SUB-CYCLOTOMICS

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1. Introduction

We restrict our attention to subfields of cyclotomic fields $\mathbb{Q}(\zeta_m)$, where we assume m is *odd and squarefree*. The Galois group $Gal(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ is canonically isomorphic to $(\mathbb{Z}/m\mathbb{Z})^*$.

Notation: for each subgroup H of $G = (\mathbb{Z}/m\mathbb{Z})^*$, we use $K_{m,H}$ to denote the fixed field

$$K_{m,H} := \mathbb{Q}(\zeta_m)^H.$$

The extension $K_{m,H}/\mathbb{Q}$ is Galois of degree $n = \frac{\varphi(m)}{|H|}$; a prime q splits completely in $K_{m,H}$ if and only if $q \pmod m \in H$. In general, the degree of a prime q in $K_{m,H}$ is equal to the order of $[q]$ in the quotient group G/H.

Every field of form $K_{m,H}$ comes with a canonical*normal integral basis*, whose embedding matrix is easy to compute. More precisely, let C denote a set of coset representatives of the group G/H. For $c \in C$, set

$$w_c = \sum_{h \in H} \zeta_m^{hc}.$$

Then we have

Proposition 1.1. *$w = (w_c)_{c \in C}$ is a $\mathbb{Z}$-basis of $R = \mathcal{O}_K$. Let $\zeta = \exp(2\pi i/m)$. Then the canonical embedding embedding matrix of w is*

$$(A_w)_{i,j} = \sum_{h \in H} \zeta^{hij}.$$

Proposition 1.2. *By spherical symmetry and the property of the normal integral basis, the error distribution $D \pmod{\mathfrak{q}}$ is independent of the choice of $\mathfrak{q}$.*

2. Searching

We search for vulnerable instances among fields of form $K_{m,H}$. The search is done by generating actual RLWE samples from the instance and run the chi-square attack (Algorithm) on these samples. Success of the attack would indicate vulnerability. Our field search requires sampling efficiently from a discete Gaussian $D_{\Lambda,\sigma}$ for which we choose the method outlined in [GPV].

In the first table, we list some instances, for which the attack have succeeded. The columns are as follows. Note that we ommited the prime ideal $\mathfrak{q}$ due to Lemma . and t denotes the running time in seconds.

In the second table, we list some vulnerable instances we found, for which the attack is likely to succeed based on the theorem in chisquare test, but will take a long time to finish. Hence instead of running the actual attack, we first run the chi-square test on the correct error samples, and then run a few chisuqare tests on some random guesses of $s \pmod{\mathfrak{q}}$. We then estimate the success rate using the theorem. More

Table 2.1. Attacked sub-cyclotomic RLWE instances

m	generators of H	n	q	f	σ_0	no. samples used	running time of attack (in secs)
2805	[1684, 1618]	40	67	2	1	22445	12569.2
90321	[90320, 18514, 43405]	80	67	2	1	26934	17323.4
15015	[12286, 2003, 11936]	60	43	2	1	11094	3813

TABLE 2.2. Some Vulnerable sub-cyclotomic RLWE instances

m	generators of H	n	q	f	σ_0	no. samples used	est.runtime (h)	$\hat{p}$
255255	[97943, 83656, 77351, 78541, 129403]	90	463	2	1	21436	1786.41	1.0 (*)

precisely, suppose $\hat{\chi^2}$ is the chi-square value of the sample errors from $D_{\mathcal{R},\mathtt{q}}$. We replace λ by $\hat{\chi^2}$ in the formula and compute

$$\hat{p} = 0.904\left(1 - \Phi\left(\frac{\Phi^{-1}(1-\frac{1}{20N})\sqrt{2(N-1)} - \hat{\chi^2}}{\sqrt{2(N-1)+4\hat{\chi^2}}}\right)\right).$$

The value $\hat{p}$ is then our estimate of the sucess rate of our attack. In addition, we estimate the runtime based on the average time taken for the tests we've done.