &= \int_0^t E[1 + N(t-x)]dF(x) \\ &= F(t) + \int_0^t E[N(t-x)]dF(x) \\ &= F(t) + \int_0^t m(t-x)dF(x) \end{aligned}$$

□

Exercise

Consider a miner trapped in a room that contains three doors. Door 1 leads her to freedom after two-days travel. Door 2 returns her to the begin room after four-days. Door 3 returns her to the begin room after eight-days. Suppose at all times she is equally to choose the three doors, and T denote the time it takes the miner to become free.

1. Define a sequence of independent and identically distributed random variables $X_1, X_2, \cdots$ and a stopping time N such that

$$T = \sum_{n=1}^{N} X_n$$

2. Use Wald's equation to find $E[t]$.
3. Compute $E[\sum_{i=1}^{N} X_i \mid N = n]$, and note that it is not equal to $E[\sum_{i=1}^{n} X_i]$.
4. Use part (iii) to compute $E[T]$.

Solution.

(i) We define X as

$$P\{X = 2\} = P\{X = 4\} = P\{X = 8\} = \frac{1}{3}$$

and

$$N = \min\{n : X_n = 2\}$$

Then N is stopping time of $\{X_n\}$ since it's depend on $X_1, \cdots, X_n$ and is independent of $X_{n+1}, \ldots$.

(ii) As we can check, N has a geometric distribution with mean 3. Then $E[N] = 3$, and $E[X] = \frac{1}{3}(2 + 4 + 8) = \frac{14}{3}$, hence by Wald's equation we have

$$E[T] = E[\sum_{i=1}^{N} X_i] = E[N]E[X] = 14.$$

(iii)

$$
\begin{aligned}
E[\sum_{i=1}^{N} X_i \mid N = n] &= E[2 + \sum_{i=1}^{n-1} X_i \mid N = n] \\
&= 2 + E[\sum_{i=1}^{n-1} X_i \mid X_i > 2] \\
&= 2 + (n-1) \times \frac{4+8}{2} \\
&= 6n - 4
\end{aligned}
$$

However $E[\sum_{i=1}^{n} X_i] = \dfrac{14}{3} n$

(iv)

$$
\begin{aligned}
E[T] &= E[\sum_{i=1}^{n} X_i] \\
&= \sum_{n=1}^{\infty} E[\sum_{i=1}^{N} X_i \mid N = n] P\{N = n\} \\
&= \sum_{n=1}^{\infty} (6n-3) \left(\frac{2}{3}\right)^{n-1} \left(\frac{1}{3}\right) \\
&= 14
\end{aligned}
$$

□

Exercise

For a renewal process, let $A(t)$ and $Y(t)$ denote the age and recess life of in the time t. That is

1. $A(t) = t - S_{N(t)}$
2. $Y(t) = S_{N(t)+1} - t$

compute

1. $P\{Y(t) > x \mid A(t) = s\}$
2. $P\{Y(t) > x \mid A(t + x/2) = s\}$
3. For a poisson process, compute $P\{Y(t) > x \mid A(t + x) > s\}$
4. $P\{Y(t) > x \,, A(t) > y\}$
5. $\mu < \infty$, prove that $A(t)/t \to 0$ as $t \to \infty$ with probability 1.

Solution. Sorry I'm too lazy.

12. 以 $A(t)$ 和 $Y(t)$ 分别记在 t 时刻的年龄和剩余寿命. 求

(a) $P\{Y(t) > x \mid A(t) = s\}$

$P\{S_{N(t)+1} - t > x \mid t - S_{N(t)} = s\}$

$= P\{S_{N(t)+1} > t+x \mid S_{N(t)} = t-s\}$

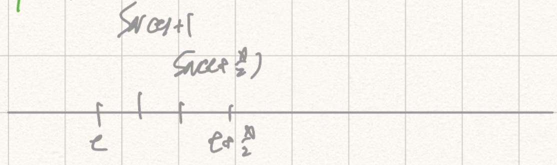
$= P\{X_{N(t)+1} > x+s \mid S_{N(t)} = t-s\}$

$= P\{X > x+s \mid X > s$, 再了[?]发生在 $t-s\}$

$$= \frac{P\{X > x+s,\ X > s,\ \text{再了[?]发生在 } t-s\}}{P\{X > s,\ \text{在[?]发生在 } t-s\}} = \frac{P\{X > x+s\}}{P\{X > s\}}$$

$= \bar{F}(x+s)/\bar{F}(s)$

(b) $P\{Y(t) > x \mid A(t+\frac{x}{2}) = s\}$.



$S_{N(t+\frac{x}{2})} = t-s+\frac{x}{2}$

① $t+\frac{x}{2}-t-s>0$ 时

$\frac{x}{2}-s>0$ 时

$S_{N(t)+1} \le S_{N(t)}+\frac{x}{2} = t-s+\frac{x}{2}$

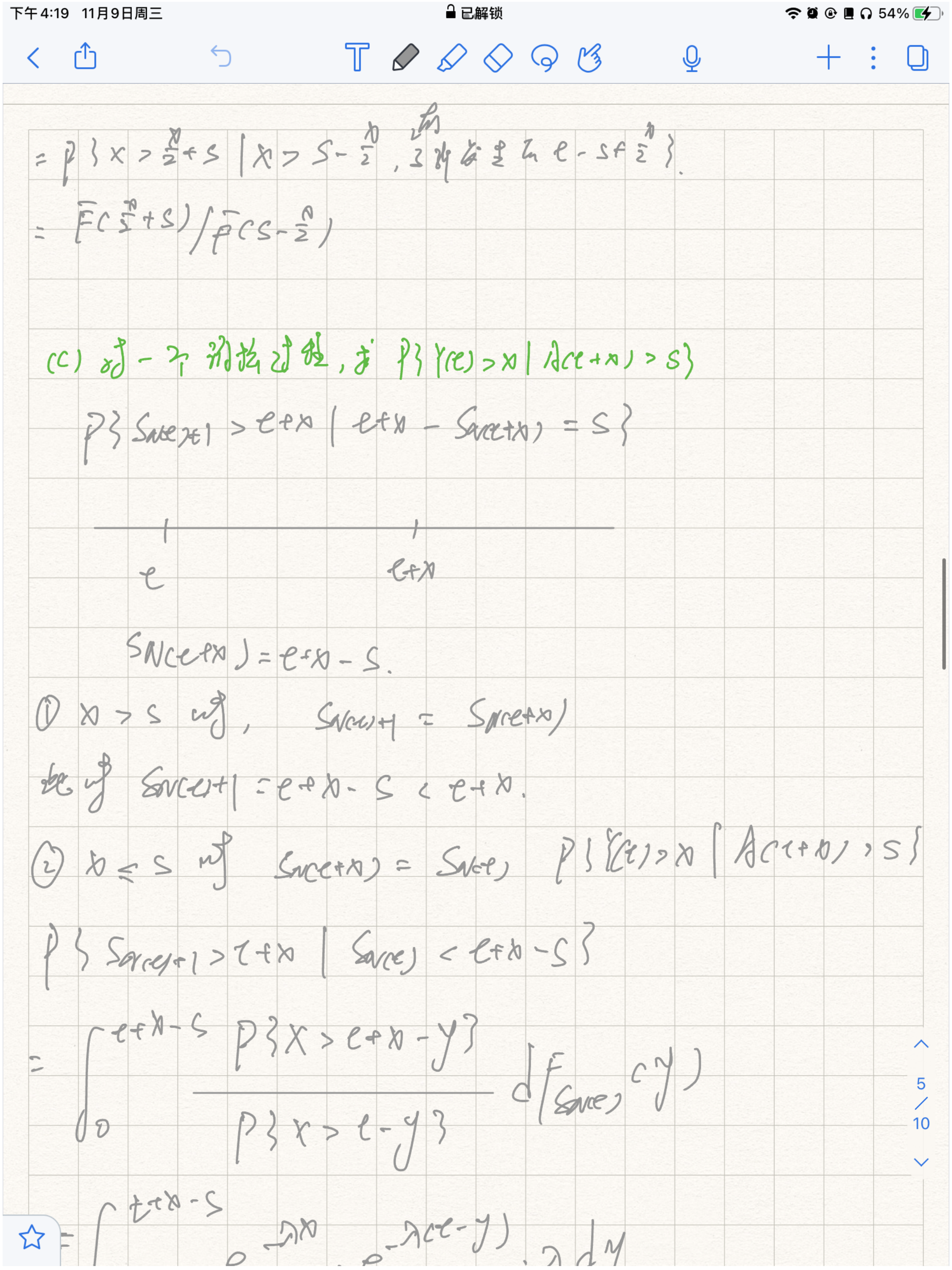
于是 $Y(t) = S_{N(t)+1} - t < x$

② $\frac{x}{2}-s \le 0$ 时.

此时 $S_{N(t+\frac{x}{2})} = S_{N(t)}$ $S_{N(t)} = t-s+\frac{x}{2}$

$P\{S_{N(t)+1} > t+x \mid S_{N(t)} = t-s+\frac{x}{2}\}$

$P\{X_{N(t)+1} > \frac{x}{2}+s \mid S_{N(t)} = t-s+\frac{x}{2}\}$



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$= P\{X > \frac{h}{2}+s \mid X > s-\frac{h}{2}$，3 时发生在 $t-s+\frac{h}{2}\}$.

$= \bar{F}(\frac{h}{2}+s)/\bar{F}(s-\frac{h}{2})$

(c) 对一个泊松过程，求 $P\{Y(t) > x \mid A(t+x) > s\}$

$P\{S_{N(t)+1} > t+x \mid t+x - S_{N(t+x)} = s\}$

t $\quad t+x$

$S_{N(t+x)} = t+x-s$.

① $x > s$ 时，$S_{N(t)+1} = S_{N(t+x)}$

此时 $S_{N(t)+1} = t+x-s < t+x$.

② $x \le s$ 时 $S_{N(t+x)} = S_{N(t)}$ $P\{Y(t) > x \mid A(t+x) > s\}$

$P\{S_{N(t)+1} > t+x \mid S_{N(t)} < t+x-s\}$

$$= \int_0^{t+x-s} \frac{P\{X > t+x-y\}}{P\{X > t-y\}} \, dF_{S_{N(t)}}(y)$$

$$= \int^{t+x-s} e^{-\lambda x} \cdots e^{-\lambda(t-y)} \lambda \, dy$$

5/10

$$= e^{-\lambda(x+t)} \cdot \int_0^{t+x-s} e^{\lambda y} \, d\lambda y$$

$$= e^{-\lambda(x+t)} \cdot e^{\lambda(t+x-s)} = e^{-\lambda s}$$

(d) $P\{Y(t) > x, A(t) > y\}$

$$P\{S_{N(t)+1} > t+x, \; S_{N(t)} < t-y\}$$

$$= \int_0^{t-y} \frac{P\{X > t+x-z\}}{P\{X > t-z\}} \, dF_{S_{N(t)}}(z)$$

$$= \int_0^{t-y} e^{-\lambda x} \, dF_{S_{N(t)}}(z) = e^{-\lambda x} \int_0^{t-y} \bar{F}(t-z) \, dm(z)$$

若是 Poisson，则 $= e^{-\lambda x} \cdot e^{-\lambda t} \cdot e^{\lambda(t-y)} = e^{-\lambda(x+y)}$

(e) $\mu < \infty$，结论 w.p. 1 概率，$t \to \infty$ 时，$A(t)/t \to 0$.

$$A(t) = t - S_{N(t)}$$

故

$$\frac{A(t)}{t} = 1 - \frac{S_{N(t)}}{t}$$

$$= 1 - \frac{S_{N(t)}}{N(t)} \frac{N(t)}{t}$$

$$\to 1 - \mu \cdot \frac{1}{\mu} \quad (t \to \infty \text{ 时, w.p. 1 概率})$$

$$= 0.$$

□

Chapter 3

Markov Chains

3.1 Basic concept

3.1.1 definition

Definition. Consider a stochastic process $\{X_n, n \geq 0\}$, that takes on a finite or countable number of possible values. Usually we denote the possible values as the set of nonnegative integers $\{0, 1, 2, \cdots\}$. If

$$\begin{aligned} &P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_1 = i_1\} \\ =&P\{X_{n+1} = j \mid X_n = i\} \\ :=&p_{ij} \geq 0 \end{aligned}$$

In other words, if we let X_n be the present state, and $X_{n-1}, \cdots, X_1$ be the past state, and X_{n+1} be the future state, then the future state only depend on present state but independent of past state. Then we call stochastic process is a **Markov Chains**.

Remark. :

1. The property that the future state only depend on the present state is called the Markov property.
2. Clearly there must have
$$\sum_{j=0}^{\infty} P_{ij} = 1$$
Since whatever the present state is, the future state must make a transition into some state.

Naturally we define
$$\begin{pmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$
As the matrix of one-step transition probabilities P_{ij}

Example 9 (Markov chain in M/G/1 queue)**.** :
Recall: In a M/G/1 queue, customers come to a service center according to a Poisson process with rate λ. There is a single server and the service time of each customers are independent and identically distributed to G.

Let $\{X_n\}$ denote the number of customers left behind by the nth departure. Y_n denote the number of customers arriving during the service period of the $(n+1)$st customer. Then we have
$$X_{n+1} = \begin{cases} X_n - 1 + Y_n & \text{if } X_n \geq 0 \\ Y_n & \text{if } X_n = 0 \end{cases}$$
Since Y_n represent the number of arrivals in non overlapping service intervals, it follows, the arrival process being a Poisson process, that they are independent and

$$\begin{aligned} P\{Y_n = j\} &= E[P\{Y_n = j \mid X_{n+1}] \\ &= \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^j}{j!} dG(x) \end{aligned}$$

Since Y_n is independent of $X_{n-1}, X_{n-2}, \cdots$, therefore X_{n+1} only relative to X_n, $\{X_n, n > 0\}$ is a Markov chain. Moreover, the one-step transition probability given by

$$\begin{aligned}
P_{0j} &= \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^j}{j!}dG(x) \quad j \geq 0 \\
P_{ij} &= \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^(j-i+1)}{(j-i+1)!}dG(x) \quad j \geq i-1, i \geq 1 \\
0 &\quad \text{otherwise}
\end{aligned}$$

3.1.2 Random Walk

Definition. Consider a sequence of variable which be independent and identically distributed with

$$P\{X_i = j\} = a_j \quad j = 0, \pm 1, \pm 2, \cdots$$

1. If we let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$, then $\{S_n, n \geq 0\}$ is a Markov chain for which $P_{ij} = a_{j-i}$. And $\{S_n\}$ is called the **general random walk**.
2. Specially, if $P\{X = 1\} = p, P\{X = -1\} = 1 - p := q$. Then at this time $\{S_n, n \geq 0\}$ is called a **simple random walk**. It's just a special case of general random walk, hence it still a Markov chain.

There has a surprising result that $\{|S_n|, n \geq 0\}$ is still a Markov chain for a simple random walk. To see this, we need some preparation.

Proposition 2. If $\{S_n, n \geq 1\}$ is a simple random walk, then

$$P\{S_n = i||S_n| = i, |S_{n-1}| = i_{n-1}, \ldots, |S_1| = i_1\} = \frac{p^i}{p^i + q^i}$$

Proof. If we let $i_0 = 0$ and define

$$j = \max\{k : 0 \leq k \leq n, i_k = 0\},$$

then, since we know the actual value of S_j, it is clear that

$$\begin{aligned}
&P\{S_n = i \mid |S_n| = i, |S_{n-1}| = i_{n-1}, \cdots, |S_1| = i_1\} \\
=&P\{S_n = i \mid |S_n| = i, \cdots, |S_{j+1}| = i_{j+1}, |S_j| = 0\}
\end{aligned}$$

Now there are two possible values of the sequence $S_{j+1}, \cdots, S_n$ for which $|S_{1+1}| = i_{1+1}, \cdots, |S_n| = i$ The first of which results in $S_n = i$ and has probability

$$p^{\frac{n-1}{2}+\frac{l}{2}}q^{\frac{n-1}{2}-\frac{1}{2}}$$

and the second results in $S_n = -i$ and has probability

$$p^{\frac{n-1}{2}-\frac{1}{2}}q^{\frac{n-1}{2}+\frac{1}{2}}$$

Hence

$$P\{S_n = i||S_n| = i, |S_{n-1}| = i_{n-1}, \ldots, |S_1| = i_1\} = \frac{p^i}{p^i + q^i}$$

$\square$

Proposition 3. The absolute value of simple random walk $\{S_n\}$ is still a Markov chain.

Proof. Consider

$$\begin{aligned}
&P\{|S_{n+1}| = i+1 \mid |S_n| = i, |S_{n-1}| = i_{n-1}, \cdots, |S_1| = i_1\} \\
=&P\{|S_{n+1}| = i+1 \mid S_n = i\}P\{S_n = i \mid |S_n| = i, |S_{n-1}| = i_{n-1}, \cdots, |S_1| = i_1\} \\
+&P\{|S_{n+1}| = i+1 \mid S_n = -i\}P\{S_n = -i \mid |S_n| = i, |S_{n-1}| = i_{n-1}, \cdots, |S_1| = i_1\} \\
=&\frac{p^{i+1} + q^[i+1]}{p^i + q^i}
\end{aligned}$$

Hence, the one-step transition probability of $\{|S_n|\}$ is

$$\begin{aligned}
P_{i,i+1} &= \frac{p^{i+1} + q^{i+1}}{p^i + q^i} = 1 - P_{i,i-1} \\
P_{01} &= 1
\end{aligned}$$

$\square$

3.2 n-step transition probability

3.2.1 Chapman-Kolmogorov equation

Definition. Still consider a Markov chain $\{X_n, n \geq 0\}$.

$$P_{ij}^n := P\{X_{n+m} = j \mid X_m = i\}$$

Which define as the probability that a process in state i will be in state j after n additional transitions.

To calculate the probability of n step transition from i to j, we often use the Chapman-Kolmogorov equations.

Theorem 3.2.1 (Chapman-Kolmogorov equation)**.** For a Markov chain,

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \quad \text{for all } n, m \geq 0, \text{all } i, j$$

Proof. It's easy to check by taking condition on the middle state k. □

Corollary 3.2.1. If we let $P^{(n)}$ denote the matrix of n-step transition probability P_{ij}^n, then from the equation above, we can get

$$P^{(n+m)} = P^{(n)} P^{(m)}$$

Where the multiplication above represents the matrix multiplication. Hence $P^{(n)} = P^n$.

3.2.2 Communicate relation between states

State j is said to be accessible from state i if for some $n \geq 0$, the probability $P_{ij}^n > 0$. If two state accessible to each other, we say they are communicate, denote as $i \leftrightarrow j$.

1. The communication is an equivalence relation. All state can be classified as different equivalence class.

2. If there are only one equivalence class in a Markov chain, we say that this chain is irreducible.

Definition. State i is said to have period d, if $P_{ii}^n = 0$ whenever n is not divisible by d, and d is the greatest integer with this property, which means the state i can only return to itself when the process transit dk times. Let $d(i)$ denote it's period.

Proposition 4. If $i \leftrightarrow j$, then $d(i) = d(j)$.

Proof. Since $i \leftrightarrow j$, there is n, m, $P_{ij}^n P_{ji}^m > 0$. Because

$$P_{jj}^{n+m} \geq P_{ji}^m + P_{ij}^n > 0$$

by 3.2.1. From the definition of $d(j)$, we can conclude that

$$d(j) \mid m + n$$

Moreover,

$$P_{jj}^{m+n+d(i)} \geq P_{ji}^n P_{ii}^{d(i)} P_{ij}^m > 0$$

Hence

$$d(j) \mid d(i)$$

A similar argument yields that $d(i) \mid d(j)$, thus $d(i) = d(j)$. □

3.2.3 Recurrent and transient

For any state i, j, define f_{ij}^n to be the probability that, staring in i, and the first transition into j occurs at time n. Then

$$f_{ij}^0 = 0$$
$$f_{ij}^n = P\{X_n = j, X_k \neq j, k = 1, 2, \cdots, n-1 \mid X_0 = i\}$$

Let

$$f_{ij} = \sum_{n=1} f_{ij}^n$$

Then f_{ij} is the probability of ever making a transition into state j, given that the process starts in i. The probability $f_{ij} > 0$ if and only if j is accessible to i.

Definition. State i is said to be recurrent if $f_{ii} = 1$, and transient otherwise.

Remark. This means, for a recurrent state i, with probability 1, it will return to state i by finite times of transition. Also, for a transient state, with probability $p > 0$, it will return to itself by infinite times of transition.

To understand the recurrent state. Consider if a state i is transient, what's the probability that it will occurs infinite times? Let $N(i)$ denote the times that state i occurs in the infinite process. Then

$$P\{N(i) \geq k\} = (f_{ii})^k$$

Since $f_{ii} < 1$, then

$$\lim_{k\to\infty} P\{N(i) \geq k\} = (f_{ii})^k = 0$$

Hence if a state i is not recurrent, with probability 1 it will not occurs from some time. In other words, $E[N(i)] < \infty$. Formally, we have

Proposition 5. For a Markov chain,

$$\text{state } i \text{ is recurrent} \iff E[\text{In the infinite process, the number that state } i \text{ occurs} \mid X_0 = i] = \infty$$

Proof. :

$\Leftarrow$:) This was proved above.

$\Rightarrow$:) Since i is recurrent, with probability 1 it will return to i. By the Markovian property it follows that the process probabilistically restarts itself upon returning to i. Hence it still will return to i with probability 1. Repeating this argument, we see that with probability 1, the number of visits to i will be infinite and thus the expectation will be infinite. □

From now on, we can get a equivalent condition for a state i is recurrent.

Theorem 3.2.2. For a Markov chain, a state i is recurrent, if and only if

$$\sum_{n=1}^{\infty} P_{ii}^n = \infty$$

Proof. Let

$$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{else} \end{cases}$$

Then

$$\begin{aligned}
&E[\text{In the infinite process, the number that state } i \text{ occurs} \mid X_0 = i] \\
=&E[\sum_{n=1}^{\infty} I_n \mid X_0 = i] \\
=&\sum_{n=1}^{\infty} E[I_n \mid X_0 = i] \\
=&\sum_{n=1}^{\infty} P_{ii}^n \\
=&\infty
\end{aligned}$$

By the proposition above. □

Corollary 3.2.2. For a Markov chain which has finite state, it must have at least one recurrent state.

Corollary 3.2.3. If $i \leftrightarrow j$, then i is recurrent $\iff$ j is recurrent

Proof. Suppose i is recurrent. Since $i \leftrightarrow j$, then there have n, m, $P_{ij}^n > 0$, $P_{ji}^m > 0$.

$$\begin{aligned}
\sum_{k\geq 0} P_{jj}^k \geq \sum_{k\geq 0} P_{jj}^{n+m+k} \geq &\sum_{k\geq 0} (P_{ji}^m)(P_{ij}^n)(P_{ii}^k) \\
&= (P_{ji}^m)(P_{ij}^n)\sum_{k\geq 0}(P_{ii}^k) \\
&= \infty
\end{aligned}$$

□

Corollary 3.2.4. If $i \leftrightarrow j$, and j is recurrent, then $f_{ij} = 1$.

Proof. By the corollary 3.2.3, we know that with probability 1, state i will return to itself by finite times of transition.

Suppose $X_0 = i$, let n be such that $p_{ij}^{(n)} > 0$. If $X_n \neq j$, we say that we loss opportunity 1. However, by finite times of transition, it will return to state i with probability 1, let T_1 denote the next time we enter state i. We say we loss opportunity 2, if $X_{T_1+n} \neq j$, and so on we can get a time sequence $\{T_n\}$ and an opportunity sequence $\{n\}$. We can view the number of opportunity that we miss is a random variable, then it's a geometric random variable with success probability $P_{ij}^{(n)}$.

Now if $f_{ij} < 1$, means it has a positive probability $p > 0$, that began from state i, and never make transition into state j. Moreover, it means with probability $p_0 > 0$, we will miss all opportunity. However, since $0 < P_{ij}^{(n)} < 1$, we know that $p_0 = \lim_{k\to\infty}(P_{ij}^{(n)})^k = 0$. Hence here is a contradiction, and we have $f_{ij} = 1$. □

3.3 Limit theorem in Markov chain

3.3.1 Delayed renewal process in Markov

We denote $N_j(t)$ as the number of transitions into j by time t.

1. If j is recurrent and $X_0 = j$, then $\{N_j(t), t \geq 0\}$ is a renewal process with interarrival time $\{X_n\}$ has the same distribution
$$P\{X = n\} = f_{jj}^n.$$

2. If we start at $X_0 = i$, then $\{N_j(t)\}$ is a delay renewal process, with the first interarrival time has distribution
$$P\{X_1 = n\} = f_{ij}^n,$$
others have distribution
$$P\{X = n\} = f_{jj}^n.$$

Then we can apply the limit theorem on renewal process to Markov chain.

In a same way we should first define the expectation of the interarrival time. Let μ_{jj} denote the expected number of transitions needed to return to state j. Then

$$\mu = \begin{cases} \infty & \text{if } j \text{ is transient} \\ \sum_{n=1} n f_{jj}^n & \text{if } j \text{ is recurrent} \end{cases}$$

Remark. :

1. What we should know is that since a state j is transient, the expectation of the number of transitions needed must be infinite, that's because $f_{jj} < 1, 1 - f_{jj} > 0$, hence it has a positive probability to be infinite, by the definition of expectation, $\mu_{jj} = \infty$.

2. If a state j is recurrent, we define $\mu = \sum_{n=1} n f_{jj}^n$ by the definition of expectation. However, it doesn't means μ_{jj} must be finite.

Now we can obtain the following theorem from the former theorem.

Theorem 3.3.1. If state i, j communicate, and the first state $X_0 = i$, then $\{N_j(t), t \geq 0\}$ is a delay renewal process, we have

1. With probability 1,
$$\frac{N_j(t)}{t} = \frac{1}{\mu_{jj}} \quad \text{as } t \to \infty$$

2. With probability 1,
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{P_{ij}^k}{n} = \frac{1}{\mu_{jj}}$$

3. If j is aperiodic, then
$$\lim_{n\to\infty} P_{ij}^n = \frac{1}{\mu_{ij}}$$

4. If j has period d, then
$$\lim_{n\to\infty} P_{ij}^{nd} = \frac{d}{\mu_{ij}}$$

Proof. i) It's obvious by 1.
ii) First notice that, if we let

$$I_n = \begin{cases} 1 & X_n = j \\ 0 & \text{else} \end{cases}$$

Then
$$P\{I_n = 1\} = P_{ij}^n, P\{I_n = 0\} = 1 - P_{ij}^n$$

Since $N_j(t) = \sum_{k=1}^{[t]} I_k$, then

$$m_j(t) = E[N_j(t)] = \sum_{k=1}^{[t]} P_{ij}^k$$

Since renewal can only take place when time t is positive integer, hence we can assume the time t only take from positive integer, then

$$\lim_{t\to\infty} \frac{m_j(t)}{t} = \lim_{n\to\infty} \frac{m_j(n)}{n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^k}{n}$$

By fundamental renewal process in delayed renewal process, we know

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^k}{n} \to \frac{1}{\mu_{jj}}$$

iii) By Blackwell theorem in delayed renewal process (see 1), and ii)

$$\lim_{n\to\infty} P_{ij}^n = \lim_{n\to\infty} m_j(n+1) - m_j(n) = \frac{1}{\mu_{jj}}$$

iv) Is just the same as iii). □

3.3.2 Positive recurrent and null recurrent

Recall. If a state j is recurrent, we know the expectation of it's return time is define as

$$\mu_{jj} = \sum_{n=1}^{\infty} n f_{jj}^n$$

Since there may have two condition, one is that μ_{jj} is infinite, one is that it's just finite. To verify them, we say

Definition. :

1. A recurrent state j is null recurrent, if $\mu_{jj} = \infty$.
2. A recurrent state j is positive recurrent, if $\mu_{jj} < \infty$.
3. A positive recurrent aperiodic state is called ergodic.

Remark. Recall that a recurrent state, with probability 1, it will return to itself with a finite transition. Is it contradict to null recurrent? The answer is no. Since an almost everywhere finite function may have infinite integral.

How to verify a recurrent state is null or positive?

Proposition 6. If a state j is recurrent, and Markov chain start at $X_0 = i$. Let

$$\pi_j = \lim_{n\to\infty} P_{ij}^{nd(j)}$$

where $d(j)$ is the period of j. Then

1. $\pi_j > 0 \iff j$ is positive.
2. $\pi_j = 0 \iff j$ is null.

Proof. By the theorem 3.3.1, we know that

$$\lim_{n\to\infty} P_{jj}^{nd} = \frac{d}{\mu_{jj}}$$

Hence if $\mu_{jj} = \infty$,$\pi_j = 0$, $\mu_{jj} < \infty$, $\pi_j > 0$. □

3.3.3 Two class of irreducible aperiodic Markov chain

Definition. A probability distribution $\{P_j, j \geq 0, j \in N^+\}$ is called stationary for the Markov chain, if the Markov chain has one-step transition probability $\{P_{ij}\}$, and

$$P_j = \sum_{i=0}^{\infty} P_i P_{ij}, \quad j \geq 0$$

Proposition 7. If the probability distribution of X_0 is stationary for the Markov chain, then X_n will have the same distribution as X_0. Then $\{X_n, n \geq 0\}$ will be a stationary process.

Proof. We will prove this by induction. First

$$\begin{aligned} P\{X_1 = j\} &= \sum_{i=0}^{\infty} P\{X_1 = j \mid X_0 = i\} P\{X_0 = i\} \\ &= \sum_{i=0} \infty P_{ij} P_i \\ &= P_j \end{aligned}$$

Hence X_1 has the same distribution of X_0. By induction

$$\begin{aligned} P\{X_n = j\} &= \sum_{i=0}^{\infty} P\{X_n = j \mid X_{n-1} = i\} P\{X_{n-1} = i\} \\ &= \sum_{i=0} \infty P_{ij} P_i \\ &= P_j \end{aligned}$$

Hence we prove the proposition. □

Remark. A stochastic process $\{X_t\}$, and $F_X(x_{t_1+\tau}, \cdots, X_{t_n+\tau})$ represent the cumulative distribution function of the unconditional joint distribution of $\{X_t\}$ at times $t_1+\tau, \cdots, t_n+\tau$. Then $\{X_n\}$ is said to be strictly stationary if

$$F_X(x_{t_1+\tau}, \cdots, X_{t_n+\tau}) = F_X(x_{t_1}, \cdots, X_{t_n})$$

holds for all t_i and n and τ.

Theorem 3.3.2 (classification of irreducible aperiodic Markov chain)**.** An irreducible aperiodic Markov chain belongs to one of the following two classes.

1. Either the states are all transient or all null recurrent, in this cases $P_{ij}^n \to 0$ as $n \to \infty$ for all i, j, and there exists no stationary distribution.

2. All state are positive recurrent, that is
$$\pi_j = \lim_{n\to\infty} P_{ij}^n > 0$$

Proof. We first prove (ii). Note that

$$\sum_{j=0}^{M} P_{ij}^n \leq \sum_{j=0}^{\infty} P_{ij}^n = 1 \quad \forall M$$

Letting $n \to \infty$, yields

$$\sum_{j=0}^{M} \pi_j \leq 1 \quad \forall M$$

Letting $M \to \infty$, yields

$$\sum_{j=0}^{\infty} \pi_j \leq 1$$

Now

$$P_{ij}^{n++1} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj} \geq \sum_{k=0}^{M} P_{ik}^n P_{kj} \quad \forall M$$

Letting $n \to \infty$, we get

$$\pi_j \geq \sum_{k=0}^{M} \pi_k P_{kj} \quad \forall M$$

Which leads to

$$\pi_j \geq \sum_{k=0}^{\infty} \pi_k P_{kj} \quad \forall M$$

We claim that this inequality can only takes equality. Otherwise, suppose that the inequality is strict for some j. Then

$$\sum_{j=0}^{\infty} \pi_j > \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \pi_k P_{kj} = \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{\infty} P_{kj} = \sum_{k=0}^{\infty} \pi_k$$

Which is a contradiction. Therefore,

$$\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}$$

Now putting $P_j = \pi_j / \sum_{k=0}^{\infty} \pi_k$, then $\{P_j, j \geq 0\}$ is a stationary dsitribution. □