Lecture 02

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$\mathcal{F}$ is a σ-algebra of subset of Ω if it is non-empty, closed under countable union and and closed under complements.
Suppose $A \in \mathcal{F} \implies A^c \in \mathcal{F}$ $A \cup A^c = \Omega \in \mathcal{F} \implies \Omega^c = \varnothing \in \mathcal{F}$

1.3 Probability defined on $(\Omega, \mathcal{F})$

Two ways to motivate/define probability:

- Frequentist: $A \in \mathcal{F}$, then we think of the probability as the long-range proportion of the times that event A occurs.
$$\frac{N(A)}{N} \xrightarrow{N\to\infty} \mathbb{P}(A)$$
- Bayesian/subjective: Suppose you were to win \$1 if A occurs, then $\mathbb{P}(A)$ is the amount you would bet for A to happen.

Defn: $\mathbb{P}$ is a probability on $(\Omega, \mathcal{F})$ if $\mathbb{P}$ maps $\mathcal{F}$ to $[0, 1]$ such that

1. $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$
2. If $A_1, A_2, \cdots \in \mathcal{F}$ are pairwise disjoint, then
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$
$(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

(Properties of probability) For an arbitrary sequence in $\mathcal{F}$, $B_1, B_2, \ldots$ (not necessarily disjoint),
$$\bigcup_{i=1}^{\infty} B_i \in \mathcal{F}$$
So,
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) \in [0, 1]$$
$$\leq \sum_{i=1}^{\infty} \mathbb{P}(B_i)$$
which is not necessarily equal to the sum of probability

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ $A \cap A^c = \varnothing$, so they are disjoint. Thus,
$$\mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c),$$
where $\mathbb{P}(\Omega) = 1$.
$$\implies 1 = \mathbb{P}(A) + \mathbb{P}(A^c)$$
$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$
Aside: If we only require $\mathbb{P}(\Omega) = 1$, then
$$\mathbb{P}(\Omega) = \mathbb{P}(\Omega^c) = 1 - \mathbb{P}(\Omega) = 1 - 1 = 0.$$
Thus, $\mathbb{P}(\Omega) = 0$ follows from $\mathbb{P}(\Omega) = 1$ and (countable) additivity of $\mathbb{P}$.
2. If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$
$$B = A \cup (B - A) = A \cup (B \cap A^c)$$
(disjoint union), where $\mathbb{P}(B \cap A^c) \in [0, 1]$
$$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c) \geq \mathbb{P}(A)$$

3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) (\leq \mathbb{P}(A) + \mathbb{P}(B))$

$$A \cup B = A \cup (B \cap (A \cap B)^c)$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap (A \cap B)^c)$$

Aside from (b):

$$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$$

substituting all A with $A \cap B$, we have

$$\mathbb{P}(B \cap (A \cap B)^c) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$\mathbb{P}(A \cup B) \overset{Aside}{=} \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

4. More generally, if $A_1, A_2, \ldots, A_n \in \mathcal{F}$, then an induction proof shows that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n).$$

E.g.: $A, B \in \mathcal{F}$, $\mathbb{P}(A) = 3/4$, $\mathbb{P}(B) = 1/3$, Show that $1/12 \leq \mathbb{P}(A \cap B) \leq 1/3$

$$A \cap B \subset A \text{ and } A \cap B \subset B$$

$$\therefore \mathbb{P}(A \cap B) \leq \min(\mathbb{P}(A), \mathbb{P}(B)) = \frac{1}{3}$$

$$A \cup B \subset \Omega \implies \mathbb{P}(A \cup B) \leq 1$$

$$\implies \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq 1$$

$$\implies \mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 = \frac{1}{12}$$