The Implementation of Black-Schole and Binomial Model to value American Option Prices

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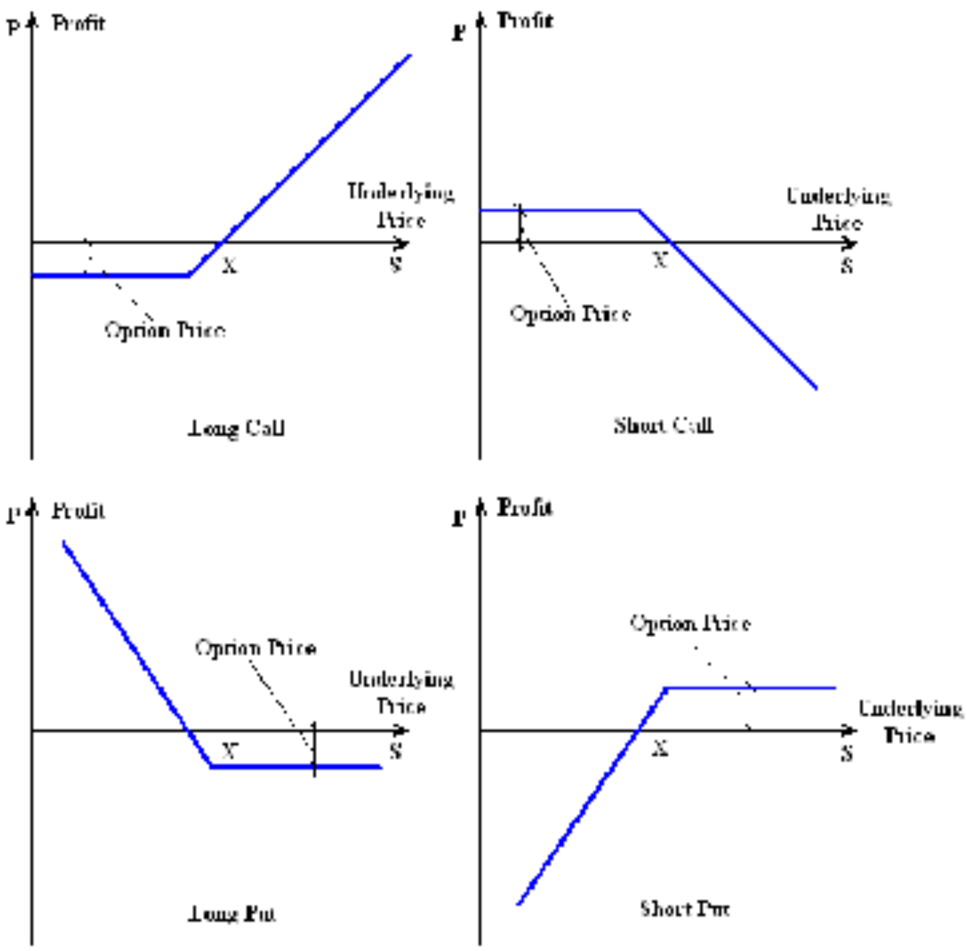
Abstract

Option is a widely used financial derivatives which provides the option holders the right but not the obligation to exercise their options to buy or sell certain amount of financial assets at certain price. Although frequently traded in the current financial market, the pricing of the option, especially the American options, which can be exercised at any time before expiration, is still a complex and inaccurate process. In order to improve the evaluation of options, many efforts have been devoted by researchers. This paper will first provide a theoretical insight of pricing options then introduce the most traditional and famous method which is the Black-Schole Model. A parallel computer program is also established to measure the accuracy of the calculation by this model. Then a new and widely used model which is the Binomial Tree Model will be introduced to evaluate the American options. The computer program will measure its accuracy and compare the results with the results generated by Black-Schole Model. Several other alternative methods will be mentioned which may serve as an insight to the audience about the future research direction of pricing the options.

1.Introduction

The option is one types of financial derivatives which specifies the legal contract between option buyers and sellers. Technically, the option enables its holders to possess the right, but not the obligation, to buy or sell certain underlying financial assets such as securities at certain prices within certain range of period. The option which offers option holders to purchase certain financial assets at certain price is defined as call options, while the option which provides option holders to sell certain financial assets at certain price is defined as put option. Currently, there are two types of options being exercised in the financial market which are European option and American option. The difference between these two types of options is easy to be classified by the distinctions between the execution dates. The European option allows the option holder to exercise the option only at the day which is the maturity date of the option, American option on the other hand, allows the option holder to exercise the option at any time between the purchase of the option and the maturity date. Although there exist other types of differences, the above features is the critical one to affect the pricing of these two types of options. Since the America options provide investor with opportunities to exercise the option any time before the expiration date, the investors need to pay for such “opportunities”, therefore, it is reasonable to detect that generally the America options will have higher price than European options for a certain underlying financial assets. In the current market in the United State, the majority options traded are the financial options for major corporations and almost all of these options are America options. The option which sets financial market index as its underlying financial assets will be more likely European options. Since the investors are focus more on the America option, it is meaningful to research and verify the pricing mechanism for this type of option.

Since the facts of early exercise can be ignored during the pricing of European option, the evaluation of the European option is much easier than American option. The payoff for the European option therefore, highly depends on the price of underlying financial assets at the expiration date. Take the payoff for a call option as an example, if the underlying financial assets have higher value than the striking price for the call option, then the option holder will exercise the option. The operation that the option holder going to conduct is to buy certain amount of underlying financial assets at striking price as specified by the call option then sell it to the market at the market price. The profits are derived from the price differences between market price and striking price. If on the expiration day, the market price is lower than the striking price, then the option will not be exercised and therefore the payoff will be zero. The payoff for the call and put options can be visualize by the following two figures:



The long and short stand for long and short position. Here, long position is the option buyers and short position is the option sellers.

The traditional financial pricing models tend to consider the present value of a financial asset as its price. Theoretically, the value of an option can be derived from its payoff from long position by discounting the money flows by risk-free interest rates. However, the payoff shows above cannot served as a standard benchmark for general references since different underlying financial assets tend to have various risk levels. With various volatility values, it is almost impossible to judge if or not a certain financial asset will arrive at a certain price that can be exercised by its option holders. However, the volatile underlying stock prices can provide the investors with general ideas about the range of values of the call options. Specifically, if the underlying financial assets have higher price, then their call option tend to have higher price. Since the higher the stock price is, the more possible that the option will be exercised. The lower the price of the underlying financial asset, the lower the call option price and at certain point, if the price of the underlying financial assets is low enough, the price of the call option can reach zero since there is no possibility that the option will be exercised. Assume the call option is going to be exercised for sure, then its value is equal to the stock price at the maturity date minus the price of a treasury bond with the face value equal to the striking price. The insight behind this above statement is that you can simply use certain about of money today to purchase a treasury bond and receive its payback which is equal to the striking price of the call option, then use this price to exercise the call option and sell the stock by the price on the exercise day. Therefore, the value of a call option can be calculate by the following formula If the stock price on the expiration day is participated:

$$P = K - \frac{S}{(1+r)^t}$$

In the above formula, the P is the price for the call option, the K stands for the underlying asset price at the exercise date, S is the striking price and r is the risk-free interest rate and t is the maturity. As shown by the right hand side of the formula, if the maturity is very long, then the current price of the treasury bond can be very low or even nearly zero. Then in this case, the value of the call option which can be very close to the price of the underlying assets at the exercise date. Therefore, a diagram can be drawn to illustrate the relationship between the striking price, underlying stock price as well as the call option price as follows:

However, all of this illustration mentioned above can only serve as a theoretical guidance for the pricing of options. The evaluation of option price in the real financial market is far more complex then the theoretical model shown above, especially for American options. Thanks to the contribution for the continuous efforts committed by researchers, there are several mathematic models established to provide professional evaluation for the option price and the most famous one is the Black-Schole Model raised by economists Fischer Black and Myron Scholes. Their model provides a standard method to calculate the price for options however, several limits in their model actually constrain the accuracy of the evaluations. More importantly, the Black-Schole model is base on the pricing for the European option which assume the option can only be exercised on maturity date. This features further restrict the functionality of this model since nowadays, most investors in the financial market are more familiar with American options. Some further research is conducted to searching for solution of the evaluation of the American options. For this paper, a parallel computer program has been established to implement the Black-Schole model and measure its accuracy of calculating American options. Then several other option pricing methods are also researched and realized by parallel computing program to verify their effectiveness. Finally, several other edge-cutting methods such as using neural networking in deep learning to evaluate option will also be mentioned to provide the audience with more ideas about the research tendency in this field.

2.Black-Schole Model and its implementation

According to the original research paper written by Black and Schole, the model is based on seven assumptions. Among all these assumptions, several of the assumptions are established for the convenience of calculation such as the stable interest rate and no dividends paid during the maturity, etc. There are two critical assumptions however should be paid much attention. The first essential assumption is that the Black-Schole will only be used to calculate the price for European options or in other words, the model assumes there is no early exercise for the options. The second important assumption is that the price of the underlying financial assets follows a random work with the variance rate proportional to the square of the price. This means the distribution of the price of the underlying financial asset at the end of maturity follows a log-normal distribution.

As discussed in the introduction section of this paper, the most straightforward way of calculating the price of an option is through the payoff of the option. If we can lock on the price of underlying financial assets at the end of the maturity, then the price of option can be calculated by following the formula 1. Actually, the general logic of the Black-Schole model is fairly simple as shown in formula 1. It tries to lock on the parameters on the right hand side of the equation by making series of assumptions, then calculate the price by this formula.

Here in this paper, only the calculation of the call option by the Black-Schole Model will be reviewed. The formula for call option calculation is shown as follows:

$$C = SN(d_1) - N(d_2)Ke^{-rt}$$

C = Call premium
S = Current stock price
t = Time until option exercise
K = Option striking price
r = Risk-free interest rate
N = Cumulative standard normal distribution
e = Exponential term

s = St. Deviation
ln = Natural Log

$$d_1 = \frac{\ln\left(S/K\right) + \left(r + s^2/2\right)t}{s \cdot \sqrt{t}}$$

$$d_2 = d_1 - s \cdot \sqrt{t}$$

As shown in the figure, the formula is generally derived from the formula 1 introduced above. The S and K represents current stock price and striking price. The portion e up to -rt simply serves as the discounting factor to discount the striking price to the present value. The N(d2) is the probability that the current stock price will go up to be higher than the striking price at the expiration date. The definition of N(d1) is much complicated. Mathematically, it is equal to the following expression:

$$E(S|S > K) * P(S > k)$$

The portion before the multiplication is the conditional expected value for the stock price at the expiration date. Plainly speaking, it presents the expected value of the stock price at the expiration date given the fact that the stock price at the expiration date is higher than the striking price. The portion behind the multiplication is much easier to understand.

It represents the the probability that the stock price at the expiration date will be higher than the striking price. Therefore, the N(d1) represents the future value of the stock price if and only if the price is higher than striking price.

Since the other critical assumption of the Black-Schole Model is that the stock price follows the random walks. Then the increase of the stock price can be summarized by a simple model with growth rate following the random walk model with drift. The formula is shown below:

$$FS = S * e^{(r-\frac{\sigma^2}{2})*t+\sigma*W_t}$$

The FS is the future stock price and S is the current stock price. The portion behind the multiplication is the increase rate. For the exponent of e, it follows a standard model of stochastic random walk with drift.

$$(r - \frac{\sigma^2}{2}) * t + \sigma * W_t$$

The portion of the expression before the addition sign is a constant drift. It simply represents that without risk, the price of stock will grow at the rate which is equal to the risk-free interest rate. The portion of the expression behind the addition sign is the random stochastic component which counts the risk of this stock. Generally speaking, the random walk growth model basically split the facts of price growth into two categories. The first categories is the dominant factor for a stock price to growth which is the risk-free interest rate. It simply assume that without other external impacts, the stock price will growth as the effect of economy growth. The other source of price change comes from the risk of a certain stock. Risk may contribute to increase in the stock price and may also cause price drops. If we use letter U to substitute the random walk expression, then the above growth model can be simplify to the following formula:

$$FS = S * e^U$$

If the periodic length of this model is set to one day, which means FS stands for the price of the stock for tomorrow and S is today's stock price, then by central limit theory, if adequate amount of periodic daily returns are graphed, the distribution of the price for tomorrow will follows a normal distribution with the mean to be the risk-free interest rate. The d2 in the Black-Schole Model tries to normalize this normal distribution and finds the rate of growth which will enable the stock price to go beyond the striking price at the expiration. Remember the normalization for a normal distribution is:

$$z = \frac{x - \mu}{\sigma}$$

And the comparison here is the growth rate instead of absolute growth value. Therefore the x parameter here is supposed to be K/S instead of S. Therefore, we can substitute x and μ by the following two expressions:

$$ln(K/S)$$

$$(r - \frac{\mu^2}{2})$$

Finally by substituting the variance, we can arrive at the expression for d2. The generation of d1 is bit more complex then d1 and the majority of steps are duplicated, therefore, details will not be mentioned here. The general meaning of d1 and d2 have been introduced above. Overall, the Black-Shcole Model follows the general idea introduced in introduction section but further expand the expression by using stochastic random walk and assuming the price of stock for the next periodic range follows a normal distribution which is generated by the implementation of central limit theory.

In order to verify the effectiveness of the Black-Schole Model, a parallel computing computer program has been developed to run on the historical option price date set. The data set is provided by https://www.discountoptiondata.com/home/productdetails. For the risk-free interest rate, this program choose to use the interest rate of the US 1 year treasury bill as the benchmark. The data set which is used in this program can be found at the official site of Federal Reserve Bank. The purpose of the implementation of parallel computation is to speed up the large volume of data and has no further relations with the results. The mechanism of the program is to gather information from all these data sets, and then feed the date to the Black-Scholde Model to calculate option prices and compare them with the actual prices found in these data sets. The accuracy of calculating American option prices by Black-Schole Model is very low. Given the confidence level of 90%, there are only 36% of valid estimations. The low accuracy is due to the fact that Black-Schole Model doesn't take the probability of early exercises into consideration. Besides, for the majority of the case, the stock price for tomorrow is influenced by many factors and doesn't follow the log normal distribution.

3.Binomial Model and Its Implementation

Since Black-Shcole Model alone cannot serve as an effective model of calculating the prices of American Options, the research tendency to solve this problem is to find a method to include the possibility of early exercise into consideration. This paper will focus on the Binomial Tree Model for pricing American options.

The Binomial Tree Model of calculating American option prices is basically focus on the American put option. The generally assumption that this model is built upon is that every investor is risk-neutral. The risk neutral is a definition which refers to the investment preference for a person. If an investor is only focus on the profits investments can bring and ignore the different risk levels behind different portfolio, then this investor is a risk-neutral investor.

Under this assumption, the model first asserts that the American call option has no difference than European call option. When an option is at a point, profitable to be exercised, the option holder can choose to exercise immediately or wait longer. If exercise immediately, the profit is simply S(t) - K where S(t) is the stock price at time t and K is the striking price. The motivation for any option holder to wait longer is that there may exist possibility that the stock price will increase faster than the discount rate therefore, a late exercise will bring even more profits. In a risk neutral world, the choice for the above situation will always be latter. Since the discounted value for a stock price tomorrow will always equal to the current stock price if the investment environment is risk neutral. In other words:

$$e^{-r\Delta t} * S(t + \Delta t) = S(t)$$

However, since the stinking price is a fixed number and is discounted, the striking price tomorrow will always smaller than today's striking price.

$$K > e^{-r\Delta t} * K$$

Therefore, the profit tomorrow will always higher than the profit of immediate exercise.

$$e^{-r\Delta t} * S(t + \Delta t) - e^{-r\Delta t} * K > S(t) - K$$

If the investment environment is risk neutral then we can arrive at the conclusion that the American options is no difference than European options.

For the American put options, the decision that the option holder needs to make can be expressed by the following formula:

$$E[e^{-r\Delta t} * (K - S(t + \Delta t)^{+}] \geq e^{-r\Delta t}K - S(t)$$

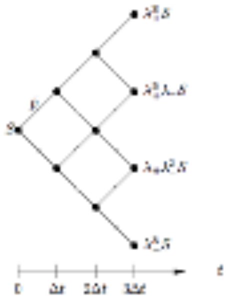
The right hand side of this expression is strictly less than the profit of immediate exercise which is

$$K - S(t)$$

Therefore, it will be hard to judge if or not the option holder will exercise the option or wait longer.

The Binomial Tree Model which is established under the risk neutral environment but divide the total expiration period into small pieces like one day to avoid the problem above. Specifically, it uses binomial distribution to set two results for the stock price at the next period which are moving up and moving down. For each of the different price changing direction, it sets parameters to calculate the scale of movements. These two parameters are introduced above, which are derived from the random walk model. These two parameters are defined as λ_+ and λ_- in this paper.

Then with the binomial results of the stock price in the next period (either increase or decrease for a certain amount), a binomial tree can be established to illustrate the possible stock prices in the future. The figure below is a sample binomial tree:



Since we can calculate the price of stock at any node of this tree by using the two parameters, we can set up a computer algorithm to calculate the option value at any points. The algorithm is established following a bottom-up fashion. Since at the leaves of this tree(the expiration date of the option), the decision to make is either exercise the option if profitable or do nothing. We can calculate the price for the leaves first. Then for the layer right before the leaves. For each node at this layer, we need to calculate two values. The first one is the profit of immediate exercise, which can be computed by following the binomial tree with two movement parameters. The corresponding mathematic formula is:

$$K - \lambda_+^k \lambda_-^{I-1-k} S$$

In this expression, K is the striking price, i is the number of total periods from the beginning of the option up to this layer and k is the number of periods within total periods that has price goes up, S as usual is the price of the stock at current stage.
The second value is the value of holding the option, which can by computed by discounting the expected future value of the option. Since we have mentioned above, the prerequisite of this model is that the market is risk neural, the probability for the stock price to move up is calculated by the following formula:

$$p = \frac{e^{r\Delta t} - \lambda_-}{\lambda_+ - \lambda_-}$$

Once we know the probability of the price to move up, we can calculate the value of holding a put option by computing the expected profit of holding the option. Assume the value of exercise the option at next period is represent as V(i, k) where i is the total number of periods and k is the number of periods within total periods that have the price goes up. Then the holding value at period i-1 can be represented as:

$$V\,Holding(I-1,k) = e^{-r\Delta t}(p * V(I,k+1) + (1-p) * V(I,k))$$

Then by comparing the value of holding the option with the profit of immediate exercise, the option value at every stage can be precisely pin down. When the algorithm finally arrive at the root of the binomial tree with is the origin V(0,0), the value is exactly the current price of the option.

A parallel computing program is also developed for this model with the same data set used above as the input data set. The results generated by this model is much more accurate than the results calculated by traditional Black-Schole Model.

However, besides the accuracy this model can reach, the overall accuracy for this model is still not ideal. The typical constraint for this model is the assumption that the market is risk neutral. Since the majority of the investor in the current market is risk averse which means if this type of investors are given two portfolio with low risk for one of them and high risk for the other one, then this type of investors will not be indifferent towards these two portfolio. They will prefer the portfolio with the low risk. Since most of the investors are not risk neutral, this can probably explain why the given the same condition, the American options will always have higher price than European options.

4.Alternative Methods

Since both models mentioned above cannot accurately pricing the American option, these year, much research has been conducted to searching for appropriate method of option evaluation. Currently, there are two general research tendencies for improving the accuracy of pricing American options. One of these research directions tries to improve the current Black-Schole Model. Since the critical assumptions of the Black-Schole Model restricts the functionalities, the relaxation of these restrictions is generally one method of improving this model.

Some researchers tries to pricing the American options by applying Black-Schole Model with a non-linear volatility function. Basically, for this model, the volatility, which in the traditional Balck-Schole Model is a parameter, is defined as a function which depends on the value of asset price and second derivative of option price. The purpose of modifying the volatility to a function is trying to make the model more flexible to the early exercise of the option. The model also includes series of improvements to take transaction costs in to consideration. Besides this model, there are numerous of research papers which focus on similar improvements such as the Jumping Volatility Model developed by Avellaneda, Levy and Paras, the nonlinear Black-Schole model which is derived from imperfect replication and investors preferences by Barles and Soner etc.

The other direction focuses on applying new technology rather than traditional mathematic model to pricing options. The most typical one is using the neural network to improve the pricing of American options. Neural networking is a computing system made up of a number of simple, highly interconnected processing elements which response to external inputs. Currently, neural networking has a wide application in artificial intelligence. The general idea for this model established by Ulrich Anders, Olaf Kom and Christian Schmitt is to gather several features as the inputs to the model. Then train the model to generate reliable results. According to their research paper, the pricing accuracy achieved by neural networking model is much higher than the results generated by traditional Black-Schole Model. However, since neural networking has a typical feature called inexplainable feature which means the processing of training the model and modifying the parameters and weights on each node is a process that cannot be controlled by programmer, people cannot exclude the coincidence of the correctness of the model. In other words, no one can provide valid proof to the reason why the model works well. Therefore, like many other fields which neural networking has achieved excellent results, the pricing of option using neural networking still cannot be used in the real market for people’s lack of control of the model.

5.Summary

Although the option is an usual type of financial derivatives which has been widely used in the current financial market, its evaluation is still a complicated process. Especially for American options, the option holder can choose to exercise the option at any time before the expiration date, the pricing of them is even more complex. Although many models have been come up in the last several decades trying to provide accurate evaluation for the pricing of options. The most famous one is the Black-Schole Model, which primarily provide a valid method of pricing European options, its functionalities are largely limited by its rigid assumptions. In order to provide accurate evaluation to American options, many new models such as the binomial model mentioned in this paper is developed. They can partially explain the pricing of American options currently traded in the financial market. However, majority of them, with many restricted assumptions, still fail to provide accurate evaluations for the American options prices. It is believed that the new methods like artificial neural networking or even the traditional machine learning methods can serve as ideal alternatives since all of these model can capture more features so that more factors which may influence option prices can be taken into consideration. Besides, with the improvement of relevant algorithms and computational capacities, the training of this model will become more reliable. In this case, it is reasonable to foresee a high connection between computer technology with traditional financial theory in the near future.