Lecture 10

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Expectation of a r.v.

Defn: The expectation of X : real-valued r.v. equals

$$\mathbb{E}X = \int_{-\infty}^{\infty} x \, dF(x)$$

if it exists

Aside: Riemann-Stieltjes integral

$$\begin{aligned} &\int_b^a f(x) \, dg(x) \\ &= \lim_{\|\pi\| \to 0} \sum_{i=1}^{n} f(x_i^*)(g(x_i) - g(x_{i-1})) \end{aligned}$$

where the norm of partition π

$$\|\pi\| = max|x_i - x_{i-1}|, \qquad \text{if } b = \infty$$

Back to expectation:

$$\begin{aligned} \mathbb{E}X &= \int_{-\infty}^{\infty} x \, dF(x) \\ &= \int_{-\infty}^{\infty} \int_0^x 1 \, dt \, dF(x) \\ &= \int_0^{-\infty} \int_{-\infty}^t 1 \, dF(x) \, dt + \int_0^{\infty} \int_t^{\infty} 1 \, dF(x) \, dt \\ &= \int_0^{-\infty} F(t) \, dt + \int_0^{\infty} (1 - F(t)) \, dt \\ &= -\int_{-\infty}^{0} F(t) \, dt + \int_0^{\infty} (1 - F(t)) \, dt \end{aligned}$$

Defn: Expectation for a Discrete r.v.

$\implies$ F is a step function and only jumps at discrete points $x_1, x_2, \ldots$.

$\implies$

$$\begin{aligned} \mathbb{E}X &= \int_{-\infty}^{\infty} x \, dF(x) \\ &= \sum_{k=1}^{\infty} x_k (F(x_k) - F(x_k^-)) \\ &= \sum_{k=1}^{\infty} x_k \mathbb{P}(X = x_k) \end{aligned}$$

E.g.: $X \sim$ Bernoulli(p), $0 < p < 1$

$$\mathbb{P}(X = k) = \begin{cases} 1 - p & k = 0 \\ p & k = 1 \end{cases}$$

$$\mathbb{E}X = 0 \cdot (1 - p) + 1 \cdot p = p$$

E.g.: $X \sim$ Binomial(n,p) $X \sim X_1 + \cdots + X_n$, X_i are independent Bernoulli(p).

Later, we will show that $\mathbb{E}$ is a linear operator, meaning

$$\mathbb{E}(aX + bY) = a\mathbb{E}X + b\mathbb{E}Y$$

where $a, b \in \mathbb{R}$, X, Y are r.v.s.

$$\therefore \mathbb{E}(X) = n\mathbb{E}(X_i) = np$$

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n$$

$$\begin{aligned}
\mathbb{E}X &= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k} \\
&= np \cdot \sum_{k=1}^{n} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \cdot p^{k-1} (1-p)^{n-k} \\
&= np \cdot \sum_{j=0}^{n-1} \cdot \frac{(n-1)!}{j!((n-1)-j)!} \cdot p^j (1-p)^{(n-1)-j}, \qquad \text{For } j = k-1 \\
&= np
\end{aligned}$$

The sum $\sum_{j=0}^{n-1} \cdot \frac{(n-1)!}{j!((n-1)-j)!} \cdot p^j (1-p)^{(n-1)-j} = 1$ because it's the pmf of Binomial($n-1, p$).

E.g.: $X \sim$ Geometric(p)

$$\mathbb{P}(X = k) = \mathbb{P}(X = k) = (1-p)^{k-1} \cdot p, \quad k = 1, 2, \ldots$$

$$\begin{aligned}
\mathbb{E}X &= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p \\
&= -p \sum_{k=1}^{\infty} \frac{d}{dp} (1-p)^k \\
&= -p \frac{d}{dp} \left(\sum_{k=1}^{\infty} (1-p)^k \right) \\
&= -p \frac{d}{dp} \frac{(1-p) - 0}{1 - (1-p)} \\
&= -p \frac{d}{dp} (\frac{1-p}{p}) \\
&= -p \frac{d}{dp} (\frac{1}{p} - 1) = \frac{-p}{-p^2} = \frac{1}{p}
\end{aligned}$$

Intuitively,

$$p = \frac{\text{success}}{\text{trial}}, \qquad \mathbb{E}X = \frac{\text{trial}}{\text{success}} = \frac{1}{p}$$

E.g.: $X \sim$ Poisson(λ)

$$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$\begin{aligned}
\mathbb{E}X &= \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\
&= \lambda \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \\
&= \lambda
\end{aligned}$$

Ad hoc pf that

$$\text{Binomial}(n, p) \xrightarrow{n \to \infty} \text{Poisson}(\lambda)$$

For $np = \lambda$,

$$\lim_{n \to \infty} \mathbb{E}(\text{Binomial}(n, p)) = \mathbb{E}(\text{Poisson}(\lambda))$$

Defn: Expectation for a Continuous r.v.
X = continuous r.v. means $dF(x) = f(x)\,dx$ where $f(x)$ is pdf

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\,dF(x) = \int_{-\infty}^{\infty} xf(x)\,dx$$

E.g.: $X \sim U(a,b)$

$$f(x) = \frac{1}{b-a}, \qquad a < x < b$$

$$\begin{aligned}\mathbb{E}X &= \int_a^b \frac{x}{b-a}\,dx \\ &= \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}\end{aligned}$$