

©Copyright 2011-2016

Hao Chen

Computational aspects of modular parametrizations of elliptic curves

Hao Chen

A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2011-2016

Reading Committee:

William Arthur Stein, Chair

First committee member

Next committee member

etc

Program Authorized to Offer Degree:
UW Mathematics

University of Washington

Abstract

Computational aspects of modular parametrizations
of elliptic curves

Hao Chen

Chair of the Supervisory Committee:
Professor William Arthur Stein
Department of Mathematics

Abstract goes here.

TABLE OF CONTENTS

	Page
Glossary	ii
Chapter 1: Introduction	1
Chapter 2: Computing the Mazur Swinnerton-Dyer critical subgroup of elliptic curves	2
Chapter 3: Chow-Heegner points computations	3
Chapter 4: Fourier expansions of modular forms forms at all cusps	4
4.1 Preliminaries	4
4.2 Reducing to the case of newforms	6
4.3 Twists of newforms	7
4.4 Pseudo-eigenvalues	10
4.5 Formula for the Fourier expansion of f at width one cusps: Part 1	13
4.6 Formula for the Fourier expansion of f at width one cusps: Part 2	14
4.7 A Converse Theorem	18
4.8 Fields of definitions	20
4.9 Denominators	21
4.10 Examples	21
4.11 Applications	21
4.12 Automorphic representations; norm of first terms	22
4.13 Norm of first terms computations	24
Chapter 5: Things I tried to do but did not end up giving a nice result	27
Bibliography	28

GLOSSARY

ARGUMENT: replacement text which customizes a \LaTeX macro for each particular usage.

DEDICATION

to all of you

Chapter 1

INTRODUCTION

Chapter 2

COMPUTING THE MAZUR SWINNERTON-DYER CRITICAL SUBGROUP OF ELLIPTIC CURVES

Chapter 3

CHOW-HEEGNER POINTS COMPUTATIONS

Chapter 4

FOURIER EXPANSIONS OF MODULAR FORMS AT ALL CUSPS

Let k be a positive even integer and let $f \in S_k(\Gamma_0(N))$ be a nonzero cusp form. Then f has a Fourier expansion at the cusp infinity:

$$f = \sum_{n \geq 1} a_n q^n$$

where a_n are complex numbers and $q = e^{2\pi i\tau}$. We are concerned with the problem of computing the Fourier expansion of f at other cusps. When N is square-free, this problem is solved by Asai [Asa76]. The problem is studied in the Ph.D. thesis of Christophe Delaunay and in [Edixhoven], where a numerical algorithm is proposed. We will give a numerical algorithm to compute such expansions. Our approach is different from the one proposed in [Ed], for they require working at a higher level: to compute expansions at cusps of denominator Q , one needs to compute period matrices for forms of level NR^2 , where $R = \gcd(Q, \frac{N}{Q})$. As a contrast, our algorithm works at levels dividing N .

The main results of this chapter are Theorem 4.6.7 and Algorithm 3. The former gives a formula for the Fourier expansion of a newform $f \in S_k(\Gamma_0(N))$ at any cusp z of width one, and the latter describes how to use the formula to explicitly compute such expansion. Along the way, we will develop algorithms to compute the twists $f \otimes \chi$ and the pseudo-eigenvalue of newforms under the Fricke involution.

Section contains some examples.

4.1 Preliminaries

Let $N \geq 1$ be an integer and let $X_0(N)$ be the modular curve of level N .

Definition 4.1.1. Let z be a cusp on $X_0(N)$. If $z \neq \infty$, write $z = [a/c]$ with $\gcd(a, c) = 1$.

The *denominator* of z is

$$d_z = \gcd(c, N).$$

. If $z = \infty$, we set $d_\infty = N$. Choose $\alpha \in SL_2(\mathbb{Z})$ such that $\alpha(\infty) = z$. The *width* of z is

$$h_z = \left| \frac{SL_2(\mathbb{Z})_\infty}{(\alpha^{-1}\{\pm I\}\Gamma_0(N)\alpha)_\infty} \right|$$

where the subscript ∞ means taking the isotropy subgroup of ∞ in the corresponding group.

The width of a cusp can be computed in terms of its denominator. In fact, we have

Lemma 4.1.2. *If z is a cusp on $X_0(N)$, then*

$$h_z = \frac{N}{\gcd(d_z^2, N)}.$$

Proof. When $z = [\infty]$, we have $d_\infty = N$ and $h_\infty = 1$, so the formula holds trivially. Otherwise, write $z = [\frac{a}{c}]$ and find $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. For $N' \in \mathbb{Z}$ we compute

$$\alpha \begin{pmatrix} 1 & N' \\ 0 & 1 \end{pmatrix} \alpha^{-1} = \begin{pmatrix} * & * \\ -c^2 N' & * \end{pmatrix}.$$

Hence $\begin{pmatrix} 1 & N' \\ 0 & 1 \end{pmatrix} \in (\alpha^{-1}\{\pm I\}\Gamma_0(N)\alpha)_\infty \iff N \mid c^2 N' \iff \frac{N}{\gcd(d_z^2, N)} \mid N'$. This completes the proof. \square

In particular, the width of a cusp z is one if and only if $N \mid d_z^2$.

Suppose f is a modular form on $\Gamma_0(N)$ of positive even weight k and $\alpha \in GL_2(\mathbb{Q})$. Recall the weight- k action is defined as

$$f|_\alpha(z) = (\det(\alpha))^{k/2} (cz + d)^{-k} f(\alpha z), \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In particular, if $\alpha \in SL_2(\mathbb{Z})$, then $f|_\alpha$ is a modular form on $\Gamma(N)$. So $f|_\alpha$ has a q -expansion, which is a power series in $q^{\frac{1}{N}}$. A natural thing to do is to define the expansion of f at the cusp z as the expansion of $f|_\alpha$. However, note that this may not be well-defined: in general the expansion depends on the choice of α . Nonetheless, when the cusp z has width one, the expansion is indeed well-defined as a power series in q .

Lemma 4.1.3. *Let z be a cusp on $X_0(N)$ with $h_z = 1$. Choose $\alpha \in SL_2(\mathbb{Z})$ such that $\alpha(\infty) = z$. Then $f|_\alpha$ is a cusp form on $\Gamma_1(N)$. Moreover, the function $f|_\alpha$ is independent of the choice of α .*

Proof. It is easy to verify that $\Gamma_1(N) \subseteq \alpha^{-1}\Gamma_0(N)\alpha$, hence the first claim holds. Now suppose $\beta \in SL_2(\mathbb{Z})$ is such that $\beta(\infty) = z$. Then $\alpha^{-1}\beta \in SL_2(\mathbb{Z})_\infty$. Since z has width one, we have $\alpha^{-1}\beta \in \alpha^{-1}\Gamma_0(N)\alpha$. Hence $\beta \in \Gamma_0(N)\alpha$, and it follows that $f|[\beta] = f|[\alpha]$. \square

In light of the lemma above, we define the q -expansion of f at a width one cusp z to be the q -expansion of $f|[\alpha]$, and denote it by f_z .

Assume further that f is an eigenform under the Atkin-Lehner operators. We will show that in order to compute the expansion of $f|[\alpha]$ for any $\alpha \in SL_2(\mathbb{Z})$, it suffices to do so for $\alpha = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$, where $0 \leq m < N$ and $N \mid \gcd(m, N)^2$. In particular, it suffices to compute the expansions of f at a some cusps of width one.

Lemma 4.1.4. *For any $\alpha \in SL_2(\mathbb{Z})$, there exists a matrix $w_Q \in W_N$ and an upper triangular matrix $u \in GL_2(\mathbb{Q})$ such that $w_Q\alpha = \alpha'u$, where $\alpha' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$ satisfies $N \mid \gcd(N, c')^2$.*

Indeed, one may find Q using Lemma. Now $f|[\alpha] = f|[w_Q][w_Q\alpha] = f|[w_Q][\alpha'][u] = \lambda_Q(f)f[\alpha'][u] = \lambda_Q(f)f[\alpha'']|u|$, where α'' is of form $\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$. Note that for an upper triangular matrix $u = \begin{pmatrix} u_0 & u_1 \\ 0 & u_2 \end{pmatrix}$, we have $f|u|(q) = f(q^{u_0/u_2}e^{2\pi i u_1/u_2})$.

4.2 Reducing to the case of newforms

The space $S_k(\Gamma_0(N))$ is spanned by elements of form $g(q^d)$, where g is newform of level $M \mid N$ and d is a divisor of $\frac{N}{M}$. Note that $g(q^d) = d^{-k/2}g| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$. For any $\alpha \in SL_2(\mathbb{Z})$, we can find $\alpha' \in SL_2(\mathbb{Z})$ and $u \in GL_2(\mathbb{Q})$ such that $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \alpha = \alpha'u$. Hence to compute all expansions $f|[\alpha]$, it suffices to give an algorithm for newforms.

In the rest of this chapter, we will restrict ourselves to solving the following problem:

Problem 4.2.1. Let f be a normalized newform in $S_k(\Gamma_0(N))$ and z be a cusp on $X_0(N)$ of width one. Compute the q -expansion of f_z .

4.3 Twists of newforms

For $f \in S_k(\Gamma_1(N), \epsilon)$ a newform with expansion $f = \sum_n a_n(f)q^n$ and χ a Dirichlet character, the *twist* f_χ is a modular form with expansion $f_\chi(q) = \sum a_n(f)\chi(n)q^n$.

Lemma 4.3.1. [AWL78, Proposition 3.1] *Let $F \in S_k(\Gamma_1(N), \epsilon)$, where ϵ is a character of conductor N' . Let χ be a character modulo M . Put $\tilde{N} = \text{lcm}(N, N'M, M^2)$. Then $f_\chi \in S_k(\Gamma_1(\tilde{N}), \epsilon\chi^2)$.*

In particular, when ϵ is the trivial character and the conductor M of χ satisfies $M^2 \mid N$, we have $F_\chi \in S_k(\Gamma_1(N), \chi^2)$.

We write $f \otimes \chi$ for the unique newform such that $a_p(f \otimes \chi) = a_p(f_\chi)$ for all but finitely many primes p . From now, we refer to $f \otimes \chi$ as *the twist of f by χ* .

We quote two more results from [AWL78], which we will use extensively. First, we recall the definitions of U_d and B_d operators. For a modular form $f = \sum a_n q^n$ and a positive integer d , we put

$$f|U_d = \sum a_{nd} q^n, \quad f|B_d = \sum a_n q^{nd}.$$

It is easy to see that for any positive integers d, d' , we have U_d commutes with $B_{d'}$.

Lemma 4.3.2. [AWL78, Theorem 3.1] *Let $q \mid N$ and Q be the q -primary part of N . Write $N = QM$. Let F be a newform in $S_k(\Gamma_1(N), \epsilon)$ with $\text{cond}(\epsilon_Q) = q^\alpha, \alpha \geq 0$. Let χ be a character with conductor $q^\beta, \beta \geq 1$. Put $Q' = \max\{Q, q^{\alpha+\beta}, q^{2\beta}\}$. Then*

- (1) *For each prime $q' \mid M$, F_χ is not of level $Q'M/q$.*
- (2) *The exact level of F_χ is $Q'M$ provided (a) $\max\{q^{\alpha+\beta}, q^{2\beta}\} < Q$ if $Q' = Q$, or (b) $\text{cond}(\epsilon_Q\chi) = \max\{q^\alpha, q^\beta\}$ if $Q' > Q$.*

Lemma 4.3.3. [AWL78, Theorem 3.2] *Let $q \mid N$ and Q be the q -primary part of N . Write $N = QM$. Let χ be a character whose conductor equals a power of q . Let f be a newform in $S_k(\Gamma_1(N), \epsilon)$. Then $f \otimes \chi$ is a newform in $S_k(\Gamma_1(Q'M, \epsilon\chi^2))$, where Q' is a power of q . Moreover, we have*

$$f_\chi = f \otimes \chi - (f \otimes \chi)|U_q|B_q.$$

Since our goal is to compute expansions of newforms on $\Gamma_0(N)$, we will make the following assumptions: from now, unless otherwise noted, we assume f has trivial character, and that $\text{cond}(\chi)^2 \mid N$.

Next, we consider the problem of identifying the newform $f \otimes \chi$. This includes finding its level and its q -expansion to arbitrarily many terms. We will assume that we have an oracle which, given weight k and level N , computes the expansions of all newforms in $S_k(\Gamma_1(N))$ to arbitrarily many terms (for example, use the algorithm in [Steb]).

Now we proceed on how to recognise the level of $f \otimes \chi$ from the coefficients of f . One potential obstacle is that we do not know all Fourier coefficients of $f \otimes \chi$: we only know that $a_n(f \otimes \chi) = a_n(f)\chi(n)$ when $\gcd(n, N) = 1$. This can be overcome using a variant of Sturm's argument. First we prove a lemma.

Lemma 4.3.4. *Let $f \in S_k(N, \epsilon)$ be a normalized newform and q be any positive integer. Then $f|U_q|B_q \in S_k(Nq^2, \epsilon)$.*

Proof. It is a standard fact that for any integer $d \geq 1$, the map $f \mapsto f|B_d$ takes $S_k(N, \epsilon)$ to $S_k(Nd, \epsilon)$. To prove the lemma, we consider two separate cases. First, assume $q \nmid N$, then we have $T_q = U_q + q^{k-1}\epsilon(q)B_q$. By our assumption, we have $f|T_q = a_q(f)f$. Therefore, we have $f|U_q|B_q = f|(T_q - q^{k-1}\epsilon(q)B_q)|B_q = a_q(f)f|B_q - q^{k-1}\epsilon(q)f|B_q^2$. Hence $f|U_q|B_q \in S_k(Nq^2, \epsilon)$. Now assume $q \mid N$, so $U_q = T_q$. Hence $f|U_q|B_q = a_q(f)f|B_q \in S_k(Nq, \epsilon) \subseteq S_k(Nq^2, \epsilon)$. \square

The next proposition generalised the usual Sturm bound argument for modular forms.

Proposition 4.3.5. *Let g_1, g_2 be two normalised newforms of levels $N_1 \mid N_2$ and the same nybentypus character ϵ . Assume ϵ has prime power conductor $Q = q^\beta$ such that $Q^2 \mid N_1$. Let B be the Sturm bound for the congruence subgroup $\Gamma_1(Nq^2)$. Suppose*

$$a_n(g_1) = a_n(g_2), \text{ for all } 1 \leq n \leq B \text{ such that } \gcd(n, q) = 1.$$

Then $g_1 = g_2$.

Proof. Following [AWL78], we define the operator K_q on the space of modular forms by

$$g|K_q = g - g|U_q|B_q.$$

Then the assumption is equivalent to the statement that $\delta = (g_1 - g_2)|K_q$ has $a_n(\delta) = 0$ for all $1 \leq n \leq B$. Since $\delta \in S_k(Nq^2, \epsilon)$, Sturm's theorem implies $\delta = 0$. We then know from [DS06, Theorem 5.7.1] that $g_1 - g_2 \in S_k(N_2, \epsilon)^{old}$. Suppose $N_1 < N_2$, then g_1 is in the old subspace, hence so is g_2 , a contradiction. Therefore we must have $N_1 = N_2$. It follows that $g_1 - g_2 \in S_k(N_2, \epsilon)^{new}$, since g_1, g_2 are newforms. Since the new subspace and the old subspace intersect trivially, we must have $g_1 - g_2 = 0$. \square

Now we are ready to describe the algorithm.

Algorithm 1 Identifying $f \otimes \chi$

Input: k – a positive even integer; $f \in S_k(\Gamma_0(N))$ a normalized newform; χ a Dirichlet character of prime power conductor $Q = q^\beta$; $Q^2 \mid N$; B – a positive integer

Output: The level M_χ of $f \otimes \chi$ and the Fourier expansion of $f \otimes \chi$ up to q^B .

```

1: if  $Q = 1$  then
2:   return  $N$ .
3: end if
4:  $Q' := \text{cond}(\chi^2)$ ;  $N_0 := \frac{N}{q^{v_q(N)}}$ ;  $M_0 := Q'N_0$ ;  $t := \frac{N}{M_0} \in \mathbb{Z}$ .
5: for each positive divisor  $d$  of  $t$  do
6:   Set  $V_d := S_k(M_0d, \chi^2)$ .
7:   Compute a basis of newforms  $\{g_1^{(d)}, \dots, g_{s_d}^{(d)}\}$  of  $V_d$ .
8:   Set  $B_d :=$  the Sturm bound for  $\Gamma_1(M_0dq^2)$ .
9:   for  $1 \leq j \leq s_d$  do
10:    if  $a_n(g_j^{(d)}) = a_n(f)\chi(n)$  for all  $1 \leq n \leq B_d, \gcd(n, q) = 1$  then
11:      return  $M_0d$ .
12:    end if
13:   end for
14: end for
```

We give some sample computations applying the above algorithm.

Example 4.3.6. Let f be the normalised newform attached to the elliptic curve

$$E : y^2 + xy + y = x^3 - x - 2$$

of Cremona label **50a**. Then $f \otimes \chi$ is new of level 50 for all Dirichlet characters χ with modulus 5. In other words, f is 5-minimal.

As another example, we demonstrate a newform which is not p -minimal.

Example 4.3.7. Let f be the normalised newform attached to the elliptic curve

$$E : y^2 + xy = x^3 + x^2 - 25x - 111$$

of label **98a**. Let χ be the Dirichlet character modulo 7 defined by $\chi(3 \pmod{7}) = -1$. We found that $f \otimes \chi$ is a newform of level 14, with q -expansion

$$(f \otimes \chi)(q) = q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + q^9 - 2q^{12} - 4q^{13} - q^{14} + O(q^{15}).$$

4.4 Pseudo-eigenvalues

Let ϵ be a Dirichlet character modulo N and let f be a newform in $S_k(N, \epsilon)$. For any divisor Q of N with $\gcd(Q, \frac{N}{Q}) = 1$, there is an algebraic number $w_Q(f)$ of absolute value one and a newform g in $S_k(N, \overline{\epsilon_Q} \epsilon_{N/Q})$ such that

$$W_Q(f) = w_Q(f)g,$$

Definition 4.4.1. The number $w_Q(f)$ is called the *pseudo-eigenvalue* of W_Q on f .

For ease of notations, we write $w(f) = w_N(f)$.

For a power series $f = \sum_{n \geq 0} a_n q^n$, its complex conjugate, denoted by f^* , is

$$f^*(q) = \sum \overline{a_n} q^n.$$

From [AWL78] we have $W_N(f) = w(f)f^*$. In the rest of this section, we describe an algorithm to efficiently compute $w(f)$ numerically. For a positive even integer k , let $\mathbb{M}(k)$ denote the space of weight- k modular symbols defined in [Steb]. The space $\mathbb{M}(k)$ is a quotient of $\mathbb{Z}[X, Y]_{k-2} \otimes \mathbb{P}^1(\mathbb{Q})^2$, and $GL_2(\mathbb{Q})$ acts on $\mathbb{M}(k)$ via the following rule

$$g(P(X, Y) \otimes \{\alpha, \beta\}) = P(g^{-1}(X, Y)^T) \{g(\alpha), g(\beta)\}.$$

Most importantly, there is a pairing between $\mathbb{M}(k)$ and the space of modular forms of weight k , defined as

$$\langle f, P(X, Y) \otimes \{\alpha, \beta\} \rangle_k = \int_{\alpha}^{\beta} f(z) P(z, 1) dz.$$

We will suppress the subscript k if its value is clear from context.

Lemma 4.4.2. *Let $M \in \mathbb{M}(k)$ and $f \in S_k(\Gamma_1(N))$. Then*

$$N^{\frac{k}{2}-1} \langle f | W_N, M \rangle = \langle f, W_N M \rangle.$$

Proof. See proof of [Steb, Proposition 8.17]. Note that the extra factor $N^{\frac{k}{2}-1}$ is due to the different constants involved in the definition of the weight- k action of $GL_2(\mathbb{Q})$ on modular forms. \square

The map

$$* : P(x, y) \{\alpha, \beta\} \mapsto P(-x, y) \{-\bar{\alpha}, -\bar{\beta}\}$$

defines the *star involution* on the space $\mathbb{M}(k)$. We have $\langle f^*, M \rangle = \overline{\langle f, M^* \rangle}$.

Lemma 4.4.3. *Let f be a normalised newform on $\Gamma_1(N)$ with positive even weight k and let $M \in \mathbb{M}(k)$ be such that $W_N(M) = N^{k/2-1} M^*$. Assume $\langle f, M \rangle \neq 0$. Then*

$$w(f) = \frac{\langle f, M \rangle}{\overline{\langle f, M \rangle}}.$$

Proof. Since $W_N^2(M) = N^{k-2}M$ for all $M \in \mathbb{M}(k)$, the assumption implies $W_N(M^*) = N^{k/2-1}M$. Now

$$\begin{aligned}
& N^{k/2-1}\langle f|W_N, M^*\rangle = \langle f, W_N(M^*)\rangle \\
& \implies N^{k/2-1}w(f)\langle f^*, M^*\rangle = N^{k/2-1}\langle f, M\rangle \\
& \implies w(f) = \frac{\langle f, M\rangle}{\langle f^*, M^*\rangle} \\
& \implies w(f) = \frac{\langle f, M\rangle}{\langle f, M\rangle}.
\end{aligned}$$

□

Suppose α, β are distinct points on the arc $\{z \in \mathbb{C} | \text{Im}(z) > 0, |z| = 1/\sqrt{N}\}$. Then it is easy to verify that $M = (xy)^{k/2-1} \otimes \{\alpha, \beta\}$ satisfies $W_N(M) = M^*$. Finally, we arrive at the algorithm to compute $w(f)$.

Algorithm 2 Computing the pseudo-eigenvalue of newforms.

Input: k – a positive even integer. $f \in S_k(\Gamma_1(N))$ a normalized newform.

Output: a numerical approximation of $w(f)$.

- 1: $n_0 := 10, z_0 := \frac{i}{\sqrt{N}}, \delta = 10^{-3}$.
 - 2: Randomly generate n_0 points $\{z_1, \dots, z_{N_0}\} \subseteq \{z | 0 < \text{Im}(z) < \frac{1}{2\sqrt{N}}, |z| = \frac{1}{\sqrt{N}}\}$.
 - 3: **for** $1 \leq i \leq n_0$ **do**
 - 4: compute the period integral $c_i = \int_{z_0}^{z_i} 2\pi i f(z) z^{\frac{k-2}{2}} dz$.
 - 5: $w_i \leftarrow c_i / \bar{c}_i$.
 - 6: **end for**
 - 7: **if** the standard deviation of w_1, \dots, w_{n_0} is less than δ **then**
 - 8: $w \leftarrow \frac{1}{n_0}(\sum_i w_i)$.
 - 9: **return** w .
 - 10: **else**
 - 11: **return** FAIL.
 - 12: **end if**
-

4.5 Formula for the Fourier expansion of f at width one cusps: Part 1

First we recall some notations from [AWL78].

Definition 4.5.1. For a positive integer c' , let $S'_c = \begin{pmatrix} 1 & \frac{1}{c'} \\ 0 & 1 \end{pmatrix}$. If χ is a character modulo c' , we define the operator on modular forms

$$f|R_\chi(c') = \sum_{u=0}^{c'-1} \bar{\chi}(u) f|S_{c'}^u.$$

Write R_χ in short for $R_\chi(\text{cond}(\chi))$. Note that $f|R_\chi = g(\bar{\chi})f_\chi$. Conversely, if $(a, M) = 1$, we have

$$\phi(c')S_{c'}^u = \sum_{\chi: \text{cond}(\chi)|c'} \chi(u) R_\chi(c'). \quad (4.5.1)$$

For our convenience, we define some operators, which are essentially the conjugates of S'_c and $R_\chi(c')$ by W_N . Let $A'_c = \begin{pmatrix} 1 & 0 \\ c' & 1 \end{pmatrix}$. Then it is easy to verify the following matrix identity.

Fact 4.5.2. $-N \cdot A_{N/c'}^{-1} = W_N S_{c'} W_N$.

From now on, we assume c is a divisor of N and $c' = \frac{N}{c}$. Then as operators on modular forms,

$$A_c^{-1} = W_N S_{c'} W_N.$$

Since $W_N^2 = id$ as operators, we have

$$A_c^{-u} = W_N S_{c'}^u W_N, \forall u \in \mathbb{Z}.$$

Parallel to the notion of $R_\chi(c')$, let $\Phi_\chi(c) = \sum_{u=0}^{c'-1} \bar{\chi}(u) A_c^{-u}$. Then $\Phi_\chi(c) = W_N R_\chi(c') W_N$. Similar to Formula 4.5.1, we have

$$\varphi(c') A_c^{-a} = \sum_{\text{cond}(\chi)|c'} \chi(a) \Phi_\chi(c) = \sum_{\text{cond}(\chi)|c'} \chi(a) W_N R_\chi(c') W_N. \quad (4.5.2)$$

Applying Formula 4.5.2 to f , we arrive at

$$f|_{[\frac{a}{c}]}(q) = \frac{1}{\varphi(c')} \sum_{\text{cond}(\chi)|c'} \chi(-a) f|W_N R_\chi(c') W_N. \quad (4.5.3)$$

$$= \frac{w(f)}{\varphi(c')} \sum_{\text{cond}(\chi)|c'} \chi(-a) f|R_\chi(c') W_N. \quad (4.5.4)$$

Now it left to compute the expansions of each $f|R_\chi(c') W_N$ in the sum.

4.6 Formula for the Fourier expansion of f at width one cusps: Part 2

In this section, we describe how to compute the expansion of $f|R_\chi(c')W_N$. First note that $T_p = U_p + \epsilon(p)p^{\frac{k}{2}}B_p$ as operators on $S_k(\Gamma_1(N), \epsilon)$. It follows that T_p commutes with B_d for any positive integer d .

We recall some notations and a result from [Del02].

Definition 4.6.1. [Del02, Definition III.2.4] For a Dirichlet character χ modulo $b = \prod_{j \in J} p_j^{\alpha_j}$. Let $r = |J|$. Decompose χ uniquely as $\chi = \chi_1 \cdots \chi_r$, where χ_i is a character modulo $p_j^{\alpha_j}$. We define $\text{cond}'(\chi)$ multiplicatively, by putting

$$\text{cond}'(\chi_j) = \begin{cases} \text{cond}(\chi_j) & \text{if } \text{cond}(\chi_j) > 1 \\ p_j & \text{else} \end{cases} \quad (4.6.1)$$

Also, if $I = \{j \in J : \chi_j \text{ is trivial character modulo } p_j^{\alpha_j}\}$, we put $tr = \prod_{j \in I} p_j^{\alpha_j}$, $nt = b/tr$, $\chi_{tr} = \prod_{j \in I} \chi_j$, and $\chi_{nt} = \chi/\chi_{tr}$. Then we set

$$g'(\chi) = (-1)^{|I|} \chi_{nt}(tr) g(\chi_{nt}). \quad (4.6.2)$$

Here $g(\chi)$ is the usual Gauss sum of χ : if χ is a character modulo d , then $g(\chi) = \sum_{a=1}^d e^{\frac{2\pi ia}{d}} \chi(a)$.

If $\chi = \chi_0$ is the trivial character, we set $g(\chi_0) = 0$.

Lemma 4.6.2. [Del02, Prop 2.6] Let c' be an integer such that $c'^2 \mid N$. For a Dirichlet character χ mod c' , we have

$$f|R_\chi(c') = \begin{cases} g'(\bar{\chi}) f_{\chi_{nt}} & \text{if } \text{cond}'(\chi) = c' \\ 0 & \text{else.} \end{cases}$$

Using this lemma, we can simplify formula 4.5.3 to

$$f|_{\left[\frac{a}{c'}\right]} = \frac{w(f)}{\varphi(c')} \sum_{\text{cond}'(\chi)=c'} \chi(-a) g'(\bar{\chi}) f_{\chi_{nt}}|W_N. \quad (4.6.3)$$

Next, we compute $f_{\chi_{nt}}$ by the following: suppose $g = f \otimes \chi_{nt}$. Then

$$f_{\chi_{nt}} = g| \prod_{i=1}^r K_{p_i}. \quad (4.6.4)$$

Moreover, we have

$$K_p = 1 - U_p B_p = \begin{cases} 1 - (T_p - \chi_{nt}^2(p) p^{\frac{k}{2}} B_p) | B_p & p \nmid M \\ 1 - T_p | B_p & p \mid M \end{cases}. \quad (4.6.5)$$

Using the commutativity of T_* and B_* , we can write $f_{\chi_{nt}}$ in the form $\sum c_i(f \otimes \chi)(q^{d_i})$, where c_i and d_i are constants. To give a precise formula, we use the following notation. For a finite set S of integers, let $\pi(S) = \prod_{s \in S} s$ denote the product of all elements in S . For a Dirichlet character χ of conductor d , let S_χ be the set of prime divisors of d . For any positive integer M and any finite set of integers S , define

$$\mathcal{B}_{S,M} = \{(S_1, S_2) \in (2^{\mathbb{Z}})^2 \mid S_1, S_2 \subseteq S, S_1 \cap S_2 = \emptyset, \gcd(M, \pi(S_2)) = 1\} \quad (4.6.6)$$

Proposition 4.6.3. *Let $k \geq 2$ be an even integer and let f be a newform in $S_k(\Gamma_0(N))$. Then*

$$f_{\chi_{nt}} = \sum_{(S_1, S_2) \in \mathcal{B}_{S_\chi, M}} (-1)^{|S_1|} a_{\pi(S_1)}(g_\chi) \pi(S_2)^{k/2} \chi_{nt}^2(\pi(S_2)) g_\chi | B_{\pi(S_1) \pi(S_2)^2}.$$

Here $g_\chi = f \otimes \chi$, M is the level of g_χ and $\mathcal{B}_{S_\chi, M}$ is as in 4.6.6.

Proof. This is a direct consequence of multiplying out 4.6.4 using 4.6.5, using the fact that T_p commutes with B_d , and noting that T_p acts as multiplication by $a_p(g_\chi)$ on g_χ . \square

Theorem 4.6.3 will be our starting point of computing the expansion of f at width one cusps. We will use it to compute $f_{\chi_{nt}} | W_N$. First we prove two lemmas.

Lemma 4.6.4. *Let f be a newform of even weight k on $\Gamma_1(M)$ and suppose d, N are positive integers such that $Md \mid N$. Then*

$$f | B_d | W_N = \left(\frac{N}{Md^2} \right)^{k/2} w(f) (f | B_{\frac{N}{Md}})^*.$$

Proof. Straightforward computation.

$$\begin{aligned}
f|B_d|W_N &= d^{-k/2} f| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \\
&= d^{-k/2} f| \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \begin{pmatrix} N/md & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \\
&= \left(\frac{N}{Md^2} \right)^{k/2} f|W_M|B_{N/Md} \\
&= \left(\frac{N}{Md^2} \right)^{k/2} w(f)f^*|B_{N/Md} \\
&= \left(\frac{N}{Md^2} \right)^{k/2} w(f)(f|B_{N/Md})^*.
\end{aligned}$$

□

Before stating the second lemma, we quote another result in [Li75] on the coefficients of a newform at primes dividing the level.

Lemma 4.6.5. [Li75, Theorem 3 (iii)] *Let $f = \sum_{n \geq 1} a_n(f)q^n$ be a normalized newform in $S_k(\Gamma_1(N), \epsilon)$ and let p be a prime dividing N . Then*

- (1) *If ϵ is a character modulo N/p and $p^2 \mid N$, then $a_p(f) = 0$.*
- (2) *If ϵ is a character modulo N/p and $p^2 \nmid N$, then $a_p(f)^2 = \epsilon(p)p^{k-2}$.*
- (3) *If ϵ is not a character modulo N/p , then $|a_p(f)| = p^{\frac{k-1}{2}}$.*

Lemma 4.6.6. *Keep the notations in Proposition 4.6.3. If $(S_1, S_2) \in \mathcal{B}_{S_\chi, M_\chi}$ is such that $a_{\pi(S_1)}(g_\chi) \neq 0$. Then $M\pi(S_1)\pi(S_2)^2 \mid N$.*

Proof. Let p be a prime divisor of $N' := M\pi(S_1)\pi(S_2)^2$. If $p \nmid M$, then $\text{ord}_p(N') \leq \text{ord}_p(\text{cond}(\chi)^2) \leq \text{ord}_p(N)$. So we assume $p \mid M$, hence $p \nmid p(S_2)$. If $p \nmid p(S_1)$, then there's nothing to prove; if $p \mid \pi(S_1)$, we want to show that $\text{ord}_p(M) < \text{ord}_p(N)$. Suppose not, then $\text{ord}_p(M) = \text{ord}_p(N) \geq 2\text{ord}_p(\text{cond}(\chi))$. Since $\text{cond}(\chi^2) \leq \text{cond}(\chi)$, we know χ^2 is a character modulo M/p . Applying case (1) of Lemma 4.6.5 to the newform g_χ , we see that $a_p(g_\chi) = 0$, hence $a_{\pi(S_1)}(g_\chi) = 0$ by multiplicativity. □

Now we can state our main theorem from this chapter.

Theorem 4.6.7. *Let $k \geq 2$ be an even integer and let f be a normalized newform in $S_k(\Gamma_0(N))$. Let z be a cusp on $X_0(N)$ of width one. Write $z = [\frac{a}{d}]$ such that $\gcd(a, d) = 1$, $d \mid N$ and $N \mid d^2$. Let $d' = \frac{N}{d}$. Then the Fourier expansion of f at the cusp z is*

$$f_z(q) = \frac{w(f)}{\varphi(d')} \sum_{\chi: \text{cond}'(\chi)=d'} \chi(-a) g'(\bar{\chi}) w(f \otimes \chi) f_\chi^!(q).$$

Here

- $w(f)$ and $w(f \otimes \chi)$ are the pseudo-eigenvalues.
- $g'(\chi)$ is the modified Gauss sum defined in 4.6.2 .
- cond' is the modified conductor of a Dirichlet character in 4.6.1.
- $f_\chi^!$ is as follows: let M_χ denote the level of $f \otimes \chi$. Then

$$f_\chi^! = \sum_{(S_1, S_2) \in \mathcal{B}_{S_{\chi_{nt}}, M_\chi}} (-1)^{|S_1|} a_{\pi(S_1)}(f \otimes \chi) \left(\frac{N}{M_\chi \pi(S_1)^2 \pi(S_2)^3} \right)^{k/2} \chi^2(\pi(S_2)) (f \otimes \chi | B_{\frac{N}{M_\chi \pi(S_1) \pi(S_2)^2}})^*$$

where the notations follow 4.6.3.

Proof. We start from formula 4.6.3:

$$f_{[\frac{a}{c}]} = \frac{w(f)}{\varphi(c')} \sum_{\text{cond}'(\chi)=c'} \chi(-a) g'(\bar{\chi}) f_{\chi_{nt}} | W_N.$$

From 4.6.3, we have

$$f_{\chi_{nt}} = \sum_{(S_1, S_2) \in \mathcal{B}_{S_\chi, M_\chi}} (-1)^{|S_1|} a_{\pi(S_1)}(f \otimes \chi) \pi(S_2)^{k/2} \chi_{nt}^2(\pi(S_2)) f \otimes \chi | B_{\pi(S_1) \pi(S_2)^2}.$$

To simplify notations, let $c(f, \chi, S_1, S_2) = (-1)^{|S_1|} a_{\pi(S_1)}(f \otimes \chi) \pi(S_2)^{k/2} \chi_{nt}^2(\pi(S_2))$. Then

$$\begin{aligned} f_{\chi_{nt}} | W_N &= \sum_{(S_1, S_2) \in \mathcal{B}_{S_\chi, M_\chi}} c(f, \chi, S_1, S_2) f \otimes \chi | B_{\pi(S_1) \pi(S_2)^2} W_N \\ &= \sum_{(S_1, S_2) \in \mathcal{B}_{S_\chi, M_\chi}} c(f, \chi, S_1, S_2) \left(\frac{N}{M_\chi (\pi(S_1) \pi(S_2)^2)^2} \right)^{k/2} w(f \otimes \chi) (f \otimes \chi | B_{\frac{N}{M_\chi \pi(S_1) \pi(S_2)^2}})^* \\ &= w(f \otimes \chi) f_\chi^!. \end{aligned}$$

Note that we applied Lemma 4.6.4 to obtain the penultimate equality, and we could do that because of Lemma 4.6.6. Now the result follows. \square

Theorem 4.6.7 gives us an algorithm to compute the expansion of f_z , which we will describe below. But first, we take a closer look at what ingredients goes into the expansion. Given a newform $f \in S_k(\Gamma_0(N))$ and a width one cusp z of denominator c . We need to consider the twist of f by all Dirichlet characters of conductor dividing c . For each such character χ , we then need to determine the level M_χ and q -expansion of the newform $f \otimes \chi$, the latter boils down to knowing $a_p(f \otimes \chi)$ for all primes $p \mid \text{cond}(\chi)$. Then we need to compute the pseudo-eigenvalues of $f \otimes \chi$. Finally, we combine these information together and apply Throem 4.6.7 to compute f_z .

Algorithm 3 Computing Fourier coefficients of f at width one cusps

Input: $f \in S_k(\Gamma_0(N))$ a newform; a, c – coprime integers such that $N \mid c^2$; B – a positive integer.

Output: The first B Fourier coefficients of $f_{[\frac{a}{c}]}(q)$.

- 1: $c' \leftarrow N/c$. $X \leftarrow$ The set of all Dirichlet characters χ such that $\text{cond}'(\chi) = c'$.
 - 2: compute $w(f)$ using Algorithm 2.
 - 3: **for** χ in X **do**
 - 4: Using Algorithm 1, compute the level M_χ and the q -expansion of $g_\chi := f \otimes \chi$ to B terms.
 - 5: Compute $w(g_\chi)$ using Algorithm 2.
 - 6: **end for**
 - 7: Apply Theorem 4.6.7 to compute f_z to B terms.
-

4.7 A Converse Theorem

Given the work in previous sections, it is a natural question then to ask whether the information on twists of f is uniquely determined by the expansion of f at width one cusps. The answer is yes, and the precise statement is in the following theorem.

Theorem 4.7.1. *Let f be a normalized newform in $S_k(\Gamma_0(N))$. Assume the eigenvalue $w_N(f)$ is known. Suppose c is a positive divisor of N such that $N \mid c^2$. Then the expansions of f_z , where z runs through all cusps of denominator c , uniquely determines the following: for each Dirichlet character χ of such that $\text{cond}'(\chi) = c'$, the level M_χ , the pseudo-eigenvalue w_{M_χ} and the q -expansion of the newform $f \otimes \chi$.*

Proof. By plug in different a 's. We can solve for t_χ . Consider the first nonzero term of t_χ . Suppose

$$t_\chi = u_\chi q^{v_\chi} + O(q^{v_\chi+1}), \quad u_\chi \neq 0.$$

Assuming that χ has prime power conductor $p^\beta > 1$, we claim that

$$\left| \frac{v^{k/2}}{u} \right| = \begin{cases} p^{k/2} & \text{if } p \nmid M_\chi \\ p^{1/2} & \text{if } p \mid M_\chi \text{ and } a_p(g) \neq 0 \\ 1 & \text{else} \end{cases}$$

Proof of claim: the first and third case are easy to verify using Theorem 4.6.7. Now assume $p \mid M$ and $a_p(g_\chi) \neq 0$. By Lemma 4.6.5, we have $|a_p(g_\chi)| = p^{k/2-1/2}$ or $p^{k/2-1}$. However, $|a_p(g_\chi)| = p^{k/2-1}$ only if $p \parallel M_\chi$ and χ^2 is a character modulo M_χ/p . This means χ^2 is the trivial character. By Lemma 4.3.2, we compute the p -level of $f = g_\chi \otimes \bar{\chi}$: note that $\max p, p^{\alpha+\beta}, p^{2\beta} > p$, so (ii) applies and the p -level of f is equal to $\max(p^\alpha, p^\beta) = p^\beta$, i.e., $\text{ord}_p(N) = \beta$. This is impossible since we have $p^{2\beta} = \text{cond}(\chi)^2 \mid N$.

Therefore, we have $|a_p(g_\chi)| = p^{k/2-1/2}$ and the claim follows.

Since $k \geq 2$, we could determine which case we are in. Then we can read off M_χ and $w_M(g_\chi)$. For example, if we are in the second case, then the level can be computed via $M_\chi = \frac{N}{v_\chi p}$. Now the N/M_χ 's coefficient of t_χ is

$$\begin{aligned} a_{\frac{N}{M}}(t_\chi) &= w(g_\chi) \left(\frac{N}{M}\right)^{k/2} (1 - |a_p(g_\chi)|^2 \chi^2(p) p^{-k/2}) \\ &= w(g_\chi) \left(\frac{N}{M}\right)^{k/2} (1 - p^{k/2-1} \chi^2(p)). \end{aligned}$$

This allows us to solve $w(g_\chi)$. Finally, we compute $a_p(g_\chi)$ by $a_p(g) = \frac{-u_\chi}{w(g_\chi) \chi^2(p) \left(\frac{N}{Mp}\right)^{k/2}}$. The

value $a_p(g)$ determines the expansion of g_χ . Recursively, we could solve for all pn -coefficients of g_χ , from which we deduce its complete q -expansion.

In the general case, we consider the following subsets of S_χ . Let $S_1^* = \{p \in S_\chi : p \mid M\}$, $S_2^* = S_\chi \setminus S_1^*$, and $\widetilde{S}_1^* = \{p \in S_1^* : a_p(g_\chi) \neq 0\}$.

It follows that the leading term of t_χ belongs to the summand corresponding to $(\widetilde{S}_1^*, S_2^*)$ in Theorem 4.6.7. Still writing the leading term as $u_\chi q^{v_\chi}$, we have

$$u_\chi = w(g_\chi) \chi^2(p(S_2)) a_{p(\widetilde{S}_1^*)}(g_\chi) p(\widetilde{S}_1^*)^{-k} (p(S_2^*))^{-3k/2} \left(\frac{N}{M_\chi} \right)^{k/2}, \quad v_\chi = \frac{N}{M_\chi p(\widetilde{S}_1^*) p(S_2^*)^2}.$$

Similar to the prime power conductor case above, we have $|a_{p(\widetilde{S}_1^*)}(g_\chi)| = p(\widetilde{S}_1^*)^{k/2-1/2}$. So

$$|v_\chi^k u_\chi^{-2}| = p(\widetilde{S}_1^*) p(S_2^*)^2. \quad (4.7.1)$$

Hence we can factor $|v_\chi^k u_\chi^{-2}|$ and obtain $p(\widetilde{S}_1^*)$ and $p(S_2^*)$. Then M_χ can be solved using v_χ . Plug it back into u_χ , we obtain $a_{p(\widetilde{S}_1^*)} w(g_\chi)$. Finally, for each $p \in \widetilde{S}_1^*$, the $v_\chi p$'s coefficient of t_χ allows us to compute $a_{p(\widetilde{S}_1^*)/p}(g_\chi) w(g_\chi)$. These together determine $w(g_\chi)$ and $a_{p(\widetilde{S}_1^*)}$. The other Fourier coefficients of g_χ can then be computed recursively. \square

4.8 Fields of definitions

In the previous sections, we have described an algorithm to compute the Fourier coefficients of f_z . In fact, the Fourier coefficients are algebraic numbers. More precisely, if c is the denominator of z and $c' = N/c$, then $f_z(q) \in K_f(\zeta_{c'})[[q]]$. Here K_f is the number field generated by the Fourier coefficients of f (at the cusp ∞). Although this result is well-known, we include a proof for the reader's convenience.

Lemma 4.8.1. *Let c be a cusp of denominator d and let $d' = N/d$. Then*

$$\mathbb{Q}(\{a_n(f, c)\}) \subseteq \mathbb{Q}(\{a_n(f)\}, \zeta_{d'}).$$

(fixme: add proof)

4.9 Denominators

(fixme)

4.10 Examples

Let $E = \mathbf{50a}$ and consider the 4 cusps of denominator 10 on $X_0(50)$. The corresponding first terms of q -expansions at these cusps are

$$\begin{aligned} a_1(f, \frac{1}{10}) &= \frac{1}{5}\zeta_5^3 - \frac{3}{5}\zeta_5^2 + \frac{3}{5}\zeta_5 - \frac{1}{5} \\ a_1(f, \frac{3}{10}) &= \frac{3}{5}\zeta_5^3 + \frac{6}{5}\zeta_5^2 + \frac{4}{5}\zeta_5 + \frac{2}{5} \\ a_1(f, \frac{7}{10}) &= \frac{2}{5}\zeta_5^3 - \frac{1}{5}\zeta_5^2 - \frac{4}{5}\zeta_5 - \frac{2}{5} \\ a_1(f, \frac{9}{10}) &= -\frac{6}{5}\zeta_5^3 - \frac{2}{5}\zeta_5^2 - \frac{3}{5}\zeta_5 - \frac{4}{5} \end{aligned}$$

As another examples, let $E = \mathbf{98a}$ and $z = [\frac{1}{14}]$. We computed numerically that

$$\begin{aligned} f_z(q) &= (-0.755001687308946 - 0.172324208281817i)q + (0.441471704846525 - 0.916725441095080i)q^2 \\ &\quad + (1.39294678431094 + 1.11083799261729i)q^3 + (0.696473392155471 - 0.555418996308649i)q^4 \\ &\quad + (1.51000337461789 - 0.344648416563641i)q^6 + (-3.80647894157196 \times 10^{-16} - 3.02371578407382i)q^7 \\ &\quad + (0.755001687308946 + 0.172324208281817i)q^8 + (-0.441471704846525 + 0.916725441095080i)q^9 + \\ &\quad (-0.882943409693050 - 1.83345088219016i)q^{12} + (-3.02000674923578 + 0.689296833127282i)q^{13} \\ &\quad + (3.80647894157196 \times 10^{-16} + 3.02371578407382i)q^{14} + O(q^{15}) \end{aligned}$$

4.11 Applications

One applications of the computation done in this chapter is the norm method to the computation of j -polynomials introduced in Chapter . Recall that the issue with the norm method for non-square free level is computing the expansions of form $f|\gamma$, where γ runs over the set of right coset representatives of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$. As we have seen, it suffices to compute the expansions of f at all width one cusps.

4.12 Automorphic representations; norm of first terms

In this section, we will restrict ourselves to the case when the Fourier coefficients of f are rational numbers. Then f induces an admissible representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$. We will see that the expansion of f at all cusps can also be computed from the local component $\pi_{f,p}$. Loeffler and Weinstein gave an algorithm to compute such local components.

We will restrict ourselves to the simplest case when f is twist-minimal, which means that the conductor of π_f is the smallest among all twists $\pi_{f \otimes \chi}$.

We will follow the notations of David Loeffler and use the formula of Francois Brunault. Also, I will use Jacquet-Langlands, Gelbart, and Bushnell-Henniart.

Okay, what is my heuristics for general k ? What is it for $\Gamma_1(N)$? What happens on the automorphic side?

Also there's the question about normalization, which was never specified.

Raw data?

Let z be a width one cusp of denominator c . Then the first coefficient $a_1(f_z)$ is an element in $K_f(\zeta_{c'})$. For simplicity, we assume that $c' = p^\alpha$ is a prime power. It can be proved using automorphic representations + local langlands correspondence that there exists β such that $p^\beta a_1(f_z) \in \bar{\mathbb{Z}}$. One question is: what prime ideals appears in the prime factorisation of $(a_1(f, z))$? It seems from our numerical data, that

$$\text{ord}_{\mathfrak{q}}(a_1(f_z)) > 0 \implies \mathfrak{q} \cap \mathbb{Z} \equiv \pm 1 \pmod{p}.$$

The following is a table of data.

(fix: add table)

4.12.1 Cuspidal local constants

We keep the assumptions that f is a newform attached to an elliptic curve E/\mathbb{Q} and f is twist-minimal. Assume p is a prime dividing the conductor N of E such that $v_p(N) = 2$. Then there exists a character $\varphi : \mathbb{F}_{p^2}^\times \rightarrow \mathbb{C}^\times$ which determines $\pi_{f,p}$. We will prove

Lemma 4.12.1. *Let $\psi : \mathbb{Q}_p \rightarrow \mathbb{C}^\times$ be a character of level one (e.g. $\psi(x) = e(\{\frac{x}{p}\}_p)$). Then*

$$\epsilon(\pi_{f,p}, 1/2, \psi) = \frac{-1}{p} \sum_{x \in \mathbb{F}_{p^2}^\times} \psi(x + x^p) \varphi(x).$$

If χ is a Dirichlet character such that the $f \otimes \chi$ has the same level as f . Then

$$\epsilon(\pi_{f \otimes \chi, p}, 1/2, \psi) = \frac{-1}{p} \sum_{x \in \mathbb{F}_{p^2}^\times} \psi(x + x^p) \varphi(x) \bar{\chi}(x^{p+1}).$$

Proof. By [BH], taking $n = r = 1$, we have

$$p^2 \epsilon(\pi_{f,p}, 1/2, \psi) \cdot \text{id} = \sum_{x \in GL_2(\mathbb{F}_p)} \psi(\text{tr}(x)) \pi_{f,p}^\vee(x). \quad (4.12.1)$$

where $\pi_{f,p}^\vee$ denotes the contragredient representation. The representation $\pi_{f,p}$ has dimension $(p-1)$. Taking traces, we obtain

$$p^2(p-1) \epsilon(\pi_{f,p}, 1/2, \psi) \cdot \text{id} = \sum_{x \in GL_2(\mathbb{F}_p)} \psi(\text{tr}(x)) \text{Tr}(\pi_{f,p}^\vee(x)). \quad (4.12.2)$$

By assumption, $\pi_{f,p}$ arises from a cuspidal representation of the finite group $GL_2(\mathbb{F}_p)$, which is in turn induced from φ . (See Fulton-Harris), we have formulae for $\text{Tr}(\pi_{f,p}^\vee(x))$. Splitting the sum corresponding to four types of conjugacy classes, we computed $S_1 = (p-1) \sum_{x \in \mathbb{F}_p^\times} \psi(2x)$, $S_2 = (p^2-1) \sum_{x \in \mathbb{F}_p^\times} \psi(2x)(-1)$, $S_3 = 0$, and $S_4 = (p^2-p)/2 \sum_{x \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p} \psi(\text{tr}(x))(\overline{\varphi(x) + \varphi(x^p)})$. So the sum on the right hand side of 4.12.2 equals $(p-p^2) \sum_{x \in \mathbb{F}_{p^2}^\times} \psi(\text{tr}(x)) \overline{\varphi(x)}$. Dividing by $p^2(p-1)$ gives the formula.

□

Moreover, since E is defined over \mathbb{Q} , the character of $\pi_{f,p}$ takes rational values. Hence the order of φ is 3, 4 or 6.

Lemma 4.12.2 (See Kraus?). *Let Δ denote the minimal discriminant of E . Then for $p \geq 5$, the order of φ is equal to $\frac{12}{\gcd(12, v_p(\Delta))}$.*

The local Langlands correspondence claims that the order of φ is equal to the order of the inertia subgroup of $\text{Gal}(L/\mathbb{Q})$, where L is the smallest number field over which E acquires good reduction (to-do: check this).

For $p = 2$ or 3 , the order of φ can be determined using Kraus's result <http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=GDZPPN002231468&IDDOC=219018>, or Dokchitser's paper Euler factors determine local Weil representations.

We remark that for elliptic curves, $v_2(N)$ is at most 8 and $v_3(N)$ is at most 5. For the sake of simplicity, we do not treat the case when $v_p(N) > 2$ here, but we point out the local constants can be also computed from formula in [BH], once the local component is determined using [DW].

Example 4.12.3. An example with trivial central character. Let f be the newform attached to $E = \mathbf{121a}$. Using Sage, we computed $w(f) = -1$. Since the weight of f is 2, we know $\epsilon_\infty = -1$ (since the central character of π_f is trivial, the level of the additive character ψ_∞ does not matter). The discriminant of E is $\Delta = -121$, so φ has order 6. Using Lemma 4.12.1, we computed that $\epsilon_{11}(\pi_{f,11}, 1/2) = -1$. This verifies $w(f) = -\prod_{p \leq \infty} \epsilon_p$.

Example 4.12.4. We give an example with nontrivial central character. Let f be as in the previous example, and let χ be the Dirichlet character of \mathbb{F}_{11}^\times defined by $\chi(2) = e^{2\pi i/10}$. Lemma 4.12.1 gives

$$\epsilon_{11}(\pi_{f \otimes \chi, 11}, 1/2) = 0.64.. + 0.76..i$$

an algebraic number with minimal polynomial $x^{20} + 109/121x^{15} + 2861/1331x^{10} + 109/121x^5 + 1$. So $w = -\epsilon_{11}\epsilon_\infty = \epsilon_{11}$. Using the numerical algorithm 2, we compute $w(f \otimes \chi) = 0.642573377564283 + 0.766224154177894i$. This confirms the computation.

4.13 Norm of first terms computations

We keep the assumptions from the previous section, that f is a newform in $S_2(\Gamma_0(N))$, attached to an elliptic curve E/\mathbb{Q} . We assume f is twist-minimal and $p \geq 5$ is a prime dividing the conductor N such that $v_p(N) = 2$. In this case, the cusp $z_p = [\frac{-p}{N}]$ is of width

one, and the q -expansion of f at z_p takes an especially simple form. We summarize this in the lemma below.

Lemma 4.13.1. *With the assumptions above, there exists a Galois-invariant set of numbers $\{b_1, \dots, b_{p-1}\} \subseteq \mathbb{Q}(\zeta_p)$, such that*

$$f_{z_p}(q) = \sum_{n \geq 1} a_n(f) b_n \pmod{pq^n}.$$

More precisely, the b_j are given by

$$b_j = w(f) \sum_{\chi: \text{cond}(\chi)=p} g(\bar{\chi}) w(f \otimes \chi) \chi(n)$$

Proof. First, the assumptions imply that $a_n(f) = 0$ if $p \mid n$. So the right hand side of the formula is well-defined. The formulae then follow directly from Theorem 4.6.7. We have $b_j \in \mathbb{Q}(\zeta_p)$ since the cusp z_p is defined over $\mathbb{Q}(\zeta_p)$. (fixme: check this). Moreover, the cusps $\{z_p^{(j)} = \frac{-jp}{N} : 1 \leq j \leq p-1\}$ form a Galois orbit on $X_0(N)$, and one has

$$a_n(f_{z_p^{(j)}}) = a_{jn}(f_{z_p}), \forall n \geq 1, 1 \leq j \leq p-1.$$

In particular, we have $\{b_j\} = \{a_1(f_{z_p^{(j)}})\}$. Since the latter set is Galois-invariant, so is the former. \square

We remark that it is clear from the formula of b_j that they are algebraic number. However, the formula does not imply directly that they lie in $\mathbb{Q}(\zeta_p)$.

We give another formula of $a_1(f_{z_p})$ in light of the previous section.

Lemma 4.13.2. *Keeping the assumptions in the previous two sections, we have*

$$a_1(f_{z_p}) = \frac{\sum_{x \in \mathbb{F}_{p^2}^\times} \psi(x + x^p + x^{p+1}) \varphi(x)}{\sum_{x \in \mathbb{F}_{p^2}^\times} \psi(x + x^p) \varphi(x)}.$$

Example 4.13.3. Let f be the newform attached to $E = \mathbf{49a}$. One checks that f is twist-minimal and φ has order 4. Using Lemma 4.13.2, we computed

$$a_1(f_{-1/7}) = -\frac{5}{7}\zeta_7^5 - \frac{3}{7}\zeta_7^4 - \frac{1}{7}\zeta_7^3 + \frac{1}{7}\zeta_7^2 + \frac{3}{7}\zeta_7 - \frac{2}{7} = 0.623489\dots + 1.29468\dots i.$$

The numerical algorithm gives $a_1(f_{-1/7}) = 0.623489801858733... + 1.29468991410431...i$. Hence our formulae are consistent for this example.

It is of interest to determine the factorization of $a_1(f_{z_p})$ as a principal fractional ideal in $\mathbb{Q}(\zeta_p)$. From the above discussion, we see that there are at most three possibilities for each p , corresponding to the order of φ being 3, 4 or 6.

Chapter 5

THINGS I TRIED TO DO BUT DID NOT END UP GIVING A NICE RESULT

generalizing the “congruence number” definition using other cusps.

Prove the “ $\pm 1 \bmod p$ ” guess.

Generalize another paper by William on computing order of component groups. (The original paper uses a trick which William fails to remember).

Prove even index for Chow-Heegner points.

Computing the critical subgroup for 5077a (multimodular is not practical).

Critical points of reduction of modular parametrization.

BIBLIOGRAPHY

- [AO03] Scott Ahlgren and Ken Ono. Weierstrass points on $X_0(p)$ and supersingular j -invariants. *Mathematische Annalen*, 325(2):355–368, 2003.
- [Asa76] Tetsuya Asai. On the fourier coefficients of automorphic forms at various cusps and some applications to rankin’s convolution. *Journal of the Mathematical Society of Japan*, 28(1):48–61, 1976.
- [AWL78] AOL Atkin and Wein-Ch’ing Winnie Li. Twists of newforms and pseudo-eigenvalues of w -operators. *Inventiones mathematicae*, 48(3):221–243, 1978.
- [BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over \mathbb{Q} : wild 3-adic exercises. *Journal of the American Mathematical Society*, pages 843–939, 2001.
- [BFH90] Daniel Bump, Solomon Friedberg, and Jeffrey Hoffstein. Nonvanishing theorems for L -functions of modular forms and their derivatives. *Inventiones mathematicae*, 102(1):543–618, 1990.
- [Che] Hao Chen. Computing Fourier expansion of $\Gamma_0(N)$ newforms at non-unitary cusps. In preparation.
- [Cre] J.E Cremona. Elliptic curve data. <http://www.maths.nott.ac.uk/personal/jec/ftp/data/INDEX.html>.
- [Del02] Christophe Delaunay. Formes modulaires et invariants de courbes elliptiques définies sur \mathbb{Q} . *Thèse de doctorat, Université Bordeaux 1*, décembre 2002.
- [Del05] Christophe Delaunay. Critical and ramification points of the modular parametrization of an elliptic curve. *J. Théor. Nombres Bordeaux*, 17:109–124, 2005.
- [DS06] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228. Springer Science & Business Media, 2006.
- [GZ86] Benedict H Gross and Don B Zagier. Heegner points and derivatives of L -series. *Inventiones mathematicae*, 84(2):225–320, 1986.

- [Li75] Wen-Ch'ing Winnie Li. Newforms and functional equations. *Mathematische Annalen*, 212(4):285–315, 1975.
- [Lig75] Gérard Ligozat. Courbes modulaires de genre 1. *Mémoires de la Société Mathématique de France*, 43:5–80, 1975.
- [Mah74] Kurt Mahler. On the coefficients of transformation polynomials for the modular function. *Bulletin of the Australian Mathematical Society*, 10(02):197–218, 1974.
- [MS04] Thom Mulders and Arne Storjohann. Certified dense linear system solving. *Journal of Symbolic Computation*, 37(4):485–510, 2004.
- [MSD74] B. Mazur and P. Swinnerton-Dyer. Arithmetic of Weil curves. *Invent. Math.*, 25:1–61, 1974.
- [S⁺14] W. A. Stein et al. *Sage Mathematics Software (Version 6.4)*. The Sage Development Team, 2014. <http://www.sagemath.org>.
- [Sil09] Joseph H Silverman. *The arithmetic of elliptic curves*, volume 106. Springer, 2009.
- [Stea] William Stein. Algebraic number theory, a computational approach. <https://github.com/williamstein/ant>.
- [Steb] William A Stein. *Modular forms, a computational approach*, volume 79.
- [Yan06] Yifan Yang. Defining equations of modular curves. *Advances in Mathematics*, 204(2):481–508, 2006.