Lecture 09

Professor Virginia R. Young
Transcribed by Hao Chen

September 19, 2022

Recap: Continuous r.v. is one for which $\exists f$ defined on $\mathbb{R}$ such that

$$F(x) = \int_{-\infty}^{x} f(y)\, dy, \qquad F'(x) = f(x)$$

Defn: $X \sim U(a, b)$

$$f(x) = \frac{1}{b-a}, \qquad a < x < b$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$$

If X is a continuous r.v., then $\mathbb{P}(X = x) = 0$, $\forall x \in \mathbb{R}$

Defn: $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\, dy$$

Let $z = \frac{y-\mu}{\sigma}$, $dz = \frac{dy}{\sigma}$,

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}z^2}\, dz$$

which is similar to pdf of $N(0, 1)$.

Let Φ denote the cdf of $N(0, 1)$. Then, if $X \sim N(\mu, \sigma^2)$, then

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Let $Y = e^X$, then $Y \sim \text{lognormal}(\mu, \sigma)$. So, $Y \sim \text{logN}(\mu, \sigma) \iff \ln Y \sim N(\mu, \sigma^2)$

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) \\ &= \mathbb{P}(X \le \ln y) = F_X(\ln y) \\ &= \Phi\left(\frac{(\ln y) - \mu}{\sigma}\right) \end{aligned}$$

for $y > 0$,

$$\frac{d}{dy} F_Y(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{y} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2}.$$

Defn: $X \sim \text{Gamma}(\alpha, \lambda)$

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad x > 0$$

where α, λ are positive parameters.

$$\begin{aligned} \Gamma(\alpha) &= \text{gamma function} \\ &= \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \alpha > 0 \end{aligned}$$

Properties of Gamma function:

1.
$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1$$

2. For $\alpha > 0$,

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty x^\alpha e^{-x}\, dx \\ &= -x^\alpha e^{-x}\Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x}\, dx \\ &= \alpha \int_0^\infty x^{\alpha-1} e^{-x}\, dx \\ &= \alpha\Gamma(\alpha)\end{aligned}$$

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1) = 1 \\ \Gamma(3) &= 2\Gamma(1) = 2 \\ \Gamma(4) &= 3\Gamma(3) = 3! \\ &\vdots \\ \Gamma(n) &= (n-1)!, \qquad n \in \mathbb{N}\end{aligned}$$

Claim:

$$\int_0^\infty \frac{\lambda^\alpha}{\Gamma(x)} x^{\alpha-1} e^{-\lambda x}\, dx = 1$$

Let $y = \lambda x$, $dy = \lambda dx$,

$$\begin{aligned}\text{lhs} &= \frac{1}{\Gamma(\alpha)} \int_0^\infty (\lambda x)^{\alpha-1} e^{-\lambda x} (\lambda\, dx) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y}\, dy = 1 = \text{rhs}\end{aligned}$$

Special case of Gamma(α, λ) occurs when $\alpha = 1$,

$$\text{Gamma}(1, \lambda) \sim \text{Exp}(\lambda).$$

Defn: $X \sim \text{Exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x}$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Exp(λ) arises as a "waiting time" r.v.. Suppose that a sequence of Bernoulli(p) trials is performed at times $\delta, 2\delta, 3\delta, \ldots$, and let W be the waiting time until the first success. Then

$$\mathbb{P}(W > k\delta) = (1-p)^k.$$

Now, fix a time t, by this time, approximately $k = \frac{t}{\delta}$ trials have occurred.

Let $\delta \to 0^+$ and assume that $p \to 0^+$ such that $\frac{p}{\delta} = \lambda$ is fixed. Then,

$$\begin{aligned}\mathbb{P}(W > t) &= \mathbb{P}(W > \frac{t}{\delta} \cdot \delta) \\ &= (1-p)^{t/\delta} = (1 - \lambda\delta)^{t/\delta}\end{aligned}$$

$$\begin{aligned}\lim_{\delta\to 0} (1-\lambda\delta)^{t/\delta} &= L \\ \lim_{\delta\to 0} \frac{t}{\delta} \ln(1-\lambda\delta) &= \ln L \\ \xrightarrow{L'H} t \lim_{\delta\to 0} \frac{\frac{-\lambda}{1-\lambda\delta}}{1} &= \ln L \\ \implies -\lambda t &= \ln L \\ \implies L &= e^{-\lambda t}\end{aligned}$$

$$\implies F(x) = 1 - \mathbb{P}(W > t) = 1 - e^{-\lambda x}$$

Therefore, W is approximate to Exp(λ).

Increasing (decreasing) sequence $\{A_n\}_{n:1,2,\ldots}$ of events in $\mathcal{F}$, then $\mathbb{P}(\lim_{n\to\infty} A_n) = \lim_{n\to\to} \mathbb{P}(A_n)$. (Read/work through Section 1.7)