Lecture 32

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5.3.4 Conditional distribution of arrival times for a $\mathcal{PP}(\lambda)$

Defn:

$$\begin{aligned}
\mathbb{P}(T_1 \le s \mid N(t)=1) &= \frac{\mathbb{P}(T_1 \le s, N(t)=1)}{\mathbb{P}(N(t)=1)}, \qquad s \le t \\
&= \frac{\mathbb{P}(N(s)=1, N(t)-N(s)=0)}{\mathbb{P}(N(t)=1)} \\
&= \frac{\mathbb{P}(N(s)=1)\mathbb{P}(N(t)-N(s)=0)}{\mathbb{P}(N(t)=1)} \\
&= \frac{e^{-\lambda s} \cdot \frac{(\lambda s)^1}{1!} \cdot e^{-\lambda(t-s)}}{e^{-\lambda t} \cdot \frac{(\lambda t)^1}{1!}} \\
&= \frac{s}{t} \\
\implies T_1 \mid N(t) = 1 &\sim \mathcal{U}(0,t)
\end{aligned}$$

Thm: Given that $N(t)=n$, the n arrival times $S_1,\ S_2,\ \ldots,\ S_n$ have the same distribution as the order statistics corresponding to n iid $\mathcal{U}(0,t)$ rv's.

Proof: To obtain the conditional pdf of $S_1,\ S_2,\ \ldots,\ S_n$ given $N(t)=n$, note that for $0 < S_1 < \cdots < S_n < t$, the event that $S_1 = s_1,\ S_2 = s_2,\ \ldots,\ S_n = s_n,\ N(t)=n$, is equivalent to the event that the first $n+1$ interarrival times satisfy $T_1 = s_1$, $T_2 = s_2 - s_1$, $\ldots$, $T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$. Thus, because the T_i are iid $\text{Exp}(\lambda)$, we have the conditional joint pdf

$$\begin{aligned}
f(s_1, s_2, \ldots, s_n \mid N(t)=n) &= \frac{f(s_1, s_2, \ldots, s_n, N(t)=n)}{\mathbb{P}(N(t)=0)} \\
&= \frac{\lambda e^{-\lambda s_1} \cdot \lambda e^{-\lambda(s_2-s_1)} \ldots \lambda e^{-\lambda(s_n - s_{n-1})} \cdot e^{-\lambda(t-s_n)}}{e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}} \\
&= \frac{n!}{t^n}, \qquad 0 < s_1 < \cdots < s_n < t
\end{aligned}$$

Now, $n!/t^n$ si the joint pdf of the order statistics for n iid $\mathcal{U}(0,t)$ rv's. (See argument on P317)

The joint pdf of n iid $\mathcal{U}(0,t)$ rv's is $1/t^n$ on $[0,t]^n$. Each order of the n rv's has probability $1/n!$ of occurring. Therefore, to scale $1/t^n$ appropriately so that it has probability mass of 1 on the set

$$\mathcal{D} = \{(s_1, \ldots, s_n) \in [0,1]^n : s_1 < \cdots < s_n\}$$

we have to multiply $1/t^n$ by $n!$.

$$\int_{\mathcal{D}} \frac{1}{t^n}\, ds_1 \ldots d_n = \frac{1}{n!} \implies \int_{\mathcal{D}} \frac{n!}{t^n}\, ds_1 \ldots d_n = 1$$

n iid rv's with common pdf $f(x)$, then the joint pdf of the order statistics is $n!(f(x))^n$.

E.g.: Cars pass a certain intersection according to a $\mathcal{PP}(\lambda)$. A person wants to cross the street at the location, and they wait to cross until they can see that no cars will come by in the next T time units.

(a) Find the probability that their waiting time is 0.

$$\mathbb{P}(N(T)=0) = e^{-\lambda T}$$

(b) Find the person's expected waiting time, $\mathbb{E}W$.

$$\begin{aligned}
\mathbb{E}W &= \mathbb{E}(\mathbb{E}(W \mid S_1)) \\
&= \mathbb{E}(W \mid S_1 > T)\mathbb{P}(S_1 > T) + \mathbb{E}(W \mid S_1 \le T)\mathbb{P}(S_1 \le T) \\
&= 0 \cdot \mathbb{P}(S_1 > T) + \mathbb{E}(W \mid S_1 \le T)\mathbb{P}(S_1 \le T) \\
&= (\mathbb{E}(S_1 \mid S_1 \le T) + \mathbb{E}W) \cdot \mathbb{P}(S_1 \le T)
\end{aligned}$$