Computational aspects of modular parametrizations of elliptic curves

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Elliptic curves over $\mathbb{Q}$

Definition

An **elliptic curve** over $\mathbb{Q}$ is a nonsingular projective curve $E \subseteq \mathbb{P}^2$ with defining equation

$$y^2 z = x^3 + Axz^2 + Bz^3,$$

where $A, B \in \mathbb{Q}$ and $4A^3 + 27B^2 \neq 0$.

Theorem (Mordell-Weil)

$E(\mathbb{Q})$ is a finitely generated abelian group, i.e.,

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus T,$$

for some $r \geq 0$ and T finite.

r is called the rank. T is the torsion subgroup.

The BSD conjecture

There is an entire function $L(E, s)$ called the L-function of E.

The rank part of the Birch and Swinnerton-Dyer (BSD) conjecture is:

$$rank(E(\mathbb{Q})) = \operatorname{ord}_{s=1} L(E, s).$$

RHS is the analytic rank, denoted by $r_{an}(E)$.
The BSD conjecture is open when $r_{an}(E) > 1$.

The proof of BSD for $r_{an}(E) = 1$ uses a construction called Heegner points.

Modular curves

Let $N \geq 1$ be an integer, consider the group

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) : N \mid c \right\}.$$

Let $\mathcal{H}^* = \{z \in \mathbb{C} : im(z) > 0\} \cup \mathbb{P}^1\mathbb{Q}$.
$\Gamma_0(N)$ acts on $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$.

Definition

$X_0(N) = \Gamma_0(N) \backslash \mathcal{H}^*$.

- $X_0(N)$ has the structure of a nonsingular projective curve.
- Rational functions on $X_0(N)$ are called modular functions. They have *q-expansions* at infinity:

$$u(q) = \sum_{n \geq -m} b_n q^n, \; q = e^{2\pi i z}.$$

The modularity theorem

Theorem (Modularity)

For every elliptic curve $E/\mathbb{Q}$, there exists an integer $N > 1$ and a surjective morphism $\varphi : X_0(N) \to E$ defined over $\mathbb{Q}$.

The smallest N is called the conductor of E.

Let $\omega = \varphi^*(\frac{dx}{y})$. Then $\omega = cf(z)\mathrm{d}z$, where f is the modular form attached to E.

We assume E is optimal. Then φ is unique up to sign.

The critical subgroup $E_{crit}(\mathbb{Q})$

Let $R_\varphi = \sum(e_\varphi(z) - 1)[z]$ be the ramification divisor of φ.

Definition (Mazur, Swinnerton-Dyer)

The critical subgroup of E is

$$E_{crit}(\mathbb{Q}) = \langle tr(\varphi([z])) : [z] \in \operatorname{supp} R_\varphi \rangle \subseteq E(\mathbb{Q}),$$

where $tr(P) = \sum_{\sigma:\mathbb{Q}(P)\to\bar{\mathbb{Q}}} P^\sigma$.

- $R_\varphi = \operatorname{div}(\omega)$. In particular, $\deg R_\varphi = 2g(X_0(N)) - 2$.

Question

Is there an elliptic curve $E/\mathbb{Q}$ with $r_{an}(E) \geq 2$ and $\operatorname{rank}(E_{crit}(\mathbb{Q})) > 0$?

Critical j-polynomial

Definition

Write $\operatorname{div}(\omega) = \sum n_z[z]$. The critical j-polynomial of E is

$$F_{E,j}(x) = \prod_{z \in \operatorname{supp} \operatorname{div}(\omega), j(z) \neq \infty} (x - j(z))^{n_z}.$$

$F_{E,j}(x) \in \mathbb{Q}[x], \deg F_{E,j} \leq 2g - 2$. Equality holds if N is square free. For $h \in \mathbb{Q}(X_0(N))$, can define $F_{E,h}(x)$.

Polynomial Relation (I)

Let $r, u \in \mathbb{Q}(C)$, a minimal polynomial relation of r and u is an irreducible polynomial $P(x, y) \in \mathbb{Q}[x, y]$, such that $P(r, u) = 0$.
Say $P(x, y) = f_n(y)x^n + \cdots + f_1(y)x + f_0(y)$.

Proposition (C.)

If $\mathbb{Q}(C) = \mathbb{Q}(r, u)$ *and* $\gcd(f_0(y), f_n(y)) = 1$, *then*

$$f_0(y) = c \prod_{z \in \mathrm{div}_0(r) \setminus \mathrm{div}_\infty(u)} (y - u(z))^{mult_z(\mathrm{div}_0(r))}.$$

Polynomial Relation (II)

Set

$$r = j(j-1728)\frac{\omega}{\mathrm{d}j},\ u = \frac{1}{j}.$$

Then $r, u \in \mathbb{Q}(X_0(N))$, and $\mathrm{div}_0(r) = \mathrm{div}(\omega) + D_0$, where points in supp D_0 have j-value 0 or 1728.

Proposition (C.)

For $T \gg 0$, let $P(x,y) = f_n(y)x^n + \cdots + f_1(y)x + f_0(y)$ is a minimal polynomial relation of rj^T and u. Then

$$F_{E,j}(x) = f_0(1/x)\cdot x^A(x-1728)^B$$

where A, B are explicitly computable integers.

The Algorithm

Input: The newform f_E attached to elliptic curve $E/\mathbb{Q}$;
Output: $F_{E,j}$.

$r = \frac{j(j-1728) f_E dz}{dj}, \ u = \frac{1}{j}$.

Compute the q-expansions of r and u to q^M.

Solve the system $\sum c_{a,b} r(q)^a u(q)^b = 0 \pmod{q^M}$

Set $P(x,y) = \sum c_{a,b} x^a y^b$ and apply the theorem.

Example

$F_{\mathbf{44a},j}(x) = H_{-44}(x)^2$. $F_{\mathbf{37a},j}(x) = H_{-148}(x)$. $F_{\mathbf{37b},j}(x) = H_{-16}(x)^2$.

The critical subgroup $E_{crit}(\mathbb{Q})$

For $i = 2, 3$, let $\mathscr{E}_i(N)$ be the set of elliptic points on $X_0(N)$ of period i.

Lemma (C.)

$6P_{all} = -3\sum_{c\in\mathscr{E}_2(N)}\varphi(c) - 4\sum_{d\in\mathscr{E}_3(N)}\varphi(d).$

Theorem (C.)

Suppose $r_{an}(E) \geq 2$, and assume at least one of the following holds:
(1) $F_{E,j} = \prod_{m=1}^{k} H_{D_m}^{s_i} \cdot F_0$, where $\mathbb{Q}(\sqrt{D_m}) \neq \mathbb{Q}(\sqrt{D_n})$ for all $m \neq n$, and F_0 is irreducible.
(2) $F_{E,h}$ is irreducible for some non-constant function $h \in \mathbb{Q}(X_0(N))$.
Then $\mathrm{rank}(E_{crit}(\mathbb{Q})) = 0$.

Critical polynomials for elliptic curves of rank 2 and conductor < 1000 (I)

E^1	$g(X_0(N))$	h	Factorization of $F_{E,h}(x)$
389a	32	j	$H_{-19}(x)^2(x^{60}+\cdots)$
433a	35	j	$x^{68}+\cdots$
446d	55	j	$x^{108}+\cdots$
563a	47	j	$H_{-43}(x)^2(x^{90}-\cdots)$
571b	47	j	$H_{-67}(x)^2(x^{90}-\cdots)$
643a	53	j	$H_{-19}(x)^2(x^{102}-\cdots)$
664a	81	$\frac{\eta_4\eta_8^2\eta_{332}^5}{\eta_{166}\eta_{664}^6\eta_2}$	$x^{160}-\cdots$
655a	65	j	$x^{128}-\cdots$
681c	75	j	$x^{148}-\cdots$
707a	67	j	$x^{132}-\cdots$

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Critical polynomials for elliptic curves of rank 2 and conductor < 1000 (II)

E	$g(X_0(N))$	h	Factorization of $F_{E,h}(x)$
709a	58	j	$x^{114} - \cdots$
718b	89	j	$H_{-52}(x)^2(x^{172} - \cdots)$
794a	98	j	$H_{-4}(x)^2(x^{192} - \cdots)$
817a	71	j	$x^{140} - \cdots$
916c	113	j	$H_{-12}(x)^8(x^{216} + \cdots)$
944e	115	$\frac{\eta_{16}^4\eta_4^2}{\eta_8^6}$	$x^{224} - \cdots$.[2]
997b	82	j	$H_{-27}(x)^2(x^{160} - \cdots)$
997c	82	j	$x^{162} - \cdots$

[1]Here 4 of the critical points are cusps, so $\deg F = 2g - 6$.

Discussion

Corollary

For all elliptic curves E of rank 2 and conductor $N < 1000$, the rank of $E_{crit}(\mathbb{Q}))$ is 0.

Future work

- Compute $E_{crit}(\mathbb{Q})$ for $E =$ **5077a**.
- Prove or disprove that $\text{rank}(E_{crit}(\mathbb{Q})) = 0$ whenever $r_{an}(E)$ is even.

q-expansion of newforms at non-unitary cusps

Modular forms

Let f be a function $f : \mathcal{H} \to \mathbb{C}$, $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and let $k \in \mathbb{Z}$. The weight-k action of α on f is defined by

$$f|[\alpha]_k(z) := (cz+d)^{-k} f(\alpha z).$$

Definition

A **modular form** of weight k and level N is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ s.t.
(1) $f(z) = f|[\alpha]_k(z)$, $\forall \alpha \in \Gamma_0(N)$ $(\Gamma_1(N))$.
(2) f has holomorphic extension to all cusps of $X_0(N)$ $(X_1(N))$.

Cusp forms $=$ modular forms that are zero at all cusps.
Modular forms have q-expansions: $f(q) = \sum_{n \geq 0} a_n q^n$, $q = exp(2\pi i z)$.
The space of cusp forms $= S_k(N)$.

Operators on modular forms

- Hecke operators: a family $\{T_n, n \geq 1\} \cup \{\langle d \rangle : (d, N) = 1\}$ of commuting linear operators on $S_k(N)$.
- B_d and U_d operators: $B_d(\sum a_n q^n) = \sum a_n q^{nd}$, $U_d(\sum a_n q^n) = \sum a_{nd} q^n$.
- The Atkin-Lehner involution W_N. If f is a newform on $\Gamma_1(N)$, then

$$f|W_N = w(f)\bar{f}$$

$w(f) \in \mathbb{C}_1$ is called the pseudo-eigenvalue of f.

Newforms

- When $M \mid N$, $\exists$ degeneracy maps $S_k(M) \to S_k(N)$.
- Old subspace $=$ span of images of all degenaracy maps.
- New subspace $=$ (Old subspace)$^\perp$.
- $S_k(N)^{new}$ has a basis of simultaneous eigenforms for all Hecke operators. These eigenforms are called newforms.

Fourier expansion

Let $f \in S_k(\Gamma_0(N))$ be a newform and let $\mathfrak{c} \in X_0(N)$ be a cusp other than ∞.
Goal: compute the expansion of f at $\mathfrak{c}$.
First, only well-defined for $denom(\mathfrak{c})^2 \mid N$.
Equivalent to computing the expansion of

$$f \left| \left[\begin{pmatrix} 1 & 0 \\ ld & 1 \end{pmatrix} \right] \right.$$

at ∞ for all $d^2 \mid N, l \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Idea of computing

Let $S'_c = \begin{pmatrix} 1 & \frac{1}{c'} \\ 0 & 1 \end{pmatrix}$ and $A'_c = \left(\begin{smallmatrix} 1 & 0 \\ c' & 1 \end{smallmatrix}\right)$. Then

$$A_c^{-u} = W_N S_{c'}^u W_N, \; \forall u \in \mathbb{Z}.$$

For a character χ modulo c', put

$$f|R_\chi(c') := \sum_{u=0}^{c'-1} \bar{\chi}(u) f|S_{c'}^u.$$

$f|R_\chi(\operatorname{cond}\chi) = g(\bar{\chi}) f_\chi$. ($f_\chi(q) = \sum a_n(f)\chi(n)q^n$ is a modular form of level N'.) We have

$$\varphi(c') A_c^{-a} = \sum_{\operatorname{cond}(\chi)|c'} \chi(a) W_N R_\chi(c') W_N. \tag{2.1}$$

Applying to f, we arrive at

$$\boxed{f_{[\frac{a}{c}]}(q) = \frac{w(f)}{\varphi(c')} \sum_{\operatorname{cond}(\chi)|c'} \chi(-a) f|R_\chi(c') W_N}. \tag{2.2}$$

Idea (ctnd)

Now it left to compute the expansions of each $f|R_\chi(c')W_N$ in the sum. $f \otimes \chi :=$ the unique newform such that $a_p(f \otimes \chi) = a_p(f_\chi)$ for almost all p. (We call $f \otimes \chi$ the twist of f by χ).
If $c' = \mathrm{cond}(\chi)$ and $f_\chi = f \otimes \chi$, then $f|R_\chi(c')W_N = w(f \otimes \chi)f_\chi$.

Otherwise, $f_\chi = (f \otimes \chi)|(1 - U_d|B_d)$, and we use

Lemma

$$f|B_d|W_N = \left(\frac{N}{Md^2}\right)^{k/2} w(f)(f|B_{\frac{N}{Md}})^*.$$

Conclusion: suffices to compute $f \otimes \chi$ and $w(f \otimes \chi)$.

Algorithm for twists

Lemma

Let ϵ be the character of f. Then the level of $f \otimes \chi$ divides $lcm(N, \mathrm{cond}(\epsilon)\,\mathrm{cond}(\chi), \mathrm{cond}(\chi)^2)$.

Lemma

For every $N \geq 1$, there exists an integer $B = B(N)$ such that if g_1, g_2 be two normalised newforms of levels N_1, N_2 dividing N and

$$a_n(g_1) = a_n(g_2), \text{ for all } 1 \leq n \leq B \text{ such that } \gcd(n, N) = 1,$$

then $g_1 = g_2$.

Algorithm to compute $f \otimes \chi$

Algorithm 1 Identifying $f \otimes \chi$

Input: $f \in S_k(\Gamma_0(N))$ a normalized newform; χ – Dirichlet character of prime power conductor $Q = q^\beta$ $(Q^2 \mid N)$.

Output: The newform $f \otimes \chi$.

```
for each M | N do
    Compute a basis {g_1, ..., g_s} of S_k(M, χ^2)^new.
    B := the Sturm bound for Γ_1(MQ^2).
    for 1 ≤ j ≤ s do
        if a_n(g_i) = a_n(f)χ(n) for all 1 ≤ n ≤ B, gcd(n, q) = 1 then
            return g_i.
        end if
    end for
end for
return Error.
```

Computing the pseudo-eigenvalue

Recall that modular symbols are linear combinations of $\{\alpha, \beta\}$, $\alpha, \beta \in \mathbb{Q} \cup \infty$, and we put

$$\langle f, \{\alpha, \beta\} \rangle := \int_{\alpha}^{\beta} f \mathrm{d}z.$$

Lemma

There exists a weight-k modular symbol M be such that $W_N(M) = N^{k/2-1} M^*$. Moreover, if $\langle f, M \rangle \neq 0$, then

$$w(f) = \frac{\langle f, M \rangle}{\overline{\langle f, M \rangle}}.$$

Examples (I)

Example

$E = \mathbf{50a}$. f_E is twist-minimal. Let $\mathfrak{c} = [\frac{1}{10}]$, write $\alpha_0 = 0$ and

$$x^4 + x^3 + x^2 - x + \frac{1}{5} = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4).$$

Then

$$f_{\mathfrak{c}}(q) = \sum_{n \geq 1} \alpha_{n \mod 5} a_n(f) q^n.$$

Example

Let $E = \mathbf{48a}$ and let $\mathfrak{c} = \left[\frac{1}{12}\right]$. We computed that

$$f_{\mathfrak{c}}(q) = -2iq^2 + 2iq^6 + O(q^7).$$

Since the first coefficient vanishes, we conclude that the modular parametrization $\varphi : X_0(48) \to E$ is ramified at the cusp $\mathfrak{c}$.

Examples (II)

Definition

A newform f is *twist-minimal* if it is not a twist of a newform of lower level.

Example

Let $E = \mathbf{98a}$ and $\mathfrak{c} = [\frac{1}{14}]$. Then f_E is not twist-minimal. More precisely, if χ is the quadratic character modulo 7, then

$$f \otimes \chi(q) = q - q^2 - 2q^3 + q^4 + O(q^6)$$

is a newform of level 14. We computed numerically that

$$\begin{aligned} f_{\mathfrak{c}}(q) = (-0.755 - 0.172i)\, q + (0.441 - 0.916i)\, q^2 + (1.392 + 1.110i)\, q^3 \\ + (0.696 - 0.555i)\, q^4 + (1.510 - 0.344i)\, q^6 - 3.023iq^7 + O(q^8) \end{aligned}$$

Further work

Assume $E/\mathbb{Q}$, $f = f_E$ is minimal by twist.

- relation to automorphic side:
  psuedo-eigenvalues relates to epsilon factors of $\pi_{f\otimes\chi}$. Another way to determine the local components of π_f.
- Let $\mathfrak{c}$ be a cusp of prime denominator $p \geq 5$. Seems that $a_1(f_{\mathfrak{c}})$ is only divisible by primes that are $\pm 1 \mod p$. Can we prove this?

Definition: Chow-Heegner points

$$\begin{tikzcd} & X_0(N) \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ E & & F \end{tikzcd}$$

E, F: two non-isogeneous elliptic curves of same conductor N.
φ, ψ: modular parametrisations of E, F.
The Chow-Heegner point associated to the pair (E, F) is

$$P_{E,F} = \sum \varphi(\psi^*(Q)), \forall Q \in F(\mathbb{C})$$

Facts: (1) $P_{E,F}$ is independent of the choice of Q;
(2) $P_{E,F} \in E(\mathbb{Q})$.

Even index of Chow-Heegner points

Numerical evidence suggests that the index $i_{E,F}$ is always divisible by 2.

Theorem (C.)

Let $\sigma_0(N)$ denote the number of distinct prime factors of N. If

$$\sigma_0(N) > \log_2(\#E[2](\mathbb{Q})) + \log_2(\#F[2](\mathbb{Q})) + 2,$$

then $P_{E,F} \in 2E(\mathbb{Q})$.

Computing Chow-Heegner points: previous work

There exist numerical algorithms to compute Chow-Heegner points.

- Darmon, Daub, Lichtenstein and Rotger – using (complex and p-adic) iterated integrals.
- Stein – using complex integration to lift points via modular parametrization.

My algorithm and an example

We present an algorithm that either computes the Chow-Heegner point $P_{E,F}$ or outputs fail. Let x_E, y_E, x_F, y_F be the compositions of φ, ψ with the x and y coordinate functions on E and F, respectively. Note that there exists an algorithm to compute the q-expansions of x_E, x_F, y_E and y_F.

Algorithm: computing Chow-Heegner points

Input: E, F = non-isogeneous elliptic curves of conuductor N; q-expansions of x_E, y_E, x_F, y_F.
Output: $P_{E,F}$.

$u_E := (x_E)^{-1}$ and $u_F := (x_F)^{-1}$.
Compute a polynomial $F(x, y)$ such that $F_{E,F}(u_E, u_F) = 0$.
$f_{ch,x}(x) := F_{E,F}(u_E, 0)$.
Repeat for $v_E = (y_E)^{-1}$ and u_F, get $f_{ch,y}(x)$.
Factor $f_{ch,x} = \prod(x - a_i)$ over $\bar{\mathbb{Q}}$.
for each a_i **do**
  if $f_{ch,y}(b_i) = 0$ **then** $P_i = (a_i, b_i)$
  else $P_i = -(a_i, b_i)$.
  end if
end for
Output $P_{E,F} = \sum_i P_i$.

Example

Example

Consider $E = \mathbf{89a}$ and $F = \mathbf{89b}$. Here $\deg(\varphi) = 2$ and $\deg(\psi) = 5$. Let $D = D_{\varphi,\psi} = \varphi(\psi^*(\infty)) \in \mathrm{div}(E)$. Define $G_1(x) = \prod_{P \in D}(x - x(P))$ and $G_2(y) = \prod_{P \in D}(y - y(P))$. We computed

$$G_1(x) = x^4 + \frac{13}{4}x^3 + \frac{17}{4}x^2 + \frac{21}{4}x + \frac{9}{2}, \; G_2(y) = y^4 + \frac{1}{8}y^3 + \frac{21}{4}y^2 + \frac{7}{2}y + 3.$$

$G_1(x)$ is irreducible over $\mathbb{Q}$. Write $G_1(x) = \prod(x - a_i)$ with $a_i \in \bar{\mathbb{Q}}$. We found $b_i = -\frac{8}{9}a_i^3 - \frac{20}{9}a_i^2 - \frac{28}{9}a_i - \frac{10}{3}$ is the root of G_2 such that $(a_i, b_i) \in E$. Hence

$$P_{E,F} = \sum_{i=1}^{4} P_i, \quad \text{where } P_i = (a_i, b_i).$$

Carrying out the summation, we obtain $P_{E,F} = (\frac{3}{4}, -\frac{15}{8})$.

Future work

- Prove even index in all cases.
- Verify the data of Chow-Heegner points in Stein's paper and extend the table.

Thank you!