Lecture 30

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November 28, 2022

5.3 Poisson Process

5.3.1 Counting Processes

Defn: A stochastic process $\{N(t) : t \geq 0\}$ is a counting process if $N(0) = 0$, $N(t) =\in \{0, 1, 2, 3, \ldots\}$ and $s \leq t$ implies $N(s) \leq N(t)$. And $N(t) - N(s)$ represents the number of events during $(s, t]$.

Defn: A stochastic process $\{X(t) : t \geq 0\}$ has independent increments if $X(a) - X(b)$ is independent of $X(c) - X(d)$ for all $b \leq a$, $d \leq c$ with $(b, a] \cap (d, c] = \emptyset$. Also, $\{X(t) : t \geq 0\}$ has stationary increments if $X(t + h) - X(t)$ has the same distribution as $X(s + h) - X(s)$ for all $s, t, h \geq 0$.

5.3.2 Definition of a Poisson Process

Defn: A counting process $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda(\lambda$ is a positive constant) if
(i) the process has independent and stationary increments.
(ii) $N(t) \sim \mathcal{P}(\lambda t)$

Claim: For $s < t$, $N(t) - N(s) \sim\sim \mathcal{P}(\lambda(t - s))$.

Proof:

$$\begin{aligned} N(t) - N(s) &\sim N(t - s) - N(0) \\ &\sim \mathcal{P}(\lambda(t - s)) \end{aligned}$$

Aside: $N(t) - N(s)$ is independent of $N(s)$ because $(s, t] \cap [0, s] = \emptyset$

$$\begin{aligned} \overset{s<t}{\Longrightarrow} &\mathbb{E}(N(t) \mid N(s) = n) \\ =&\mathbb{E}((N(t) - N(s)) + N(s) \mid N(s) = n) \\ =&\mathbb{E}(N(t) - N(s) \mid N(s) = n) + \mathbb{E}(N(s) \mid N(s) = n) \\ =&\mathbb{E}(N(t) - N(s)) + n \\ =&\lambda(t - s) + n \end{aligned}$$

As a comparison, $\mathbb{E}(N(t)) = \lambda t$

E.g.: Interarrival and waiting time distributions:

Let T_1 to be the time of the first event, For $n = 2, 3, \ldots$, let T_n to be elapsed time between the $(n - 1)^{\text{th}}$ and n^{th} events. $T_1, T_2, T_3, \ldots$ is the sequence of interarrival times.

Prop 5.4: $T_1, T_2, \ldots$ are iid $\text{Exp}(\lambda)$.

First, $\forall t \geq 0$

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N(t) = 0) = e^{-\lambda t}$$

where $N(t) \sim \mathcal{P}(\lambda t)$.

$$\Longrightarrow T_1 \sim \text{Exp}(\lambda)$$

Next, $\forall s$, $s \geq t_1$

$$\begin{aligned} \mathbb{P}(T_2 > t \mid T_1 = t_1) &= \mathbb{P}(N(t_1 + t) - N(t_1) = 0 \mid N(t_1) = 1) \\ &= \mathbb{P}(N(t) = 0) = e^{-\lambda t} \end{aligned}$$

$$\Longrightarrow T_2 \Longrightarrow \text{Exp}(\lambda)$$

Continuing this argument

$$\begin{aligned} &\mathbb{P}(T_n > t \mid T_1 = t_1, T_2 = t_2, \ldots, T_{n-1} = t_{n-1}) \\ =&\mathbb{P}(N(t + t_1 + \cdots + t_{n-1}) - N(t_1 + \cdots + t_{n-1}) = 0 \mid N(t_1 + \cdots + t_{n-1}) = n - 1) \\ =&\mathbb{P}(N(t) = 0) = e^{-\lambda t} \end{aligned}$$

which is independent of $T_1, \dots, T_{n-1}$

$$\implies T_n \sim \text{Exp}(\lambda)$$