Lecture 18

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Moment Generating Functions

Defn: Moment Generating Functions $M_X(t) = \mathbb{E}(e^{Xt})$, in which t is any real number for which this finite.

If $X \geq 0$, then $M_X(t)$ exists for $t \leq 0$. If $M_X(t)$ exists in an open interval about $t = 0$, then

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{Xt}] \\ &= \mathbb{E}[1 + Xt + \frac{1}{2}(Xt)^2 + \dots] \\ &= 1 + t\mathbb{E}X + \frac{1}{2}t^2\mathbb{E}(X^2) + \dots + \frac{1}{n!}t^n\mathbb{E}(X^n) + \dots \end{aligned}$$

$$\implies M_X^{(n)}(t)\Big|_{t=0} = \mathbb{E}(X^n)$$

The Pareto r.v. does have a MGF that is finite for any $t > 0$ because $\mathbb{E}(X^k)$ is finite only for $0 \leq k < \alpha$.

$$S_X(x) = \left(\frac{\theta}{\theta + x}\right)^\alpha, \qquad x \geq 0$$

E.g.: $X \sim \text{Binomal}(n, p)$

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{Xt}) \\ &= \sum_{x=0}^{n} e^{xt} \cdot \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\ &= (pe^t + (1-p))^n, \qquad t \in \mathbb{R} \end{aligned}$$

$$\begin{gathered} M_X'(t) = n(pe^t + (1-p))^{n-1} \cdot pe^t \\ \implies \mathbb{E}X = M_X'(0) = np \\ M_X''(t) = n(n-1)(pe^t + (1-p))^{n-2}(pe^t)^2 + n(pe^t + (1-p))^{n-1}pe^t \\ \mathbb{E}(X^2) = M_X''(0) = n(n-1)p^2 + np \\ \implies \operatorname{Var} X = n(n-1)p^2 + np - (np)^2 = np - np^2 = np(1-p) \end{gathered}$$

E.g.: $X \sim \text{Poisson}(\lambda)$

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{Xt}) \\ &= \sum_{x=0}^{\infty} e^{xt} \cdot e^{-\lambda} \frac{\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} e^{-\lambda} \cdot \frac{(\lambda e^t)^x}{x!} \\ &= e^{-\lambda} \cdot e^{\lambda e^t} \\ &= e^{\lambda(e^t - 1)}, \qquad t \in \mathbb{R} \end{aligned}$$

$$\begin{gathered} M_X'(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t \\ \implies \mathbb{E}X = M_X'(0) = \lambda \\ M_X''(t) = e^{\lambda(e^t-1)}(\lambda e^t)^2 + e^{\lambda(e^t-1)} \cdot \lambda e^t \\ \mathbb{E}(X^2) = M_X''(0) = \lambda^2 + \lambda \\ \implies \operatorname{Var} X = \lambda^2 + \lambda - \lambda^2 = \lambda \end{gathered}$$

E.g.: $X \sim \text{Gamma}(\alpha, \lambda)$

$$\begin{aligned}
M_X(t) &= \mathbb{E}(e^{Xt}) \\
&= \int_0^\infty e^{xt} \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x(\lambda-t)}\, dx \\
&= \left(\frac{\lambda}{\lambda-t}\right)^\alpha, \qquad t < \lambda
\end{aligned}$$

where $\int_0^\infty x^{\alpha-1}e^{-x(\lambda-t)}\,dx = \frac{\Gamma(\alpha)}{(\lambda)}$. From here, we will have $\mathbb{E}X = \frac{\alpha}{\lambda}$ and $\text{Var}\, X = \frac{\alpha}{\lambda^2}$.

Cor: Second application of MGFs. If $X_1, X_2, \ldots, X_n$ are independent r.v.s with MGFs $M_1, M_2, \ldots, M_n$, then the MGF of $X = X_1 + X_2 + \cdots + X_n$ is

$$\begin{aligned}
M_X(t) &= \mathbb{E}(e^{Xt}) \\
&= \mathbb{E}(e^{(X_1+X_2+\cdots+X_n)t}) \\
&= \mathbb{E}(e^{X_1 t} \cdot e^{X_2 t} \ldots e^{X_n t}) \\
&= \mathbb{E}(e^{X_1 t}) \cdot \mathbb{E}(e^{X_2 t}) \ldots \mathbb{E}(e^{X_n t}) \\
&= M_1(t) M_2(t) \ldots M_n(t)
\end{aligned}$$

If these MGFs exist in an open interval about 0, then $M_X(t)$ determines the distribution of X.

E.g.: $X_i \sim \text{Binomal}(n_i, p)$, $i = 1, \ldots, m$, independent.

Let $X = X_1 + \cdots + X_m$

$$\begin{aligned}
&\implies M_X(t) = \prod_{i=1}^m (pe^t + (1-p))^{n_i} = (pe^t + (1-p))^{\sum_{i=1}^m n_i} \\
&\implies X \sim \text{Binomal}\left(\sum_{i=1}^m n_i, p\right)
\end{aligned}$$

E.g.: $X_i \sim \text{Poisson}(\lambda_i)$, $i = 1, \ldots, n$, independent.

Let $X = X_1 + \cdots + X_n$

$$\begin{aligned}
&\implies M_X(t) = \prod_{i=1}^n e^{\lambda_i(e^t-1)} = e^{(\sum_{i=1}^n \lambda_i)(e^t-1)} \\
&\implies X \sim \text{Poisson}\left(\sum_{i=1}^n \lambda_i\right)
\end{aligned}$$

E.g.: Aside: $\alpha = 1$, then $X \sim \text{Exp}(\lambda)$, $M_X(t) = \frac{\lambda}{\lambda-t}$

$X_i \sim \text{Gamma}(\alpha_i, \lambda)$, $i = 1, 2, \ldots, n$, independent.

$$X = \sum_{i=1}^n X_i$$

$$\begin{aligned}
M_X(t) &= \prod_{i=1}^n \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_i} \\
&= \left(\frac{\lambda}{\lambda-t}\right)^{\sum_{i=1}^n \alpha_i}
\end{aligned}$$

$$X \sim \Gamma(\sum_{i=1}^n \alpha_i, \alpha)$$

E.g.: Geom(p)

$\mathbb{P}(X = x) = (1-p)^{x-1}p$, $x = 1, 2, \ldots$

$$\begin{aligned} M_X(t) &= \sum_{x=1}^{\infty} e^{xt} \cdot (1-p)^{x-1} \cdot p \\ &= \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t(1-p))^x \\ &= \frac{p}{1-p} \cdot \frac{e^t(1-p) - 0}{1 - e^t(1-p)} \cdot \left(\frac{e^{-t}}{e^{-t}}\right) \\ &= \frac{p}{e^{-t} - (1-p)} \end{aligned}$$

where $e^t(1-p) \in (-1, 1)$. $e^t < \frac{1}{1-p}$, $t < -\ln(1-p)$.