Lecture 16

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$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mathbb{E}X)(Y=\mathbb{E}Y)] \\ &= \mathbb{E}(XY) - \mathbb{E}X\cdot\mathbb{E}Y\end{aligned}$$

Assume $\mathrm{Var}\,X > 0$, $\mathrm{Var}\,Y > 0$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\,X\cdot\mathrm{Var}\,Y}}$$

$\rho_{X,Y} = \pm 1 \iff \exists a, b \in \mathbb{R} (a \neq 0)$ such that $\mathbb{P}(X = aY + b) = 1$
Proof:

1. $\rho_{X,Y} = 1 \implies \mathbb{P}(X = aY + b) = 1$ where $a > 0$, $\exists a, b$

$$\begin{aligned}1 = \rho_{X,Y} &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\,X\cdot\mathrm{Var}\,Y}} \\ 1 &= \frac{\mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)]}{\sqrt{\mathrm{Var}\,X\cdot\mathrm{Var}\,Y}} \\ 1 &= \mathbb{E}\left[\frac{X-\mathbb{E}X}{\sqrt{\mathrm{Var}\,X}}\cdot\frac{Y-\mathbb{E}Y}{\sqrt{\mathrm{Var}\,Y}}\right]\end{aligned}$$

Define $Z = \frac{X-\mathbb{E}X}{\sqrt{\mathrm{Var}\,X}}$ and $V = \frac{Y-\mathbb{E}Y}{\sqrt{\mathrm{Var}\,Y}}$

$$\mathbb{E}Z = 0 = \mathbb{E}V$$

$$\begin{aligned}\mathrm{Var}\,Z &= \mathbb{E}(Z^2) = 1 \\ \mathrm{Var}\,V &= \mathbb{E}(V^2) = 1\end{aligned}$$

$$\mathbb{E}(ZV) = 1$$

Consider $\mathbb{E}[(Z-V)^2] \geq 0$

$$\begin{aligned}&\implies \mathbb{E}[Z^2 - 2ZV + V^2] \\ &=\mathbb{E}(Z^2) - 2\mathbb{E}(ZV) + \mathbb{E}(V^2) \\ &=1 - 2 + 1 = 0\end{aligned}$$

$$\begin{aligned}\mathbb{E}[(Z-V)^2] = 0 &\implies \mathbb{P}[(Z-V)^2 = 0] = 1 \\ &\implies \mathbb{P}[(Z-V = 0] = 1 \\ &\implies \mathbb{P}[Z = V] = 1\end{aligned}$$

(Similarly if $\mathrm{Var}\,X = \mathbb{E}[(X-\mathbb{E}X)^2] = 0$, then $\mathbb{P}(X = \mathbb{E}X) = 1$)

$$\begin{aligned}1 &= \mathbb{P}\left(\frac{X-\mathbb{E}X}{\sqrt{\mathrm{Var}\,X}} = \frac{Y-\mathbb{E}Y}{\sqrt{\mathrm{Var}\,Y}}\right) \\ &= \mathbb{P}\left(X = \mathbb{E}X + \frac{\sqrt{\mathrm{Var}\,X}}{\sqrt{\mathrm{Var}\,Y}}(Y-\mathbb{E}Y)\right) \\ &= \mathbb{P}(X = aY + b), \qquad a = \frac{\sqrt{\mathrm{Var}\,X}}{\sqrt{\mathrm{Var}\,Y}} > 0\end{aligned}$$

2. $\rho_{X,Y} = -1$ then $\exists a < 0, b \in \mathbb{R}$ such that $\mathbb{P}(X = aY + b) = 1$.

   HW:

   (a) rewrite $\rho_{X,Y} = \mathbb{E}(ZV) = -1$
   (b) consider $\mathbb{E}[(Z+V)^2]$

If $\mathbb{P}(X = aY + b) = 1$ for some $a \neq 0, b \in \mathbb{R}$ then $\rho_{X,Y} = \pm 1$.

$$\begin{aligned}\rho_{X,Y} &= \frac{\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]}{\sqrt{\operatorname{Var} X \cdot \operatorname{Var} Y}} \\ &= \frac{\mathbb{E}[(aY + b - \mathbb{E}(aY + b))(Y - \mathbb{E}Y)]}{\sqrt{\operatorname{Var}(aY + b) \cdot \operatorname{Var} Y}} \\ &= \dots \\ &= \pm 1\end{aligned}$$

Sum of two independent random variables

Defn: Suppose X and Y are continuous, $Z = X + Y$. Then the joint pdf of X, Y is

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

to get the pdf of Z, first calculate the cdf of Z:

$$\begin{aligned}F_Z(z) &= \mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq Z) \\ &= \iint\limits_{x+y \leq z} f_{X,Y}(x, y)\ dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x, y)\ dxdy \\ &= \int_{-\infty}^{\infty} F_X(z - y) f_Y(y)\ dy\end{aligned}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\ dy$$

If $X, Y \geq 0$ rvs, then

$$f_Z(z) = \int_0^z f_X(z - y) f_Y(y)\ dy$$

E.g.: X, Y are independent $\operatorname{Exp}(\lambda)$ rvs $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$

$$\begin{aligned}f_Z(z) &= \int_0^z \lambda e^{-\lambda(z-y)} \cdot \lambda e^{-\lambda y}\ dy, \qquad z \geq 0 \\ &= \lambda^2 e^{-\lambda z} \int_0^z 1\ dy \\ &= \lambda^2 z e^{-\lambda z} \implies Z \sim \operatorname{Gamma}(2, \lambda)\end{aligned}$$

HW: Show $X + Y \sim \operatorname{Gamma}(\alpha + \beta, \lambda)$

$$\left.\begin{array}{l} X \sim \operatorname{Gamma}(\alpha, \lambda) \\ Y \sim \operatorname{Gamma}(\beta, \lambda) \end{array}\right\} \text{independent rvs}, \quad \alpha, \beta > 0$$