Lecture 28

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Defn: Stochastic Process

$\{X_t\}_{t\in I}$ where X_t is a random variable, or the function from $\Omega \to \mathbb{R}$ such that $\{\omega : X_t(\omega) \leq x\} \in \mathcal{F}_t$, $\forall x \in \mathbb{R}$, $I = \{0, 1, 2, 3, \dots\}$, or $I = \{0, 1, 2, \dots, T\}$, or $I = [0, T]$, or $I = [0, \infty)$.

Intuition: $\mathcal{F}_t$ represents info we have at time t, As time t progresses, we anticipate that we will have more info so that $\mathcal{F}_t \subset \mathcal{F}_s$, $t \leq s$.

Special class of stochastic processes is the class of Markov processes:

$$\mathbb{P}(X_t \leq x \mid X_s, \forall s \in [a, b]) = \mathbb{P}(X_t \leq x \mid X_b), \qquad a < b < t$$

Random walk: Suppose $X_0 = 2$, then

$$\mathbb{P}(X_1 = 3 \mid X_0 = 2) = p$$
$$\mathbb{P}(X_1 = 1 \mid X_0 = 2) = 1 - p$$

More generally,

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \end{cases}$$

Defn: Gambler's Ruin Problem

For each play of the game, the gambler wins \$1 w.p. p, and loses \$1 w.p. $1 - p$. Let X_n is gambler's fortune at time $n = 0, 1, 2, \dots$. Find the probability P_i that if the gambler starts with $\$i$, then gambler's fortune will reach $\$N$ before reaching \$0, $\forall i = 0, 1, \dots, N$.

Boundary condition: $P_0 = 0$, $P_N = 1$.

For $i = 1, 2, \dots, N-1$,

$$\begin{aligned}
P_i &= \mathbb{P}(\text{reaching } N \text{ before } 0 \mid X_0 = i) \\
&= \mathbb{E}(\mathbb{P}(\text{reaching } N \text{ before } 0 \mid X_0 = i, X_1)) \\
&= \mathbb{P}(\text{reaching } N \text{ before } 0 \mid X_0 = i, X_1 = i+1)\mathbb{P}(X_1 = i+1 \mid X_0 = i) \\
&+ \mathbb{P}(\text{reaching } N \text{ before } 0 \mid X_0 = i, X_1 = i-1)\mathbb{P}(X_1 = i-1 \mid X_0 = i) \\
&= \mathbb{P}(\text{reaching } N \text{ before } 0 \mid X_1 = i+1) \cdot p + \mathbb{P}(\text{reaching } N \text{ before } 0 \mid X_1 = i-1) \cdot (1-p)
\end{aligned}$$

$$\begin{aligned}
P_i &= p \cdot P_{i+1} + (1-p) \cdot P_{i-1} \\
(p + (1-p)) \cdot P_i &= p \cdot P_{i+1} + (1-p) \cdot P_{i-1} \\
(1-p)(P_i - P_{i-1}) &= p(P_{i+1} - P_i)
\end{aligned}$$

$$\begin{aligned}
\implies P_{i+1} - P_i &= \frac{1-p}{p}(P_i - P_{i-1}) \\
&= \left(\frac{1-p}{p}\right)^2 (P_{i-1} - P_{i-2}) \\
&\vdots \\
&= \left(\frac{1-p}{p}\right)^i (P_1 - P_0)
\end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow P_{i+1} - P_i = \left(\frac{1-p}{p}\right)^i P_1 \\
&\Longrightarrow \sum_{j=0}^{i}(P_{j+1} - P_j) = \left(\sum_{j=0}^{i}\left(\frac{1-p}{p}\right)^j\right) P_1 \\
&\Longrightarrow P_{i+1} = \left(\sum_{j=0}^{i}\left(\frac{1-p}{p}\right)^j\right) \cdot P_1
\end{aligned}
$$

1).

$$
p = \frac{1}{2} \Longrightarrow \frac{1-p}{p} \Longrightarrow P_{i+1} = (i+1)P_1
$$

$$
1 = P_N = N \cdot P_1 \Longrightarrow P_1 = \frac{1}{N} \Longrightarrow P_i = \frac{i}{N}
$$

2).

$$
p \neq \frac{1}{2} \Longrightarrow \frac{1-p}{p} \neq 1 \Longrightarrow P_{i+1} = \frac{1 - \left(\frac{1-p}{p}\right)^{i+1}}{1 - \frac{1-p}{p}} \cdot P_1
$$

$$
1 = P_N = \frac{1 - \left(\frac{1-p}{p}\right)^{N}}{1 - \frac{1-p}{p}} \cdot P_1
$$

$$
\Longrightarrow P_1 = \frac{1 - \frac{1-p}{p}}{1 - \left(\frac{1-p}{p}\right)^{N}}
$$

$$
\begin{aligned}
\Longrightarrow P_i &= \frac{1 - \left(\frac{1-p}{p}\right)^{i}}{1 - \frac{1-p}{p}} \cdot \frac{1 - \frac{1-p}{p}}{1 - \left(\frac{1-p}{p}\right)^{N}} \\
&= \frac{1 - \left(\frac{1-p}{p}\right)^{i}}{1 - \left(\frac{1-p}{p}\right)^{N}}, \qquad i = 0, 1, 2, \ldots, N
\end{aligned}
$$

Suppose $N \to \infty$,

1). for any fixed i

$$
p = \frac{1}{2} \Longrightarrow \lim_{N\to\infty} P_i = \lim_{N\to\infty} \frac{1}{N} = 0
$$

2a).

$$
p < \frac{1}{2}, \ \frac{1-p}{p} > 1 \Longrightarrow \lim_{N\to\infty} P_i = 0
$$

2b).

$$
p > \frac{1}{2}, \ \frac{1-p}{p} < 1 \Longrightarrow \lim_{N\to\infty} P_i = 1 - \left(\frac{1-p}{p}\right)^i
$$

$$
\begin{aligned}
i \uparrow &\Longrightarrow \left(\frac{1-p}{p}\right)^i \downarrow \\
&\Longrightarrow 1 - \left(\frac{1-p}{p}\right)^i \uparrow
\end{aligned}
$$