Lecture 20

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Conditional Probability and Conditional Expectation

Defn: Conditional probability of discrete r.v.s

Recall $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$

Suppose we have jointly distribution discrete r.v.s X and Y with pmf

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

Define

$$\begin{aligned} p_{X|Y}(x \mid y) &= \mathbb{P}(X = x \mid Y = y) \\ &= \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} \\ &= \frac{p_{X,Y}(x,y)}{p_Y(y)} \end{aligned}$$

in which $p_Y(y) = \sum_{x'} p_{X,Y}(x', y)$

Or, if we are given $p_{X|Y}$ and p_Y, then the joint pmf

$$p_{X,Y}(x,y) = p_{X|Y}(x \mid y) \cdot p_Y(y)$$

Iterated expectation:

$$\mathbb{E}_Y(\mathbb{E}_{X|Y}(g(X) \mid Y)) = \mathbb{E}(g(X))$$

$$\begin{aligned} \text{lhs} &= \mathbb{E}\left(\sum_x g(x) p_{X|Y}(x|Y)\right) \\ &= \sum_y \left(\sum_x g(x) p_{X|Y}(x|y)\right) p_Y(y) \end{aligned}$$

or

$$\begin{aligned} \text{lhs} &= \sum_y \mathbb{E}(g(X) \mid y) p_Y(y) \\ &= \sum_y \left(\sum_x g(x) p_{X|Y}(x|y)\right) p_Y(y) \end{aligned}$$

$$\begin{aligned} \text{lhs(cont'd)} &= \sum_y \sum_x g(x) p_{X,Y}(x,y) \\ &= \sum_x g(x) \sum_y p_{X,Y}(x,y) \\ &= \sum_x g(x) p_X(x) = \mathbb{E}(g(X)) = \text{rhs} \end{aligned}$$

E.g.: A hen lays N eggs, in which $N \sim \mathcal{P}(\lambda)$. Each egg hatches with probability p, independently of the other eggs.

Let K = the number of chicks. Find $\mathbb{E}(K \mid N)$, $\mathbb{E}K$ and $\mathbb{E}(N \mid K)$.

Solution: If $N = n$, then

$$K \mid n \sim \text{Binom}(n,p) \implies \mathbb{E}(K \mid N = n) = np$$

$$\mathbb{E}(K) = \mathbb{E}_N(\mathbb{E}_{K|N}(K \mid N)) = \mathbb{E}_N(N \cdot p) = \lambda p$$

To compute $\mathbb{E}(N \mid K)$, we need the conditional pmf of $N \mid K = k$

$$\begin{aligned} p_{N|K}(n \mid k) &= \mathbb{P}(N = n \mid K = k) \\ &= \frac{\mathbb{P}(N = n, K = k)}{\mathbb{P}(K = k)} \end{aligned}$$

Given:

$$\begin{aligned} p_N(n) &= e^{-\lambda}\frac{\lambda^n}{n!}, \qquad n = 0, 1, 2, \ldots \\ p_{K|N}(k \mid n) &= \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, 2, \ldots, n \\ &\implies p_{N,K}(n, k) = p_{K|N}(k \mid n) p_N(n) \end{aligned}$$

and

$$\begin{aligned} p_K(k) = \mathbb{P}(K = k) &= \sum_{m=k}^{\infty} p_{N,K}(m, k) \\ &= \sum_{m=k}^{\infty} p_{K|N}(k \mid m) p_N(m) \\ &= \sum_{m=k}^{\infty} \binom{m}{k} p^k (1-p)^{m-k} \cdot e^{-\lambda}\frac{\lambda^m}{m!} \\ &= e^{-\lambda} \sum_{m=k}^{\infty} \frac{m!}{k!(m-k)!} p^k (1-p)^{m-k} \frac{\lambda^m}{m!} \\ &= e^{-\lambda}\frac{1}{k!} p^k \lambda^k \sum_{m=k}^{\infty} \frac{(\lambda(1-p))^{m-k}}{(m-k)!} \\ &= e^{-\lambda}\frac{1}{k!} p^k \lambda^k \sum_{m'=0}^{\infty} \frac{(\lambda(1-p))^{m'}}{(m')!} \\ &= e^{-\lambda}\frac{1}{k!} p^k \lambda^k e^{\lambda(1-p)} \end{aligned}$$

$$\begin{aligned} p_{N|K}(n \mid k) &= \frac{e^{-\lambda}\frac{1}{k!} p^k \lambda^k \frac{(\lambda(1-p))^{n-k}}{(n-k)!}}{e^{-\lambda}\frac{1}{k!} p^k \lambda^k e^{\lambda(1-p)}} \\ &= e^{-\lambda(1-p)} \frac{(\lambda(1-p))^{n-k}}{(n-k)!}, \qquad n = k, k+1, \ldots \\ &\therefore (N - k) \mid (K = k) \sim \mathcal{P}(\lambda(1-p)) \end{aligned}$$

$$\begin{aligned} \therefore \mathbb{E}(N \mid K = k) &= \mathbb{E}((N - k) + k \mid K = k) \\ &= \mathbb{E}(N - k \mid K = k) + \mathbb{E}(k \mid K = k) \\ &= \lambda(1-p) + k \end{aligned}$$

Defn: Conditional probability of continuous r.v.s

Conditional probability:

$$\begin{aligned} \mathbb{P}(X \le x \mid y < Y \le y + \Delta y) &= \frac{\mathbb{P}(X \le x, y < Y \le y + \Delta y)}{\mathbb{P}(y < Y \le y + \Delta y)} \\ &= \int_{-\infty}^{x} \int_{y}^{y+\Delta y} f_{X,Y}(u, v)\, dv du \\ &= \int_{y}^{y+\Delta y} f_Y(v)\, dv \end{aligned}$$