Lecture 15

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Covariance

Recap:

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2 \sum_{i \le j \le n} \operatorname{Cov}(X_i, X_j)$$

If the X_i's are independent, then

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i)$$

E.g.: In Section 2.6, we will show that if $Y \sim \text{Gamma}(n, \lambda)$, $n \in \mathbb{N}$, then $Y \sim X_1 + X_2 + \cdots + X_n$ in which $X_i \sim \text{Exp}(\lambda)$ and the X_is are independent.

$$\begin{aligned}
\operatorname{Var} Y &= \operatorname{Var}(X_1 + \cdots + X_n) \\
&= \sum_{i=1}^{n} \operatorname{Var} X_i \\
&= \sum_{i=1}^{n} \frac{1}{\lambda^2} = \frac{n}{\lambda^2}
\end{aligned}$$

E.g.: Suppose $X \sim \text{Bern}(p)$

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$\mathbb{E}(X^k) = 1^k \cdot p + 0^k(1-p) = p, \qquad k > 0$$

$$\implies \operatorname{Var} X = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = p - p^2 = p(1-p)$$

Aside: max when $p = \frac{1}{2}$

Defn: If $Y \sim \text{Bin}(n, p)$, $n \in \mathbb{N}$

The number of successes in n independent trials in which the probability of success is p for any trial.

Let X_i = outcome of the i^{th} trial so that

$$X_i = \begin{cases} 1 & \text{if success,} & \text{w.p. } p \\ 0 & \text{if failure,} & \text{w.p. } 1-p \end{cases}$$

$$Y = X_1 + \cdots + X_n,$$

where X_i is independent and identically distributed(IID) $\text{Bern}(p)$.

$$\mathbb{E}Y = \sum_{i=1}^{n} \mathbb{E}X_i = \sum_{i=1}^{n} p = np$$

$$\operatorname{Var} Y = n \operatorname{Var} X_1 = np(1-p)$$

Defn: More properties of the covariance:

Last time we showed that Covariance is symmetric and bilinear.

$$\operatorname{Cov}(X, Y) = \operatorname{Cov}(Y, X)$$

$$\operatorname{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \operatorname{Cov}(X_i, Y_j)$$

The same is true for dot product of two vectors $\vec{x}$, $\vec{y}$:

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$$

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$$
$$|\vec{x} \cdot \vec{y}|^2 \le (\vec{x} \cdot \vec{x})(\vec{y} \cdot \vec{y})$$
$$|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \times \|\vec{y}\| \times \cos\theta$$

Proof: $|\operatorname{Cov}(X,Y)| \le \sqrt{\operatorname{Var} X \cdot \operatorname{Var} Y}$

1. $\operatorname{Var} Y = 0$

$$\implies \mathbb{E}[(Y - \mathbb{E}Y)^2] = 0$$

   because $(Y - \mathbb{E}Y)^2 \ge 0$,

$$\implies Y - \mathbb{E}Y = 0, \qquad \text{w.p. } 1$$
$$\implies \mathbb{P}(Y = a) = 1 \qquad \exists\, a \in \mathbb{R}$$

   For example, $X \sim U(0,1)$

$$Y = \begin{cases} \frac{1}{2} & \text{if} \quad X < \frac{1}{2} \text{ or } X > \frac{1}{2} \\ -\frac{1}{2} & \text{if} \quad X = \frac{1}{2} \end{cases}$$

$$\therefore Y = \frac{1}{2} \qquad \text{w.p. } 1$$

2. $\operatorname{Var} Y > 0$

   Define $Z = X - \frac{\operatorname{Cov}(X,Y)}{VarY} \cdot Y$, $\operatorname{Var} Z \ge 0$,

$$\begin{aligned} \operatorname{Var} Z &= \operatorname{Cov}(Z, Z) \\ &= \operatorname{Cov}\left(X - \frac{\operatorname{Cov}(X,Y)}{VarY} \cdot Y, X - \frac{\operatorname{Cov}(X,Y)}{VarY} \cdot Y\right) \\ &= \operatorname{Cov}(X,X) - 2\frac{\operatorname{Cov}(X,Y)}{VarY} \cdot \operatorname{Cov}(X,Y) + \left(\frac{\operatorname{Cov}(X,Y)}{VarY}\right)^2 \cdot \operatorname{Cov}(Y,Y) \end{aligned}$$

$$\implies 0 \le \operatorname{Var} X - \frac{(\operatorname{Cov}(X,Y))^2}{\operatorname{Var} Y}$$
$$\implies (\operatorname{Cov}(X,Y))^2 \le \operatorname{Var} X \cdot \operatorname{Var} Y$$
$$\implies \operatorname{Cov}(X,Y) \le \sqrt{\operatorname{Var} X \cdot \operatorname{Var} Y}$$

Cor:

$$\rho_{X,Y} = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var} X \cdot \operatorname{Var} Y}} \in [-1, 1]$$