Functional Analysis Class Note

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This note is taken for the Functional Analysis course, lectured by Professor Yong Jiao. This note contains my personal thoughts, so all errors in this notes should be mine.

Here are some conventions:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are fields learnt in Mathematical Analysis. $\mathrm{Re}, \mathrm{Im}\colon \mathbb{C} \to \mathbb{R}$ is the real part and imaginary part of a complex number respectively. $\mathbb{K}$ is one of $\mathbb{R}$ and $\mathbb{C}$, usually used to state different cases conveniently. $\mathbb{N}$ is the set of **positive** integers.
- $\mathbb{K}^{n\times n}$ means the matrix space containing all $n \times n$ matrices.
- $\forall, \exists$ and $\exists!$ means "for all, there is and there is unique" respectively.
- Formula $A := B$ means A is defined as B. For example, $\mathbb{C} := \mathbb{R}[x]/(1+x^2)$ means $\mathbb{C}$ is defined as the quotient ring $\mathbb{R}[x]/(1+x^2)$.
- For each set A, the identity map is $\mathrm{id}_A\colon A \to A, a \mapsto a$.
- For a mapping $f\colon A \to B$, we write $A = \mathrm{dom}(f), B = \mathrm{cod}(f)$.
- For $a, b \in \mathbb{R}$, define minimum function

$$\wedge\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto \frac{a + b - |a - b|}{2},$$

  and maximum function

$$\vee\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto \frac{a + b + |a - b|}{2}.$$

- Subtraction of sets A, B is $A \setminus B := \{x \in A : x \notin B\}$.
- For a sets A, $\mathcal{P}(A)$ means the power set of A.

- A sequence in X is a map $x\colon \mathbb{N} \to X, n \mapsto x_n$, and $x\colon \mathbb{N} \to X$ is usually denoted as $(x_n)_{n\in\mathbb{N}} \subseteq X$.

  If X is a topological space, the definition of limit is just the definition of a net in a topological space.

  Furthermore, limit of a double indexed sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ is defined as the limit for the product directed set $\mathbb{N} \times \mathbb{N}$.

- For proposition p, q, we use $p \wedge q$ to mean the proposition "p and q", $\wedge$ has the truth table as follows.

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

  Similarly, we define $p \vee q$.

- Addition of real/complex-valued functions is defined pointwisely. That is: let $f, g\colon X \to \mathbb{K}$, we define a funtion $f + g\colon X \to \mathbb{K}$ by $x \mapsto f(x) + g(x)$.

- For $f\colon X \to \mathbb{K}$ and $k \in \mathbb{K}$, we define that function $f + k$ by $x \mapsto f(x) + k$. That is, respect k as a constant function $x \mapsto k$.

- $\lim_n$ means $\lim_{n\to\infty}$ for short.

- Given a linear map $f\colon X \to Y$ where X, Y are linear spaces. Then
$$\ker f := f^{-1}(0) = \{x \in X : f(x) = 0\}.$$

- We say a diagram commutes, if all the morphisms (and their possible compositions) with the same domain and same codomain coincide.

- If an arrow is unique/injective/surjective, we denote the arrow by $\dashrightarrow$/ $\hookrightarrow$ / $\twoheadrightarrow$ respectively.

- The Kronecker symbol on a set is defined as
$$\delta\colon X \times X \to \{0, 1\}, (x, y) \mapsto \delta^x_y := \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}.$$

  It is also denoted by $\delta_{x,y}$.

- The sign function is defined as follows

$$\mathrm{sign}\colon \mathbb{K} \to \mathbb{K}, z \mapsto \frac{\overline{z}}{|z|}.$$

- Let $(\Omega, \mathcal{F})$ be a measurable space. We say a function $f\colon \Omega \to \mathbb{R}$ is measurable, if the preimage of Borel subsets of $\mathbb{K}$ under f is $\mathcal{F}$-measurable. That is, assume $\mathbb{K}$ is equipped with Borel σ-algebra.

- Somewhere you can see color different, that is reminding you to think about what here should be. (Just like $1 + 1 = 2$.)

Contents

0 Introduction

Not done...

Syllabus

This lecture note contains topics as follows:

- Linear normed space, Bounded linear map.
- Banach space, completion of a linear normed space.
- Baire category Theorem, Banach-Steinhaus Theorem, Open mapping theorem, Closed graph theorem.

1 Week 1

1.1 Week 1, Lecture 1

We begin from **Banach Space** and **Metric Space**. Before the definition of **Banach space**, we should recall the definition of Vector spaces(or Linear Spaces). Given a set X, a vector space is a triple $(X, +, \cdot)$ where $+ \colon X \times X \to X$ is called the addition on X, and $\cdot : \mathbb{K} \times X \to X$ is called scalar-multiplication on X, satisfying 8 axioms.

Recall

An isomorphism between vector space means a bijection that keeps the linear structure, that is $\varphi \colon X \to Y$ satisfies: $\forall k, l \in \mathbb{K}, \forall x, x' \in X$ we have $\varphi(kx + lx') = k\varphi(x) + l\varphi(x')$. Isomorphism in categories should be in mind:

Category	Grp	$\mathsf{Lin}_{\mathbb{K}}$	Top
Isomorphism	Group isomorphism	$\mathbb{K}$-Linear isomorphism	Homeomorphism

1.1.1 Linear Normed space

Definition (Linear Normed space)**.** Let X be a linear space. Define a map $\| \ \| \colon X \to \mathbb{R}_{\geq 0}$ satisfying:

(i) $\|x\| = 0 (\in \mathbb{K}) \iff x = 0 (\in X)$;

(ii) $\|kx\| = |k| \cdot \|x\| (\forall k \in \mathbb{K}, x \in X)$;

(iii) $\|x + y\| \leq \|x\| + \|y\| (\forall x, y \in X)$.

Then $\| \ \|$ is called a **norm** over X, and $(X, \| \ \|$ is called a **linear normed space**.

Remark. There is some similar weaker definitions:

- If (only) **(i)** is not satisfied, we call $\| \ \|$ a **semi-norm**.
- If (only) **(iii)** becomes $\|x + y\| \leq C(\|x\| + \|y\|)$ for some $C \in \mathbb{R}_{>1}$, we call $\| \ \|$ a **quasi-norm**.

Equivalently, we can change the codomain of $\| \ \|$ to $\mathbb{R}$ and **(i)** to $(\forall x \in X, \|x\| \geq 0) \wedge \|x\| = 0 \iff x = 0$.

Example 1 (Euclidean Spaces). $(\mathbb{R}^n, \| \|)$ is a linear normed space, who-se norm is defined as follow:

$$\| \|\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}, x = (x_1, \dots, x_n) \mapsto \Big(\sum_{j=1}^{n} x_j^2\Big)^{1/2} (= d(x, 0)).$$

Triangle inequality for this norm comes to be the particular triangle inequality for the metric, which can be shown by Cauchy-Schwarz inequality for real numbers.

Example 2 (Continuous Functions Spaces). $\big(C([a,b], \mathbb{K}), \max_{[a,b]}| |\big)$ is a linear normed space. Recall the definition of $C([a,b], \mathbb{K})$ the family of continuous function from $[a,b]$ to $\mathbb{K}$. whose norm is defined as follow:

$$\max_{[a,b]}| |\colon (C([a,b], \mathbb{K}) \to [0, \infty), f \mapsto \max_{x \in [a,b]} |f(x)|.$$

Recall why $C([a,b], \mathbb{K})$ is a vector space. What is needed to show is just "addition of continuous functions is continuous", and there is lots of ways to do this, see remark. Notice that $[a,b]$ is compact and so is $f([a,b])$, guaranteeing the existence of $\max_{x \in [a,b]} |f(x)|$. Compatibility with multiplication and triangle inequality is trivial.

Remark. We have many methods for proving "addition of continuous functions is continuous". They give the same result with different standpoints. Suppose $f, g \in C([a,b], \mathbb{K})$

1. By the definition of continuity: We prove pointwisely: Fix $x \in [a,b]$. $\forall \varepsilon > 0$,we can find $\delta_1, \delta_2 > 0$ such that $\forall y : 0 < |y - x| < \delta_1, |f(y) - f(x)| < \varepsilon/2$ and $\forall y : 0 < |y - x| < \delta_2, |g(y) - g(x)| < \varepsilon/2$. Therefore, let $\delta := \delta_1 \wedge \delta_2$ and we have $\forall y : 0 < |y - x| < \delta$,

$$\begin{aligned} |(f+g)(y) - (f+g)(x)| =& |f(y) + g(y) - f(x) - g(x)| \\ \leq& |f(y) - f(x)| + |g(y) - g(x)| \\ <& \varepsilon/2 + \varepsilon/2 \\ =& \varepsilon. \end{aligned}$$

   Therefore, $f + g$ is continuous at x.

2. By sequence: We prove pointwisely: Fix $x \in [a,b]$. Suppose sequence $(x_n)_{n \in \mathbb{N}} \subseteq [a,b]$ converges to x, then:

$$\begin{aligned} \lim_{n\to\infty} (f+g)(x_n) =& \lim_{n\to\infty} \big(f(x_n) + g(x_n)\big) \\ =& \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) \\ =& f(x) + g(x) \\ =& (f+g)(x). \end{aligned}$$

Therefore, $f+g$ is continuous at x.

3. By the topological definition($\mathbb{K}=\mathbb{R}$ case): an observation :

$$(f+g)^{-1}(t,\infty)=\bigcup_{r\in\mathbb{R}}\left(f^{-1}(t-r,\infty)\cap g^{-1}(r,\infty)\right),$$

which should be prove by $A\subseteq B\wedge B\subseteq A \implies A=B$. Right hand side is union of intersection of two open sets, and similarly for $(f+g)^{-1}(-\infty,t)$. We're done.

4. By the continuity of addition($\mathbb{K}=\mathbb{R}$ case): We decompose $f+g$ as following communicative diagrams

$$\begin{tikzcd}
{[a,b]} \arrow[r, "\langle f,g\rangle"] \arrow[dr, "f+g"'] & \mathbb{R}\times\mathbb{R} \arrow[d, "+"] \\
& \mathbb{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dr, mapsto] & (f(x),g(x)) \arrow[d, mapsto] \\
& f(x)+g(x)
\end{tikzcd}$$

The right diagram explains what the functions in the left diagram mean. By the property of product topology and continuity of f and g, we know $f\times g$ is continuous. Continuity of $+\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is trivial. Therefore $f+g=+\circ(f\times g)$ is continuous.

To get rid of the assumption $\mathbb{K}=\mathbb{R}$, use the fact that $f\colon X\to\mathbb{C}$ is continuous if and only if both $\operatorname{Re}(f),\operatorname{Im}(f)$ are continuous.

Example 3 (p-summable sequence spaces)**.** Given $p\in[1,\infty]$ we define $(\ell_p,\|\ \|_p)$, where

$$\begin{aligned}
\ell_p &:=\Big\{(a_n)_{n\in\mathbb{N}}:\sum_{n\geq 1}|a_n|^p<\infty\Big\}, \text{(for } p<\infty)\\
\ell_\infty &:=\Big\{(a_n)_{n\in\mathbb{N}}:\sup_{n\in\mathbb{N}}|a_n|<\infty\Big\}.
\end{aligned}$$

And norms are

$$\|a\|_p:=\Big(\sum_{n\geq 1}|a_n|^p\Big)^{1/p}, \qquad \text{(for } p<\infty)$$

$$\|a\|_\infty:=\sup_{n\in\mathbb{N}}|a_n|. \qquad \text{(Here } a \text{ means } (a_n)_{n\in\mathbb{N}})$$

Proposition 1.1. $(\ell_\infty,\|\ \|_\infty)$ is a normed space.

Proof. Clearly ℓ_∞ is a vector space. Now we prove $\|\ \|_\infty$ is a norm.

1. $\|a\|_\infty \geq 0$ and $\|a\|_\infty = 0 \iff a = 0$: $\|a\|_\infty \geq 0$ is trivial. Suppose $\|a\|_\infty = 0$, that is $\sup_{n\in\mathbb{N}}|a_n| = 0$. By definition of supremum, $|a_n| \leq 0 (\forall n \in \mathbb{N})$. Therefore, $a = 0$.

2. $\forall k \in \mathbb{K}$, by property of absolute value we know $\|ka\|_\infty = |k|\|a\|_\infty$.

3. Let $a, b \in \ell_\infty$ and $M_a = \|a\|_\infty, M_b = \|b\|_\infty$. Now from definition of supremum

   $$\forall n \in \mathbb{N} : |a_n + b_n| \leq |a_n| + |b_n| \leq M_a + M_b$$

   Again using definition of supremum, we get $\|a + b\|_\infty \leq M_a + M_b$, which was what we wanted.

□

Theorem 1.2 (Minkowski's Inequality)**.** For each measure space $(\Omega, \mathcal{F}, \mu)$ and $f, g \in \mathcal{L}_p (1 \leq p \leq \infty)$, we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Remark. In general, the inequality $\|f + g\|_p \leq \|f\|_p + \|g\|_p (p \geq 1)$ is called the Minkowski's inequality.

Example 4. ℓ_∞ has linear subspaces: $c_0 \subseteq c \subseteq \ell_\infty$, where

$$c := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N} : (x_n)_{n\in\mathbb{N}} \text{ is a convergent sequence}\},$$
$$c_0 := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N} : (x_n)_{n\in\mathbb{N}} \text{ is a convergent sequence, with limit } 0\}.$$

1.2 Week 1, Lecture 2

1.2.1 Lebesgue integrable function spaces

Recall the left problem: Minkowski's inequality, which makes $(\ell_p, \| \ \|_p)$ a normed space. Now, we need a lemma.

Lemma 1.3 (Hölder's Inequality). Let $a \in \ell^p, b \in \ell^q$ for $p \in (1, \infty)$ and $q \in (1, \infty)$ satisfy $1/p + 1/q = 1$, we have:

$$\|ab\|_1 \leq \|a\|_p \|b\|_q, \tag{1}$$

Remark. $q = p/(p-1)$ is also called the dual index of p, usually denoted by p'.

Remark. Before start of the proof, we have a look at (1). Recall what we have learned in mathematical analysis, and have a problem in mind: is there anything similar? That is Cauchy-Schwarz Inequality, since they coincide when $p = q = 2$. Now we have a direct goal.

Aim. Prove (1) by imitating the proof of Cauchy-Schwarz Inequality.

Now, recall all the proofs of Cauchy-Schwarz Inequality you know and think: Which would be useful in this case? [5] Lagerange's Idendity, Schwarz's argument(inner product $\langle x + ty, x + ty \rangle \geq 0$), or just $2xy \leq x^2 + y^2$? When $p \neq 2$, Schwarz's argument is a nonstarter since there is no quadratic polynomial in sight. Similarly, the absence of a quadratic form means that one is unlikely to find an effective analog of Lagrange's identity.

This brings us to our most robust proof of Cauchy-Schwarz Inequality, the one that starts with the so-called "humble bound,"

$$xy \leq \frac{x^2}{2} + \frac{y^2}{2}, \forall x, y \in \mathbb{R}. \tag{2}$$

(2) proves Cauchy's inequality as follows.

Proof of Cauchy's inequality from (2). Without lost of generality, suppose that $\sum_{n\geq 1} a_n^2 = A^2 \neq 0$ and $\sum_{n\geq 1} b_n^2 = B^2 \neq 0$. Let

$$a'_j = a_j/A, b'_j = b_j/B, \forall j \in \mathbb{N}.$$

Notice that $\sum_{n\geq 1} a'_n = \sum_{n\geq 1} b'_n = 1$. Now (2) implies

$$\sum_{n\geq 1} a_n b_n \leq \sum_{n\geq 1} (a_n^2 + b_n^2)/2 = \sum_{n\geq 1} a_n^2/2 + \sum_{n\geq 1} b_n^2/2 = 1.$$

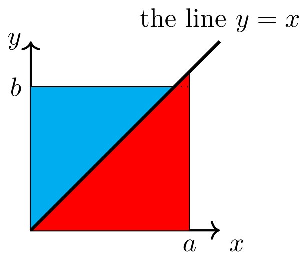

Figure 1: Area meaning of (2)

And, in terms of $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$, and multiply AB on both sides, we have

$$\sum_{n\geq 1} a_n b_n \leq \Big(\sum_{n\geq 1} a_n^2\Big)^{1/2}\Big(\sum_{n\geq 1} b_n^2\Big)^{1/2}. \qquad \square$$

This bound may now remind us that the general AM-GM inequality

$$x^p y^q \leq \frac{x}{p} + \frac{y}{q} \quad \text{for all } x, y \geq 0 \text{ and } q = p' (p, q > 1). \tag{3}$$

(3) is the perfect analog of the "humble boun"(2).

Proof of (2). There is many ways to to this, see[5]. We choose the way by area of regions. Consider the region under the function $x \mapsto x$:

$$\begin{aligned} A :=& \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq a\}, \\ B :=& \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq b\}. \end{aligned}$$

Then Figure 1 shows that $m(A) + m(B) \geq m([0, a] \times [0, b])$, where m denotes the Lebesgue measure on $\mathbb{R}^2$. $\square$

Now, by imitating the proof of (2), we need to get the x^p/p as area of some region under a function, so consider the function $x \mapsto x^{p-1}$.

Proof of (3). It's easy to verify that

$$m(A) = \int_{[0,a]} f \,\mathrm{d}m, m(B) = b^{\frac{p}{p-1}} - \int_{[0,b^{p/(p-1)}]} f \,\mathrm{d}m,$$

where m is the Lebesgue measure on $\mathbb{R}$. By simple calculation, we have $m(A) = a^p/p, m(B) = b^q/q$. Notice that $A \cup B$ contains $[0, a] \times [0, b]$, we're done. $\square$

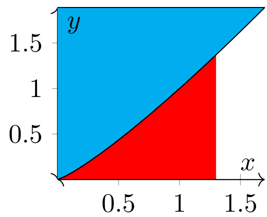

Figure 2: Area meaning of (3)

Proof of (1). Without loss of generality, suppose $\|a+b\|_p \neq 0$. And suppose $a \neq 0 (\in \ell_p), b \neq 0 (\in \ell_q)$. As what we do in the proof of Cauchy's inequality, let

$$a'_j = a_j / \|a\|_p, b'_j = b_j / \|b\|_p, \forall j \in \mathbb{N}.$$

Notice that $\|a'\|_p = \|b'\|_q = 1$. Now, apply (3) to $|a_j b_j|$, we have

$$\sum_{n\geq 1} |a'_n b'_n| \leq \sum_{n\geq 1} |a'_n|^p / p + \sum_{n\geq 1} |b'_n|^q / q = 1/p + 1/q = 1,$$

which implies

$$\|ab\|_1 \leq \|a\|_p \|b\|_p.$$

□

Proof of Minkowski's inequality.

$$\begin{aligned}
\|x+y\|_p^p =& \sum_{n\geq 1} |(x+y)_n|^p \\
=& \sum_{n\geq 1} |x_n + y_n|^{p-1} |x_n + y_n| \\
\leq& \sum_{n\geq 1} |x_n + y_n|^{p-1} (|x_n| + |y_n|) \text{(Triangle inequality on } \mathbb{R}) \\
=& \sum_{n\geq 1} |x_n + y_n|^{p-1} |x_n| + \sum_{n\geq 1} |x_n + y_n|^{p-1} |y_n| \\
=& \left\|(x+y)^{p-1} x\right\|_1 + \left\|(x+y)^{p-1} y\right\|_1 \text{(def of norm)} \\
\leq& \left\|(x+y)^{p-1}\right\|_q \|x\|_p + \left\|(x+y)^{p-1}\right\|_q \|y\|_p \text{(see (1)) } (*) \\
=& \|(x+y)\|_p^{p/q} (\|x\|_p + \|y\|_p) ((p-1)q = p),
\end{aligned}$$

and divide $\|x+y\|_p^{p/q} (\neq 0)$ from both sides, getting

$$\|x+y\|_p^{p-p/q} \leq \|x\|_p + \|y\|_p.$$

We're done, since $p - p/q = 1$. □

To summarize what we have done, we need the language of measure.

Definition (σ-algebra)**.** A **σ-algrbra** on a set Ω is a subset Ω, satisfying:

1. $\Omega, \varnothing \in \mathcal{F}$;
2. $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$;
3. $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F} \implies \bigcup_{n\geq 1} A_n \in \mathcal{F}$.

Definition (Measurable Space)**.** A **measurable space** is a double $(\Omega, \mathcal{F})$ where Ω is an aritrary set and $\mathcal{F}$ is a σ-algebra over Ω. Elements of $\mathcal{F}$ is called **measurable sets** of $(\Omega, \mathcal{F})$.

Definition (Measure, Measure space)**.** A **measure** is a σ-additive function from $\mathcal{F}$ to $[0, \infty]$. A triple $(\Omega, \mathcal{F}, \mu)$ is called a **measure space**, if $(\Omega, \mathcal{F})$ is a measurable space and μ is a measure.

Definition (Integral with respect to measure)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We have a glance at "how to define integral with respect to measure". For the detail, see [2].

Step 1: Define **integral** $\int$ for measurable simple nonnegative function:

$$\sum_{k=1}^{n} a_k \chi_{A_k} \longmapsto \sum_{k=1}^{n} a_k \mu(A_k).$$

Step 2: Define **integral** $\int$ for measurable nonnegative function:

$$f \longmapsto \sup\left\{ \int \varphi : \varphi \leq f, \varphi \text{ is nonnegative simple function}\right\}.$$

Step 3: Define **integral** $\int$ for measurable function:

$$f \longmapsto \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu,$$

where $f^+ = f\chi_{f^{-1}[0,\infty)}, f^- = -f\chi_{f^{-1}(-\infty,0]}$.

Definition (p-integrable space)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, then the p-integrable space over $\mathcal{L}_p(\Omega, \mathcal{F}, \mu)$ is defined as

$$\mathcal{L}_p(\Omega, \mathcal{F}, \mu) := \left\{ f \in \mathbb{K}^\Omega : f \text{ is measurable and } \int |f|^p \,\mathrm{d}\mu < \infty \right\}.$$

Fact. The proof of Minkowski's inequalityover ℓ_p actually proved the Minkowski's inequalityof **every** p-integrable space $\mathcal{L}_p(\Omega, \mathcal{F}, \mu)$.

To understand this fact, we should have another way to illustrate $\sum$. That is, $\sum$ is a kind of integral.

Definition (Counting Measure)**.** Given a measurable space $(\Omega, \mathcal{F})$. Define $\mu\colon \mathcal{F} \to [0, \infty], A \mapsto \sharp A$. Where $\sharp A = \infty$ if A is an infinite set, and $\sharp A = n$ if A has exactly n elements. μ is called the **counting measure** over $(\Omega, \mathcal{F})$.

Remark. It can be shown that,[1] for real sequence $(a_n)_{n\in\mathbb{N}}$ (equivalent to a function $a\colon \mathbb{N} \to \mathbb{R}$), we have

$$\sum_{n\geq 1} a_n = \int a \,\mathrm{d}\mu.$$

That's why we can respect $\sum$ as $\int$. And hence, the fact above is just regard $\sum$ as integral with respect to coungting measure, and the proof works for arbitrary measure space.

Remark. We can also prove Minkowski's inequalityof L^p by using the $L^{p'}$. Since

$$\|f\|_p = \sup\left\{\left|\int fg \,\mathrm{d}\mu\right| : g \in L^{p'}(\Omega, \mathcal{F}, \mu), \|g\|_{p'} \leq 1\right\}.$$

2 Week 2

2.1 Week 2, Lecture 1

2.1.1 Quotient Spaces

Let X be a vector space with a linear subspace X_0, denoted as $X_0 \hookrightarrow X$.

Definition (Coset). $\forall x \in X$, the coset of x (with respect to X_0), denoted as $[x]$ or $x + X_0$ is defined as

$$[x] = x + X_0 := \{x + y : y \in X_0\}.$$

Definition (Quotient Space). ${}^{X}\!/_{X_0} := \{[x] : x \in X\}$, called the quotient space of X (with respect to X_0).

We want ${}^{X}\!/_{X_0}$ to be a vector space, so we define operations as follows:

$$\oplus : {}^{X}\!/_{X_0} \times {}^{X}\!/_{X_0} \to {}^{X}\!/_{X_0}, ([x],[y]) \mapsto [x+y];$$
$$\odot : \mathbb{K} \times {}^{X}\!/_{X_0} \to {}^{X}\!/_{X_0}, ([x],k) \mapsto [kx].$$

Where $[x+y]$ means the addtion (and take the coset), and the $[kx]$ means the scalar multiplication of X (and take the coset). You should verify that the operations are well defined. For simplicity, we write $+, \cdot$ instead of $\oplus, \odot$.

Claim. $({}^{X}\!/_{X_0}, +, \cdot)$ is a vector space.

Question. Think this questions:

1. Clearly, the zero element in ${}^{X}\!/_{X_0}$ is $[0]$. But, $[0] =?$;
2. If $[x] \neq [y]$, what is $[x] \cap [y]$?
3. Show that $x \in [y] \iff x - y \in X_0$.

Answers are as follows:

1. $[0] = X_0$, from definition of coset.
2. $\varnothing$. Since (3) implies $[x] \cap [y] \neq \varnothing$ means $\exists z : z - x, z - y \in X_0$, therefore $x - y = (z - y) - (z - x) \in X_0$ since X_0 is a linear subspace. Now, $\forall a \in [x]$, from $a = x + w (w \in X_0)$, we have $a = y + \big(w + (x - y)\big)$ and $\big(w + (x - y)\big) \in X_0$ so $a \in [y]$. Above all, $[x] \subseteq [y]$. It is the same to know $[y] \subseteq [x]$.

3. Since

$$\begin{aligned} x \in [y] &\iff x = y + z \text{ for some } z \in X_0 \\ &\iff x - y = z (= 0 + z) \text{ for some } z \in X_0 \\ &\iff x - y \in [0] = X_0. \end{aligned}$$

Let's see a simple example:

Example 5. From previous example, $c_0 \hookrightarrow c \hookrightarrow \ell_\infty$. And we introduce a new notion:

Definition (Codimension)**.** Suppose X a vector space and $X_0 \hookrightarrow X$. Then the codimension of X_0, is $\operatorname{codim}_X X_0 := \dim {}^{X}\!/_{X_0}$. Also denoted by just $\operatorname{codim}(X_0)$ if there is no confusion.

Claim. $\operatorname{codim}_c c_0 = 1$.

Proof. Let $(1_n)_{n\in\mathbb{N}}$ be the sequence with all elements 1. We want to show that $\{(1_n)_{n\in\mathbb{N}}\}$ is a basis of ${}^{c}\!/_{c_0}$. Let $(x_n)_{n\in\mathbb{N}} \in c$, and suppose $\lim_n x_n = x \in \mathbb{K}$. We have $[(x_n)_{n\in\mathbb{N}}] = [x(1_n)_{n\in\mathbb{N}}]$, since $x(1_n)_{n\in\mathbb{N}}$ is just the sequence with all elements x, and clearly $\lim_n(x_n - x) = 0 \implies (x_n)_{n\in\mathbb{N}} - x(1_n)_{n\in\mathbb{N}} \in c_0$. That is, $[(x_n)_{n\in\mathbb{N}}] = [x(1_n)_{n\in\mathbb{N}}] = x[(1_n)_{n\in\mathbb{N}}]$. We're done. $\square$

Remark. There is an isomorphism from ${}^{c}\!/_{c_0}$ to $\mathbb{K}$, that is $[(x_n)_{n\in\mathbb{N}}] \mapsto \lim_n x_n$.

Example 6. Consider $X = \mathbb{R}^2$, $X_0 \hookrightarrow X$ with $\dim X_0 = 1$. It is easy to see that $\forall x \in \mathbb{R}$, the coset containing x is just translating X_0 such that $0 (\in X_0)$ is translated to x. And

$${}^{X}\!/_{X_0} = \{X_0\} \cup \{\text{all lines that are parallel to } X\}.$$

Now we want to define a norm on ${}^{X}\!/_{X_0}$. An intuitive norm is the distance between X_0 and the coset.

Definition (Norm on ${}^{X}\!/_{X_0}$)**.** Define

$$\| \ \| \colon {}^{X}\!/_{X_0} \to \mathbb{R}_{\geq 0}, [x] \mapsto \inf_{y \in X_0} \|x - y\|.$$

The norm in green color is the usual norm in $\mathbb{R}^2$, see previous example.

We should verify that $\| \ \|$ is actually a norm. That is

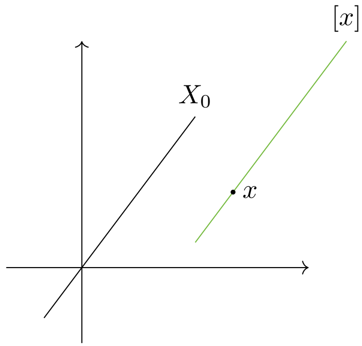

Figure 3: X, X_0 and points of ${}^{X}\!/\!_{X_0}$

Question. Verify that :

1. $\forall [x] \in {}^{X}\!/\!_{X_0} : \|[x]\| \geq 0$ and $\|x\| = 0 \iff x = X_0$;
2. $\forall [x] \in {}^{X}\!/\!_{X_0} : \|k[x]\| = |k| \cdot \|x\|$;
3. $\|[x] + [y]\| \leq \|[x]\| + \|[y]\|$.

Proof. For (1): Only needed is to show that $\|x\| = 0 \iff x = X_0$. Here we use a Theorem (in the below remark) and a trivial fact:

Fact. X_0 is a closed subset of X.

Now suppose $[x] \in {}^{X}\!/\!_{X_0}$ satisfying $\|[x]\| = 0$. By definition, we have $\inf_{y \in X_0} \|x - y\| = 0$. From the definition of infimum : $\forall n \in \mathbb{N} \exists y_n \in X_0$ such that $\|x - y_n\| < 1/n$, therefore we have a sequence $(y_n)_{n \in \mathbb{N}} \subseteq X_0$ converging to x. From the theorem below, we know $x \in X_0$, so $[x] = X_0$ as we wanted.

(2): It holds naturally when $k = 0$. If $k \neq 0$, it just follows from property of norm and $k^{-1} X_0 = X_0$.

(3): Intuitively , we have

$$
\begin{aligned}
\|[x] + [y]\| &= \|[x + y]\| \\
&= \inf_{z \in X_0} \|x + y - 2z\| \\
&\leq \inf_{z \in X_0} (\|x - z\| + \|y - z\|) \text{(triangle inequality of norm)} \\
&\leq \inf_{z \in X_0} \|x - z\| + \inf_{z \in X_0} \|y - z\| \\
&= \|[x]\| + \|[y]\|.
\end{aligned} \tag{4}
$$

So easy, isn't it? However, look at the $\leq$, this inequality is non-trivial and we should prove. By simple application of definition of infimum, we find: the inequality is **reversed**! But (4) can be corrected: $\forall \varepsilon > 0$, $\exists z_\varepsilon \in X_0, w_\varepsilon \in X_0$ such that

$$\|x - z_\varepsilon\| < \inf_{z \in X_0} \|x - z\| + \varepsilon/2 = \|x\| + \varepsilon/2,$$
$$\|y - w_\varepsilon\| < \inf_{z \in X_0} \|y - z\| + \varepsilon/2 = \|y\| + \varepsilon/2.$$

Therefore we have

$$\inf_{z \in X_0} (\|x - z\| + \|y - z\|) \leq \|x\| + \|y\| + \varepsilon.$$

Since ε is arbitrary, we know $\inf_{z \in X_0}(\|x - z\| + \|y - z\|) \leq \|x\| + \|y\|$ and then $\|x + y\| \leq \|x\| + \|y\|$.

However, this is wrong again. Since z_ε may not coincide with w_ε. To fix this, write

$$\|[x + y]\| = \inf_{z, w \in X_0} \|x + y - (z + w)\|. \tag{5}$$

By (5), and $\|x + y - (z + w)\| \leq \|x - z\| + \|y - w\|$, we use the definition of inf for $\inf_{z \in X_0} \|x - z\|, \inf_{w \in X_0} \|y - w\|$. We can find $z_\varepsilon, w_\varepsilon$ as above and get $\|[x + y]\| \leq \|[x]\| + \|[y]\| + \varepsilon$, we're done.

Above all, $\| \ \|$ is actually a norm. □

Remark. We define the topology of linear normed space as follows:

Definition (Topology of linear normed space)**.** Let $(X, \| \ \|)$ be a linear normed space. Then there is a natural metric on X, that is $d \colon X \times X \to \mathbb{R}_{\geq 0}, (x, y) \mapsto \|x - y\|$. The topology induced by this metric is called the (usual) topology of $(X, \| \ \|)$.

Now we have a topology of X, and we have a result characterizing the closed subsets of X.

Theorem. Given a linear normed space X with $X_0 \hookrightarrow X$. Then, X is closed **if and only if** for all $(x_n)_{n \in \mathbb{N}} \subseteq X_0$ such that $\lim_n x_n = x \in X$, we have $x \in X_0$.

Remark. A quotient semi-norm in ${}^{X}\!/_{X_0}$ is a norm if and only if X_0 is closed.

2.2 Week 2, Lecture 2

2.2.1 Metric Spaces

Definition (Metric, Metric Spaces)**.** Let X be a set. $d\colon X \times X \to \mathbb{R}$ is called a metric, if d satisfies:

1. $\forall x, y \in X\colon d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$.
2. $\forall x, y \in X\colon d(x, y) = d(y, x)$.
3. $\forall x, y, z \in X\colon d(x, y) + d(y, z) \geq d(x, z)$.

The ordered pair (X, d) is called a metric space.

Remark. Every metric space has a topology, we will discuss this later.

Remark. Let's compare normed spaces and metric spaces: normed space need linear structures but metric spaces don't need. A normed space $(X, \|\ \|)$ is naturally a metric space by the metric induced by norm $d\colon X \times X \to \mathbb{R}, (x, y) \mapsto \|x - y\|$.

Remark. Let X be an arbitrary set, we can define a metric on X by the Kronecker symbol δ.

Example 7. $(\mathbb{R}^n, d)$ is a metric space, where

$$d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) \mapsto \Big(\sum_{j=1}^{n} (x_j - y_j)^2\Big)^{1/2}.$$

Example 8. $(\mathbb{R}^{\mathbb{N}}, d)$ is a metric space, where

$$d\colon \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}, \big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\big) \mapsto \sum_{j\geq 1} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}.$$

d is well-defined, since the series can be dominated by $\sum_{j=1}^{\infty} 1/2^j$. To verify the triangle inequality, we use the monotone function $f\colon [0, \infty) \to [0, 1), x \mapsto x/(1 + x)$. So, $|x_j - y_j| + |y_j - z_j| \geq |x_j - z_j|$ implies

$$\frac{|x_j - y_j| + |y_j - z_j|}{1 + |x_j - y_j| + |y_j - z_j|} \geq \frac{|x_j - z_j|}{1 + |x_j - z_j|},$$

and clearly the left-hand side is no more than $f(|x_j - y_j|) + f(|y_j - z_j|)$. Sum for $j \in \mathbb{N}$ and we're done.

Example 9. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let $\mathcal{L}_0(\Omega)$ be the space of all $\mathcal{F}$-measurable functions from Ω to $\mathbb{K}$, written $\mathcal{L}_0$ for short. Define

$$\mathcal{Z} := \{f \in \mathcal{L}_0(\Omega) : f(x) = 0 \text{ for } \mu\text{-almost every } x \in \Omega\},$$

the (linear) subspace containing all functions equal 0 μ-almost everywhere. Now consider the quotient space $\mathcal{L}_0/_{\mathcal{Z}}$. We define

$$\begin{aligned} d\colon \mathcal{L}_0/_{\mathcal{Z}} \times \mathcal{L}_0/_{\mathcal{Z}} &\longrightarrow \mathbb{R} \\ (f+\mathcal{Z}, g+\mathcal{Z}) &\longmapsto \int_\Omega \frac{|f-g|}{1+|f-g|}\,\mathrm{d}\mu. \end{aligned} \tag{6}$$

Integrand on the right-hand side can be dominated by $1_\Omega (= 1)$, hence the integral is finite. The definition of d involves the selection of representative element, so we should verify that d is well-defined. Suppose $f + \mathcal{Z} = f' + \mathcal{Z}, g = g' + \mathcal{Z}$, and suppose f, g is finite everywhere, then

$$\begin{aligned} &\exists A_1 : \mu(A_1) = 0\ \forall x \in A_1^c\ f(x) = f'(x); \\ &\exists A_2 : \mu(A_2) = 0\ \forall x \in A_1^c\ g(x) = g'(x). \end{aligned} \tag{7}$$

Then $f(x)-g(x) = f'(x)-g'(x)$ for all $x \in (A_1 \cup A_2)^c$ and $\mu(A_1 \cup A_2) = 0$. Therefore $f - g = f' - g'$ μ-almost everywhere, and hence $\frac{|f-g|}{1+|f-g|} = \frac{|f'-g'|}{1+|f'-g'|}$ μ-almost everywhere, implying that their integration coincide. Above all, $d(f + \mathcal{Z}, g + \mathcal{Z}) = d(f' + \mathcal{Z}, g' + \mathcal{Z})$ whenever $f - f' \in \mathcal{Z}, g - g' \in \mathcal{Z}$.

Proof of "d is a metric" is the same as the previous example.

Example 10. These are all metric spaces, since they are linear normed spaces: $\ell_p, c_0, c, C([a, b], \mathbb{K}), L_p, \mathbb{R}^n$.

Definition (Convergence in metric space)**.** Let (X, d) be a metric space. A sequence in X, say $(x_n)_{n\in\mathbb{N}} \subseteq X$. We say $(x_n)_{n\in\mathbb{N}}$ is convergent to $x \in X$, if $\lim_n d(x_n, x) = 0$(limit of real sequence). $(x_n)_{n\in\mathbb{N}}$ is convergent to x is usually denoted by $(x_n)_{n\in\mathbb{N}} \xrightarrow{d} x$ or $(x_n)_{n\in\mathbb{N}} \to x$ if there is no ambiguity.

Example 11. Suppose X is an arbitrary set. (X, δ) is a metric space, where δ means the Kronecker symbol. Then

$$(x_n)_{n\in\mathbb{N}} \to x \iff \exists N \in \mathbb{N}\ \forall n \geq N\ x_n = x.$$

Example 12. Consider $\big((C[a, b], \mathbb{K}), d\big)$, where

$$d\colon (C[a, b], \mathbb{K}) \times (C[a, b], \mathbb{K}) \to \mathbb{R}, (f, g) \mapsto \max_{[a,b]}|f - g|.$$

Then $(f_n)_{n\in\mathbb{N}} \xrightarrow{d} f \iff (f_n)_{n\in\mathbb{N}}$ converge to f uniformly, as we learned in Mathematical Analysis.

Example 13. Recall $({}^{L_0}\!/_{\mathcal{Z}}, d)$, $(f+\mathcal{Z}_n)_{n\in\mathbb{N}} \xrightarrow{d} f+\mathcal{Z} \iff (f_n)_{n\in\mathbb{N}} \xrightarrow{\mu} f$.

Proof. Necessity:$(f+\mathcal{Z}_n)_{n\in\mathbb{N}} \xrightarrow{d} f+\mathcal{Z}$ means

$$\lim_n \int_\Omega \frac{|f_n - f|}{1+|f_n-f|}\,\mathrm{d}\mu = 0.$$

Given $\sigma > 0$. Define a set $E_n^\sigma := \{x\in\Omega : |f_n(x)-f(x)|>\sigma\}$, we need to show $\lim_n \mu(E_n^\sigma) = 0$. By Chebyshev's inequality:

$$\begin{aligned}
\mu(E_n^\sigma) =&\mu\{x\in\Omega : |f_n(x)-f(x)|>\sigma\}\\
=&\mu\Big\{x\in\Omega : \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} > \frac{\sigma}{1+\sigma}\Big\}\\
\le&\frac{1+\sigma}{\sigma}\int_{E_\sigma^n}\frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|}\,\mathrm{d}\mu\\
\le&\frac{1+\sigma}{\sigma}\int_{\Omega}\frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|}\,\mathrm{d}\mu\\
=&\frac{1+\sigma}{\sigma}d(f_n+\mathcal{Z}, f+\mathcal{Z}).
\end{aligned}$$

$\lim_n d(f_n+\mathcal{Z}, f+\mathcal{Z}) = 0$ implies $\lim_n \mu(E_n^\sigma) = 0$, that is $f_n \xrightarrow{\mu} f$.

Sufficiency: Given $\sigma \in (0,1)$, we know:

$$\Big\{x\in\Omega : \frac{|f_n-f|}{1+|f_n-f|} > \sigma\Big\} = \{x\in\Omega : |f_n-f| > \frac{\sigma}{1-\sigma}\}.$$

This implies that $\frac{|f_n-f|}{1+|f_n-f|} \xrightarrow{\mu} 0$.

Now, from the dominated convergence theorem (1_Ω being the dominated function), we have:

$$\begin{aligned}
\lim_n d(f_n+\mathcal{Z}, f+\mathcal{Z}) =& \lim_n \int_\Omega \frac{|f_n-f|}{1+|f_n-f|}\,\mathrm{d}\mu\\
=& \int_\Omega \lim_n \frac{|f_n-f|}{1+|f_n-f|}\,\mathrm{d}\mu\\
=& \,0.
\end{aligned}$$

□

Remark. I didn't get this solution for sufficiency at the class. So, it is meaningful to have a look after class.

Topology of metric spaces

Definition (Topology of metric space)**.** The topology of a metric space (X, d) is generated by the base

$$\mathcal{B} = \{B(x, r) \colon x \in X, r \in (0, \infty)\},$$

where $B(x, r) := \{y \in X \colon d(y, x) < r\}$.

Remark. Now we can define these things for metric spaces:

- Interior points of a set.
- Interior of sets.
- Limit points of a set.
- Derived sets.
- Closure.
- Isolated point.
- Boundary.

Fact. For a metric space (X, d):

1. A set G is open $\iff \forall x \in G \, \exists r > 0 \; B(x, r) \subseteq G$.

   Proof. Sufficiency is trivial. For necessity, since each open set is union of bases, then $x \in G$ must lie in a open ball contained in G, and we can find some $r > 0$ such that $B(x, r)$ is contained in the open ball. □

2. Intersection of open sets may not be open. For example,

$$\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}.$$

Definition (Continuity for Met)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces. We say $f \colon X \to Y$ is continuous at $x \in X$, if $\forall \varepsilon > 0 \, \exists r > 0$ such that $f\big(B(x, r)\big) \subseteq B\big(f(x), \varepsilon\big)$(two balls are in X and Y respectively). f is continuous if f is continuous at every $x \in X$.

Theorem (Continuity's equivalent conditions)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces. A map $f \colon X \to Y$ is continuous at x if and only if $\forall (x_n)_{n \in \mathbb{N}} \subseteq X (\lim_n x_n = x \implies \lim_n f(x_n) = f(x))$.

Proof. Suppose f is continuous at x and $(x_n)_{n\in\mathbb{N}} \to x$. $\forall\varepsilon > 0$, by continuity of f at x, $\exists r > 0$ such that $f\big(B(x,r)\big) \subseteq B\big(f(x),\varepsilon\big)$. For this $r > 0$, by convergence of $(x_n)_{n\in\mathbb{N}}$, $\exists N \in \mathbb{N}$ such that $\forall n > N\ x_n \in B(x,r)$ and hence $\forall n > N\ f(x_n) \in B\big(f(x),\varepsilon\big)$. Therefore, $\lim_n f(x_n) = f(x)$.

Suppose $\forall (x_n)_{n\in\mathbb{N}} \subseteq X(\lim_n x_n = x \implies \lim_n f(x_n) = f(x))$. If f is not continuous at x, by definition of continuity,

$$\exists\varepsilon_0 > 0\forall\delta > 0\exists y \in B(x,\delta) f(y) \notin B\big(f(x),\varepsilon_0\big).$$

In particular, take $\delta_n = 1/n$. Then there is $y_n \in B(x,1/n)$ and $f(y_n) \notin B\big(f(x),\varepsilon_0\big)$. Now we have a sequence $(y_n)_{n\in\mathbb{N}}$ converge to x but $\lim_n f(y_n) \neq x$, contradiction. Therefore, f must be continuous at x. $\square$

Definition (Continuity for Top)**.** Let $(X,\mathcal{T})$, $(Y,\mathcal{U})$ be two topological spaces. We say $f\colon X \to Y$ is continuous if $\forall O \in \mathcal{U}\ f^{-1}(O) \in \mathcal{T}$.

Theorem (Equivalence of definitions of continuity)**.** $f\colon (X,d) \to (Y,d)$ is continuous if and only if $f\colon (X,\mathcal{T}_{d_X}) \to (Y,\mathcal{T}_{d_Y})$ is continuous.

Remark. Here we mean $f\colon (X,d) \to (Y,d)$ is continuous, if it satisfies the definition of continuous maps between metric spaces. And "$f\colon (X,\mathcal{T}_{d_X}) \to (Y,\mathcal{T}_{d_Y})$ is continuous" means it satisfies the definition of continuous maps between topological spaces.

Proof. Suppose $f\colon (X,d) \to (Y,d)$ is continuous. Since $(Y,\mathcal{T}_{d_Y})$ has the topology base

$$\mathcal{B}_Y = \{B(y,r) : y \in Y, r \in (0,\infty)\},$$

it suffices to show that $\forall B(y,r) \in \mathcal{B}_Y$ we have $f^{-1}\big(B(y,r)\big) \in \mathcal{T}_{d_X}$. Suppose $f^{-1}\big(B(y,r)\big) \neq \varnothing$, else it's automatically open. Since $f(x_1) \in B(y,r)$, $\exists r_1 > 0$ such that $B(f(x_1),r_1) \subseteq B(y,r)$. Using the continuity of f at x_1, $\exists\delta > 0$ such that $f\big(B(x_1,\delta)\big) \subseteq B\big(f(x_1),r_1\big) \subseteq B(y,r)$. Therefore $B(x_1,\delta) \subseteq f^{-1}\big(B(y,r)\big)$. This means $f^{-1}\big(B(y,r)\big)$ contains a neighbourhood for each point of itself, and hence $f^{-1}\big(B(y,r)\big)$ is open.

Suppose $f\colon (X,\mathcal{T}_{d_X}) \to (Y,\mathcal{T}_{d_Y})$ is continuous. Then $\forall x \in X$, $f^{-1}\big(B(f(x),r)\big)$ is open for all $r > 0$. $x \in f^{-1}\Big(B\big(f(x),r\big)\Big)$ and $f^{-1}\big(B(f(x),r)\big)$ is union of sets like $B(x_0,\delta_0)$, so we can suppose $x \in B(x_0,\delta_0)$ for some $x_0 \in X, \delta_0 > 0$. Now choose $\delta > 0$ such that $B(x,\delta) \subseteq B(x_0,\delta_0)$ and we have

$$f\big(B(x,\delta)\big) \subseteq f\big(B(x_0,\delta_0)\big) \subseteq f\Big(f^{-1}\big(B(f(x),r)\big)\Big) \subseteq B(f(x),r).$$

We're done. $\square$

3 Week 3

3.1 Week 3, Lecture 1

Recall

Every linear normed space $(X, \| \ \|)$ has a metric (induced by its norm) $d\colon X \times X \to \mathbb{R}, (x, y) \mapsto \|x - y\|$. This is surely a metric, ensured by the properties of norm. However, a metric space (X, d) need not to be a linear normed space, since it is possible that X has no linear structure.

Now, suppose (X, d) a metric space, where X is a linear space. We have a question: is there some norm $\| \ \|$ such that d is induced from $\| \ \|$? If there is a norm that we want, it is clear that $\| \ \|\colon X \to \mathbb{R}, x \mapsto \|x\| := d(x, 0)$. We want $\| \ \|$ is a norm, so it should satisfy:

1. $\| \ \| \geq 0$ and $\|x\| = 0 \iff x = 0$. This holds, since d is a metric.
2. $\forall k \in \mathbb{K}, x \in X, d(kx, 0) = |k| d(x, 0)$. This should be satisfied.
3. $d(x, 0) + d(y, 0) \geq d(x + y, 0)$ as the triangle inequality.

Moreover, d should satisfy $d(x + z, y + z) = d(x, y)$, since $(x + z) - (y + z) = x - y$. In fact, the following conditions ensure that d is induced by a norm:

Condition 1. $d(kx, 0) = |k| d(x, 0)$.

Condition 2. d is translation-invariant, that is $d(x+z, y+z) = d(x, y)$.

Suppose d satisfies condition 1 and condition 2, then it is enough to show that $\| \ \|$ satisfies the triangle inequality.

Proof.

$$
\begin{aligned}
\|x + y\| &= d(x + y, 0) \\
&= d(x + y, -y + y) \\
&= d(x, -y) && \text{(condition 2)} \\
&\leq d(x, 0) + d(0, -y) && \text{(triangle inequality of } d\text{)} \\
&= d(x, 0) + d(-y, 0) && (d \text{ is symmetric}) \\
&= d(x, 0) + d(y, 0) && \text{(condition 1)} \\
&= \|x\| + \|y\|.
\end{aligned}
$$

We're done. □

Here comes an important notion of functional analysis.

3.1.1 Banach Space

Definition (Banach Space). A **complete** linear normed space $(X, \| \|)$ is called a **Banach Space**.

Here the word "complete" should be defined.

Definition (Completeness). A metric space (X, d) is complete if every Cauchy sequence in X converges.

Definition (Cauchy sequence). Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be a Cauchy sequence, if

$$\lim_{m,n} \|x_m - x_n\| = 0.$$

Remark. Here $\{\|x_m - x_n\|\}_{m,n\in\mathbb{N}}$ is a double index real sequence, and "the double index limit is 0 " should be interpreted as

$$\forall \varepsilon > 0 \exists M \in \mathbb{N} \exists N \in \mathbb{N} (\forall m > M \forall n > N \ |\|x_m - x_n\| - 0| < \varepsilon).$$

Warning. Convergent sequence must be Cauchy sequence (from definition), while Cauchy sequence may not converge (as the following examples).

Example 14. Let $d \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}, (x, y) \mapsto |x - y|$ be the normal metric on $\mathbb{R}$. Consider $(\mathbb{Q}, d|_{\mathbb{Q}\times\mathbb{Q}})$. This is not a complete metric space, since $\mathbb{Q}$ is dense in $\mathbb{R}$ and for arbitrary $x \in \mathbb{R}$ we can find a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x in $\mathbb{R}$. Consider $x \in \mathbb{R} \setminus \mathbb{Q}$ and we get a sequence in $\mathbb{Q}$, that is Cauchy in $(\mathbb{Q}, d|_{\mathbb{Q}\times\mathbb{Q}})$ and doesn't converge to any $r \in \mathbb{Q}$.

Example 15. Consider $\big(C[0, 1], \| \|_{L_1}\big)$, where $\| \|_{L_1}$ means the norm

$$\| \|_{L_1} : C[0, 1] \to \mathbb{R}, f \mapsto \int_{[0,1]} |f| \,\mathrm{d}m.$$

This is a norm, since $\|f\|_{L_1} = 0 \iff |f| = 0$ m-a.e, and continuity of f ensures $f = 0$. Other conditions for norm is trivial. And this is a incomplete normed vector space, since $C[0, 1]$ is dense (with respect to the norm $\| \|_{L_1}$) in L_1.

From now on, $C_p[a, b]$ means $(C[0, 1], \| \|_{L_p})$.

Remark. The completion (which will be defined the next class) of $C_p[a, b], 1 \le p < \infty$ is $L_p[a, b]$, since $C[a, b]$ is dense in $L_1[a, b]$

Example 16. Let $P[a, b] := \{\text{Polynomial functions defined on}[a, b]\}$, then the linear normed space $(P[a, b], \max_{[a,b]}| \,|)$ is incomplete. Since

$\exists f \in C[a,b]$ such that f is not a polynomial, such as $f = \exp|_{[a,b]}$. Suppose $\exp\colon \mathbb{R} \to \mathbb{R}$ is defined as the power series for convenience. Then by **Weierstrass Approximation Theorem**, for each fixed $\varepsilon > 0$, there is some $p \in P[a,b]$ such that $\max_{[a,b]} p - f < \varepsilon$.

In fact, for $f = \exp|_{[a,b]}$, it is enough to take

$$p_n\colon [a,b] \to \mathbb{R}, x \mapsto \sum_{j=1}^{n} \frac{x^j}{j!}.$$

By the result in power series theory, we know $p_n \xrightarrow{\max_{[a,b]}|\ |} f$.

Now we compare two normed spaces sharing the underlying set $C[a,b]$. $C[a,b]$ means the normed space $(C[a,b], \max_{[a,b]}|\ |)$ somewhere. And we will prove the completeness of $C[a,b]$.

Normed space	$C[a,b]$	$C_p[a,b]$
Underlying set	$C[a,b]$	$C[a,b]$
Norm	$\max_{[a,b]}\|\ \|$	$\\|\ \\|_p$
Completeness	complete	incomplete

Proof of completeness. Let $(f_n)_{n\in\mathbb{N}} \subseteq C[a,b]$ be a Cauchy sequence. That is

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \max_{[a,b]} |f_m - f_n| < \varepsilon.$$

Therefore, given any $x \in [a,b]$ we have

$$|f_m(x) - f_n(x)| \leq \max_{[a,b]} |f_m - f_n| < \varepsilon.$$

That is the sequence $\big(f_n(x)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of $\mathbb{R}$, $\big(f_n(x)\big)_{n\in\mathbb{N}}$ converge. Then we can define a function

$$f\colon [a,b] \to \mathbb{R}, x \mapsto \lim_n f_n(x).$$

$\lim_n f_n(x)$ is surely a real number, as explained above. And we have two claims.

Claim. $f_n \xrightarrow{\max_{[a,b]}|\ |} f$.

$\forall n > N$, we have

$$\max_{\in[a,b]} |f_m - f_n| < \varepsilon.$$

It's equivalent to

$$|f_m(x) - f_n(x)| < \varepsilon(\forall x \in [a, b]),$$

and let $m \to \infty$, using the continuity of $|\ |$ (to change the order of $\lim_m$ and $|\ |$)

$$|f(x) - f_n(x)| < \varepsilon(\forall x \in [a, b]),$$

which is equivalent to

$$\max_{\in[a,b]} |f - f_n| < \varepsilon.$$

Therefore, $f_n \xrightarrow{\max_{[a,b]}|\ |} f$.

Claim. $f \in C[a, b]$.

It suffices to show that f is uniformly continuous. Given arbitrary $\varepsilon > 0$, by the convergence of $(f_n)_{n\in\mathbb{N}}$

$$\exists N \forall n \geq N \max_{[a,b]} |f_n - f| < \varepsilon/3.$$

Fix this N, and the continuity (equivalent to uniform continuity for functions on $[a, b]$) of f_N ensures that $\exists \delta > 0$ such that

$$\forall x \forall y (|x - y| < \delta \implies |f_N(x) - f_N(y)| < \varepsilon/3).$$

And $\forall x \forall y$ such that $|x - y| < \delta$, we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &\leq \max_{[a,b]} |f_N - f| + \varepsilon/3 + \max_{[a,b]} |f_N - f| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

Thus f is uniformly continuous. □

Example 17. Suppose $1 \leq p \leq \infty$. then $L_p(\Omega, \mathcal{F}, \mu)$ is a Banach space.

Proof. First, suppose $1 \leq p < \infty$. Here is a proof different from our textbook. Suppose $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure (by Chebyshev's Inequality). By the lemma,

$\exists$ a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ such that $f_{n_j} \to f$ μ-a.e.. Therefore, by **Fatou's Lemma**:

$$\begin{aligned}
\lim_j \left\| f_{n_j} - f \right\|_p^p &= \lim_j \int_\Omega \left| f_{n_j} - f \right|^p \mathrm{d}\mu \\
&\leq \int_\Omega \liminf_j \left| f_{n_j} - f \right|^p \mathrm{d}\mu && \text{(Fatou's Lemma)} \\
&= 0. && (f_{n_j} \to f \ \mu\text{-a.e.})
\end{aligned}$$

While the inequality should be reversed. This can be corrected:

$$\begin{aligned}
\left\| f_{n_j} - f \right\|_p^p &= \int_\Omega \lim_n \left| f_{n_j} - f_n \right|^p \mathrm{d}\mu \\
&\leq \liminf_j \int_\Omega \left| f_{n_j} - f \right|^p \mathrm{d}\mu, && \text{(Fatou's Lemma)}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n_j} \left\| f_{n_j} - f \right\|_p^p &= \lim_{n_j} \int_\Omega \lim_n \left| f_{n_j} - f_n \right|^p \mathrm{d}\mu \\
&\leq \lim_{n_j} \liminf_n \int_\Omega \left| f_{n_j} - f_n \right|^p \mathrm{d}\mu && \text{(Fatou's Lemma)} \\
&= 0. && \text{(Cauchy sequence)}
\end{aligned}$$

So $f_{n_j} \xrightarrow{\| \ \|_{L_p}} f$. Minkowski's inequality shows

$$\|f_n - f\| \leq \left\| f_n - f_{n_j} \right\| + \left\| f - f_{n_j} \right\|.$$

Let $n_j, n \to \infty$ and use the fact that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in norm, we have $f_n \xrightarrow{\| \ \|_{L_p}} f$.

If $f \in L_p$, we are done. f is a μ- a.e. limit of $(f_{n_j})_{j\in\mathbb{N}}$ and hence is measurable. Minkowski's inequality shows

$$\|f\|_p \leq \left\| f - f_{n_j} \right\|_p + \left\| f_{n_j} \right\|_p.$$

The first term is bounded (since the real sequence has limit 0), and the second term is finite since $f_{n_j} \in L_p$.

Then, suppose $p = \infty$. There is $(A_{m,n})_{m,n\in\mathbb{N}} \in \mathcal{F}$ such that $\mu(A_{m,n}) = 0 \forall m, n \in \mathbb{N}$ and

$$\forall \omega \in A_{m,n}^c \ |f_m(\omega) - f_n(\omega)| \leq \|f_n - f_m\|_\infty.$$

Clearly for $A := \cup_{m,n\geq 1} A_{m,n}$, we have$\mu() = 0$. And we have

$$\forall \omega \in A^c |f_n(\omega) - f_m(\omega)| \leq \|f_n - f_m\|_\infty.$$

Let $m \to \infty$

$$\forall \omega \in A^c |f_n(\omega) - f(\omega)| \leq \lim_m \|f_n - f_m\|_\infty,$$

and hence

$$\|f_n - f\|_\infty \leq \lim_m \|f_n - f_m\|_\infty.$$

Let $n \to \infty$ and use

$$\lim_n \lim_m \|f_n - f_m\|_\infty = 0.$$

We're done. ☐

Lemma. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}_0(\Omega)$ is Cauchy in measure, where

$$\mathcal{L}_0(\Omega) := \{f \colon \Omega \to (\mathbb{K}, \mathcal{B}(\mathbb{K})) \text{ that is measurable}\}.$$

Then there is a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $g_n \to f$ $\mu-$a.e.. Here $f \in \mathcal{L}_0(\Omega)$.

Proof. We can choose a subsequence $(g_n)_{n\in\mathbb{N}} = \left(f_{n_j}\right)_{j\in\mathbb{N}}$ such that if $E_j := |g_j - g_{j+1}|^{-1}[2^{-j}, \infty)$ then $\mu(E_j) \leq 2^{-j}$. Because

$$\forall j \in \mathbb{N} \lim_{m,n\to\infty} |f_m - f_n|_* \mu[2^{-j}, \infty) = 0.$$

And pick n_j inductively, such that $n_{j+1} > n_j$ and

$$\mu_* |f_m - f_n| [2^{-j}, \infty) < 2^{-j} \; \forall m, n \geq n_j.$$

Set $F_k := \bigcup_{j\geq k} E_j$ then $\mu(F_k) \leq \sum_{j\geq k} 2^{-j} = 2^{1-k}$. Continuity from above is allowed! If $x \notin F_k$, for $i \geq j \geq k$ we have

$$|g_i(x) - g_j(x)| \leq \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq \sum_{l=j}^{i-1} 2^{-l} \leq 2^{1-j},$$

which ensures that $\forall x \in F_k^c$, $\left(g_j(x)\right)_{j\in\mathbb{N}}$ is a Cauchy sequence. Let

$$F = \bigcap_{j\geq 1} F_j = \limsup_j E_j,$$

we have $\mu(F) = \mu(\lim_j F_j) = \lim_j \mu(E_j) = 0$. ☐

Exercise. Prove that ℓ_p is complete when $1 \leq p < \infty$.

Suppose $(X, \| \ \|)$ is a linear normed space. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ satisfies $\sum_{n\geq 1}\|x_n\| < \infty$, and we can define the infinite sum for this sequence as

$$\sum_{n\geq 1} x_n := \lim_{n\to\infty} S_n, \text{ where } S\colon \mathbb{N} \to X, j \mapsto \sum_{j=1}^{N} x_j.$$

Theorem 3.1. $(X, \| \ \|)$ is a Banach space if and only if $\forall (x_n)_{n\in\mathbb{N}} \subseteq X$,

$$\sum_{n\geq 1}\|x_n\| < \infty \implies \sum_{n\geq 1} x_n < \infty.$$

Here $\sum_{n\geq 1} x_n < \infty$ means $\sum_{n\geq 1} x_n$ exists for short.

Proof. Necessity: suppose X is a Banach space, then $\sum_{n\geq 1}\|x_n\| < \infty$ implies

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \Big(\sum_{j=1}^{p} \|x_{n+j}\| < \varepsilon (\forall p \in \mathbb{N}) \Big),$$

and therefore $\forall n > N \|S_{n+p} - S_n\| \leq \sum_{j=1}^{p}\|x_{n+j}\| < \varepsilon$, this means that $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. X is complete, so $(S_n)_{n\in\mathbb{N}}$ converges. That is $\sum_{n\geq 1} x_n < \infty$.

Sufficiency: suppose X satisfies the condition above. If X is not complete, then $\exists (x_n)_{n\in\mathbb{N}} \subseteq X$ that is Cauchy but has no limit in X. Now, select a subsequence of $(x_n)_{n\in\mathbb{N}}$, say $\left(x_{n_j}\right)_{j\in\mathbb{N}}$ such that

$$\forall j \in \mathbb{N} \left\| x_{n_{j+1}} - x_{n_j} \right\| < 2^{-j}.$$

Define $y\colon \mathbb{N} \to X, j \mapsto x_{n_{j+1}} - x_{n_j}$, then $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, satisfying

$$\forall j \in \mathbb{N} \ \|y_j\| < 2^{-j}.$$

Therefore, $\sim_{n\geq 1} \|y_j\| < \infty$. Then X satisfies the condition, which implies that $\sum_{n\geq 1} y_n < \infty$. Equivalently, $\lim_j x_{n_j}$ exists in X. While $(x_n)_{n\in\mathbb{N}}$ is Cauchy, so $\lim_n x_n = \lim_j x_{n_j}$ exists, that's a contradiction (see how we selected $(x_n)_{n\in\mathbb{N}}$). $\square$

3.2 Week 3, Lecture 2

Recall

1. $L_p(\Omega)(1 \le p \le \infty)$ is complete. The outline of proof for $p < \infty$ is here:

   Step 1. Show that if $(f_n)_{n\in\mathbb{N}}$ is Cauchy (in norm), then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure.

   Step 2. Show that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, then $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ that converges to a measurable function f μ-a.e..

   Step 3. Use Fatou's lemma to show that $(f_{n_j})_{j\in\mathbb{N}} \xrightarrow{\|\ \|_p} f$.

   Step 4. Show that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\|\ \|_p} f$ and $f \in L_p$

2. About quotient space. Given a normed space $(X, \|\ \|)$ and a closed subspace $X_0 \hookrightarrow X$. We can define the quotient space

$$X/_{X_0} := \{[x] = x + X_0 : x \in X\},$$

   whose norm is

$$\|[x]\| = \inf_{y\in X_0} \|x - y\| = \inf_{y\in[x]} \|y(-0)\|.$$

   The second equality can be verified by change $y \in [x] \iff y = x + x_0, x_0 \in X_0$.

3. Norm and semi-norm (p, $p(x) = 0 \not\Rightarrow x = 0$). Let X be a linear semi-normed space, with the semi-norm p. A familiar linear semi-normed is $\mathcal{L}_p(1 \le p \le \infty)$. Let $X_0 := \{x \in X : p(x) = 0\} \hookrightarrow X$.

   Claim. X_0 is closed subspace of X (so, $X/_{X_0}$ is allowed, see this remark.

   Proof. X_0 is a linear subspace, since p is a semi-norm.

   p is a continuous map, since the triangle inequality holds. Then $N = p^{-1}(0)$ must be closed. □

Now, the remark ensures that $\|\ \|: X/_{X_0}, [x] \mapsto p(x)$ is a norm on $X/_{X_0}$.

Proof. It should be verified that p is well-defined (though this should have been proved in the remark). Suppose $[x] = [y]$, that is $[x-y] = [y-x] = [0]$. Since p is a semi-norm, we have the triangle inequality

$$p(x) + p(y-x) \geq p(y), p(y) + p(x-y) \geq p(x),$$

and $[x-y] = [y-x] = 0 \implies p(x-y) = p(y-x) = 0$, that is $p(x) = p(y)$. Thus, $[x] \mapsto p(x)$ is well-defined. And

(1) $\|[x]\| = 0 \iff p(x) = 0 \iff x \in X_0 = [0] \iff [x] = [0] \left(\in X\diagup X_0\right)$.

(2) $\|k[x]\| = \|[kx]\| = p(kx) = |k|p(x) = |k|\|x\|$.

(3) $\|[x] + [y]\| = \|[x+y]\| = p(x+y) \leq p(x) + p(y) = \|[x]\| + \|[y]\|$.

Above all, $\| \ \|$ is a norm on $[X]$. □

3.2.1 Completion

In this class, X is a linear noremd space, unless otherwise specified.

Definition (Isometry)**.** Suppose X, Y are two linear normed spaces. We say X is isomtetric with Y, if there is a linear surjection $T \colon X \to Y$ such that

$$\|Tx\| = \|x\| (\forall x \in X),$$

or equivalently $\| \ \|_Y \circ T = \| \ \|_X$.

Remark. Isometry is automatically injective, since $Tx = 0 \iff \|Tx\| = \|x\| = 0 \iff x = 0$. That is $\ker T = \{0\}$. Therefore, T is automatically injective and hence bijective as we want.

Definition (Density)**.** Let $(X, \| \ \|)$ be a liner normed space and $X_0 \hookrightarrow X$. X_0 is said to be dense in X, if $\overline{X_0} = X$.

Question. How to verify $\overline{X_0} = X$?

$\overline{X_0} = X$, if

$$\forall x \in X \forall \varepsilon > 0 \exists x_\varepsilon \in X_0 (\|x_\varepsilon - x\| < \varepsilon.)$$

And equivalently

$$\forall x \in X \forall n \in \mathbb{N} \exists x_n \in X_0 (\|x_\varepsilon - x\| < 1/n.)$$

That is, $\exists (x_n)_{n \in \mathbb{N}} \subseteq X_0$ that converges to x.

Theorem 3.2 (Completion thm)**.** Let $(X, \| \ \|)$ be a linear normed space. There is a Banach space $(\widehat{X}, \| \ \|)$ such that X is isometric to a dense subspace of $\widehat{X}$.

Remark. in fact, the completion $\widehat{X}$ is unique up to an isometry.

Definition (Completion)**.** $\widehat{X}$ (together with the isometric inclusive mapping) is called the completion of X.

Proof. We will construct a complection of X. Let

$$\mathcal{E} := \{(x_n)_{n\in\mathbb{N}} \subseteq X : (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence}\},$$

and define $p\colon \mathcal{E} \to \mathbb{R}, x (= (x_n)_{n\in\mathbb{N}}) \mapsto \lim_n \|x_n\|$. Here $\lim_n \|x_n\|$ exists in $\mathbb{R}$, because $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence implies that $= (\|x_n\|)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and $\mathbb{R}$ is complete. Moreover, p is a semimorn on $\mathcal{E}$. Now define $N := p^{-1}(0)$. Then $N \hookrightarrow \mathcal{E}$ and N is closed (by the continuity of p). Therefore we can consider $\widehat{X} := \mathcal{E}/N$, with the norm $\| \ \|\colon \widehat{X} \to \mathbb{R}, x + N \mapsto p(x)$.

Now, we prove this thm in 3 steps.

Step 1. X is isometric to a subspace of $\widehat{X}$. Let $X_0 := \{[(x)_{n\in\mathbb{N}}] : x \in X\}$ and

$$T\colon X \to X_0, x \mapsto [(x)_{n\in\mathbb{N}}] = (x)_{n\in\mathbb{N}} + N,$$

where $(x)_{n\in\mathbb{N}}$ means the constant sequence $(x, \ldots, x, \ldots)$. That is, $T(x) = (x, \ldots, x, \ldots) + N$. Clearly T is a linear surjection. We want to show T is isometric, that is $\forall x \in X, \|T(x)\| = \|x\|$. By definiton

$$\begin{aligned}
\|T(x)\| &= \|[(x)_{n\in\mathbb{N}}]\| && \text{(def of } T\text{)} \\
&= p\big((x)_{n\in\mathbb{N}}\big) && \text{(def of } \| \ \|_{\widehat{X}}\text{)} \\
&= \lim_n \|x\| && \text{(def of } p\text{)} \\
&= \|x\|.
\end{aligned}$$

To sum up, T is a linear isometric surjection as we want.

Step 2. $X_0 \hookrightarrow \widehat{X}$ is dense. As discussed above, it suffices to show that $\forall [x] = (x_1, \ldots, x_n, \ldots) + N \in \widehat{X}$, there is a sequence in X_0 converge to X. Let

$$[x]^{(m)}\colon \mathbb{N} \to [(x_m)_{n\in\mathbb{N}}] = (x_m, \ldots, x_m, \ldots) + N,$$

and we prove that the sequence $([x^{(m)}])_{m\in\mathbb{N}}$ is convergent to $[x]$.

$$\begin{aligned}
&\lim_m \left\| [x]^{(m)} - [x] \right\| \\
=\ &\lim_m \|(x_m - x_1, \dots, x_m - x_n, \dots) + N\| && \text{(def of } \pm\text{)} \\
=\ &\lim_m p\big((x_m - x_n)_{n\in\mathbb{N}}\big) && \text{(def of } \|\ \|\text{)} \\
=\ &\lim_m \lim_n \|x_m - x_n\| && \text{(def of } p\text{)} \\
=\ &0. && \text{(see remark)}
\end{aligned}$$

Step 3. $\widehat{X}$ is a Banach space. That is $\widehat{X}$ is complete. Let $([x]^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widehat{X}$. By the density of $X_0 = TX$, we have a sequence $(y_n)_{n\in\mathbb{N}} \subseteq X$ such that

$$\forall n \in \mathbb{N} \left\| T(y_n) - [x]^{(n)} \right\| \le 1/n.$$

Claim. $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned}
&\|y_m - y_n\| \\
=\ &\|T(y_m) - T(y_n)\| \\
\le\ &\left\| T(y_m) - [x]^{(m)} \right\| + \left\| [x]^{(m)} - [x]^{(n)} \right\| + \left\| T(y_n) - [x]^{(n)} \right\| \\
\le\ &1/m + \left\| [x]^{(m)} - [x]^{(n)} \right\| + 1/n.
\end{aligned}$$

Apply $\limsup_{m,n}$ on both sides and we have

$$\limsup_{m,n} \|y_m - y_n\| \le 0.$$

Therefore, $(y_n)_{n\in\mathbb{N}}$ is Cauchy, and $(y_n)_{n\in\mathbb{N}} \in \mathcal{E}$. Now we show that $([x]^{(n)})_{n\in\mathbb{N}} \to [y] = (y_1, \dots, y_n, \dots) + N$. By definition of $\|\ \|_{\widehat{X}}$

$$\begin{aligned}
\|[x]^m - [y]\| &\le \|[x]^m - T(y_m) + T(y_m) - [y]\| \\
&\le \|[x]^m - T(y_m)\| + \|T(y_m) - [y]\| \\
&\le 1/m + p\big((y_n - y_m)_{n\in\mathbb{N}}\big) \\
&= 1/m + \lim_n \|y_n - y_m\|,
\end{aligned}$$

and let $m \to \infty$, we have

$$\limsup_m \|[x]^m - [y]\| \leq \limsup_m 1/m + \limsup_m \lim_n \|y_n - y_m\|.$$

The second limit must be 0, since $\lim_m \lim_n \|y_n - y_m\| = 0$ (see remark) .

$\square$

Remark. Here we explain why $\lim_m \lim_n \|x_m - x_n\| = 0$. We may wan to write: suppose $\lim_n x_n = x$, then

$$\lim_m \lim_n \|x_m - x_n\| = \lim_m \|x_m - x\| = 0,$$

where the first equality is using the continuity of $\| \ \|$ and the second equality follows from the definition of $\lim_n x_n = x$. Everything makes sense, except $\lim_n x_n = x$. Notice that is a sequence in X and none said that X is complete.

So, why $\lim_m \lim_n \|x_m - x_n\| = 0$ holds? It suffices to show that we have

$$\lim_m \lim_n d(x_m, x_n) = \lim_{m,n} d(x_m, x_n) = 0.$$

whenever $(x_n)_{n\in\mathbb{N}}$ is Cauchy. See https://math.stackexchange.com/a/633595/1061247.

Theorem (Uniqueness of completion)**.** The completion of a linear normed space X is unique up to an unique isometry (that conincides with the two inclusions). That is, if $\widehat{X}, Y$ with isometric inclusion map ι, ι' respectively are completions of X, then the following diagram commutes

$$\begin{tikzcd} & X \arrow[dl, hook', "\iota"'] \arrow[dr, hook, "\iota'"] & \\ \widehat{X} \arrow[rr, dashed, "\exists\varphi"'] & & Y \end{tikzcd}$$

Proof. Consider the corestriction of ι, that is $\iota_0 := \iota|^{\iota(X)}$. Clearly ι_0 is an isometry from X to $\iota(X)$ $\Big($which is dense in $\widehat{X}\Big)$. Now we define a map φ_0 by the following diagram (i.e. $\varphi_0 := \iota' \circ (\iota_0)^{-1}$)

$$\begin{tikzcd} & X \arrow[dr, "\iota'"] & \\ \iota(X) \arrow[ur, "(\iota_0)^{-1}"] \arrow[rr, "\varphi_0"] & & Y \end{tikzcd}$$

Now φ_0 is linear and keeps norm. Since $\iota(X)$ is dense in $\widehat{X}$, Y is complete and φ_0 is uniformly continuous (φ_0 keeps norm and hence is uniformly continuous), we can extend φ_0 to a uniformly continuous map $\varphi\colon \widehat{X} \to Y$ (see the textbook, Thm 2.3.4).

To show that φ is an isometry, we should show that:

1. φ is linear;
2. φ keeps norm.
3. φ is surjective;

First, we prove that φ is linear. Since $\varphi|_{\iota(X)} = \varphi_0$ and $\iota(X)$ is dense in $\widehat{X}$, $\forall x, y \in \widehat{X}$ $\exists (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq \iota(X)$ such that $(x_n)_{n\in\mathbb{N}} \to x, (y_n)_{n\in\mathbb{N}} \to y$. Then $\forall k \in \mathbb{K}$, we have

$$\begin{aligned}
\varphi(kx+y) &= \varphi\big(\lim_n (kx_n + y_n)\big) \\
&= \lim_n \varphi(kx_n + y_n) && \text{(continuity of } \varphi) \\
&= \lim_n \varphi_0(kx_n + y_n) && \Big(\varphi|_{\iota(X)} = \varphi_0\Big) \\
&= \lim_n \big(k\varphi_0(x_n) + \varphi_0(y_n)\big) && (\varphi_0 \text{ is linear}) \\
&= k\lim_n \varphi_0(x_n) + \lim_n \varphi_0(y_n) && \Big(\lim_n \text{ is linear}\Big) \\
&= k\lim_n \varphi(x_n) + \lim_n \varphi(y_n) && \Big(\varphi|_{\iota(X)} = \varphi_0\Big) \\
&= k\varphi(x) + \varphi(y). && \text{(continuity of } \varphi)
\end{aligned}$$

Therefore, φ is linear.

Second, φ keeps norm. $\iota(X)$ is dense in $\widehat{X}$, so $\forall x \in \widehat{X}$, $\exists (x_n)_{n\in\mathbb{N}} \subseteq \iota(X)$ such that $(x_n)_{n\in\mathbb{N}} \to x$.

$$\begin{aligned}
\|x\| &= \Big\|\lim_n x_n\Big\| \\
&= \lim_n \|x_n\| && \text{(continuity of } \|\ \|) \\
&= \lim_n \|\varphi_0(x_n)\| && (\varphi_0 \text{ is an isometry}) \\
&= \lim_n \|\varphi(x_n)\| && \Big(\varphi|_{\iota(X)} = \varphi_0\Big) \\
&= \Big\|\lim_n \varphi(x_n)\Big\| && \text{(continuity of } \|\ \|) \\
&= \|\varphi(x)\| && \text{(continuity of } \varphi).
\end{aligned}$$

Thirdly, φ is surjective. $\forall y \in Y$, by the density of $\iota'(X)$, $\exists (y_n)_{n\in\mathbb{N}} \subseteq \iota'(X)$ such that $(y_n)_{n\in\mathbb{N}} \to y$. And $\forall n \in \mathbb{N}$, let $x_n := \varphi_0^{-1}(y_n)$ then

$(x_n)_{n\in\mathbb{N}} \subseteq \iota(X) \subseteq \widehat{X}$ is well-defined and Cauchy (since $(y_n)_{n\in\mathbb{N}}$ is Cauchy and φ keeps norm). Now

$$y = \lim_n y_n = \lim_n \varphi(x_n) = \varphi(\lim_n x_n) = \varphi(x).$$

The last equality used the completeness of $\widehat{X}$. Therefore, φ is surjective.

Above all, φ is an isometry. If there is another isometry $\phi\colon \widehat{X} \to Y$ such that the diagram commutes, then $\varphi|_{\iota(X)} = \phi|_{\iota(X)} = \varphi_0$. φ and ϕ conincide on a dense subset of $\widehat{X}$ and hence $\varphi = \phi$. $\square$

4 Week 4

4.1 Week 4, Lecture 1

Recall

No recall today.

4.1.1 Exercise course

We have only 3 exercises this course.

Question. Let $(X, \| \ \|)$ be a linear normed space, $X_0 \hookrightarrow X$. If X is complete and X_0 is closed then X_0 is complete.

Question. Let (X, d) be a metric space. $T\colon X \to X$ such that $\exists\lambda \in (0, 1)$

$$d(T(x), T(y)) \leq \lambda d(x, y), \forall x, y \in X.$$

Prove that $\exists! x_0 \in X$ such that $Tx_0 = x_0$.

Remark. This result doesn't hold when $\lambda = 1$. To see this, consider

$$(X, d) = ([0, \infty), d), T\colon X \to X, x \mapsto \sqrt{1 + x^2}.$$

And completeness is necessary too, consider $(X, d) = ((0, \infty), d)$ and $T\colon X \to X, x \mapsto x/2$. Other examples can be found.

Question. Let $(X, \| \ \|)$ be a linear normed space. Then X is a Banch space if and only if for each closed decreasing non-empty subsets sequence $(A_n)_{n\in\mathbb{N}}$, $\bigcap_{n\geq 1} A_n$ is a singleton set whenever $\lim_n \operatorname{diam}(A_n) = 0$.

There are answers in the next section.

4.2 Week 4, Lecture 2

Recall

For all l.n.s $(X, \| \ \|)$, there is a Banach space $\widehat{X}$ such that $X \cong X_0 \hookrightarrow \widehat{X}$, where X_0 is a dense subspace of $\widehat{X}$. It's ok to say $X = X_0 \hookrightarrow \widehat{X}$, and hence $\overline{X} = \widehat{X}$. The proof has 3 steps: construction of $\widehat{X}$, embedding X to $\widehat{X}$ and showing the completeness.

Remark. In the final exam and Phd qualifying exam, stating this theorem and its proof is common.

Review of exercise class

Here are the proofs of the questions of the exercise class.

Proof of the first. Suppose $(x_n)_{n\in\mathbb{N}} \subseteq X_0$ is a Cauchy sequence in X_0, then $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X. X is complete so $\exists x \in X$ such that $(x_n)_{n\in\mathbb{N}} \to x$. Now, X_0 is closed and hence $x \in X_0$. Thus, $(x_n)_{n\in\mathbb{N}} \to x \in X_0$. That is every Cauchy sequence in X_0 is convergent to some point $x \in X_0$, which is equivalent to X_0's completeness. □

Proof of the second. Let a be an arbitrary point in X. Define a sequence inductively:

$$(x_n)_{n\in\mathbb{N}} \colon \mathbb{N} \mapsto X, n \mapsto x_n := \begin{cases} a, & n = 1; \\ T(x_{n-1}), & n \geq 2. \end{cases}$$

Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy, because for all $n \geq 2$

$$d(x_{n+1}, x_n = d(T(x_n), T(x_{n-1}) \leq \lambda(x_n, x_{n-1}).$$

By induction, we have $d(x_{n+1}, x_n) \leq \lambda^{n-1} d(x_2, x_1)$, and hence

$$\sum_{n\geq 1} d(x_{n+1}, x_n) \leq \sum_{n\geq 1} \lambda^{n-1} d(x_2, x_1) = \frac{1}{1-\lambda} d(x_2, x_1) < \infty.$$

Therefore, the sequence $(S_n)_{n\in\mathbb{N}}$ is Cauchy, where

$$S \colon \mathbb{N} \to \mathbb{R}, n \mapsto S_n := \sum_{j=1}^{n} d(x_j, x_{j+1}).$$

The triangle inequality implies that

$$\forall m, n > 1 \big(S_{m\vee n} - S_{m\wedge n - 1} \geq d(x_m, x_n) \big),$$

which ensures that $(x_n)_{n\in\mathbb{N}}$ is Cauchy (let $S_0 = 0$ and then the inequality above always holds). By the completeness of X, $\exists! x_0 \in X$ such that $(x_n)_{n\in\mathbb{N}} \to x$. Now, the continuity (from $d(T(x), T(y)) \leq \lambda d(x, y)$) of T implies

$$T(x_0) = \lim_n T(x_n) = \lim_n x_{n+1} = x_0.$$

This proves the existence. Supppose there is $y \in X$ such that $T(y) = y$, then

$$d(y, x_0) = d(T(y), T(x_0)) \leq \lambda d(y, x_0).$$

$\lambda < 1$ implies that $d(y, x_0) = 0$. Equivalently, $x_0 = y$. This proves the uniqueness. □

Proof of the third. I think this proof is similar to the proof of [4, Chapter 5, Thm 2].

Necessity: suppose X is a Banach space. Given a closed decreasing non-empty subsets sequence $(A_n)_{n\in\mathbb{N}}$, choose $x_n \in A_n$ for each $n \in \mathbb{N}$. This is possible since $\forall n \in \mathbb{N}\ A_n \neq \varnothing$. Since $(A_n)_{n\in\mathbb{N}}$ is decreasing, we have

$$\forall m, n \in \mathbb{N}(x_m \in A_{m\wedge n}, x_n \in A_{m\wedge n}),$$

and hence

$$d(x_m, x_n) \leq \operatorname{diam} A_{m\wedge n} \to 0(m, n \to \infty).$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then the completeness of X ensures that $\exists a \in X$ such that $(x_n)_{n\in\mathbb{N}} \to a$. $\forall n \in \mathbb{N}$, since A_n is closed and $x_j \in A_n$ for all except for finite $j \in \mathbb{N}$, we have $a \in A_n$. Therefore, $a \in \bigcap_{n\geq 1} A_n$. Clearly $\bigcap_{n\geq 1} A_n$ cann't have more than 1 elements. If so, $\exists y \in A_n \forall n \in \mathbb{N}$ and hence $\operatorname{diam}(A_n) \geq d(x, y) \geq 0$. That's a contradiction.

Sufficiency: suppose X satisfies the condition above. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X, define $(A_n)_{n\in\mathbb{N}}$ as follows

$$\forall n \in \mathbb{N}, A_n := \{x_m \in X : m \geq n\}.$$

Then $(\overline{A}_n)_{n\in\mathbb{N}}$ satisfies the condition for set sequence: clearly $(\overline{A}_n)_{n\in\mathbb{N}}$ is decreasing, and $\operatorname{diam}(\overline{A_n}) = \operatorname{diam}(A) \to 0$ since $(x_n)_{n\in\mathbb{N}}$ is Cauchy. The reason of $\operatorname{diam}(\overline{A_n}) = \operatorname{diam}(A_n)$ is written in remark. Therefore, $\exists! a \in \bigcap_{n\geq 1} A_n$. Now, it suffices to show that $(x_n)_{n\in\mathbb{N}} \to a$. This follows from

$$d(x_n, a) \leq \operatorname{diam}(\overline{A_n}) \to 0(n \to \infty).$$ □

Remark. $\forall n \in \mathbb{N}$, we want to show that $\operatorname{diam}(\overline{A_n}) = \operatorname{diam}(A_n)$. Since n is fixed, we can omit the index. Given $A \subseteq X$ and $\varepsilon > 0$, $\forall x, y \in \overline{A}$, there is $x_\varepsilon, y_\varepsilon \in A$ such that

$$\|x - x_\varepsilon\| < \varepsilon/2, \|y - y_\varepsilon\| < \varepsilon/2.$$

Therefore

$$\|x - y\| \leq \|x - x_\varepsilon\| + \|x_\varepsilon - y_\varepsilon\| + \|y_\varepsilon - y\| \leq \|x_\varepsilon - y_\varepsilon\| + \varepsilon,$$

and use $\|x_\varepsilon - y_\varepsilon\| \leq \operatorname{diam}(A)$,

$$\|x - y\| \leq \operatorname{diam}(A) + \varepsilon.$$

Since $x, y \in \overline{A}$ are arbitrary, we have

$$\operatorname{diam}(\overline{A}) \leq \operatorname{diam}(A) + \varepsilon.$$

And ε is arbitrary, so

$$\operatorname{diam}(\overline{A}) \leq \operatorname{diam}(A).$$

The reversed inequality is trivial.

4.2.1 Banach fixed-point theorem

Here we introduce a classical result about Banach spaces.

Definition (Contraction mapping)**.** Given a metric space (X, d). Then a mapping $T\colon X \to X$ is called a contraction if $\exists \lambda \in (0, 1)$ such that $d(T(x), T(y)) \leq \lambda d(x, y)$.

Remark. Every linear normed space $(X, \| \ \|)$ has the natural metric $d(x, y) = \|x - y\|$ and hence a contraction on $(X, \| \ \|)$ means $T\colon X \to X$, such that $\exists \lambda \in (0, 1), \forall x, y \in X$

$$\|T(x) - T(y)(\neq T(x - y))\| \leq \lambda \|x - y\|.$$

The $\neq$ above means that T may not be a linear map.

It is easy to verify that each contraction is continuous.

Theorem 4.1 (Banach fixed-point theorem)**.** Suppose (X, d) is a complete metric space and T is a contraction on X. Then $\exists! x_0 \in X$ such that $Tx_0 = x_0$.

Proof. See the the proof of the second question. ☐

Let's have some applications. Suppose X is a Banach space and $U: X \to X$. We want to solve the equation $U(x) = y$.

Proof. To use 4.2.1, we should rewrite the equation $U(x) = y$ as $T(x) = x$ for some T.

$$U(x) = y \iff U(x) - y = 0 \iff U(x) + x - y = x,$$

thus consider $T: X \to X, x \mapsto U(x) + x - y$. And

$$\|T(u) - T(u)\| = \|U(u) + u - y - U(v) - v + y\|.$$

If it's verified that T is a contraction, then 4.2.1 (Banach fixed-point theorem) implies that T has a unique fixed-point, i.e. $U(x) = y$ has a unique solution. □

Example 18. X is a Banach space, on which U is a contraction. Prove that $U(x) = x + y$ has a unique solution.

Proof. We want solve $U(x) - x = y$, i.e. $(U - \mathrm{id})(x) - y = 0$. So the discussion above tells us that we should consider $T = U - \mathrm{id} + \mathrm{id} - y = U - y$. Let $x_1 \in X$ be an arbitrary point. Define $x_{n+1} = T(x_n) = U(x_n) - y$ for all $n \in \mathbb{N}$. Then T is a contraction since

$$\|T(a) - T(b)\| = \|U(a) - U(b)\|,$$

and U is a contraction. Then use Theorem 4.2.1 (Banach fixed-point theorem) and we're done. □

5 Week 5

5.1 Week 5, Lecture 1

In this part, X, Y are supposed to be two linear normed spaces $(X, \| \|)$, $(Y, \| \|)$.

Recall

A map $T\colon X \to Y$ is said to be continuous, if

$$\forall x \in X \forall (x_n)_{n\in\mathbb{N}} \xrightarrow{\| \|_X} x, (Tx_n)_{n\in\mathbb{N}} \xrightarrow{\| \|_Y} Tx.$$

5.1.1 Bounded linear operator/map

Here is the definition of Bounded linear operator/map

Definition (Bounded linear operator/map)**.** $T\colon X \to Y$ is said to be bounded, if $\exists C > 0$ such that $\| \|_Y \circ T \leq \| \|_X$, equivalently $\|Tx\|_Y \leq \|x\|_X$, $\forall x \in X$. The set of all bounded linear operators from X to Y is denoted as $\mathcal{B}(X, Y)$. If $Y = X$, $\mathcal{B}(X, X)$ is also written as $\mathcal{B}(X)$.

Remark. $\exists C > 0 : \|Tx\|_Y \leq \|x\|_X$, $\forall x \in X$ is **not** equivalent to $\forall x \in X \exists C > 0 : \|Tx\|_Y \leq \|x\|_X$.

Remark. Usually we don't distinguish map and operator, but a functional should be distinguished (see the definition of Bounded linear functional).

It is easy to verify: a bounded map is continuous. Then it's natural to consider the inverse proposition. To do this, we define bounded sets.

Definition (Bounded set)**.** Suppose $A \subseteq X$. If $\exists M > 0$ such that $\sup_{x\in A}\|x\| \leq M$, then A is said to be bounded.

Remark. T is a bounded map $\iff$ T maps bounded sets to bounded sets.

Proposition 5.1. The following statements are equivalent.

1. T is continuous;
2. T is continuous at some point $x_0 \in X$;
3. T is continuous at 0;
4. T is bounded.

Proof. We prove in the following order

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ \uparrow & & \downarrow \\ 4 & \longleftarrow & 3 \end{array}$$

1→2: is done automatically.

2→3: Suppose T is continuous at x_0, then $\forall (x_n)_{n\in\mathbb{N}} \to x_0$ we have $(Tx_n)_{n\in\mathbb{N}} \to Tx_0$. Now $\forall (y_n)_{n\in\mathbb{N}} \to 0$, we have $(y_n + x_0)_{n\in\mathbb{N}} \to x_0$ since

$$\|(y_n + x_0) - x_0\| = \|y_n\| \to 0(n \to \infty).$$

Thus, $T(y_n + x_0) \to T(x_0)$ by T 's continuity at x_0 and hence

$$\|T(y_n) - 0\| = \|T(y_n + x_0) - T(x_0)\| \to 0.$$

Therefore, $(Ty_n)_{n\in\mathbb{N}} \to 0$ as we wanted.

3→4: Given T that is continuous at 0. If T isn't bounded, then there is a bounded subset of X, denoted by A, such that TA is unbounded. **Replace** A with $\bigcup_{0\le t\le 1} tA$, still denoted by A. By the definition of unboundedness:

$$\forall n \in \mathbb{N} \exists x_n \in A : \|Tx_n\| > n.$$

Now we want a seuqnce $(y_n)_{n\in\mathbb{N}} \subseteq A$ satisfying $(\|Ty_n\|)_{n\in\mathbb{N}}$ is unbounded. Take $y_n = x_n/\sqrt{n}$, and we're done. Since $\{y_n : n \in \mathbb{N}\}$ is a bounded subset of A whose image under T is unbounded. That's a contradiction.

4→1: T is bounded, then T is uniformly continuous. □

Remark. There is another proof of $3 \to 4$, see the textbook.

Now, we have a set and it's naturally to consider it's linear structure and topology. There is a natural linear structure on $\mathcal{B}(X, Y)$ as follows

$$\begin{aligned} +\colon \mathcal{B}(X,Y) \times \mathcal{B}(X,Y) &\to \mathcal{B}(X,Y) \\ (S,T) &\mapsto S + T := (x \mapsto S(x) + T(x)), \end{aligned}$$

and

$$\begin{aligned} \cdot\colon \mathcal{B}(X,Y) \times \mathbb{K} &\to \mathcal{B}(X,Y) \\ (S,k) &\mapsto k \cdot S := (x \mapsto k \cdot S(x)). \end{aligned}$$

Definition (Operator norm)**.** The operator norm on $\mathcal{B}(X, Y)$ is defined as follows

$$\| \ \| \colon \mathcal{B}(X, Y) \to \mathbb{R}_{\geq 0}, T \mapsto \sup_{\|x\| \leq 1} \|Tx\|.$$

It's easy to verify that the operator norm is a norm on $\mathcal{B}(X, Y)$.

Remark. $(\mathcal{B}(X, Y), \| \ \|)$ is a linear normed space.

Remark. Equivalent definitions:

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

Proof. Since

$$\begin{aligned} \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} &= \sup_{x \neq 0} \left\| \frac{1}{\|x\|} Tx \right\| \\ &= \sup_{x \neq 0} \left\| T\left(\frac{x}{\|x\|} \right) \right\| \\ &= \sup_{\|x\|=1} \|Tx\| \\ &= \sup_{\|x\| \leq 1} \|Tx\| \\ &= \sup_{0 < \|y\| \leq \delta} \frac{1}{\delta} \|Ty\| (y = \delta x) \\ &\leq \sup_{0 < \|y\| \leq \delta} \frac{1}{\|y\|} \|Ty\| \\ &= \sup_{y \neq 0} \frac{\|Ty\|}{\|y\|}. \end{aligned}$$

And

$$\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{y \neq 0} \frac{\|Ty\|}{\|y\|},$$

which ensures that the $\leq$ above can be replaced with $=$. ☐

Definition (Bounded linear functional)**.** An element of $\mathcal{B}(X, \mathbb{K})$ is called a linear functional on X. $\mathcal{B}(X, \mathbb{K})$ is also called the dual space of X, denoted by X^*.

Remark. Discontinuous linear functionals exist (but only when X is infinite dimensional. See this post).

Example 19. Fix $a = (a_n)_{n\in\mathbb{N}} \in \ell_1$. Define

$$T\colon c_0 \to \ell_1, x = (x_n)_{n\in\mathbb{N}} \mapsto a \cdot x = (a_n x_n)_{n\in\mathbb{N}}. \tag{8}$$

Show that:

1) T is bounded;

2) $\|T\| = \|a\|_1$.

Proof. **1)** Recall that $c_0 \hookrightarrow \ell_\infty$ is equipped with the norm $\|\ \|_\infty = \sup_{\mathbb{N}}|\ |$. $\forall x \in c_0$, we have

$$\begin{aligned} \|Tx\|_1 &= \|a \cdot x\|_1 \\ &= \sum_{n\geq 1} |a_n x_n| \\ &\leq \sum_{n\geq 1} |a_n| \|x\|_\infty \\ &= \|a\|_1 \|x\|_\infty. \end{aligned}$$

Thus pick $C = \|a\|_1$, we have $\|Tx\|_{\ell_1} \leq C\|c\|_\infty$. This means $T \in \mathcal{B}(c_0, \ell_1)$.

2) We have proved $\|T\| \leq \|a\|_1$. Thus it suffices to show the reversed inequality. From the definition of $\|\ \|_1$

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : \sum_{j=1}^{n} |a_j| > \|a\|_1 - \varepsilon.$$

In particular

$$\sum_{j=1}^{N} |a_j| > \|a\|_1 - \varepsilon.$$

Now consider

$$c_0 \ni x_N := (\underbrace{1, \ldots, 1}_{N \text{ terms}}, 0, 0, \ldots),$$

whose image under T is

$$\|Tx_N\| = \sum_{j=1}^{N} |a_j| > \|a\|_1 - \varepsilon.$$

$\|x\|_\infty = 1$ ensures that

$$\|T\| \geq \|Tx_N\|_{\ell_1} > \|a\|_1 - \varepsilon.$$

$\varepsilon > 0$ is arbitrary, therefore $\|T\| \geq \|a\|_1$.

$\square$

Remark. In fact, $c_0^* \cong \ell_1$. Here $\cong$ means "isometrically isomorphic".

Here is a left exercise:

Exercise. Consider $X = C[0,1]$, with the norm $x \mapsto \max_{[0,1]}|x|$. Define the linear functional

$$f\colon X \to \mathbb{K}, x \mapsto \int_0^{1/2} x \,\mathrm{d}m - \int_{1/2}^1 x \,\mathrm{d}m.$$

Here m is the Lebesgue measure on $\mathbb{R}$. Show that:

1) f is a bounded linear functional (i.e. $f \in \big(C[0,1]\big)^*$);

2) $\|f\| = 1$.

5.2 Week 5, Lecture 2

Here is a remark for the previous exercise. We want to find $x \in C[0,1], \|x\| = 1, |f(x)| = 1$, i.e.

$$\int_0^{1/2} x \,\mathrm{d}m = 1/2, \int_{1/2}^1 x \,\mathrm{d}m = -1/2.$$

But this is impossible, by $\max_{[0,1]}|x| = 1$ and the continuity of x. Now, consider the approximation of x: $\forall \varepsilon \in (0, 1/2)$, let

$$x_\varepsilon \colon [0,1] \to \mathbb{R}, t \mapsto \begin{cases} 1, & t \in [0, 1/2 - \varepsilon] \\ l(t), & t \in (1/2 - \varepsilon, 1/2 + \varepsilon) \\ -1, & [1/2 + v\varepsilon, 1] \end{cases},$$

where l is the unique affine function determined by

$$l(1/2 - \varepsilon) = 1, l(1/2 + \varepsilon) = -1.$$

Since $|f(x_\varepsilon)| = 1 - \varepsilon$ and $|x_\varepsilon| = 1$, we have $\|f\| \geq 1 - \varepsilon$. Therefore, $\|f\| \geq 1$.

5.2.1 Some exercises

Here are exercises for this class.

Exercise. Given a measure space $(\Omega, \mathcal{F}, \mu)$ and $\alpha \in L_1(\Omega)$. Let

$$T_\alpha \colon L_\infty(\Omega) \to L_1(\Omega), x \mapsto \alpha \cdot x,$$

where $\alpha \cdot x$ means pointwise product. Try to find $\|T_\alpha\|$.

Solution: It's natural to guess that $\|T_\alpha\| = \|\alpha\|_1$. Hölder's inequality implies that

$$\|T_\alpha(x)\|_1 = \|\alpha \cdot x\|_1 \leq \|\alpha\|_1 \|x\|_\infty.$$

Thus, $\|T_\alpha\| \leq \|\alpha\|_1$. On the other hand,

$$L_\infty(\Omega) \ni x \colon \Omega \to \mathbb{K}, \omega \mapsto 1$$

then $\|x\|_\infty = 1$, and $T_\alpha(x) = \alpha$, hence $\|T_\alpha\| \geq \|\alpha\|_1$. □

Remark. We have proved this for μ being the counting measure, see the previous example.

Fact. A matrix (with respect to the normal base) $T \in \mathbb{K}^{n \times n}$ considered as a linear map $T \colon \mathbb{K}^n \to \mathbb{K}^n, x \mapsto Tx$ is bounded.

Proof. Since $\mathbb{K}$ is equipped with the norm $\| \|\colon x \mapsto (\sum_{j=1}^{n}|x_j|^2)^{1/2}$ that is not very convenient. It ban be proved that $\| \|_\infty \le \| \| \le \sqrt{n}\| \|_\infty$. So it suffices to show that $T\colon (\mathbb{K}^n, \| \|_\infty) \to (\mathbb{K}^n, \| \|_\infty)$ is continuous. Suppose $T = (a_{i,j})_{n\times n}$. Now $\forall x = (x_1, \dots, x_n)^t \in \mathbb{K}^n$

$$\begin{aligned}
\|Tx\|_\infty &= \left\| \Big(\sum_{j=1}^{n} a_{1,j}x_1, \dots, \sum_{j=1}^{n} a_{n,j}x_n\Big)^t \right\|_\infty \\
&\le \sum_{k=1}^{n} \left\| \Big(\sum_{j=1}^{n} a_{1,j}x_1\delta_{k,j}, \dots, \sum_{j=1}^{n} a_{n,j}x_n\delta_{k,j}\Big)^t \right\|_\infty \\
&\le \sum_{k=1}^{n}\sum_{j=1}^{n} \left\| \Big(a_{1,j}x_1\delta_{k,j}, \dots, a_{n,j}x_n\delta_{k,j}\Big)^t \right\|_\infty \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} |a_{k,j}| \cdot |x_k| \\
&\le \Big(\sum_{k=1}^{n}\sum_{j=1}^{n} |a_{k,j}|\Big) \|x\|_\infty .
\end{aligned} \tag{9}$$

Thus, let $C := \sum_{j=1}^{n}\sum_{k=1}^{n}|a_{k,j}|$ and we have proved $\| \|_\infty \circ T \le C\| \|_\infty$, i.e. T is bounded. □

Claim. Each finite dimensional linear normed space X is linear homeomorphic to $\mathbb{K}^n$.

Proof. Suppose $\mathbb{K}$ is equipped with $\| \|_\infty$ and $\{\alpha_1, \dots, \alpha_n\}$ is a base of X. Thus there is a map

$$\varphi\colon \mathbb{K}^n \to X, (x_1, \dots, x_n)^t \mapsto \sum_{j=1}^{n} x_j\alpha_j,$$

which is a bijection from definition of base. And φ is bounded, since

$$\begin{aligned}
\|\varphi(x_1, \dots, x_n)\|_X \le & \sum_{j=1}^{n} |x_j| \|\alpha_j\|_X \\
& \le \Big(\sum_{j=1}^{n} \|\alpha_j\|_X\Big) \|(x_1, \dots, x_n)\|_\infty .
\end{aligned} \tag{10}$$

Let $C := \sum_{j=1}^{n}\|\alpha_j\|_X$, then $\| \|_\infty \circ \varphi \le C\| \|_\infty$ and thus φ is bounded.

Now we prove that $\Phi := \varphi^{-1}$ is bounded. Given $(x_1, \ldots, x_n) \in \mathbb{K}^n$ such that

$$\left\| \sum_{j=1}^{n} x_j \alpha_j \right\|_X \leq 1,$$

i.e. an element in the unit ball of X. We prove that $\Phi(\sum_{j=1}^n x_j \alpha_j) = (x_1, \ldots, x_n)$ lies in some ball of $\mathbb{K}^n$. $\{\alpha_1, \ldots, \alpha_n\}$ is a base for X, thus $\alpha_j \neq 0 (\forall j)$ and let $\delta = \min_{1 \leq j \leq n} \|\alpha_j\| > 0$. Now

$$1 \geq \left\| \sum_{j=1}^{n} x_j \alpha_j \right\|_X \geq \sum_{j=1}^{n} |x_j| \|\alpha_j\| \geq \delta \sum_{j=1}^{n} |x_j| \geq \delta \left\| (x_1, \ldots, x_n)^t \right\|_\infty.$$

Therefore $\|(x_1, \ldots, x_n)^t\|_\infty \leq 1/\delta$, i.e. $\left\| \Phi(\sum_{j=1}^n x_j \alpha_j) \right\|_\infty \leq 1/\delta$. This means that Φ is bounded. $\square$

Exercise. Given a measure space $(\Omega, \mathcal{F}, \mu)$ and $\alpha \in L_\infty(\Omega)$. Let p be a real number fixed in $(1, \infty)$. Define

$$T_\alpha \colon L_p(\Omega) \to L_p(\Omega), x \mapsto \alpha \cdot x.$$

Try to find $\|T_\alpha\|$.

Proof. Let T denotes T_α for short. First, $\|T\| \leq \|\alpha\|_\infty$: since $|\alpha(\omega)| \leq \|\alpha\|_\infty$ for a.e. $\omega \in \Omega$, and

$$\|Tx\|_p = \Big(\int_\Omega |\alpha|^p |x|^p \,\mathrm{d}\mu \Big)^{1/p} \leq \|\alpha\|_\infty \Big(\int_\Omega |x|^p \,\mathrm{d}\mu \Big)^{1/p} = \|\alpha\|_\infty \|x\|_p.$$

The reversed inequality needs a lemma: Now, $\forall \varepsilon > 0$, consider the set $E_\varepsilon := \{\omega \in \Omega : |\alpha(\omega)| > \|\alpha\|_\infty - \varepsilon\}$.

Case 1: $\mu(E_{\varepsilon_1}) < \infty$ for some $\varepsilon_1 > 0$. Since $0 < a < b$ implies $E_a \subseteq E_b$, by considering $\varepsilon < \varepsilon_1$ we have $\mu(E_\varepsilon) < \infty$. Then $\chi_{E_\varepsilon} \in L_p$. And hence

$$\|T\| \geq \frac{\|T\chi_{E_\varepsilon}\|}{\|\chi_{E_\varepsilon}\|} \geq \frac{(\|\alpha\|_\infty - \varepsilon)\Big(\int_{E_\varepsilon} \chi_{E_\varepsilon}^p \,\mathrm{d}\mu \Big)^{1/p}}{\|\chi_{E_\varepsilon}\|} = \|\alpha\|_\infty - \varepsilon.$$

Since $\varepsilon \in (0, \varepsilon_1)$ is arbitrary, we have $\|T\| \leq \|\alpha\|_\infty$.

Case 2: $\mu(E_\varepsilon) = \infty$ for all $\varepsilon > 0$. If $\exists A_\varepsilon \subseteq E_\varepsilon$ such that $0 < \mu(A_\varepsilon) < \infty$ and hence $\chi_{A_\varepsilon} \in L_p$, then

$$\|T\| \geq \frac{\|T\chi_{A_\varepsilon}\|}{\|\chi_{A_\varepsilon}\|} \geq \frac{(\|\alpha\|_\infty - \varepsilon)\Big(\int_{A_\varepsilon} \chi_{A_\varepsilon}^p \,\mathrm{d}\mu \Big)^{1/p}}{\|\chi_{A_\varepsilon}\|} = \|\alpha\|_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\|T\| \leq \|\alpha\|_\infty$.

For the case that there is some $\varepsilon > 0$ such that $\mu(E_\varepsilon) = \infty$ and $\mu(A) \in \{\infty, 0\}$ for all $A \subseteq E_\varepsilon$, we can't prove that $\|T\| \geq \|\alpha\|_\infty$ and there is a example such that $\|T\| \neq \|\alpha\|_\infty$ in this case.

Example 20. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ where μ is defined as the unique measure such that

$$\nu(\{1\}) = \infty, \nu(A) = \operatorname{card}(A)(\forall 1 \notin A),$$

where $\operatorname{card}(A)$ is the number of elements of the set A when A is finite, and ∞ when A is infinite. Now the function $\alpha = \chi_{\{1\}} \in L_\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ and $\forall f \in L_p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ we have $f(1) = 0$. Therefore

$$T_\alpha \colon L_p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu) \to L_p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu), f \mapsto \alpha \cdot f$$

is just a zero operator and hence $\|T_\alpha\| = 0 \neq \|\alpha\|_\infty$.

Therefore, the operator

$$T \colon L_\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu) \to \mathcal{B}\big(L_p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)\big)$$

has a nontrivial kernel $\ker T \ni \alpha \neq 0$. □

6 Week 6

6.1 Week 6, Lecture 1

6.1.1 Compactness, Relative Compactness and total boundedness

Definition (Open Cover)**.** Given a topological space $(X, \mathcal{T})$. $A \subseteq X$ is said to have an open cover $(O_i)_{i \in I}$ if

$$A \subseteq \bigcup_{i \in I} O_i.$$

Definition (Compact)**.** A topological space $(X, \mathcal{T})$ is said to be compact, if each open cover of X has a finite subcover.

Remark. Compactness is topological invariant.

Definition (Relatively Compact)**.** Let $(X, \mathcal{T})$ be a topological space. A subset F of $(X, \mathcal{T})$ is said to be relatively compact, if its closure $\overline{F}$ is compact.

Definition (Sequentially Compact)**.** Let $(X, \mathcal{T})$ be a topological space. A subset F of $(X, \mathcal{T})$ is said to be sequentially compact, if every sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$ there is a subsequence $(x_{n_k})_{k \in \mathbb{N}} \subseteq (x_n)_{n \in \mathbb{N}}$ such that $(x_{n_k})_{k \in \mathbb{N}} \to x \in A$.

Definition (ε-net)**.** Let (X, d) be a metric space. $E \subseteq X$ is called an ε-net of A, if $A \subseteq \bigcup_{x \in E} B(x, \varepsilon)$.

Definition (Totally bounded)**.** Let (X, d) be a metric space. $A \subseteq X$ is said to be totally bounded, if $\forall \varepsilon > 0$ there is a finite ε-net of A.

Remark. This is a not topological invariant (since it needs a metric), but is invariant under bi-Lipschitz mappings.

Now, we will compare the following notions in metric space: compact sets, relatively compact sets and totally bounded sets.

Theorem 6.1. Let (X, d) be a metric space and $A \subseteq X$. The following statements are equivalent:

1. A is compact.

2. A is sequentially compact.

Proof. 1 $\Rightarrow$ 2: Suppose A is compact while not sequentially compact. Then $\exists (x_n)_{n\in\mathbb{N}} \subseteq A$ such that $\forall a \in A$, a is not a limit point of $(x_n)_{n\in\mathbb{N}}$. Thus

$$\forall a \in A \exists \varepsilon_a > 0 (\exists N_a \in \mathbb{N}\ \forall n \geq N_a\ d(x_{N_a}, a) > \varepsilon_a).$$

Now we have an open cover of A, $\{B(a, \varepsilon_a) : a \in A\}$. Since A is compact, there is $a_1, \ldots, a_m \in A$ such that

$$A \subseteq \bigcup_{k=1}^{n} B(a_k, \varepsilon_{a_k}),$$

Let $N := N_{a_1} \vee \cdots N_{a_m}$ then $x_N \notin B(a_k, \varepsilon_{a_k}) \forall 1 \leq k \leq m$. But $x_N \in A = \bigcup_{k=1}^{m} B(x_k, \varepsilon_{a_k})$. That's a contradiction.

2 $\Rightarrow$ 1: Let $(O_i)_{i\in I}$ be an open covering of A. First, we prove that $\exists \lambda > 0$ such that $\forall 0 < r < \lambda \forall x \in A$, $B(x, r) \subseteq O_i$ for some $i \in I$ (This constant λ is called an Lebesgue number of the open covering $(O_i)_{i\in I}$).

If there is no Lebesgue number for $(O_i)_{i\in I}$, then $\forall n \in \mathbb{N} \exists x_n \in A$ such that $B(x_n, 1/n)$ is not contained in any element of $(O_i)_{i\in I}$. Therefore we have a sequence $(x_n)_{n\in\mathbb{N}}$. 2 ensures that $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with its limit x_0. Notice that $x_0 \in O_{i_0}$ for some $i_0 \in I$ and O_{i_0} is open, so $\exists r > 0$ such that $B(x_0, r) \subseteq O_{i_0}$. From the definition of convergence, $\exists K$ such that $\forall k \geq K\ d(x_{n_k}, x_0) < r/2$. WLOG, suppose $n_K > 2/r$. Now, $\forall y \in B(x_{n_K}, 1/n_K)$, we have

$$d(y, x_0) \leq d(y, x_{n_K}) + d(x_{n_K}, x_0) < \frac{1}{n_K} + \frac{r}{2} < r.$$

This means $B(x_{n_K}, 1/n_K) \subseteq B(x_0, r)$. Since $B(x_0, r) \subseteq O_{i_0}$, we get $B(x_{n_K}, 1/n_K) \subseteq O_{i_0}$. That's a contradiction with the selection of $(x_n)_{n\in\mathbb{N}}$. Therefore, there is a Lebesgue number.

Let λ be a Lebesgue number, whose existence is proved above. Then A has an open cover $\{B(x, \lambda/2) : x \in A\}$. Take arbitrary $x_1 \in A$. If $A \subseteq B(x_1, \lambda/2)$ we're done. Else, it's possible to take $x_2 \in A \setminus B(x_1, \lambda/2)$. Similarly we can take $x_3, \ldots, x_n, \ldots$ if possible. This process must end in finite steps, i.e. we can only get a finite sequence as above. If we get a infinite sequence $(x_n)_{n\in\mathbb{N}}$ as above, then

$$d(x_m, x_n) \geq \frac{\lambda}{2}, \forall m \neq n.$$

That's a contradiction since A is supposed to be sequentially compact. Suppose we get a sequence having only m terms and then

$$A \subseteq \bigcup_{k=1}^{n} B\Big(x_k, \frac{\lambda}{2}\Big).$$

Recall the selection of λ, x_k ensures that $B(x_k, \lambda/2)$ lies in an element of $(O_i)_{i\in I}$ for each k. Therefore $(O_i)_{i\in I}$ has a finite subcover. $\square$

Theorem 6.2. Let (X, d) be a metric space and $A \subseteq X$. The following statements are equivalent:

1. A is relatively compact.

2. $\forall (x_n)_{n\in\mathbb{N}} \subseteq A$, $\exists (x_{n_k})_{k\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}$ such that $(x_{n_k})_{k\in\mathbb{N}} \xrightarrow{d} x \in X$.

Remark. Notice that $(x_{n_k})_{k\in\mathbb{N}} \xrightarrow{d} x \in X$ but not $(x_{n_k})_{k\in\mathbb{N}} \xrightarrow{d} x \in A$.

Proof. We use Theorem 6.1 to prove this theorem.

1 $\Rightarrow$ 2: Suppose 1 holds, then $\overline{A}$ is compact, Theorem 6.1 implies $\overline{A}$ is sequentially compact and hence 2 holds.

2 $\Rightarrow$ 1: Suppose 2 holds, then clearly $x \in \overline{A}$. Now we want to prove that $\overline{A}$ is compact. 1 means that it suffices to show that $\overline{A}$ is sequentially compact. Given an arbitrary sequence $(x_n)_{n\in\mathbb{N}} \subseteq \overline{A}$, we want to show that there is a subsequence ${x_{n_k}}_{k\in\mathbb{N}} \to x$ for some $x \in X$. Since $(x_n)_{n\in\mathbb{N}} \subseteq \overline{A}$ doesn't mean that $(x_n)_{n\in\mathbb{N}} \subseteq A$, we should find a sequence $(y_n)_{n\in\mathbb{N}} \subseteq A$ such that ${x_{n_k}}_{k\in\mathbb{N}} \to x$ whenever ${y_{n_k}}_{k\in\mathbb{N}} \to x$. By the property of closure: we can define $(y_n)_{n\in\mathbb{N}} \subseteq A$ such that

$$\forall n \in \mathbb{N}, y_n := \begin{cases} x_n, & x_n \in A; \\ x'_n, & x_n \notin A, x'_n \in A, d(x'_n, x_n) < 1/n. \end{cases}$$

Now 2 implies that $\exists (y_{n_k})_{k\in\mathbb{N}}$ such that $(y_{n_k})_{k\in\mathbb{N}} \to x \in X$, and hence $(x_{n_k})_{k\in\mathbb{N}} \to x \in X$ as we want. $\square$

Theorem 6.3. Let (X, d) be a metric space and $A \subseteq X$. The following statements are equivalent:

1. A is totally bounded.

2. $\forall (x_n)_{n\in\mathbb{N}} \subseteq A$, $\exists (x_{n_k})_{k\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence.

Proof. 1 $\Rightarrow$ 2: proof given by our professor is omitted here and should be found in your notes. And the "another proof" is not very different from this.

"Another proof" of 1 $\Rightarrow$ 2: suppose A is totally bounded. Given an arbitrary sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$. WLOG, suppose $(x_n)_{n\in\mathbb{N}}$ has infinite distinct terms, else we're done. $\forall \varepsilon > 0$ there is a finite ε-net of A. Thus for each $k \in \mathbb{N}$ there is a finite ε-net F_k for A. Let $J_0 = \mathbb{N}$

and define $J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$ inductively as follows. Suppose J_k is defined. Since F_{k+1} is finite and J_k is infinite, for each $n \in J_k$ there is an element $p_{k+1} \in F_{k+1}$ such that the ball $B(p_{k+1}, 1/(k+1))$ contains infinite elements of $\{x_n : n \in J_k\}$. Let

$$J_{k+1} := \{n \in J_k : d(x_n, p_{k+1}) < 1/(k+1)\}.$$

Now, let $n_1 \in J_1$ be an arbitrary element. And inductively select $n_{k+1} \in J_{k+1}$ such that $n_{k+1} > n_k$. We have defined a subsequence $(x_{n_k})_{k\in\mathbb{N}}$. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $2/N < \varepsilon$ and hence $\forall j, k \geq N$ we have $d(x_{n_j}, p_N) < 1/n_j < 1/N$, $d(x_{n_k}, p_N) < 1/N$. Therefore

$$d(x_{n_k}, x_{n_j}) \leq d(x_{n_k}, p_N) + d(x_{n_j}, p_N) < 1/N + 1/N < \varepsilon$$

by the triangle inequality. Now $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ that is Cauchy.

2 $\Rightarrow$ 1: Suppose A satisfies 1. If A isn't totally bounded, then $\exists \varepsilon_0 > 0$ such that A has no finite ε_0-net. Thus pick an arbitrary point $x_1 \in \mathbb{N}$ and $X \setminus B(x_1, \varepsilon_0) \neq \varnothing$ (since A has no finite ε_0-net). Pick an arbitrary point $x_2 \in X \setminus B(x_1, \varepsilon_0)$ and pick x_3 similarly. We have defined a sequence $(x_n)_{n\in\mathbb{N}}$ inductively, satisfying

$$d(x_m, x_n) \geq \varepsilon_0 (\forall m \neq n),$$

which implies that $(x_n)_{n\in\mathbb{N}}$ has no Cauchy subsequence. That's a contradiction. Therefore, A must be totally bounded. □

Corollary 6.4. Let (X, d) be a metric space and $A \subseteq X$. Then

1. A is compact $\implies$ A is relatively compact $\implies$ A is totally bounded.

2. A is compact $\implies$ A is closed and bounded.

3. Suppose A is closed. Then A is compact $\iff$ A is relatively compact.

4. Suppose X is complete. Then A is relatively compact $\iff$ A is totally bounded.

5. X is compact $\iff$ X is complete and totally bounded.

6. $X = \mathbb{K}^n$, then A is bounded $\iff$ A is totally bounded $\iff$ A is relatively compact.

Proof. We imply some results from the **point set topology** course.

1: (X, d) is a metric space and hence a Hausdorff space. Compact sets in Hausdorff space is closed. Therefore $A = \overline{A}$ and $\overline{A}$ is compact, i.e. A is relatively compact. The definition of compactness ensures that A is totally bounded.

2: A is closed as talked above. To see that A is bounded, consider an arbitrary point $x_0 \in X$ and the open covering

$$\{B(x, r) : r > 0\}. \tag{11}$$

(11) is an open cover of A. Compactness of A means that there is a finite subcover of (11), which ensures that A is bounded.

3: Since $\overline{A} = A$.

4: A is totally bounded if and only if for all $(x_n)_{n\in\mathbb{N}} \subseteq A$, $(x_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence i.e. a convergent subsequence. Therefore, A is totally bounded if and only if $\overline{A}$ is sequentially compact i.e. $\overline{A}$ is compact.

5: Necessity follows from 1 and 6.3. Apply 4 for sufficiency.

6: Heine-Borel theorem [4, Chapter 5, Thm 14] implies this. □

Remark. The inverse proposition of 2 is **incorrect**. Consider $(\mathbb{R}, d_1)$ where d_1 is defined as

$$d_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto |\phi(x) - \phi(y)|,$$

where

$$\phi : \mathbb{R} \to (-1, 1), x \mapsto \frac{x}{1 + |x|}.$$

Then $(\mathbb{R}, d_1)$ has a closed and bounded subset that is not compact: $\mathbb{R}$, itself. But clearly $(\mathbb{R}, d_1)$ has the same topology as the usual topological space $\mathbb{R}$. Therefore $(\mathbb{R}, d_1)$ is not compact since the open covering $\{(n, -n) : n \in \mathbb{N}\}$ has no finite subcover.

In fact, $(\mathbb{R}, d_1) \cong \mathbb{R}$. Here $\cong$ means there is a homeomorphism. Thus "boundedness" is not topological invariant.

6.2 Week 6, Lecture 2

Recall

Let X, Y be two linear normed spaces.

- $T\colon X \to Y$ is said to be bounded/continuous, if $\exists C > 0$ such that $\| \ \|_Y \circ T \leq C \| \ \|_X$ (i.e. $\|T\| \leq C$).
- $X \cong Y$ means that X is isometric to Y, i.e. $\exists T\colon X \to Y$ such that T is linear, surjective and satisfies $\| \ \|_Y \circ T = \| \ \|_X$.

Definition (Isomorphism)**.** X is **isomorphic** to Y, if there is a linear surjection T and $C_1, C_2 > 0$ such that

$$C_1 \| \ \|_X \leq \| \ \|_Y \leq C_2 \| \ \|_X,$$

and this T is called an **isomorphism** from X to Y. X is isomorphic to Y is denoted by $X \simeq Y$.

Remark. In the category $\mathsf{Vect}_{\mathbb{K}}$, an isomorphism is a linear bijection and vice versa. In the category Nor: $\mathrm{Ob}(\mathsf{Nor})$ are normed spaces and $\mathrm{Mor}(\mathsf{Nor})$ are bounded linear maps. An isomorphism in Nor is a linear homeomorphism. In the category Nor_1: $\mathrm{Ob}(\mathsf{Nor}_1)$ are normed spaces and $\mathrm{Mor}(\mathsf{Nor}_1)$ are contraction operators. An isomorphism in Nor_1 is an isometry. In this notes, $X \cong Y$ means that X is isometric to Y and $X \simeq Y$ means that X is isomorphic to Y.

6.2.1 Finite Dimensional Linear Normed Spaces

Definition (Equivalent norms)**.** Let $(X, \| \ \|_1)$, $(X, \| \ \|_2) \in \mathrm{Ob}(\mathsf{Nor})$. We say $\| \ \|_1$ is equivalent to $\| \ \|_2$, if $\exists a, b > 0$ such that

$$a \| \ \|_2 \leq \| \ \|_1 \leq b \| \ \|_2.$$

Remark. $\sim$ is an equivalent relation between norms on X, as you should verify.

See the definition of Isomorphism and we get $\| \ \|_1 \sim \| \ \|_2$ if and only if

$$\mathrm{id}\colon (X, \| \ \|_2) \to (X, \| \ \|_1), x \mapsto x$$

is an isomorphism.

Example 21. Consider $(\mathbb{R}^n, \| \ \|_2)$ and $(\mathbb{R}^n, \| \ \|_\infty)$. Clearly

$$\| \ \|_\infty \leq \| \ \|_2 \sqrt{n} \| \ \|_\infty,$$

and hence $\| \ \|_2 \sim \| \ \|_\infty$.

Theorem 6.5 (Classification of finite dimensinal sapces)**.** Let $X \in \mathrm{Ob}(\mathsf{Nor})$ with $\dim(X) = n < \infty$, then $X \simeq \mathbb{K}^n$.

Proof. WLOG, suppose $\mathbb{K}^n$ is equipped with $\| \ \|_\infty$. Consider

$$\varphi\colon \mathbb{K}^n \to X, (x_1, \ldots, x_n) \mapsto \sum_{j=1}^{n} x_j \alpha_j,$$

where $\{\alpha_1, \ldots, \alpha_n\}$ is a base of X. φ is proved to be continuous because

$$\begin{aligned} \|\varphi(x_1, \ldots, x_n)\|_X &\le \sum_{j=1}^{n} |x_j| \|\alpha_j\| \\ &\le \sum_{j=1}^{n} \|(x_1, \ldots, x_n)\|_\infty \|\alpha_j\| \\ &= \le \Big(\sum_{j=1}^{n} \|\alpha_j\|\Big) \|(x_1, \ldots, x_n)\|_\infty. \end{aligned}$$

Then let

$$\Phi\colon \mathbb{K}^n \to \mathbb{R}, (x_1, \ldots, x_n) \mapsto \|\varphi(x_1, \ldots, x_n)\|_X.$$

Now $\Phi = \| \ \|_X \circ \varphi$ is continuous. Hence Φ obtains a minimal value on $S = \{x \in \mathbb{K}^n : \|x\|_\infty = 1\}$. Suppose $\delta = \min \Phi|_S$ (such δ exists, since S is compact). Then $\delta > 0$ since $\| \ \|_X$ is a norm and $0 \notin S$. Now we have $\forall 0 \neq (x_1, \ldots, x_n) \in \mathbb{K}^n$,

$$\begin{aligned} \Phi(x_1, \ldots, x_n) &= \|(x_1, \ldots, x_n)\|_\infty \Phi\Big(\frac{(x_1, \ldots, x_n)}{\|(x_1, \ldots, x_n)\|_\infty}\Big) \\ &\ge \delta \|(x_1, \ldots, x_n)\|_\infty, \end{aligned}$$

i.e.

$$\|\varphi(x_1, \ldots, x_n)\|_X \ge \delta \|(x_1, \ldots, x_n)\|_\infty. \tag{12}$$

(12) holds for $\forall (x_1, \ldots, x_n) \in \mathbb{K}^n$ and means that φ^{-1} is continuous. Above all, φ is a linear homeomorphism, i.e. an isomorphism. $\square$

Remark. Consider $\min \Phi|_S$ is natural, just like

$$\|T\| = \sup_{\|x\|_X = 1} \|Tx\|_Y.$$

Corollary 6.6. Let $(X, \| \ \|) \in \mathrm{Ob}(\mathsf{Nor})$.

1) $\dim X = n$ implies that X is complete.

2) X is an arbitrary linear normed space and $X_0 \hookrightarrow X$ such that $\dim(X_0) < \infty$. Then X_0 is closed.

3) $\dim(X) < \infty$ implies that $\mathcal{L}(X) = \mathcal{B}(X)$.

Theorem 6.5 implies that: if $\dim(X) < \infty$, then $A \subseteq X$ is compact if and only if A is closed and bounded. But it is not true for some (all, in fact, see Theorem 6.8) infinite dimensional normed spaces.

Example 22 (A closed bounded set that is not compact)**.** Consider ℓ_2 and its base $\{e_n : n \in \mathbb{N}\}$, where

$$e_n := (\underbrace{0, \ldots, 0}_{n-1 \text{ terms}}, 1, 0, \ldots), \forall n \in \mathbb{N}.$$

Proof. $B := \{e_n : n \in \mathbb{N}\}$ is what we want.

- B is closed: consider an arbitrary convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq B$, then there is some $m \in \mathbb{N}$ such that $x_n = e_m$ for all but finite many $n \in \mathbb{N}$, because $\|e_m - e_n\| = \sqrt{2}\delta_n^m$. Thus $(x_n)_{n\in\mathbb{N}} \to e_m \in B$.

- B is bounded: since $\operatorname{diam}(B) = \sqrt{2}$.

- B is not compact: since $(e_n)_{n\in\mathbb{N}} \subseteq B$ is a sequence having no convergent subsequence. Thus B is not sequentially compact and hence not compact. □

Lemma 6.7 (Riesz)**.** Let X be a linear normed space and $X_0 \hookrightarrow X, X_0 \neq X$ is a closed subspace. Then

$$\forall \varepsilon \in (0,1) \exists x_\varepsilon \in X (\|x_\varepsilon\| = 1 \wedge d(x_\varepsilon, X_0) > \varepsilon.)$$

Proof. Taking arbitrary $x' \in X \setminus X_0$, then $d(x', X_0) > 0$. Let $d = d(x', X_0)$, now $d/\varepsilon > d$ and hence

$$\exists \bar{x} \in X_0 \|\bar{x} - x'\| < d/\varepsilon.$$

Taking $x_\varepsilon := \frac{\bar{x}-x'}{\|\bar{x}-x'\|}$, then $\|x_\varepsilon\| = 1$ and $\forall x \in X_0$

$$\begin{aligned}
\|x_\varepsilon - x\| &= \left\| \frac{\bar{x} - x' - \|\bar{x} - x'\| x}{\|\bar{x} - x'\|} \right\| \\
&= \frac{1}{\|\bar{x} - x'\|} \|\bar{x} - x' - \|\bar{x} - x'\| x\| \\
&> \varepsilon.
\end{aligned}$$

The last inequality comes from $\|\bar{x} - x'\| < d/\varepsilon$ and $\|y - x'\| \geq d$, where $y = \bar{x} - \|\bar{x} - x'\| x \in X_0$. □

Theorem 6.8. Let X be a linear normed space and $\overline{B(0,1)}$ is its closed unit ball. The following statements are equivalent:

1. X is finite dimensional.
2. $\partial B(0,1)$ is compact.
3. $\overline{B(0,1)}$ is compact.
4. $\forall A \subseteq X$, A is closed and bounded if and only if A is compact.

Proof. We want to show that

$$\begin{array}{ccc} 4 & \Longleftarrow & 1 \\ \Downarrow & & \Uparrow \\ 3 & \Longrightarrow & 2 \end{array}$$

We get $1 \implies 4$ from Theorem 6.5, $4 \implies 3$ is trivial and $3 \implies 2$ since a closed subset of a compact set is compact.

It suffices to prove that $2 \implies 1$. Consider proof by contradiction. Suppose $\dim(X) = \infty$. Let $\forall x_1 \in X$ such that $x_1 \neq 0$. Consider the closed linear subspace $\operatorname{span}\{x_1\}$ (this is a closed linear subspace, see the third corollary of Theorem 6.5). From Lemma 6.7, there is $x_2 \in X \setminus \operatorname{span}\{x_1\}$ such that $\|x_2\| = 1$ and $d(x_2, \operatorname{span}\{x_1\}) > 1/2$. Then consider the closed linear subspace $\operatorname{span}\{x_1, x_2\}$ (that is closed by the same reason as $\operatorname{span}\{x_1\}$), $\operatorname{span}\{x_1, x_2\} \neq X$ and Lemma 6.7 implies that there is $x_3 \in X \setminus \operatorname{span}\{x_1, x_2\}$ such that $\|x_3\| = 1$ and $d(x_3, \operatorname{span}\{x_1, x_2\}) > 1$. Thus, We can define a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \partial B(0,1)$ inductively such that

$$\forall m \neq n, d(x_m, x_n) > 1/2.$$

Therefore, $\partial B(0,1)$ is not sequentially compact and hence not compact.

Above all, X is infinite dimensional implies that $\partial B(0,1)$ is not compact. Thus $2 \implies 1$. $\square$

Summary

We have proved that

1. $\dim X < \infty \implies X \simeq \mathbb{K}^n$ and hence:
   (a) X is complete
   (b) Every finite dimensional subspace of an arbitrary linear normed space is closed.

(c) $\mathcal{L}(X) = \mathcal{B}(X)$.

2. Riesz's Lemma $\implies$ Theorem 6.8 which gives equivalent descriptions of finite dimensions.

7 Week 7

7.1 Week 7, Lecture 1

7.1.1 Construct more linear normed spaces

Let $(X_i, \| \|_{X_i}) \in \mathrm{Ob}(\mathsf{Nor}), 1 \leq i \leq n$. Define

$$\bigotimes_{i=1}^{n} X_i := \prod_{i=1}^{n} X_i$$

with operations

$$k(x_1, \ldots, x_n) + l(y_1, \ldots, y_n) = (kx_1 + ly_1, \ldots, kx_n + ly_n).$$

$\forall p \in [1, \infty]$, define a norm on $X = \bigotimes_{i=1}^{n} X_i$

$$\| \|_X : X \to \mathbb{R}, (x_1, \ldots, x_n) \mapsto \left(\sum_{i=1}^{n} \|x_i\|_{X_i}^p \right)^{1/p}.$$

At the case of $p = \infty$, $\|x\|$ should be interpreted like $\| \|_\infty$. To see that $\| \|$ is a norm, it suffices to show that

1. $\| \|$ is positive definite;
2. $\| \|$ is homogeneous;
3. Triangle inequality holds. And this follows from the Minkowski's Inequality for ℓ_p, since

$$\begin{aligned} \|x + y\| &= \left(\sum_{i=1}^{n} \|x_i + y_i\|^p \right)^{1/p} \\ &\leq \left(\sum_{i=1}^{n} (\|x_i\| + \|y_i\|)^p \right)^{1/p} \\ &\leq \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} + \left(\sum_{i=1}^{n} \|y_i\|^p \right)^{1/p}. \end{aligned}$$

The first inequality comes from the triangle inequality of $\| \|_X$ and the second inequality comes from the Minkowski's Inequality:

$$\begin{aligned} &\|(\|x_1\|, \ldots, \|x_n\|, 0, \ldots)\|_{\ell_p} + \|(\|y_1\|, \ldots, \|y_n\|, 0, \ldots)\|_{\ell_p} \\ \geq\ &\|(\|x_1\| + \|y_1\|, \ldots, \|x_n\| + \|y_n\|, 0, \ldots)\|_{\ell_p}. \end{aligned}$$

Now, we have some questions

Question.

1) Let $p_i \colon X \to X_i, (x_1, \ldots, x_n) \mapsto x_i$ be the projection to the i- coordinate. Show that p_i is continuous and $\|p_i\| = 1$.

2) $X = \bigotimes_{i=1}^n X_i$ is complete $\iff$ all X_i is complete.

Proof. **1)** On the one hand: $\forall x = (x_1, \ldots, x_n) \in X$, we have $\|p_i(x)\| = \|x_i\|$ and

$$\|x\|_X = \Big(\sum_{i=1}^n \|x_i\|_{X_i}^p\Big)^{1/p} \geq \|x_i\|_{X_i}.$$

Thus $\|p_i\| \leq 1$. On the other hand: taking $x = (0, \ldots, x_i, \ldots, 0) \in X$ with $\alpha_i \neq 0 \in X_i$, we get $\|p_i(x)\|_{X_i} = \|x_i\|_{X_i} = \|x\|_X$ which implies $\|p_i\| \geq 1$.

2) **Sufficiency**: taking an arbitrary Cauchy sequence

$$(x_m)_{m\in\mathbb{N}} = \big((x_m^{(1)}, \ldots, x_m^n)\big)_{m\in\mathbb{N}}$$

in X. Then

$$\max_{1\leq i\leq n} \Big\|x_p^{(i)} - x_q^{(i)}\Big\|_{X_i} \leq \|x_p - x_q\| \to 0 (p, q \to \infty),$$

which means that $(x_m^{(i)})_{m\in\mathbb{N}}$ is Cauchy in X_i and hence converges to some $y^{(i)} \in X_i$. Then

$$\lim_m x_m = (y^{(1)}, \ldots, y^{(n)}) =: y \in X$$

Because

$$\lim_m \|x_m - y\|_X = \lim_m \Big(\sum_{i=1}^n \Big\|x_m^{(i)} - y^{(i)}\Big\|^p\Big)^{1/p} = 0.$$

Therefore X is complete.

Necessity: $\forall 1 \leq i \leq n$, X_i is isometric to $E_i \hookrightarrow X$, where $E_i := \{(0, \ldots, x_i, 0, \ldots) : x_i \in X_i\}$. The isometry is

$$\iota_i \colon X_i \to E_i, x \mapsto (0, \ldots, x, 0, \ldots).$$

E_i is closed, since

$$E_i = \bigcap_{j\neq i} p_j^{-1}(0)$$

is a finite intersection of closed sets. Thus E_i is complete since X is complete. And now, X_i is isometric to a Banach space. We're done. ∎

7.1.2 Unbounded linear functional

This proposition gives a description of unbounded linear functional.

Lemma 7.1. Let $X \in \mathrm{Ob}(\mathsf{Nor})$ and $f \in \mathcal{L}(X, \mathbb{K})$ is a unbounded linear functional. Then

$$f\big(B(0,r)\big) = \mathbb{K}, \forall r > 0.$$

Proof. Given an arbitrary $\alpha \in \mathbb{K}$, $\alpha \neq 0$ there is $x' \in B(0,r)$ such that $|f(x')| \geq |\alpha|$ (else, f maps a bounded set to a bounded set and hence f is bounded). Taking $x = \frac{\alpha}{f(x')}x'$, we're done since

$$f(x) = f\Big(\frac{\alpha}{f(x')}x'\Big) = \frac{\alpha}{f(x')}f(x') = \alpha$$

and $x \in B(0,r)$ since

$$\|x\| = \frac{|\alpha|}{|f(x')|}\|x'\| \leq \|x'\| < r.$$

□

By lemma 7.1, we have

Proposition 7.2. Suppose $f \in X^*$ and $f \neq 0$. The following statements are equivalent:

1) f is continuous;

2) $\ker f$ is closed.

Proof. **1)** $\Longrightarrow$ **2)** $\{0\}$ is closed in $\mathbb{K}$ and **2)** follows from the topological definition of continuity.

2) $\Longrightarrow$ **1)** $\ker f$ is closed and hence is not dense in X since $f \neq 0$. Therefore

$$\exists x_0 \in X \exists r > 0 (B(x_0, r) \cap \ker f = \varnothing.) \tag{13}$$

You can check (13) by denying the proposition "$\ker f$ is dense in X". If f is not continuous, then Lemma 7.1 ensures that $f\big(B(0,r)\big) = \mathbb{K}$. Thus, $\exists y \in B(0,r)$ such that $f(y) = -f(x_0)$. And now

$$f(y + x_0) = f(y) + f(x_0) = 0,$$

i.e. $y + x_0 \in \ker f$ while $y + x_0 \in B(x_0, r)$. This is a contradiction since $B(x_0, r) \cap \ker f = \varnothing$.

□

Exercise. Determine which of the following sets are closed

1) $M := \{x \in \ell_2 : \sum_{n\geq 1} x_n/\sqrt{n} = 0\}$;

2) $M := \{x \in \ell_2 : \sum_{n\geq 1} x_n/n = 0\}$.

Solution. In fact, **2)** is simpler.

1) We will not prove this. $f\colon \ell_2 \to \mathbb{K}$ is not well-defined. Since

$$x\colon \mathbb{N} \to \mathbb{K}, n \mapsto \begin{cases} 0 & n = 1 \\ \frac{1}{\sqrt{n}\log n} & n \geq 2 \end{cases}$$

lies in ℓ_2 while $f(x) \notin \mathbb{K}$.

The set in **1)** can be proved to be not closed by the theory of **Hilbert Space**. See this post.

2) Let

$$f\colon \ell_2 \to \mathbb{K}, x \mapsto \sum_{n=1}^{\infty} \frac{1}{n} x_n. \tag{14}$$

Clearly f is well-defined. Furthermore, $\forall x \in \ell_2$, we have

$$|f(x)| \leq \sum_{n\geq 1} \frac{1}{n}|x_n| \leq \Big(\sum_{n\geq 1} 1/n^2\Big)^{1/2} \|x\|_2$$

And hence $M = f^{-1}(0)$ is closed.

$\square$

Remark. We have $\|f\| = \pi/\sqrt{6}$ by taking

$$\ell_2 \ni x = (1, 1/2, \ldots, 1/n, \ldots),$$

since $\|x\|_{\ell_2} = \pi/\sqrt{6}$.

Remark. In fact,

$$\ell_p^* \cong \ell_q, \tag{15}$$

where $p \in [1, \infty)$ and $q = p/(p-1)$.

7.2 Week 7, Lecture 2

7.2.1 Theorems about Banach space

Here are some topics of this lecture:

1. Open mapping theorem, see Theorem 7.3;
2. Banach-Steinhaus Theorem, see Theorem 7.6;
3. Hahn-Banach Theorem, see Theorem 9.1 and Theorem 9.2.

To state Theorem 7.3 better, we need a topological notion:

Definition (Open mapping)**.** Let $(X, \mathcal{T}), (Y, \mathcal{T})$ be two topological spaces and $f : X \to Y$ be an arbitrary map (not continuous possibly). f is said to be an open mapping, if $\forall O \in \mathcal{T}_X, f(O) \in \mathcal{T}_Y$.

And then we have

Theorem 7.3 (Open mapping theorem)**.** Let X, Y be two Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is surjective then T is open.

Proof of Theorem 7.3 is delayed to next (maybe) course.

Theorem 7.4 (Boundedness of inverse mapping)**.** Let X, Y be two Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is bijective, then $T^{-1} \in \mathcal{B}(Y, X)$.

Proof. Theorem 7.3 implies that T is an open mapping and equivalently T^{-1} is continuous. $\square$

Theorem 7.4 implies

Corollary 7.5. Let $(X, \| \ \|_1), (X, \| \ \|_2)$ be two Banach spaces. If $\exists C > 0$ such that $\| \ \|_1 \leq C \| \ \|_2$, then $\| \ \|_1 \sim \| \ \|_2$.

Proof. Consider $\mathrm{id}_X : (X, \| \ \|_2) \to (X, \| \ \|_1)$. $\| \ \|_1 \leq C \| \ \|_2$ implies that id_X is continuous. Apply Theorem 7.4 to id_X and we get that id_X^{-1} is bounded. $\square$

Theorem 7.6 (Banach-Steinhaus)**.** Let $(X, \| \ \|_X)$ be a Banach space, $(Y, \| \ \|_Y) \in \mathrm{Ob}(\mathsf{Nor})$ and $\{T_\lambda\}_{\lambda \in \Gamma} \subseteq \mathcal{B}(X, Y)$. If

$$\forall x \in X, \sup_{\lambda \in \Gamma} \|T_\lambda x\|_Y < \infty,$$

then $\sup_{\lambda \in \Gamma} \|T_\lambda\| < \infty$.

The other name of this theorem is "the uniform boundedness principle".

Proof. Here is a proof using the Corollary above.

Let $\| \|_I$ be a new norm on X, defined as

$$\| \|_I \colon X \to \mathbb{R}_{\geq 0}, x \mapsto \|x\|_X + \sup_{\lambda \in \Gamma} \|T_\lambda x\|_Y.$$

It's easy to verify that $\| \|_I$ is actually a norm. Clearly $\mathrm{id}_X \colon (X, \| \|_I) \to (X, \| \|_X)$ is continuous. If $(X, \| \|_I$ is a Banach space, then Corollary can be applied and we're done. Now, taking an arbitrary Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subseteq (X, \| \|_I)$, i.e.

$$\lim_{m,n} \|x_n - x_m\|_I = 0. \tag{16}$$

And (16) is equivalent to

$$\lim_{m,n} \|x_n - x_m\|_X = 0, \tag{17}$$

$$\lim_{m,n} \sup_{\lambda \in \Gamma} \|T_\lambda x_n - T_\lambda x_m\|_Y = 0. \tag{18}$$

Since $(X, \| \|_X)$ is a Banach space, (17) implies that $(x_n)_{n\in\mathbb{N}} \xrightarrow{\| \|_X} x \in X$. Now we prove that $(x_n)_{n\in\mathbb{N}} \xrightarrow{\| \|_I} x \in X$ and it suffices to show that $\lim_n \sup_{\lambda\in\Gamma} \|T_\lambda x_n - T_\lambda x\|_Y = 0$. And this proof is similar to the proof of the completeness of $C[a, b]$.

To see this, from the definition of limit of double indexed sequence:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall m, n > N, \sup_{\lambda \in \Gamma} \|T_\lambda x_n - T_\lambda x_m\|_Y < \varepsilon.$$

The definition of sup implies that

$$\forall m, n > N, \|T_\lambda x_n - T_\lambda x_m\|_Y < \varepsilon (\forall \lambda \in \Gamma).$$

Let $m \to \infty$, the continuity of $\| \|_Y$ and T_λ (for each $\lambda \in \Gamma$) implying that

$$\forall n > N, \|T_\lambda x_n - T_\lambda x\|_Y \leq \varepsilon (\forall \lambda \in \Gamma).$$

Therefore,

$$\forall n > N, \sup \|T_\lambda x_n - T_\lambda x\|_Y \leq \varepsilon.$$

Equivalently,

$$\lim_n \sup \|T_\lambda x_n - T_\lambda x\| = 0,$$

which was what we wanted. $\square$

7.2.2 Baire category Theorem

Definition (G_δ-set, F_σ-set)**.** Let $(X, \mathcal{T}) \in \mathrm{Ob}(\mathsf{Top})$.

- A set of the form $\bigcap_{n=1}^{\infty} G_n$ is called a G_δ-set, where $(G_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}$.
- A set of the form $\bigcup_{n=1}^{\infty} F_n$ is called a G_δ-set, where $(F_n^c)_{n\in\mathbb{N}} \subseteq \mathcal{T}$.

Remark. Here "G" is German (Gebiet) and "F" is French (Fermé).

Definition (First category set)**.** Let $(X, \mathcal{T}) \in \mathrm{Ob}(\mathsf{Top})$. $A \subseteq X$ is called a **set of the first category**, if $A \subseteq B$ for some F_σ set B.

Definition (Second category set)**.** Let $(X, \mathcal{T}) \in \mathrm{Ob}(\mathsf{Top})$. $A \subseteq X$ is called a **set of the second category**, if A is not of the first category.

Definition (Baire space)**.** Let $(X, \mathcal{T}) \in \mathrm{Ob}(\mathsf{Top})$. $(X, \mathcal{T})$ is called a **Baire space**, if each countable intersection of dense open sets is dense in X.

Here is an equivalent definition of Baire space

Definition (Baire space)**.** Let $(X, \mathcal{T}) \in \mathrm{Ob}(\mathsf{Top})$. $(X, \mathcal{T})$ is called a **Baire space**, if each countable union of closed sets with empty interior has empty interior.

And now we can talk about Baire category Theorem.

Theorem 7.7 (Baire category Theorem)**.** If $(X, \mathcal{T})$ is a topological space whose topology $\mathcal{T}$ can be induced by a complete metric, then X is a Baire space.

Proof. Suppose (X, d) is the metric space whose topology induced by d is $\mathcal{T}$. Let $(O_n)_{n\in\mathbb{N}}$ be a sequence of dense open sets in (X, d). It suffices to show that $O = \bigcap_{n\geq 1} O_n$ is dense in X. Taking an arbitrary open set $\varnothing \neq U \in \mathcal{T}$, now we show that $O \cap U \neq \varnothing$.

Since O_1 is dense in X, we have $O_1 \cap U \neq \varnothing$ and thus $\exists x_1 \in O_1 \cap U$. Moreover, $\exists r > 0$ such that $B(x_1, r) \subseteq O_1 \cap U$ since $O_1 \cap U \in \mathcal{T}$. Let $F_1 := \overline{B(x_1, r/2)}$. Then $\mathring{F_1} \neq \varnothing$, $F_1 \subseteq O_1 \cap U$ and $\operatorname{diam} F_1 = r =: r_1$.

Since O_2 is dense in X and $\mathring{F_1} \neq \varnothing$, we have $O_2 \cap \mathring{F_1} \neq \varnothing$ and thus $\exists x_2 \in O_1 \cap U$. Moreover, $\exists r_2 > 0 \land r_2 < r_1/2$ such that $B(x_2, r_2) \subseteq O_2 \cap \mathring{F_1}$. Let $F_2 := \overline{B(x_2, r_2)}$. Then $\mathring{F_2} \neq \varnothing$, $F_2 \subseteq O_2 \cap U$ and $\operatorname{diam} F_2 < r_1/2$.

Analogically, we can define a sequence of decreasing closed sets $(F_n)_{n\in\mathbb{N}}$ such that $\mathring{F_n} \neq \varnothing (\forall n \in \mathbb{N})$, $F_n \subseteq O_n \cap U (\forall n \in \mathbb{N})$ and

$\operatorname{diam} F_n \leq 2^{1-n} r (\forall n \in \mathbb{N})$. Then the third question of Week 4, Lecture 1 implies that $\exists! x_0 \in X$ such that

$$\{x\} = \bigcap_{n \geq 1} F_n.$$

Therefore,

$$O \cap U = \Big(\bigcap_{n\geq 1} O_n\Big) \bigcap U = \bigcap_{n\geq 1} (O_n \cap U) \supseteq \bigcap_{n\geq 1} F_n = \{x\},$$

and hence $O \cap U \neq \varnothing$. Since U is arbitrary, O is dense in X. □

Here is some results about Baire space:

Theorem 7.8. Let X be a Baire space. Then

1) Each open subset of X with the subspace topology is a Baire sapce;

2) Suppose $(F_n)_{n\in\mathbb{N}}$ is a sequence of closed subsets of X with $X = \bigcup_{n\geq 1} F_n$, then $\bigcup_{n\geq 1} \mathring{F}_n$ is dense in X.

Proof.

1) For $A \subseteq X$, let cl_A means the closure operator with respect to the subspace topology of A. Similarly, int means the interior operator.

Suppose $\Omega \subseteq X$ is open. Given $(O_n)_{n\in\mathbb{N}} \subseteq \Omega$ such that $\mathrm{cl}_\Omega(O_n) = \Omega (\forall n \in \mathbb{N})$, i.e. O_n is dense in Ω for all $n \in \mathbb{N}$. Since $\mathrm{cl}_\Omega(O_n) = \Omega \cap \overline{O_n}$, we have $\overline{O_n} \supseteq \Omega$ and hence $\overline{O_n} \supseteq \overline{\Omega}$. Since the closure of union is the union of closure, we have $O_n \cup \left(\overline{\Omega}\right)^c$ is dense in X for all $n \in \mathbb{N}$. Therefore,

$$(U_n)_{n\in\mathbb{N}} := \left(O_n \cup \left(\overline{\Omega}\right)^c\right)_{n\in\mathbb{N}}$$

is a sequence of dense open sets in X. Now X is a Baire space, which means $\bigcap_{n\geq 1} U_n$ is dense in X. While

$$\bigcap_{n\geq 1} U_n = \Big(\bigcap_{n\geq 1} O_n\Big) \cup \left(\overline{\Omega}\right)^c,$$

and hence

$$\overline{\bigcap_{n\geq 1} U_n} = X = \overline{\bigcap_{n\geq 1} O_n} \cup \overline{\left(\overline{\Omega}\right)^c}.$$

To prove that $\mathrm{cl}_\Omega(\bigcap_{n\geq 1} O_n) = \Omega$, we want to show that $\overline{(\overline{\Omega})^c} \subseteq \Omega^c$. And this holds

$$(\overline{\Omega})^c \subseteq \Omega^c \implies \overline{(\overline{\Omega})^c} \subseteq \Omega^c.$$

since $\overline{(\overline{\Omega})^c}$ is the smallest closed set containing $(\overline{\Omega})^c$.

Above all, Ω is a Baire space.

2) Let $\Omega \neq \varnothing$ be an arbitrary open set in X. Then 7.8 implies that Ω is a Baire space. And

$$\Omega = \Omega \cap X = \bigcup_{n\geq 1} (\Omega \cap F_n),$$

the definition of Baire space ensures that there is some $n \in \mathbb{N}$ such that $\mathrm{int}(\Omega \cap F_n) \neq \varnothing$. Since "the interior of intersection is the intersection of union" and Ω is open, we have $\Omega \cap \mathring{F}_n \neq \varnothing$. Therefore

$$\Omega \cap \left(\bigcup_{j\geq 1} \mathring{F}_j \right) \supseteq \Omega \cap \mathring{F}_n \neq \varnothing.$$

Then $\bigcap_{n\geq 1} F_n$ is dense in X since Ω is arbitrary. □

Now we give another proof of Theorem 7.6 by the Baire category Theorem.

Proof of Theorem 7.6. Let

$$M \colon X \to \mathbb{R}, x \mapsto \sup_{\lambda\in\Gamma} \|T_\lambda x\|_Y,$$

which is well-defined by the assumption. For all $n \in \mathbb{N}$

$$F_n := M^{-1}[0,n] = \bigcap_{\lambda\in\Gamma} (\| \ \|_Y \circ T_\lambda)^{-1}[0,n],$$

and $\| \ \|_Y, T_\lambda (\forall \lambda \in \Gamma)$ is continuous. Therefore, F_n is closed. Now X is a Banach space (hence a Baire space) and

$$X = \bigcup_{n\geq 1} F_n.$$

Theorem 7.8 shows that there is some $k \in \mathbb{N}$ such that $\mathring{M}_k \neq \varnothing$. There is $x_0 \in \mathring{F}_k$ and $r > 0$ such that $B_X(x_0, r) \subseteq \mathring{F}_k$. Now $\forall x \in B_X(x_0, r)$, $x + x_0 \in B_X(x_0, r)$ and hence $\forall \lambda \in \Gamma$, we have

$$\|T_\lambda(x)\|_Y \leq \|T_\lambda(x + x_0)\|_Y + \|T_\lambda(x_0)\|_Y \leq k + M(x_0).$$

Thus $T\big(B_X(x_0, r)\big) \subseteq \overline{B_Y(0, k + M(x_0))}$ holds for all $\lambda \in \Gamma$, which implies

$$\|T_\lambda\| \leq \frac{k + M(x_0)}{r}, \forall \lambda \in \Gamma.$$

Above all, $\sup_{\lambda \in \Gamma} \|T_\lambda\| \leq (k + M(x_0))/r < \infty$. $\square$

Remark. To give $X = \bigcup_{n \geq 1} F_n$, we need the assumption

$$\forall x \in X, M(x) < \infty.$$

8 Week 8

8.1 Week 8, Lecture 1

Recall

We have proved Theorem 7.8 by Open mapping theorem. We used the corollary 7.5 and proved that $\mathrm{cod}(\mathrm{id}_X)$ is a Banach space, where

$$\mathrm{id}_X\colon (X, \| \ \|_X) \to (X, \| \ \|_X + \sup_{\lambda\in\Gamma} \| \ \|_Y \circ T_\lambda)$$

is continuous.

Moreover, we proved Baire category Theorem and applied it to prove Banach-Steinhaus Theorem.

8.1.1 Application of Banach-Steinhaus Theorem

Definition (Strong convergence)**.** Let X, Y be two normed spaces, $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$. We say that $(T_n)_{n\in\mathbb{N}}$ converges to T strongly, if

$$\forall x \in X, (T_n x)_{n\in\mathbb{N}} \xrightarrow{\| \ \|_Y} Tx,$$

denoted as $(T_n)_{n\in\mathbb{N}} \xrightarrow{s} T$.

Remark. The relation between $(T_n)_{n\in\mathbb{N}} \xrightarrow{s} T$ and $(T_n)_{n\in\mathbb{N}} \xrightarrow{\| \ \|_{\mathcal{B}(X,Y)}} T$ is similar to the pointwise convergence and uniform convergence of function sequence.

We use Banach-Steinhaus to prove the following theorem about strong convergence.

Theorem 8.1. Let X be a linear normed space, Y be a Banach space, and $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(X,Y)$ is a sequence of operators. Suppose

1. $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$;
2. $\exists G \subseteq X$ such that $\overline{G} = X$ and $\forall x \in G$, $(T_n x)_{n\in\mathbb{N}}$ converges in Y.

Then there is a $T \in \mathcal{B}(X,Y)$ with $\|T\| \le \liminf_{n\to\infty} \|T_n\|$ such that

$$T_n \xrightarrow{s} T (n \to \infty).$$

Proof. Let $M := \sup_{n\in\mathbb{N}}\|T_n\|$. Since G is dense in X, $\forall x \in X$ and $\forall \varepsilon > 0$, there is $y \in G$ such that $\|y - x\| < \varepsilon$. Then

$$\begin{aligned}\|T_n x - T_m x\| &\le \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m y - T_m x\| \\ &\le \|T_n\|\|x - y\| + \|T_n y - T_m y\| + \|T_m\|\|y - x\| \\ &\le 2M\varepsilon + \|T_n y - T_m y\|.\end{aligned}$$

Let $m, n \to \infty$ and use the strong convergence of $(T_n)_{n\in\mathbb{N}}$, we find

$$\limsup_{m,n}\|T_n x - T_m x\| \le 2M\varepsilon.$$

From arbitrariness of $\varepsilon > 0$, we get $\lim_{m,n}\|T_n x - T_m x\| = 0$, i.e. $(T_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y and hence converges to some point in Y, because Y is a Banach space. Therefore, we can define

$$T\colon X \to Y, x \mapsto \lim_n T_n x,$$

which is linear since both $T_n (\forall n)$ and $\lim_n$ is linear (i.e. T is a composition of two linear maps, $f_1\colon X \to \mathcal{E}, x \mapsto (T_n x)_{n\in\mathbb{N}}$ and $f_2\colon \mathcal{E} \to Y, (y_n)_{n\in\mathbb{N}} \mapsto \lim_n y_n$, where $\mathcal{E}$ is the set of all Cauchy sequences in Y. Then $T = f_2 \circ f_1$ is linear.

Now we show that, T is what we want. For all $x \in X$,

$$\|Tx\| = \left\|\lim_n T_n x\right\| = \lim_n\|T_n x\|,$$

since $\| \ \|$ is continuous. And $\forall n \in \mathbb{N}, \|T_n x\| \le \|T_n\|\|x\|$, take $\liminf$ on both sides and we get

$$\liminf\|T_n x\| \le \liminf\|T_n\|\|x\|. \tag{19}$$

And $(\|T_n x\|)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, thus $\lim\|T_n x\|$ exists in $\mathbb{R}$. Then 19 implies

$$\|Tx\| = \lim\|T_n x\| \le \liminf\|T_n\|\|x\| \quad (\forall x \in X)$$

so

$$\|T\| \le \liminf_n\|T_n\|,$$

which ensures that $T \in \mathcal{B}(X, Y)$. □

If X is also a Banach space, then the inverse proposition holds. That is:

Proposition 8.2. Let X, Y be two Banach spaces. Suppose there is some $T \in \mathcal{B}(X, Y)$ such that $(T_n)_{n\in\mathbb{N}} \xrightarrow{s} T$, then

1. $\sup_{n\in\mathbb{N}}\|T_n\| < \infty$;
2. $\exists G \subseteq X$ such that G is dense in X and $\forall x \in G$, $(T_n x)_{n\in\mathbb{N}}$ converges in Y.

Proof. For all $x \in X$, $(\|T_n x\|)_{n\in\mathbb{N}}$ is a Cauchy sequence (and hence bounded) in $\mathbb{R}$ by strong convergence. Then Theorem 7.8 implies that $\sup_{n\in\mathbb{N}}\|T_n\|$ is finite. Let $G = X$ then $\overline{G} = X$ and $\forall x \in G, (T_n x)_{n\in\mathbb{N}}$ converges in Y, since $(T_n)_{n\in\mathbb{N}} \xrightarrow{s} T$. □

To state the next theorem better, there is an essential exercise.

Exercise. If Y is a Banach space and X is a linear normed space, then $\mathcal{B}(X, Y)$ is a Banach space. Especially, X^* is a Banach space.

Proof of this exercise is written in the Appendix B. Note that the exercise is just saying that $\mathcal{B}(X, Y)$ is complete in the meaning of the metric induced by the norm, then you can see that the next theorem is just saying that $\mathcal{B}(X, Y)$ is complete in the meaning for "strongly Cauchy sequence converges to some operator strongly".

Theorem 8.3. If X, Y are Banach spaces, then $\forall (T_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(X, Y)$ such that

$$T_n - T_m \xrightarrow{s} 0(m, n \to \infty), \tag{20}$$

we have

$$T_n - T \xrightarrow{s} 0(n \to \infty)$$

for some $T \in \mathcal{B}(X, Y)$.

Proof. Since (20) implies that $\forall x \in X$, $(T_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Since Y is a Banach space, $(T_n x)_{n\in\mathbb{N}}$ converges to some point in Y. Notice that $\forall x \in X$, $(\|T_n x\|)_{n\in\mathbb{N}}$ in bounded in $\mathbb{R}$, thus Theorem 7.6 implies that $\sup_{n\in\mathbb{N}}\|T_n\| < \infty$, so let $G = X$, apply Theorem 8.1 and we're done. □

Inverse of Hölder's inequality

We have learnt the Hölder's inequality (especially, for the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$): $\forall p \in [1, \infty]$, $\forall a \in \ell_p, b \in \ell_q$,

$$\|ab\|_1 \leq \|a\|_p \|b\|_q$$

holds, where $q = p'$. Now we're going to show that if $p \in (1, \infty)$, $\forall x \in \ell_p$ we have $\sum_{n\geq 1} \alpha_n x_n < \infty$, then $(\alpha_n)_{n\in\mathbb{N}} \in \ell_q$, where $q = p'$. For convenience, consider $\mathbb{K} = \mathbb{R}$.

Proof. Let

$$(\forall k \in \mathbb{N}) f_k \colon \ell_p \to \mathbb{R}, (x_n)_{n\in\mathbb{N}} \mapsto \sum_{j=1}^{k} x_j \alpha_j .$$

Then f_k is linear, and bounded since Hölder's inequality implies

$$\begin{aligned} |f_k x| &\le \sum_{j=1}^{k} |x_j \alpha_j| \\ &\le \Big(\sum_{j=1}^{k} |x_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{k} |\alpha_j|^q\Big)^{1/q} \\ &\le \Big(\sum_{j=1}^{k} |\alpha_j|^q\Big)^{1/q} \|x\|_p, \end{aligned}$$

i.e. $\|f_k\| \le \big(\sum_{j=1}^{k} |\alpha_j|^q\big)^{1/q}$. And the reversed inequality holds, to see this, consider the equality condition of Hölder's inequality (and triangle inequality) and hence pick

$$\ell_p \ni x^{(k)}, (x_n^{(k)})_{n\in\mathbb{N}} := (\operatorname{sign}(\alpha_1)|\alpha_1|^{q/p}, \ldots, \operatorname{sign}(\alpha_k)|\alpha_k|^{q/p}, 0, \ldots).$$

And

$$\left| f_k(x^{(k)}) \right| = \sum_{j=1}^{k} |\alpha_j|^q, \left\| x^{(k)} \right\|_p = \Big(\sum_{j=1}^{k} |\alpha_j|^q\Big)^{1/p},$$

implies

$$\|f_k\| \ge \Big(\sum_{j=1}^{k} |\alpha_j|^q\Big)^{1-1/p} = \Big(\sum_{j=1}^{k} |\alpha_j|^q\Big)^{1/q}.$$

Above all, $\|f_k\| = \big(\sum_{j=1}^{k} |\alpha_j|^q\big)^{1/q}$.

By assumption, we have $\forall x \in \ell_p, \big(f_n(x)\big)_{n\in\mathbb{N}}$ converges, and hence is bounded. Now apply Theorem 7.6, we get $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$. While $(\|f_n\|)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is a non decreasing sequence, thus

$$\sup_{n\in\mathbb{N}} \|f_n\| = \lim_{n\to\infty} \|f_n\| = \lim_{n\to\infty} \Big(\sum_{j=1}^{k} |\alpha_j|^q\Big)^{1/q} = \|\alpha\|_q .$$

Therefore, $\|\alpha\|_q < \infty$, i.e. $\alpha \in \ell_p$. $\square$

Remark. We can drop the restriction $\mathbb{K} = \mathbb{R}$.

Fourier series's divergence

First, we introduce some notions for convenience.

Definition. Let $C_{2\pi}$ be the normed space whose underlying set is

$$\{f: \mathbb{R} \to \mathbb{C} \mid f \text{ is continuous and } 2\pi \text{ -periodic}\},$$

with the norm

$$\| \, \|_\infty : C_{2\pi} \to \mathbb{R}, f \mapsto \sup_{x \in \mathbb{R}} |f(x)|.$$

Remark. The norm max is well-defined since f is 2π-periodic implies that

$$\sup_{\mathbb{R}} |f| = \sup_{[0,2\pi]} |f| = \max_{[0,2\pi]} |f|.$$

For clarity, here is the definition of period of a real function.

Definition (Period, Periodic function)**.** Let f be a function $\mathbb{R} \to \mathbb{R}$. A number $T \in \mathbb{R}$ is called a **period** of f, if

$$f = \tau_T f,$$

where

$$\tau_T f : \mathbb{R} \to \mathbb{C}, x \mapsto f(x - T).$$

Function f has a period T is called a **T-periodic function**, or a **periodic function** for short.

Remark. Let

$$\operatorname{per}(f) := \{T \in \mathbb{R} : \tau_T f = f\},$$

then $\operatorname{per}(f)$ is a subgroup of the additive group $\mathbb{R}$. The structure of $\operatorname{per}(f)$ has only 3 possibilities:

1. $\operatorname{per}(f)=\{0\}$, i.e. f is not a periodic function.
2. $\operatorname{per}(f) = T_0\mathbb{Z} = \{T_0 k : k \in \mathbb{Z}\}$ for some $T_0 > 0$. And such T_0 is usually called the fundamental period or the minimum period.
3. $\operatorname{per}(f)$ is a dense subgroup of $\mathbb{R}$, equivalently f has no fundamental period. For example, $\operatorname{per}(\chi_{\mathbb{Q}}) = \mathbb{Q}$.

The Fourier series of a 2π-periodic function is defined as follows

Definition (Fourier transform, Fourier coefficient)**.** Given $f \in C_{2\pi}$, the **Fourier transform** of f is the sequence defined as follows

$$\widehat{f} : \mathbb{Z} \to \mathbb{C}, n \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \mathrm{e}^{-in\theta} \, \mathrm{d}\theta.$$

We use the notation

$$f(x) \sim \sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{inx}$$

to mean that $a_n = \widehat{f}(n), \forall n \in \mathbb{Z}$. The n-th term of $\widehat{f}$, $\widehat{f}(n)$ is called the n-th **Fourier coefficient** of f.

And define the partial sum of Fourier series

Definition. The n-th partial sum of f's Fourier series, denoted by $S_n(f)$ is

$$\sum_{k=-n}^{n} \widehat{f}(k)\mathrm{e}^{-ikx} = \sum_{k=-n}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{e}^{ik(\theta-x)}\,\mathrm{d}\theta.$$

To understand the partial sum better, we have to notations: convolution and Dirichlet kernel.

Definition (Convolution)**.** Given the measure space $([-\pi,\pi],\mathcal{B},\mu)$ and two measurable function f, g, define

$$f * g \colon [-\pi,\pi] \to \mathbb{C}, t \mapsto \begin{cases} \int_{-\pi}^{\pi} f(t-x)g(x)\,\mathrm{d}\mu(x), & \text{if integral is finite} \\ 0. & \text{else} \end{cases}$$

Here μ is an arbitrary Borel measure on $(\mathbb{R},\mathcal{B})$.

Fubini's Theorem implies that, if $f, g \in \mathcal{L}_1[-\pi,\pi]$, then

$$\int_{[-\pi,\pi]} f(t-x)g(x)\,\mathrm{d}\frac{m}{2\pi}(x) < \infty$$

for almost every $t \in \mathbb{R}$.

Definition (Dirichlet kernel)**.** Given $n \in \mathbb{N}$, the function

$$D_n \colon \mathbb{R} \to \mathbb{C}, x \mapsto \sum_{k=-n}^{n} \mathrm{e}^{ikx}$$

is called the n-th Dirichlet kernel.

By this definitions, we have a convolution formula for the n-th partial sum of Fourier series

$$S_n(f)(x) = \int_{-\pi}^{\pi} f(\theta) \sum_{k=-n}^{n} \mathrm{e}^{ik(x-\theta)} \frac{\mathrm{d}m}{2\pi}(x) = D_n * f(x),$$

where the convolution is taking integration with respect to the measure $\frac{m}{2\pi}$, i.e. Lebesgue measure multiplied by $1/2\pi$.

Now, we have the following result.

Proposition 8.4. The set $\{f \in C_{2\pi} : \sup_{n\in\mathbb{N}}|S_n(f)(0)| = \infty\}$ is a dense G_δ subset of $C_{2\pi}$.

This means that, there are lots of functions in $C_{2\pi}$ whose Fourier series diverges at 0.

First, we have a overlook about the proof

Sketch of proof. We have the following steps:

Step 1. Define a linear functional sequence $(u_n)_{n\in\mathbb{N}}$ as follows

$$u_n\colon C_{2\pi} \to \mathbb{C}, f \mapsto S_n(f)(0). \tag{21}$$

Check that $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(C_{2\pi}, \mathbb{C})$.

Step 2. Show that $\sup_{n\in\mathbb{N}}\|u_n\| = \infty$.

Step 3. Apply the following theorem.

Theorem 8.5 (Principle of concentration of singularity)**.** Let X be a Banach space, Y be a linear normed space and $\{u_i \in \mathcal{B}(X, Y) : i \in I\}$ such that

$$\sup_{i\in I}\|u_i\| = \infty.$$

Then $\{x \in X : \sup_{i\in I}\|u_i(x)\| = \infty\}$ is a dense G_δ set in X.

□

And now we give the detailed proof.

Proof. **Step 1.** Define $(u_n)_{n\in\mathbb{N}}$ as (21). Given $n \in \mathbb{N}$, we have

$$\begin{aligned}
|u_n(f)| &= |S_n(f)(0)| \\
&\le \int_{-\pi}^{\pi} \left| f(t) \sum_{k=-n}^{n} \mathrm{e}^{ikt} \right| \frac{\mathrm{d}m}{2\pi}(t) \\
&\le \|f\|_\infty \|D_n\|_1 \text{(Hölder's inequality)}
\end{aligned}$$

Thus $\forall n \in \mathbb{N}, u_n \in \mathcal{B}(C_{2\pi}, \mathbb{C})$ and $\|u_n\| \le \|D_n\|_1$.

The reversed inequality holds. To see this, $\forall \varepsilon > 0$, take a finite union of intervals (denote the union by I) such that $m(I) \le \pi\varepsilon/(2n+1)$. This is possible since D_n has only finite zeros in $[0, 2\pi]$ (consider $D_n(t) = \frac{\sin(n+1/2)x}{\sin(x/2)}$). Now Define f

$$f\colon [0, 2\pi] \to \mathbb{C}, x \mapsto \begin{cases} 1, & x \notin I \wedge D_n(x) > 0; \\ -1, & x \notin I \wedge D_n(x) < 0; \\ l(x), & x \in I. \end{cases}$$

Here l is the affine mapping on each subinterval of I such that f is continuous.

$$\begin{aligned}
|u_n(f)| &\geq \left|\int_{[-\pi,\pi]\setminus I} f(t)D_n(t)\frac{\mathrm{d}m}{2\pi}(t)\right| - \left|\int_I f(t)D_n(t)\frac{\mathrm{d}m}{2\pi}(t)\right| \\
&= \int_{[-\pi,\pi]\setminus I} |D_n(t)|\frac{\mathrm{d}m}{2\pi}(t) - \left|\int_I f(t)D_n(t)\frac{\mathrm{d}m}{2\pi}(t)\right| \\
&= \int_I |D_n(t)|\frac{\mathrm{d}m}{2\pi}(t) - 2\left|\int_I f(t)D_n(t)\frac{\mathrm{d}m}{2\pi}(t)\right| \\
&\geq \|D_n\|_1 - \|f\|_\infty \varepsilon.
\end{aligned}$$

The first inequality is just $|x+y| \geq |x| - |y|$, and the last inequality follows from

$$\begin{aligned}
\left|\int_I f(t)D_n(t)\frac{\mathrm{d}m}{2\pi}(t)\right| &\leq \int_I |f\cdot D_n|\frac{\mathrm{d}m}{2\pi} \\
&\leq \|f\|_\infty \int_I |D_n|\frac{\mathrm{d}m}{2\pi} \\
&\leq \|f\|_\infty (2n+1)\cdot \frac{m}{2\pi}(I) \\
&\leq \|f\|_\infty (2n+1)\cdot \frac{1}{2\pi}\frac{\pi\varepsilon}{2n+1}.
\end{aligned}$$

since $\|D_n\|_\infty \leq 2n+1$. From $\|f\|_\infty = 1$, we get

$$\|u_n\| \geq \|D_n\|_1 - \varepsilon \ (\forall \varepsilon > 0),$$

which implies $\|u_n\| \geq \|D_n\|$.

Above all, $\|u_n\| = \|D_n\|$. Now we show that $(\|D_n\|)_{n\in\mathbb{N}}$ is unbounded. Let $\mathrm{d}x$ denote $\mathrm{d}m(x)$ for short

$$\begin{aligned}
\|D_n\|_1 &= \int_{-\pi}^{\pi} \frac{|\sin(n+1/2)x|}{|\sin(x/2)|}\frac{\mathrm{d}x}{2\pi} \\
&\geq \frac{1}{2\pi}\int_0^{\pi} \frac{|\sin(n+1/2)x|}{x/2}\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\frac{2n+1}{2}\pi} \frac{|\sin x|}{x}\,\mathrm{d}x \\
&> \frac{2}{\pi}\int_0^{\pi} \frac{|\sin x|}{x}\,\mathrm{d}x \\
&= \frac{2}{\pi}\sum_{k=0}^{n-1}\int_0^{\pi} \frac{|\sin x|}{x+k\pi}\,\mathrm{d}x,
\end{aligned}$$

and

$$\int_0^\pi \frac{|\sin x|}{x+k\pi}\,\mathrm{d}x \geq \frac{1}{(k+1)\pi}\int_0^\pi |\sin x|\,\mathrm{d}x = \frac{2}{(k+1)\pi}.$$

Therefore

$$\|D_n\|_1 \geq \frac{4}{\pi^2}\sum_{k=1}^n \frac{1}{k} \sim \frac{4}{\pi^2}\log n \to \infty (n\to\infty).$$

Then, Theorem 8.5 implies the result.

□

Here is the proof of Theorem 8.5.

Proof of Theorem 8.5. Let

$$M\colon X \to [0,\infty], x \mapsto \sup_{i\in I}\|u_i(x)\|.$$

Define

$$F_n = \{x \in X : M(x) \leq n\} (\forall n \in \mathbb{N}),$$

then F_n is closed in X and hence $\Omega_n := F_n^c$ is open, for all $n \in \mathbb{N}$. Now if $\mathring{F}_m = \varnothing$ for some $m \in \mathbb{N}$, then

$$\{x \in X : M(x) = \infty\} = \bigcap_{n\geq 1} \Omega_n,$$

hence $\{x \in X : M(x) = \infty\}$ is a G_δ-set. If $\{x \in X : M(x) = \infty\}$ is not dense in X, i.e. $\bigcap_{n\geq 1}\Omega_n$ is not dense in X, then there is some $m \in \mathbb{N}$ such that Ω_m is not dense in X, since X is a Banach space (hence a Baire space). But $\overline{\Omega_m} \neq X$ implies that $\mathring{F}_m \neq \varnothing$, thus the proof of Theorem 7.8 works (but we should replace $\|T_\lambda(x_0)\| \leq M(x_0)$ with $\|T_\lambda(x_0)\| \leq k$), contradiction with $\sup_{i\in I}\|u_i\| = \infty$. □

8.2 Week 8, Lecture 2

The aim of this lecture is

Aim. Prove the open mapping theorem.

Though we have proved Theorem 7.3, which needs a strong condition: T is surjective. We will change this restriction weaker.

Recall

- A set B is called a set of first category, if $B \subseteq \mathring{F}$, where F is a F_σ set.
- A topological space $(X, \mathcal{T})$ is said to be a Baire space, if for all open set sequence $(O_n)_{n\in\mathbb{N}}$ such that $\overline{O_n} = X (\forall n \in \mathbb{N})$, we have $\overline{\bigcap_{n\geq 1} O_n} = X$.
    1. A set O is open and dense in X if and only if O^c is closed and $(O^c)^\circ = \varnothing$.
    2. Let X be a topological space. Then X is a Baire space if and only if for all closed set sequence $(F_n)_{n\in\mathbb{N}}$ such that $\mathring{F}_n = \varnothing$, we have $\left(\bigcup_{n\geq 1} F_n\right)^\circ = \varnothing$.

8.2.1 Open mapping Theorem (general version)

Theorem 8.6 (Open mapping theorem)**.** Let X, Y be two Banach spaces and $T \in \mathcal{B}(X, Y)$. If $T(X) \hookrightarrow Y$ is a set of 2 category, then

1. there is $c > 0$ such that $B_Y \subseteq cT(B_X)$. Here B_X, B_Y means the unit ball in X, Y respectively.
2. T is an open mapping.

Proof. In this theorem, for $k \in \mathbb{K}$ and $A, B \subseteq X$,

$$kA := \{kx \in X : x \in A\} A + B := \{x + y : x \in A, y \in B\}.$$

Similarly for $A, B \subseteq Y$.

From $X = \bigcup_{n\geq 1} nB_X$, we have

$$T(X) = \bigcup_{n\geq 1} T(nB_X) \subseteq \bigcup_{n\geq 1} \overline{T(nB_X)},$$

and hence $T(X) \subseteq \bigcup_{n\geq 1} \overline{T(nB_X)}$. Thus $T(X)$ is a F_σ-set in Y. So $\left(\bigcup_{n\geq 1} \overline{T(nB_X)}\right)^\circ \neq \varnothing$ (else, $\bigcup_{n\geq 1} \overline{T(nB_X)}$ is a F_σ-set containing $T(X)$

while having empty interior, contradiction with $T(X)$ is not a set of 2 category). While Y is a Banach space, hence a Baire space, and there is some $m \in \mathbb{N}$ such that $\left(\overline{T(mB_X)}\right)^\circ \neq \varnothing$. Thus, there is $y_0 \in \left(\overline{T(mB_X)}\right)^\circ$ and $r > 0$ such that

$$y_0 + rB_Y \subseteq \left(\overline{T(mB_X)}\right)^\circ \subseteq \overline{T(mB_X)}.$$

Then

$$rB_Y \subseteq \overline{T(mB_X)} - y_0 \subseteq \overline{T(mB_X)} - \overline{T(mB_X)} \subseteq \overline{T(2mB_X)}.$$

The second $\subseteq$ follows from $y_0 \in \overline{T(mB_X)}$ and the last $\subseteq$ can be easily checked by taking sequences convergent to each point. Now we have

$$B_Y \subseteq \overline{T(cB_X)},$$

where $c := 2m/r$. And we want $B_Y \subseteq T(cB_X)$. To see this, given arbitrary $y_1 \in B(Y) \subseteq \overline{T(cB_X)}$, there is $x_1 \in cB_X$ such that

$$\|y_1 - T(x_1)\| < 1/2,$$

from the definition of closure. Then $y_2 := 2(y_1 - T(x_1)) \in B_Y$, since $\|y_1\| < 1$. Using the definition of closure again, there is $x_2 \in cB_X$ such that

$$\|y_2 - T(x_2)\| < 1/2.$$

Now we define $y_3 := 2(y_2 - T(x_2)) \in B_Y$. And we can define two sequences $(x_n)_{n\in\mathbb{N}} \subseteq cB_X, (y_n)_{n\in\mathbb{N}} \subseteq B_Y$ inductively, such that

$$\forall n \in \mathbb{N} \begin{cases} \|y_n - T(x_n)\| < 1/2, \\ y_{n+1} = 2(y_n - T(x_n)). \end{cases}$$

Hence, $\forall n \in \mathbb{N}$

$$\begin{aligned} y_1 &= y_2/2 + T(x_1) \\ &= y_3/2^2 + T(x_2)/2 + T(x_1) \\ &= \cdots \\ &= y_{n+1}/2^n + \sum_{j=1}^{n} T(x_j)/2^j \\ &= y_{n+1}/2^n + T\Big(\sum_{j=1}^{n} x_j/2^j\Big), \end{aligned}$$

where $\|y_{n+1}/2^n\| < 2^{-n}$ since $y_{n+1} \in B_Y$, $\sum_{j=1}^n x_j/2^j$ is absolutely convergent since

$$\left\|\sum_{j=1}^n x_j/2^j\right\| \leq \sum_{j=1}^n \|x_j/2^j\| < \sum_{j=1}^n \frac{c}{2^j} < c(\forall n \in \mathbb{N}).$$

and is convergent to some $x_0 \in cB_X$ because X is a Banach space. Therefore, by the continuity of T

$$\begin{aligned} y_1 = & \lim_{n\to\infty} y_{n+1}/2^n + T\Big(\sum_{j=1}^n x_j/2^j\Big) \\ &= \lim_{n\to\infty} T\Big(\sum_{j=1}^n x_j/2^j\Big) \\ &= T\Big(\lim_{n\to\infty} \sum_{j=1}^n x_j/2^j\Big) \\ &= T(x_0). \end{aligned}$$

Above all, $B_Y \subseteq T(cB_X) = cT(B_X)$. Then

$$Y = \bigcup_{n\geq 1} nB_Y \subseteq \bigcup_{n\geq 1} nT(cB_X) = T(\bigcup_{n\geq 1} ncB_X) = TX,$$

i.e. T is surjective.

To see that T is open, it suffices to show that $T(x+\delta B_X)$ is open in Y since $\{x+\delta B_X : x \in X, \delta > 0\}$ is a topology base for X. While T is linear, WLOG, it suffices to show $T(B_X)$ is open. Given $x \in B_X$, $T(x) \in T(B_X)$, there is $r_x > 0$ such that $x + r_x B_X \subseteq B_X$ and hence

$$T(B_X) \supseteq T(x) + r_x T(B_X) \supseteq T(x) + r_x c^{-1} B_Y,$$

thus $T(B_X)$ is open. □

Remark. This implies Theorem 7.3, since $T(X) = Y$ and Y is a Banach space (hence a Baire space) implies $T(X) = Y$ is of 2 category.

Proof. If $T(X) = Y$ is of 1 category, then $Y \subseteq (\bigcup_{n\geq 1} F_n)^\circ$ where F_n is closed. If $\mathring{F}_n = \varnothing$ for all n, Y is a Baire space implies $(\bigcup_{n\geq 1} F_n)^\circ = \varnothing$, contradiction with $Y \neq \varnothing$. Thus there is m such that $\mathring{F}_m \neq \varnothing$ and hence $(\bigcup_{n\geq 1} F_n)^\circ \supseteq \mathring{F}_m \neq \varnothing$.

Thus, Y can't be contained in a F_σ-set with empty interior, i.e. Y is a set of 2 category. □

8.2.2 Closed graph Theorem

Some results in subsection 7.1 are used here.

This graph tells the relation between theorems.

$$\begin{tikzcd} \text{Thm 8.6} \arrow[r, Rightarrow] & \text{Thm 7.4} \arrow[d, Rightarrow] \arrow[dl, Rightarrow] & \\ \text{Thm 8.8} & \text{Cor 7.5} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & \text{Thm 7.6} \end{tikzcd}$$

Now we talk about the Closed Graph Theorem. Here is the natural definition of the graph of a operator.

Definition (Graph)**.** Let X, Y be two sets (allowed to have structures such as topology, norm and so on) and $T\colon X \to Y$ is a map. The graph of T, denoted by $G(T)$ is defined as

$$G(T) := \{(x, y) \in X \times Y : y = Tx\} = \{(x, Tx) \in X \times Y : x \in X\}.$$

Remark. In this lecture, we assume $X \times Y$ is a linear normed space, the norm of $X \times Y$ is

$$\| \ \| \colon X \times Y \to \mathbb{R}, (x, y) \mapsto \|x\|_X + \|y\|_Y,$$

if nothing else is mentioned. Equivalently, pick $p = 1$ by default.

Furthermore, we need this notion.

Definition (Closed operator)**.** Suppose X, Y are two sets, $T\colon X \to Y$ is a map. Then T is said to be closed, if $G(T)$ is closed.

Recall that, a sequence $\big((x_n, y_n)\big)_{n\in\mathbb{N}} \subseteq X \times Y$ converges to $(x, y) \in (X \times Y, \| \ \|_p)$, if

$$\lim_n \|(x_n, y_n) - (x, y)\|_p = \lim_n \Big(\|x_n - x\|^p + \|y_n - y\|^p\Big)^{1/p} = 0,$$

i.e. $(x_n)_{n\in\mathbb{N}} \xrightarrow{\| \ \|_X} x \wedge (y_n)_{n\in\mathbb{N}} \xrightarrow{\| \ \|_Y} y$.

Proposition 8.7. Let $X, Y \in \mathrm{Ob}(\mathsf{Nor})$ and $T \in \mathcal{L}(X, Y)$. Then

1. T is closed iff $\forall (x_n)_{n\in\mathbb{N}} \subseteq X, \forall (y_n)_{n\in\mathbb{N}} \subseteq y, \forall x \in X, \forall y \in Y$ such that $\lim_n x_n = x, \lim_n T(x_n) = y$, we have $T(x) = y$.

2. If T is bounded, then T is closed.

Proof. For necessity of 1: suppose T is closed. Then $\forall (x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\lim_n x_n = x \wedge \lim_n Tx_n = y$, we have $\big((x_n, Tx_n)\big)_{n\in\mathbb{N}} \subseteq G(T)$ converges to $(x,y) \in X \times Y$. While $G(T)$ is closed, we get $(x,y) \in G(T)$ thus $y = Tx$.

For sufficiency of 1: suppose T satisfies the latter condition in 1. Given an arbitrary convergent sequence $\big((x_n, y_n)\big)_{n\in\mathbb{N}} \subseteq G(T)$, i.e. $(x_n)_{n\in\mathbb{N}} \subseteq X$, $y_n = T(x_n)$ for all $n \in \mathbb{N}$ and $(x_n)_{n\in\mathbb{N}} \to x \in X$. Then the latter condition implies $\lim_n y_n = \lim_n Tx_n$. And continuity of T implies $T(x) = \lim_n y_n$. Thus the limit of $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$, i.e. $(x, \lim_n y_n) \in X \times Y$ lies in $G(T)$. Therefore, $G(T)$ is closed.

For 2, suppose T is bounded. Then the continuity of T implies that T satisfies the latter condition in 1. ☐

Now, the Closed Graph Theorem is

Theorem 8.8 (Closed Graph)**.** Let $X, Y \in \mathrm{Ob}(\mathsf{Ban})$. If $T\colon X \to Y$ is closed, then T is bounded.

Proof. We know that $X \times Y$ is a Banach space, and hence $G(T) \hookrightarrow X \times Y$ being a closed subspace of $X \times Y$ is also a Banach space. Consider the projection mapping

$$\begin{aligned} p\colon G(T) &\to X \\ (x, Tx) &\mapsto x, \end{aligned}$$

which is a linear bijection and $\|p\| \le 1$ since

$$\forall x \in X, \big\|p\big((x,Tx)\big)\big\|_X = \|x\| \le \|x\| + \|Tx\| = \|(x,Tx)\|_{X\times Y}.$$

Theorem 7.4 implies $p^{-1} \in \mathcal{B}(X, G(T))$. Therefore,

$$\begin{aligned} \forall x \in X, \|Tx\| &\le \|x\| + \|T_x\| \\ &= \|(x,Tx)\|_{X\times Y} \\ &= \big\|p^{-1}(x)\big\|_{X\times Y} \\ &\le \big\|p^{-1}\big\|\|x\|_X, \end{aligned}$$

i.e. $\|T\| \le \big\|p^{-1}\big\| < \infty$.

Above all, $T \in \mathcal{B}(X,Y)$. ☐

Remark. This can also be proved by Corollary 7.5: since $(X, \|\ \|_X)$ is a Banach space, and

$$(X, \|\ \|_X + \|\ \|_Y \circ T) \cong (G(T), \|\ \|_{X\times Y})$$

is also a Banach space, where the isometry is just $T\colon X \to G(T)$. Then Corollary 7.5 ensures Theorem 8.8.

9 Week 9

9.1 Week 9, Lecture 1

In this subsection, I will use the notation $f \le g$ lots of times, whose meaning can be found here.

Recall

We have proved the relation between Theorems on Banach spaces, see the graph in section 8.2.2.

9.1.1 Hahn-Banach Theorem

Today, here is going to prove Theorem 9.1 and Theorem 2. Zorn's lemma is needed here, see Appendix A.

There is an important object related to Hahn-Banach Theorem:

Definition (Sub-linear functional)**.** Let X be a **real** linear space. A real-valued function $p\colon X \to \mathbb{R}$ is called a **sub-linear functional**, if

1. For all $x \in X, \lambda \ge 0$, $p(\lambda x) = \lambda p(x)$ holds;
2. For all $x, y \in X$, $p(x+y) \le p(x) + p(y)$ holds.

Definition (Linear dual space)**.** For a linear space $X \in \mathrm{Ob}(\mathsf{Lin}_{\mathbb{K}})$, the dual space, denoted by $X^\sharp$, is

$$X^\sharp := \{f\colon X \to \mathbb{K} \text{ that is } \mathbb{K}\text{-linear}\}.$$

The first theorem is irrelevant to topology, considering only linear space.

Theorem 9.1 (Hahn-Banach)**.** Let X be a **real** vector space and $X_0 \hookrightarrow X$ is a subspace. Suppose $f_0 \in X_0^\sharp$ and $p\colon X \to \mathbb{R}$ is a sub-linear functional such that $f_0 \le p|_{X_0}$. Then there is (at least one) $f \in X^\sharp$ such that $(f|_{X_0} = f_0) \wedge (f \le p)$.

Proof. We will prove this Theorem by Zorn's lemma. Thus we need to construct an partially ordered set whose maximal element is the function we want.

Step 1: We construct the partially ordered set. Let

$$\mathcal{F} := \bigcup_{X_0 \hookrightarrow D \hookrightarrow X} \{g \in D^\sharp\colon\ g|_{X_0} = f_0 \wedge g \le p|_D\},$$

where the union is taken over all subspaces D such that $X_0 \hookrightarrow D \hookrightarrow X$. Define an order on $\mathcal{F}$ as follows

$$g_1 \leq g_2 \iff \operatorname{dom}(g_1) \hookrightarrow \operatorname{dom}(g_2) \wedge g_2|_{\operatorname{dom}(g_1)} = g_1,$$

i.e. $g_1 \leq g_2$ iff g_2 is an extension of g_1 in the sense above.

Exercise. Check that $(\mathcal{F}, \leq)$ is a partially ordered set.

Step 2: We prove that $\mathcal{F}$ satisfies the condition of Zorn's lemma. Given an arbitrary linearly ordered subset $\mathcal{F}_0 \subseteq \mathcal{F}$. We prove that $\mathcal{F}_0$ has an upper bound in $\mathcal{F}$. Consider the set $\bigcup_{g \in \mathcal{F}_0} \operatorname{dom}(g) = \operatorname{dom}\left(\bigcup \mathcal{F}_0\right)$. Define the linear structure as follows

$$\begin{aligned} +\colon \operatorname{dom}\left(\bigcup \mathcal{F}_0\right) \times \operatorname{dom}\left(\bigcup \mathcal{F}_0\right) &\to \operatorname{dom}\left(\bigcup \mathcal{F}_0\right) \\ (v_1, v_2) &\mapsto v_1 +_{V_1 \cup V_2} v_2, \end{aligned}$$

where $V_j \in \operatorname{dom}\left(\bigcup \mathcal{F}_0\right)$ is a space containing v_j, $V_1 \cup V_2$ is a subspace of $\operatorname{dom}\left(\bigcup \mathcal{F}_0\right)$ since $\operatorname{dom}\left(\bigcup \mathcal{F}_0\right)$ is linearly ordered and $+_{V_1 \cup V_2}$ is the natural addition of the subspace $V_1 \cup V_2$ (this is surely a subspace by the linear order). Though it's possible that $v_1 \in V_1 \cap V_1', v_2 \in V_2 \cap V_2'$ for some $V_1, V_1' \in \mathcal{F}_0$, the summation $+$ is well-defined. WLOG, we suppose $V_1 \hookrightarrow V_1', V_2 \hookrightarrow V_2'$ by the linear order and then

$$v_1 +_{V_1 \cup V_2} v_2 = v_1 +_{V_1' \cup V_2'} v_2$$

since $V_1 \cup V_2 \hookrightarrow V_1' \cup V_2'$. And the scalar multiplication is just

$$\cdot \colon \operatorname{dom}\left(\bigcup \mathcal{F}_0\right) \times \mathbb{R} \to \operatorname{dom}\left(\bigcup \mathcal{F}_0\right), (v, k) \mapsto k \cdot_V v,$$

where $V \in \mathcal{F}_0$ is a subspace of X containing v. We can prove that $\cdot$ is well-defined similarly. Above all, $\operatorname{dom}\left(\bigcup \mathcal{F}_0\right)$ is a vector space of X and contains X_0. Now we define a linear functional on $\operatorname{dom}\left(\bigcup \mathcal{F}_0\right)$ as follows

$$h\colon \operatorname{dom}\left(\bigcup \mathcal{F}_0\right) \to \mathbb{R}, v \mapsto g(v),$$

whenever g is an element of $\mathcal{F}_0$ such that $v \in \operatorname{dom}(g)$. This is well-defined by the property of $\mathcal{F}_0$. And $h \in \mathcal{F}$ is an upper bound for $\mathcal{F}_0$.

Step 3: apply Zorn's lemma, thus there is a maximal element in $\mathcal{F}$ and let f be the maximal element. We prove that f is what we want. Equivalently, we prove that

1. $f \in X^\sharp$, i.e. $\operatorname{dom}(f) = X$;

2. $f|_{X_0} = f_0$;

3. $f \leq p$.

Since $f \in \mathcal{F}$, we get $f|_{X_0} = f_0$ and $f \leq p|_{\mathrm{dom}(f)}$. Thus it suffices to show that $\mathrm{dom}(f) = X$. Suppose there is an element $x_0 \in X \setminus \mathrm{dom}(f)$, then

$$\mathrm{dom}(f) + \mathbb{R}x_0 = \{y + kx_0 \colon y \in \mathrm{dom}(f) \wedge k \in \mathbb{R}\}$$

is a subspace of X strictly bigger than $\mathrm{dom}(f)$. We prove that f can be extended to a linear functional $\widetilde{f}$ on $\mathrm{dom}(f) + \mathbb{R}x_0$ such that $\widetilde{f} \in \mathcal{F}$, which is a contradiction with f being a maximal element. In order to define $\widetilde{f}$, it suffices to check that $\widetilde{f}(x_0)$ can be defined, since

$$\widetilde{f}(y + kx_0) = \widetilde{f}(y) + k\widetilde{f}(x_0) = f(y) + k\widetilde{f}(x_0)$$

is determined by $\widetilde{f}(x_0)$. The only thing restricts the value of $\widetilde{f}(x_0)$ is $\widetilde{f} \leq p|_{\mathrm{dom}(\widetilde{f})}$, i.e. $\forall y \in \mathrm{dom}(f), \forall k \in \mathbb{R} \setminus \{0\}:$

$$\begin{cases} \widetilde{f}(y + kx_0) \leq p|_{\mathrm{dom}(\widetilde{f})}(y + kx_0), & k > 0; \\ \widetilde{f}(y + kx_0) \leq p|_{\mathrm{dom}(\widetilde{f})}(y + kx_0), & k < 0; \end{cases}$$

i.e. $\forall y \in \mathrm{dom}(f), \forall k \in \mathbb{R} \setminus \{0\}:$

$$\begin{cases} \widetilde{f}(x_0) \leq p(y + kx_0)/k - \widetilde{f}(y)/k, & k > 0; \\ \widetilde{f}(x_0) \geq p(y + kx_0)/k - \widetilde{f}(y)/k, & k < 0; \end{cases}$$

i.e. $\forall y, z \in \mathrm{dom}(f), \forall k > 0, k' < 0:$

$$\begin{cases} \widetilde{f}(x_0) \leq p(y/k + x_0) - f(y/k); \\ \widetilde{f}(x_0) \geq -p(-z/k' - x_0) - f(z/k'). \end{cases}$$

Here I don't care the case $k = 0$ since $\widetilde{f}(y) \leq p|_{\mathrm{dom}(\widetilde{f})}(y)$ can be deduced from $f \leq p|_{\mathrm{dom}(f)}$. Therefore, it suffices to show that $\forall y, z \in \mathrm{dom}(f), \forall k > 0, k' < 0:$

$$-p(-z/k' - x_0) - f(z/k') \leq p(y/k + x_0) - f(y/k), \tag{22}$$

And (22) holds. To see this, we can set $k = -k' = 1$ and then

$$p(z - x_0) + p(y + x_0) \geq p(y + z) \geq f(y + z) = f(y) + f(z).$$

Then (22) implies

$$\sup S^- \leq \inf S^+.$$

where

$$\begin{aligned} S^- &:= \{-p(-z/k' - x_0) - f(z/k') \in \mathbb{R} \colon z \in \operatorname{dom}(f), k' < 0\}, \\ S^+ &:= \{p(y/k + x_0) - f(y/k) \in \mathbb{R} \colon y \in \operatorname{dom}(f), k > 0\}. \end{aligned}$$

Then $\widetilde{f}(x_0)$ can be taken as an arbitrary number in the interval

$$[\sup S^-, \inf S^+].$$

We're done. $\square$

Back to linear normed space.

Theorem 9.2 (Hahn-Banach, general version)**.** Let X be a linear normed space over the field $\mathbb{K}$ and $X_0 \hookrightarrow X$ is a subspace. Suppose $f \in X_0^*$, then there is $f \in X^*$ such that

1. $f|_{X_0} = f_0$;
2. $\|f\|_{X^*} = \|f_0\|_{X_0^*}$.

In other words, f is an extension of f_0 with the same norm.

Remark. Before the proof, we should have an observation: a complex vector space can be viewed as a real vector space. For the detail, see Proposition A.2.

Proof. To use Theorem 9.1, I will prove the case $\mathbb{K} = \mathbb{R}$ first, which can be applied for the case $\mathbb{K} = \mathbb{C}$.

Case 1: $\mathbb{K} = \mathbb{R}$. Let p be the norm defined as follows

$$p \colon X \to \mathbb{R}, x \mapsto \|f_0\|_{X_0^*} \|x\|.$$

Then p is a semi-norm such that $f_0 \leq p|_{X_0}$. Thus, Theorem 9.1 implies that there is a function f, an extension of f_0 that satisfies $f \leq p$. For all $x \in X$

$$\begin{aligned} f(x) &\leq p(x), \\ f(x) = -f(-x) &\geq -p(x). \end{aligned}$$

Thus $|f(x)| \leq p(x)$ and $\|f\|_{X^*} \leq \|f_0\|_{X_0^*}$, hence $f \in X^*$. The reversed inequality holds since $f|_{X_0} = f_0$.

Case 2: $\mathbb{K} = \mathbb{C}$. It can be shown that $\forall h \in \mathcal{B}(X, \mathbb{C})$, h is uniquely determined by $\operatorname{Re} \circ h = \operatorname{Re}(h) \in \mathcal{B}(X, \mathbb{R})$. For all $x \in X$,

$$h(ix) = i\big(\operatorname{Re} h(x) + i \operatorname{Im} h(x)\big) = -\operatorname{Im} h(x) + i \operatorname{Re} h(x),$$

take real parts for both sides and get $-\operatorname{Im} h(x) = \operatorname{Re} h(ix)$. Thus

$$\forall x \in X : h(x) = \operatorname{Re} h(x) - i \operatorname{Re} h(ix).$$

Now, view X as a real vector space and suppose $\forall x \in X, f_0(x) = g_0(x) - ig_0(ix)$, where $g_0 \in \mathcal{B}(X, \mathbb{R})$. Define

$$p\colon X \to \mathbb{R}, x \mapsto \|g_0\|_{\mathcal{B}(X,\mathbb{R})}\|x\|.$$

Then apply the result in 9.1.1 and we get $\exists g \in \mathcal{B}(X, \mathbb{R})$ such that

$$\big(\, g|_{X_0} = g_0\big) \wedge \big(g \leq p\big) \wedge \big(\|g\|_{\mathcal{B}(X,\mathbb{R})} = \|g_0\|_{\mathcal{B}(X_0,\mathbb{R})}\big).$$

Then $f\colon X \to \mathbb{C}, x \mapsto g(x) - ig(ix)$ satisfies

$$\big(f|_{X_0} = f_0\big) \wedge \big(\|f\|_{X^*} = \|f_0\|_{X_0^*}\big).$$

The first is trivial and the second is true if $\|f\|_{X^*} \leq \|f_0\|_{X_0^*}$, equivalently, $\forall x \in X : |f(x)| \leq p(x)$. To see this, let

$$\theta\colon X \to \mathbb{C}, x \mapsto (\operatorname{sign} \circ f)(x).$$

Notice that $|\,| \circ \theta\colon X \to \mathbb{C}$ is a constant function. Then $\forall x \in X$

$$\begin{aligned}
|f(x)| &= f(x) \cdot \theta(x) \\
&= f\big(\theta(x) \cdot x\big) \\
&= g\big(\theta(x) \cdot x\big) && (f(y) \in \mathbb{R} \iff f(y) = g(y)) \\
&\leq \|g\|_{\mathcal{B}(X,\mathbb{R})} \cdot \|\theta(x) \cdot x\|_X \\
&= \|g\|_{\mathcal{B}(X,\mathbb{R})} \cdot |\theta(x)| \cdot \|x\|_X \\
&= \|g\|_{\mathcal{B}(X,\mathbb{R})} \cdot \|x\|_X.
\end{aligned}$$

And

$$\|g\|_{\mathcal{B}(X,\mathbb{R})} = \|g_0\|_{\mathcal{B}(X_0,\mathbb{R})} \leq \|f_0\|_{X_0^*}.$$

Thereby, $\forall x \in X : |f(x)| \leq \|f_0\|_{X_0^*}\|x\|$, i.e. $\|f\|_{X^*} \leq \|f_0\|_{X_0^*}$. □

Theorem 9.2 is of great importance in the theory of "dual space of Banach space", which can be seen later.

9.2 Week 9, Lecture 2

Recall

We have studied

1. Theorem 7.6 (Resonance Theorem/ Uniformly bounded principle):

   Let X be a Banach space and Y be a linear normed space. Suppose $\{T_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{B}(X,Y)$ satisfies: $\forall x \in X \exists M_x > 0$ such that $\sup_{\lambda\in\Lambda}\|T_\lambda x\|_Y < M_x$. Then there is $M > 0$ such that $\sup_{\lambda\in\Lambda}\|T_\lambda\| < M$.

2. Theorem 7.3(Open mapping Theorem):

   (a) Let X, Y be two Banach spaces and $T \in \mathcal{B}(X,Y)$ is a surjection. Then T is an open mapping.

   (b) Theorem 8.6.

3. Theorem 8.8(Closed graph Theorem):

   For a mapping $T\colon X \to Y$. the graph of T is $G(T) := \{(x,Tx) : x \in X\} \hookrightarrow X \times Y$. A mapping T is said to be closed if $G(T)$ is closed.

   Let X, Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$. Then T is a closed operator implies $T \in \mathcal{B}(X,Y)$.

4. Theorem 9.1 and Theorem 9.2 (Hahn-Banach Theorem):

   Remark. This is one of the most important theorems for functional analysis.

Here is an exercise that explains the name "Resonance Theorem".

Exercise. Let X be a Banach space and Y be a linear normed space. Suppose $\{T_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{B}(X,Y)$ satisfies $\sup_{\lambda\in\Lambda}\|T_\lambda\| = \infty$. Show that $\exists x_0 \in X$ such that $\sup_{\lambda\in\Lambda}\|T_\lambda y\|_Y = \infty$.

9.2.1 Review

Recall the definition of semi-norm

Definition (Semi-norm)**.** Let X be a linear normed space. A function $p\colon X \to \mathbb{R}$ is said to be a semi-norm, if it satisfies:

1. For all $x \in X$, $p(x) \geq 0$;

2. For all $x, y \in X$, $p(x+y) \leq p(x) + p(y)$;

3. For all $x \in X, k \in \mathbb{K}$, $p(kx) = |k| p(x)$.

Here is another way to state “semi-norm”.

Definition (Sub-additive)**.** Let X be a linear normed space. A function $f\colon X \to \mathbb{R}$ is said to be sub-additive if

$$\forall x, y \in X\colon f(x) + f(y) \leq f(x) + f(y).$$

Definition (Positive-homogeneity)**.** Let X be a linear normed space. A function $f\colon X \to \mathbb{R}$ is said to be positive-homogeneous if

$$\forall x \in X, \alpha \in [0, \infty)\colon f(\alpha x) = \alpha f(x).$$

Thus, a function $p\colon X \to \mathbb{R}$ is a semi=norm if and only if (p is sub additive $\wedge$ p is positive-homogeneous). Let

$$X'_+ := \{f\colon X \to \mathbb{R} \mid f \text{ is sub-additive} \wedge \text{ homogeneous}\}.$$

Theorem 9.3. Let X be a linear normed space over $\mathbb{R}$ and $p \in X'_+$. Suppose $X_0 \hookrightarrow X$ and $f \in \mathcal{L}(X, \mathbb{R})$. Then

1. $\exists f \in \mathcal{L}(X, \mathbb{R})$ such that $f|_{X_0} = f$;

2. If $f_0 \leq p|_{X_0}$, then $f \leq p$.

1 is equivalent to the following commutative diagram

$$\begin{tikzcd} X_0 \arrow[rr, "f_0"] \arrow[dr, hook, "\iota"'] & & \mathbb{R} \\ & X \arrow[ur, "\exists f"'] & \end{tikzcd}$$

Remark. We've prove this theorem, see 9.1. Why we need Zorn's lemma here? Suppose $X_0 \neq X$ and $x_0 \in X \setminus X_0$. Let $M := \operatorname{span}(\{x_0\} \cup X_0) \hookrightarrow X$. Notice that $\dim\big(M \setminus X_0\big) = 1$. Thus, bu Mathematical Induction, we can prove the case $\dim\big(X \setminus X_0\big) < \infty$. To get rid of the assumption $\dim\big(X \setminus X_0\big) < \infty$, we need “” in some sense, which is relevant to the Axiom of Choice (equivalent to Zorn's lemma). And how do we apply Zorn's lemma? Recall how do we define the partially ordered set $(\mathcal{F}, \leq)$.

Theorem 9.4. Let X be a linear normed space over $\mathbb{K}$, $X_0 \hookrightarrow X$ and $f_0 \in \mathcal{L}(X_0, \mathbb{K})$. Then

1. $\exists f \in \mathcal{L}(X, \mathbb{K})$ such that $f|_{X_0} = f$, i.e. the following diagram commutes
2. $\|f\|_{X^*} = \|f_0\|_{X_0^*}$.

Remark. Proof of this theorem ($\mathbb{K} = \mathbb{C}$ case) needs an observation: $\forall f \in \mathcal{L}(X, \mathbb{C})$, f is uniquely determined by $\operatorname{Re} f$ since

$$\forall x \in X, f(x) = f_1(x) - i f_1(ix),$$

where $f_1 = \operatorname{Re}(f)$.

Here is some corollaries of Theorem 9.4.

Corollary 9.5. Let X be a linear normed space over $\mathbb{K}$, $x_0 \in X$ and $x_0 \neq 0$. Then there is $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.

Proof. It suffices to show that there is an $f \in X^*$ such that $|f(x_0)| = \|x_0\|$, since we can multiply f by a constant $\operatorname{sign}\big(f(x_0)\big)$. Let $X_0 := \operatorname{span}\{x_0\}$, a subspace of X and

$$f_0 \colon X_0 \to \mathbb{K}, k \cdot x_0 \mapsto k \cdot \|x_0\|.$$

Clearly $\forall k \in \mathbb{K}, |f_0(kx_0)| \leq |k||x_0|$ and $\|kx_0\| = |k|\|x_0\|$, thus $\|f_0\|_{X_0^*} \leq 1$. And the inverse inequality holds, since

$$f\big(x_0/\|x_0\|\big) = 1, \|x_0/\|x_0\|\| = 1.$$

Then apply Theorem 9.4, there is a function $f \colon X \to \mathbb{C}$ such that $f|_{X_0} = f_0 \wedge \|f\|_{X^*} = \|f_0\|_{X_0^*}$. And

$$f(x_0) = f|_{X_0}(x_0) = f_0(x_0) = \|x_0\|. \qquad \square$$

For both Theorem 9.3 and Theorem 9.4, the extension doesn't need to be unique. But there is a result for some unique extension.

Exercise. Let X be a linear normed space and $X_0 \hookrightarrow X$ is a dense subspace. Suppose $f_0 \in X_0^*$, then there is a unique $f \in X^*$ such that $f|_{X_0} = f$. In other words, the following diagram commutes.

$$\begin{tikzcd}
X_0 \arrow[rr, "f_0"] \arrow[dr, hook, "\iota"'] & & \mathbb{K} \\
& X \arrow[ur, dashed, "\exists f"'] &
\end{tikzcd}$$

Proof. $\square$

Corollary 9.6. Let X be a linear normed space and $x_1, x_2 \in X$ satisfy $x_1 \neq x_2$. Then there is $f \in X^*$ such that $f(x_1) \neq f(x_2)$. In other words, X^* separates points of X.

Proof. Let $x_0 := x_1 - x_2$ and apply Corollary 9.5. □

Corollary 9.7. Let X be a linear normed space and $x_0 \in X$. If $\forall f \in X^* : f(x_0) = 0$ then $x_0 = 0$.

Proof. If $x_0 \neq 0$, apply Corollary 9.6 and get a contradiction. □

A Hamel basis

Definition (Partial order, Partially ordered set)**.** A binary relation on X is called an **partial order** on X if it satisfies

1. $x \prec y \wedge y \prec z \implies x \prec z$;
2. $\forall x \in X \; x \prec x$;
3. $x \prec y \wedge y \prec x \implies x = y$.

A set with an order is called an **partially ordered set**.

Remark. In fact, an order on X can be defined as a binary relation, i.e. a subset of $X \times X$. But we don't care this now.

Definition (Total order, Totally ordered set)**.** An order is said to be **linear**, if $\forall x, y \in X (x \prec y \vee y \prec x)$. A set with a linear order is called a **totally ordered set**.

Definition (Bound, Bounded set, Maximal element)**.** Let X be an ordered set and $Y \subseteq X$. An element $x \in X$ is called a **bound** for Y if $y \prec x (\forall y \in Y)$ and at the same time Y is called a **bounded set**. An element $m \in X$ is called a **maximal element** if $\forall y \in X \neg(m \prec y)$.

Axiom (Zorn's lemma)**.** Let X be an ordered set with the following property: every totally ordered subset of X (in the sense of the order induced by the initial order of X) is bounded. Then there is at least one maximal element in X.

This is equivalent to the **Axiom of Choice**, which cannot be proved from the other axioms of set theory. To define base, we need the notion of linear independence.

Definition (Linearly Independent, Hamel Base)**.** Let V be a linear space over $\mathbb{K}$. A system of vectors of V is called **linearly independent** if every finite subsystem of this system is linearly independent (i.e. every finite combination gives 0 if and only if all coefficients are 0).

A family of vectors $\{e_i \in V : i \in I\}$ is called a **Hamel basis** of V, if $\forall x \in V, x \neq 0$ can be uniquely represented as a (finite) linear combination of vectors in $\{e_i : i \in I\}$.

Theorem A.1 (Existence of Hamel base)**.** Each linear space V (over an arbitrary field) has a Hamel Base.

Proof. To use Zorn's lemma, we need to construct an ordered set whose maximal element can be a Hamel basis of V. Thus, consider

$$\mathcal{D} := \{X \subseteq V : X \text{ is linearly independent}\}$$

with the order: $\forall A, B \in \mathcal{D} : A \prec B \iff A \subseteq B$. Given an arbitrary totally ordered set $\mathcal{A} \subseteq \mathcal{D}$, we have a bound for $\mathcal{A}$ — $\bigcup \mathcal{A}$. To show that $\bigcup \mathcal{A} \in \mathcal{D}$, taking arbitrary $e_1, e_2, \ldots, e_n \in \bigcup \mathcal{A}$ such that $e_j \in X_j \in \mathcal{A}$ for all $j = 1, 2, \ldots, n$. Since $\mathcal{A}$ is totally ordered, we can suppose $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$ and hence $e_j \in X_n (\forall j = 1, 2, \ldots, n)$. Since $X_n \in \mathcal{A} \subseteq \mathcal{D}$, X_n is linear independent and hence $e_1, e_2, \ldots, e_n$ is linearly independent. Therefore, $\bigcup \mathcal{A} \in \mathcal{D}$. Now apply Zorn's lemma and we know there is a maximal element B in $\mathcal{D}$. And B is a Hamel basis. To show this, it suffices to prove that every element in V lies in $\operatorname{span}(B)$. If there is an element $v \in V$ such that $v \notin \operatorname{span}(B)$, i.e. $B \cup \{v\}$ is linearly independent. This is impossible by the definition of the maximal element. □

In fact, we can define Hamel bases for an arbitrary linear space over an arbitrary field such as $\mathbb{Q}$ and $\mathbb{F}_p$ for some prime p.

Here I explain why we can view a vector space over $\mathbb{C}$ as a vector space over $\mathbb{R}$.

Proposition A.2. Let V be a vector space over $\mathbb{C}$, then there is a real vector space W and a $\mathbb{R}$-linear mapping $\varphi \colon V$ such that $\mathbb{R}$ is a bijection.

Proof. Let $\{v_\alpha\}_{\alpha \in I}$ be a base for V. Consider the set $W := V \times iV$, where $iV = \{iv : v \in V\}$ with the natural real linear structure. Now I define a linear structure on W such that W is a $\mathbb{R}$-linear space. Then, define the mapping $\varphi \colon V \to W$ by

$$\forall \alpha \in I, \forall z \in \mathbb{C} : \varphi(z v_\alpha) := \big(\operatorname{Re}(z) v_\alpha, \operatorname{Im}(z) v_\alpha\big).$$

Extend φ to V keeping $\mathbb{R}$-linear. Then $\varphi|^{\operatorname{Im}(\varphi)}$ is what we wanted. □

B Banach functor

This appendix comes from [3].

Recall the exercise:

Exercise. If Y is a Banach space and X is a linear normed space, then $\mathcal{B}(X,Y)$ is a Banach space. Especially, X^* is a Banach space.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(X,Y)$ be a Cauchy sequence. Thus

$$\lim_{m,n} \|u_n - u_m\|_{\mathcal{B}(X,Y)} = 0.$$

Taking an arbitrary $x \in X$, we have

$$\begin{aligned} \|u_n x - u_m x\|_Y &\le \|(u_n - u_m)x\|_Y \\ &\le \|u_n - u_m\|_{\mathcal{B}(X,Y)} \|x\|_X \\ &\to 0 (m, n \to \infty). \end{aligned} \tag{23}$$

Therefore, $(u_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Since Y is a Banach space, we know $(u_n x)_{n\in\mathbb{N}}$ converges to some point in Y. Thus we can define a map

$$u\colon X \to Y, x \mapsto \lim_n u_n(x).$$

And now we prove that $(u_n)_{n\in\mathbb{N}} \to u$ in $\mathcal{B}(X,Y)$. This proof is similar to the proof of "uniform limit of a continuous function sequence is continuous", see this proof.

By definition, $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $\forall m, n \ge N$ we have

$$\|u_n - u_m\|_{\mathcal{B}(X,Y)} < \varepsilon,$$

which implies

$$\|u_n x - u_m x\|_Y = \|(u_n - u_m)x\|_Y \le \varepsilon \|x\|, \forall x \in X.$$

Let $m \to \infty$, by the continuity of $\| \ \|_Y$, we have

$$\|(u_n - u)x\| = \|u_n x - ux\|_Y \le \varepsilon \|x\|, \forall x \in X.$$

Therefore, $\|u_n - u\| \le \varepsilon$ holds for all $n > N$. That is $(u_n)_{n\in\mathbb{N}} \to u$. $\square$

Now we can define

Definition. The contravariant functor

$$\begin{aligned} *\colon \mathsf{Nor} &\longrightarrow \mathsf{Ban}, \\ \mathrm{Ob}(\mathsf{Nor}) \ni X &\longmapsto X^*, \\ \mathrm{Mor}(\mathsf{Nor}) \ni \varphi\colon X_1 \to X_2 &\longmapsto \varphi^*\colon X_2^* \to X_1^*. \end{aligned} \tag{24}$$

Where $\varphi^*\colon X_2^* \to X_1^*, f \mapsto f \circ \varphi$.

Remark. Banach Functor is a special case of the functor $\mathcal{B}(\ ,Y)$ where Y is a Banach space, defined as

$$\begin{aligned} *\colon \mathsf{Nor} &\longrightarrow \mathsf{Ban}, \\ \mathrm{Ob}(\mathsf{Nor}) \ni X &\longmapsto \mathcal{B}(X,Y), \\ \mathrm{Mor}(\mathsf{Nor}) \ni \varphi\colon X_1 \to X_2 &\longmapsto \mathcal{B}(\varphi,Y)\colon \mathcal{B}(X_1,Y) \to \mathcal{B}(X_2,Y). \end{aligned}$$

Here $\mathcal{B}(\varphi,Y)\colon \mathcal{B}(X_1,Y) \to \mathcal{B}(X_2,Y), f \mapsto f \circ \varphi$.

Banach functor is surely a functor.

Proof. It suffices to prove that $(\mathrm{id}_X)^* = \mathrm{id}_{X^*}$ and $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$. To prove two maps are the same, we should prove that they coincide at every point.

- Given $X \in \mathrm{Ob}(\mathsf{Nor})$, we have

  $$\forall f \in X^* : (\mathrm{id}_X)^*(f) = f \circ \mathrm{id}_X = f = \mathrm{id}_{X^*}(f).$$

  Thus $(\mathrm{id}_X)^* = \mathrm{id}_{X^*}$, since id_{X^*} is uniquely determined by this property.

- Given $X_1 \xleftarrow{\varphi} X_2 \xleftarrow{\psi} X_3$. Notice that $\mathrm{dom}((\varphi \circ \psi)^*) = X_1^*$ and for all $f \in X_1^*$, we have

  $$\begin{aligned} (\varphi \circ \psi)^*(f) &= f \circ (\varphi \circ \psi) \\ &= (f \circ \varphi) \circ \psi \\ &= \big(\varphi^*(f)\big) \circ \psi \\ &= \psi^*\big(\varphi^*(f)\big) \\ &= (\psi^* \circ \varphi^*)(f). \end{aligned}$$

  This means $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$.

Now, What is needed to check is just $\varphi^* \in \mathcal{B}(X_2^*, X_1^*)$, and this is true since $\forall f \in X_2^*$

$$\|\varphi^*(f)\|_{X_1^*} = \|f \circ \varphi\|_{\mathcal{B}(X_1,\mathbb{K})} \le \|f\|_{X_2^*} \|\varphi\|_{\mathcal{B}(X_1,X_2)},$$

hence $\|\varphi^*\|_{\mathcal{B}(X_2^*,X_1^*)} \le \|\varphi\|_{\mathcal{B}(X_1,X_2)}$. □

If we restrict $*\colon \mathsf{Nor} \to \mathsf{Ban}$ to the full subcategory Ban, and call it **Banach adjointness functor**, then we can consider the composition of $*\colon \mathsf{Ban} \to \mathsf{Ban}$, i.e. $**\colon \mathsf{Ban} \to \mathsf{Ban}$, which is covariant. On object, X^{**} is the usual second dual space; on morphism, T^{**} is the usual second dual operator. This is the main topic about "the dual theory of Banach space".

Similarly we can define a covariant functor

$$\begin{aligned}
\mathcal{B}(\mathbb{K},\ )\colon \mathsf{Ban} &\longrightarrow \mathsf{Ban}\\
X &\longmapsto \mathcal{B}(\mathbb{K}, X),\\
(f\colon X\to Y) &\longmapsto (\mathcal{B}(\mathbb{K}, f)\colon \mathcal{B}(\mathbb{K}, X)\to \mathcal{B}(\mathbb{K}, Y)),
\end{aligned}$$

where

$$\mathcal{B}(\mathbb{K}, f)\colon \mathcal{B}(\mathbb{K}, X)\to \mathcal{B}(\mathbb{K}, Y), \psi\mapsto f\circ\psi.$$

Exercise. Show that $\mathcal{B}(\mathbb{K},\ )$ is naturally equivalent to $\mathrm{id}_{\mathsf{Ban}}$, the identity functor of Ban.

Solution. Define the natural transformation $\theta = \{\theta_X\colon X\in \mathrm{Ob}(\mathsf{Ban})\}$ as follows

$$\theta_X\colon \mathcal{B}(\mathbb{K}, X)\to X, \varphi\mapsto \varphi(1),$$

where 1 is just the multiplicative identity of $\mathbb{K}$. We can check that $\ker\theta_X = \{0\}$ and $\mathrm{Im}\,\theta_X = X$: $\theta_X(\varphi) = 0$ if and only if $\varphi(1) = 0$, while $\varphi\in\mathcal{B}(\mathbb{K}, X)$ so $\varphi(1) = 0$ if and only if $\varphi = 0$; for arbitrary $x\in X$, define $\varphi_x\colon \mathbb{K}\to X, z\mapsto zx$ and then $\theta_X(\varphi_x) = x$ as we want. Thus θ_X is an isomorphism for each $X\in\mathrm{Ob}(\mathsf{Ban})$.

Then we check the diagram commutes

$$\begin{tikzcd}
\mathcal{B}(\mathbb{K}, X) \arrow[r, "\theta_X"] \arrow[d, "\mathcal{B}(\mathbb{K},f)"'] & \mathrm{id}_{\mathsf{Ban}}(X) \arrow[d, "\mathrm{id}_{\mathsf{Ban}}(f)"] \\
\mathcal{B}(\mathbb{K}, Y) \arrow[r, "\theta_Y"'] & \mathrm{id}_{\mathsf{Ban}}(Y)
\end{tikzcd}$$

And we ignore $\mathrm{id}_{\mathsf{Ban}}$ since it makes nothing different. Given $\varphi\in\mathcal{B}(X, C)$, on the one hand

$$(f\circ\theta_X)(\varphi) = f\big(\theta_X(\varphi)\big) = f\big(\varphi(1)\big).$$

On the other hand

$$\big(\theta_Y\circ\mathcal{B}(f,\mathbb{K})\big)(\varphi) = \theta_Y\Big(\big(\mathcal{B}(f,\mathbb{K})\big)(\varphi)\Big) = \theta_Y(f\circ\varphi) = (f\circ\varphi)(1).$$

We're done since $f(\varphi(1)) = (f\circ\varphi)(1)$. $\square$

Remark. The definition of θ_X is natural since this is the simplest element of $\mathcal{B}(\mathcal{B}(\mathbb{K}, X), X)$.

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