Solution to 2013-1 problem 2

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1 Introduction

1.1 Problem description

You are allowed to transform positive integers n in the following way. Write n in base 2. Write plus signs between the bits at will, and then perform the additions of binary numbers. For example, $123_{10} = 1111011$ can become $11 + 1 + 10 + 11 = 9_{10}$.
Prove that it is possible to reduct arbitrary integer to 1 in a bounded number of steps. That is the existence of a constant C such that for any n there is a sequence of C transformations that starts with n and ends with 1.

1.2 Result

We prove in section 2 that the constant C is 2.

2 Proof

2.1 Notations

Throughout n will be a positive binary integer.

- Let $H(n)$ denote the number of 1's in n, for example $H(11101) = 4$
- we write $n \to m$ if there exists a transformation of n into m. Write $n = \overline{ab}$ if n is the concatenation of a, and b. For example, a = 11, b = 10, then $\overline{ab} = 1110$.
- Let $C(n)$ be the minimum number of transformations it takes to reduce n to 1.

2.2 lemmas and a theorem

It's clear that if $n = (2^s)_{10}$ for some $s \geq 0$, then $H(n) = C(n) = 1$. Our idea is to show C = 2 by showing that for all n, $n \to 2^s$ for some s.

The first lemma handles the case when $H(n) \leq 3$

Lemma 2.1. $H(n) \leq 3 \implies C(n) \leq 2$

Proof. When H(n) = 1(resp. 2), simply adding up all digits will give 1(resp. 2). And $2 \to 1$ by transforming 10 to 1+0.
When H(n) = 3, then n either starts with 10 or 11, so we have
$n \to 11 + 0 + ... + 1 + ... + 1 = 4_{10}$ or $n \to 10 + ... + 1 + ... + 1 = 4_{10}$
and $4_{10} = 100 \to 1 + 0 + 0 = 1$ □

Lemma 2.2. *if* $H(n) \in \{4, 5, 6, 7, 8\}$*, then we have* $n \to 8_{10}$ *unless* $n =$ *11111. In all cases we have* $C(n) \leq 2$

Proof. Below we present a case-by-case discussion, most of which are easy exercises.

H(n) = 4 : If n = ...111..., then 111+1 = 8_{10};
else If n = ...110..., then 110+1+1 = 8_{10}
else If n = ...101..., then 101+10+1 = 8_{10} or 101+11 = 8_{10}
else we must have n = ...100..., then 100+10+1+1 = 8_{10}
(some trivial details are ommited).

$H(n) = 5$: First we assume $n \neq 11111$.
If n = ...101..., then 101+1+1+1 = 8_{10}.
else we must have n = ...100..., then 100+1+1+1+1 = 8_{10}.
else we must have n = ...11..10, then 11+10+1+1 +1 = 8_{10}

If $n = 11111$, then $n \to 1111 + 1 = 10000$, hence $C(n) \leq 2$.

H(n) = 6 :

If n = ...11..., then 11+11+1+1 = 8_{10}.
else we must have n = ...10..10..., then 10+10+1+1+1+1 = 8_{10}

H(n) = 7 : n = ...11... , 11+1+1+1+1+1+1 = 8_{10}
otherwise n = ...10..., and 10+1+1+1+1+1+1 = 8_{10}.

8 : trivial.

□

The third and last lemma deals with the case $H(n) \in [8, 16]$

Lemma 2.3. *If $H(n) \in \{8 \cdots 16\}$, then $n \to 16_{10}$.*

Proof. Notice that $\{4, 6, 7, 8\} + \{4, 6, 7, 8\} = \{8 \cdots, 16\} \setminus \{9\}$. If $H(n) \neq 9$, we could write $n = \overline{ab}$, where $H(a), H(b) \in \{4, 6, 7, 8\}$, and lemma 2.2 shows that $a, b \to 8_{10}$, hence $n \to 16_{10}$.
When $H(n) = 9$, we can do the above except n starts and ends with 11111, which implies n = 111111111, but then $n \to 11 + 11 + 111 + 11 = 16_{10}$. So all cases are covered.

□

Now we are ready for the theorem:

Theorem 2.4. *Let $n \geq 1$,$s \geq 4$ be integer with $2^{s-1} \leq H(n) \leq 2^s$, then $n \to 2^s$. Therefore, $C(n) \leq 2$.*

Proof. When $s = 4$, this is lemma 2.3.
Suppose it's true for s, now for $s + 1$, $2^s \leq H(n) \leq 2^{s+1}$, it's clear that we can write $n = \overline{ab}$ with $2^{s-1} \leq H(a) \leq 2^s$,$2^{s-1} \leq H(b) \leq 2^s$. Now it follows by induction. □

At last, combining all the results in this section, we have $C(n) \leq 2$ for all n, since $C(2) = 2$, we identify the constant C to be 2.

3 acknowledgements

I would like to mention that the intuition to this proof is based on the data of C(n) of n up to 30000 computed using the SAGE software.