# Computational aspects of modular parametrizations of elliptic curves

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# Critical subgroups of elliptic curves

# Elliptic curves over $\mathbb{Q}$

## Definition

An elliptic curve over $\mathbb{Q}$ is a nonsingular projective curve $E \subseteq \mathbb{P}^2$ with defining equation

$$y^2 z = x^3 + Axz^2 + Bz^3,$$

where $A, B \in \mathbb{Q}$ and $4A^3 + 27B^2 \neq 0$.

## Theorem (Mordell-Weil)

$E(\mathbb{Q})$ is a finitely generated abelian group, i.e.,

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus T,$$

for some $r \geq 0$ and $T$ finite.

$r$ is called the rank. $T$ is the torsion subgroup.

# The BSD conjecture

There is an entire function $L(E,s)$ called the $L$-function of $E$.

The rank part of the Birch and Swinnerton-Dyer (BSD) conjecture is:

$$rank(E(\mathbb{Q})) = \mathrm{ord}_{s=1} L(E,s).$$

RHS is the analytic rank, denoted by $r_{an}(E)$.
The BSD conjecture is open when $r_{an}(E) > 1$.

The proof of rank BSD for $r_{an}(E) \leq 1$ uses Heegner points.

# Modular curves

Let $N \geq 1$ be an integer, consider the group

$$\Gamma_0(N) = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) : N \mid c \right\}.$$

Let $\mathcal{H}^* = \{ z \in \mathbb{C} : im(z) > 0 \} \cup \mathbb{P}^1(\mathbb{Q})$.
$\Gamma_0(N)$ acts on $\mathcal{H}^*$ by fractional linear transformations.

## Definition

$X_0(N) = \Gamma_0(N) \backslash \mathcal{H}^*$.

- $X_0(N)$ has the structure of a nonsingular projective curve.
- Rational functions on $X_0(N)$ are called modular functions. They have *q-expansions* at infinity:

$$u(q) = \sum_{n \geq -m} b_n q^n, \ q = e^{2\pi i z}.$$

# The modularity theorem

## Theorem (Modularity)

*For every elliptic curve $E/\mathbb{Q}$, there exists an integer $N > 1$ and a surjective morphism $\varphi : X_0(N) \to E$ defined over $\mathbb{Q}$.*

The smallest $N$ is called the conductor of $E$.

Let $\omega = \varphi^*(\frac{dx}{y})$. Then $\omega = cf(z)\mathrm{d}z$, where $f$ is the modular form attached to $E$.

We assume $E$ is optimal. Then $\varphi$ is unique up to sign.

Idea: use $\varphi$ to find points on $E$.

– rational points on $X_0(N)$ – cusps. – Heegner points. – Ramification points. – Others??

Note: up to now, no known construction in $\geq 2$.

# The critical subgroup $E_{crit}(\mathbb{Q})$

Let $R_\varphi = \sum(e_\varphi(z) - 1)[z]$ be the ramification divisor of $\varphi$.

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**Definition (Mazur, Swinnerton-Dyer)**

The critical subgroup of $E$ is

$$E_{crit}(\mathbb{Q}) = \langle tr(\varphi([z])) : [z] \in \text{supp } R_\varphi \rangle \subseteq E(\mathbb{Q}),$$

where $tr(P) = \sum_{\sigma:\mathbb{Q}(P)\to\bar{\mathbb{Q}}} P^\sigma$.

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- $R_\varphi = \text{div}(\omega)$. In particular, $\deg R_\varphi = 2g(X_0(N)) - 2$.

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**Question (Mazur and Swinnerton-Dyer, 1974)**

*Is there an elliptic curve $E/\mathbb{Q}$ with $r_{an}(E) \geq 2$ and $rank(E_{crit}(\mathbb{Q})) > 0$?*

# Critical *j*-polynomial

To help compute $E_{crit}(\mathbb{Q})$, we make the following definition.

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**Definition**

Write $\operatorname{div}(\omega) = \sum n_z[z]$. The critical j-polynomial of $E$ is

$$F_{E,j}(x) = \prod_{z \in \operatorname{supp} \operatorname{div}(\omega), j(z) \neq \infty} (x - j(z))^{n_z}.$$

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$F_{E,j}(x) \in \mathbb{Q}[x]$ and $\deg F_{E,j} \leq 2g - 2$ (equality holds if $N$ is square free).
For $h \in \mathbb{Q}(X_0(N))$, can define $F_{E,h}(x)$.

# Polynomial Relation (I)

Let $C$ be a curve and let $r, u \in \mathbb{Q}(C)$.

A minimal polynomial relation of $r$ and $u$ is an irreducible polynomial $P(x, y) \in \mathbb{Q}[x, y]$, such that $P(r, u) = 0$.

Say $P(x, y) = f_n(y)x^n + \cdots + f_1(y)x + f_0(y)$.

## Lemma (C.)

If $\mathbb{Q}(C) = \mathbb{Q}(r, u)$ and $\gcd(f_0(y), f_n(y)) = 1$, then

$$f_0(y) = c \prod_{z \in \text{supp div}_0(r) \setminus \text{supp div}_\infty(u)} (y - u(z))^{mult_z(\text{div}_0(r))}.$$

# Polynomial Relation (II)

Set

$$r = j(j - 1728)\frac{\omega}{dj}, \ u = \frac{1}{j}.$$

Then $r, u \in \mathbb{Q}(X_0(N))$, and $\mathrm{div}_0(r) = \mathrm{div}(\omega) + D_0$, where points in supp $D_0$ have $j$-value 0 or 1728.

## Proposition (C.)

For $T \gg 0$, let $P(x, y) = f_n(y)x^n + \cdots + f_1(y)x + f_0(y)$ be a minimal polynomial relation of $rj^T$ and $u$. Then

$$F_{E,j}(x) = f_0(1/x) \cdot x^A(x - 1728)^B$$

where $A, B$ are explicitly computable.

## The Algorithm

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**Input:** The newform $f_E$ attached to elliptic curve $E/\mathbb{Q}$;

**Output:** $F_{E,j}$.

  $r = \frac{j(j-1728)f_E \, dz}{dj}$, $u = \frac{1}{j}$.

  Compute the q-expansions of $r$ and $u$ to precision $q^M$.

  Solve the linear equations $\sum c_{a,b} r(q)^a u(q)^b = 0 \pmod{q^M}$ for $c_{a,b}$.

  Set $P(x,y) = \sum c_{a,b} x^a y^b$ and apply the proposition.

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For a discriminant $d < 0$, let $H_d$ be the Hilbert class polynomial of disc $d$.

### Example

$F_{\mathbf{44a},j}(x) = H_{-44}(x)^2$. $F_{\mathbf{37a},j}(x) = H_{-148}(x)$. $F_{\mathbf{37b},j}(x) = H_{-16}(x)^2$.

# The critical subgroup $E_{crit}(\mathbb{Q})$

Elliptic points: a finite set of points on $X_0(N)$ corresponding to elliptic curves with extra endormorphisms.

For $i = 2, 3$, let $\mathscr{E}_i(N)$ be the set of elliptic points on $X_0(N)$ of period $i$.

## Lemma (C.)

Let $P_{all} = \sum_{z \in \operatorname{supp} R_\varphi} mult_{R_\varphi}(z)\varphi(z)$. Then
$6P_{all} = -3\sum_{c \in \mathscr{E}_2(N)} \varphi(c) - 4\sum_{d \in \mathscr{E}_3(N)} \varphi(d)$.

## Theorem (C.)

Suppose $r_{an}(E) \geq 2$, and assume at least one of the following holds:
(1) $F_{E,j} = \prod_{m=1}^{k} H_{D_m}^{s_i} \cdot F_0$, where $\mathbb{Q}(\sqrt{D_m}) \neq \mathbb{Q}(\sqrt{D_n})$ for all $m \neq n$, and $F_0$ is irreducible.
(2) $F_{E,h}$ is irreducible for some non-constant function $h \in \mathbb{Q}(X_0(N))$.
Then $rank(E_{crit}(\mathbb{Q})) = 0$.

# Critical polynomials for elliptic curves of rank 2 and conductor < 1000 (I)

| $E$ | $g(X_0(N))$ | $h$ | Factorization of $F_{E,h}(x)$ |
|------|------|------|------|
| 389a | 32 | $j$ | $H_{-19}(x)^2(x^{60} + \cdots)$ |
| 433a | 35 | $j$ | $x^{68} + \cdots$ |
| 446d | 55 | $j$ | $x^{108} + \cdots$ |
| 563a | 47 | $j$ | $H_{-43}(x)^2(x^{90} - \cdots)$ |
| 571b | 47 | $j$ | $H_{-67}(x)^2(x^{90} - \cdots)$ |
| 643a | 53 | $j$ | $H_{-19}(x)^2(x^{102} - \cdots)$ |
| 664a | 81 | $\dfrac{\eta_4\eta_8^2\eta_{332}^5}{\eta_{166}\eta_{664}^6\eta_2}$ | $x^{160} - \cdots$ |
| 655a | 65 | $j$ | $x^{128} - \cdots$ |
| 681c | 75 | $j$ | $x^{148} - \cdots$ |
| 707a | 67 | $j$ | $x^{132} - \cdots$ |

# Critical polynomials for elliptic curves of rank 2 and conductor $< 1000$ (II)

| $E$ | $g(X_0(N))$ | $h$ | Factorization of $F_{E,h}(x)$ |
|------|-------------|-----|-------------------------------|
| 709a | 58 | $j$ | $x^{114} - \cdots$ |
| 718b | 89 | $j$ | $H_{-52}(x)^2(x^{172} - \cdots)$ |
| 794a | 98 | $j$ | $H_{-4}(x)^2(x^{192} - \cdots)$ |
| 817a | 71 | $j$ | $x^{140} - \cdots$ |
| 916c | 113 | $j$ | $H_{-12}(x)^8(x^{216} + \cdots)$ |
| 944e | 115 | $\frac{\eta_{16}^4 \eta_4^2}{\eta_8^6}$ | $x^{224} - \cdots.^1$ |
| 997b | 82 | $j$ | $H_{-27}(x)^2(x^{160} - \cdots)$ |
| 997c | 82 | $j$ | $x^{162} - \cdots$ |

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[1] Here 4 of the critical points are cusps, so $\deg F = 2g - 6$.

# Discussion

## Corollary

*For all elliptic curves E of rank 2 and conductor $N < 1000$, the rank of $E_{crit}(\mathbb{Q})$ is 0.*

Future work:

- Compute $E_{crit}(\mathbb{Q})$ for $E = $ **5077a**.
  Current method will take roughly 5500/(number of cpus) hours.

- Prove or disprove that rank$(E_{crit}(\mathbb{Q})) = 0$ whenever $r_{an}(E)$ is even. (For infinitely many?)

# *q*-expansion of newforms at non-unitary cusps

# Modular forms

Let $f$ be a function $f : \mathcal{H} \to \mathbb{C}$, $\alpha = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z})$, and let $k \in \mathbb{Z}$. The weight-k action of $\alpha$ on $f$ is defined by

$$f|[\alpha]_k(z) := (cz + d)^{-k} f(\alpha z).$$

## Definition

A **modular form** of weight $k$ and level $N$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ s.t.
(1) $f(z) = f|[\alpha]_k(z)$, $\forall \alpha \in \Gamma_0(N)$ $(\Gamma_1(N))$.
(2) $f$ has holomorphic extension to all cusps of $X_0(N)$ $(X_1(N))$.

Cusp forms = modular forms that are zero at all cusps.
Modular forms have $q$-expansions: $f(q) = \sum_{n \geq 0} a_n q^n$, $q = exp(2\pi i z)$.
The space of cusp forms = $S_k(N)$.

# Operators on modular forms

- Hecke operators: a family $\{T_n, n \geq 1\} \cup \{\langle d \rangle : (d, N) = 1\}$ of commuting linear operators on $S_k(N)$.
- $B_d$ and $U_d$ operators: $B_d(\sum a_n q^n) = \sum a_n q^{nd}$, $U_d(\sum a_n q^n) = \sum a_{nd} q^n$.
- The Atkin-Lehner involution $W_N$. If $f$ is a newform on $\Gamma_1(N)$, then

$$f | W_N = w(f) \bar{f}$$

$w(f) \in \mathbb{C}_1$ is called the pseudo-eigenvalue of $f$.

# Newforms

- When $M \mid N$, $\exists$ degeneracy maps $S_k(M) \to S_k(N)$.
- Old subspace $=$ span of images of all degeneracy maps.
- New subspace $=$ (Old subspace)$^{\perp}$.
- $S_k(N)^{new}$ has a basis of simultaneous eigenforms for all Hecke operators. These eigenforms are called newforms.

## Fourier expansion

Let $f \in S_k(\Gamma_0(N))$ be a newform and let $\mathfrak{c} \in X_0(N)$ be a cusp other than $\infty$.

Goal: compute the expansion of $f$ at $\mathfrak{c}$.

First, only well-defined for $denom(\mathfrak{c})^2 \mid N$.

Equivalent to computing the expansion of

$$f\left| \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ ld & 1 \end{pmatrix} \end{bmatrix} \right.$$

at $\infty$ for all $d^2 \mid N, l \in (\mathbb{Z}/N\mathbb{Z})^\times$.

## Idea of computing

Let $S'_c = \begin{pmatrix} 1 & \frac{1}{c'} \\ 0 & 1 \end{pmatrix}$ and $A'_c = \begin{pmatrix} 1 & 0 \\ c' & 1 \end{pmatrix}$. Then

$$A_c^{-u} = W_N S_{c'}^u W_N, \ \forall u \in \mathbb{Z}.$$

For a character $\chi$ modulo $c'$, put

$$f | R_\chi(c') := \sum_{u=0}^{c'-1} \bar{\chi}(u) f | S_{c'}^u.$$

$f | R_\chi(\operatorname{cond} \chi) = g(\bar{\chi}) f_\chi$. ($f_\chi(q) = \sum a_n(f) \chi(n) q^n$ is a modular form of level $N'$.) We have

$$\varphi(c') A_c^{-a} = \sum_{\operatorname{cond}(\chi)|c'} \chi(a) W_N R_\chi(c') W_N. \tag{2.1}$$

Applying to $f$, we arrive at

$$\boxed{f_{[\frac{a}{c}]}(q) = \frac{w(f)}{\varphi(c')} \sum_{\operatorname{cond}(\chi)|c'} \chi(-a) f | R_\chi(c') W_N.} \tag{2.2}$$

# Idea (ctnd)

It suffices to compute the expansions of each $f|R_\chi(c')W_N$ in the sum.
$f \otimes \chi :=$ the unique newform such that $a_p(f \otimes \chi) = a_p(f_\chi)$ for almost all
$p$. (We call $f \otimes \chi$ the twist of $f$ by $\chi$).
If $c' = \mathrm{cond}(\chi)$ and $f_\chi = f \otimes \chi$, then $f|R_\chi(c')W_N = w(f \otimes \chi)f_\chi$.

Otherwise, $f_\chi = (f \otimes \chi)|(1 - U_d|B_d)$, and we use

## Lemma

$$f|B_d|W_N = \left(\frac{N}{Md^2}\right)^{k/2} w(f)(f|B_{\frac{N}{Md}})^*.$$

Conclusion: suffices to compute $f \otimes \chi$ and $w(f \otimes \chi)$.

# Algorithm for twists

### Lemma

*Let $\epsilon$ be the character of $f$. Then the level of $f \otimes \chi$ divides $lcm(N, \text{cond}(\epsilon)\,\text{cond}(\chi), \text{cond}(\chi)^2)$.*

### Lemma

*For every $N \geq 1$, there exists an integer $B = B(N)$ such that if $g_1$, $g_2$ be two normalised newforms of levels $N_1, N_2$ dividing $N$ and*

$$a_n(g_1) = a_n(g_2), \text{ for all } 1 \leq n \leq B \text{ such that } \gcd(n, N) = 1,$$

*then $g_1 = g_2$.*

# Algorithm to compute $f \otimes \chi$

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**Algorithm 1** Identifying $f \otimes \chi$

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**Input:** $f \in S_k(\Gamma_0(N))$ a normalized newform; $\chi$ – Dirichlet character of prime power conductor $Q = q^\beta$ ($Q^2 \mid N$).

**Output:** The newform $f \otimes \chi$.

    **for** each $M \mid N$ **do**

        Compute a basis $\{g_1, \ldots, g_s\}$ of $S_k(M, \chi^2)^{new}$.

        $B :=$ the Sturm bound for $\Gamma_1(MQ^2)$.

        **for** $1 \le j \le s$ **do**

            **if** $a_n(g_i) = a_n(f)\chi(n)$ for all $1 \le n \le B, \gcd(n, q) = 1$ **then**

                **return** $g_i$.

            **end if**

        **end for**

    **end for**

    **return** Error.

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# Computing the pseudo-eigenvalue

Recall that modular symbols are linear combinations of $\{\alpha, \beta\}$, $\alpha, \beta \in \mathbb{Q} \cup \infty$, and we put

$$\langle f, \{\alpha, \beta\} \rangle := \int_{\alpha}^{\beta} f \, \mathrm{d}z.$$

### Lemma (C.)

*There exists a weight-$k$ modular symbol $M$ be such that $W_N(M) = N^{k/2-1} M^*$. Moreover, if $\langle f, M \rangle \neq 0$, then*

$$w(f) = \frac{\langle f, M \rangle}{\overline{\langle f, M \rangle}}.$$

## Examples (I)

### Example

$E = \textbf{50a}$. $f_E$ is twist-minimal. Let $\mathfrak{c} = [\frac{1}{10}]$, write $\alpha_0 = 0$ and

$$x^4 + x^3 + x^2 - x + \frac{1}{5} = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4).$$

Then

$$f_{\mathfrak{c}}(q) = \sum_{n \geq 1} \alpha_{n \mod 5} a_n(f) q^n.$$

### Example

Let $E = \textbf{48a}$ and let $\mathfrak{c} = \left[\frac{1}{12}\right]$. We computed that

$$f_{\mathfrak{c}}(q) = -2iq^2 + 2iq^6 + O(q^7).$$

Since the first coefficient vanishes, we conclude that the modular parametrization $\varphi : X_0(48) \to E$ is ramified at the cusp $\mathfrak{c}$.

# Examples (II)

## Definition
A newform $f$ is *twist-minimal* if it is not a twist of a newform of lower level.

## Example
Let $E = \mathbf{98a}$ and $\mathfrak{c} = [\frac{1}{14}]$. Then $f_E$ is not twist-minimal. More precisely, if $\chi$ is the quadratic character modulo 7, then

$$f \otimes \chi(q) = q - q^2 - 2q^3 + q^4 + O(q^6)$$

is a newform of level 14. We computed numerically that

$$\begin{aligned}
f_{\mathfrak{c}}(q) = (-0.755 - 0.172i)\, q + (0.441 - 0.916i)\, q^2 + (1.392 + 1.110i)\, q^3 \\
+ (0.696 - 0.555i)\, q^4 + (1.510 - 0.344i)\, q^6 - 3.023iq^7 + O(q^8)
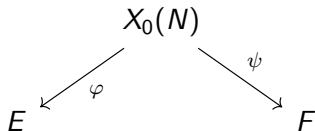\end{aligned}$$

# Further work

Assume $E/\mathbb{Q}$, $f = f_E$ is minimal by twist.

- relation to automorphic side:
  psuedo-eigenvalues relates to epsilon factors of $\pi_{f \otimes \chi}$. Another way to determine the local components of $\pi_f$.

- Let $\mathfrak{c}$ be a cusp of prime denominator $p \geq 5$. Seems that $a_1(f_{\mathfrak{c}})$ is only divisible by primes that are $\pm 1 \mod p$. Can we prove this?

# Chow-Heegner points

# Definition: Chow-Heegner points



$E, F$: two non-isogeneous elliptic curves of same conductor $N$.
$\varphi, \psi$: modular parametrisations of $E, F$.
The Chow-Heegner point associated to the pair $(E, F)$ is

$$P_{E,F} = \sum \varphi(\psi^*(Q)), \forall Q \in F(\mathbb{C})$$

Facts: (1) $P_{E,F}$ is independent of the choice of $Q$;
(2) $P_{E,F} \in E(\mathbb{Q})$.

# Even index of Chow-Heegner points

## Theorem (Yuan-Zhang-Zhang)

*Assume some technical conditions. Then*

$$\hat{h}(P_{E,F}) = (\star) \cdot L'(E, F, F, \frac{1}{2}),$$

*where $(\star)$ is nonzero.*

When $E$ has rank 1, numerical evidence suggests that the index $i_{E,F} := [E(\mathbb{Q})/tors : \mathbb{Z}P_{E,F}]$ is even, when it is finite.

## Theorem (C.)

*Let $\sigma_0(N)$ denote the number of distinct prime factors of $N$. If*

$$\sigma_0(N) > \log_2(\#E[2](\mathbb{Q})) + \log_2(\#F[2](\mathbb{Q})) + 2,$$

*then $P_{E,F} \in 2E(\mathbb{Q})$.*

# Computing Chow-Heegner points: previous work

There exist numerical algorithms to compute Chow-Heegner points.

- Darmon, Daub, Lichtenstein and Rotger – using (complex and $p$-adic) iterated integrals.
- Stein – using complex integration to lift points via modular parametrization.

## My algorithm and an example

We present an algorithm that either computes the Chow-Heegner point $P_{E,F}$.

Let $x_E, y_E, x_F, y_F$ be the compositions of $\varphi, \psi$ with the $x$ and $y$ coordinate functions on $E$ and $F$, respectively. Note that there exists an algorithm to compute the q-expansions of $x_E, x_F, y_E$ and $y_F$.

## Algorithm: computing Chow-Heegner points

**Input:** $E, F$ = non-isogeneous elliptic curves of conuductor $N$; $q$-expansions of $x_E, y_E, x_F, y_F$.

**Output:** $P_{E,F}$.

$\quad u_E := (x_E)^{-1}$ and $u_F := (x_F)^{-1}$.

$\quad$ Compute a polynomial $F(x, y)$ such that $F_{E,F}(u_E, u_F) = 0$.

$\quad f_{ch,x}(x) := F_{E,F}(u_E, 0)$.

$\quad$ Repeat for $v_E = (y_E)^{-1}$ and $u_F$, get $f_{ch,y}(x)$.

$\quad$ Factor $f_{ch,x} = \prod(x - a_i)$ over $\bar{\mathbb{Q}}$.

$\quad$ **for** each $a_i$ **do**

$\quad\quad$ **if** $f_{ch,y}(b_i) = 0$ **then** $P_i = (a_i, b_i)$

$\quad\quad$ **else** $P_i = -(a_i, b_i)$.

$\quad\quad$ **end if**

$\quad$ **end for**

$\quad$ Output $P_{E,F} = \sum_i P_i$.

## Example

### Example

Consider $E = \mathbf{89a}$ and $F = \mathbf{89b}$. Here $\deg(\varphi) = 2$ and $\deg(\psi) = 5$. Let $D = D_{\varphi,\psi} = \varphi(\psi^*(\infty)) \in \operatorname{div}(E)$. Define $G_1(x) = \prod_{P \in D}(x - x(P))$ and $G_2(y) = \prod_{P \in D}(y - y(P))$. We computed

$$G_1(x) = x^4 + \frac{13}{4}x^3 + \frac{17}{4}x^2 + \frac{21}{4}x + \frac{9}{2}, \ G_2(y) = y^4 + \frac{1}{8}y^3 + \frac{21}{4}y^2 + \frac{7}{2}y + 3.$$

Write $G_1(x) = \prod(x - a_i)$ with $a_i \in \bar{\mathbb{Q}}$. We found
$b_i = -\frac{8}{9}a_i^3 - \frac{20}{9}a_i^2 - \frac{28}{9}a_i - \frac{10}{3}$ is the root of $G_2$ such that $(a_i, b_i) \in E$.
Hence

$$P_{E,F} = \sum_{i=1}^4 P_i, \ \text{where } P_i = (a_i, b_i).$$

Carrying out the summation, we obtain $P_{E,F} = (\frac{3}{4}, -\frac{15}{8})$.

# Future work

- Compute Chow-Heegner points for curves of different conductors.
- Prove even index in all cases.
- Verify the data of Chow-Heegner points in Stein's paper and extend the table.

# Thank you!