Quotient Category and Sub-Category

A short story

Hao Fan

Faculty of Mathematics
Undergraduate

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Recall

We have studied a special category (hTop), defined as follows:

$$\mathrm{Ob}\,(\mathrm{hTop}) := \mathrm{Ob}\,(\mathrm{Top})$$
$$\mathrm{Hom}_{(\mathrm{hTop})}(X, Y) := \mathrm{Hom}_{(\mathrm{Top})}(X, Y)/\simeq,$$

where $\simeq$ satisfies:

$$f_0 \simeq f_1, g_0 \simeq g_1 \implies g_0 \circ f_0 \simeq g_1 \circ f_1.$$

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Quotient Category

As a generalization:

Definition

Let a category $\mathcal{C}$ and (a family of) equivalence relations $\overset{A,B}{\sim}$ on each set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, where $A, B \in \mathrm{Ob}(\mathcal{C})$ be given. We define a new category $\mathcal{C}/\sim$ as follows:

$$\mathrm{Ob}(\mathcal{C}/\sim) := \mathrm{Ob}(\mathcal{C}),$$

$$\mathrm{Hom}_{\mathcal{C}/\sim}(A, B) := \mathrm{Hom}_{\mathcal{C}}(A, B)/\overset{A,B}{\sim}.$$

where $\simeq$ satisfies: $f_0 \simeq f_1, g_0 \simeq g_1 \implies g_0 \circ f_0 \simeq g_1 \circ f_1$. It automatically has the identity and associativity holds.

Quotient Category

Remark

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Corollary

If $A \cong B$ in the category $\mathcal{C}$, then $A \cong B$ in $\mathcal{C}/\sim$.

Examples of Quotient Categories

Example

The category (hTop) $=$ (Top)$/\simeq$, where $\simeq$ is the homotopy relation.

Example

The category ($\mathrm{Lin}_{\mathbb{C}}$), with equivalence relations $\sim$ defined as follows: for $f, g \in \mathcal{L}(X, Y)$, $f \sim g \iff \exists k \in \mathbb{C} \setminus \{0\}$ such that $f = kg$. Then we have a quotient category $(\mathrm{Lin}_{\mathbb{C}})/\sim$.

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Sub-Category

Definition

Let a category $\mathcal{C}$ be given. If another category $\mathcal{C}'$ satisfies:

1. $\mathrm{Ob}(\mathcal{C}') \subseteq \mathrm{Ob}(\mathcal{C})$;
2. $\forall A, B \in \mathrm{Ob}(\mathcal{C}') : \mathcal{C}'(A, B) := \mathrm{Hom}_{\mathcal{C}'}(A, B) \subseteq \mathcal{C}(A, B)$;
3. compositions in $\mathcal{C}'$ are birestrictions of compositions in $\mathcal{C}$, i.e. the following diagram commutes;

$$\begin{tikzcd} \mathcal{C}'(A,B) \times \mathcal{C}'(B,C) \arrow[r] \arrow[d] & \mathcal{C}'(A,C) \arrow[d] \\ \mathcal{C}(A,B) \times \mathcal{C}(B,C) \arrow[r] & \mathcal{C}(A,C) \end{tikzcd}$$

4. $\forall A \in \mathrm{Ob}(\mathcal{C}')$, the identity map of A (in $\mathcal{C}$), i.e. $1_A^{\mathcal{C}}$ lies in $\mathcal{C}'(A, A)$;

then we say $\mathcal{C}'$ is a sub-category of $\mathcal{C}$. denoted by $\mathcal{C}' \subset \mathcal{C}$.

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2. $(\mathrm{FinLin}_{\mathbb{C}}) \subset (\mathrm{Lin}_{\mathbb{C}})$, it is full;
3. (Ab) $\subset$ (Grp), it is full;
4. (Haus) $\subset$ (Top), it is full;

Sub-Category

Example

Let a fixed category $\mathcal{C}$ satisfying $\mathrm{Aut}(X, Y) \neq \mathrm{Hom}(X, Y)$ for some $X, Y \in \mathrm{Ob}(\mathcal{C})$ be given. Consider the sub-category $B\mathcal{C}$ defined as follows:

$$\mathrm{Ob}(B\mathcal{C}) := \mathrm{Ob}(\mathcal{C}), \mathrm{Hom}_{B\mathcal{C}}(X, Y) := \mathrm{Isom}_{\mathcal{C}}(X, Y).$$

Then is is a sub-category that is not full.