

TITLE OF DOCUMENT

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1. THE χ^2 TEST FOR UNIFORM DISTRIBUTION

We briefly recall the property and usage of the χ^2 test for uniform distributions over a finite set S . Suppose M samples $y_1, \dots, y_M \in S$. We partition S into r subsets

$$S = \sqcup_{j=1}^r S_j,$$

For each $1 \leq j \leq r$, we compute the expected number of samples that would fall in the j -th subset: $c_j := |S_j|M/|S|$. Then we compute the actual number of samples, $t_j := |\{1 \leq i \leq M : y_i \in S_j\}|$. Finally, the χ^2 value is computed as

$$\chi^2(S, y) = \sum_{j=1}^r \frac{(t_j - c_j)^2}{c_j}.$$

Note that degree of freedom in this test is $d = r - 1$. To decide whether the samples are from a uniform distribution, we can either look up a table of χ^2 values, or use an approximation rule: when d is large, the χ^2 distribution can be well-approximated by a normal distribution $N(d, 2d)$; for example, if it turns out that $\chi^2 \notin (d - c\sqrt{2d}, d + c\sqrt{2d})$, then the confidence we have that the samples are not taken from a uniform distribution is $2\Phi(c) - 1$.

If P, Q are two probability distributions on S , then their *statistical difference* is defined as

$$d(P, Q) = \frac{1}{2} \sum_{t \in S} |P(t) - Q(t)|,$$

and their l_2 distance is

$$d_2(P, Q) = \left(\sum_{t \in S} |P(t) - Q(t)|^2 \right)^{\frac{1}{2}}.$$

We have $d(P, Q) \leq \frac{\sqrt{|S|}}{2} d_2(P, Q)$.

2. THE χ^2 ATTACK ON $SRLWE(\mathcal{R}, \mathfrak{q})$

Let \mathcal{R} be an RLWE instance with error distribution $D_{\mathcal{R}}$ and \mathfrak{q} be a prime ideal above q . Our attack relies on the assumption that the distribution $D_{\mathcal{R}} \pmod{\mathfrak{q}}$ is distinguishable from the uniform distribution on the finite field $F = R/\mathfrak{q}$. More precisely, the attack loop through all q^f possibilities of $\bar{s} = s \pmod{\mathfrak{q}}$. For each guess s' , it computes the values $\bar{e}' = \bar{b} - \bar{a}s' \pmod{\mathfrak{q}}$ for every sample $(a, b) \in S$. If the guess is wrong, or if the samples are taken from the uniform distribution in $(R_q)^2$ instead of an RLWE instance, the values \bar{e}' would be uniformly distributed in F and it is likely to pass the χ^2 test. On the other hand, if the guess is correct, then we expect the errors \bar{e}' to fail the χ^2 test.

Let $N = q^f$ denote the cardinality of F . Note that N is also the number of χ^2 tests we run in the attack. For the attack to be successful, we need the $(N - 1)$ tests corresponding to wrong guess of $s \pmod{\mathfrak{q}}$ to pass, and the one test corresponding to the correct guess to fail. Therefore, we need to choose the confidence interval of our χ^2 test so that it is unlikely for a set of samples coming from uniform distribution to fail the test. In practice, we choose the confidence level to be $\alpha = 1 - \frac{1}{10N}$. Let β denote the probability that the sample errors fails the uniform test with probability α . Then the probability that our algorithm will success is $p = (1 - \frac{1}{10N})^{N-1} \beta$. Note that when N is large, $(1 - \frac{1}{10N})^{N-1}$ is about $e^{-1/10} \approx 0.904$.

Algorithm 1 χ^2 -test attack of $SRLWE(\mathcal{R}, \mathfrak{q})$

Require: $R = (K, q, \sigma, s)$ – an RLWE instance.

$R = \mathcal{O}_K$ – the ring of integers of K .

n : the degree of K .

\mathfrak{q} : a prime ideal in K above q .

N : the cardinality of R/\mathfrak{q} .

S : a collection of M ($M = \Omega(N)$) RLWE samples $(a, b) \sim \mathcal{R}$.

Ensure: a guess of the value $s \pmod{\mathfrak{q}}$, or **NON-RLWE**, or **INSUFFICIENT-SAMPLES**

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1:  $\alpha \leftarrow 1 - \frac{1}{10N}$ .
2:  $\omega \leftarrow \Phi^{-1}((1 + \alpha)/2)$ 
3:  $G = \emptyset$ 
4: for  $s$  in  $F$  do
5:   for  $a, b$  in  $S$  do
6:      $E \leftarrow \emptyset$ .
7:      $\bar{a}, \bar{b} \leftarrow a \pmod{\mathfrak{q}}, b \pmod{\mathfrak{q}}$ .
8:      $\bar{e} \leftarrow \bar{b} - \bar{a}s$ .
9:     add  $e$  to  $E$ .
10:   end for
11:   Run  $\chi^2$  test on  $E$  and obtain the value  $\chi^2(E)$ .
12:   if  $|\chi^2(E) - B - 1| > \omega\sqrt{2B - 2}$  then
13:     add  $s$  to  $G$ 
14:   end if
15: end for
16: if  $G = \emptyset$  then
17:   return NOT RLWE
18: else if  $G = \{g\}$  then
19:   return  $g$ 
20: else
21:   return INSUFFICIENT-SAMPLES
22: end if
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The time complexity of the attack is $O(q^{2f})$.

Proposition 2.1. Assume $q \gg 1$. Let Δ denote the statistical distance between the distribution $E = D_{R, \sigma} \pmod{\mathfrak{q}}$ and the uniform distribution U on R/\mathfrak{q} . Then the above attack succeeds with probability at least

$$p = 0.904(1 - \Phi(\frac{\omega\sqrt{2(N-1)} - \lambda}{\sqrt{2(N-1) + 4\lambda}})),$$

where Φ is the cumulative distribution function for the standard Gaussian distribution, $\omega = \Phi^{-1}(1 - \frac{1}{20N})$, and $\lambda = 4M\Delta$.

Proof. The chi-square value for uniformity on samples from E follow a noncentral chisquare distribution with the same degree of freedom and a parameter λ given by

$$\lambda = MNd_2(E, U)^2.$$

In particular, we have $\lambda \geq 4M\Delta$. We approximate a noncentral chi-square distribution with degree of freedom k and parameter λ with a Gaussian distribution of mean $k + \lambda$ and variance $2k + 4\lambda$. The result now follows from the fact that the argument of Φ is a decreasing function of λ . \square

To get a sense of how the constants behave, we give a table containing some p values.

N	Δ	p
4093	0.02	0.58