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Predator and Prey's Population Model

Abstract

This paper constructs a population growth model between predators and prey. The net growth rates are interrelated: the growth of prey depends on the amount eaten by the predator and the growth of the predator depends on the amount of food source, which is prey.

Besides, the growth of predators and prey is constrained by the environment's carrying capacity. We assume only the food chain will affect the population of predators and prey to find out the stable equilibrium population.

Problem Description

In the Lotka-Volterra equations (P214-215, Topics in Mathematical Modeling), only the natural rate of reproduction and food chain is considered as factors that affect the population growth rate of prey and predators. However, the resources in nature are limited, so the carrying capacity of each environment is also limited. Thus, when the population of prey and predators change, the environment also affected on the result: as the number of the prey and predators are far below the environment's carrying capacity, the environment has little affection on the growth rate, which is primarily decided by the natural growth rate; as the number of prey and predators are closed to the carrying capacity, the environment's affection is huge and will decrease the natural growth rate a lot. Thus, to find out a valid and stable population equilibrium that describes what nature really reaches, it is necessary to put the Lotka-Volterra into the Logistic Population Model.

There is one similar model created by the Tung, K. K. (P222, Topics in Mathematical Modeling):

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - axy$$

$$\frac{dy}{dt} = -ky + bxy$$

In this model, Tung, K. K. considered the growth rate and death rate of the prey separately. Thus, when fitting the Lotka-Volterra equations into the Logistic Population Model, Tung, K. K. let the prey's natural growth rate equals to the growth rate of the prey, r. However, in this model, the growth rate and death rate of the prey is considered as a whole, letting the prey's natural growth rate equals to the rate that prey's population change.

Besides, Tung, K.K. assumed only the number of prey is constrained by environment. But in this model, both the number of prey and predators is constrained by the environment.

Simplification

In this case, suppose the carrying capacity of both prey and predators are the same by the environment, which is represented by K.

Furthermore, suppose the population of the prey and predator is only related to the food chain, that is the predators eat prey and cause the population of prey to decrease. In the natural environment, there are many other reasons that affect the population growth rates of predators and prey, such as fishing, environmental pollution (Pollution's Effect on Animals), and climate change (Climate Change and Its Impact on Animals). But in this case, these factors are ignored to simplify the model.

Moreover, the age structure has a huge impact on the natural growth rate, since the number of adult prey and predators can change the growth rate of them, and the number of old prey and predators can also change the death rate of them. In this model, the impact of age structure of the prey and predators is ignored.

The sex ratio also has a huge impact on the natural growth rate, since only when female and male appear in pairs are they able to reproduce. If the ratio is not balanced, their reproduction rate will decrease a lot. To simplify the model, the sex ratio is always balanced, which means reproductive rate does not change as time varies (5).

Mathematical Model

According to the Logistic Population Model, the population growth rate is shown as the following (Wikipedia):

$$\frac{dP}{dt} = RP(1 - \frac{P}{K})$$

P: Population size

R: population growth rate

K: carrying capacity

Let x to represent the population size of the prey, and let y to represent the population size of the predators.

According to the Lotka-Volterra equations, the natural growth rate of the predators and prey are shown as the following (P214-215, Topics in Mathematical Modeling):

$$\frac{dx}{dt} = rx - axy,\ \frac{dy}{dt} = bxy - ky$$

r: growth rate of prey

ay: per capita death rate of prey

bx: growth rate of predators at a per capita rate

k: per capita death rate of predators

To fit the natural growth rate model into the Logistic Population Model, substitute R by $rx - axy$ and $bxy - ky$:

$$\frac{dx}{dt} = (rx - axy)x(1 - \frac{x}{K})$$
$$\frac{dy}{dt} = (bxy - ky)y(1 - \frac{y}{K})$$

Solution of the Mathematical Model

The process of solving the Predator and Prey's Population Model is shown below (P215, Topics in Mathematical Modeling):

$$\frac{dx}{dt} = (rx - axy)\,x\,(1 - \frac{x}{K}) = f(x, y)$$

$$\frac{dy}{dt} = (bxy - ky)\,y(1 - \frac{y}{K})$$

$x(t)$ and $y(t)$ are the interaction of two species. f and g are in general nonlinear functions of x and y.

The equilibrium point (x^*, y^*) could be found by solving the simultaneous algebraic equations (P.209 9.3, Topics in Mathematical Modeling):

$$f(x^*, y^*) = 0$$

$$g(x^*, y^*) = 0$$

Both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are equal to zero, which means that both the population of preys and predators are not changing.

To have $f(x^*, y^*) = 0$, $x^* = 0$, or $x^* = K$, or $(rx^* - ax^*y^*) = 0$, or $(bx^*y^* - ky^*) = 0$.

When $x^* = 0$, plug it into $g(x^*, y^*) = 0$:

$$(-ky^*)\,y^*\,(1 - \frac{y^*}{K}) = 0$$

$$y^* = 0 \text{ or } y^* = K$$

The equilibria are:

$$(x^*, y^*) = (0,0) \qquad ①$$

$$(x^*, y^*) = (0, K) \qquad ②$$

When $x^* = K$, plug it into $g(x^*, y^*) = 0$:

$$y^*(bK - k)\,y^*\,(1 - \frac{y^*}{K}) = 0$$

$$y^* = 0 \text{ or } y^* = K \text{ or } bK - k = 0\ (y^* \textit{ could be any real number})$$

In this condition (the population of prey equals the carrying capacity K), if $bK - k = 0$, the population of predators could be any real number to reach the equilibrium state for the system. However, the population of predators cannot exceed the carrying capacity. Thus, this condition is meaningless. Otherwise, the equilibria are:

$$(x^*, y^*) = (K, 0) \qquad ③$$

$$(x^*, y^*) = (K, K) \qquad ④$$

When $(rx^* - ax^*y^*) = 0$, $y^* = \frac{r}{a}$, plug it into $g(x^*, y^*) = 0$:

$$(bx^* - k)(\frac{r}{a})^2(1 - \frac{r}{aK}) = 0$$

Since

$$\frac{r}{a} \neq 0$$

Then

$$x^* = \frac{k}{b} \text{ or } K = \frac{r}{a}$$

In this condition,

$$y^* = \frac{r}{a}$$

If $K = \frac{r}{a}$, the population of predators could be any real number to reach the equilibrium state for the system. However, the population of predators cannot exceed the carrying capacity. Thus, this condition is meaningless. Otherwise, the equilibrium is:

$$(x^*, y^*) = (\frac{k}{b}, \frac{r}{a}) \qquad ⑤$$

When $(bx^*y^* - ky^*) = 0$, $x^* = \frac{k}{b}$, plug it into $f(x^*, y^*) = 0$:

$$(r - ay^*)(\frac{k}{b})^2(1 - \frac{k}{bK}) = 0$$

Since

$$\frac{k}{b} \neq 0$$

Then

$$y^* = \frac{r}{a} \text{ or } K = \frac{k}{b}$$

If $K = \frac{k}{b}$, the population of predators could be any real number to reach the equilibrium state for the system. However, the population of predators cannot exceed the carrying capacity. Thus, this condition is meaningless. Otherwise, the equilibrium is:

$$(x^*, y^*) = (\frac{k}{b}, \frac{r}{a})$$

The result is same with ⑤.

After calculating the equilibria, linearize the nonlinear equations about the equilibrium solutions to use the small perturbations from the equilibrium to determine equilibria's stability (P.209 9.5, Topics in Mathematical Modeling):

$$x(t) = x^* + u(t)$$

$$y(t) = y^* + v(t)$$

$$\frac{dx}{dt} = \frac{d}{dt}x^* + u = \frac{d}{dt}u$$

$$\frac{dy}{dt} = \frac{d}{dt}v$$

Expand f and g about the equilibrium in a Taylor Series (P.209 9.6, Topics in Mathematical Modeling):

$$f(x, y) = f(x^*, y^*) + \frac{df}{dx}(x^*, y^*)\,u + \frac{df}{dx}(x^*, y^*)\,v$$
$$+ \text{ terms involving } u^2, v^2, uv, u^3, v^3, \text{ etc}$$

$$a_{11} = \frac{df}{dx}(x^*, y^*) = (r - ay)(2x - \frac{3x^2}{K})$$
$$a_{12} = \frac{df}{dy}(x^*, y^*) = -ax^2(1 - \frac{x}{K})$$

$$f(x, y) \cong a_{11}u + a_{12}v$$

Similarly,

$$a_{21} = \frac{dg}{dx}(x^*, y^*) = by^2(1 - \frac{y}{K})$$

$$a_{22} = \frac{dg}{dy}(x^*, y^*) = (bx - k)(2y - \frac{3y^2}{K})$$

$$g(x, y) \cong a_{21}u + a_{22}v$$

After linearization, the nonlinear system transforms to linear system:

$$\frac{du}{dt} = a_{11}u + a_{12}v,$$
$$\frac{dv}{dt} = a_{21}u + a_{22}v$$

Try exponential solutions for this linear system with constant coefficients, substituting in the above linear system:

$$u(t) = u_0e^{\lambda t}, v(t) = v_0e^{\lambda t}$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = 0$$

To have nontrivial solutions:

$$det\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = 0$$

Obtaining its characteristic equation:

$$\lambda^2 - (a_{11} + a_{12})\lambda + (a_{11}a_{22} \quad a_{12}a_{21}) = 0$$

$$\lambda^2 - p\lambda + q = 0$$

Where

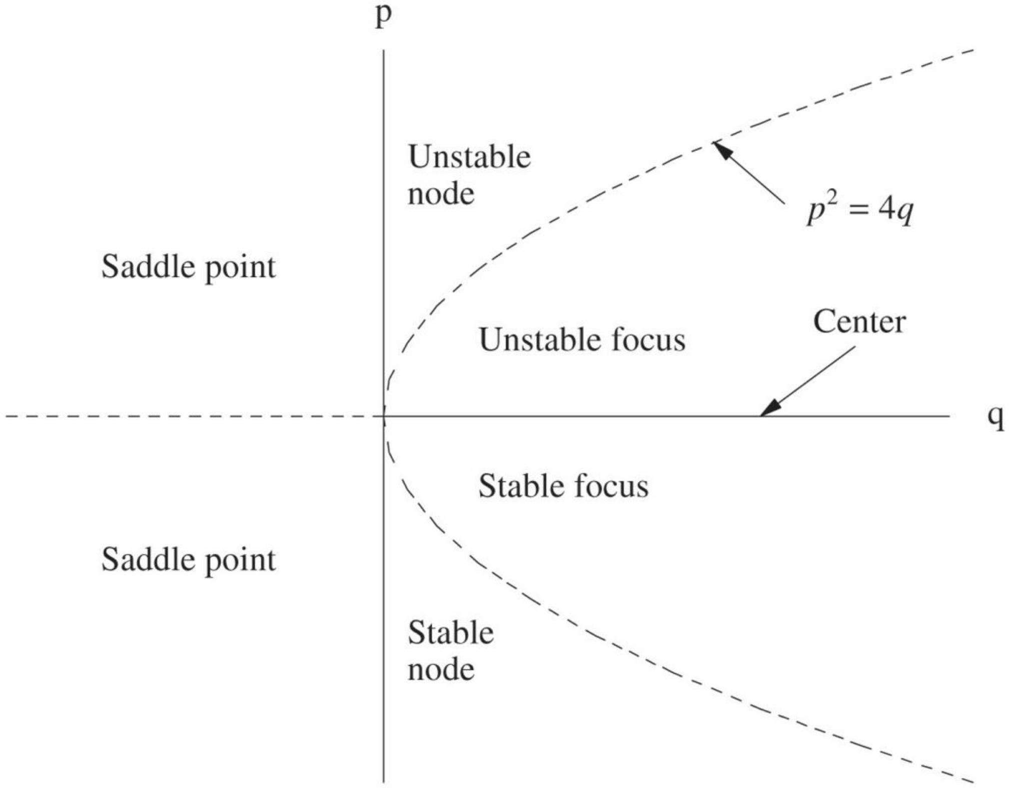
$$p = a_{11} + a_{22} = T_r\{A\},$$

$$q = a_{11}a_{22} - a_{12}a_{21} = detA$$

are the trace and determinant of matrix A. After solving the characteristic equation, there are two roots (P.211 9.11, Topics in Mathematical Modeling):

$$\lambda_1 = \frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2}, \lambda_1 = \frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}$$

The two parameters p and q determine the stability of the system. Refer to the graph below (p.212, Topics in Mathematical Modeloing):



For the equilibrium ①, when x = 0 and y = 0,

$$\Rightarrow \ a_{11} = \ a_{12} = \ a_{21} = \ a_{22} = 0,$$

$$\Rightarrow p = q = 0.$$

According to the above figure, (0, 0) is an unstable equilibrium.

For the equilibrium ②, when x = 0 and y = K,

$$\Rightarrow\ a_{11} = a_{12} = a_{21} = 0, \text{ and } a_{22} = kK,$$

$$\Rightarrow p = kK \text{ and } q = 0.$$

Since $k > 0$ and $K > 0$, $p > 0$.

According to the above figure, (0, K) is an unstable equilibrium.

For the equilibrium ③, when x = K and y = 0,

$$\Rightarrow\ a_{12} = a_{21} = a_{22} = 0, \text{ and } a_{11} = -rK,$$

$$\Rightarrow p = -rK \text{ and } q = 0.$$

Since $r > 0$ and $K > 0$, $p < 0$

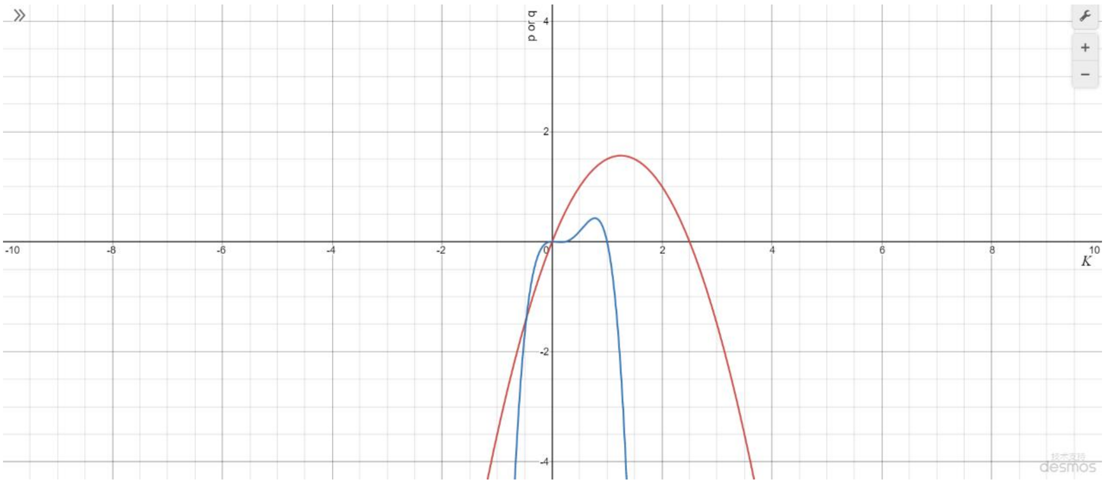
According to the above figure, (K, 0) is an unstable equilibrium.

For the equilibrium ④, when x = y = K,

$$\Rightarrow\ a_{12} = a_{21} = 0,\ a_{11} = -rK + aK^2 \text{ and } a_{22} = -bK^2 + kK,$$

$$\Rightarrow p = (a-b)K^2 + (k-r)K = \text{and } q = K^3(rb + ak) - K^2(rk + abK^2)$$

Since the parameters are too closely related and equations are too complicated, it is hard to write out a simple expression for p and q. In this situation, a graph is better to show the result.



Suppose $r = 0.5$, $k = 3$, $a = 2$, and $b = 3$. The graph is like the above one.

To check for the different result when r, k, a, and b changed, go over

https://www.desmos.com/calculator/bllswtv7bi?lang=zh-CN

For equilibrium ⑤, when x = k/b and y = r/a,

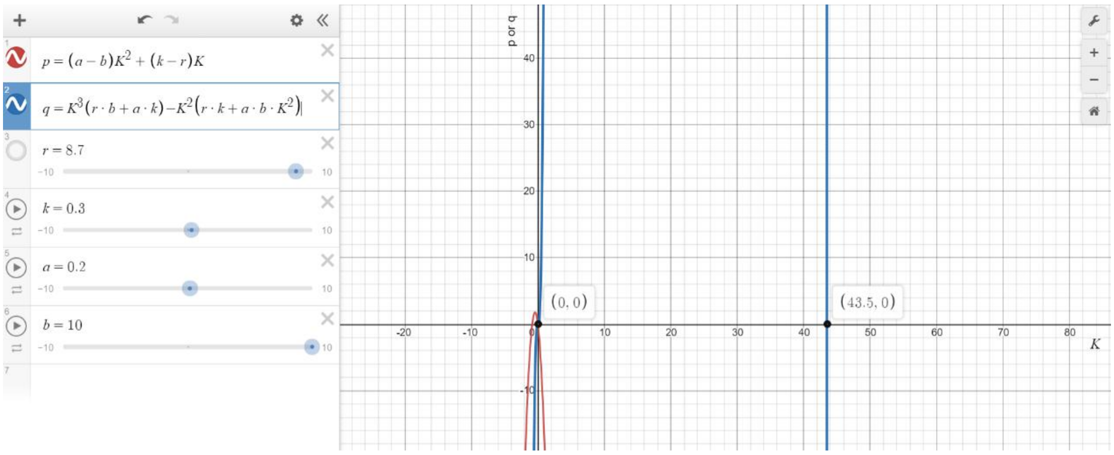
$$\Rightarrow a_{11} = a_{22} = 0 \text{ and } a_{12} = \frac{ak^3}{Kb^3} - \frac{ak^2}{b^2} \text{ and } a_{21} = \frac{br^2}{a^2} - \frac{br^3}{Ka^3},$$

$$\Rightarrow p = 0 \text{ and } q = \frac{r^2k^2}{ab}(1 - \frac{k}{bK})(1 - \frac{r}{aK})$$

Since $p = 0$ holds true, $(\frac{k}{b}, \frac{r}{a})$ is an unstable equilibrium.

Result Discussion

According to the model, when $(x, y) = (0,0)$, or $(x, y) = (0, K)$, or $(x, y) = (K, 0)$, or $(x, y) = (\frac{k}{b}, \frac{r}{a})$, at least one of p and q has a value of 0. This means that it is impossible for these four equilibria to be stable. When $(x, y) = (K, K)$, since there are plenty of variables involved, there are many possible combinations of variables' values that could make the equilibrium stable (to make $p < 0$ and $q > 0$). For example, when $r = 8.7$, $k = 0.3$, $a = 0.2$, $b = 10$, the system can be stable if $0 < K < 43.5$.



Improvement

There are some faults in this model.

First, since both predators and prey are living in the open environment, the effect of disease on both species must be considered because disease could not only affect the number of predators and prey but also the pregnancy rate of predators and prey. Thus, as the disease spreads, the birth rate and death rate of both predators and prey start changing as time varies. Then, in this situation, the growth rate of prey, r, and the death rate of predator, k, is no longer a constant. It is necessary to build equations to monitor r and k and find the stable situation.

Second, juvenile predators are not able to prey until they become mature, so the time that requires juvenile predators to become mature needs to be considered. There should be an extra term representing the transferring rate of baby prey to adult prey. Also, not all juvenile predators could prey when they become mature (some of them might die before becoming mature, others might be born with serious defects), and the factors that could lead to this phenomenon may vary. When the predator problem is taken into account, the proportionality of per capita death rate of prey, a, and the proportionality of growth rate of predators, b, are not constant anymore. It is necessary to build equations to monitor a and b and find the stable situation.

Third, since the influential environment factors are different with different species, the carrying capacity to the two species prey and predator should be different. If adding this improvement to the original Predator and Prey's Population model, the nonlinear differential equation system becomes:

$$\frac{dx}{dt} = (rx - axy)x(1 - \frac{x}{K_1})$$
$$\frac{dx}{dt} = (bxy - ky)y(1 - \frac{y}{K_2})$$

Due to the same condition that the population of predators or preys cannot exceed the carrying capacity, the modified system also have five different equilibria:

$$(x^*, y^*) = (0,0) \qquad ①$$

$$(x^*, y^*) = (0, K_2) \qquad ②$$

$$(x^*, y^*) = (K_1, 0) \qquad ③$$

$$(x^*, y^*) = (K_1, K_2) \qquad ④$$

$$(x^*, y^*) = (\frac{k}{b}, \frac{r}{a}) \qquad ⑤$$

Comparing to the original Predator and Prey's Population Model, the carrying capacities in the equilibria are different. Other than that, it is all the same. According to the solution above:

For the equilibrium ①, when x = 0 and y = 0,

$$\Rightarrow a_{11} = a_{12} = a_{21} = a_{22} = 0,$$

$$\Rightarrow p = q = 0.$$

According to the above figure, (0, 0) is an unstable equilibrium.

For the equilibrium ②, when x = 0 and y = K_2,

$$\Rightarrow a_{11} = a_{12} = a_{21} = 0, \text{ and } a_{22} = kK_2,$$

$$\Rightarrow p = kK_2 \text{ and } q = 0.$$

Since k > 0 and $K_2 > 0$, p > 0.

According to the above figure, (0, K_2) is an unstable equilibrium.

For the equilibrium ③, when x = K_1 and y = 0,

$$\Rightarrow a_{12} = a_{21} = a_{22} = 0, \text{ and } a_{11} = -rK_1,$$

$$\Rightarrow p = -rK_1 \text{ and } q = 0.$$

Since r > 0 and $K_1 > 0$, p < 0

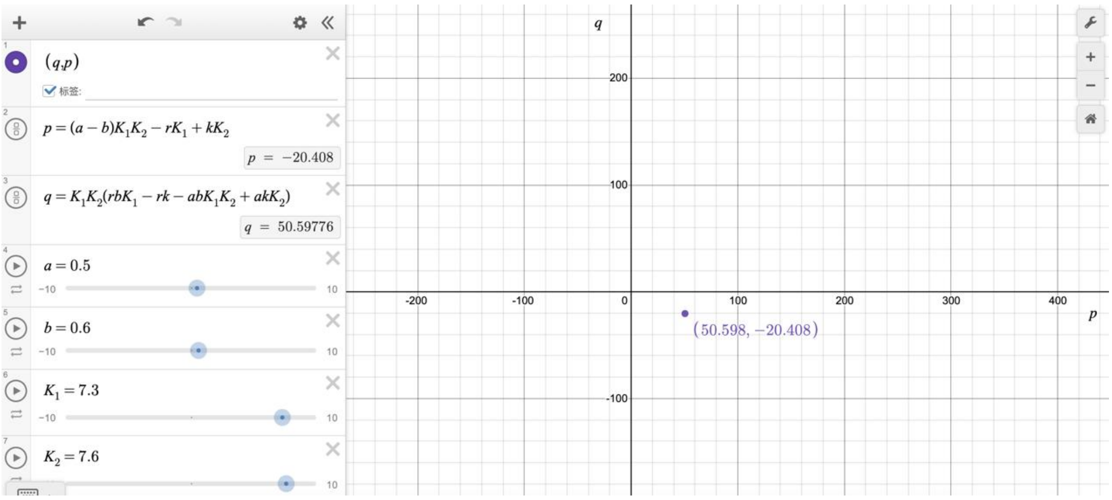
According to the above figure, (K_1, 0) is an unstable equilibrium.

For the equilibrium ④, when x = K_1 y = K_2,

$$\Rightarrow a_{12} = a_{21} = 0,\ a_{11} = -rK_1 + aK_1K_2 \text{ and } a_{22} = -bK_1K_2 + kK_2,$$

$$\Rightarrow p = (a-b)K_1K_2 - rK_1 + kK_2 \text{ and } q = K_1K_2(rbK_1 - rk - abK_1K_2 + akK_2)$$

Since the parameters are too closely related and equations are too complicated, it is hard to write out a simple expression for p and q. In this situation, a graph is better to show the result.



Suppose $r = 6.2$, $k = 4$, $a = 0.5$, $b = 0.6$, $K_1 = 7.3$, and $K_2 = 7.6$. The graph is like the above one. In this result, q > 0 and p < 0. It is a stable equilibrium.

To check for the different result when r, k, a, and b changed, go over https://www.desmos.com/calculator/umhrhbdywz?lang=zh-CN

For equilibrium ⑤, when x = k/b and y = r/a,

$$\Rightarrow a_{11} = a_{22} = 0 \text{ and } a_{12} = \frac{ak^3}{K_1 b^3} - \frac{ak^2}{b^2} \text{ and } a_{21} = \frac{br^2}{a^2} - \frac{br^3}{K_2 a^3},$$

$$\Rightarrow p = 0,\ q = -\frac{k^3 r^2}{K_1 b^2 a} + \frac{k^3 r^3}{K_1 K_2 b^2 a^2} + \frac{k^2 r^2}{ba} - \frac{k^2 r^3}{b K_2 a^2}$$

Since $p = 0$ holds true, $(\frac{k}{b}, \frac{r}{a})$ is an unstable equilibrium.

Thus, the system can be stable when the population of predators and preys reach equilibrium ④ $(x^*, y^*) = (K_1, K_2)$ with certain condition.

Conclusion

Overall, the Predator and Prey's Population model is fully built in the paper. The model exists five equilibria, and only the equilibrium $(x^*, y^*) = (K, K)$ exists the condition that the populations of prey and predator are stable.

By making multiple changes on the value of a, b, r, k and analyzing the equation of p and the equation of q, we could find out that as long as $a < b$ and $k < r$, $p < 0$ and q > 0 will hold true before K becomes too large: this is because the highest power of K is 4 in the equation of q and the factor it is $-ab$, which is always smaller than 0. This means if K gets too large, q will change from positive to negative. A possible explanation for this phenomenon is that the natural source for species is limited. If there are too many animals in one species to stay alive, the total resource cannot satisfy the requirement, which will lead to a decrease, even extinction.

Citation

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