Math 381

Homework 5

Nov 5, 2021

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I design a game. I start with a score of zero and set a goal score of M. Set M = 10. On each turn, I roll a six-sided, fair die. If my score is greater than zero and my roll divides my score, I subtract the roll from my score. Otherwise, I add the roll to my score. If the score is M or greater, I won the game. Otherwise, if the score is a perfect square greater than 4, I lost the game. I continue rolling and changing my score until I either win or lose.

Problem:

- I will investigate the game using Markov chains and generate matrices with code.
- Find the expected value of the number of rolls made before the game ends. Report both the exact value (a rational number) and its decimal approximation.
- Write a simulation of the game to support my expected value computations.
- Find the probability to win this game.

Think about how I should do to solve this problem:

I will create a Markov chain to study this game, and will introduce how I create it.

There are only 11 states for this game: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, where states 0 through 9 represent my current score and state 10 indicates my current score exceeds 9.

By definition, absorbing states are states I never leave once I enter. In this game, I will enter an absorbing state when I win or lose, so states 9 (lose) and 10 (win) are absorbings states.

Define matrix A as the transition matrix for this Markov chain and each entry (a_{ij}) as the possibility of transiting from state i-1 to j-1 in one step for i, j ≤ M-1. Losing state and winning state will be put at the end of this transition matrix.

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Since there are 11 states and initial score is 0, A should be a 12*12 matrix and I start at state 0.

Define Q as a non-negative matrix that arises from the transition probabilities between non-absorbing states. Since there are 2 absorbing states, Q should be a 10*10 matrix.

Define I as the identity matrix with the same size as matrix Q and matrix N = $(I - Q)^{-1}$.

Define R as a matrix and each entry (r_{ij}) as the possibility of transiting from a non-absorbing state to the absorbing states. Since there are 9 non-absorbing states, R is a 9*2 matrix.

By theorem from lecture, we have the following true statements:

- The ij-th entry of N is the expected number of times that the chain will be in state j after starting in state i.
- The sum of the i-th row of N gives the mean number of steps until absorption when the chain is started in state i.
- The ij-th entry of the matrix B = NR is the probability that I will win the game after starting in the non-absorbing state i and ending in the absorbing state j.

Now, I will move on to the coding part to define the matrix introduced above.

Solving the problem:

I will use python to write the code below.

<u>Below is the code I write generate the transition matrix A and matrix Q:</u>

```
import numpy as np
import fractions
import math
np.set_printoptions(formatter={'all': lambda x: str(fractions.Fraction(x).limit_denominator())})
m = 10
def issquare(num):
```

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```
    return ((int(math.sqrt(num)))*math.sqrt(num)) == num and num > 4
A = [[0 for i in range (m+2)]for j in range (m+2)]
A[m+1][m+1] = 1
for i in range (0, m):
    for j in range (1, 7):
        if issquare(i+j) and i%j!=0:
            A[i][m] = 1/6
        elif issquare(i):
            for k in range (0, m+2):
                A[k][i] = 0
        elif (i+j < m) and ((i==0) or (i%j!=0)):
            A[i][i+j] = 1/6
        elif i!=0 and i%j==0:
            A[i][i-j] = 1/6
        elif i+j >= m:
            A[i][m+1] += 1/6
if m >= 10:
    A[m][m] = 1
```

<u>Below is the transition matrix that is generated :</u>

$$\begin{bmatrix} 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\ 1/6 & 1/6 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 1/6 & 0 & 1/6 & 0 & 0 & 1/6 & 0 & 1/6 & 1/6 & 0 & 1/6 & 0 \\ 1/6 & 0 & 1/6 & 1/6 & 0 & 0 & 0 & 1/6 & 0 & 0 & 1/6 & 1/6 \\ 1/6 & 0 & 0 & 0 & 1/6 & 0 & 0 & 1/6 & 1/6 & 0 & 1/6 & 1/6 \\ 1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 0 & 0 & 0 & 1/6 & 2/3 \\ 0 & 0 & 0 & 0 & 1/6 & 0 & 1/6 & 1/6 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Below is the matrix Q that is generated :</u>

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$$
\begin{bmatrix}
0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\
1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \\
1/6 & 1/6 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\
1/6 & 0 & 1/6 & 0 & 0 & 1/6 & 0 & 1/6 & 1/6 & 0 \\
1/6 & 0 & 1/6 & 1/6 & 0 & 0 & 0 & 1/6 & 0 & 0 \\
1/6 & 0 & 0 & 0 & 1/6 & 0 & 0 & 1/6 & 1/6 & 0 \\
1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/6 & 0 & 1/6 & 1/6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Then, by the definition of N above, I have N = $(I - Q)^{-1}$ =

$$
\begin{bmatrix}
\frac{1157637}{807644} & \frac{492165}{1615288} & \frac{159429}{403822} & \frac{722763}{1615288} & \frac{788259}{1615288} & \frac{818133}{1615288} & \frac{92610}{201911} & \frac{637023}{1615288} & \frac{181551}{807644} & 0 \\
\frac{833985}{1615288} & \frac{3632841}{3230576} & \frac{186405}{807644} & \frac{1352823}{3230576} & \frac{1452927}{3230576} & \frac{1460409}{3230576} & \frac{85200}{201911} & \frac{1539627}{3230576} & \frac{296571}{1615288} & 0 \\
\frac{793665}{1615288} & \frac{900009}{3230576} & \frac{953181}{807644} & \frac{837687}{3230576} & \frac{1067871}{3230576} & \frac{1434777}{3230576} & \frac{91932}{201911} & \frac{1511211}{3230576} & \frac{507099}{1615288} & 0 \\
\frac{670575}{1615288} & \frac{388407}{3230576} & \frac{247323}{807644} & \frac{3793641}{3230576} & \frac{800961}{3230576} & \frac{1226727}{3230576} & \frac{52992}{201911} & \frac{1366773}{3230576} & \frac{500805}{1615288} & 0 \\
\frac{324657}{807644} & \frac{194073}{1615288} & \frac{128781}{403822} & \frac{522135}{1615288} & \frac{1919295}{1615288} & \frac{375129}{1615288} & \frac{46266}{201911} & \frac{626859}{1615288} & \frac{117699}{807644} & 0 \\
\frac{554157}{1615288} & \frac{259797}{3230576} & \frac{112617}{807644} & \frac{512139}{3230576} & \frac{1082355}{3230576} & \frac{3722757}{3230576} & \frac{38898}{201911} & \frac{1134735}{3230576} & \frac{390447}{1615288} & 0 \\
\frac{174555}{403822} & \frac{84195}{807644} & \frac{39015}{201911} & \frac{283149}{807644} & \frac{304101}{807644} & \frac{305667}{807644} & \frac{240372}{201911} & \frac{209553}{807644} & \frac{62703}{403822} & 0 \\
\frac{51815}{807644} & \frac{28065}{1615288} & \frac{13005}{403822} & \frac{94383}{1615288} & \frac{103167}{1615288} & \frac{101889}{1615288} & \frac{40062}{201911} & \frac{1685139}{1615288} & \frac{20691}{807644} & 0 \\
\frac{30498}{201911} & \frac{8136}{201911} & \frac{18318}{201911} & \frac{24642}{201911} & \frac{54768}{201911} & \frac{22674}{201911} & \frac{54450}{201911} & \frac{56898}{201911} & \frac{212850}{201911} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

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By definition above, the sum of the first row of N is the expected value of the number of rolls made before the game ends. The number, after using calculation, is 7515315/1615288, and the decimal approximation is 4.6526.

I also wrote a simulation for this game in Matlab. The code is below:

```
clear; close all; clc
% Generate simulation for 100000 times and calculate the average number of rolls made before the game ends.
% Record the number of games played.
number = 0;
% Record the number of rolls in a game before the game ends.
total = 0;
while number < 100000
    number = number + 1;
    m = 10;
    n = 0;
    % Record the number of rolls that have been made.
    count = 0;
    % Continue playing if the total score is smaller than the goal score.
    while n < m
        j = randi([1 6], 1);
        % Stop automatically if the total score is a perfect square greater than 4.
        if issquare(n)
            break;
        % Otherwise subtract the number on the dice from the total score if the score is not 0 and the number on the dice is a factor of the score.
        elseif mod(n, j) == 0 && n > 0
            n = n - j;
        % Otherwise add the number on the dice to the total score.
        else
            n = n + j;
        end
        % Increase the number of rolls by 1 after rolling the dice once.
        count = count + 1;
    end
    total = total + count;
end
% Calculate the average number of rolls in a game before the game ends for 100000 games.
average = total/100000.0;
% Return true only if the input integer is a perfect square greater than 4.
function y = issquare(x)
    y = (mod(sqrt(x), 1) == 0 && x > 4);
end
```

I did the simulation for ten rounds and got a mean number of average number of rolls in a game as 4.6492. There is a difference of 0.0034.

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The result I get from the simulation isn't the same as the expected number. That's because 4.6526 is an expected number only and the result for the number of rolls in a game depends on the situation. Thus, I could assume that there is evidence to show that the expected value I get is true.

By definition, B = N*R. I calculated it and listed the table below:

	1	2
1	0.3061	0.6939
2	0.2995	0.7005
3	0.2503	0.7497
4	0.3708	0.6292
5	0.3553	0.6447
6	0.3329	0.6671
7	0.2275	0.7725
8	0.2046	0.7954
9	0.1312	0.8688

By definition of matrix B, as $M > 9$ (there will be no losing state if $M \leq 9$), for $1 \leq i \leq 9$, ($b_{(i-1)1}$) from matrix B represents the probability of losing if I start from a score of i-1 and ($b_{(i-1)2}$) from matrix B represents the probability of winning if I start from a score of i-1.

Thus, the probability of winning this game if I start at a score of 0 is about 0.6939.

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Challenge:

Calculate this probability of winning with various M with my transition matrix.

Think about how I should do to solve this problem by using Markov chains:

Recall: Absorbing states are states that I will never leave once I enter.

I will lose if my score is a perfect square greater than 4 and is not my goal score.

I will win if my score becomes no smaller than my goal score after a roll.

Note: If the largest possible integer in my set of states is a perfect integer which is greater than 4, I will succeed if my final score is equal to that integer.

In general, absorbing states are states with a positive square integer greater than 4 or the goal score. Since I want my transition matrix always in the canonical form, there will be only two absorbing states in my transition matrix: one represents the losing state (my score is a perfect square greater than 4 and is not my goal score), and the other one represents the winning state (my score becomes no smaller than my goal score after a roll).

The definition of matrix A (transition matrix), Q, I, R, N, B are all the same as I described above.

Here are some clarification for the size of these matrices and helpful information:

There are always M+1 states (0, 1, 2, ... M) and state M+1 indicates my score exceeds M-1.

Since there are M+1 states and initial score is 0, A is a (M+2)*(M+2) matrix and I start at state 0.

Suppose there are a total of n absorbing states in my Markov Chains, then the size of matrix I, Q, and N are all (M+1-n) by (M+1-n), and the size of matrix B and R are both (M+1-n) by 2.

The three theorems I described at page 2 stay correct.

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Table:

First, define the probability of winning as P(M) when my goal score is M.

I change the value of M to 5, 7, 10, 14, 19, 25, 31, 39, 48, 58, 70, 85, 100, 200, 300 separately, since I want to include a large range of M to find the behavior of the probability of winning with different M. I recorded the related data and results in the table below:

M	P(M) (Keep 8 significant digits)
5	1
7	1
10	0.69394746
14	0.59051089
19	0.34849448
25	0.30042595
31	0.17450393
39	0.12958651
48	0.09946952
58	0.04724791
70	0.02749194
85	0.01492614
100	0.01263408
200	0.00061159
300	6.8475491e-05

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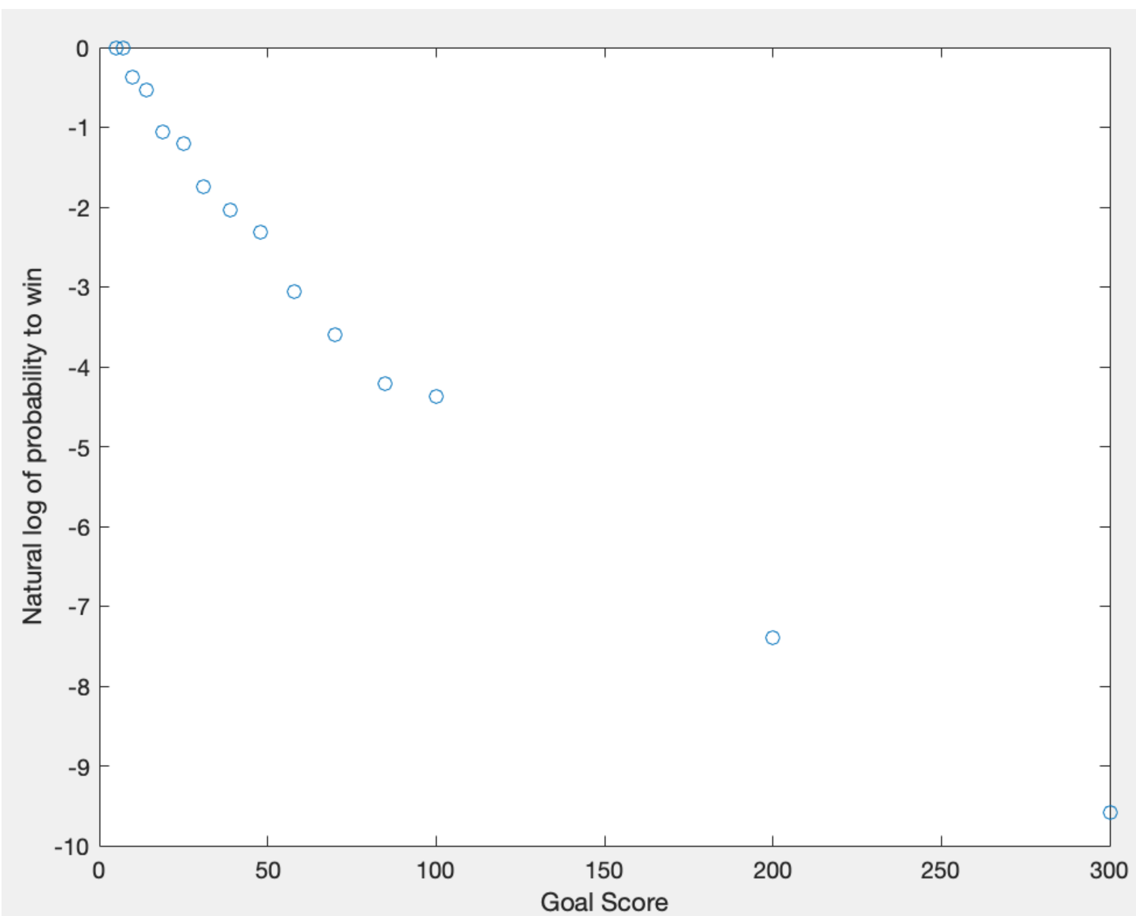
Some interesting things I found by the table and rules:

- When $M \leq 9$, $P(M)$ is always 1 since there is no state to lose.
- When $M \geq 9$, $P(M)$ is always less than 1 since there is at least one state to lose.
- $P(M)$ will decrease as M increases (phenomenon appears in the graph).

Find the relationship between M and P(M):

I will find $r(M) < P(M) < s(M)$ for $5 \leq M \leq 300$ for some $r(M)$ and $s(M)$ that are "close together".

I create a plot for the relationship between M and P(M) according to the data from the table:



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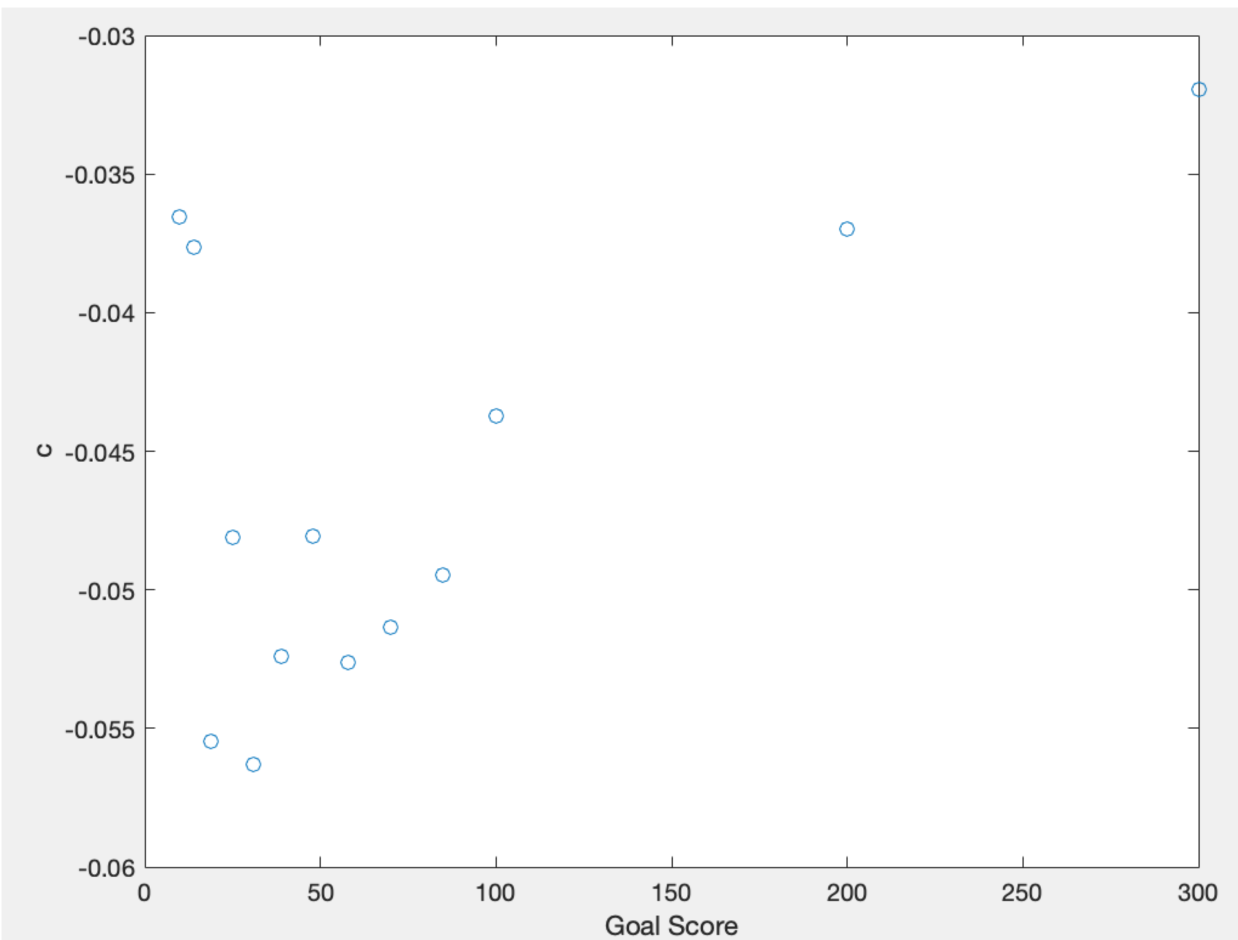
Now, I want to model P(M) with an exponential function for $M > 9$ ($P(M) = 1$ for all $M \leq 9$) where c is a positive constant.

My next goal is to find the interval that includes c for $P(M) = e^{-cM}$ when $5 \leq M \leq 300$.

To find an appropriate c to fit the function above, I create a plot for the relationship between M and -ln(P(M))/M (represented by c) for $M > 9$ according to the data from the table above:



We could see that the value of c is in the interval of (-0.06, -0.03).

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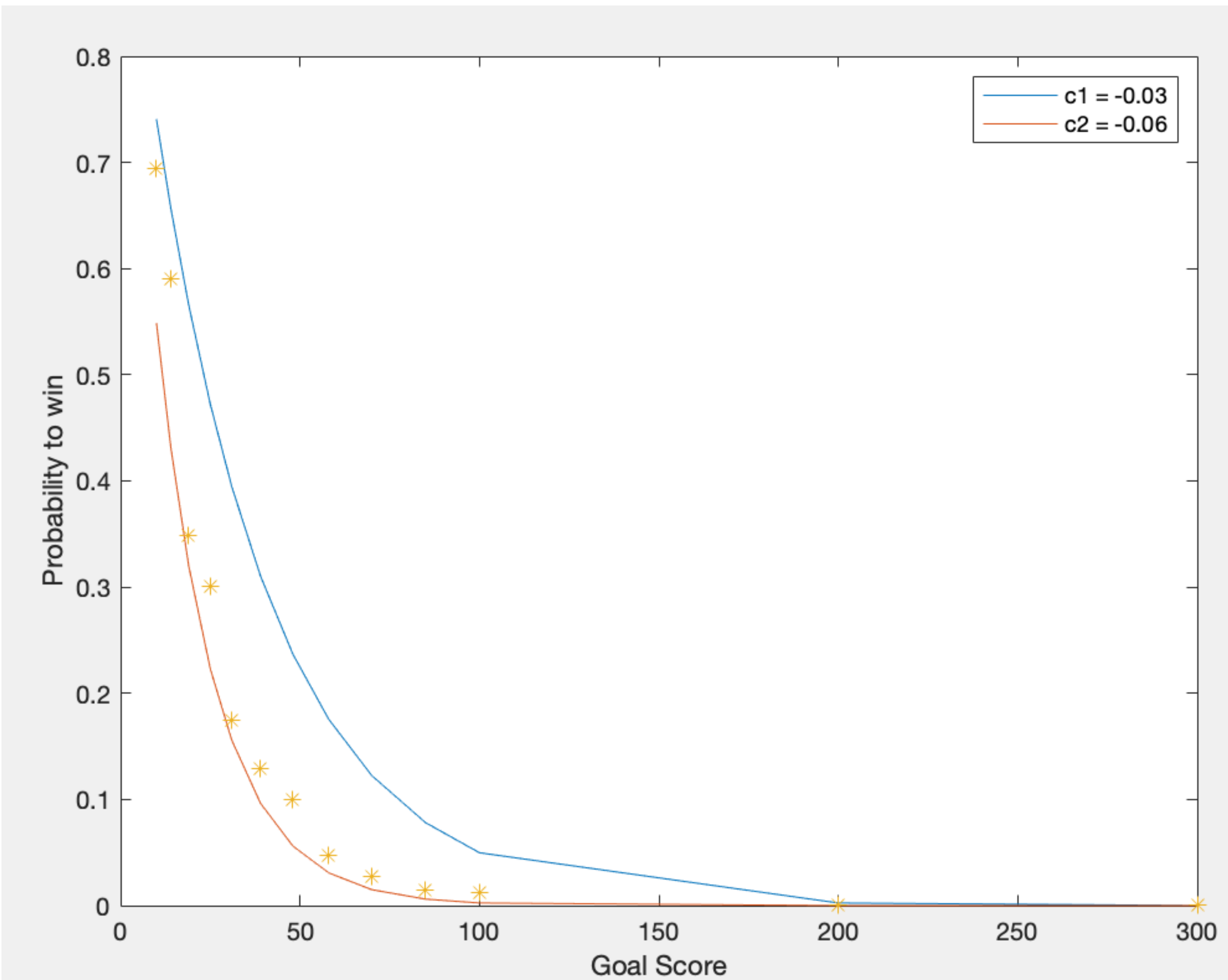
Finally, I plot the goal score greater than 9 in my table above with related winning probability, function $P(M) = e^{-0.03M}$, and function $P(M) = e^{-0.06M}$ in a same graph:



Final conclusion:

Define P(M) as the possibility of winning the game when the goal score is M, then:

- $P(M) = 1$ when $1 \leq M \leq 9$.
- We could have $e^{-0.06M} \leq P(M) \leq e^{-0.03M}$ when $10 \leq M \leq 300$.