Lecture 24

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E.g.: 3.12: without memory $\mathbb{E}X = 10$, with memory $\mathbb{E}X = 6$

E.g.: 3.15: X is the number trials required to get k successes

$\mathbb{P}(X = n \mid k)$ — compute by conditioning on outcome of first trial.

$$\text{1st trial} = \begin{cases} \text{success} & \text{w.p. } p \\ \text{failure} & \text{w.p. } 1-p \end{cases}$$

$$\begin{aligned} P_{n,k} &= \mathbb{P}(X = n \mid k) \\ &= p \cdot \mathbb{P}(X = n-1 \mid k-1) + (1-p)\mathbb{P}(X = n-1 \mid k) \\ &= p \cdot \mathbb{P}_{n-1,k-1} + (1-p)\mathbb{P}_{n-1,k} \end{aligned}$$

where $\mathbb{P}(X = n-1 \mid k) = 0$ if $n = k$.

Technique for solving:

∂ conditions:

$$P_{n,1} = (1-p)^{n-1} \cdot p, \qquad P_{k,k} = p^k$$

Generating function:

$$G(x,y) = \sum_{1 \le k \le n} P_{n,k} \cdot x^n y^k$$

$$G(x,y) = \frac{p \cdot xy}{1 - p \cdot xy - (1-p)x}$$

Solution:

$$\mathbb{P}(X = n \mid k) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

which is negative binomial and $n = k, k+1, \ldots$

Computing probabilities by conditioning

Defn:

$$\mathbb{P}(A) = \mathbb{E}(1_A) = \mathbb{E}(\mathbb{E}(1_A \mid Y))$$
$$\mathbb{P}(A) = \mathbb{E}(\mathbb{P}(A \mid Y))$$

E.g.: 3.23: An insurance co. supposes the number of accidents $N \mid \lambda$ that each of its policy holders(ph) has in a year is Poisson(λ) with λ varying from ph to ph.

(a) If $\lambda \sim \text{Gamma}(\alpha, \beta)$, what is the probability that a randomly chosen ph has exactly n accidents next year?

Solution: $N \mid \lambda \sim \text{Poisson}(\lambda)$, $\lambda \sim \text{Gamma}(\alpha, \beta)$

$$\begin{aligned} \mathbb{P}(N = n) &= \mathbb{E}(\mathbb{P}(N = n \mid \lambda)) \\ &= \mathbb{E}\left(e^{-\lambda}\frac{\lambda^n}{n!}\right) \\ &= \int_0^\infty e^{-\lambda}\frac{\lambda^n}{n!}\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\,d\lambda \\ &= \frac{\beta^\alpha}{n!\Gamma(\alpha)}\int_0^\infty \lambda^{(\alpha+n)-1}e^{-(\beta+1)\lambda}\,d\lambda \\ &= \frac{\beta^\alpha}{n!\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+n)}{(\beta+1)^{\alpha+n}} \\ &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!}\left(\frac{\beta}{\beta+1}\right)^\alpha \left(1 - \frac{\beta}{\beta+1}\right)^n, \qquad n = 0, 1, 2, \ldots \end{aligned}$$

(b) Calculate $\mathbb{E}N$ and $\operatorname{Var} N$ via iteration.

Solution:

$$\begin{aligned}
\mathbb{E}N &= \mathbb{E}(\mathbb{E}(N \mid \lambda)) \\
&= \mathbb{E}(\lambda) \\
&= \frac{\alpha}{\beta}
\end{aligned}$$

where $N \mid \lambda \sim \text{Poisson}(\lambda)$, $\lambda \sim \text{Gamma}(\alpha, \beta)$.

$$\begin{aligned}
\operatorname{Var} N &= \mathbb{E}(\operatorname{Var}(N \mid \lambda)) + \operatorname{Var}(\mathbb{E}(N \mid \lambda)) \\
&= \mathbb{E}(\lambda) + \operatorname{Var}(\lambda) \\
&= \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} = \frac{\alpha}{\beta^2}(\beta + 1) \\
&> \mathbb{E}N
\end{aligned}$$