Lecture 12

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Jointly distributed r.v.s

Defn: (X, Y) is a bivariate real-valued r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$ if

$$(X, Y) : \Omega \to \mathbb{R}^2, \qquad \omega \to (X(\omega), Y(\omega))$$

such that $\{\omega \in \Omega : X(\omega) \leq x, Y(\omega) \leq y\}$ is in $\mathcal{F}$, for all $x, y \in \mathbb{R}$.

Joint cdf of X and Y, or cdf of (X, Y)

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$$

From $F_{X,Y}$ we can retrieve the cdfs of X and Y.

Marginal cdf of X:

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, Y < \infty) = \lim_{y \to \infty} F_{X,Y}(x, y)$$

Similarly, the marginal cdf of Y:

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y)$$

1. If (X, Y) is discrete then its pmf

$$\begin{aligned} p(x, y) &= \mathbb{P}(X = x, Y = y) \\ &= \mathbb{P}(X \leq x, Y \leq y) - \mathbb{P}(X < x, Y \leq y) - \mathbb{P}(X \leq x, Y < y) + \mathbb{P}(X < x, Y < y) \\ &= F_{X,Y}(x, y) - F_{X,Y}(x^-, y) - F_{X,Y}(x, y^-) + F_{X,Y}(x^-, y^-) \end{aligned}$$

In the discrete case, to get the marginal pmf of X

$$p_X(x) = \mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y p_{X,Y}(x, y)$$

2. If (X, Y) is continuous, then $\exists$pdf $f_{X,Y}(x, y)$ such that

$$F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v)\, du dv$$

If f is continuous (where it is non-zero), then

$$\begin{aligned} \frac{\partial F_{X,Y}}{\partial x} &= \int_{-\infty}^{y} f_{X,Y}(x, v) dv \\ \frac{\partial^2 F_{X,Y}}{\partial y \partial x} &= f_{X,Y}(x, y) \end{aligned}$$

In the continuous case, to get the marginal pdf of X, say,

$$F_X(x) = F_{X,Y}(x, \infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{X,Y}(u, v)\, du dv$$

Differentiate with respect to x

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$

which we call $f_X(x)$ as the marginal pdf of X.

E.g.: Bivariate normal has pdf

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho \left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 \right] \right\}$$

for some $\rho \in (-1, 1)$, $x, y \in \mathbb{R}$.

Let $z = \frac{y-\mu_2}{\sigma_2}$ and $dz = \frac{dy}{\sigma_2}$,

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \\
&= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)z + z^2\right]}\, dz \\
&= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1\sqrt{1-\rho^2}} \cdot e^{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)z + z^2\right]}\, dz \\
&= \frac{e^{-\frac{1-\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left[z-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2}\, dz
\end{aligned}$$

Integrand is pdf of $\mathrm{N}(\rho\frac{x-\mu_1}{\sigma_1}, 1-\rho^2)$, Therefore,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}, \quad x \in \mathbb{R} \qquad \Longrightarrow \qquad X \sim \mathrm{N}(\mu_1, \sigma_1^2)$$

Similarly, $Y \sim \mathrm{N}(\mu_2, \sigma_2^2)$.

Independent r.v.s

Defn: X and Y are independent if

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

for all $x, y \in \mathbb{R}$.

Or equivalently, if (X, Y) is discrete,

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

or if (X,Y) is continuous,

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

E.g.: Bivariate normal X and Y are independent if and only if $\rho = 0$.

$$\begin{aligned}
f_{X,Y}(x, y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \\
&= \frac{1}{\sqrt{2\pi\sigma_1}\sqrt{2\pi\sigma_2}} \exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \\
&= f_X(x) \cdot f_Y(y)
\end{aligned}$$

HW: Prove that, if (X, Y) is continuous, then $F_{X,Y}(x, y) = F_X(x)F_Y(y) \iff f_{X,Y}(x, y) = f_X(x)f_Y(y)$.