Lecture 26

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Defn: Decomposition Theorem $N \sim \text{Poisson}(\lambda)$

- m types – independent
- Probabilities of the types p_j such that $\sum_{j=1}^{m} p_j = 1$
- N_j is the number of events of type j

then,

1. N_j are all independent of each other
2. $N = \sum_{j=1}^{m} N_j$
3. $N_j \sim \text{Poisson}(\lambda p_j)$

E.g.: Applications: the number of eggs is $N \sim \text{Poisson}(\lambda)$, each egg hatches into a chick w.p. p. Calculate $\mathbb{E}(N \mid K = k)$, in which K is the number of eggs that hatch.

$$N = K + (N - K)$$

where $K \sim \text{Poisson}(\lambda p)$ and $N - K \sim \text{Poisson}(\lambda(1-p))$ are independent.

$$\begin{aligned}
\Longrightarrow\ & \mathbb{E}(N \mid K = k) \\
=& \mathbb{E}(K + (N - K) \mid K = k) \\
=& \mathbb{E}(K \mid K = k) + \mathbb{E}(N - K \mid K = k) \\
=& k + \mathbb{E}(N - K) \\
=& k + \lambda(1-p)
\end{aligned}$$

E.g.: Aggregate insurance claims:

$N \sim \text{Poisson}(\lambda)$ is the number of insurance claims, or claim frequency. X_i is the size of the ith claim. $X_1, X_2, \ldots$ are iid, independent of N. (Compound Poisson model)

Additional assumption: X takes values in $\{1, 2, \ldots, m\}$ with probabilities $p_1, p_2, \ldots, p_m$.

Define N_j is the number of claims of size j, $j = 1, 2, \ldots, m$.

$$\Longrightarrow\ S = 1 \cdot N_1 + 2 \cdot N_2 + \cdots + m \cdot N_m$$

where $N_j \sim \text{Poisson}(\lambda p_j)$

Specific ex: For a compound Poisson(6) distribution, individual claims have pmf

$$\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \mathbb{P}(X = 4) = \frac{1}{3}$$

Use the Decomposition Theorem to compute $\mathbb{P}(S = x)$ for $x = 0, 1, 2, \ldots, 6$.

$$N_1 \sim \text{Poisson}(6 \cdot \frac{1}{3} = 2) \sim N_2 \sim N_4$$

$$S = 1 \cdot N_1 + 2 \cdot N_2 + 4 \cdot N_4$$

$$\mathbb{P}(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

x	$\mathbb{P}(2 \cdot N_2 = x)$	$\mathbb{P}(4 \cdot N_4 = x)$	$\mathbb{P}(2 \cdot N_2 + 4 \cdot N_4 = x)$	$\mathbb{P}(1 \cdot N_1 = x)$	$\mathbb{P}(S = x)$
0	e^{-2}	e^{-2}	e^{-4}	e^{-2}	e^{-6}
1	0	0	0	$2e^{-2}$	$2e^{-6}$
2	$2e^{-2}$	0	$2e^{-4}$	$2e^{-2}$	$4e^{-6}$
3	0	0	0	$\frac{4}{3}e^{-2}$	$\frac{16}{3}e^{-6}$
4	$\frac{2^2}{2!}e^{-2} = 2e^{-2}$	$2e^{-2}$	$4e^{-4}$	$\frac{2}{3}e^{-2}$	$\frac{26}{3}e^{-6}$
5	0	0	0		
6	$\frac{2^3}{3!}e^{-2} = \frac{4}{3}e^{-2}$	0	$\frac{16}{3}e^{-4}$		

$$
\begin{aligned}
M_S(t) &= \mathbb{E}(e^{St}) \\
&= \mathbb{E}(\mathbb{E}(e^{(X_1+\cdots+X_n)t} \mid N = n)) \\
&\overset{X_i \text{ ind't}}{=} \mathbb{E}_N\left(\prod_{i=1}^{n} \mathbb{E}(e^{X_i t}), N = n\right) \\
&= \mathbb{E}_N((M_X(t))^N) \\
&= \mathbb{E}_N(e^{N \ln M_X(t)})
\end{aligned}
$$

where $M_X(t) = \mathbb{E}(e^{Xt})$, $M_N(t) = e^{\lambda(e^t-1)}$. Given that $X = 1, 2, 4$ w.p. $\frac{1}{3}$ and $\lambda = 6$,

$$
\begin{aligned}
M_S(t) &= e^{\lambda(M_X(t)-1)} \\
&= e^{6\left(\frac{1}{3}(e^t+e^{2t}+e^{4t})-1\right)} \\
&= e^{2(e^t-1)} \cdot e^{2(e^{2t}-1)} \cdot e^{2(e^{4t}-1)} \\
&= M_{N_1} \cdot M_{2 \cdot N_2} \cdot M_{4 \cdot N_4}
\end{aligned}
$$