Lecture 04

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Defn: The Monty Hall Problem

$$\begin{aligned}
\mathbb{P}(\text{win car if switch}) &= \sum_{i=1}^{3} \mathbb{P}(\text{win car if switch} \mid \text{choose Door i}) \cdot \mathbb{P}(\text{choose Door i}) \\
&= \frac{1}{3} \cdot 3 \cdot \mathbb{P}(\text{win car if switch} \mid \text{Door 1}) \\
&= \mathbb{P}(\text{win car if switch} \mid \text{Door 1}) \\
&= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}
\end{aligned}$$

Consider choosing Door 1,

Location of car	Host opens	Outcome if switch
Door 1	Door 2 or 3	no car
Door 2	Door 3	win car
Door 3	Door 2	win car

Independent Events

Defn: $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space two events $A, B \in \mathcal{F}$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
If A, B are independent, then

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

E.g.: Let $\Omega = \{1, 2, \ldots, p\}$; p is a prime number. $\mathcal{F} = \{0,1\}^{\Omega}$, which $A \subset \Omega$, $\mathbb{P}(A) = \frac{|A|}{p}$, $A \in \mathcal{F}$ means the map $\Omega \to \{0,1\}$ such that $A(\omega) = 1$ if and only if $\omega \in A$. Show that if A and B are independent events, then at least one of A and B is either $\varnothing$ or Ω.
Pf: Suppose $A, B \subset \Omega$ are independent. Then,

$$P(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \Rightarrow \frac{|A \cap B|}{p} = \frac{|A|}{p} \cdot \frac{|B|}{p}$$

$$p|A \cap B| = |A| \cdot |B|.$$

Because p is prime, it divides $|A|$ or $|B|$. Without loss of generality, p divides $|A|$. Because $|A| \in \{0, 1, \ldots, p\}$, either $|A| = 0$ or $|A| = p \Rightarrow \varnothing$ or Ω.

Defn: $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are independent if for every subset $\{A_{1'}, A_{2'}, \ldots, A_{r'}\}$ s.t. $r \leq n$, we have

$$\mathbb{P}(A_{1'} \cap \cdots \cap A_{r'}) = \mathbb{P}(A_{1'}) \ldots \mathbb{P}(A_{r'}).$$

E.g.: Suppose $\Omega = \{abc, acb, cab, cba, bca, bac, aaa, bbb, bbb\}$, $\mathcal{F} = \{0,1\}^{\Omega}$, and each of the nine element in Ω occurs with equal probability. Let A_k be the event that the k^{th} is "a", $k = 1, 2, 3$. Show that A_1, A_2, A_3 are pairwise independent but not independent.
Pf:

$$A_1 = \{abc, acb, aaa\}, A_2 = \{cab, bac, aaa\}, A_3 = \{cba, bca, aaa\}$$

$$\mathbb{P}(A_k) = \frac{3}{9} = \frac{1}{3}$$

$$A_1 \cap A_2 = \{aaa\} = A_1 \cap A_3 = A_2 \cap A_2$$

when $i \neq j$,

$$\mathbb{P}(A_k \cap A_j) = \frac{1}{9} = \frac{1}{3} \cdot \frac{1}{3} = \mathbb{P}(A_k)\mathbb{P}(A_j)$$

$$A_1 \cap A_2 \cap A_3 = \{aaa\}$$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{9} \neq \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3) = \frac{1}{27}$$

E.g.: In a class, there are 4 freshmen boys, 6 freshmen girls, and 6 sophomore boys. How many sophomore girls must be present if gender and class are to be independent if a student is chosen at random?
Ans: Let $x = \#$ sophomore girls. Let F, S, B, G be the events of choosing a freshmen, a sophomore, a boy, or a girl, respectively. Then,

$$\begin{aligned}
&\mathbb{P}(F \cap B) = \mathbb{P}(F)\mathbb{P}(B) \\
\Rightarrow &\frac{4}{16+x} = \frac{10}{16+x} \cdot \frac{10}{16+x} \\
\Rightarrow &16 + x = 25 \\
\Rightarrow &x = 9
\end{aligned}$$

Suppose we only know $\frac{FB}{FG} = \frac{2}{3}$ and we know $SB = 6$

$$FB + FG = y, \quad \frac{FB}{FG} + 1 = \frac{y}{FG} \Rightarrow \frac{5}{3} = \frac{y}{FG} = \frac{2}{3} \cdot \frac{y}{FB}, \quad FG = \frac{3}{2}FB$$

$$\begin{aligned}
\mathbb{P}(F \cap B) &= \mathbb{P}(F)\mathbb{P}(B) \\
\frac{\frac{2}{5}y}{y+6+x} &= \frac{y}{y+6+x} \cdot \frac{\frac{2}{5}y+6}{y+6+x} \\
\frac{2}{5} &= \frac{\frac{2}{5}y+6}{y+6+x} \\
\frac{2}{5}y + \frac{2}{5}(6+x) &= \frac{2}{5}y + 6 \\
6 + x &= \frac{5}{2} \cdot 6 \\
x &= 9
\end{aligned}$$