Lecture 31

Professor Virginia R. Young

Transcribed by Hao Chen

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Recap: $\{N(t) : t \geq 0\}$ is a $\mathcal{PP}(\lambda)$ if

- counting process
- independent, stationary increment
- $N(t) \sim \mathcal{P}(\lambda t)$

Showed that $T_1, T_2, \ldots$ of interarrival times is iid $\text{Exp}(\lambda)$

- $\mathbb{P}(N(t) = 0) = \mathbb{P}(T_1 > t) = e^{-\lambda t}$

$$\therefore N(t) \sim \mathcal{P}(\lambda t) \implies T_1 \sim \text{Exp}(\lambda)$$

- independent increments of $\{N(t)\}$ drives the independence of $T_1, T_2, \ldots$
- stationary increments of $\{N(t)\}$ drives the identical distributions of $T_1, T_2, \ldots$

Noted that:

T_i is random between event $i - 1$ and i.

$N(t)$ is the number of events during $[0, t]$ or $(0, t]$ because $N(0) = 0$.

$S_n = T_1 + \cdots + T_n$ is the time of events n.

E.g.: Another quantity of interest is $S_n = T_1 + T_2 + \cdots + T_n$ because T_i are iid $\text{Exp}(\lambda)$, $S_n \sim \text{Gamma}(n, \lambda)$.

For $n = 1, 2, \ldots$

$$\begin{aligned}
F_{S_n}(t) &= \mathbb{P}(S_n \leq t) \\
&= \mathbb{P}(N(t) \geq n) \\
&= \sum_{k=n}^{\infty} \mathbb{P}(N(t) = k) \qquad (N(t) \sim \mathcal{P}(\lambda t)) \\
&= \sum_{k=n}^{\infty} e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}
\end{aligned}$$

$$\begin{aligned}
f_{S_n}(t) &= F'_{S_n}(t) \\
&= \sum_{k=n}^{\infty} \left(-\lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + e^{-\lambda t} \cdot \frac{\lambda^k \cdot t^{k-1}}{(k-1)!} \right) \\
&= e^{-\lambda t} \sum_{k=n}^{\infty} \left(\frac{\lambda^k \cdot t^{k-1}}{(k-1)!} - \frac{(\lambda t)^k}{k!} \right) \\
&= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \qquad \text{(telescoping sum)} \\
&= \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t} \\
&\implies S_n \sim \text{Gamma}(n, \lambda)
\end{aligned}$$

E.g.: 5.13: Suppose people immigrate into a territory at a Poisson rate of $\lambda = 1$ per day.

1. What is the expectation time until the 10^{th} immigrant arrives?

   Answer: $\mathbb{E}S_{10} = \frac{10}{1} = 10$, $S_{10} \sim \text{Gamma}(10, 1)$

2. What is the probability that the elapsed time between the 10^{th} and 11^{th} arrival exceeds two days?

   Answer: $\mathbb{P}(T_{11} > 2) = e^{-\lambda t} = e^{-1 \cdot 2} = e^{-2}$

3. Given that four people have arrived by the end of the sixth day, what is the expected time until the tenth immigrant arrives?

   Answer: $S_{10} = \{T_1 + \cdots + T_4\}(6) + T_5 + \cdots + T_{10}$

   $$\mathbb{E}(S_{10} \mid N(6) = 4) = 6 + \mathbb{E}(T_5 + \cdots + T_{10}) = 6 + 6 = 12$$

   where $(T_5 + \cdots + T_{10}) \sim S_6$.

4. $\operatorname{Var}(N(10) \mid N(6) = 4)$

   Answer:

   $$\begin{aligned}
   &\operatorname{Var}(N(10) \mid N(6) = 4) \\
   =&\operatorname{Var}((N(10) - N(6)) + N(6) \mid N(6) = 4) \\
   =&\operatorname{Var}(N(10) - N(6) \mid N(6) = 4) + \operatorname{Var}(N(6) \mid N(6) = 4) \\
   =&\operatorname{Var}(N(10) - N(6)) + \operatorname{Var}(4) \\
   =&\operatorname{Var}(N(10 - 6)) \\
   =&\operatorname{Var}(N(4)) \qquad (N(4) \sim \mathcal{P}(4\lambda)) \\
   =&4\lambda = 4
   \end{aligned}$$

5. $\operatorname{Var}(S_{10} \mid N(6) = 4)$ Answer:

   $$\begin{aligned}
   \operatorname{Var}(S_{10} \mid N(6) = 4) &= \operatorname{Var}(6 + S_6) \\
   &= \operatorname{Var} S_6 = \frac{6}{1^2} = 6
   \end{aligned}$$

5.3.3 Further properties of $\mathcal{PP}$s

Defn: Decomposing a $\mathcal{PP}$: Suppose that each event can be categorized (independently of other events) into one of m types with probabilities p_j such that $\sum_{j=1}^{m} p_j = 1$. Let $N_j(t)$ to be the number of events of type j during $(0, t]$, $j = 1, 2, \ldots, m$. Then, $\{N_j(t)\}$ is a $\mathcal{PP}(\lambda p_j)$ and the processes $\{N_1(t), N_2(t), \ldots, N_m(t)\}$ are independent.

E.g.: Continue with 5.13. Immigrants arrive according to a $\mathcal{PP}(1)$. Suppose we can categorize immigrants according to country of origin, and suppose an immigrant arriving from one country is independent of an immigrant arriving from another country.

Country 1 vs Everyone else (m=2) with corresponding probabilities 2/3 and 1/3. Then, immigrants from country 1 arrive according to a $\mathcal{PP}(2/3 \cdot 1) = \mathcal{PP}(2/3)$.