Lecture 11

Professor Virginia R. Young
Transcribed by Hao Chen

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Expectations

Recap:

$$\mathbb{E}X = \int_{-\infty}^{\infty} x \, dF(x)$$

E.g.: $X \sim U(a,b)$

$$\mathbb{E}X = \int_a^b x f(x) \, dx = \frac{b+a}{2}$$

E.g.: $X \sim \text{Gamma}(\alpha, \lambda)$

$$\begin{aligned}
\mathbb{E}X &= \int_0^\infty x f(x) \, dx \\
&= \int_0^\infty x \cdot \frac{x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \lambda^\alpha \, dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{(\alpha+1)-1} e^{-\lambda x} \, dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \int_0^\infty \frac{\lambda^{\alpha+1} x^{(\alpha+1)-1} e^{-\lambda x}}{\Gamma(\alpha+1)} \, dx
\end{aligned}$$

where we have

$$\int_0^\infty \frac{\lambda^{\alpha+1} x^{(\alpha+1)-1} e^{-\lambda x}}{\Gamma(\alpha+1)} \, dx = 1$$

Thus,

$$\mathbb{E}X = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{1}{\lambda} = \frac{\alpha}{\lambda}$$

E.g.: Special case of the $\text{Gamma}(\alpha, \lambda)$ with $\alpha = 1$ is $\text{Exp}(\lambda)$ Another way to compute $\mathbb{E}$. Let $S(x) = 1 - F(x) = \mathbb{P}(X > x)$, or survival function of X.

$$\begin{aligned}
\mathbb{E}X &= \int_0^\infty (1 - F(x)) \, dx \\
&= \int_0^\infty S(x) \, dx \\
&= \int_0^\infty e^{-\lambda x} \, dx \\
&= -\frac{1}{\lambda} e^{-\lambda x} \bigg|_0^\infty = \frac{1}{\lambda}
\end{aligned}$$

Therefore, $\mathbb{E}X = \frac{1}{\lambda}$.

E.g.: $X \sim \text{Pareto}(\alpha, \theta)$

$$S(x) = \begin{cases} 1 & x < 0 \\ \left(\frac{\theta}{\theta+x}\right)^\alpha & x \geq 0 \end{cases}$$

$$f(x) = F'(x) = -S'(x)$$

$$\begin{aligned}
\mathbb{E}X &= \int_0^\infty \left(\frac{\theta}{\theta+x}\right)^\alpha dx \\
&= \theta^\alpha \frac{(\theta+x)^{-\alpha+1}}{-\alpha+1} \bigg|_{x=0}^\infty
\end{aligned}$$

$$\mathbb{E}X = \begin{cases} \infty & \alpha \leq 1 \\ \frac{\theta}{\alpha-1} & \alpha > 1 \end{cases}$$

E.g.: $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$f(x)$ symmetric with respect to $x = \mu \implies \mathbb{E}X = \mu$.

$$\mathbb{E}X = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx$$

Let $z = \frac{x-\mu}{\sigma}$, then $dz = \frac{dx}{\sigma}$,

$$\begin{aligned}
\mathbb{E}X &= \int_{-\infty}^{\infty} (\sigma z + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2}\, dz + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \\
&= \left(-e^{-\frac{1}{2}z^2}\right)\Big|_{z=-\infty}^{\infty} + \mu \\
&= \mu
\end{aligned}$$

Defn: Expectation of a function g of X

$$\begin{aligned}
\mathbb{E}(g(X)) &= \int_{-\infty}^{\infty} g(x)\, dF(x) \\
&= \lim_{\|\pi\| \to 0} \sum_{i=-\infty}^{\infty} g(x_i^*)(F(x_i) - F(x_{i-1}))
\end{aligned}$$

where $x_{-\infty} = -\infty$ and $x_\infty = \infty$.

If X is discrete with pmf p,

$$\mathbb{E}(g(X)) = \sum_{x_k} g(x_k) p(x_k)$$

If X is continuous with pdf f,

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$\mathbb{E}$ is a linear operator

$$g(x) = ax + b, \qquad a, b \in \mathbb{R}$$

$$\begin{aligned}
\mathbb{E}(aX + b) &= \int_{-\infty}^{\infty} (ax + b)\ F(x) \\
&= a \int_{-\infty}^{\infty} x\, dF(x) + b \int_{-\infty}^{\infty} 1\, dF(x) \\
&= a\mathbb{E}X + b
\end{aligned}$$

Defn: Variance of X: $g(x) = (x - \mathbb{E}X)^2$

$$\begin{aligned}
\operatorname{Var} X &= \mathbb{E}[(X - \mathbb{E}X)^2] \\
&= \mathbb{E}[X^2 - 2\mathbb{E}X \cdot X + (\mathbb{E}X)^2] \\
&= \mathbb{E}(X^2) - 2\mathbb{E}X \cdot \mathbb{E}X + (\mathbb{E}X)^2 \\
&= \mathbb{E}(X^2) - (\mathbb{E}X)^2
\end{aligned}$$

Typically, we call $\mathbb{E}(X^k)$ as $k^{\underline{th}}$ moment, so $\mathbb{E}(X^2)$ is second moment.
Similarly, we call $\mathbb{E}((X - \mathbb{E}X)^k)$ as $k^{\underline{th}}$ central moment, so $\operatorname{Var} X$ is 2^{nd} central moment.

For $a, b \in \mathbb{R}$,

$$\begin{aligned}
&\operatorname{Var}(aX + b) \\
=&\mathbb{E}((aX + b - \mathbb{E}(aX + b))^2) \\
=&\mathbb{E}((aX + b - (a\mathbb{E}X + b))^2) \\
=&\mathbb{E}((aX - a\mathbb{E}X)^2) \\
=&\mathbb{E}(a^2(X - \mathbb{E}X)^2) \\
=&a^2\mathbb{E}((X - \mathbb{E}X^2) \\
=&a^2 \operatorname{Var} X
\end{aligned}$$

E.g.: $X \sim N(\mu, \sigma^2)$ Given that $\mathbb{E}X = \mu$,

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\ dx$$

Let $z = \frac{x-\mu}{\sigma}$, then $dz = \frac{dx}{\sigma}$,

$$\begin{aligned}\mathrm{Var}\, X &= \int_{-\infty}^{\infty} \sigma^2 z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\ dz \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \left(z e^{-\frac{1}{2}z^2}\ dz \right)\end{aligned}$$

Consider that $u = z$ and $dt = z e^{-\frac{1}{2}z^2}\ dz$,

$$\begin{aligned}\mathrm{Var}\, X &= \sigma^2 \frac{1}{\sqrt{2\pi}} \left[-z e^{-\frac{1}{2}z^2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\ dz \right] \\ &= \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\ dz = \sigma^2\end{aligned}$$