Lecture 01

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Chapter 1 Intro to Prob Theory

1.2 Sample spaces and events

Defn: The set of all possible outcomes of an experiment is known as the sample space. We denote it by Ω. Also called the state space. $\Omega = \{\omega\}$

E.g.:

- Ex1: Flip a coin once. $\Omega = \{H, T\}$.
- Ex2: Roll a single die. $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Ex3: Flip two coins. $\Omega = \{(H, H), (H, T), (T, T)\}$ (unordered pair)
- Ex5: Measure the lifetime of an individual $\Omega = (0, \infty)$

Events of interest

- Ex1" The outcome is not a tail $= \{T\}^c = \Omega - \{T\} = \{H\}$
- Ex2" $A =$ outcome is even $= \{2, 4, 6\}$
- $B =$ the outcome does not exceed $3 = \{1, 2, 3\} = \{4, 5, 6\}^c$
- Ex5" A = the individual lives at least ten years $=(10, \infty)$
- B = the individual dies before age 20 $= (0, 20)$

Defn: A collection $\mathcal{F}$ of subset of Ω is called a σ-algebra if it satisfies the following properties:

1. $\varnothing \in \mathcal{F}$
2. if $A_1, A_2, \cdots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$
3. if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

E.g.:

- (i) $\mathcal{F} = \varnothing, \Omega$ is a σ-algebra.
- (ii) if $A \subset \Omega$, then $\mathcal{F} = \{\varnothing, A, A^c, \Omega\}$ is a σ-algebra.
- (iii) Borel σ-algebra of $\mathbb{R} = \sigma$-algebra generated by (a, b), $a < b$, $a, b \in \mathbb{R}$.

$A + B = A \cup B$
$A - B = A \cap B^c$
$\Omega - B = \Omega \cap B^c = B^c$

Proof:
(a) $\varnothing \in \mathcal{F}$ by definition of $\mathcal{F}$.
(b) Consider a sequence of up to 4 elements in $\mathcal{F}$,

- If A and A^c are in the sequence, then the union is $\omega \in \mathcal{F}$.
- If Ω is the sequence, then the union is $\Omega \in \mathcal{F}$.
- Otherwise, the union is either $\varnothing$, A, or $A^c \in \mathcal{F}$.

(c) $\varnothing^c = \Omega$, $\{A\}^c = \{A^c\}$, $\Omega^c = \varnothing$, $\{A^c\}^c = \{A\}$ Therefore, $\mathcal{F}$ is closed under ^{c}.
Let $\mathcal{F}$ be a σ-algebra, and consider a sequence $A_1, A_2, \cdots \in \mathcal{F}$, then, $A_i^c \in \mathcal{F}$, $i = 1, 2, \ldots$.
Let $\omega \in \cup_{i=1}^{\infty} A_i^c$; thus, $\exists j = 1, 2, \ldots$ s.t. $\omega \in A_j^c \implies \omega \notin A \implies \omega \notin \cap_{i=1}^{\infty} A_i \implies \omega \in (\cap_{i=1}^{\infty} A_i)^c$. Prove that $\cup_{i=1}^{\infty} A_i^c \subset (\cap_{i=1}^{\infty} A_i)^c$. $\therefore \cap_{i=1}^{\infty} A_i \in \mathcal{F}$

E.g.: $\cap_{n=1}^{\infty} (a - 1/n, b + 1/n) = [a, b]$