Lecture 05

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Bayes' Formula

Recap: From 1.4, we defined conditional probability. If $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \mathbb{P}(B \mid A)$$
$$\implies \mathbb{P}(A \cap B) = \mathbb{P}(B \mid A)\mathbb{P}(A)$$

Defn: Bayes' formula

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

E.g.: A random number N of dice are thrown. A_i is the event that $N = i$ and $\mathbb{P}(A_i) = \frac{1}{2^i}$, $i = 1, 2, 3, \ldots$. Then sum of the dice is S. Assume that each die is fair with $\mathbb{P}(\text{die} = j) = \frac{1}{6}$, $j = 1, 2, \ldots, 6$, independent of A_i.

1. Calculate $\mathbb{P}(N = 2 \mid S = 4)$

$$\begin{aligned}
&= \frac{\mathbb{P}(S = 4 \mid N = 2)\mathbb{P}(N = 2)}{\mathbb{P}(S = 4)} \\
&= \frac{\mathbb{P}(S = 4 \mid N = 2)\mathbb{P}(N = 2)}{\sum_{i=1}^{\infty} \mathbb{P}(S = 4 \mid N = i)\mathbb{P}(N = i)} \\
&= \frac{\frac{3}{36} \cdot \frac{1}{2^2}}{(\frac{1}{6} \cdot \frac{1}{2} + \frac{3}{36} \cdot \frac{1}{2^2} + \frac{3}{6^3} \cdot \frac{1}{2^3} + \frac{1}{6^4} \cdot \frac{1}{2^4})} \\
&= \frac{432}{2197}
\end{aligned}$$

2. Calculate $\mathbb{P}(S = 4 \mid N = \text{even})$

$$\begin{aligned}
&= \frac{\mathbb{P}(S = 4 \text{ and } N = \text{even})}{\mathbb{P}(N = \text{even})} \\
&= \frac{\mathbb{P}(S = 4, N = 2) + \mathbb{P}(S = 4, N = 4)}{\mathbb{P}(N = \text{even})} \\
&= \frac{\frac{3}{36} \cdot \frac{1}{2^4} + \frac{1}{6^4} \cdot \frac{1}{2^4}}{\frac{1}{2^2} + \frac{1}{2^4} + \ldots} \\
&= \frac{433}{6912}
\end{aligned}$$

3. Calculate $\mathbb{P}(N = 2 \mid S = 4 \text{ and first die} = 1)$

$$
\begin{aligned}
&= \frac{\mathbb{P}(N = 2 \text{ and } S = 4 \text{ and } X_1 = 1)}{\mathbb{P}(S = 4 \text{ and } X_1 = 1)} \\
&= \frac{\mathbb{P}(X_1 = 1 \text{ and } X_2 = 3 \text{ and } N = 2)}{\mathbb{P}(S = 4 \text{ and } X_1 = 1)} \\
&= \frac{\mathbb{P}(X_1 = 1 \text{ and } X_2 = 3 \mid N = 2)\mathbb{P}(N = 2)}{\mathbb{P}(S = 4, X_1 = 1)} \\
&= \frac{\mathbb{P}(X_1 = 1, X_2 = 3)\mathbb{P}(N = 2)}{\mathbb{P}(S = 4, X_1 = 1)} \\
&= \frac{\mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 3)\mathbb{P}(N = 2)}{\mathbb{P}(S = 4, X_1 = 1)} \\
&= \frac{\mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 3)\mathbb{P}(N = 2)}{\sum_{i=1}^{\infty} \mathbb{P}(S = 4, X_1 = 1 \mid N = i)\mathbb{P}(N = i)} \\
&= \frac{\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{2^2}}{(\frac{1}{6^2} \cdot \frac{1}{2^2} + \frac{2}{6^3} \cdot \frac{1}{2^3} + \frac{1}{6^4} \cdot \frac{1}{2^4})} \\
&= \frac{144}{181}
\end{aligned}
$$

4. Calculate $\mathbb{P}_r := \mathbb{P}(\text{largest \# show by any die is } r)$

$$
P_r = \mathbb{P}(\text{all die} \leq r) - \mathbb{P}(\text{all die} \leq r - 1)
$$

$$
\begin{aligned}
r = 1 : \ & \mathbb{P}_1 = \mathbb{P}(\text{all die} = 1) \\
& = \sum_{i=1}^{\infty} \mathbb{P}(X_1 = 1 = \cdots = X_i \mid N = i)\mathbb{P}(N = i) \\
& = \sum_{i=1}^{\infty} \frac{1}{6^i} \cdot \frac{1}{2^i} = \sum_{i=1}^{\infty} \frac{1}{12^i} = \frac{\frac{1}{12}}{1 - \frac{1}{12}} = \frac{1}{11} \\
r \geq 2 : \ & \mathbb{P}(\text{all die} \leq r) = \sum_{i=1}^{\infty} \mathbb{P}(\text{all die} \leq r \mid N = i)\mathbb{P}(N = i) \\
& = \sum_{i=1}^{\infty} (\frac{r}{6})^i \cdot \frac{1}{2^i} = \frac{r}{12 - r} \\
& \implies \mathbb{P}_2 = \frac{2}{12 - 2} - \frac{1}{11} = \frac{6}{55} \\
& \vdots \\
& \implies \mathbb{P}_6 = \frac{2}{7}
\end{aligned}
$$