Lecture 03

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1.4 Conditional prob

Defn: Let $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability place. Let $B \in \mathcal{F}$ be a fixed set such that $\mathbb{P}(B) > 0$. Define the conditional probability given B by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

for any $A \in \mathcal{F}$, where $\mathbb{P}$ is defined by $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\sqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

HW: Show that $\mathbb{P}(\cdot \mid B)$ defines a probability on $(\Omega, \mathcal{F})$. ($\cdot$ means any element in $\mathcal{F}$)

Defn: **Law of total probability:** Let $A_1, A_2, \dots, A_n$ be a partition of $\Omega \in \mathcal{F}$, A_i are pairwise disjoint and $\cup_{i=1}^{n} A_i = \Omega$. Assume $\mathbb{P}(A) > 0$,

$$A_1 \cup A_2 \cup \cdots \cup A_n = \Omega.$$

Let $B \in \mathcal{F}$,

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(B \cap \Omega) \\
&= \mathbb{P}(B \cap (A_1 \cup A_2 \cup \cdots \cup A_n)) \\
&= \mathbb{P}((B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n))
\end{aligned}$$

which $B \cap A_i$ are pointwise disjoint for $\forall i$, or

$$(B \cap A_i) \cap (B \cap A_j) = B \cap (A_i \cap A_j) = \varnothing$$

for $i \neq j$. Thus,

$$\begin{aligned}
\mathbb{P}(B) &= \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) \\
&= \sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)
\end{aligned}$$

E.g.: Bob possesses five coins, 2 of which are double-headed, 1 is double-tailed, and 2 are normal. Bob shuts his eyes, picks a coin at random, and tosses it.

1. What is the prob that the lower face of the coin is a head?

$$\begin{aligned}
\mathbb{P}(\text{lower face} = H) &= \mathbb{P}(\text{lower face} = H \mid \text{2-headed})\mathbb{P}(\text{2-headed}) \\
&\quad + \mathbb{P}(\text{lower face} = H \mid \text{2-tailed})\mathbb{P}(\text{2-tailed}) \\
&\quad + \mathbb{P}(\text{lower face} = H \mid \text{normal})\mathbb{P}(\text{normal}) \\
&= 1 \cdot \frac{2}{3} + 0 \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{5}
\end{aligned}$$

2. Bob opens his eyes and sees that the coin that showing heads. What is the probability the lower face is a head?

$$\begin{aligned}
\mathbb{P}(\text{lower face} = H \mid \text{upper face} = H) &= \frac{\mathbb{P}(\text{lower face} = H, \text{upper face} = H)}{\mathbb{P}(\text{upper face} = H)} \\
&= \frac{\mathbb{P}(\text{both faces} = H)}{\mathbb{P}(\text{upper face} = H)} \\
&= \frac{2/5}{3/5} = \frac{2}{3}
\end{aligned}$$

3. Bob shuts his eyes again and re-tosses the same coin. What is the prob that the lower face is a head?

$$\begin{aligned}
&\mathbb{P}(\text{lower face} = H \mid \text{the coin has an H}) \\
=&\mathbb{P}(\text{lower face} = H, \text{coin is 2-H})\mathbb{P}(\text{coin is 2-H} \mid \text{the coin has an H}) \\
+&\mathbb{P}(\text{lower face} = H, \text{coin is normal})\mathbb{P}(\text{coin is normal} \mid \text{the coin has an H}) \\
=&1 \cdot \frac{2}{3} + \frac{1}{2}(1 - \frac{2}{3}) \\
=&\frac{2}{3} + \frac{1}{6} \\
=&\frac{5}{6}
\end{aligned}$$

4. Bob open his eyes and sees the coin is showing heads. What is the prob that the lower face is a head?

$$\begin{aligned}
\mathbb{P}(\text{lower face} = H \mid \text{this coin upper face} = H) &= \frac{\mathbb{P}(\text{this coin is 2-H})}{\mathbb{P}(\text{this coin lower face} = H)} \\
&= \frac{2/3}{5/6} = \frac{4}{5}
\end{aligned}$$

$$\mathbb{P}(A \mid B) = \sum_{i=1}^{n} \mathbb{P}(A \mid B \cap C_i)\mathbb{P}(C_i \mid B)$$