# Lecture 22

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## Calculating expectation and variance by conditioning

**Defn:**

$$\mathbb{E}_X X = \mathbb{E}_Y(\mathbb{E}_{X|Y}(X \mid Y))$$

Also,

$$\mathbb{E}_X g(X) = \mathbb{E}_Y(\mathbb{E}_{X|Y}(g(X) \mid Y))$$

$$\overset{g(X)=X^2}{\Longrightarrow} \mathbb{E}_X X^2 = \mathbb{E}_Y(\mathbb{E}_{X|Y}(X^2 \mid Y))$$

$$\begin{aligned}
\operatorname{Var} X &= \mathbb{E}(X^2) - (\mathbb{E}X)^2 \\
&= \mathbb{E}(\mathbb{E}(X^2 \mid Y)) - (\mathbb{E}(\mathbb{E}(X \mid Y)))^2 \\
&= \mathbb{E}(\operatorname{Var}(X \mid Y) + (\mathbb{E}(X \mid Y))^2) - (\mathbb{E}(\mathbb{E}(X \mid Y)))^2 \\
&= \mathbb{E}(\operatorname{Var}(X \mid Y)) + \mathbb{E}((\mathbb{E}(X \mid Y))^2) - (\mathbb{E}(\mathbb{E}(X \mid Y)))^2 \\
&= \mathbb{E}(\operatorname{Var}(X \mid Y)) + \operatorname{Var}(\mathbb{E}(X \mid Y))
\end{aligned}$$

in stats, called ANOVA.

**Defn:** Compound model: $S = X_1 + X_2 + \cdots + X_N$, $N =$ the number of events $\in \{0, 1, 2, \dots\}$, $X_i$ are iid and independent of $N$.

$$\begin{aligned}
\mathbb{E}S &= \mathbb{E}(\mathbb{E}(S \mid N)) \\
&= \mathbb{E}(\mathbb{E}(X_1 + \cdots + X_N \mid N)) \\
&= \mathbb{E}(N \cdot \mathbb{E}X) \\
&= \mathbb{E}N \cdot \mathbb{E}X
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var} S &= \mathbb{E}(\operatorname{Var}(S \mid N)) + \operatorname{Var}(\mathbb{E}(S \mid N)) \\
&= \mathbb{E}(\operatorname{Var}(X_1 + \cdots + X_N \mid N)) + \operatorname{Var}(\mathbb{E}(S \mid N)) \\
&= \mathbb{E}(N \cdot \operatorname{Var}(X)) + \operatorname{Var}(N \cdot \mathbb{E}(X)) \\
&= \mathbb{E}N \cdot \operatorname{Var} X + (\mathbb{E}X)^2 \cdot \operatorname{Var} N
\end{aligned}$$

Suppose $N = \mathbb{E}N \implies \operatorname{Var} S = \mathbb{E}N \cdot \operatorname{Var} X$
Suppose $X = \mathbb{E}X \implies S = \mathbb{E}X \cdot N \implies \operatorname{Var} S = (\mathbb{E}X)^2 \operatorname{Var} N$

**E.g.:** Special case: $N \sim \text{Poisson}(\lambda)$; then $S$ follows the compound Poisson model

$$\mathbb{E}N = \lambda = \operatorname{Var} N$$

$$\implies \mathbb{E}S = \mathbb{E}N \cdot \mathbb{E}X = \lambda \mathbb{E}X$$

$$\begin{aligned}
\operatorname{Var} S &= \mathbb{E}N \cdot \operatorname{Var} X + (\mathbb{E}X)^2 \operatorname{Var} N \\
&= \lambda \operatorname{Var} X + \lambda(\mathbb{E}X)^2 \\
&= \lambda \mathbb{E}(X^2)
\end{aligned}$$

If $\lambda$ is large, then we often use the normal distribution to approximate the compound Poisson.

$$\mathbb{P}(S \le s) = \mathbb{P}\left(\frac{S - \mathbb{E}S}{\sqrt{\operatorname{Var} S}} \le \frac{s - \mathbb{E}S}{\sqrt{\operatorname{Var} S}}\right)$$

$$\begin{aligned}
\lambda \uparrow \implies \mathbb{P}(S \le s) &\approx \mathbb{P}\left(Z \le \frac{s - \mathbb{E}S}{\sqrt{\operatorname{Var} S}}\right) \\
&= \Phi\left(\frac{s - \lambda \mathbb{E}X}{\sqrt{\lambda \mathbb{E}(X^2)}}\right)
\end{aligned}$$

**E.g.:** Exs (not the compound model)

Last time Ex.3.11 mean of the geometric distribution, we computed it by conditioning on $Y = 1$ if first experiment was a success; 0, otherwise. $\implies \mathbb{E}N = \frac{1}{p}$

Ex3.19: Variance of the geometric:

Independent trials, each with probability $p$ of success, $N$ is the number of trials until $1^{\text{st}}$ success. Calculate $\operatorname{Var} N$.

Solution:

$$\begin{aligned}\operatorname{Var} N &= \mathbb{E}(N^2) - (\mathbb{E}N)^2 \\ &= \mathbb{E}(N^2) - \frac{1}{p^2}\end{aligned}$$

$$\begin{aligned}\mathbb{E}(N^2) &= \mathbb{E}(\mathbb{E}(N^2 \mid Y)) \\ &= \mathbb{E}(N^2 \mid Y = 1)\mathbb{P}(Y = 1) + \mathbb{E}(N^2 \mid Y = 0)\mathbb{P}(Y = 0) \\ &= 1^2 \cdot p + \mathbb{E}((1 + N)^2)(1 - p) \\ &= p + \mathbb{E}(1 + 2N + N^2)(1 - p) \\ &= p + (1 + 2 \cdot \frac{1}{p} + \mathbb{E}(N^2))(1 - p)\end{aligned}$$

$$\begin{aligned}p\mathbb{E}(N^2) &= p + (1 - p)\left(1 + \frac{2}{p}\right) \\ \mathbb{E}(N^2) &= 1 - \left(1 - \frac{1}{p}\right)\left(1 + \frac{2}{p}\right) \\ &= 1 - \left(1 + \frac{1}{p} - \frac{2}{p^2}\right) \\ &= -\frac{1}{p} + \frac{2}{p^2} \\ \operatorname{Var} N &= -\frac{1}{p} + \frac{2}{p^2} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1 - p}{p^2}\end{aligned}$$