Lecture 06

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Random Variables

Defn: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X is a real-valued random variable(r.v.) if $X : \Omega \to \mathbb{R}$ such that $\{\omega : X(\omega) \le x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$. Then Cumulative Distribution Function(CDF) is

$$F_X(x) = \mathbb{P}(X \le x) \overset{\text{means}}{=} \mathbb{P}(\{\omega : X(\omega) \le x\}).$$

E.g.: $X \sim$ Bernoulli(p) r.v., $\Omega = \{S, F\}$, $\mathcal{F}$ = power set = σ-algebra generated by $\{S\}$.

$$\mathbb{P}(\{S\}) = p \qquad \mathbb{P}(\{F\}) = \mathbb{P}(\{S\}^c) = 1 - p$$

$$X : \{S, F\} \to \mathbb{R}$$

$$X(S) = 1, \qquad X(F) = 0$$

$$F_X(x) = \mathbb{P}(X \le x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

E.g.: $X \sim$ Geometric(p)
Suppose we flip a coin until the first head appears. Let X denote the r.v. that counts the # of flips required to get the first head. Let p =the prob that H appears on any flip.

$$\Omega = \{(T_1, T_2, \dots, T_{n-1}, H) : n = 1, 2, \dots\}$$

$$\mathbb{P}(T_1, T_2, \dots, T_{n-1}, H) = (1-p)^{n-1} \cdot p, \qquad n = 1, 2, \dots$$

$$X : \Omega \to \mathbb{R}$$

$$X(T_1, T_2, \dots, T_{n-1}, H) = n$$

Probability Mass Function(PMF):

$$\mathbb{P}(X = n) = (1-p)^{n-1} \cdot p, \qquad n = 1, 2, \dots$$

$$\mathbb{P}(X \le n) - \mathbb{P}(X < n) = F(x) - F(x^-)$$

A CDF F has the following properties:
Proof: If $x < y$, then $F(x) \le F(y)$.

$$\begin{aligned} x < y &\implies \{\omega : X(\omega) \le x\} \subset \{\omega : X(\omega) \le y\} \\ &\implies \mathbb{P}(\{\omega : X(\omega) \le x\}) \le \mathbb{P}(\{\omega : X(\omega) \le y\}) \\ &\implies \mathbb{P}(X \le x) \le \mathbb{P}(X \le y) \\ &\implies F(x) \le F(y) \end{aligned}$$

Proof: $\lim_{x \to \infty} F(x) = 1$
Define $A_n = \{\omega \in \Omega : X(\omega) \le n\}$ where $n = 1, 2, \dots$. Then,

$$\Omega = A_1 \cup (A_2 - A_1) \cup (A_3 - (A_1 \cup A_2)) \cup \dots$$

is a pairwise disjoint union of sets in $\mathcal{F}$. Thus,

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A_1) + \mathbb{P}(A_2 - A_1) + \mathbb{P}(A_3 - A_2) + \dots$$

Aside: Consider $A_n \subset A_{n+1}$,

$$A_{n+1} = A_n \cup (A_{n+1} \cup A_n^c) \implies \mathbb{P}(A_{n+1}) = \mathbb{P}(A_n) + \mathbb{P}(A_{n+1} - A_n).$$

Then,

$$\mathbb{P}(A_{n+1} - A_n) = \mathbb{P}(A_{n+1} \cap A_n^c)$$

$$= \mathbb{P}(A_{n+1}) - \mathbb{P}(A_n)$$

Therefore, because $\mathbb{P}(A_1) + \sum_{i=1}^{n}(\mathbb{P}(A_{i+1}) - \mathbb{P}(A_i)) = \mathbb{P}(A_{n+1})$,

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A_1) + (\mathbb{P}(A_2) - \mathbb{P}(A_1)) + (\mathbb{P}(A_3) - \mathbb{P}(A_2)) + \cdots = \lim_{n\to\infty} \mathbb{P}(A_n)$$

$$\implies \lim_{n\to\infty} F(n) = 1$$

Since F is non-decreasing,

$$\lim_{x\to\infty} F(x) = 1$$

Proof: $\lim_{x\to-\infty} F(x) = 0$

Define $B_n = \{\omega \in \Omega : X(\omega) \leq -n\}$ where $n = 1, 2, \ldots$. Then $B_n^c = \{X > -n\}$. A similar argument as in part (b) shows

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}(B_n^c) = \mathbb{P}(\Omega) = 1 \\
&\implies \lim_{n\to\infty}(1 - \mathbb{P}(B_n)) = 1 \\
&\implies \lim_{n\to\infty} \mathbb{P}(B_n) = 0 \\
&\implies \lim_{n\to\infty} F(-n) = 0 \\
&\overset{F \text{ is non-decreasing}}{\implies} \lim_{n\to\infty} F(-x) = 0 \\
&\implies \lim_{n\to-\infty} F(x) = 0
\end{aligned}$$

Proof: $\mathbb{P}(X > x) = 1 - F(x)$

$$\begin{aligned}
\mathbb{P}(X > x) &= \mathbb{P}(\{\omega : X(\omega) > x\}) \\
&= \mathbb{P}(\{\omega : X(\omega) \leq x\}^c) \\
&= 1 - \mathbb{P}(\{\omega : X(\omega) \leq x\}) \\
&= 1 - F(x)
\end{aligned}$$

Proof: $\mathbb{P}(x < X \leq y) = F(y) - F(x)$, $x \leq y$

$$\begin{aligned}
&\mathbb{P}(\{\omega : X(x < \omega) \leq y\}) \\
=&\mathbb{P}(\{\omega : X(\omega) \leq y\} - \{\omega : X(\omega) \leq x\}) \\
=&\mathbb{P}(X \leq y) - \mathbb{P}(X \leq x) \\
=&F(y) - F(x)
\end{aligned}$$

Proof: $\mathbb{P}(X = x) = F(x) - \lim_{h\to0^+} F(x - h)$

Proof: F is right-continuous, or $F(x) = \lim_{h\to0^+} F(x + h)$