Lecture 14

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Recap: $1_A : \Omega \to \mathbb{R}$, $A \in \mathcal{F}$

$$1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Claim: $\mathbb{E}(1_A) = \mathbb{P}(A)$

$$1_A(\omega) = \begin{cases} 1 & \text{w.p. } \mathbb{P}(A) \\ 0 & \text{w.p. } 1 - \mathbb{P}(A) \end{cases}$$

$1_A \sim \text{Bernoulli}(\mathbb{P}(A)) \implies \mathbb{E}(1_A) = \mathbb{P}(A)$

Aside:

$$1_{A \cap B} = 1_A \cdot 1_B \qquad 1_{A \cup B} = 1_A + 1_B - 1_{A \cap B}$$
$$\mathbb{E}(1_{A \cap B}) = \mathbb{E}(1_A) + \mathbb{E}(1_B) - \mathbb{E}(1_{A \cap B})$$
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

If X and Y are independent, then $\text{Cov}(X, Y) = 0$. But if $\text{Cov}(X, Y) = 0$, then we cannot that X and Y are independent.

E.g.: $X \sim \text{U}(-1, 1)$, $Y = X^2$

$$\begin{aligned} \text{Cov}(X, Y) &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \\ &= \mathbb{E}(X^3) = \int_{-1}^{1} x^3 \, \frac{dx}{2} \\ &= \left.\frac{x^4}{8}\right|_{-1}^{1} = 0 \end{aligned}$$

Detailed proof that X and Y are not independent,

$$\begin{aligned} F_{X,Y}(x, y) &= \mathbb{P}(X \le x, Y \le y) \\ &= \mathbb{P}(X \le x, X^2 \le y) \\ &= \mathbb{P}(-1 \le X \le x, -\sqrt{y} \le X \le \sqrt{y}) \\ &= \mathbb{P}(-\sqrt{y} \le X \le \min(x, \sqrt{y})) \\ &= \mathbb{P}(-\sqrt{y} < X \le \min(x, \sqrt{y})) \\ &= F_X(\min(x, \sqrt{y})) - F_X(-\sqrt{y}) \\ &= \frac{\min(x, \sqrt{y}) + \sqrt{y}}{2} \end{aligned}$$

which cannot be factor into $F_X(x)F_Y(y)$.

$$F_{X,Y}(x, y) = \begin{cases} \mathbb{P}(-\sqrt{y} < X \le x) & -\sqrt{y} < x \\ 0 & -\sqrt{y} \ge x \end{cases}$$

Properties of the covariance

Proof: 1. $\text{Cov}(X, X) = \text{Var}\, X$

$$\begin{aligned} \text{Cov}(X, X) &= \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)] \\ &= \mathbb{E}[(X - \mathbb{E}X)^2] \\ &= \text{Var}\, X \end{aligned}$$

Proof: 2. $\text{Cov}(X, Y) = \text{Cov}(Y, X)$

$$\begin{aligned} \text{Cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] \\ &= \mathbb{E}[(Y - \mathbb{E}Y)(X - \mathbb{E}X)] \\ &= \text{Cov}(Y, X) \end{aligned}$$

Proof: 3. $\mathrm{Cov}(cX, Y) = c\,\mathrm{Cov}(X, Y)$, $c \in \mathbb{R}$

$$\begin{aligned}
\mathrm{Cov}(cX, Y) &= \mathbb{E}(cX \cdot Y) - \mathbb{E}(cX)\mathbb{E}Y \\
&= c\mathbb{E}(XY) - c\mathbb{E}X\mathbb{E}Y \\
&= c\{\mathbb{E}(XY) - \mathbb{E}X\mathbb{E}Y\} \\
&= c\,\mathrm{Cov}(X, Y)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{Cov}(X, cY) &= \mathrm{Cov}(cY, X) \\
&= c\,\mathrm{Cov}(Y, X) \\
&= c\,\mathrm{Cov}(X, Y)
\end{aligned}$$

Proof: 4. $\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$

$$\begin{aligned}
\mathrm{Cov}(X, Y + Z) &= \mathbb{E}(X(Y + Z)) - \mathbb{E}X \cdot \mathbb{E}(Y + Z) \\
&= \mathbb{E}(XY) + \mathbb{E}(XZ) - \mathbb{E}X\{\mathbb{E}Y + \mathbb{E}Z\} \\
&= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(XZ) - \mathbb{E}X \cdot \mathbb{E}Z \\
&= \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{Cov}(X + Y, Z) &= \mathrm{Cov}(Z, X + Y) \\
&= \mathrm{Cov}(Z, X) + \mathrm{Cov}(Z, Y) \\
&= \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)
\end{aligned}$$

Proof: By mathematical induction, properties 2-4 imply

$$\mathrm{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j\,\mathrm{Cov}(X_i, Y_j)$$

$\therefore$ We say that covariance is a symmetric, bi-linear operator.

$$\begin{bmatrix}
x_1^2 & x_1x_2 & \cdots & x_1x_n \\
x_2x_1 & x_2^2 & \cdots & x_2x_n \\
\vdots & \vdots & \ddots & \vdots \\
x_nx_1 & x_nx_2 & \cdots & x_n^2
\end{bmatrix}$$

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) &= \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} \mathrm{Cov}(X_i, X_i) + \sum_{i \neq j}^{n} \mathrm{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} \mathrm{Var}\,X_i + 2\sum_{i \leq j \leq n} \mathrm{Cov}(X_i, X_j)
\end{aligned}$$

Specialize $n = 2$:

$$\mathrm{Var}(X + Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y + 2\,\mathrm{Cov}(X, Y)$$

Specialize X_is are independent:

$$\mathrm{Var}(\mathrm{sum}) = \mathrm{sum}(\mathrm{Var})$$