Lecture 33

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E.g.: (b) Expected waiting time $\mathbb{E}W$

$$\begin{aligned}
\mathbb{E}W &= \mathbb{E}(\mathbb{E}(W \mid S_1)) \\
&= 0 \cdot \mathbb{P}(S_1 > T) + \mathbb{E}(S_1 + \mathbb{E}W) \mid S_1 \le T) \cdot \mathbb{P}(S_1 \le T) \\
&= \mathbb{E}(S_1 \mid S_1 \le T) \cdot \mathbb{P}(S_1 \le T) + \mathbb{E}W \cdot \mathbb{P}(S_1 \le T) \\
&= \int_0^T \frac{s \cdot \lambda e^{-\lambda s}\, ds}{1 - e^{-\lambda T}} \cdot (1 - e^{-\lambda T}) + \mathbb{E}W(1 - e^{-\lambda T})
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}W \cdot e^{-\lambda T} &= \int_0^T s \cdot \lambda e^{-\lambda s}\, ds \\
&= -Te^{-\lambda T} + \frac{1 - e^{-}\lambda T}{\lambda} \\
\mathbb{E}W &= -T + \frac{e^{\lambda T} - 1}{\lambda}
\end{aligned}$$

E.g.: Compute the conditional probability $\mathbb{P}(N(s) = m \mid N(t) = n)$ for $0 \le m \le n$, $0 \le s \le t$.

$$\begin{aligned}
\mathbb{P}(N(s) = m \mid N(t) = n) &= \frac{\mathbb{P}(N(s) = m, N(t) = n)}{\mathbb{P}(N(t) = n)} \\
&= \frac{\mathbb{P}(N(s) = m, N(t) - N(s) = n - m)}{\mathbb{P}(N(t) = n)}
\end{aligned}$$

where $N(t) - N(s) \sim N(t - s) \sim \mathcal{P}(\lambda(t - s))$. Then

$$\begin{aligned}
\mathbb{P}(N(s) = m \mid N(t) = n) &= \frac{e^{-\lambda s} \frac{(\lambda s)^m}{m!} \cdot e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-m}}{(n-m)!}}{e^{\lambda t} \frac{(\lambda t)^n}{n!}} \\
&= \frac{s^m}{m!} \cdot \frac{(t-s)^{n-m}}{(n-m)!} \cdot \frac{n!}{t^n} \\
&= \frac{n!}{m!(n-m)!} \cdot \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m} \\
&= \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}
\end{aligned}$$

$$N(s) \mid (N(t) = n) \sim \text{Binom}\left(n, \frac{s}{t}\right)$$

E.g.: Compute the conditional probability $\mathbb{P}(N(t) = n \mid N(s) = m)$ for $0 \le m \le n$, $0 \le s \le t$.

$$\begin{aligned}
\mathbb{P}(N(t) = n \mid N(s) = m) &= \frac{\mathbb{P}(N(s) = m, N(t) = n)}{\mathbb{P}(N(s) = m)} \\
&= \frac{\mathbb{P}(N(s) = m, N(t) - N(s) = n - m)}{\mathbb{P}(N(s) = m)} \\
&= \mathbb{P}(N(t) - N(s) = n - m) \\
&= e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-m}}{(n-m)!}
\end{aligned}$$

where $n - m = 0, 1, 2, \ldots$ such that $n = m, m+1, m+2, \ldots$

$$N(t) \mid (N(s) = m) \sim m + \mathcal{P}(\lambda(t - s))$$

$$\implies \mathbb{E}(N(t) \mid N(s) = m) = m + \lambda(t - s)$$

Recap: Friday:

$$\mathbb{P}(T_1 \leq s \mid N(t) = 1) = \frac{s}{t}$$

E.g.: Compute $\mathbb{E}(N(s) \mid N(t) = n)$ for $0 \leq m \leq n$, $0 \leq s \leq t$.

$$\mathbb{E}(N(s) \mid N(t) = n) = n \cdot \frac{s}{t}$$

E.g.: Compute $\mathbb{E}(N(s) \mid N(r) = l, N(t) = n)$ for $r \leq s \leq t$, $0 \leq l \leq n$.

$$\begin{aligned}\mathbb{E}(N(s) \mid N(r) = l, N(t) = n) &= \mathbb{E}(N(s) - N(r) \mid N(t) - N(r) = n - l) + L \\ &= l + (n - l)\frac{s - r}{t - r}\end{aligned}$$