Lecture 21

Professor Virginia R. Young
Transcribed by Hao Chen

October 28, 2022

Defn: Conditional probability of continuous r.v.s

With discrete r.v.s, it makes sense to consider

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} = \mathbb{P}(X = x \mid Y = y)$$

where $p_{X|Y}(x \mid y)$ is the conditional pmf.

Continuous r.v.s joint pdf $f_{X,Y}(x,y)$

$$\begin{aligned}\mathbb{P}(X \le x \mid y < Y \le y + \Delta y) &= \frac{\mathbb{P}(X \le x, y < Y \le y + \Delta y)}{\mathbb{P}(y < Y \le y + \Delta y)} \\ &= \frac{\int_{-\infty}^{x} \int_{y}^{y+\Delta y} f_{X,Y}(u,v)\, dv du}{\int_{y}^{y+\Delta y} f_Y(v)\, dv}\end{aligned}$$

Recall

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

To get what we might label the conditional cdf of X given $Y = y$, take a limit as $\Delta y \to 0^+$

$$\begin{aligned}\mathbb{P}(X \le x \mid y < Y \le y + \Delta y) &\overset{\text{def'n}}{=} \lim_{\Delta y \to 0^+} \frac{\int_{-\infty}^{x} \int_{y}^{y+\Delta y} f_{X,Y}(u,v)\, dv du}{\int_{y}^{y+\Delta y} f_Y(v)\, dv} \\ &\overset{\text{L'H}}{=} \lim_{\Delta y \to 0^+} \frac{\int_{-\infty}^{x} f_{X,Y}(u, y + \Delta y)\, du}{f_Y(y + \Delta y)} \\ &= \frac{\int_{-\infty}^{x} f_{X,Y}(u,y)\, du}{f_Y(y)}\end{aligned}$$

where $\mathbb{P}(X \le x \mid y < Y \le y + \Delta y)$ is the conditional cdf. Thus, conditional pdf of X given $Y = y$

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

conditional expectation of X given $Y = y$

$$\mathbb{E}(X \mid Y = y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x \mid y)\, dx$$

As in Section 3.2, $\mathbb{E}X = \mathbb{E}(\mathbb{E}(X \mid Y)) = \int_{-\infty}^{\infty} \mathbb{E}(X \mid Y = y) f_Y(y)\, dy$.

E.g.: Suppose $f_{X,Y}(x,y) = 6xy(2 - x - y)$, $0 < x, y < 1$; 0, otherwise. Compute $\mathbb{E}(X \mid Y = y)$.

Solution: First calculate $f_{X|Y}(x \mid y)$. Need,

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \\ &= \int_{0}^{1} 6xy(2 - x - y)\, dx \\ &= 4y - 3y^2\end{aligned}$$

$0 < x, y < 1$

$$\begin{aligned}\implies f_{X|Y}(x \mid y) &= \frac{6xy(2 - x - y)}{y(4 - 3y)} \\ &= \frac{6x(2 - x - y)}{4 - 3y}\end{aligned}$$

$0 < y < 1$

$$\begin{aligned}
\implies \mathbb{E}(X \mid Y = y) &= \int_0^1 x \cdot \frac{6x(2-x-y)}{4-3y}\, dx \\
&= \frac{5/2 - 2y}{4-3y}
\end{aligned}$$

Aside:

$$\begin{aligned}
\mathbb{E}X &= \int_0^1 \frac{5/2-2y}{4-3y} \cdot y(4-3y)\, dy \\
&= \int_0^1 \left(\frac{5}{2}y - 2y^2\right)\, dy = \frac{5}{4} - \frac{2}{3} = \frac{15-8}{12} = \frac{7}{12}
\end{aligned}$$

or:

$$\mathbb{E}X = \int_0^1 \int_0^1 x \cdot 6xy(2-x-y)\, dxdy = \cdots = \frac{7}{12}$$

Aside:

$$\frac{5}{2} - 2y < 4 - 3y \iff y < 4 - \frac{5}{2} = \frac{3}{2}$$

E.g.: Suppose $f_{X,Y}(x,y) = 4y(x-y)e^{-(x+y)}$, if $0 \le y \le x < \infty$, $f_{X,Y}(x,y) = 0$, otherwise. Compute $\mathbb{E}(Y \mid X = x)$.

Solution: First, compute

$$\begin{aligned}
f_X(x) &= \int_0^x 4y(x-y)e^{-(x+y)}\, dy \\
&= 4e^{-x} \int_0^x (xy - y^2)e^{-y}\, dy
\end{aligned}$$

Let $u = (xy - y^2)$ and $dv = e^{-y}\, dy$

$$\begin{aligned}
f_X(x) &= 4e^{-x}\left[-(xy-y^2)e^{-y}\Big|_{y=0}^{x} + \int_0^x (x-2y)e^{-y}\, dy\right] \\
&= 4e^{-x}\left[-(x-2y)e^{-y}\Big|_{y=0}^{x} - 2\int_0^x e^{-y}\, dy\right] \\
&= 4e^{-x}[xe^{-x} + x - 2(1-e^{-x})] \\
&= 4e^{-x}[(x-2) + (x+2)e^{-x}]
\end{aligned}$$

$$\begin{aligned}
\implies f_{Y|X}(y \mid x) &= \frac{4y(x-y)e^{-(x+y)}}{4e^{-x}[(x-2)+(x+2)e^{-x}]}, \qquad 0 \le y \le x \\
&= \frac{y(x-y)e^{-y}}{(x-2)+(x+2)e^{-x}}
\end{aligned}$$

$$\begin{aligned}
\implies \mathbb{E}(Y \mid X = x) &= \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y \mid x)\, dy \\
&= \int_0^x y \cdot \frac{y(x-y)e^{-y}}{(x-2)+(x+2)e^{-x}}\, dy \\
&\vdots \\
&= \frac{(2x-6)+(x^2+4x+6)e^{-x}}{(x-2)+(x+2)e^{-x}}, \qquad 0 \le x < \infty
\end{aligned}$$

Computing expectation via conditioning

E.g.: 3.11: Mean of a geometric distribution.

A coin, with probability p of coming up heads, is flipped until the first head appear. What is the expected number of flips required?

Solution: Let $Y = 1$ if the first flip is a head; 0, otherwise.

$$\begin{aligned}
\mathbb{E}N &= \mathbb{E}(\mathbb{E}(N \mid Y)) \\
&= \mathbb{E}(N \mid Y = 1)\mathbb{P}(Y = 1) + \mathbb{E}(N \mid Y = 0)\mathbb{P}(Y = 0)
\end{aligned}$$

$$\mathbb{E}N = 1 \cdot p + (1 + \mathbb{E}N)(1-p)$$

$$\mathbb{E}N(1-(1-p)) = p + (1-p) \implies \mathbb{E}N = \frac{1}{p}$$

Aside: N is the number of flips until first head.

$$\mathbb{P}(N = n) = (1-p)^{n-1} \cdot p, \qquad n = 1, 2, \ldots$$