Lecture 23

Professor Virginia R. Young
Transcribed by Hao Chen

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E.g.: Ex3.12: A miner is trapped in a mine containing 3 doors. Door 1 leads to safety in 2 hours. Door 2 leads back to the three doors in 3 hours. Door 3 leads back to the three doors in 5 hours. Assuming all three are equally likely to be chosen at all times, what is the expected length of time until the miner reaches safety?

Solution: X is the time until reaches safety, Y is the door chosen initially.

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E}(\mathbb{E}(X \mid Y)) \\
&= \mathbb{E}(X \mid Y=1)\mathbb{P}(Y=1) + \mathbb{E}(X \mid Y=2)\mathbb{P}(Y=2) + \mathbb{E}(X \mid Y=3)\mathbb{P}(Y=3) \\
&= \frac{1}{3}(2 + (3 + \mathbb{E}X) + (5 + \mathbb{E}X)) \\
&= \frac{10}{3} + \frac{2}{3}\mathbb{E}X
\end{aligned}$$

$$\implies \frac{1}{3}\mathbb{E}X = \frac{10}{3} \implies \mathbb{E}X = 10$$

E.g.: Ex3.15: Generalization of Geom(p) Independent trials, each of which is a success with probability p, are performed until there a k consecutive successes. What is the expected number of trials?

Solution: N_k is the number of required trials for k consecutive successes. $M_k = \mathbb{E}N_k$
Compute M_k by conditioning on N_{k-1}.

$$\mathbb{E}(N_k \mid N_{k-1}) = \mathbb{E}_Y(\mathbb{E}(N_k \mid N_{k-1}, Y))$$

$Y = 1$ if next trial is a success; 0, otherwise.

$$\begin{aligned}
\mathbb{E}(N_k \mid N_{k-1}) &= \mathbb{E}(N_k \mid N_{k-1}, Y=1)\mathbb{P}(Y=1) + \mathbb{E}(N_k \mid N_{k-1}, Y=0)\mathbb{P}(Y=0) \\
&= (N_{k-1}+1)p + (N_{k-1} + 1 + \mathbb{E}N_k)(1-p) \\
&= N_{k-1} + 1 + (1-p)\mathbb{E}N_k
\end{aligned}$$

$$\begin{aligned}
\overset{\mathbb{E}_{N_{k-1}}}{\Longrightarrow} M_k &= M_{k-1} + 1 + (1-p)M_k \\
pM_k &= M_{k-1} + 1 \\
M_k &= \frac{1}{p} + \frac{1}{p}M_{k-1}
\end{aligned}$$

$$k = 1 \implies M_{k-1} = 0 \implies M_1 = \frac{1}{p}$$

$$M_2 = \frac{1}{p} + \frac{1}{p}\cdot\frac{1}{p} = \frac{1}{p} + \frac{1}{p^2}$$

$$M_3 = \frac{1}{p} + \frac{1}{p}\left(\frac{1}{p} + \frac{1}{p^2}\right) = \sum_{i=1}^{3}\frac{1}{p^i}$$

$$\begin{aligned}
M_k = \sum_{i=1}^{k}\frac{1}{p^i} &= \frac{\frac{1}{p} - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}}\cdot\frac{p}{p} \\
&= \frac{1 - \frac{1}{p^k}}{p-1} = \frac{\frac{1}{p^k} - 1}{1-p}
\end{aligned}$$

E.g.:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\cdot\exp\left(-\frac{1}{2(1-p^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}\right)$$

(a) Compute $f_{X|Y}(x \mid y)$ and identify this pdf.

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

with $Y \sim N(\mu_Y, \sigma_Y^2)$.

$$\implies X \mid (Y = y) \sim N\left(\mu_Y + \frac{\rho\sigma_X}{\sigma_Y}(y - \mu_y), \sigma_X^2(1 - p^2)\right)$$

vs $X \sim N(\mu_X, \sigma_X^2)$

(b) Compute $f_{X|X+Y}(x \mid z)$ and identify this pdf.
(i) Compute $Z = X + Y$'s pdf.

$$\overset{\text{conv}}{\implies} X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2)$$

(ii) X and $X + Y$ are bivariate normal with correlation coefficient $\frac{\sigma_X + \rho\sigma_Y}{\sqrt{\sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2}}$
use (a) to get

$$X \mid (X + Y = Z) \sim N\left(\mu_X + \frac{\sigma_X(\sigma_X + \rho\sigma_Y)}{\sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2}(z - (\mu_X + \mu_Y)), \sigma_X^2\sigma_Y^2\frac{1 - \rho^2}{\sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2}\right)$$