Lecture 08

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Defn: A discrete r.v. is one for which the cdf is a right-continuous step function $F(x)$. Instead of defining the the cdf for discrete r.v., we usually define its pmf p,

$$\begin{aligned} p(x) &= F(x) - F(x^-) \\ &= \mathbb{P}(X \le x) - \mathbb{P}(X < x) \end{aligned}$$

Common r.v.:
For $p \in (0,1)$, $n \in \mathbb{N}$

$$\begin{array}{ll} \mathrm{Bern}(p) & \mathbb{P}(X=k) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases} \\ \mathrm{Bin}(n,p) & \mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0,1,2,\ldots,n \\ \mathrm{Geom}(p) & \mathbb{P}(X=k) = (1-p)^{n-1}p, \quad n = 1,2,\ldots \\ \mathrm{Poisson}(\lambda) & \mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0,1,2,\ldots \end{array}$$

$$\sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

Continuous r.v.

Defn: A r.v. X is continuous if $\exists$ a non-negative function f defined on $\mathbb{R}$ such that

$$F(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f(t)\, dt.$$

f is called the probability density function (pdf) of X.

Fundamental theorem of calculus

$$F'(x) = f(x).$$

f must satisfy $\int_{-\infty}^{\infty} f(t)\, dt = 1$ because $\lim_{x\to\infty} F(x) = 1$.

For $a < b$,

$$\mathbb{P}(a < X \le b) = \mathbb{P}(X \le b) - \mathbb{P}(X \le a)$$

because

$$\mathbb{P}(X \le a) + \mathbb{P}(a < X \le b) = \mathbb{P}(X \le b).$$

$$\begin{aligned} \mathbb{P}(a < X \le b) &= F(b) - F(a) \\ &= \int_{-\infty}^{b} f(x)\, dx - \int_{-\infty}^{a} f(x)\, dx \\ &= \int_{a}^{b} f(x)\, dx \end{aligned}$$

Defn: Uniform: $X \sim U(a,b)$ for $a < b$,

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{else} \end{cases}$$

Defn: $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

If $\mu = 0$, $\sigma^2 = 1$, then we have the standard normal $X \in \mathbb{R}$ r.v..