Lecture 29

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Exponential Distribution and Poisson Process

Defn: Exponential $X \sim \text{Exp}(\lambda)$

$$f_X(x) = \lambda e^{-\lambda x},\ x \geq 0$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$S_X(x) = \begin{cases} e^{-\lambda x} & x \geq 0 \\ 1 & x < 0 \end{cases}$$

$$\mathbb{E}X = \int_0^\infty S_X(x)\, dx = \frac{1}{\lambda}, \qquad \text{Var}\, X = \frac{1}{\lambda^2}$$

$$M_X(t) = \frac{\lambda}{\lambda - t}, \qquad t < \lambda$$

Memoryless Property:

$$\begin{aligned} \mathbb{P}(X > s + t \mid X > t) &= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > t)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ &= e^{-\lambda s}, \qquad t \text{ is independent} \end{aligned}$$

E.g.: 5.2 $X \sim \text{Exp}(\frac{1}{10})$, $\mathbb{E}X = 10$

1. $\mathbb{P}(X > 15) = e^{-\frac{15}{10}} = e^{-\frac{3}{2}}$
2. $\mathbb{P}(X > 15 \mid X > 10) = e^{-\frac{5}{10}} = e^{-\frac{1}{2}}$

Defn: Hazard rate function for a continuous random variable X with pdf f and cdf F.

$$\begin{aligned} &\lim_{\Delta t \to 0^+} \frac{\mathbb{P}(t < X \leq t + \Delta t \mid X > t)}{\Delta t} \\ =&\lim_{\Delta t \to 0^+} \frac{\mathbb{P}(t < X \leq t + \Delta t)}{\Delta t \cdot \mathbb{P}(X > t)} \\ =&\frac{1}{\mathbb{P}(X > t)} \lim_{\Delta t \to 0^+} \frac{\mathbb{P}(t < X \leq t + \Delta t)}{\Delta t} \\ =&\frac{f(t)}{1 - F(t)} := r(t) \qquad \text{(hazard rate fn)} \end{aligned}$$

$$X \sim \text{Exp}(\lambda) : r(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \qquad \text{(constant)}$$

Recap: From Chapter 2, if $X_1, X_2, \ldots, X_n$ are iid $\text{Exp}(\lambda)$ then $X_1 + X_2 + \cdots + X_n \sim \text{Gamma}(n, \lambda)$ can show via mgf's because mgf of $\text{Gamma}(\alpha, \lambda)$ is $\left(\frac{\lambda}{\lambda - t}\right)^\alpha$.

E.g.: Suppose $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, 2$ independent. What is $\mathbb{P}(X_1 < X_2)$?

$$\begin{aligned} \mathbb{P}(X_1 < X_2) &= \mathbb{E}_{X_1}(\mathbb{P}(X_1 < X_2 \mid X_1)) \\ &= \int_0^\infty \mathbb{P}(X_1 < X_2 \mid X_1 = x) \lambda_1 e^{-\lambda_1 x}\, dx \\ &= \int_0^\infty \mathbb{P}(X_2 > x) \lambda_1 e^{-\lambda_1 x}\, dx \\ &= \int_0^\infty e^{-\lambda_2 x} \cdot \lambda_1 e^{-\lambda_1 x}\, dx \\ &= \lambda_1 \int_0^\infty e^{-(\lambda_1 + \lambda_2)x}\, dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

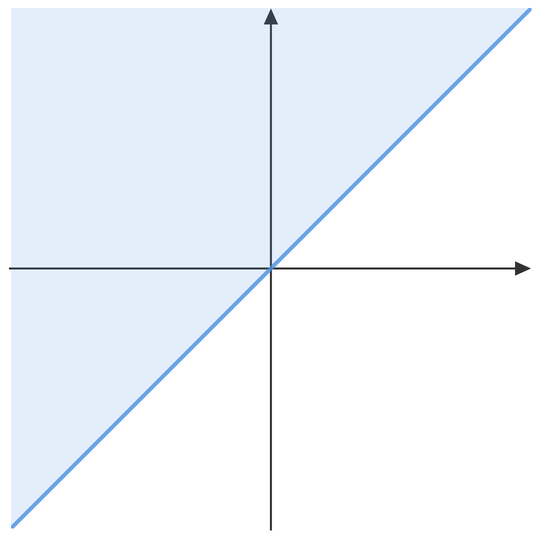

Figure 1: $x_2 = x_1$, integrate joint pdf over shaded region

$$\begin{aligned}
\text{large } \lambda_1 &\implies \mathbb{E}(X_1) = \frac{1}{\lambda_1} \text{ is small} \\
&\implies X_1 < X_2 \text{ is more likely} \\
&\implies \frac{\lambda_1}{\lambda_1 + \lambda_2} \text{ is close to 1}
\end{aligned}$$

E.g.: Suppose $X_i \sim \text{Exp}(\lambda_i)$, independent, $i = 1, 2, \ldots, n$, Let $Y = \min(X_1, X_2, \ldots, X_n)$.

$$\begin{aligned}
\mathbb{P}(Y > t) &= \mathbb{P}(\min(X_1, X_2, \ldots, X_n > t)) \\
&= \mathbb{P}(X_1 > t, \ldots, X_n > t) \\
&= \prod_{i=1}^{n} \mathbb{P}(X_i > t) \\
&= \prod_{i=1}^{n} e^{-\lambda_i t} \\
&= e^{-(\lambda_1 + \cdots + \lambda_n)t}
\end{aligned}$$

$$\implies Y \sim \text{Exp}(\lambda_1 + \cdots + \lambda_n)$$

E.g.: 5.8 Suppose you arrive at a post office with two clerks who are both busy, but no one else is waiting in line. If service time for clerk i is $\text{Exp}(\lambda_i)$, $i = 1, 2$, find $\mathbb{E}T$, in which T is the amount of time you spend in the post office.

Solution: Let R_i to be the remaining service time of the customer currently with clerk i, $i = 1, 2$, and let S be your service time. Then,

$$T = \min(R_1, R_2) + S$$

$$\implies \mathbb{E}T = \frac{1}{\lambda_1 + \lambda_2} + \mathbb{E}S$$

$$\begin{aligned}
\mathbb{E}S &= \mathbb{E}(S \mid R_1 < R_2)\mathbb{P}(R_1 < R_2) + \mathbb{E}(S \mid R_1 > R_2)\mathbb{P}(R_1 > R_2) \\
&= \frac{1}{\lambda_1} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2} \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2} \\
&= \frac{2}{\lambda_1 + \lambda_2}
\end{aligned}$$

$$\therefore \mathbb{E}T = \frac{3}{\lambda_1 + \lambda_2}$$

E.g.: 5.1: Receive money at random rate $\mathbb{R}(t)$ as long as you are alive. Suppose your time of death $T \sim \text{Exp}(\lambda)$. Then, your expected total amount received is $\mathbb{E}(\int_0^T R(t)\ dt)$ Show the expectation equals

$$\int_0^\infty e^{-\lambda t} \mathbb{E}(R(t))\ dt$$

Solution:

$$\begin{aligned}\int_0^T R(t)\,dt &= \int_0^T R(t)\cdot 1\,dt + \int_T^\infty R(t)\cdot 0\,dt \\ &= \int_0^\infty R(t)\cdot \mathbf{1}_{\{t<T\}}\,dt\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\int_0^T R(t)\,dt\right) &= \int_0^\infty \mathbb{E}\left(R(t)\mathbf{1}_{\{t<T\}}\right)\,dt \\ &= \int_0^\infty \mathbb{E}(R(t))\mathbb{E}\left(\mathbf{1}_{\{t<T\}}\right)\,dt \\ &= \int_0^\infty \mathbb{E}(R(t))\mathbb{P}(T>t)\,dt \\ &= \int_0^\infty e^{-\lambda t}\mathbb{E}(R(t))\,dt\end{aligned}$$