```
> read "C:/Users/DELL/Desktop/ISSAC2025CR/src/AddDecomp.txt";
> read "C:/Users/DELL/Desktop/ISSAC2025CR/src/DField.txt";
> read "C:/Users/DELL/Desktop/ISSAC2025CR/src/LogPart.txt";
> read "C:/Users/DELL/Desktop/ISSAC2025CR/src/PrimitiveCR.txt";
> read "C:/Users/DELL/Desktop/ISSAC2025CR/src/PrimitiveInt.txt";
> read "C:/Users/DELL/Desktop/ISSAC2025CR/src/Rational.txt";
> read "C:/Users/DELL/Desktop/ISSAC2025CR/src/Reduction.txt";
```

```
> # load modules
> with(AddDecomp); with(DField); with(LogPart);with(PrimitiveTower);with
  (PrimitiveInt);with(Rational);with(Reduction);
                                [SPrimitive]
[AlgebraicDerivative, AssociatedMatrix, Derivative, FromFunctionalToRational,
   FromRationalToFunctional, Tower]
[CompleteLogarithmicPart, IsACompleteLogarithmicPart, LogarithmicPart,
   VerifyLogPart]
[AuxiliaryReduction, Basis, CollectTowerInfo, CompleteReduction, Projection,
   Verification]
[AlgRaab, ConstantCoefficient, DecompositionOfGenerators,
   DecompositionOfRemainder, DifferentiateResidue, LocalResidue,
   LogarithmicPartI, PolynomialMatrix, PrimInt]
[BasisElement, CoeffLaurent, Coefficient, EEA, ExtendedEuclidean,
   ExtendedEuclideanOld, HdegAndHcoeff, HeadTerm, LaurentCoeffAndMonomial,
   LaurentMatryoshka, LinearReduction, Matryoshka, NormalizedBasis,
   NormalizedElement, NormallyProperAndLaurentPolynomialParts,
   NumerAndMonicDenom, PadicExpand, ProperAndPolynomialParts, TdegAndTcoeff]
[GCanonicalForm, GKernelAndShellReduction, GKernelReduction, GShellReduction,      (1)
   GShellReduction1, HermiteReduce, Indicator, LaurentPolynomialReduction,
   LaurentPolynomialReduction1, LaurentPolynomialReduction2, ResidualForm0]
```

Example 3.13

```
> # set up a tower
> K313 := x, [t], [1/x];
```

$$K313 := x,\ [t],\ \left[\frac{1}{x}\right] \tag{1.1}$$

```
> # associate pairs
> CollectTowerInfo(K313);
```

$$\left[\left[0,\ \frac{1}{x},\ \frac{1}{x},\ 1,\ \left[\left[t,\ \frac{1}{x}\right]\right]\right]\right] \tag{1.2}$$

```
> # input an integrand
> f := ((x+1)*t^2+(x^2+2*x+2)*t+x+1)/(x*(t+1));
```

$$f := \frac{(x+1)\ t^2 + (x^2+2\ x+2)\ t + x + 1}{x\ (t+1)} \tag{1.3}$$

```
> # initial reduction
> (g, s, p) := op(HermiteReduce(K313, f));
```

$$g,\ s,\ p := 0,\ -\frac{x}{t+1},\ \frac{t\ x + x^2 + t + x + 1}{x} \tag{1.4}$$

```
> # auxiliary reduction
> (q,r) := AuxiliaryReduction(K313, p);
```

$$q,\ r := t\ x + \frac{1}{2}\ x^2,\ \frac{t}{x} + \frac{1}{x} \tag{1.5}$$

```
> # projection
> (u,v) := Projection(K313, r, _TowerInfo[1][3], _TowerInfo[1][4]);
```

$$u,\ v := \frac{1}{2}\ t^2 + t,\ 0 \tag{1.6}$$

```
> # verification
> normal(Derivative(K313, g+q+u)+ (s+v) - f);
```

$$0 \tag{1.7}$$

```
> # complete reduction
> (I_Part, NI_Part) := CompleteReduction(K313, f);
```

$$I\_Part,\ NI\_Part := t\ x + \frac{1}{2}\ x^2 + \frac{1}{2}\ t^2 + t,\ -\frac{x}{t+1} \tag{1.8}$$

```
> # verification
> Verification(K313, f, I_Part, NI_Part);
```

$$correct \tag{1.9}$$

Example 4.4

```
> # set up a tower
> K44 := x, [t1, t2], [1/(x-1), -t1/x];
```

$$K44 := x,\ [t1,\ t2],\ \left[\frac{1}{x-1},\ -\frac{t1}{x}\right] \tag{2.1}$$

```
> # associate pairs
> CollectTowerInfo(K44);
```

$$\left[\left[0,\ \frac{1}{x-1},\ \frac{1}{x-1},\ 1,\ \left[\left[t1,\ \frac{1}{x-1}\right],\ \left[\frac{1}{2}\ t1^2,\ \frac{t1}{x-1}\right]\right]\right],\ \left[0,\ -\frac{t1}{x},\ \frac{t1}{x},\ -1,\ \left[\left[t2,\ -\frac{t1}{x}\right]\right]\right]\right] \tag{2.2}$$

```
> # input an integrand
> f := (((x-1)^2*t1+x)*t2^3+x*(x-1)*t1)/(x^2*(x-1)*t2^2); ft := t2^2;
```

$$f := \frac{\left((x-1)^2\ t1 + x\right)\ t2^3 + x\ (x-1)\ t1}{x^2\ (x-1)\ t2^2}$$

$$ft := t2^2 \tag{2.3}$$

```
> # initial reduction
> (g, s, p) := op(HermiteReduce(K44,f)); (gt, st, pt) := op(HermiteReduce
  (K44,ft));
```

$$g,\ s,\ p := \frac{1}{t2},\ 0,\ \frac{\left(t1\ x^2 - 2\ t1\ x + t1 + x\right)\ t2}{x^2\ (x-1)}$$

$$gt,\ st,\ pt := 0,\ 0,\ t2^2 \tag{2.4}$$

```
> # auxiliary reduction
> (q,r) := AuxiliaryReduction(K44, p); (qt,rt) := AuxiliaryReduction(K44,
  pt);
```

$$q,\ r := \frac{t1\ t2}{x} - \frac{t1^2}{x} + t1^2,\ \frac{t1\ t2}{x} - \frac{2\ t1}{x}$$

$$qt,\ rt := x\ t2^2 + (2\ t1\ x - 2\ t1 - 2\ x)\ t2 + 2\ x\ t1^2 - 2\ t1^2 - 6\ x\ t1 + 6\ t1 + 6\ x,\ -\frac{2\ t1^2}{x} \tag{2.5}$$

```
> # projection
> (u,v) := Projection(K44, r, _TowerInfo[2][3], _TowerInfo[2][4]); (ut,vt)
  := Projection(K44, rt, _TowerInfo[2][3], _TowerInfo[2][4]);
```

$$u,\ v := -\frac{1}{2}\ t2^2 + 2\ t2,\ 0$$

$$ut,\ vt := 0,\ -\frac{2\ t1^2}{x} \tag{2.6}$$

```
> # verification
> normal(Derivative(K44, g+q+u)+ (s+v) - f); normal(Derivative(K44, gt+qt+
  ut)+ (st+vt) - ft);
```

$$0$$

$$0 \tag{2.7}$$

```
> # complete reduction
> (I_Part, NI_Part) := CompleteReduction(K44, f); (I_Part_t, NI_Part_t) :=
  CompleteReduction(K44, ft);
```

$$I\_Part,\ NI\_Part := \frac{1}{t2} + \frac{t1\ t2}{x} - \frac{t1^2}{x} + t1^2 - \frac{1}{2}\ t2^2 + 2\ t2,\ 0$$

$$I\_Part\_t,\ NI\_Part\_t := x\ t2^2 + (2\ t1\ x - 2\ t1 - 2\ x)\ t2 + 2\ x\ t1^2 - 2\ t1^2 - 6\ x\ t1 + 6\ t1 + 6\ x,\ -\frac{2\ t1^2}{x} \tag{2.8}$$

```
> # verification
> Verification(K44, f, I_Part, NI_Part); Verification(K44, ft, I_Part_t,
  NI_Part_t);
```

$$correct$$

$$correct \tag{2.9}$$

Example 4.5 (not yet available because there is no complete reduction for algebraic functions in Maple)

Example 5.4

```
> # set up a tower
> K54 := x, [t1, t2, t3], [1/(x-1), (1-t1)/x, 1/t1+1/x];
```

$$K54 := x,\ [t1,\ t2,\ t3],\ \left[\frac{1}{x-1},\ \frac{-t1+1}{x},\ \frac{1}{t1}+\frac{1}{x}\right] \tag{4.1}$$

```
> # associate pairs
> CollectTowerInfo(K54);
```

$$\left[\left[0,\ \frac{1}{x-1},\ \frac{1}{x-1},\ 1,\ \left[\left[t1,\ \frac{1}{x-1}\right],\ \left[\frac{1}{2}\ t1^2,\ \frac{t1}{x-1}\right]\right]\right],\ \left[0,\ -\frac{t1}{x}+\frac{1}{x},\ \frac{t1}{x},\ -1,\ \left[\left[t2,\ -\frac{t1}{x}+\frac{1}{x}\right]\right]\right],\ \left[0,\ \frac{x+t1}{x\ t1},\ \frac{1}{x},\ 1,\ \left[\left[t3,\ \frac{x+t1}{x\ t1}\right]\right]\right]\right] \tag{4.2}$$

```
> # input an integrand
> f := (x+(x-1)*t2)/((x-1)*t1)+(t2+(1-t1)*t3)/x;
```

$$f := \frac{x+(x-1)\ t2}{(x-1)\ t1}+\frac{t2+(-t1+1)\ t3}{x} \tag{4.3}$$

```
> # determine elementary integrability
> g := PrimInt(K54,f);
```

$$g := \ln(t1)+t2\ t3-\ln(x)+t3 \tag{4.4}$$

```
> # verification
> F := subs({t1 = log(1-x), t2 = log(x)+polylog(2, x), t3 = ln(x)-Li(1-x)}
  , f); G := subs({t1 = log(1-x), t2 = log(x)+polylog(2, x), t3 = ln(x)-Li
  (1-x)},g);
```

$$F := \frac{x+(x-1)\ (\ln(x)+\mathrm{polylog}(2,\ x))}{(x-1)\ \ln(-x+1)}+\frac{\ln(x)+\mathrm{polylog}(2,\ x)+(-\ln(-x+1)+1)\ (\ln(x)-\mathrm{Li}(-x+1))}{x}$$

$$G := \ln(\ln(-x+1))+(\ln(x)+\mathrm{polylog}(2,\ x))\ (\ln(x)-\mathrm{Li}(-x+1))-\mathrm{Li}(-x+1) \tag{4.5}$$

```
> normal(diff(G,x)-F);
```

$$0 \tag{4.6}$$

```
> # try int
> int(F,x);
```

$$\int \left(\frac{x + (x-1)\ (\ln(x) + \mathrm{polylog}(2,\ x))}{(x-1)\ \ln(-x+1)} + \frac{\ln(x) + \mathrm{polylog}(2,\ x) + (-\ln(-x+1)+1)\ (\ln(x) - \mathrm{Li}(-x+1))}{x} \right) dx \tag{4.7}$$

Example 5.4+ (removed from the submitted version due to lack of space)

```
> # set up a tower
> K := x, [t1, t2, t3], [1/x, 1/t1-1/(x+1), 1/(t1+1)+1/(x+2)];
```

$$K := x,\ [t1,\ t2,\ t3],\ \left[\frac{1}{x},\ \frac{1}{t1} - \frac{1}{x+1},\ \frac{1}{t1+1} + \frac{1}{x+2}\right] \tag{5.1}$$

```
> # associate pairs
> CollectTowerInfo(K);
```

$$\left[\left[0,\ \frac{1}{x},\ \frac{1}{x},\ 1,\ \left[\left[t1,\ \frac{1}{x}\right]\right]\right],\ \left[0,\ \frac{1}{t1} - \frac{1}{x+1},\ \frac{1}{x+1},\ -1,\ \left[\left[t2,\ \frac{1}{t1} - \frac{1}{x+1}\right]\right]\right],\ \left[0,\ \frac{x+3+t1}{(t1+1)\ (x+2)},\ \frac{1}{x+2},\ 1,\ \left[\left[t3,\ \frac{x+3+t1}{(t1+1)\ (x+2)}\right]\right]\right]\right] \tag{5.2}$$

```
> # input an integrand
> f := (2*t1^2*t3*x+2*t1*t3*x^2+6*t1*t3*x+2*t1*x^2+4*t1*x+x^2+x-2)/((t1+1)
  *t1*x*(x+2));
```

$$f := \frac{2\ t1^2\ t3\ x + 2\ t1\ t3\ x^2 + 6\ t1\ t3\ x + 2\ t1\ x^2 + 4\ t1\ x + x^2 + x - 2}{(t1+1)\ t1\ x\ (x+2)} \tag{5.3}$$

```
> # determine elementary integrability
> g := PrimInt(K,f);
```

$$g := \ln(t1+1) - \ln(t1) + t3^2 + \ln(x+1) - \ln(x+2) + t2 + t3 \tag{5.4}$$

```
> # verification
> F := subs([t1 = log(x), t2 = -log(x+1)+Li(x), t3 = log(x+2)+int(1/(log
  (x)+1), x)],f); G:=subs([t1 = log(x), t2 = -log(x+1)+Li(x), t3 = log
  (x+2)+int(1/(log(x)+1), x)],g);
```

$$F := \frac{1}{(\ln(x)+1)\ \ln(x)\ x\ (x+2)} \left(2\ \ln(x)^2\ \left(\ln(x+2) - e^{-1}\ \mathrm{Ei}(1,\ -\ln(x) - 1)\right)\ x + 2\ \ln(x)\ \left(\ln(x+2) - e^{-1}\ \mathrm{Ei}(1,\ -\ln(x) - 1)\right)\ x^2 + 6\ \ln(x)\ \left(\ln(x+2) - e^{-1}\ \mathrm{Ei}(1,\ -\ln(x) - 1)\right)\ x + 2\ \ln(x)\ x^2 + 4\ x\ \ln(x) + x^2 + x - 2\right)$$

$$G := \ln(\ln(x)+1) - \ln(\ln(x)) + \left(\ln(x+2) - e^{-1}\ \mathrm{Ei}(1,\ -\ln(x) - 1)\right)^2 + \mathrm{Li}(x) - e^{-1}\ \mathrm{Ei}(1,\ -\ln(x) - 1) \tag{5.5}$$

```
> simplify(evala(normal(diff(G,x)-F)));
```

$$0 \tag{5.6}$$

```
> # try int
> int(F,x);
```

$$\int \frac{1}{(\ln(x)+1)\ln(x)\,x\,(x+2)}\left(2\ln(x)^2\left(\ln(x+2)-e^{-1}\,\mathrm{Ei}(1,-\ln(x)-1)\right)x+2\ln(x)\left(\ln(x+2)-e^{-1}\,\mathrm{Ei}(1,-\ln(x)-1)\right)x^2+6\ln(x)\left(\ln(x+2)-e^{-1}\,\mathrm{Ei}(1,-\ln(x)-1)\right)x+2\ln(x)\,x^2+4\,x\ln(x)+x^2+x-2\right)dx \tag{5.7}$$

Example 5.7

```
> # set up a tower

> K57 := y, [t], [1/(x+y)];
```

$$K57 := y,\ [t],\ \left[\frac{1}{x+y}\right] \tag{6.1}$$

```
> # associate pairs
> CollectTowerInfo(K57);
```

$$\left[\left[0,\ \frac{1}{x+y},\ \frac{1}{x+y},\ 1,\ \left[\left[t,\ \frac{1}{x+y}\right]\right]\right]\right] \tag{6.2}$$

```
> # input a function
> f := 2*x/(t^2*x+t^2*y-x^2-x*y);
```

$$f := \frac{2\,x}{t^2\,x+t^2\,y-x^2-x\,y} \tag{6.3}$$

```
> # complete reduction
> (g0, r0) := CompleteReduction(K57, f);
```

$$g0,\ r0 := 0,\ \frac{2\,x}{(x+y)\,(t^2-x)} \tag{6.4}$$

```
> # complete reduction
> (g1,r1) := CompleteReduction(K57, Derivative(x, [t], [1/(x+y)], f));
```

$$g1,\ r1 := \frac{\left(\frac{1}{2}\,x+\frac{1}{2}\,y\right)\left(-\frac{2\,t}{x+y}+\frac{4\,x}{(x+y)^2}\right)}{t^2-x},\ \frac{1}{(x+y)\,(t^2-x)} \tag{6.5}$$

```
> # look for a C(x)-linear relation between r0 and r1
> normal(2*x*r1-r0);
```

$$0 \tag{6.6}$$

```
> # the minmal telescoper is 2*x*Dx - 1
```

```
> # find a certificate
> g := 2*x*g1-g0;
```

$$g := \frac{2\left(\frac{1}{2}\ x + \frac{1}{2}\ y\right)\left(-\frac{2\ t}{x+y} + \frac{4\ x}{(x+y)^2}\right)\ x}{t^2 - x} \tag{6.7}$$

```
> # verification
> normal(2*x*Derivative(x, [t], [1/(x+y)], f) - f - Derivative(K57,g));
```

$$0 \tag{6.8}$$