Lecture 34

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E.g.: 5.18 $X_1 \sim \text{Exp}(\mu)$, $X_2 \sim \text{Exp}(\lambda)$, independent. Let $X_{(1)}$, $X_{(2)}$ be the order stats, that is, $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$. Calculate mean/var of $X_{(i)}$, $i = 1, 2$. Start with $X_{(1)}$

$$\begin{aligned}
\mathbb{P}(X_{(1)} > t) &= \mathbb{P}(X_1 > t, X_2 > t) \\
&= \mathbb{P}(X_1 > t)\mathbb{P}(X_2 > t) \\
&= e^{-\mu t} e^{-\lambda t} \\
&= e^{-(\mu+\lambda)t}
\end{aligned}$$

$$X_{(1)} \sim \text{Exp}(\mu + \lambda)$$

$$\mathbb{E}X_{(1)} = \frac{1}{\mu + \lambda}, \qquad \operatorname{Var} X_{(1)} = \frac{1}{(\mu + \lambda)^2}$$

$$X_{(1)}^k + X_{(2)}^k = X_1^k + X_2^k$$

$$\begin{aligned}
\mathbb{E}X_{(2)} &= \mathbb{E}X_1 + \mathbb{E}X_2 - \mathbb{E}X_{(1)} \\
&= \frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\mu - \lambda}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X_{(2)}^2) &= \mathbb{E}(X_1^2) + \mathbb{E}(X_2^2) - \mathbb{E}(X_{(1)}^2) \\
&= \frac{2}{\mu^2} + \frac{2}{\lambda^2} - \frac{2}{(\mu + \lambda)^2}
\end{aligned}$$

$$\operatorname{Var} X_{(2)} = \frac{2}{\mu^2} + \frac{2}{\lambda^2} - \frac{2}{(\mu + \lambda)^2} - \left(\frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\mu - \lambda}\right)^2$$

E.g.: 5.45 $\{N(t)\} \sim \mathcal{PP}(\lambda)$ independent of a non-negative r.v. T with mean μ and variance σ^2

(a) $\operatorname{Cov}(T, N(T))$

$$\begin{aligned}
\operatorname{Cov}(T, N(T)) &= \mathbb{E}(TN(T)) - \mathbb{E}T \cdot \mathbb{E}N(T) \\
&= \mathbb{E}_T(\mathbb{E}(TN(T) \mid T)) - \mu\mathbb{E}_T(\mathbb{E}(N(T) \mid T)) \\
&= \mathbb{E}(T\mathbb{E}(N(T) \mid T)) - \mu\mathbb{E}(\mathbb{E}(N(T) \mid T)) \\
&= \mathbb{E}(T \cdot \lambda T) - \lambda\mathbb{E}(\lambda T) \\
&= \lambda\mathbb{E}(T^2) - \mu\lambda\mathbb{E}T \\
&= \lambda(\sigma^2 + \mu^2) - \mu\lambda \cdot \mu \\
&= \lambda\sigma^2
\end{aligned}$$

(b) $\operatorname{Var} N(T)$

$$\begin{aligned}
\operatorname{Var} N(T) &= \mathbb{E}(\operatorname{Var}(N(T) \mid T)) + \operatorname{Var}(\mathbb{E}(N(T) \mid T)) \\
&= \mathbb{E}(\lambda T) + \operatorname{Var}(\lambda T) \\
&= \lambda\mathbb{E}T + \lambda^2 \operatorname{Var} T \\
&= \lambda\mu + \lambda^2\sigma^2
\end{aligned}$$