Rigid case

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The matrix is

$$A = \begin{pmatrix} 0 & 0.999999999239596 & -0.390573067729029 \\ 0 & 0 & -0.0511117554203621 \\ 0 & 0 & -0.0993802666255617 \end{pmatrix}. \tag{1}$$

Its eigenvalues are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -0.0993802666255617$, where eigenvalue 0 has multiplicity 2.

We know that for $\lambda = 0$, this case only has one eigenvector $\boldsymbol{v_1} = (1, 0, 0)^T$. This indicates a defect of 1, meaning that we need to fill up the multiplicity with chains of generalized eigenvector. Now, let us look for a generalized eigenvector $\boldsymbol{v_2}$:

$$(A - 0\, I)\boldsymbol{v_2} = \boldsymbol{v_1}. \tag{2}$$

Then, we find $\boldsymbol{v_2} = (0, 1, 0)^T$ (the first component can choose any value, and here we choose 0). For $\lambda = -0.0993802666255617$, we also know its eigenvector is $\boldsymbol{v_3}$, where $\boldsymbol{v_3} \neq \boldsymbol{0}$.

Therefore, we have

$$\begin{aligned} \begin{pmatrix} \hat{X}(t) \\ \hat{Y}(t) \\ \hat{\theta}(t) \end{pmatrix} &= c_1\, \boldsymbol{v_1}\, e^{\lambda_1 t} + c_2\, (\boldsymbol{v_2} + \boldsymbol{v_1} t)\, e^{\lambda_2 t} + c_3\, \boldsymbol{v_3}\, e^{\lambda_3 t}. \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t \right) + c_3\, \boldsymbol{v_3}\, e^{\lambda_3 t}. \end{aligned} \tag{3}$$