Lecture 13

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Independent random variables

Defn: X and Y are independent if

$$F_{X,Y}(x,y) = F_X(x)F_Y(y), \qquad \forall x, y \in \mathbb{R}$$

means the event $\{\omega \in \Omega : X(\omega) \le x, Y(\omega) \le y\}$ is independent with respect to $\mathbb{P}$ for all $x, y \in \mathbb{R}$.

X and Y are independent random variables

$$\implies \mathbb{E}(g(X)h(Y)) = \mathbb{E}g(X) \cdot \mathbb{E}h(Y).$$

For ex, if X and Y are continuous

$$F_{X,Y} = F_X(x)F_Y(y)$$

$$\begin{aligned} F_{X,Y}(x,y) &= F_X(x)F_Y(y) \\ \frac{\partial^2}{\partial y \partial x} \implies f_{X,Y}(x,y) &= F'_X(x)F'_Y(y) \\ &= f_X(x)f_Y(y) \end{aligned}$$

$$\begin{aligned} \implies \mathbb{E}(g(X)h(Y)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)\,dxdy \\ &= \int_{-\infty}^{\infty} g(x)f_X(x)\,dx \cdot \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy \\ &= \mathbb{E}g(X) \cdot \mathbb{E}h(Y) \end{aligned}$$

If X and Y are independent and discrete, then

$$\begin{aligned} p_{X,Y}(x,y) &= F_X(x)F_Y(y) - F_X(x)F_Y(y-) - F_X(x-)F_Y(y) + F_X(x-)F_Y(y-) \\ &= F_X(x)(F_Y(y) - F_Y(y-)) - F_X(x-)(F_Y(y) - F_Y(y-)) \\ &= (F_X(x) - F_X(x-))(F_Y(y) - F_Y(y-)) \\ &= p_X(x)p_Y(y) \end{aligned}$$

Covariance

Defn: The covariance of X and Y equals $\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$ write it as $\mathrm{Cov}(X,Y)$.

$$\begin{aligned} \mathrm{Cov}(X,Y) &= \mathbb{E}[XY - Y\mathbb{E}X - X\mathbb{E}Y + \mathbb{E}X \cdot \mathbb{E}Y] \\ &= \mathbb{E}(XY) - \mathbb{E}X \cdot \mathbb{E}Y - \mathbb{E}Y \cdot \mathbb{E}X + \mathbb{E}X \cdot \mathbb{E}Y \\ &= \mathbb{E}(XY) - \mathbb{E}X\mathbb{E}Y \end{aligned}$$

Correlation coefficient

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\,X \cdot \mathrm{Var}\,Y}} \in [-1, 1]$$

$\rho_{X,Y} = 1$ means $X \sim aY + b$, $a > 0$.

E.g.: Covariance of bivariate normal

$$\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_1)(y - \mu_2)f_{X,Y}(x,y)\,dxdy$$

Let $z = \frac{x-\mu_1}{\sigma_1}$ and $v = \frac{x-\mu_2}{\sigma_2}$,

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x-\mu_1)(y-\mu_2)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_1 z \cdot \sigma_2 v \cdot \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[z^2 - 2\rho\cdot zv + v^2\right]\right\} \\
&= \int_{-\infty}^{\infty} \frac{\sigma_1\sigma_2 v}{2\pi\sqrt{1-\rho^2}} \cdot e^{-\frac{v^2}{2(1-\rho^2)}} \cdot e^{\frac{(\rho v)^2}{2(1-\rho^2)}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2(1-\rho^2)}(z^2-2\rho zv+(\rho v)^2)}\, dzdv \\
&= \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} v e^{-\frac{v^2}{2}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2}\, dzdv \\
&= \frac{\sigma_1\sigma_2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v e^{-\frac{v^2}{2}} \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2}\, dzdv
\end{aligned}
$$

where $\int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2}\, dz$ is essentially the expectation of $N(\rho v, 1-\rho^2)$, which is ρv. Then,

$$
\begin{aligned}
&= \frac{\sigma_1\sigma_2}{\sqrt{2\pi}} \cdot \rho \int_{-\infty}^{\infty} v^2 e^{-\frac{v^2}{2}}\, dv \\
&= \sigma_1\sigma_2\rho
\end{aligned}
$$

since $\int_{-\infty}^{\infty} \frac{v^2}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\, dv$ is equivalent to $\mathbb{E}(V^2)$, $V \sim N(0,1)$.

Correlation coefficient

$$
\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\, X \cdot \mathrm{Var}\, Y}} = \frac{\sigma_1\sigma_2\rho}{\sqrt{\sigma_1^2\sigma_2^2}} = \rho
$$

For the bivariate normal, X and Y are independent iff $\rho = 0$ iff $\mathrm{Cov}(X,Y) = 0$.

For other jointly distributed X and Y, X and Y are independent,

$$
\implies \mathbb{E}(XY) = \mathbb{E}X \cdot \mathbb{E}Y \implies \mathrm{Cov}(X,Y) = 0
$$

But if $\mathrm{Cov}(X,Y) = 0$, then we cannot deduce that X and Y are independent.

E.g.: Special case: $X(\omega) = 1_A(\omega)$, $A \in \mathcal{F}$

$$
X(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}
$$

$Y = 1_B$, $B \in \mathcal{F}$. Then,

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbb{E}(XY) - \mathbb{E}X \cdot \mathbb{E}Y \\
&= \mathbb{E}[1_A \cdot 1_B] - \mathbb{E}(1_A)\mathbb{E}(1_B) \\
&= \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(X,Y) > 0 &\iff \mathbb{P}(A \cap B) > \mathbb{P}(A)\mathbb{P}(B) \\
&\iff \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} > \mathbb{P}(A) \\
&\iff \mathbb{P}(A \mid B) > \mathbb{P}(A)
\end{aligned}
$$