Lecture 27

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Recap:

$$S = X_1 + X_2 + \cdots + X_N$$
$$M_S(t) = M_N(\ln M_X(t))$$

If $S \sim$ Compound Poisson(λ), then $M_S(t) = e^{\lambda(M_X(t)-1)}$.

Recall Composition Theorem, $N_j \sim \mathcal{P}(\lambda_j)$, where $j = 1, \ldots, m$, all independent,

$$\implies \sum_{j=1}^{m} N_j \sim \mathcal{P}\left(\sum_{j=1}^{m} \lambda_j\right)$$

Defn: General Composition Theorem:

Suppose $S_j \sim$ Compound Poisson(λ_j) (all independent) with claim size cdf F_j. Then,

$$S = \sum_{i=1}^{m} S_i$$

is Compound Poisson($\sum_{j=1}^{m} \lambda_j$) with claim size cdf

$$\sum_{j=1}^{m} \left(\frac{\lambda_j}{\sum_{i=1}^{m} \lambda_i}\right) F_j = \frac{\sum_{j=1}^{m} \lambda_j F_j}{\sum_{j=1}^{m} \lambda_j}$$

$$\begin{aligned} M_S(t) &= \prod_{j=1}^{m} M_{S_j}(t) \\ &= \prod_{j=1}^{m} e^{\lambda_j (M_j(t)-1)} \\ &= e^{\sum_{j=1}^{m}(\lambda_j(M_j(t)-1))} \\ &= e^{(\sum_{j=1}^{m}\lambda_j)\left(\frac{\sum_{j=1}^{m}\lambda_j M_j(t)}{\sum_{j=1}^{m}\lambda_j} - 1\right)} \end{aligned}$$

(write M_j as the mgf corresponding to F_j)

$\implies S \sim C\mathcal{P}\left(\sum_{j=1}^{m} \lambda_j\right)$ with X's cdf.

$$F = \frac{\sum_{j=1}^{m} \lambda_j M_j(t)}{\sum_{j=1}^{m} \lambda_j}$$

Aside: Suppose X has cdf. Then,

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{Xt}) \\ &= \int_{-\infty}^{\infty} e^{xt}\, dF(x) \\ &= \int_{-\infty}^{\infty} e^{xt} \frac{\sum_{j=1}^{m} \lambda_j\, dF_j(x)}{\sum_{j=1}^{m} \lambda_j} \\ &= \frac{\sum_{j=1}^{m} \lambda_j \int_{-\infty}^{\infty} e^{xt}\, dF_j(x)}{\sum_{j=1}^{m} \lambda_j} \\ &= \frac{\sum_{j=1}^{m} \lambda_j M_j(t)}{\sum_{j=1}^{m} \lambda_j} \end{aligned}$$

E.g.: Suppose $S_A \sim C\mathcal{P}(2)$ r.v. with claim size distribution

$$\mathbb{P}(X_A = 1) = 0.6 = 1 - \mathbb{P}(X_A = 2),$$

and suppose $S_B \sim C\mathcal{P}(1)$ r.v. with claim size distribution

$$\mathbb{P}(X_B = 1) = 0.7 = 1 - P(X_B = 3).$$

(a) Compute the mgf of $S = S_A + S_B$.

$$\begin{aligned} M_{S_A}(t) &= e^{\lambda_A(M_A(t)-1)} \\ &= e^{2(0.6e^t + 0.4e^{2t} - 1)} \\ &= e^{1.2e^t + 0.8e^{2t} - 2} \end{aligned}$$

$$M_{S_B}(t) = e^{0.7e^t + 0.3e^{3t} - 1}$$

$$\begin{aligned} M_S(t) &= e^{1.9e^t + 0.8e^{2t} + 0.3e^{3t} - 3} \\ &= e^{3\left(\frac{19}{30}e^t + \frac{4}{15}e^{2t} + \frac{1}{10}e^{3t} - 1\right)} \end{aligned}$$

(b) State S's distribution.

$S \sim C\mathcal{P}(3)$ with claim size distribution $\mathbb{P}(X = 1) = \frac{19}{30}$, $\mathbb{P}(X = 2) = \frac{4}{15}$, $\mathbb{P}(X = 3) = \frac{1}{10}$.

$$\begin{aligned} \mathbb{P}(X = 1) &= \frac{2}{3} \cdot \mathbb{P}(X_A = 1) + \frac{1}{3} \cdot \mathbb{P}(X_B = 1) \\ &= \frac{2}{3}(0.6) + \frac{1}{3}(0.7) = \frac{19}{30} \end{aligned}$$

$$\begin{aligned} \mathbb{P}(X = 2) &= \frac{2}{3} \cdot \mathbb{P}(X_A = 2) + \frac{1}{3} \cdot \mathbb{P}(X_B = 2) \\ &= \frac{2}{3}(0.4) + \frac{1}{3} \cdot 0 = \frac{4}{15} \end{aligned}$$

$$\begin{aligned} M_{S_A}(t) &= e^{\lambda_A(M_A(t)-1)} \\ &= e^{2(0.6e^t + 0.4e^{2t} - 1)} \\ &= e^{1.2e^t + 0.8e^{2t} - 2} \end{aligned}$$

$$M_{S_B}(t) = e^{0.7e^t + 0.3e^{3t} - 1}$$

$$\begin{aligned} M_S(t) &= e^{1.9e^t + 0.8e^{2t} + 0.3e^{3t} - 3} \\ &= e^{3(\frac{19}{30}e^t + \frac{4}{15}e^{2t} + \frac{1}{10}e^{3t} - 1)} \end{aligned}$$

(c) Write $S = 1 \cdot N_1 + 2 \cdot N_2 + 3 \cdot N_3$ and compute $\mathbb{P}(S = x)$, $x = 0, 1, 2, 3$

$$\begin{aligned} N_1 &= \text{the number of size 1} \\ &\sim \mathcal{P}\left(3 \cdot \frac{19}{30}\right) = \mathcal{P}\left(\frac{19}{10}\right) \end{aligned}$$

$$\begin{aligned} N_2 &= \text{the number of size 2} \\ &\sim \mathcal{P}\left(3 \cdot \frac{4}{15}\right) = \mathcal{P}\left(\frac{4}{5}\right) = \mathcal{P}(0.8) \end{aligned}$$

$$\begin{aligned} N_3 &= \text{the number of size 3} \\ &\sim \mathcal{P}\left(3 \cdot \frac{1}{10}\right) = \mathcal{P}(0.3) \end{aligned}$$

x	$\mathbb{P}(2 \cdot N_2 = x)$	$\mathbb{P}(3 \cdot N_3 = x)$	$\mathbb{P}(2 \cdot N_2 + 3 \cdot N_3 = x)$	$\mathbb{P}(1 \cdot N_1 = x)$	$\mathbb{P}(S = x)$
0	$e^{-0.8}$	$e^{-0.3}$	$e^{-1.1}$	$e^{-1.9}$	e^{-3}
1	0	0	0	$1.9e^{-1.9}$	$1.9e^{-3}$
2	$0.8e^{-0.8}$	0	$0.8e^{-1.1}$	$\frac{1.9^2}{2}e^{-1.9}$	$\left(\frac{1.9^2}{3} + 0.8\right)e^{-3}$
3	0	$0.3e^{-0.3}$	$0.3e^{-1.1}$	$\frac{1.9^3}{6}e^{-1.9}$	$\left(\frac{1.9^3}{6} + 1.9(0.8) + 0.3\right)e^{-3}$

$$S_A \sim C\mathcal{P}(1), \qquad \mathbb{P}(X_A = 1) = 1$$

$$S_B \sim C\mathcal{P}(3), \qquad \mathbb{P}(X_B = 1) = \frac{1}{2} = 3\mathbb{P}(X_B = 3)$$

$$\mathbb{P}(S = 3) = \mathbb{P}(S_A = 0)\mathbb{P}(S_B = 3) + \dots$$

$$\mathbb{P}(S_B = 3) = \mathbb{P}(N_B = 1, X_B = 3) + \mathbb{P}(N_B = 3)(\mathbb{P}(X_B = 1))^3$$