Lecture 07

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September 14, 2022

Recap: R.v. $X : \Omega \to \mathbb{R}$ such that $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$, $\forall x \in \mathbb{R}$, cdf $F(x) = \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$.

Properties of the cdf F:

1. F is non-decreasing on $\mathbb{R}$
2. $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$
3. F is right-continuous

Discrete random variables

Defn: Discrete r.v. has a right-continuous, non-decreasing step function as its cdf.

$$P(X = x) = F(X \leq x) - F(X < x)$$

where $\mathbb{P}(X = x)$ is non-zero, we have what we call the probability mass function(pmf).

Defn: $X \sim$ Bernoulli(p), $0 < p < 1$

$$P_X(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \end{cases}$$

Defn: $X \sim$ Binomial(n,p)
Fact: $X \sim X_1 + X_2 + \cdots + X_n$ where $X_1, X_2, \ldots, X_n$ are independent Bernoulli(p).

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad 0! = 1.$$

To verify,

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1.$$

Defn: $X \sim$ Geometric(p)
= "time" of the first success for an independent sequence of Bernoulli experiments.

$$P_X(k) = \mathbb{P}(X = k) = (1-p)^{k-1} \cdot p, \quad k = 1, 2, \ldots$$

I have seen the Geom(p) defined as the # of failures before the first success. Call that Y,

$$P_Y(k) = (1-p)^k \cdot p$$

$$Y + 1 \sim X$$

which means that they have same distribution.
To verify,

$$\begin{aligned}
\sum_{k=0}^{\infty} P_X(k) &= \sum_{k=0}^{\infty} (1-p)^{k-1} \cdot p \\
&= p\{1 + (1-p) + (1-p)^2 + \ldots\} \\
&= p \cdot \frac{1-0}{1-(1-p)} = \frac{p}{p} = 1
\end{aligned}$$

The Geom(p) r.v satisfies a memory less property:

$$\begin{aligned}
\mathbb{P}(X = n+k \mid X > n) &= \frac{\mathbb{P}(X = n+k, X > n)}{\mathbb{P}(X > n)} \\
&= \frac{\mathbb{P}(X = n+k)}{\mathbb{P}(X > n)} \\
&= \frac{(1-p)^{n+k-1} \cdot p}{\sum_{j=n+1}^{\infty} (1-p)^{j-1} \cdot p} \\
&= \frac{(1-p)^{n+k-1} \cdot p}{\frac{(1-p)^n \cdot p - 0}{1-(1-p)}} \\
&= \frac{(1-p)^{n+k-1} \cdot p}{(1-p)^n} \\
&= (1-p)^{k-1} \cdot p \\
&= \mathbb{P}(X = k)
\end{aligned}$$

Defn: $X \sim \text{Poisson}(\lambda)$

$$p(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

The Poisson(λ) is the limiting distinction of the Bin(n, p) as $n \to \infty$ with $np = \lambda$ fixed. Intuitive calculation to show this: for a fixed $k << n$ and for $X \sim \text{Bin}(n, p)$,

$$\begin{aligned}
P(k) =& \mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \\
=& \frac{n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k} \\
=& \frac{1}{k!} \cdot \frac{n!}{(n-k)!} \cdot \left(\frac{p}{1-p}\right)^k (1-p)^n, \quad \text{where } \frac{n!}{(n-k)!} \approx n^k \text{ as } k << n \\
\approx& \frac{1}{k!} \cdot \left(\frac{np}{1-p}\right)^k (1-p)^n \\
=& \frac{1}{k!} \cdot \left(\frac{\lambda}{1-\frac{\lambda}{n}}\right)^k \left(1 - \frac{\lambda}{n}\right)^n \\
\xrightarrow{n \to \infty}& \frac{1}{k!} \cdot \lambda^k \cdot e^{-\lambda}
\end{aligned}$$

Aside:

$$\begin{aligned}
\lim_{n \to \infty} \ln \left(1 - \frac{\lambda}{n}\right)^n &= \ln y \\
\lim_{n \to \infty} \frac{\ln\left(1 - \frac{\lambda}{n}\right)}{\frac{1}{n}} &= \ln y \\
\lim_{x \to 0} \frac{\ln(1 - \lambda x)}{x} &= \ln y \\
-\lambda &= \ln y \\
y &= e^{-\lambda}
\end{aligned}$$