Lecture 25

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E.g.: 3.60: Two players alternate flipping a coin that comes up heads w.p. p. The first one to obtain a head is the winner. Calculate the probability that the first player to flip the coin ultimately wins the game. Call the probability $f(p)$.

Solution: Let $Y = 1$ if the first flip is a head; 0, else.

$$\begin{aligned} f(p) &= \mathbb{P}(\text{I wins} \mid Y=1)\mathbb{P}(Y=1) + \mathbb{P}(\text{I wins} \mid Y=0)\mathbb{P}(Y=0) \\ &= 1 \cdot p + (1-f(p))(1-p) \\ &= p + 1 - f(p) - p + p \cdot f(p) \\ &= 1 - f(p) + p \cdot f(p) \end{aligned}$$

$$f(p)(2-p) = 1 \implies f(p) = \frac{1}{2-p} > \frac{1}{2}$$

E.g.: 3.29: Let $U_1, U_2, \ldots$ be a sequence of iif Unif$(0,1)$ r.v.s. Let $N = \min\{n \geq 2 : U_n > U_{n-1}\}$ and let $M = \min\{n \geq 1 : U_1 + \cdots + U_n > 1\}$. Surprisingly, N and M have the same probability distribution with common mean e.

Solution:

$$\mathbb{P}(N > n) = \mathbb{P}(U_1 > U_2 > \cdots > U_n) = \frac{1}{n!}$$

because all $n!$ orderings of $\{U_1, U_2, \ldots, U_n\}$ are equally likely.

To show $\mathbb{P}(M > n) = \frac{1}{n!}$, we use induction. Let $M(x) = \min\{n \geq 1 : U_1 + \cdots + U_n \geq x\}$, for $0 \leq x \leq 1$. Note that $M = M(1)$. Via induction, we now show that $\mathbb{P}(M(x) > n) = \frac{x^n}{n!}$. For $n = 1$,

$$\mathbb{P}(M(x) > 1) = \mathbb{P}(U_1 \leq x) = x = \frac{x^1}{1!}$$

So, assume that for some $n \geq 1$ and for all $0 \leq x \leq 1$, we have $\mathbb{P}(M(x) > n) = \frac{x^n}{n!}$. To compute $\mathbb{P}(M(x) > n+1)$, condition on U_1:

$$\begin{aligned} \mathbb{P}(M(x) > n+1) &= \mathbb{E}(\mathbb{P}(M(x) > n+1 \mid U_1)) \\ &= \int_0^1 \mathbb{P}(M(x) > n+1 \mid U_1)\, du \\ &= \int_0^x \mathbb{P}(M(x) > n+1 \mid U_1 = u)\, du + \int_x^1 \mathbb{P}(M(x) > n+1 \mid U_1 = u)\, du \\ &= \int_0^x \mathbb{P}(M(x-u) > n)\, du \\ &= \int_0^x \frac{(x-u)^n}{n!}\, du \\ &= -\frac{(x-u)^{n+1}}{(n+1)!}\Bigg|_{u=0}^{x} \\ &= \frac{x^{n+1}}{(n+1)!} \end{aligned}$$

where $U_1 + U_2 + \cdots + U_{n+1} \leq x$ implies $U_2 + \cdots + U_{n+1} \leq x - u$ and $\mathbb{P}(M(x) > n+1 \mid U_1 = u) = 0$ since $U_1 = u$ means $M(x) = 1$.

$$\mathbb{P}(M > n) = \frac{1}{n!} = \mathbb{P}(N > n)$$

$$\implies \mathbb{E}M = \mathbb{E}N = \int_0^\infty \mathbb{P}(N > x)\, dx = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

$$\mathbb{E}M = \mathbb{E}N = \sum_{n=0}^{\infty} \frac{1^n}{n!} = e^1 = e$$

Poisson Distribution

Defn: Composition Theorem:

If $N_j \sim \text{Poisson}(\lambda_j)$, $j = 1, 2, 3, \ldots, m$ with the N_j independent, then $N = \sum_{j=1}^m N_j \sim \text{Poisson}(\lambda)$, in which $\lambda = \sum_{j=1}^m \lambda_j$. Before we prove this, let's re-calculate the mgf of $N \sim \text{Poisson}(\lambda)$

Aside:

$$\begin{aligned} M_N(t) &= \mathbb{E}(e^{Nt}) \\ &= \sum_{n=0}^{\infty} e^{nt} \cdot e^{-\lambda} \frac{\lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)} \end{aligned}$$

Proof:

$$\begin{aligned} M_N(t) &= \prod_{j=1}^{m} M_{N_j}(t) \\ &= \prod_{j=1}^{m} e^{\lambda_j (e^t - 1)} \\ &= e^{\sum_{j=1}^m (\lambda_j (e^t - 1))} \\ &= e^{(\sum_{j=1}^m \lambda_j)(e^t - 1)} \end{aligned}$$

Defn: Decomposition Theorem:

Suppose the number of events $N \sim \text{Poisson}(\lambda)$. Suppose each event can be classified m to one of m types with corresponding probabilities $p_1, p_2, \ldots, p_m$ with $\sum_{j=1}^m p_j = 1$. We assume the occurrence of one type is independent of the occurrence of any type. Then, the number of events $N_1, N_2, \ldots, N_m$ corresponding to types $1, 2, \ldots, m$ are independent Poisson r.v.s with parameters $\lambda p_1, \lambda p_2, \ldots, \lambda p_m$.

Notation: $N_j \sim \text{Poisson}(\lambda p_j)$, $j = 1, 2, \ldots, m$ and $N_1, N_2, \ldots, N_m$ are independent,

$$N = \sum_{j=1}^{m} N_j$$