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# Computational aspects of modular parametrizations of elliptic curves

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<sup>†</sup>an egocentric imitation, actually

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**Abstract**

Computational aspects of modular parametrizations  
of elliptic curves

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Department of Chair

This sample dissertation is an aid to students who are attempting to format their theses with  $\text{\LaTeX}$ , a sophisticated text formatter widely used by mathematicians and scientists everywhere.

- It describes the use of a specialized macro package developed specifically for thesis production at the University. The macros customize  $\text{\LaTeX}$  for the correct thesis style, allowing the student to concentrate on the substance of his or her text.<sup>1</sup>
- It demonstrates the solutions to a variety of formatting challenges found in thesis production.
- It serves as a template for a real dissertation.

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<sup>1</sup>See Appendix A to obtain the source to this thesis and the class file.

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## GLOSSARY

ARGUMENT: replacement text which customizes a  $\text{\LaTeX}$  macro for each particular usage.

BACK-UP: a copy of a file to be used when catastrophe strikes the original. People who make no back-ups deserve no sympathy.

CONTROL SEQUENCE: the normal form of a command to  $\text{\LaTeX}$ .

DELIMITER: something, often a character, that indicates the beginning and ending of an argument. More generally, a delimiter is a field separator.

DOCUMENT CLASS: a file of macros that tailors  $\text{\LaTeX}$  for a particular document. The macros described by this thesis constitute a document class.

DOCUMENT OPTION: a macro or file of macros that further modifies  $\text{\LaTeX}$  for a particular document. The option `[chapternotes]` constitutes a document option.

FIGURE: illustrated material, including graphs, diagrams, drawings and photographs.

FONT: a character set (the alphabet plus digits and special symbols) of a particular size and style. A couple of fonts used in this thesis are twelve point roman and *twelve point roman slanted*.

FOOTNOTE: a note placed at the bottom of a page, end of a chapter, or end of a thesis that comments on or cites a reference for a designated part of the text.

FORMATTER: (as opposed to a word-processor) arranges printed material according to instructions embedded in the text. A word-processor, on the other hand, is normally controlled by keyboard strokes that move text about on a display.

$\text{\LaTeX}$ : simply the ultimate in computerized typesetting.

MACRO: a complex control sequence composed of other control sequences.

PICA: an archaic unit of length. One pica is twelve points and six picas is about an inch.

POINT: a unit of length. 72.27 points equals one inch.

ROMAN: a conventional printing typestyle using serifs. the decorations on the ends of letter strokes. This thesis is set in roman type.

RULE: a straight printed line; e.g., \_\_\_\_\_.

SERIF: the decoration at the ends of letter strokes.

TABLE: information placed in a columnar arrangement.

THESIS: either a master's thesis or a doctoral dissertation. This document also refers to itself as a thesis, although it really is not one.



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The author wishes to express sincere appreciation to University of Washington, where he has had the opportunity to work with the T<sub>E</sub>X formatting system, and to the author of T<sub>E</sub>X, Donald Knuth, *il miglior fabbro*.

## DEDICATION

to my dear wife, Joanna

## Chapter 1

# INTRODUCTION

The utility of a clean, professionally prepared thesis is well documented<sup>1</sup> and, even if you never intend to actually print your thesis, you still ought to format it as if that were your intention.

T<sub>E</sub>X facilitates that. It is a flexible, complete and professional typesetting system. It will produce **pdf** output as required by the Graduate School.

### *1.1 The Purpose of This Sample Thesis*

This sample is both a demonstration of the quality and propriety of a L<sup>A</sup>T<sub>E</sub>Xformatted thesis and documentation for its preparation. It has made extensive use of a custom class file developed specifically for this purpose at the University of Washington. Chapter II discusses T<sub>E</sub>X and L<sup>A</sup>T<sub>E</sub>X. Chapter III describes the additional macros and functions provided by the custom thesis class file. Finally, Chapter IV hopes to tie things up.

It is impossible to predict all the formatting problems one will encounter and there will be problems that are best handled by a specialist. The Graduate School may be able to help you find help. Some departments may also be able to provide L<sup>A</sup>T<sub>E</sub>X assistance.

### *1.2 Conventions and Notations*

In this thesis the typist refers to the user of L<sup>A</sup>T<sub>E</sub>X—the one who makes formatting decisions and chooses the appropriate formatting commands. He or she will most often be the degree candidate.

---

<sup>1</sup>See, for example, W. Shakespeare[?] for a recent discussion.

This document deals with L<sup>A</sup>T<sub>E</sub>X typesetting commands and their functions. Wherever possible the conventions used to display text entered by the typist and the resulting formatted output are the same as those used by the T<sub>E</sub>Xbooks. Therefore, `typewriter type` is used to indicate text as typed by the computer or entered by the typist. It is quite the opposite of *italics*, which indicates a category rather than exact text. For example, `alpha` and `beta` might each be an example of a *label*.

### **1.3 *Nota bene***

This sample thesis was produced by the L<sup>A</sup>T<sub>E</sub>X document class it describes and its format is consonant with the Graduate School's electronic dissertation guidelines, as of November, 2014, at least. However, use of this package does not guarantee acceptability of a particular thesis.

## Chapter 2

# FOURIER EXPANSIONS OF CUSPIDAL MODULAR FORMS FORMS AT CUSPS

Let  $k$  be a positive even integer and let  $f \in S_k(\Gamma_0(N))$  be a nonzero cusp form. We are concerned with the problem of computing the Fourier expansion of  $f$  at cusps of width one other than the cusp  $[\infty]$ . Note that such cusps exist if and only if  $N$  is not square-free. We will give two algorithms, one numerical and the other exact, to compute such expansions. The question is studied in the Ph.D. thesis of Christophe Delaunay. We draw insight from another preprint by F.Brunault. The question is also studied in [Edixhoven], where numerical algorithm is given. The algorithm in [Ed] for computing expansions requires working at a higher level: to compute expansions at cusps of denominator  $Q$ , one needs to compute period matrices for forms of level  $NR^2$ , where  $R = \gcd(Q, \frac{N}{Q})$ . As a contrast, our algorithm works at levels dividing  $N$ .

### 2.1 Preliminaries

Let  $N \geq 1$  be an integer and let  $X_0(N)$  be the modular curve of level  $N$ .

**Definition 2.1.1.** Let  $z \in \mathbb{Q} \cup \{\infty\}$  be a cusp on  $X_0(N)$ . Write  $z = [a/c]$  with  $\gcd(a, c) = 1$ . The *denominator* of  $z$  is

$$d_z = \gcd(c, N).$$

As a convention, we set  $d_\infty = N$ . Choose  $\alpha \in SL_2(\mathbb{Z})$  such that  $\alpha(\infty) = z$ . The *width* of  $z$  is

$$h_z = \left| \frac{SL_2(\mathbb{Z})_\infty}{(\alpha^{-1}\{\pm I\}\Gamma_0(N)\alpha)_\infty} \right|$$

where the subscript  $\infty$  means taking the isotropy subgroup of  $\infty$  in the corresponding group.

The width of a cusp can be computed in terms of its denominator. In fact, we have

**Lemma 2.1.2.** *If  $z$  is a cusp on  $X_0(N)$ , then*

$$h_z = \frac{N}{\gcd(d_z^2, N)}.$$

*Proof.* When  $z = [\infty]$ , we have  $d_\infty = N$  and  $h_\infty = 1$ , so the formula holds trivially. Otherwise, write  $z = [\frac{a}{c}]$  and find  $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ . For  $N' \in \mathbb{Z}$  we compute

$$\alpha \begin{pmatrix} 1 & N' \\ 0 & 1 \end{pmatrix} \alpha^{-1} = \begin{pmatrix} * & * \\ -c^2 N' & * \end{pmatrix}.$$

Hence  $\begin{pmatrix} 1 & N' \\ 0 & 1 \end{pmatrix} \in (\alpha^{-1}\{\pm I\}\Gamma_0(N)\alpha)_\infty \iff N \mid c^2 N' \iff \frac{N}{\gcd(d_z^2, N)} \mid N'$ . This completes the proof.  $\square$

In particular, the width of a cusp  $z$  is one if and only if  $N \mid d_z^2$ .

Suppose  $f$  be a cusp form on  $\Gamma_0(N)$  and  $\alpha \in SL_2(\mathbb{Z})$ . Then  $f[\alpha]$  is a cusp form on  $\Gamma(N)$ . So  $f[\alpha]$  has a  $q$ -expansion, which is a power series in  $q^{\frac{1}{N}}$ . A natural thing to do is to define the expansion of  $f$  at the cusp  $z$  as the expansion of  $f[\alpha]$ . However, one must note that this may not be well-defined: the expansion depends on the choice of  $\alpha$ . Nonetheless, when the denominator of the cusp is sufficiently divisible by the prime divisors of  $N$ , the expansion is well-defined as a power series in  $q$ .

**Lemma 2.1.3.** *Let  $z$  be a cusp on  $X_0(N)$  with width one. Choose  $\alpha \in SL_2(\mathbb{Z})$  such that  $\alpha(\infty) = z$ . Then  $f|[\alpha]$  is a cusp form on  $\Gamma_1(N)$ . Moreover, the function  $f|[\alpha]$  is independent of the choice of  $\alpha$ .*

*Proof.* It is easy to verify that  $\Gamma_1(N) \subseteq \alpha^{-1}\Gamma_0(N)\alpha$ , hence the first claim holds. Now suppose  $\beta \in SL_2(\mathbb{Z})$  is such that  $\beta(\infty) = z$ . Then  $\alpha^{-1}\beta \in SL_2(\mathbb{Z})_\infty$ . Since  $z$  has width one, we have  $\alpha^{-1}\beta \in \alpha^{-1}\Gamma_0(N)\alpha$ . Hence  $\beta \in \Gamma_0(N)\alpha$ , and it follows that  $f|[\beta] = f|[\alpha]$ .  $\square$

In light of the lemma above, we define the  $q$ -expansion of  $f$  at a width one cusp  $z$  to be the  $q$ -expansion of  $f|[\alpha]$ , and denote it by  $f_z$ .

Assume further that  $f$  is an eigenform under the Atkin-Lehner operators. We will show that in order to compute the expansion of  $f|[\alpha]$  for any  $\alpha \in SL_2(\mathbb{Z})$ , it suffices to do so for  $\alpha = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ , where  $0 \leq m < N$  and  $N \mid \gcd(m, N)^2$ . In particular, it suffices to compute the expansions of  $f$  at a some cusps of width one.

**Lemma 2.1.4.** *For any  $\alpha \in SL_2(\mathbb{Z})$ , there exists a matrix  $w_Q \in W_N$  and an upper triangular matrix  $u \in GL_2(\mathbb{Q})$  such that  $w\alpha = \alpha'u$ , where  $\alpha' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$  satisfies  $N \mid \gcd(N, c')^2$ .*

Indeed, one may find  $Q$  using Lemma. Now  $f|[\alpha] = f|[w_Q][w_Q\alpha] = f|[w_Q][\alpha'][u] = \lambda_Q(f)f[\alpha'][u] = \lambda_Q(f)f[\alpha'']|u|$ , where  $\alpha''$  is of form  $\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ . Note that for an upper triangular matrix  $u = \begin{pmatrix} u_0 & u_1 \\ 0 & u_2 \end{pmatrix}$ , we have  $f[u](q) = f(q^{u_0/u_2}e^{2\pi i u_1/u_2})$ .

## 2.2 Reducing to the case of newforms and $N/d_z$ a prime power

The space  $S_k(\Gamma_0(N))$  is spanned by elements of form  $g(q^d)$ , where  $g$  is newform of level  $M \mid N$  and  $d$  is a divisor of  $\frac{N}{M}$ . Note that  $g(q^d) = d^{-k/2}g| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ . For any  $\alpha \in SL_2(\mathbb{Z})$ , we can find  $\alpha' \in SL_2(\mathbb{Z})$  and  $u \in GL_2(\mathbb{Q})$  such that  $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \alpha = \alpha'u$ . Hence to compute all expansions  $f|[\alpha]$ , it suffices to give an algorithm for newforms.

We can reduce the problem to the case when  $N/d_z$  is a prime power.

**Lemma 2.2.1.** *Let  $b$  be an integer such that  $b^2 \mid N$ . Write  $b = \prod_j Q_j$  uniquely as a product of distinct prime powers. For each  $1 \leq j \leq r$ , let  $s_j$  be an integer such that  $(s_j, N/Q_j) = 1$  and  $s_j \equiv (\frac{b}{Q_j})^{-1} \pmod{Q_j}$ . Then for all integers  $a$  such that  $(a, N/b) = 1$ , we have*

$$a_1 \left( f, \frac{a}{N/b} \right) = \prod_{j=1}^r a_1 \left( f, \frac{as_j}{N/Q_j} \right).$$

In the rest of this chapter, we will restrict ourselves to solving the following problem:

**Problem 2.2.2.** Let  $f$  be a normalized newform in  $S_2(\Gamma_0(N))$  and  $z$  be a cusp on  $X_0(N)$  of width one. Compute the  $q$ -expansion of  $f_z$ .

Add lemma  
on prime  
power cord

### 2.3 Twists of newforms

We define the twist  $f \otimes \chi$ .

Reference: [Winnie].

### 2.4 Pseudo-eigenvalues

Let  $\epsilon$  be a Dirichlet character of modulus  $N$  and let  $f$  be a newform in  $S_k(N, \epsilon)$ . For any divisor  $Q$  of  $N$  with  $\gcd(Q, \frac{N}{Q}) = 1$ , there is an algebraic number  $\lambda_Q(f)$  of absolute value one and a newform  $g$  in  $S_k(N, \overline{\epsilon_Q} \epsilon_{N/Q})$  such that

$$W_Q(f) = \lambda_Q(f)g,$$

**Definition 2.4.1.** The number  $\lambda_Q(f)$  is called the pseudo-eigenvalue of  $W_Q$  on  $f$ .

Recall that if  $f = \sum a_n q^n$  is a modular form, then  $f^*$  is defined as  $f^*(q) = \sum \bar{a}_n q^n$ . From now on, we take  $Q = N$ . Then from [Winnie] we have  $W_N(f) = \lambda_N(f)f^*$ . In the rest of this section, we describe an algorithm to efficiently compute the numerical value of  $\lambda(f) := \lambda_N(f)$ .



---

**Algorithm 1** Computing the pseudo-eigenvalue of newforms.

---

**Input:**  $k \geq 0$  an *even* integer.  $f \in S_k(\Gamma_1(N))$  a normalized newform.

**Output:** a numerical approximation of the complex number  $\lambda_N(f)$  defined by  $f|W_N = \lambda_N(f)f^*$ .

```

1:  $n_0 := 10, z_0 := \frac{i}{\sqrt{N}}, \delta = 10^{-3}$ .
2: Randomly generate  $n_0$  points  $\{z_1, \dots, z_{n_0}\} \subseteq \{z | 0 < \text{Im}(z) < 1/2\sqrt{N}, |z| = \sqrt{1}N\}$ 
3: for  $1 \leq i \leq n_0$  do
4:   compute the period integral  $c_i = \int_{z_0}^{z_i} 2\pi i f(z) z^{\frac{k-2}{2}} dz$ .
5:    $w_i := c_i / \bar{c}_i$ .
6: end for
7:  $w := (w_1, \dots, w_{n_0})$ .
8: if the standard deviation of  $w$  is less than  $\delta$  then
9:   return  $\frac{1}{n_0}(\sum_i w_i)$ .
10: else
11:   go back to step 1 and increase  $n_0$ .
12: end if
```

---

*Remark 2.4.2.* When  $f$  has trivial character, we have  $g = f$  and  $\lambda_N(f) = \pm 1$  is indeed an eigenvalue of  $W_N$ . This eigenvalue can be determined using modular symbols, which is faster than Algorithm 1. For more details, see documentation of sage.

*Remark 2.4.3.* The period integral in step 4 of Algorithm ?? is computed as follows: This approach is taken from [Cre97].

To prove the correctness of algorithm 1, we will define a space of symbols that mimics weight- $k$  modular symbols and then show that they share certain properties. The proofs follow [Stein-modform, Chapter ].

**Definition 2.4.4.**

Let any element  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$  on  $M(k)$  by

$$g(P\{\alpha, \beta\}) = P(dX - bY, -cX + aY)\{g\alpha, g\beta\}.$$

We define a pairing between  $S_k(\Gamma_1(N))$  and  $M(k)$  by the following:

$$(f, P\{\alpha, \beta\}) \stackrel{def}{=} \int_{\alpha}^{\beta} 2\pi i f(z) P(z, 1) dz.$$

**Lemma 2.4.5.** *Let  $k \geq 0$  be even. Let  $M \in M(k)$  and  $f \in S_k(\Gamma_1(N))$ . Then*

$$N^{\frac{k}{2}-1}(f|W_N, M) = (f, W_N M).$$

*Proof.* See [Stein,]. Note that the extra factor  $N^{\frac{k}{2}-1}$  is due to the different definitions the weight- $k$  action of  $GL_2(\mathbb{Q})$  on modular forms: in [Stein] it is defined by

$$blah$$

□

The map

$$P(x, y)\{\alpha, \beta\} \mapsto P(-x, y)\{-\bar{\alpha}, -\bar{\beta}\}$$

defines an involution on the space  $M(k)$  which we call the *start involution* and denote by  $*$ .

Then

**Lemma 2.4.6.**

$$(f^*, M) = \overline{(f, M^*)}.$$

*Proof.*

□

**Lemma 2.4.7.** *Let  $f$  be a normalised newform with even weight  $k \geq 0$  on  $\Gamma_1(N)$  and let  $M \in M(k)$  be such that  $W_N(M) = N^{k/2-1}M^*$ . Assume  $(f, M) \neq 0$ . Then the pseudo-eigenvalue  $\lambda_N(f)$  of  $f$  is*

$$\lambda_N(f) = \frac{(f, M)}{\overline{(f, M)}}.$$

*Proof.* Since  $W_N^2(M) = N^{k-2}M$  for all  $M \in M(k)$ , the assumption implies  $W_N(M^*) = N^{k/2-1}M$ . Now

$$\begin{aligned}
& N^{k/2-1}(f|W_N, M^*) = (f, W_N(M^*)) \text{ (Lemma ??)} \\
& \implies N^{k/2-1}\lambda_N(f)(f^*, M^*) = N^{k/2-1}(f, M) \\
& \implies \lambda_N(f) = \frac{(f, M)}{(f^*, M^*)} \\
& \implies \lambda_N(f) = \frac{(f, M)}{(f, M)}.
\end{aligned}$$

□

Finally, suppose  $\alpha, \beta \in \{z \in \mathbb{C} | \text{Im}(z) > 0, |z| = 1/\sqrt{N}\}$ . Then it is easy to verify that  $M = (xy)^{k/2-1}\{\alpha, \beta\}$  satisfies the condition of the above lemma, i.e.,  $W_N(M) = M^*$ . The correctness of Algorithm ?? now follows.

## 2.5 Identifying level of twists

We start by recalling some facts on twists of newforms from [Winnie].

**Theorem 2.5.1** (Winnie, theorem 3.2). *Let  $f \in S_k(N, \epsilon)$  be a newform. Let  $\chi$  be a Dirichlet character of modulus  $N$  s.t., the conductor of  $\chi$  is equal to  $Q = q^\beta$ , where  $q$  is a prime divisor of  $N$ . Then there is a newform  $h \in S_k(Q'M, \epsilon\chi^2)$ , such that*

$$f_\chi = h - h|U_q|B_q.$$

By the above theorem, we can compute the  $q$ -expansion of  $f_\chi|W_N$  once we identify the level and  $q$ -expansion of  $h$ .

Notation: For convenience we write  $f \otimes \chi$  for  $h$ .

We are mainly interested in the case where  $f$  has trivial nebentypus character. For such forms we have the following lemma.

**Lemma 2.5.2** ( $h$  has Level dividing  $N$ ). *Let  $f \in S_k(\Gamma_0(N))$ . Suppose  $Q = \text{cond}(\chi)$  has  $Q^2 \mid N$ . Then  $f \otimes \chi$  is new of level  $N' \mid N$ .*

*Proof.* By [Winnie, theorem 3.1], we have  $f_\chi \in S_k(N, \chi^2)$  (since  $\alpha = 0$  and  $q^{2\beta} \leq Q$ ). It now follows from theorem 3.2 that the level of  $f \otimes \chi$  is a divisor of  $N$ .  $\square$

Now we proceed on how to algorithmically recognise the level of  $f \otimes \chi$  from the coefficients of  $f$ . One potential obstacle is that we do not know all Fourier coefficients of  $f \otimes \chi$ . We only know that

$$a_n(f \otimes \chi) = a_n(f)\chi(n), \text{ if } \gcd(n, N) = 1.$$

We will overcome this using a variant of Sturm's bound argument. First we recall the usual Sturm bound. We quote theorem from [Stein]. Or rather, let's prove a lemma first.

**Lemma 2.5.3.** *Let  $f \in S_k(N, \epsilon)$  be a normalized newform. Then  $f|U_q|B_q \in S_k(Nq^2, \epsilon)$ .*

*Proof.* We use a standard fact that for any integer  $d \geq 1$ , the map  $f \mapsto f|B_d$  takes  $S_k(N, \epsilon)$  to  $S_k(Nd, \epsilon)$ . To prove the lemma, we consider two separate cases. First, assume  $q \nmid N$ , then we have  $T_q = U_q + q^{k-1}\epsilon(q)B_q$ . By our assumption, we have  $f|T_q = a_q(f)f$ . Therefore, we have  $f|U_q|B_q = f|(T_q - q^{k-1}\epsilon(q)B_q)|B_q = a_q(f)f|B_q - q^{k-1}\epsilon(q)f|B_q^2$ . Hence  $f|U_q|B_q \in S_k(Nq^2, \epsilon)$ . Now assume  $q \mid N$ , so  $U_q = T_q$ . Hence  $f|U_q|B_q = a_q(f)f|B_q \in S_k(Nq, \epsilon) \subseteq S_k(Nq^2, \epsilon)$ .  $\square$

The next proposition generalised the usual Sturm bound argument for modular forms.

**Proposition 2.5.4.** *Let  $g_1, g_2$  be two normalised newforms of levels  $N_1 \mid N_2$ . Suppose  $g_i \in S_k(N_2, \epsilon)$ , where  $\epsilon$  has conductor  $Q = q^\beta$ ,  $q$  is a prime, and  $Q^2 \mid N$ . Let  $B$  be the Sturm bound for the congruence subgroup  $\Gamma_1(Nq^2)$ . Suppose*

$$a_n(g_1) = a_n(g_2), \ 1 \leq n \leq B \text{ and } \gcd(n, q) = 1.$$

*Then  $g_1 = g_2$ .*

*Proof.* Following [Winnie], we define the operator  $K_q$  on the space of modular forms by

$$g|K_q = g - g|U_q|B_q.$$

Condition is equivalent to the statement that  $\delta = (g_1 - g_2)|K_q$  has  $a_n(\delta) = 0$  for all  $1 \leq n \leq B$ . From Lemma ?? we know  $\delta \in S_k(Nq^2, \epsilon)$ . Hence Sturm's theorem ??, implies  $\delta = 0$ , i.e.,  $g_1 - g_2|K_q = 0$ .

Theorem 5.7.1 in [DS] then implies that  $g_1 - g_2 \in S_k(N_2, \epsilon)^{old}$ . If  $N_1 < N_2$ , then  $g_1$  in the old subspace. So  $g_2 \in S_k(N_2, \epsilon)^{old}$ , a contradiction. Hence  $N_1 = N_2$ . Then  $g_1 - g_2 \in S_k(N_2, \epsilon)^{new}$ , and since the new subspace and the old subspace have trivial intersection, we have  $g_1 - g_2 = 0$ .  $\square$

Now we are ready to describe the algorithm that given  $f$  and  $\chi$ , determines the level of  $f \otimes \chi$ .

---

**Algorithm 2** Identifying level of  $f \otimes \chi$

---

**Input:**  $f \in S_k(\Gamma_0(N))$  a newform;  $\chi$  a Dirichlet character of prime power conductor  $Q = q^\beta$ ;  $Q^2 \mid N$ .

**Output:** The positive integer  $M$  such that the  $f \otimes \chi$  is new of level  $M$ .

```

1: if  $Q = 1$  then
2:   return  $N$ .
3: end if
4:  $Q' := \text{cond}(\chi^2)$ ;  $N_0 := \frac{N}{q^{v_q(N)}}$ ;  $M_0 := Q'N_0$ ;  $t := \frac{N}{M_0} \in \mathbb{Z}$ .
5: for each positive divisor  $d$  of  $t$  do
6:   Set  $V_d := S_k(M_0d, \chi^2)$ .
7:   Compute a basis of newforms  $\{g_1^{(d)}, \dots, g_{s_d}^{(d)}\}$  of  $V_d$ .
8:   Set  $B_d :=$  the Sturm bound for  $\Gamma_1(M_0dq^2)$ .
9:   for  $1 \leq j \leq s_d$  do
10:    if  $a_n(g_j^{(d)}) = a_n(f)\chi(n)$  for all  $1 \leq n \leq B_d, \gcd(n, q) = 1$  then
11:      return  $M_0d$ .
12:    end if
13:   end for
14: end for

```

---

The correctness of the algorithm is ensured by Lemma and Lemma .

It is natural to define *p-minimality* of newforms. The definition mimics that of [Brunault].

**Definition 2.5.5.** Let  $f \in S_k(\Gamma_1(N))$  be a newform. Let  $p$  be a prime such that  $p^2 \mid N$ . We say  $f$  is *p-minimal* if  $f \otimes \chi$  is new of level  $N$  for all Dirichlet character  $\chi : (\mathbb{Z}/p^{\frac{v_p(N)}{2}}\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ .

We give some example computations applying the above algorithm.

**Example 2.5.6.** Let  $f$  be the normalised newform attached to the elliptic curve

$$E :$$

of label **50a**. Then  $f \otimes \chi$  is new of level 50 for all Dirichlet characters  $\chi$  with modulus 5. In other words,  $f$  is 5-minimal.

As another example, we demonstrate a newform which is not  $p$ -minimal.

**Example 2.5.7.** Let  $f$  be the normalised newform attached to the elliptic curve

$$E : y^2 + xy = x^3 + x^2 - 25x - 111$$

of label **98a**. Let  $\chi$  be the Dirichlet character modulo 7 defined by  $\chi(3 \pmod{7}) = -1$ . Then  $f \otimes \chi$  is a newform of level 14.

## 2.6 The formula for the Fourier expansion of $f$ at width one cusps

We start with the formula

$$f_{[\frac{-a}{M}]}(q) = \frac{w_N(f)}{\varphi(M')} \sum_{\text{cond}(\chi) \mid M'} \chi(a) f|_{R_\chi(M')} W_N.$$

## 2.7 Fields of definitions

**Lemma 2.7.1.** Let  $c$  be a cusp of denominator  $d$  and let  $d' = N/d$ . Then

$$\mathbb{Q}(\{a_n(f, c)\}) \subseteq \mathbb{Q}(\{a_n(f)\}, \zeta_{d'}).$$

*Proof.* Let  $K_0 = \mathbb{Q}(\{a_n(f)\})$ . Choose a form  $0 \neq g \in S_k(\Gamma_1(N))$  with rational Fourier coefficients such that  $h = \frac{f}{g}$  is non-constant. From [Cox, ] it is easy to see that  $h \in K_0$ . Then we have  $\square$

## 2.8 Examples

Let  $E = \mathbf{50a}$  and consider the 4 cusps of denominator 10 on  $X_0(50)$ . The corresponding first terms of  $q$ -expansions at these cusps are

## 2.9 Applications

One applications of the computation done in this chapter is the norm method to the computation of  $j$ -polynomials introduced in Chapter . Recall that the issue with the norm method for non-square free level is computing the expansions of form  $f|_\gamma$ , where  $\gamma$  runs over the set of right coset representatives of  $\Gamma_0(N)$  in  $SL_2(\mathbb{Z})$ . To compute the norm of  $f$  when  $N$  is non-square free, it suffices to compute the expansions of  $f$  at all width-1 cusps. This is a consequence of the following lemma.

**Lemma 2.9.1.** *For any cusp  $z$  of  $X_0(N)$ , there exists an Atkin-Lehner involution  $w \in W(N)$  such that  $z_1 = w(z)$  has width one.*

*Proof.* Let  $z \neq [\infty]$  be a cusp. Recall that  $z$  has width one if and only if its denominator  $d(z)$  satisfies  $d(z)^2 \equiv 0 \pmod{N}$ . Let  $p$  be a prime divisor of  $N$ . Then it is easy to see that  $v_p(d(w_p(z))) = v_p(N) - v_p(d(z))$  and  $v_l(d(w_p(z))) = v_l(d(z))$  for primes  $l \neq p$ . The lemma now follows by taking  $w = \prod_{p|N: v_p(d(z)) \leq v_p(N)/2} w_p$ .  $\square$

## 2.10 Norm guess and data

## Chapter 3

### THE THESIS UNFORMATTED

This chapter describes the `uwthesis` class (`uwthesis.cls`, version dated 2014/11/13) in detail and shows how it was used to format the thesis. A working knowledge of Lamport's `LATEX` manual[?] is assumed.

#### 3.1 *The Control File*

The source to this sample thesis is a single file only because ease of distribution was a concern. You should not do this. Your task will be much easier if you break your thesis into several files: a file for the preliminary pages, a file for each chapter, one for the glossary, and one for each appendix. Then use a control file to tie them all together. This way you can edit and format parts of your thesis much more efficiently.

Figure 3.1.1 shows a control file that might have produced this thesis. It sets the document style, with options and parameters, and formats the various parts of the thesis—but contains no text of its own.

The first section, from the `\documentclass` to the `\begin{document}`, defines the document class and options. This sample thesis specifies the `proquest` style, which is now required by the Graduate School and is the default. Two other, now dated, other styles are available: `twoside`, which is similar but produces a wider binding margin and is more suitable for paper printing; and `oneside`, which is really old fashioned. This sample also specified a font size of 11 points. Possible font size options are: `10pt`, `11pt`, and `12pt`. Default is 12 points, which is the preference of the Graduate School. If you choose a smaller size be sure to check with the Graduate School for acceptability. The smaller fonts can produce very small sub and superscripts.



```

% LaTeX thesis control file

\documentclass [11pt, proquest]{uwthesis}[2014/11/13]

\begin{document}

% preliminary pages
%
\prelimpages
\include{prelim}

% text pages
%
\textpages
\include{chap1}
\include{chap2}
\include{chap3}
\include{chap4}

% bibliography
%
\bibliographystyle{plain}
\bibliography{thesis}

% appendices
%
\appendix
\include{appxa}
\include{appxb}

\include{vita}
\end{document}

```

Figure 3.1.1: A thesis control file (`thesis.tex`). This file is the input to  $\text{\LaTeX}$  that will produce a thesis. It contains no text, only commands which direct the formatting of the thesis.

Include most additional formatting packages with `\usepackage`, as describe by Lamport[?]. The one exception to this rule is the `natbib` package. Include it with the `natbib` document option.

Use the `\includeonly` command to format only a part of your thesis. See Lamport[?, sec. 4.4] for usage and limitations.

## **3.2 The Text Pages**

A chapter is a major division of the thesis. Each chapter begins on a new page and has a Table of Contents entry.

### *3.2.1 Chapters, Sections, Subsections, and Appendices*

Within the chapter title use a `\\` control sequence to separate lines in the printed title (recall Figure ??). The `\\` does not affect the Table of Contents entry.

Format appendices just like chapters. The control sequence `\appendix` instructs L<sup>A</sup>T<sub>E</sub>X to begin using the term ‘Appendix’ rather than ‘Chapter’.

Specify sections and subsections of a chapter with `\section` and `\subsection`, respectively. In this thesis chapter and section titles are written to the table of contents. Consult Lamport[?, pg. 176] to see which subdivisions of the thesis can be written to the table of contents. The `\\` control sequence is not permitted in section and subsection titles.

### *3.2.2 Footnotes*

Footnotes format as described in the L<sup>A</sup>T<sub>E</sub>X book. You can also ask for end-of-chapter or end-of-thesis notes. The thesis class will automatically set these up if you ask for the document class option `chapternotes` or `endnotes`.

If selected, `chapternotes` will print automatically. If you choose `endnotes` however you must explicitly indicate when to print the notes with the command `\printendnotes`. See the style guide for suitable endnote placement.

### 3.2.3 *Figures and Tables*

Standard L<sup>A</sup>T<sub>E</sub>X figures and tables, see Lamport[?, sec. C.9], normally provide the most convenient means to position the figure. Full page floats and facing captions are exceptions to this rule.

If you want a figure or table to occupy a full page enclose the contents in a **fullpage** environment. See figure 3.2.1.

#### *Facing pages*

Facing page captions are an artifact of traditional, dead-tree printing, where a left-side (even) page faces a right-side (odd) page.

In the **twoside** style, a facing caption is full page caption for a full page figure or table and should face the illustration to which it refers. You must explicitly format both pages. The caption part appears on an even page (left side) and the figure or table comes on the following odd page (right side). Enclose the float contents for the caption in a **leftfullpage** environment, and enclose the float contents for the figure or table in a **fullpage** environment. The first page (left side) contains the caption. The second page (right side) could be left blank. A picture or graph might be pasted onto this space. See figure 3.2.1.

You can use these commands with the **proquest** style, but they have little effect on online viewing.

### 3.2.4 *Horizontal Figures and Tables*

Figures and tables may be formatted horizontally (a.k.a. landscape) as long as their captions appear horizontal also. L<sup>A</sup>T<sub>E</sub>X will format landscape material for you.

Include the **rotating** package

```
\usepackage[figuresright]{rotating}
```

and read the documentation that comes with the package.

Figure 3.2.2 is an example of how a landscape table might be formatted.

```

\begin{figure}[p]% the left side caption
  \begin{leftfullpage}
    \caption{ . . . }
  \end{leftfullpage}
\end{figure}
\begin{figure}[p]% the right side space
  \begin{fullpage}
    . . .
    ( note.. no caption here )
  \end{fullpage}
\end{figure}

```

Figure 3.2.1: This text would create a double page figure in the two-side styles.

```

\begin{sidewaystable}
  ...
  \caption{ . . . }
\end{sidewaystable}

```

Figure 3.2.2: This text would create a landscape table with caption.

### 3.2.5 Figure and Table Captions

Most captions are formatted with the `\caption` macro as described by Lamport[?, sec. C.9]. The `uwthesis` class extends this macro to allow continued figures and tables, and to provide multiple figures and tables with the same number, e.g., 3.1a, 3.1b, etc.

To format the caption for the first part of a figure or table that cannot fit onto a single page use the standard form:

```
\caption[toc]{text}
```

To format the caption for the subsequent parts of the figure or table use this caption:

```
\caption(-){(continued)}
```

It will keep the same number and the text of the caption will be (*continued*).

To format the caption for the first part of a multi-part figure or table use the format:

```
\caption(a)[toc]{text}
```

The figure or table will be lettered (with ‘a’) as well as numbered. To format the caption for the subsequent parts of the multi-part figure or table use the format:

```
\caption(x){text}
```

where  $x$  is b, c, .... The parts will be lettered (with ‘b’, ‘c’, ...).

### 3.2.6 Line spacing

Normally line spacing will come out like it should. However, the ProQuest style allows single spacing in certain situations: figure content, some lists, and etc. Use `\uwsinglespace` to switch to single spacing within a `\begin{}` and `\end{}` block. The code examples in this document does this.

## 3.3 The Preliminary Pages

These are easy to format only because they are relatively invariant among theses. Therefore the difficulties have already been encountered and overcome by L<sup>A</sup>T<sub>E</sub>X and the thesis document classes.

Start with the definitions that describe your thesis. This sample thesis was printed with the parameters:

```
\Title{The Suitability of the \LaTeX\ Text Formatter\\
      for Thesis Preparation by Technical and\\
      Non-technical Degree Candidates}
\Author{Jim Fox}
\Program{UW Information Technology}
\Year{2012}
```

```

\Chair{Name of Chairperson}{title}{Chair's department}
\Signature{First committee member}
\Signature{Next committee member}
\Signature{etc}

```

Use two or more `\Chair` lines if you have co-chairs.

### 3.3.1 Copyright page

Print the copyright page with `\copyrightpage`.

### 3.3.2 Title page

Print the title page with `\titlepage`. The title page of this thesis was printed with<sup>1</sup>

```
\titlepage
```

You may change default text on the title page with these macros. You will have to redefine `\Degreetext`, for instance, if you're writing a Master's thesis instead of a dissertation.<sup>2</sup>

```

\Degree{degree name} defaults to "Doctor of Philosophy"
\School{school name} defaults to "University of Washington"
\Degreetext{degree text} defaults to "A dissertation submitted ..."
\textofCommittee{committee label} defaults to "Reading Committee:"
\textofChair{chair label} defaults to "Chair of the Supervisory Committee:"

```

These definitions must appear before the `\titlepage` command.

---

<sup>1</sup>Actually, it wasn't. I added a footnote—something you would not do.

<sup>2</sup>If you use these they can be included with the other information before `\copyrightpage`.

### 3.3.3 *Abstract*

Print the abstract with `\abstract`. It has one argument, which is the text of the abstract. All the names have already been defined. The abstract of this thesis was printed with

```
\abstract{This sample . . . ‘real’ dissertation.}
```

### 3.3.4 *Tables of contents*

Use the standard L<sup>A</sup>T<sub>E</sub>X commands to format these items.

### 3.3.5 *Acknowledgments*

Use the `\acknowledgments` macro to format the acknowledgments page. It has one argument, which is the text of the acknowledgment. The acknowledgments of this thesis was printed with

```
\acknowledgments{The author wishes . . . {\it il miglior fabbro}.\par}}
```

### 3.3.6 *Dedication*

Use the `\dedication` macro to format the dedication page. It has one argument, which is the text of the dedication.

### 3.3.7 *Vita*

Use the `\vita` macro to format the curriculum vitae. It has one argument, which chronicles your life’s accomplishments.

Note that the Vita is not really a preliminary page. It appears at the end of your thesis, just after the appendices.

## Chapter 4

### **RUNNING L<sup>A</sup>T<sub>E</sub>X** (*AND PRINTING IF YOU MUST*)

From a given source T<sub>E</sub>X will produce exactly the same document on all computers and, if needed, on all printers. *Exactly the same* means that the various spacings, line and page breaks, and even hyphenations will occur at the same places.

How you edit your text files and run L<sup>A</sup>T<sub>E</sub>X varies from system to system and depends on your personal preference.

#### **4.1 Running**

The author is woefully out of his depth where T<sub>E</sub>X on Windows is concerned. Google would be his resource. On a linux system he types

```
$ pdflatex uwthesis
```

and it generally works.

#### **4.2 Printing**

All implementations of T<sub>E</sub>X provide the option of **pdf** output, which is all the Graduate School requires. Even if you intend to print a copy of your thesis create a **pdf**. It will print most anywhere.



## Appendix A

### WHERE TO FIND THE FILES

The `uwthesis` class file, `uwthesis.cls`, contains the parameter settings, macro definitions, and other  $\text{\TeX}$  commands which allow  $\text{\LaTeX}$  to format a thesis. The source to the document you are reading, `uwthesis.tex`, contains many formatting examples which you may find useful. The bibliography database, `uwthesis.bib`, contains instructions to BibTeX to create and format the bibliography. You can find the latest of these files on:

- My page.

`http://staff.washington.edu/fox/tex/uwthesis.html`

- CTAN

`http://tug.ctan.org/tex-archive/macros/latex/contrib/uwthesis/`

(not always as up-to-date as my site)

## VITA

Jim Fox is a Software Engineer with UW Information Technology at the University of Washington. His duties do not include maintaining this package. That is rather an avocation which he enjoys as time and circumstance allow.

He welcomes your comments to `fox@uw.edu`.