Lecture 17

Professor Virginia R. Young
Transcribed by Hao Chen

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E.g.: $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, X, Y are independent. Show that $Z = X + Y \sim \text{Poisson}(\lambda + \mu)$.

$$\begin{aligned}
\mathbb{P}(Z = z) &= \sum_{x+y=z}\sum \mathbb{P}(X = x)\mathbb{P}(Y = y) \\
&= \sum_{x=0}^{z} \mathbb{P}(X = x)\mathbb{P}(Y = z - x) \\
&= \sum_{x=0}^{z} e^{-\lambda}\frac{\lambda^x}{x!} \cdot e^{-\mu}\frac{\mu^{z-x}}{(z-x)!} \\
&= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \cdot \lambda^x \mu^{z-x} \\
&= e^{-(\lambda+\mu)} \cdot \frac{(\lambda+\mu)^z}{z!}
\end{aligned}$$

where the binomial expansion could be simplified by $\sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \cdot \lambda^x \mu^{z-x} = (\lambda + \mu)^z$.

$$\therefore Z \sim \text{Poisson}(\lambda + \mu)$$

Defn: Order statistics

Suppose we have $X_1, X_2, \ldots, X_n$ iid rvs. For any realization (values at $\omega \in \Omega$), we can order the outcomes

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}$$

E.g.: $S_{X_{(1)}}(x) = (1 - F_X(x))^n$

$$\begin{aligned}
S_{X_{(1)}}(x) &= 1 - F_{X_{(1)}}(x) \\
&= \mathbb{P}(X_{(1)} > x) \\
&= \mathbb{P}(\min(X_1, X_2, \ldots, X_n) > x) \\
&= \mathbb{P}(X_1 > x, X_2 > x, \ldots, X_n > x) \\
&= \mathbb{P}(X_1 > x)\mathbb{P}(X_2 > x)\ldots\mathbb{P}(X_n > x) \\
&= (1 - F_X(x))^n
\end{aligned}$$

where $X_1 \sim X_2 \sim \cdots \sim X_n \sim X$ such that $\mathbb{P}(X_1 > x) = \cdots = \mathbb{P}(X_n > x) = (1 - F_X(x))$.

E.g.: $F_{X_{(n)}}(x) = (F_X(x))^n$

$$\begin{aligned}
F_{X_{(n)}}(x) &= \mathbb{P}(X_{(n)} \le x) \\
&= \mathbb{P}(\max(X_1, X_2, \ldots, X_n) \le x) \\
&= \mathbb{P}(X_1 \le x, X_2 \le x, \ldots, X_n \le x) \\
&= \mathbb{P}(X_1 \le x)\mathbb{P}(X_2 \le x)\ldots\mathbb{P}(X_n \le x) \\
&= (F_X(x))^n
\end{aligned}$$

E.g.: $F_{X_{(k)}}(x) = \sum_{i=k}^{n} \binom{n}{i} (F_X(x))^i (1 - F_X(x))^{n-1}$ for $k < n$

$$\begin{aligned}
F_{X_{(k)}}(x) &= \mathbb{P}(X_{(k)} \le x) \\
&= \sum_{i=k}^{n} \mathbb{P}(i \text{ of the } X\text{s} \le x,\ n-i \text{ of them} > x) \\
&= \sum_{i=k}^{n} \binom{n}{i} (F_X(x))^i (1 - F_X(x))^{n-1}
\end{aligned}$$

HW: 2.61(iv) $N = \min\{n : n > 1 \text{ and a record occurs at time } n\}$. Show that $\mathbb{E}N = \infty$.

$$X_1, X_2, \ldots, X_{n-1}, X_n$$

If X_n is a record, then $X_n = X_{(n)}$. If X_n is the first record after time 1, then also $X_1 = X_{(n-1)}$.

$$\mathbb{P}(N = n) = \mathbb{P}(X_n = X_{(n)}, X_1 = X_{(n-1)})$$

for $n = 2, 3, 4, \ldots$.

Aside: when X_1 and X_2 are two continuous random variables, we have

$$\mathbb{P}(X_1 = X_2) = \{\text{volume under } f_{X_1}(x) f_{X_2}(x) \text{ and over the line } x_1 = x_2\} = 0$$

Method 1:

$$\begin{aligned}
\mathbb{P}(N = n) &= \mathbb{P}(X_{(n-1)} = X_1 \mid X_n = X_{(n)}) \cdot \mathbb{P}(X_n = X_{(n)}) \\
&= \frac{1}{n-1} \cdot \frac{1}{n} \\
&= \frac{1}{n(n-1)}
\end{aligned}$$

Method 2:

$$\max(X_2, \ldots, X_{n-1}) < X_1 < X_n$$

$$\max(X_2, \ldots, X_{n-1}) \implies \text{cdf: } (F_X(x))^{n-2} \implies \text{pdf: } (n-2)(F_X(x))^{n-3} f_X(x)$$

$$\begin{aligned}
\mathbb{P}(N = n) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z} \int_{-\infty}^{y} (n-2)(F_X(x))^{n-3} f_X(x) f_X(y) f_X(z)\, dxdydz \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{z} (F_X(y))^{n-2} f_X(y) f_X(z)\, dydz \qquad u = F_X(y) \quad du = f_X(y)\, dy \\
&= \int_{-\infty}^{\infty} \frac{(F_X(z))^{n-1}}{n-1} f_X(z)\, dz \\
&= \left. \frac{(F_X(z))^n}{n(n-1)} \right|_{z=-\infty}^{\infty} \\
&= \frac{1}{n(n-1)}
\end{aligned}$$

Therefore, the expectation of N is

$$\mathbb{E}N = \sum_{n=2}^{\infty} n \cdot \mathbb{P}(N = n) = \sum_{n=2}^{\infty} \frac{n}{n(n-1)} = \infty$$

E.g.: Change of variables from 2.5.4

Lognormal:

$$X \sim N(\mu, \sigma^2) \implies e^X \sim \text{lognormal}(\mu, \sigma)$$

E.g.: $X \sim \text{Pareto}(\alpha, \theta)$, $S_X(x) = \left(\frac{\theta}{\theta + x}\right)^{\alpha}$, $x \geq 0$

Let $Y = \ln(1 + \frac{X}{\theta})$. What is Y's distribution?

$$\begin{aligned}
S_Y(y) &= \mathbb{P}(Y > y) \\
&= \mathbb{P}\left(\ln\left(1 + \frac{X}{\theta}\right) > y\right) \\
&= \mathbb{P}\left(1 + \frac{X}{\theta} > e^y\right) \\
&= \mathbb{P}(X > \theta(e^y - 1)) \\
&= S_X(\theta(e^y - 1)) \\
&= \left(\frac{\theta}{\theta + \theta(e^y - 1)}\right)^{\alpha} \\
&= e^{-\alpha y} \implies Y \sim \text{Exp}(\alpha)
\end{aligned}$$

Defn: Moment generating functions $M_X(t) = \mathbb{E}(e^{Xt})$ wherever this expectation is finite.