Log-Sobolev Inequalities for Quantum Markovian Semigroups on Finite Dimensional Spaces

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Contents

1 Introduction

This text is a survey of the recent progress on the log-Sobolev inequalities for quantum Markovian semigroups on finite dimensional spaces. Physically, we care about the steady state behaviors of a certain type of quantum dissipative dynamics which can be modeled by the Lindblad equation, whose underlying mathematical structure is quantum Markovian semigroups in the essence. Just like the case of classical Markov chain Monte Carlo (MCMC), we care about the convergence and the rate of mixing of the dynamics to the steady distribution. We also care about the rate of convergence of quantum Markov semigroups in the non-commutative settings. Similar to the classical characterization of the steady distribution by the detailed balance condition, we will also establish the quantum detialed balance condition and study the ergodic properties of quantum Markov semigroups.

Perhaps the simplest estimation of the convergence rate is the spectral gap method. Formally, we have the following convergence in L_2 space:

$$\|\mathcal{P}_t - \mathcal{P}_t \circ E\|_{L_2} \leq e^{-\lambda(\mathcal{L}_*)t}. \tag{1.1}$$

Here $\mathcal{P}_t$ is the quantum Markov semigroup, $\mathcal{L}_*$ is the generator of the semigroup, and $\lambda(\mathcal{L}_*)$ is the spectral gap. E can be viewed as the projection to the "subspace of invariant states" (in fact it is also a subalgebra). Even in the quantum settings, the spectral gap exhibits a quite valuable property called *tensorization*. Basically, it means that we can study the convergence to equilibrium for high-dimensional generators by reductions to the two-point space setting. However, this optimal property comes with a dimensional price to relate the L_2 norm to the more operationally meaningful trace distance chosen in the definition of the mixing time. This often results in suboptimal upper bounds on the mixing time.

To recover the true behavior, more powerful analytic tools are needed. One of them is the so-called hypercontractivity which asks at which $1 \leq p \leq 2$, $\|\mathcal{P}_t\|_{L_p}^{L_2} \leq 1$ holds, i.e. at which $\mathcal{P}_t$ is a contraction from L_p to L_2. The hypercontractivity property of a QMS is related to a positive constant, coined as the log-Sobolev constant $\alpha_2(\mathcal{L})$, which provides a finer characterization of the convergence to equilibrium than the spectral gap. Nonetheless, the tensorization property also holds for the log-Sobolev constant *under some special assumptions*. However, the quantum system in general does not allow us to establish the tensorization property as in the classical case. The main obstucle, as we will see, is that we lack a convinient way to interprete the *conditioning* in non-commutative settings.

As is often considered in non-commutative analysis, one possible strategy to retrieve the highly precious tensorization property is to consider the *complete* version of these constants, e.g. the complete log-Sobolev constant $\alpha_{2,c}(\mathcal{L}) = \inf_k \alpha_2(\mathcal{L} \otimes \mathrm{id}_{M_k(\mathbb{C})})$ resulting from appending a trivial "reference system" of arbitarily large dimension. Unfortunately, we will prove that it is *impossibile* for us to obtain a *non-trival* complete log-Sobolev constant.

Motivated by this, we aim to find a slighly weaker version of the log-Sobolev inequality while (1) still being able to provide the asymptotically optimal estimation of mixing time for certain systems; (2) being able to provide a non-trivial complete log-Sobolev constant. The literature on mixing times of classical Markov chains provides us with a natural candidate: the *modified log-Sobolev inequality*, which describes the rate of exponential convergence of the QMS towards its equilibrium value in relative entropy

$$H(\mathcal{P}_t(\rho)\|\mathcal{P}_t \circ E(\rho)) \leq e^{-\alpha_1(\mathcal{L})t} H(\rho\|E(\rho)). \tag{1.2}$$

Though we do not know whether $\alpha_1(\mathcal{L})$ tensorizes in general in quantum settings, it is possible for us to show that $\alpha_1(\mathcal{L})$ is lower-bounded non-trivially for a large class of reversible QMS. In fact, we can also introduce an *approximate tensorization* method to provide even tighter bound for "weak-correlated" systems. Due to space limitations, we do not elaborate on this here. We will also provide an important example arising in quantum information theory to show how these analytic tools can be useful in the study of quantum Markovian dynamics.

2 Preliminaries

In this section, we introduce the notations and basic concepts that will be used throughout the text and are intended to help readers understand the material. For most of the results presented in this section, we will state them without proof and refer to the original papers for detailed proof. This section will have a relatively dense and compact writing style.

2.1 Notations and basic concepts

Throughout this text, $\mathcal{H}$ denotes a **finite-dimensional** Hilbert space (viewed as $\mathbb{C}^d$), $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$, which can be considered as $M_d(\mathbb{C})$. We often use $\mathcal{M} \subset \mathcal{B}(\mathcal{H}) = M_d(\mathbb{C})$

to denote a von Neumann algebra, which is equivalent to a unital C^*-algebra in the finite-dimensional case.

We use $\langle\cdot,\cdot\rangle$ to denote the Hilbert-Schmidt inner product on $\mathcal{B}(\mathcal{H})$, i.e., for $A, B \in \mathcal{B}(\mathcal{H})$, we have $\langle A, B\rangle = \mathrm{Tr}(A^*B)$. Tr is the standard trace on $M_d(\mathbb{C})$, and we denote the operator 2-functional (or 2-norm) by $\|A\|_2 := [\mathrm{Tr}(A^*A)]^{\frac{1}{2}} = \mathrm{Tr}\left(|A|^2\right)^{\frac{1}{2}}$. More generally, we denote the operator p-functional by $\|A\|_p := [\mathrm{Tr}(|A|^p)]^{\frac{1}{p}}$ for $1 \le p \le \infty$. We have the trace Hölder inequality:

$$\|A_1\cdots A_m\|_r \le \|A_1\|_{p_1}\cdots\|A_m\|_{p_m}, \quad \frac{1}{p_1}+\cdots+\frac{1}{p_m} = r, \quad 0 \le p_1, \cdots, p_m, r \le \infty. \tag{2.1}$$

In fact, we have a variational formula of the p-functional $\|A\|_p = \sup\{\mathrm{Tr}(AX) : \|X\|_q = 1\}, 1 \le p \le \infty$ where q is the Hölder conjugate of p. This supremum can actually be achieved.

We say $\rho \in \mathcal{B}(\mathcal{H}) = M_d(\mathbb{C})$ is a density operator if $\rho \ge 0$ and $\mathrm{Tr}(\rho) = 1$. We denote the set of density operators on $\mathcal{H}$ by $\mathcal{D}(\mathcal{H})$, and the set of full-rank (invertible) density operators by $\mathcal{D}_+(\mathcal{H})$.

Given a full-rank density operator $\sigma \in \mathcal{D}_+(\mathcal{H})$, we define the weighted p-functional as

$$\|A\|_{p,\sigma} := \left\|\sigma^{\frac{1}{2p}}A\sigma^{\frac{1}{2p}}\right\|_p = \mathrm{Tr}\left(\left|\sigma^{\frac{1}{2p}}A\sigma^{\frac{1}{2p}}\right|^p\right)^{\frac{1}{p}} = \left[\mathrm{Tr}\left(\sigma^{\frac{1}{2p}}A^*\sigma^{\frac{1}{p}}A\sigma^{\frac{1}{2p}}\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}. \tag{2.2}$$

We often define the multiplication operator $\Gamma_\sigma(X) := \sigma^{\frac{1}{2}}X\sigma^{\frac{1}{2}}$, then we can write $\|A\|_{p,\sigma} = \|\Gamma_\sigma^{\frac{1}{p}}(A)\|_p$. We also define the class of Banach spaces $L_p(\sigma)$ $(1 \le p \le \infty)$ to be $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{p,\sigma})$. $L_p(\sigma)$ spaces form an interpolation family. That is, for $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ a linear map, σ, σ' two full-rank states, we have the so-called Riesz-Thorin inequality (see e.g. Section 3 in [Bei13])

$$\begin{aligned}&\|\Phi\|_{L_{p_\theta}(\sigma)}^{L_{q_\theta}(\sigma')} \le [\|\Phi\|_{L_{p_0}(\sigma)}^{L_{q_0}(\sigma')}]^{1-\theta}[\|\Phi\|_{L_{p_1}(\sigma)}^{L_{q_1}(\sigma')}]^{\theta},\\ &\forall\theta \in [0,1], \quad 1 \le p_0 \le p_1 \le \infty, 1 \le q_0 \le q_1 \le \infty, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \frac{1}{q_\theta} = \frac{1-\theta}{q_0}+\frac{\theta}{q_1}.\end{aligned} \tag{2.3}$$

We denote the range projection (support projection) of $A \in \mathcal{B}(\mathcal{H})$ by $\mathrm{Range}(A) := P_{\mathrm{Range}(A)}$.

Assume $\Phi, \Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are two linear maps. We say $\Phi \le_{\mathrm{cp}} \Psi$ if $\Psi - \Phi$ is completely positive. A *quantum channel* $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a completely positive trace-preserving map. We refer to duals of quantum channels, which is completely positive and unital map ($\Phi(I) = I$), as *quantum Markov maps*.

A *quantum Markov semigroup* (QMS) $(\mathcal{P}_t)_{t\ge0}$ is a family of quantum Markov maps $\mathcal{P}_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ parametrized by $t \ge 0$, such that (1) $\mathcal{P}_0 = \mathrm{id}_{\mathcal{B}(\mathcal{H})}$; (2) $\mathcal{P}_{s+t} = \mathcal{P}_s \circ \mathcal{P}_t$ for any $s, t \ge 0$. Such semigroup is characterized its generator $\mathcal{L} := \lim_{t\to0+}\frac{\mathcal{P}_t-\mathrm{id}_{\mathcal{B}(\mathcal{H})}}{t}$ which is often called the *Lindbladian*. In the finite-dimensional case, this allows us to write $\mathcal{P}_t = e^{t\mathcal{L}}$, $\forall t \ge 0$. A QMS is said to be *primitive* if it possesses a unique and full-rank invariant state σ.

Given two states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ with $\mathrm{Range}(\rho) \le \mathrm{Range}(\sigma)$, we define the *relative entropy* as

$$H(\rho\|\sigma) := \mathrm{Tr}(\rho\log\rho - \rho\log\sigma). \tag{2.4}$$

It is easy to verify that $H(\rho\otimes\rho'\|\sigma\otimes\sigma') = H(\rho\|\sigma) + H(\rho'\|\sigma')$ using $\log(A\otimes B) = \log((A\otimes I)(I\otimes B)) = \log(A\otimes I)+\log(I\otimes B)$ (see e.g. Section 4.3 of [Bha09]). The relative entropy upper bounds the 1-distance of two states by Pinkser's inequality $\|\rho-\sigma\|_1 \le \sqrt{2H(\rho\|\sigma)}$.

G. Lindblad proved in [Lin75] that the relative entropy is a quantity that exhibits *data processing*, i.e. it is monotonically non-increasing under the action of a quantum channel.

Theorem 1 (Data processing inequality)**.** *Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a quantum channel and ρ, σ are two density matrices on $\mathcal{H}$. Then we have*

$$H(\Phi(\rho)\|\Phi(\sigma)) \leq H(\rho\|\sigma). \tag{2.5}$$

Proof. This is Lemma 6 in [Lin75]. □

Physically, the relative entropy provides the rate at which one can asymptotically distinguish two quantum states. Intuitively, quantum channels often model noise processes and the data processing states that the noise can never make the discrimination task easier.

We define the quantum Fisher information metric for some full rank state $\sigma \in \mathcal{D}_+(\mathcal{H})$ as

$$\langle A, B\rangle_{\sigma^{-1}} := \int_0^\infty \mathrm{Tr}\left(A^* \frac{1}{\sigma + s} B \frac{1}{\sigma + s}\right) \mathrm{d}s. \tag{2.6}$$

It is easy to see that it is an inner product. [GR22] shows that the Fisher information metric is actually "equivalent" to the relative entropy, as stated in the following useful inequality

Lemma 2. *Let $\rho, \sigma \in \mathcal{D}_+(\mathcal{H})$, and we suppose $c\sigma - \rho \geq 0$ for some $c > 0$. Then we have*

$$\frac{c\log c - c + 1}{(c-1)^2}\|\rho - \sigma\|^2_{\sigma^{-1}} \leq H(\rho\|\sigma) \leq \|\rho - \sigma\|^2_{\sigma^{-1}}. \tag{2.7}$$

Here, we denote $k(c) = \frac{c\log c - c+1}{(c-1)^2} \leq \frac{1}{2}$ for $c \geq 1$.

We need the following standard comparison inequality for the proof of Lemma 2.

Lemma 3. *Same assumpations as Lemma 2. Then for any matrix $X \in \mathcal{B}(\mathcal{H})$ and $\mu_1, \mu_2 > 0$, we have*

$$\int_0^\infty \mathrm{Tr}\left(X^* \frac{1}{\mu_1\sigma + r} X \frac{1}{\mu_2\sigma + r}\right) \mathrm{d}r \leq c \int_0^\infty \mathrm{Tr}\left(X^* \frac{1}{\mu_1\rho + r} X \frac{1}{\mu_2\rho + r}\right) \mathrm{d}r. \tag{2.8}$$

In particular, $\|X\|_{\sigma^{-1}} \leq c\|X\|_{\rho^{-1}}$.

Proof of Lemma 3. Note that $t \mapsto -t^{-1}$ is operator monotone, we have

$$\begin{aligned}
\int_0^\infty \mathrm{Tr}\left(X^* (\mu_1\rho + r)^{-1} X (\mu_2\rho + r)^{-1}\right) \mathrm{d}r &\geq \int_0^\infty \mathrm{Tr}\left(X^* (c\mu_1\sigma + r)^{-1} X (\mu_2\rho + r)^{-1}\right) \mathrm{d}r \\
&\geq \int_0^\infty \mathrm{Tr}\left(X^* (c\mu_1\sigma + r)^{-1} X (c\mu_2\sigma + r)^{-1}\right) \mathrm{d}r \\
&= \int_0^\infty \frac{1}{c^2} \mathrm{Tr}\left(X^* \left(\mu_1\sigma + \frac{r}{c}\right)^{-1} X \left(\mu_2\sigma + \frac{r}{c}\right)^{-1}\right) \mathrm{d}r \\
&\overset{\text{change of variable}}{=} \frac{1}{c}\int_0^\infty \mathrm{Tr}\left(X^* (\mu_1\sigma + r)^{-1} X (\mu_2\sigma + r)^{-1}\right) \mathrm{d}r
\end{aligned} \tag{2.9}$$

□

Proof of Lemma 2. For the lower bound, we consider $\rho_t := (1-t)\sigma + t\rho$ for $t \in [0,1]$ and the function $f(t) = H(\rho_t\|\sigma)$. We have $f(0) = 0$, $f(1) = H(\rho\|\sigma)$. We calculate:

$$f'(t) = \mathrm{Tr}((\rho - \sigma)\log\rho_t - (\rho - \sigma)\log\sigma), \tag{2.10}$$

$$f''(t) = \int_0^\infty \mathrm{Tr}\left((\rho-\sigma)\frac{1}{\rho_t+r}(\rho-\sigma)\frac{1}{\sigma+r}\right)\mathrm{d}r = \|\rho-\sigma\|^2_{\rho_t^{-1}}. \tag{2.11}$$

Note that $f'(0) = 0$, and $\rho_t \le (1-t)\sigma + tc\sigma = (ct + (1-t))\sigma$, we have

$$\begin{aligned} H(\rho\|\sigma) &= \int_0^1 \int_0^s f''(t)\mathrm{d}t\mathrm{d}s = \int_0^1 \int_0^s \|\rho-\sigma\|^2_{\rho_t^{-1}}\mathrm{d}t\mathrm{d}s \\ &\overset{\text{Lemma 3}}{\ge} \int_0^1 \int_0^s \frac{1}{1+(c-1)t}\mathrm{d}t\mathrm{d}s\|\rho-\sigma\|^2_{\sigma^{-1}} = k(c)\|\rho-\sigma\|^2_{\sigma^{-1}}. \end{aligned} \tag{2.12}$$

For the upper bound, it follows readily by noting that $\rho_t = (1-t)\sigma + t\rho \ge (1-t)\sigma$, and then by Lemma 3,

$$H(\rho\|\sigma) \le \int_0^1 \int_0^s \frac{1}{1-t}\|\rho-\sigma\|^2_{\sigma^{-1}}\mathrm{d}t\mathrm{d}s = \int_0^1 \int_0^s \frac{1}{1-t}\mathrm{d}t\mathrm{d}s\|\rho-\sigma\|^2_{\sigma^{-1}} = \|\rho-\sigma\|^2_{\sigma^{-1}}. \tag{2.13}$$

□

Given a von Neumann subalgebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ a linear map $E_\mathcal{N} : \mathcal{B}(\mathcal{H}) \to \mathcal{N}$ is called a *conditional expectation* if (1) it is positive; (2) it is unital; (3) for any $a, b \in \mathcal{N}$ and $X \in \mathcal{B}(\mathcal{H})$, we have $E_\mathcal{N}(aXb) = aE_\mathcal{N}(X)b$. According to [Tak03], a conditional expectation is always (1) contractive; (2) completely positive. Therefore, $E_\mathcal{N}$ is a quantum Markov map and its adjoint denoted as $E_{\mathcal{N}^*}$ w.r.t. the Hilbert-Schmidt inner product is a quantum channel.

Note that according to the structure of I-type factor [Tak03], any finite dimensional von Neumann algebra can be expressed as a direct sum of matrix algebras with multiplicities. To be more explicit,

$$\mathcal{N} = \bigoplus_{i=1}^n \mathcal{B}(\mathcal{H}_i) \otimes \mathbb{C}I_{\mathcal{K}_i}, \quad \mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i. \tag{2.14}$$

Denote $P_i \in \mathcal{B}(\mathcal{H})$ as the projection onto $\mathcal{H}_i \otimes \mathcal{K}_i$. The conditional expectation $E_\mathcal{N}$ can be then written as

$$E_\mathcal{N}(X) = \bigoplus_{i=1}^n \mathrm{Tr}_{\mathcal{K}_i}[P_iXP_i(I_{\mathcal{H}_i} \otimes \tau_i)] \otimes I_{\mathcal{K}_i}. \tag{2.15}$$

Here $\mathrm{Tr}_{\mathcal{K}_i}$ denotes the partial trace with respect to $\mathcal{K}_i$. $\{\tau_i\}_{i=1}^n$ are density operators on $\mathcal{D}(\mathcal{K}_i)$ which can be chosen arbitrarily. Therefore, the conditional expectation $E_\mathcal{N}$ onto $\mathcal{N}$ is not unique and depends on the choice of $\{\tau_i\}_{i=1}^n$. The adjoint map of $E_\mathcal{N}$ can be then calculated as

$$E_{\mathcal{N}^*}(\rho) = \bigoplus_{i=1}^n \mathrm{Tr}_{\mathcal{K}_i}(P_i\rho P_i) \otimes \tau_i. \tag{2.16}$$

Physically, we care about the adjoint of the conditional expectation because it can be viewed as the operation in the Schrödinger quantum state picture. $E_\mathcal{N}$ itself, as a positive unital map, is often viewed as the operation in the Heisenberg observable picture.

We denote the set of *invariant states* of $E_\mathcal{N}$ as

$$\begin{aligned} \mathcal{D}(E_\mathcal{N}) &:= \{\sigma \in \mathcal{D}(\mathcal{H}) : E_{\mathcal{N}^*}(\sigma) = \sigma\} = \{\sigma \in \mathcal{D}(\mathcal{H}) : \mathrm{Tr}(\sigma E_\mathcal{N}(a)) = \mathrm{Tr}(\sigma a), \quad \forall a \in \mathcal{N}\} \\ &= \{\sigma \in \mathcal{D}(\mathcal{H}) : \mathrm{Tr}(\sigma E_\mathcal{N}(X)) = \mathrm{Tr}(\sigma X), \quad \forall X \in \mathcal{B}(\mathcal{H})\}. \end{aligned} \tag{2.17}$$

If there is a state $\sigma \in \mathcal{D}(E_\mathcal{N})$, then $E_\mathcal{N}$ is uniquely determined by σ. In this case, we sometimes write $E_\mathcal{N} = E_{\mathcal{N},\sigma}$ to emphasize the dependence on the invariant state σ. For instance, any state σ with the form $\sigma = \bigoplus_{i=1}^n p_i\sigma_i \otimes \tau_i$ satisfies eq. (2.17).

Remark 4. *This resembles the case of classical probability theory. Recall that on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for a σ-subalgebra $\mathcal{F}' \subset \mathcal{F}$, to define the conditional expectation $\mathbb{E}[X|\mathcal{F}']$, one has to specify the subalgebra (corresponding to the von Neumann subalgebra) and the probability measure (corresponding to the invariant state). In fact, choosing $\mathcal{H}_j = \mathbb{C}$ and $\mathcal{K}_j = \mathbb{C}^{|B_j|}$ leads to the classical decomposition $\mathbb{E}[X|\mathcal{F}'] = \sum_{j\in\mathcal{J}} \mathbb{E}[X|B_j]\mathbf{1}_{B_j}$ where we assume that $\mathcal{F}'$ is generated by a set of disjoint events $\{B_j\}_{j\in\mathcal{J}}$.*

Proposition 5. *Let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann subalgebra, $E_{\mathcal{N},\sigma}$ is a conditional expectation with respect to σ onto $\mathcal{N}$, then E is self-adjoint with respect to the Kubo-Martin-Schwinger (KMS) inner product*

$$\langle A, B\rangle_{\sigma,\frac{1}{2}} := \mathrm{Tr}\Big(\sigma^{\frac{1}{2}} A^* \sigma^{\frac{1}{2}} B\Big). \tag{2.18}$$

That is,

$$E_{\mathcal{N}^*}(\sigma^{\frac{1}{2}} A \sigma^{\frac{1}{2}}) = \sigma^{\frac{1}{2}} E_{\mathcal{N}}(A)\sigma^{\frac{1}{2}} \quad \forall A \in \mathcal{B}(\mathcal{H}). \tag{2.19}$$

Proof. The proofs can be found in [Tak03]. □

Proposition 6 (Chain rule)**.** *Let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann subalgebra, $E_{\mathcal{N}}$ is a conditional expectation with respect to $\sigma \in \mathcal{D}(E_{\mathcal{N}})$ onto $\mathcal{N}$. Given any state $\rho \in \mathcal{D}(\mathcal{H})$, we have the* chain rule *holds true:*

$$H(\rho\|\sigma) = H(\rho\|E_{\mathcal{N}^*}(\rho)) + H(E_{\mathcal{N}^*}(\rho)\|\sigma). \tag{2.20}$$

Proof. The proof can be found in e.g. Lemma 3.4 in [JLR23]. □

For conditional expectation $E_{\mathcal{N}}$ onto $\mathcal{N}$, we define the *subalgebra indices* as

Definition 7 (Subalgebra indices and completely bounded subalgebra indices)**.**

$$\begin{aligned} C(E_{\mathcal{N}}) &= \inf\{c > 0 : \rho \leq cE_{\mathcal{N}^*}(\rho), \quad \forall \rho \in \mathcal{D}(\mathcal{H})\}, \\ C_{cb}(E_{\mathcal{N}}) &= \sup_{n\in\mathbb{N}} C(E_{\mathcal{N}} \otimes id_{M_d(\mathbb{C})}). \end{aligned} \tag{2.21}$$

Here, the index cb *stands for* completely bounded*.*

Example 8 (Finiteness of subalgebra indices)**.** *We consider a special case of trace preserving conditional expectation $E_{\mathcal{N},\mathrm{Tr}}$ i.e. when $\tau_i = \frac{I_{\mathcal{K}_i}}{\dim(\mathcal{K}_i)}$ in eq. (2.15).*

$$E_{\mathcal{N},\mathrm{Tr}}(X) = \bigoplus_{i=1}^{n} \mathrm{Tr}_{\mathcal{K}_i}(P_i X P_i) \otimes I_{\mathcal{K}_i}. \tag{2.22}$$

It is easy to see that $E_{\mathcal{N},\mathrm{Tr}}$ is self-adjoint w.r.t. the Hilbert-Schmidt inner product.

The subalgebra index $C(E_{\mathcal{N},\mathrm{Tr}})$ can be explicitly calculated according to Theorem 6.1 of [PP86]

$$C(E_{\mathcal{N},\mathrm{Tr}}) = \sum_{i=1}^{n} \min\{\dim(\mathcal{H}_i), \dim(\mathcal{K}_i)\} \dim(\mathcal{K}_i), \quad C_{cb}(E_{\mathcal{N},\mathrm{Tr}}) = \sum_{i=1}^{n} \dim(\mathcal{K}_i)^2. \tag{2.23}$$

Recall that given the subalgebra $\mathcal{N}$, a conditional expectation $E_{\mathcal{N}}$ is uniquely determined by any invariant state or equivalently the density operators $\{\tau_i\}$. If we additionally assume that $\tau := \bigoplus_{i=1}^{n} I_{\mathcal{H}_i} \otimes \tau_i$ is full-rank (in this case we say that $E_{\mathcal{N}}$ is a faithful *conditional expectation), then by $\tau \in Z(\mathcal{N})$ (i.e. τ commutes with $\mathcal{N}$), we have*

$$E_{\mathcal{N}^*}(\rho) = \bigoplus_{i=1}^{n} \mathrm{Tr}_{\mathcal{K}_i}(P_i \rho P_i) \otimes \tau_i = \tau^{\frac{1}{2}} \left[\bigoplus_{i=1}^{n} \mathrm{Tr}_{\mathcal{K}_i}(P_i \rho P_i) \otimes I_{\mathcal{K}_i}\right] \tau^{\frac{1}{2}} = \tau^{\frac{1}{2}} E_{\mathcal{N},\mathrm{Tr}}(\rho)\tau^{\frac{1}{2}}, \tag{2.24}$$

and then

$$C(E_{\mathcal{N},\tau}) \le \mu_{\min}(\tau)^{-1} C(E_{\mathcal{N},\mathrm{Tr}}), \quad C_{cb}(E_{\mathcal{N},\tau}) \le \mu_{\min}(\tau)^{-1} C_{cb}(E_{\mathcal{N},\mathrm{Tr}}) \tag{2.25}$$

where $\mu_{\min}(\tau) = \min_{1\le i\le n} \mu_{\min}(\tau_i) > 0$ *is the minimal eigenvalue of* τ*. Therefore, in the finite dimensional case, both* $C(E_{\mathcal{N}})$ *and* $C_{cb}(E_{\mathcal{N}})$ *are finite if and only if* $E_{\mathcal{N}}$ *is faithful.*

2.2 Ergodic properties of quantum Markov maps

We focus on quantum channels $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with Kraus decomposition $\Phi(X) = \sum_k V_k^* X V_k$ such that $\sum_k V_k V_k^* = I$, which possess a invertible invariant state σ i.e. $\Phi(\sigma) = \sigma$. Φ^* is a quantum Markov map. We define the set of fixed points of Φ^* as

$$\mathcal{F}(\Phi^*) := \{X \in \mathcal{B}(\mathcal{H}) : \Phi^*(X) = X\}. \tag{2.26}$$

Choi [Cho74] discussed the *multiplicative domain* of Φ^*, defined as

$$\mathcal{M}(\Phi^*) := \{X \in \mathcal{B}(\mathcal{H}) : \Phi^*(X^*X) = [\Phi^*(X)]^*\Phi^*(X), \quad \Phi^*(XX^*) = \Phi^*(X)[\Phi^*(X)]^*\}. \tag{2.27}$$

Proposition 9. *$\forall A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{M}(\Phi^*)$, we have*

$$\Phi^*(AB) = \Phi^*(A)\Phi^*(B), \quad \Phi^*(BA) = \Phi^*(B)\Phi^*(A). \tag{2.28}$$

Consequently, $\mathcal{M}(\Phi^*)$ *is a von Neumann algebra.* $\Phi^*|_{\mathcal{M}(\Phi^*)}$ *is a* $*$*-homomorphism.*

Proof. The proof can be found in the Section 3 of [Cho74]. □

Definition 10. $\mathcal{N}(\Phi^*) := \bigcap_{n=1}^{\infty} \mathcal{M}(\Phi^{*n})$ *is called the decoherence-free algebra of* Φ^**. One can verify by definition that* $\mathcal{N}(\Phi^*)$ *is a von-Neumann subalgebra. It is the smallest subalgebra of* $\mathcal{B}(\mathcal{H})$ *that makes* $\Phi^*|_{\mathcal{N}}$ *a* $*$*-automorphism. The term* decoherence-free *means that* $\Phi^*|_{\mathcal{N}}$ *acts like a unitary evolution. We can see this even more clearly in the following Proposition 11.*

Proposition 11. *For the brevity of notation, we denote* $\mathcal{F}(\Phi^*)$*,* $\mathcal{M}(\Phi^*)$ *and* $\mathcal{N}(\Phi^*)$ *as* $\mathcal{F}$*,* $\mathcal{M}$ *and* $\mathcal{N}$ *respectively. Then we have*

(1) *Since* Φ *possesses a full rank invariant state,* $\mathcal{F}$ *is a* $*$*-subalgebra of* $\mathcal{N}$*.*

(2) $\mathcal{N}$ *coincides with the* reversible subalgebra *of* Φ^**, i.e. the algebra spanned by the eigenvectors of* Φ^* *corresponding to the moduli* 1 *eigenvalues.*

$$\mathcal{N} := \{X \in \mathcal{B}(\mathcal{H}) : \exists \varphi \in \mathbb{R} \text{ s.t. } \Phi^*(X) = e^{i\varphi}X\}. \tag{2.29}$$

Proof. The first claim is the Theorem 6.12 of [Wol12]. The second claim is the Lemma 1 and Theorem 1 of [CJ20]. □

These ergodic properties of quantum Markov maps are naturally extended to the case of quantum Markov semigroup. We denote $\mathcal{P}_t$ as a quantum Markov semigroup with generator $\mathcal{L}$. Then we define

$$\begin{aligned}\mathcal{F}(\mathcal{L}) &:= \{X \in \mathcal{B}(\mathcal{H}) : \forall t \ge 0, \mathcal{P}_t(X) = X\} \\ \mathcal{N}(\mathcal{L}) &:= \{X \in \mathcal{B}(\mathcal{H}) : \forall t \ge 0, \mathcal{P}_t(X^*X) = \mathcal{P}_t(X)^*\mathcal{P}_t(X)\}.\end{aligned} \tag{2.30}$$

Quite similarly, we have

Proposition 12. *Assume again that there exists a full-rank invariant state σ of $\mathcal{P}_t$. Then $\mathcal{N}(\mathcal{L})$ coincides with the largest von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ for which $\mathcal{P}_t|_N$ is a continuous group of $*$-automorphisms. $\mathcal{F}(\mathcal{L})$ is an algebra which is the range of the conditional expectation onto $\mathcal{F}(\mathcal{L})$, defined as*

$$E_{\mathcal{F}} := \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathcal{P}_s \mathrm{d}s. \tag{2.31}$$

Proof. The first claim and the second claim are Proposition 6.1.1 and Theorem 6.1.2 of [Rou19] respectively. □

2.3 Quantum detailed balance condition

In the setting of classical Markov chain on a finite state space E with transition matrix $p = (p_{ij})$, the detailed balance condition refers to the case where a probability distribution $\{\pi(i)\}_{i\in E}$ satisfies $p_{ij}\pi(j) = p_{ji}\pi(i)$ which is a sufficient condition for the existence of stationary distribution. This can also be reformulated in terms of the self-adjointness of the transition matrix with respect to the weighted inner product $\langle\cdot,\cdot\rangle_\pi$:

$$\langle fP, g\rangle_\pi := \sum_{i,j\in E} \pi(i)g(i)\overline{f(j)}p_{ji} = \sum_{i,j\in E} \pi(j)\overline{f(j)}g(i)p_{ij} = \langle f, gP\rangle_\pi, \quad \forall f,g \in \ell^2(E). \tag{2.32}$$

There are many ways to generalize the classical detailed balance condition (DBC) to the quantum setting. Basically, this depends on our choice of the weighted inner product.

- The GNS inner product: $\langle X, Y\rangle_{\sigma,s} = \mathrm{Tr}\left(X^*\sigma^{1-s}Y\sigma^s\right)$, $s \in [0,1]$, $X, Y \in \mathcal{B}(\mathcal{H})$. We denote $L_2(\sigma, s) = (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\sigma,s})$. Note that the KMS inner product is the special case of GNS inner product by taking $s = \frac{1}{2}$, that is $L_2(\sigma) = L_2(\sigma, \frac{1}{2})$.
- The Bogoliubov-Kubo-Mori (BKM) inner product is $\langle X, Y\rangle_{\sigma,\mathrm{BKM}} = \int_0^1 \langle X, Y\rangle_{\sigma,s}\mathrm{d}s$. We denote the associate Hilbert space as $L_2^{\mathrm{BKM}}(\sigma)$.

We define the multiplication operator $\Gamma_{\sigma,s} : \mathcal{B}(\mathcal{H})$ as $\Gamma_{\sigma,s}(X) = \sigma^{1-s}X\sigma^s$. Assume that $\Phi^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is self-adjoint w.r.t. $\langle\cdot,\cdot\rangle$, then

$$\langle X, Y\rangle_{\sigma,s} = \langle X, \Gamma_{\sigma,s}(Y)\rangle \Rightarrow \langle X, (\Phi\circ\Gamma_{\sigma,s})(Y)\rangle = \langle \Phi^*(X), \Gamma_{\sigma,s}(Y)\rangle = \langle X, (\Gamma_{\sigma,s}\circ\Phi^*)(Y)\rangle. \tag{2.33}$$

From the above calculations, we see that Φ^* is GNS-self adjoint if and only if $\Phi\circ\Gamma_{\sigma,s} = \Gamma_{\sigma,s}\circ\Phi^*$. The GNS self adjointness of a linear map on $\mathcal{B}(\mathcal{H})$ can be viewed as the quantum version of DBC.

Proposition 13. *Assume that $\Phi^*(a^*) = [\Phi^*(a)]^*$, then Φ^* is σ, s-DBC for some $s \in [0,1]\backslash\{\frac{1}{2}\}$ iff $\Phi^*\circ\Delta_\sigma = \Delta_\sigma\circ\Phi^*$ and is $\sigma, \frac{1}{2}$-DBC. Here $\Delta_\sigma(X) = \sigma X\sigma^{-1}$ is the modular operator of σ. Consequently, Φ^* is σ, s-DBC for all $s \in [0,1]$ and is also self-adjoint w.r.t. the BKM inner product.*

Proof. This is the Lemma 2.1 of [CM20]. □

Lemma 14. *Let Φ^* be a σ-DBC (which is the shorthand for $\sigma, 1$-DBC) quantum Markov map. σ is a full-rank state. Then Φ^* has the Kraus decomposition $\Phi^*(X) = \sum_k R_kXR_k^*$ where R_k satisfies $\Delta_\sigma(R_k) = \eta_kR_k$ for some positive constants $\eta_k > 0$ and $\sum_k R_kR_k^* = I$.*

Proof. By Proposition 13, we have $\Phi^* \circ \Delta_\sigma = \Delta_\sigma \circ \Phi^*$. We consider $\psi = \sum_{i=1}^d e_i \otimes e_i$ where $\{e_i\}_{i=1}^d$ is an orthonormal basis of $\mathcal{H}$. Then the Choi-Jamiołkowski isomorphism of Φ is

$$J = (\Phi \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})})(\psi\psi^*), \quad \psi\psi^* = \sum_{i,j=1}^d e_i e_j^* \otimes e_i e_j^* = \sum_{i,j=1}^d E_{ij} \otimes E_{ij}. \tag{2.34}$$

Note that for any matrix M we have

$$(M \otimes I)\psi = \sum_{i=1}^d M e_i \otimes e_i = \sum_{i,j=1}^d M_{ji} e_j \otimes e_i = \sum_{j=1}^d e_j \otimes \left(\sum_{i=1}^d M_{ji} e_i \right) = \sum_{j=1}^d e_j \otimes M^T e_j = (I \otimes M^T)\psi. \tag{2.35}$$

Using this, $\Phi^* \circ \Delta_\sigma = \Delta_\sigma \circ \Phi^*$ translates into the following equation under the Choi-Jamiolkowski isomorphism

$$(\sigma^{-1} \otimes \sigma^T) J = J(\sigma^{-1} \otimes \sigma^T). \tag{2.36}$$

Thus we can let $\{v_j\}_{j=1}^{d^2}$ be the orthonormal basis of $\mathcal{H} \otimes \mathcal{H}$ that diagonalizes J and $\sigma^{-1} \otimes \sigma^T$ simultaneously. We write $J v_k = \lambda_k v_k$ and $(\sigma^{-1} \otimes \sigma^T) v_k = \eta_k^{-1} v_k$ where $\lambda_k \geq 0$ and $\eta_k > 0$. Let $\widetilde{V}_k$ be the matrix such that $\widetilde{V}_k \psi = v_k$. W.L.O.G. we assume $\widetilde{V}_k = V_k \otimes I$ by eq. (2.35). Using eq. (2.35) again, $(\sigma^{-1} \otimes \sigma^T) v_k = \eta_k^{-1} v_k$ translates to

$$(\sigma^{-1} V_k \sigma \otimes I)\psi = \eta_k^{-1} (V_k \otimes I)\psi \Rightarrow \sigma^{-1} V_k \sigma = \eta_k^{-1} V_k. \tag{2.37}$$

On the other hand, we have, for $X \in \mathcal{B}(\mathcal{H})$,

$$(\Phi^* \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})})(\psi\psi^*) = J = \sum_{k=1}^{d^2} \lambda_k v_k v_k^* = \sum_{k=1}^{d^2} \lambda_k (V_k \otimes I)\psi\psi^*(V_k^* \otimes I) \Rightarrow \Phi^*(X) = \sum_{k=1}^{d^2} \lambda_k V_k X V_k^*. \tag{2.38}$$

We let $R_k = \sqrt{\lambda_k} V_k$ and then we have $\sigma R_k = \eta_k R_k \sigma$ by eq. (2.37). ◻

We next give an important property of the conditional expectation. Recall that if $\sigma \in \mathcal{D}(E_\mathcal{N})$, then $E_{\mathcal{N}^*}(\sigma) = \sigma$. Thus $E_\mathcal{N}$ satisfies the σ-DBC.

Lemma 15. *Let $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{N}$ be the set of fixed points, multiplicative domain and the decoherence-free algebra of Φ^* respectively. If Φ^* satisfies σ-DBC with respect to a full rank invariant state $\sigma = \Phi(\sigma)$, then*

1. *Φ^* is a contraction on $L_2(\sigma, s)$ for all $s \in [0,1]$ and $L_2^{BKM}(\sigma)$.*
2. *$\Phi^*|_\mathcal{M}$ is a $*$-automorphism on $\mathcal{M}$ and an isometry with respect to $L_2(\sigma, s)$ for all $s \in [0,1]$ as well as $L_2^{BKM}(\sigma)$. In particular, $\mathcal{M} = \mathcal{N}$.*
3. *Let $E_\mathcal{N}$ be the conditional expectation onto $\mathcal{N}$ with $\sigma \in \mathcal{D}(E_\mathcal{N})$, then*

$$\Phi^* \circ E_\mathcal{N} = E_\mathcal{N} \circ \Phi^*, \quad (\Phi^*)^2 \circ E_\mathcal{N} = E_\mathcal{N} \circ (\Phi^*)^2 = E_\mathcal{N}. \tag{2.39}$$

Proof. This is the Lemma 2.5 of [GR22]. ◻

Remark 16. *We see from the above lemma that under σ-DBC, Φ^* is a self-adjoint contraction on $L_2(\sigma, s)$ and $L_2^{BKM}(\sigma)$. Therefore, by Proposition 11 we know that $\mathcal{N}$ is the eigenspace of Φ^* for eigenvalues ± 1 (that of 1 is the subalgebra $\mathcal{F} \subset \mathcal{N}$). In finite dimensional case, since the spectrum of Φ^* is discrete, Φ^* restricted on the orthogonal complement of $\mathcal{N}$ must be a strict contraction which is the spectral gap condition. That is,*

$$\|\Phi^* \circ (id - E_\mathcal{N})\|_{L_2(\sigma,s)}^{L_2(\sigma,s)} < 1 - \delta, \quad \textit{for some } 0 < \delta < 1. \tag{2.40}$$

Again, the above results can be extended to the case of QMS. We say that the QMS is σ-DBC if the generator $\mathcal{L}$ satisfies σ-DBC

$$\mathcal{L}^* \circ \Gamma_\sigma = \Gamma_\sigma \circ \mathcal{L}. \tag{2.41}$$

Likewise, $\mathcal{L}$ is σ-DBC if and only if $\mathcal{L}$ and $\mathcal{P}_t$ are both self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$.

Note that $\mathcal{P}_t = e^{t\mathcal{L}}$ is unital, we have $\mathcal{L}(I) = 0$. Thus

$$\mathcal{L}^* \circ \Gamma_\sigma(I) = \mathcal{L}^*(\sigma) = \Gamma_\sigma \circ \mathcal{L}(I) = 0. \tag{2.42}$$

Thus, $\mathcal{P}_t^*(\sigma) = \sigma$. Here the adjoint $\mathcal{P}_t^*$ of $\mathcal{P}_t$ is a CPTP map, which is the associated QMS generated by $\mathcal{L}^*$ in the Schödinger picture. The original QMS $\mathcal{P}_t$ should be viewed as the operation on the space of operators (observables) in the Heisenberg picture.

If we additionally assume that $\mathcal{P}_t^*$ is primitive, then if the QMS is σ-DBC, σ would be the unique invariant state of $\mathcal{P}_t^*$ for $t \geq 0$. Extending Lemma 15 to the case of QMS, we have under DBC, $\mathcal{P}_t$ is $L_2(\sigma)$, s and $L_2^{\mathrm{BKM}}(\sigma)$-contractive and the decoherence subalgebra and the multiplicative domain coincide.

Remark 17. *Recall the case of quantum Markov semigroup eq. (2.30), we directly define the decoherence-free algebra $\mathcal{N}(\mathcal{L})$ as the "multiplicative domain" of $\mathcal{P}_t$. We may justify this by the second claim of Lemma 15.*

2.4 Spectral gap and its tensorization

Although data processing Theorem 1 is potent, it is often not enough to know that a quantity is decreasing under a quantum channel. To quantify the contraction rate of some information measures under the application of a quantum evolution and to study the dependence on the structure of the evolution and the system size, we need some finer tools.

Spectral gap method. We begin with the L_2 metric as the information measure. By Lemma 15, we know that if Φ^* satisfies σ-DBC, then Φ^* is a self-adjoint contraction on $L_2(\sigma, s)$ ($s \in [0, 1]$) and $L_2^{\mathrm{BKM}}(\sigma)$ and $\mathcal{N}$ is the eigenspace of Φ^* for ± 1 eigenvalues. To quantify the contraction rate, we define the spectral gap as

$$\lambda(\sigma, s) := \|\Phi^*(\mathrm{id} - E_{\mathcal{N}})\|_{L_2(\sigma,s)}^{L_2(\sigma,s)}, \quad \lambda(\sigma) := \|\Phi^*(\mathrm{id} - E_{\mathcal{N}})\|_{L_2^{\mathrm{BKM}}(\sigma)}^{L_2^{\mathrm{BKM}}(\sigma)}. \tag{2.43}$$

[GR22] showed that the spectral gap condition is in fact independent of $s \in [0, 1]$ and the choice of the invariant state $\sigma \in \mathcal{D}(E_{\mathcal{N}})$. This result is a consequence of Lemma 15.

Lemma 18. *Let Φ be a quantum channel and Φ^* satisfies σ-DBC with respect to some full rank invariant state σ. Then we have*

1. *If $\rho \in \mathcal{D}(E_{\mathcal{N}})$, then $(\Phi^*)^2$ satisfies ρ-DBC. If $\rho = \Phi(\rho)$ is an invariant state, then Φ^* satisfies ρ-DBC.*
2. *For any full rank state $\rho \in \mathcal{D}(E_{\mathcal{N}})$ and all $s \in [0, 1]$, $\lambda(\rho, s) = \lambda(\rho) = \lambda(\sigma, 1) = \lambda(\sigma)$. Since the spectral gap is independent of the choice of the invariant state σ and the parameter s, we denote $\lambda(\Phi^*) := \lambda(\sigma)$.*

Proof. This is the Lemma 2.6 of [GR22]. □

Note that we state the spectral gap condition in the setting of quantum Markov map Φ^* with the assumption that Φ possesses a full rank invariant state σ and Φ^* satisfies σ-DBC. In fact, this naturally extends to the case of quantum Markov semigroup. Recall the definition in eq. (2.30), we define

Definition 19 (Dirichlet form)**.** *We define the Dirchlet form associated with the generator $\mathcal{L}$ of a quantum Markov semigroup $\mathcal{P}_t$ as $\mathcal{E}_{\mathcal{L},\sigma}(X) := -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{P}_t(X - E_{\mathcal{N}}(X))\|^2_{\sigma,\frac{1}{2}}|_{t=0}$. By the theory in Section 5 of [CM17], $\mathcal{E}_{\mathcal{L},\sigma}$ takes the simple form $\mathcal{E}_{\mathcal{L},\sigma}(X) = -\langle X, \mathcal{L}(X)\rangle_{\sigma,\frac{1}{2}}$. If the weight σ in the inner product is clear by the context, we may drop the index σ and denote $\mathcal{E}_{\mathcal{L}}(X) = \mathcal{E}_{\mathcal{L},\sigma}$.*

Definition 20. *We define the spectral gap of the generator $\mathcal{L}$ using the following variational characterization*

$$\lambda(\mathcal{L}) = \inf_X \frac{-\langle X, \mathcal{L}(X)\rangle_{\sigma,\frac{1}{2}}}{\|(id - E_{\mathcal{N}})(X)\|^2_{\sigma,\frac{1}{2}}} = \inf_X \frac{\mathcal{E}_{\mathcal{L}}(X)}{\|(id - E_{\mathcal{N}})(X)\|^2_{\sigma,\frac{1}{2}}}. \tag{2.44}$$

Tensorization. Similar to the case of classial Markov semigroup, the spectral gap has the tensorization structure. We define

$$\mathcal{K}_n = \sum_{i=1}^n \widehat{\mathcal{L}}_i, \quad \widehat{\mathcal{L}}_i := \mathrm{id}^{\otimes(i-1)} \otimes \mathcal{L} \otimes \mathrm{id}^{\otimes(n-i)}. \tag{2.45}$$

Then, the eigenvalues of $\mathcal{K}_n$ are summations of eigenvalues of individual $\widehat{\mathcal{L}}_i$'s since they commute with each other. Moreover, each $\widehat{\mathcal{L}}_i$ has the same eigenvalues as those of $\mathcal{L}$. Using these, we conclude that $\lambda(\mathcal{K}_n) = \lambda(\mathcal{L})$ for any n, which means that the spectral gap is preserved under the tensorization. This gives us the first tool to characterize the convergence of product channels.

Example 21 (Depolarizing channels)**.** *We define the depolarizing channel of parameter $\eta \in (0, 1)$*

$$\mathcal{D}_\eta(\rho) := (1 - \eta)\rho + \eta \operatorname{Tr}(\rho)\frac{I}{d}, \quad \rho \in \mathcal{B}(\mathbb{C}^d). \tag{2.46}$$

Direct calculations show that $\mathcal{D}_\eta$ is unital and self-adjoint and admits $\frac{I}{d}$ as its unique invariant state. Moreover, the gap of $\mathcal{D}_\eta$ is $\lambda(\mathcal{D}_\eta) = 1 - \eta$. By tensorization, we have for all $n \in \mathbb{N}$ and $\rho \in (\mathbb{C}^d)^{\otimes n}$ (n qubit states),

$$\left\|\mathcal{D}_\eta^{\otimes n}(\rho) - \frac{I}{d^n}\right\|_1 \overset{eq.\ (2.3)}{\leq} \left\|\mathcal{D}_\eta^{\otimes n}(d^n\rho) - I\right\|_{2,\frac{I}{d^n}} \leq \lambda(\mathcal{D}_\eta)\|d^n\rho - I\|_{2,\frac{I}{d^n}} = (1-\eta)\|d^n\rho - I\|_{2,\frac{I}{d^n}} \tag{2.47}$$

Direct calculation shows that $\|d^n\rho - I\|_{2,\frac{I}{d^n}} = \sqrt{d^n \operatorname{Tr}\rho^2 - 1} \leq \sqrt{d^n - 1}$ and the bound is achieved for pure states. Therefore, the output state of $\mathcal{D}_\eta^{\otimes n}$ converges to the maximally mixed state $\frac{I}{d^n}$ at a rate of $1 - \eta = O(d^{-n})$. A natural question is whether this dependence is optimal. As we will see in the next section, it turns out that it is not the case.

3 Log-Sobolev inequalities

3.1 Introduction

Spectral gap is not the only way to quantify the convergence of a quantum Markov semigroup. In fact, it fails to capture the optimal convergence rate of the depolarizing channel, as we have seen in example 21.

The first concept we need to introduce is *hypercontractivity*. It is well-known that for a quantum channel Φ with full-rank invariant state σ, Φ^* is $L^p(\sigma)$-contractive i.e. $\|\Phi^*\|^{L_p(\sigma)}_{L_p(\sigma)} \leq 1$ (see Theorem 6 in [Bei13]). The idea of hypercontractivity is to strengthen this result in the sense that we care about the maximal $q \geq p$ (recall that $\|X\|_{p,\sigma} \leq \|X\|_{q,\sigma}$ for $q \geq p$ by eq. (2.3)) for which Φ^* still strictly contracts from $L_p(\sigma)$ to $L_q(\sigma)$. We denote the hypercontractivity index of Φ^* as $\alpha(\Phi^*)$.

Definition 22 (Hypercontractivity)**.** *Let $\Phi^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a* primitive *quantum Markov map with unique full-rank invariant state σ. Let $1 \le p \le q \le \infty$, then Φ^* is said to be (p,q)-hypercontractive if $\|\Phi^*\|_{L_p(\sigma)}^{L_q(\sigma)} \le 1$.*

Example 23 (Depolarizing channels revisited)**.** *We will show later that the product depolarizing channel $\mathcal{D}_\eta^{\otimes n}$ is $(p,2)$-hypercontractive whenever $1-\eta \le (p-1)^{\frac{5}{2}\log d + \frac{11}{2}}$. This property is independent of the number of n of copies of the system. Using this, we may enhance eq. (2.47) in example 21 as follows. In fact the depolarizing channel $\mathcal{D}_\eta$ is a semigroup parameterized by $t = -\log(1-\eta)$, which we denote as $\mathcal{P}_t := \mathcal{D}_\eta$, then we have*

$$\left\|\mathcal{D}_\eta^{\otimes n}(\rho) - \frac{I}{d^n}\right\|_1 \le \left\|\mathcal{D}_\eta^{\otimes n}(d^n\rho) - I\right\|_{2,\frac{I}{d^n}} = \left\|\mathcal{D}^{\otimes n}_{\sqrt{1-\eta}}[\mathcal{P}_{\frac{t}{2}}(X)] - I\right\|_{2,\frac{I}{d^n}} \overset{\text{spectral gap}}{\le} e^{-\frac{t}{2}}\left\|\mathcal{P}_{\frac{t}{2}}(X) - I\right\|_{2,\frac{I}{d^n}}. \quad (3.1)$$

Here we denote $X = d^n\rho$. Next we will use the hypercontractivity of $\mathcal{P}_t = \mathcal{D}_{\widetilde{\eta}}$ where $\widetilde{\eta} = 1 - \sqrt{1-\eta}$, which holds whenever $\sqrt{1-\eta} = 1 - \widetilde{\eta} \le (p-1)^{\frac{5}{2}\log d + \frac{11}{2}}$. Then we have

$$\left\|\mathcal{D}_\eta^{\otimes n}(\rho) - \frac{I}{d^n}\right\|_1 \le e^{-\frac{t}{2}}\left\|\mathcal{P}_{\frac{t}{2}}(X) - I\right\|_{2,\frac{I}{d^n}} \overset{\text{hypercontractivity}}{\le} e^{-\frac{t}{2}}\|X - I\|_{p,\frac{I}{d^n}} \lesssim e^{-\frac{t}{2}}(d^n)^{\frac{p-1}{p}}. \quad (3.2)$$

Here the last inequality follows from a similar calculation as in example 21. In particular, we choose $p = 1 + \frac{1}{n}$, then the rate of the convergence is $O(n^{-\frac{5\log d}{2} - \frac{11}{2}})$. This is an exponentially sharper result than $O(d^{-n})$ obtained from the spectral gap method.

The generator of the depolarizing semigroup $\mathcal{P}_t$ is in fact given by

$$\mathcal{L}(X) = \mathrm{Tr}(X)\frac{I}{d} - X, \quad \forall X \in \mathcal{B}(\mathbb{C}^d). \quad (3.3)$$

In fact, this generator can be generalized for arbitrary full-rank state $\sigma \in \mathcal{D}_+(\mathbb{C}^d)$ as

$$\mathcal{L}(X) = \mathrm{Tr}(\sigma X)I - X, \quad \forall X \in \mathcal{B}(\mathbb{C}^d). \quad (3.4)$$

eq. (3.3) can be viewed as a special case of eq. (3.4) with σ being the maximal mixed state $\frac{I}{d}$. The generalized depolarizing generator example will be repeatedly revisited in the following text. The adjoint $\mathcal{L}^*$ of the generator $\mathcal{L}$ w.r.t. the Hilbert-Schmidt inner product also takes a simple form:

$$\mathcal{L}^*(\rho) = \mathrm{Tr}(\rho)\sigma - \rho, \quad \forall \rho \in \mathcal{B}(\mathbb{C}^d). \quad (3.5)$$

In fact, in the case where the dual channel Φ^* is generated by a QMS $\mathcal{P}_t$ (including the depolarizing channel $\mathcal{D}_{e^{-t}}$), the essence of the hypercontractivity (HC) phenomenon is the quantum logarithmic Sobolev (log-Sobolev) inequality (LSI). The LSI can be seen as a continuous version of the HC.

Definition 24 (Log-Sobolev inequality)**.** *Let $\mathcal{L}$ be a primitive generator (Lindbladian) that is σ-DBC with respect to a full-rank state σ, an LSI is an inequality of the form*[1]

$$\begin{aligned}
&\beta \mathrm{Ent}_{2,\sigma}(X) \le \mathcal{E}_{\mathcal{L}}(X), \quad \forall X > 0,\\
&\mathrm{Ent}_{p,\sigma}(X) := \frac{1}{p}[H(\Gamma_\sigma^{\frac{1}{p}}(X)^p\|\sigma) - \mathrm{Tr}\,\Gamma_\sigma^{\frac{1}{p}}(X)^p\,\mathrm{Tr}\log\Gamma_\sigma^{\frac{1}{p}}(X)^p]\ (p\text{-}Entropy),\\
&\mathcal{E}_{\mathcal{L}}(X) = -\langle X, \mathcal{L}(X)\rangle_{\sigma,\frac{1}{2}}\ (Dirichlet\ form).
\end{aligned} \quad (3.6)$$

[1] In some literature, the p-entropy is defined as $\mathrm{Ent}_{p,\sigma}(X) = H(\Gamma_\sigma^{\frac{1}{p}}(X)^p\|\sigma) - \mathrm{Tr}\,\Gamma_\sigma^{\frac{1}{p}}(X)^p\,\mathrm{Tr}\log\Gamma_\sigma^{\frac{1}{p}}(X)^p$ (see e.g. [Rou19], Section 7.2). In this text, we use the convention in e.g. [KT13].

In particular, if $\rho \in \mathcal{D}_+(\mathcal{H})$, then we have $\operatorname{Ent}_{2,\sigma}(\rho) = \frac{1}{2}H(\Gamma_\sigma^{\frac{1}{2}}(\rho)^2\|\sigma)$. It is easy to see that the p-entropy is p-homogeneous i.e. $\operatorname{Ent}_{p,\sigma}(cX) = c^p \operatorname{Ent}_{p,\sigma}(X)$ for any $c > 0$. We refer to e.g. Section 2.2 of [BDR20] for more properties of the p-entropy functional.

The best constant β satisfying the above inequality is called the log-Sobolev constant of $\mathcal{L}$ denoted as

$$\alpha_2(\mathcal{L}) := \inf_{X\in\mathbb{H}_d^{>0},\operatorname{Ent}_{2,\sigma}(X)\neq 0} \frac{\mathcal{E}_{\mathcal{L}}(X)}{\operatorname{Ent}_{2,\sigma}(X)}. \tag{3.7}$$

The significance of the 2-entropy (in general p-entropy) function comes from the derivative formula related to the 2-norm (in general p-norm). If $p \mapsto X_p$ is a operator-valued differentiable function with $X_p \geq 0$ for all p, then we have (see [KT13, OZ99])

$$\frac{\mathrm{d}}{\mathrm{d}p}\left\|X_p\right\|_{p,\sigma} = \frac{1}{p}\left\|X_p\right\|_{p,\sigma}^{1-p}\left(\operatorname{Ent}_{p,\sigma}(X_p) + p^2 \operatorname{Tr}\left[\Gamma_\sigma^{\frac{1}{p}}(\tfrac{\mathrm{d}}{\mathrm{d}p}X_p)\Gamma_\sigma^{\frac{1}{p}}(X_p)^{p-1}\right]\right). \tag{3.8}$$

In fact, the LSI is equivalent to the HC in some sense for primitive QMS generator $\mathcal{L}$ (see [KT13]):

Theorem 25 (HC $\Longleftrightarrow$ LSI)**.** *Let $\mathcal{L}$ be a primitive Lindbladian with full-rank invariant state σ. $\alpha > 0$, then the following are equivalent:*

(1) *$\mathcal{P}_t^*$ is (p,q)-hypercontractive for $1 \leq p \leq q$ and $e^{2\alpha t} \geq \frac{q-1}{p-1}$.*

(2) *$\alpha \leq \alpha_2(\mathcal{L})$.*

A simple question is whether we can always find a LSI for a given QMS. For finite dimensional case (matrix algebra), it is true by relating it to the spectral gap. For a primitive Lindbladian $\mathcal{L}$, we have the following results:

- If $\mathcal{L}$ is KMS-symmetric (w.r.t. $\langle\cdot,\cdot\rangle_{\sigma,\frac{1}{2}}$), then we have $\frac{\lambda(\mathcal{L})}{\log(\|\sigma^{-1}\|_\infty+1)} \leq \alpha_2(\mathcal{L}) \leq \lambda(\mathcal{L})$ (see [KT13, TPK14]);
- If the semigroup generated by $\mathcal{L}^*$ is unital (i.e. $\sigma \propto I$), then $\frac{2(1-\frac{2}{d})\lambda(\mathcal{L})}{\log(d-1)} \leq \alpha_2(\mathcal{L}) \leq \lambda(\mathcal{L})$ (see [KT13, OZ99]).

3.2 Gerneal results of log-Sobolev inequalities on matrix algebras

The main material of this section follows the work [BDR20] and also Section 10.1.2 of [Rou19].

The lower bounds above do not provide advantage compared with spectral gap estimates for the convergence rate of QMS, due to the coarse-grained knowledge of the gap and the dimension of the global system d. However, if we have some tensorization structures, we can usually obtain a tighter lower bound. In this section, we will mainly focus on two cases: the first one is the direct-sum generator or the so-called *quasi-tensorization*, while the second case is the tensorization log-Sobolev constant in low dimensions such as qubit system (i.e. $\mathcal{H} = \mathbb{C}^2$).

Theorem 26 (Log-Sobolev inequality for direct-sum generators)**.** *Let $\mathcal{L}_i : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be primitive Lindbladians with full-rank invariant states σ_i, where each $\mathcal{L}_i$ is σ_i-KMS symmetric and has spectral gap λ_i. We define $\widehat{\mathcal{L}_i}$ and $\mathcal{K}_n$ as in eq. (2.45). Then we have the following bound of the log-Sobolev constant for the direct-sum generator $\mathcal{K}_n$:*

$$\frac{\lambda(\mathcal{K}_n)}{\log\left(d^4 \max_i \left\|\sigma_i^{-1}\right\|_\infty\right) + 11} \leq \alpha_2(\mathcal{K}_n) \leq \lambda(\mathcal{K}_n). \tag{3.9}$$

Here, by the tensorization of spectral gap, we have $\lambda(\mathcal{K}_n) = \min_{1\leq i\leq n} \lambda_i$.

Proof. This result is the Theorem 9 of [TPK14]. □

Example 27 (Depolarizing channels revisited). *We assume $\mathcal{L}_i(X) = \mathrm{Tr}(X)\frac{I}{d} - X$ for each i, then $\sigma_i = \frac{I}{d}$ thus $\left\|\sigma_i^{-1}\right\|_\infty = d$. The spectral gap $\lambda_i = 1$, then*

$$\frac{1}{5\log d + 11} \le \alpha_2(\mathcal{K}_n) \le 1. \tag{3.10}$$

Using Theorem 25, we know that $\mathcal{K}_n$ is $(p,2)$-hypercontractive if

$$e^{2\alpha_2(\mathcal{K}_n)(-\log(1-\eta))} \ge \frac{1}{p-1} \Leftarrow 1-\eta \le (p-1)^{-\frac{5}{2}\log d - \frac{11}{2}} \tag{3.11}$$

which is the condition of the hypercontractivity stated in example 23.

Remark 28. *In [MHSFW16b] the log-Sobolev constant is explicitly calculated for the* generalized depolarizing generator *$\mathcal{L}(X) = \mathrm{Tr}(\sigma X)I - X$, which is $\alpha_2(\mathcal{L}) = \frac{2(1-2\mu_{\min})(\sigma)}{\log(1/\mu_{\min}(\sigma)-1)}$ where $\mu_{\min}(\sigma)$ is the smallest eigenvalue of σ (the detailed proof can be found in the Appendix C of [BDR20]). In this case, we have $\alpha_2(\mathcal{K}_n) = \frac{2(1-\frac{2}{d})}{\log(d-1)}$. Compared to eq. (3.10), we see that the asymtotic dependence on d is well recovered, albeit with a slightly worse constant. Using the technique of low-dimensional tensorization, we will see that the exact expression can be retrieved for $d = 2$. Note that the $d = 2$ case is the most common scenario in quantum information theory (qubit) and quantum many-body theory (quantum spin systems).*

Theorem 29 (Tensorization of log-Sobolev constant for qubit generators). *Let $\mathcal{H} = \mathbb{C}^2$ and $\mathcal{L}(X) = \mathrm{Tr}(\sigma X) - X$ be the generalized depolarizing generator with a full-rank state σ. Also, let $\mathcal{K}$ be a Lindbladian associated with a primitive QMS that is ρ-symmetric with respect to a full-rank state ρ. Then the log-Sobolev constant of $\widehat{\mathcal{L}}_1 + \widehat{\mathcal{K}}_2$ tensorizes, i.e.*

$$\alpha_2(\mathcal{L}\otimes id + id\otimes\mathcal{K}) = \min\{\alpha_2(\mathcal{L}), \alpha_2(\mathcal{K})\}. \tag{3.12}$$

The strategy of the proof in [BDR20] is to first show the tensorization property of the entropy functional Ent_2, and then for the Dirichlet form. Combining these two results we can derive the tensorization of the log-Sobolev constant.

We first deal with the tensorization of the entropy functional.

Lemma 30. *Let $\mathcal{H}$ and $\mathcal{H}'$ be two finite-dimensional Hilbert spaces with $\dim\mathcal{H} = 2$. Let $X \in \mathbb{H}^{\ge 0}_{\mathcal{H}\otimes\mathcal{H}'}$ be a Hermitian positive semidefinite matrix with block form*

$$X = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}, \quad A, B, C \in \mathcal{B}(\mathcal{H}'). \tag{3.13}$$

Let $\rho \in \mathcal{D}_+(\mathcal{H}')$ be a full-rank density matrix. We define a 2×2 matrix M with its entries being the $L_2(\rho)$-norm of the blocks of X as follows:

$$M = \begin{pmatrix} ||A||_{2,\rho} & ||C||_{2,\rho} \\ ||C^*||_{2,\rho} & ||B||_{2,\rho} \end{pmatrix}. \tag{3.14}$$

Then M is positive semidefinite. Furthermore, let $\sigma \in \mathcal{D}_+(\mathcal{H})$ with the form $\sigma = \mathrm{diag}(\theta, 1-\theta)$ where $\theta \in (0,1)$, then we have

$$\begin{aligned} \mathrm{Ent}_{2,\sigma\otimes\rho}(X) \le\ & \mathrm{Ent}_{2,\sigma}(M) + \theta\,\mathrm{Ent}_{2,\rho}(A) + (1-\theta)\,\mathrm{Ent}_{2,\rho}(B) \\ & + \sqrt{\theta(1-\theta)}\,\mathrm{Ent}_{2,\rho}(I_{2,2}(C)) + \sqrt{\theta(1-\theta)}\,\mathrm{Ent}_{2,\rho}(I_{2,2}(C^*)) \end{aligned} \tag{3.15}$$

Here the map $I_{2,2}$ is defined as $I_{2,2}(X) := \Gamma_\rho^{-\frac{1}{2}}\left(\left|\Gamma_\rho^{\frac{1}{2}}(X)\right|\right)$.

We first recall two very simple results in matrix analysis.

Lemma 31. *Let $A, B \geq 0$, then the block matrix $\left(\begin{smallmatrix} A & C \\ C^* & B \end{smallmatrix}\right) \geq 0$ if and only if $C = A^{\frac{1}{2}} R B^{\frac{1}{2}}$ for some contraction R.*

Proof of Lemma 31. Assume first $A, B > 0$. Note that we have the following congruence transformation

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \mapsto \begin{pmatrix} A^{-\frac{1}{2}} & \\ & B^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \begin{pmatrix} A^{-\frac{1}{2}} & \\ & B^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} I & A^{-\frac{1}{2}} C B^{-\frac{1}{2}} \\ B^{-\frac{1}{2}} C^* A^{-\frac{1}{2}} & I \end{pmatrix} \geq 0. \tag{3.16}$$

We denote $R = A^{-\frac{1}{2}} C B^{-\frac{1}{2}}$, then $\left(\begin{smallmatrix} I & R \\ R^* & I \end{smallmatrix}\right) \geq 0$ if and only if $I \geq R^* I^{-1} R = R^* R$ i.e. R is a contraction. This proves the proposition when $A, B > 0$. The general case follows by a continuity argument. $\square$

Lemma 32. *Let $M = \left(\begin{smallmatrix} A & C \\ C^* & B \end{smallmatrix}\right) \geq 0$, $m = \left(\begin{smallmatrix} \|A\|_p & \|C\|_p \\ \|C^*\|_p & \|B\|_p \end{smallmatrix}\right)$. Then we $\|M\|_p \leq \|m\|_p$ for $1 \leq p \leq \infty$ and $p \in \mathbb{N}$.*

Proof of Lemma 32. We show this by brute-force calculation. From Lemma 31 and Hölder inequality we see that $m \geq 0$. The details is the same as the proof of Lemma 30 as we will see later so we do not repeat it here. We denote

$$R_{11} = A, \quad R_{12} = C, \quad R_{21} = C^*, \quad R_{22} = B, \quad E_{ij} = e_i e_j^* \quad (i, j = 1, 2) \tag{3.17}$$

and

$$r_{11} = \|A\|_p, \quad r_{12} = \|C\|_p, \quad r_{21} = \|C^*\|_p, \quad r_{22} = \|B\|_p. \tag{3.18}$$

Then

$$M = E_{11} \otimes A + E_{12} \otimes C + E_{21} \otimes C^* + E_{22} \otimes B = \sum_{i,j=1}^{2} E_{ij} \otimes R_{ij}, \quad m = \sum_{i,j=1}^{2} r_{ij} E_{ij}. \tag{3.19}$$

Thus we have

$$\|M\|_p^p = \mathrm{Tr}\,|M|^p = \mathrm{Tr}\, M^p = \sum_{i_1, j_1, \cdots, i_p, j_p = 1,2} \mathrm{Tr}\Big(E_{i_1 j_1} \cdots E_{i_p j_p}\Big) \mathrm{Tr}\Big(R_{i_1 j_1} \cdots R_{i_p j_p}\Big). \tag{3.20}$$

$$\|m\|_p^p = \mathrm{Tr}\,|m|^p = \mathrm{Tr}\, m^p = \sum_{i_1, j_1, \cdots, i_p, j_p = 1,2} r_{i_1 j_1} \cdots r_{i_p j_p} \mathrm{Tr}\Big(E_{i_1 j_1} \cdots E_{i_p j_p}\Big). \tag{3.21}$$

Since every entry of E_{ij} is non-negative, we have $\mathrm{Tr}\Big(E_{i_1 j_1} \cdots E_{i_p j_p}\Big) \geq 0$. Then the required inequality follows readily from the trace Hölder inequality $\mathrm{Tr}\Big(R_{i_1 j_1} \cdots R_{i_p j_p}\Big) \leq r_{i_1 j_1} \cdots r_{i_p j_p}$. $\square$

Remark 33. *In fact this result can be generalized to the case for p being any real value in $0 \leq p \leq \infty$ but with much more complicated proof. In fact, we have $\|M\|_p \leq \|m\|_p$ for $2 \leq p \leq \infty$ and $\|M\|_p \geq \|m\|_p$ for $1 \leq p \leq 2$. We refer to [Kin03] for more details. Moreover, for $p = 1$ and $p = 2$, the inequality becomes equality.*

Proof of Lemma 30. We denote $M_p := \left(\begin{smallmatrix} \|A\|_{p,\rho} & \|C\|_{p,\rho} \\ \|C^*\|_{p,\rho} & \|B\|_{p,\rho} \end{smallmatrix}\right)$. In fact, we will show that $M_p \geq 0$ and in particular $M = M_2 \geq 0$. Since $X \geq 0$, we have $A \geq 0$ and $B \geq 0$. Moreover,

$$\Gamma_{I \otimes \rho}^{\frac{1}{p}}(X) = \begin{pmatrix} \rho^{\frac{1}{p}} & \\ & \rho^{\frac{1}{p}} \end{pmatrix} \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \begin{pmatrix} \rho^{\frac{1}{p}} & \\ & \rho^{\frac{1}{p}} \end{pmatrix} = \begin{pmatrix} \Gamma_\rho^{\frac{1}{p}}(A) & \Gamma_\rho^{\frac{1}{p}}(C) \\ \Gamma_\rho^{\frac{1}{p}}(C^*) & \Gamma_\rho^{\frac{1}{p}}(B) \end{pmatrix} \geq 0. \tag{3.22}$$

By Lemma 31, we know that there exists a contraction $R \in \mathcal{B}(\mathcal{H}')$ such that $\Gamma_\rho^{\frac{1}{p}}(C) = \Gamma_\rho^{\frac{1}{p}}(A)^{\frac{1}{2}} R \Gamma_\rho^{\frac{1}{p}}(B)^{\frac{1}{2}}$. By trace Hölder inequality eq. (2.1), we have

$$\begin{aligned}\left\|\Gamma_p^{\frac{1}{p}}(C)\right\|_p &\le \left\|[\Gamma_p^{\frac{1}{p}}(A)]^{\frac{1}{2}}\right\|_{2p} \|R\|_\infty \left\|[\Gamma_p^{\frac{1}{p}}(B)]^{\frac{1}{2}}\right\|_{2p} \\ &\overset{R\text{ contraction}}{\le} \left\|[\Gamma_p^{\frac{1}{p}}(A)]^{\frac{1}{2}}\right\|_{2p} \left\|[\Gamma_p^{\frac{1}{p}}(B)]^{\frac{1}{2}}\right\|_{2p} \overset{\text{by defn.}}{=} \left\|\Gamma_p^{\frac{1}{p}}(A)\right\|_p^{\frac{1}{2}} \left\|\Gamma_p^{\frac{1}{p}}(B)\right\|_p^{\frac{1}{2}}\end{aligned} \tag{3.23}$$

In other words,

$$\|C\|_{p,\rho} \le \|A\|_{p,\rho}^{\frac{1}{2}} \|B\|_{p,\rho}^{\frac{1}{2}} \Rightarrow M_p \ge 0. \tag{3.24}$$

Therefore, $\mathrm{Ent}_{2,\rho}(M) = \frac{1}{2} H(\Gamma_\rho^{\frac{1}{2}}(M)^2 \| \rho)$ is well-defined. We note that

$$\begin{aligned}\|X\|_{p,\sigma\otimes\rho} &= \left\|\begin{pmatrix} \theta^{\frac{1}{p}} \Gamma_\rho^{\frac{1}{p}}(A) & (\theta(1-\theta))^{\frac{1}{2p}} \Gamma_\rho^{\frac{1}{p}}(C) \\ (\theta(1-\theta))^{\frac{1}{2p}} \Gamma_\rho^{\frac{1}{p}}(C^*) & (1-\theta)^{\frac{1}{p}} \Gamma_\rho^{\frac{1}{p}}(B)\end{pmatrix}\right\|_p, \\ \left\|M_p\right\|_{p,\sigma} &= \left\|\begin{pmatrix} \theta^{\frac{1}{p}} \|\Gamma_\rho^{\frac{1}{p}}(A)\|_p & (\theta(1-\theta))^{\frac{1}{2p}} \left\|\Gamma_\rho^{\frac{1}{p}}(C)\right\|_p \\ (\theta(1-\theta))^{\frac{1}{2p}} \left\|\Gamma_\rho^{\frac{1}{p}}(C^*)\right\|_p & (1-\theta)^{\frac{1}{p}} \left\|\Gamma_\rho^{\frac{1}{p}}(B)\right\|_p\end{pmatrix}\right\|_p\end{aligned} \tag{3.25}$$

We denote $\psi(p) := \left\|M_p\right\|_{p,\sigma} - \|X\|_{p,\sigma\otimes\rho}$. By Lemma 32 (generally, Theorem 1 in [Kin03]), we have $\psi(p) \ge 0$ for $p \ge 2$. Moreover, since the inequality in Lemma 32 becomes an equality for $p = 1, 2$, we have $\psi(2) = 0$. Therefore $\psi'(2) = \lim_{\delta\to 0+} \frac{\psi(2+\delta)}{\delta} \ge 0$. On the other hand, using eq. (3.8) repeatedly, we can compute the derivative of $\psi(p)$ as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\|X\|_{p,\sigma\otimes\rho}\Big|_{p=2} &= \frac{1}{2}\|X\|_{2,\sigma\otimes\rho}^{-1} \mathrm{Ent}_{2,\sigma\otimes\rho}(X), \\ \frac{\mathrm{d}}{\mathrm{d}p}\|M\|_{p,\sigma}\Big|_{p=2} &= \frac{1}{2}\|M\|_{2,\sigma}^{-1}\left(\mathrm{Ent}_{2,\sigma}(M) + 4\,\mathrm{Tr}\left[\Gamma_\sigma^{\frac{1}{2}}(M_2')\Gamma_\sigma^{\frac{1}{2}}(M)\right]\right).\end{aligned} \tag{3.26}$$

Here,

$$M_2' = \frac{\mathrm{d}}{\mathrm{d}p} M_p\Big|_2 = \frac{1}{2}\begin{pmatrix} \|A\|_{2,\rho}^{-1}\,\mathrm{Ent}_{2,\rho}(A) & \|C\|_{2,\rho}^{-1}(\frac{1}{2}\,\mathrm{Ent}_{2,\rho}[I_{2,2}(C)] + \frac{1}{2}\,\mathrm{Ent}_{2,\rho}[I_{2,2}(C^*)]) \\ \|C\|_{2,\rho}^{-1}(\frac{1}{2}\,\mathrm{Ent}_{2,\rho}[I_{2,2}(C)] + \frac{1}{2}\,\mathrm{Ent}_{2,\rho}[I_{2,2}(C^*)]) & \|B\|_{2,\rho}^{-1}\,\mathrm{Ent}_{2,\rho}(B)\end{pmatrix} \tag{3.27}$$

We conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\left\|M_p\right\|_{p,\sigma}\Big|_{p=2} = \frac{1}{2}\|M\|_{2,\sigma}^{-1}\cdot\Big(&\mathrm{Ent}_{2,\sigma}(M) + \theta\,\mathrm{Ent}_{2,\rho}(A) + (1-\theta)\,\mathrm{Ent}_{2,\rho}(B) \\ &+ \sqrt{\theta(1-\theta)}\,\mathrm{Ent}_{2,\rho}\left(I_{2,2}(C)\right) + \sqrt{\theta(1-\theta)}\,\mathrm{Ent}_{2,\rho}\left(I_{2,2}(C^*)\right)\Big).\end{aligned} \tag{3.28}$$

The desired inequality follows using $\|X\|_{2,\sigma\otimes\rho} = \|M\|_{2,\sigma}$ and comparing eq. (3.15) with eq. (3.28). □

The second technical lemma concerns a similar tensorization result for the Dirichlet form.

Lemma 34. *For any Lindbladian $\mathcal{K}$ that is ρ-symmetric for some $\rho \in \mathcal{D}_+(\mathcal{H}')$, we have*

$$\mathcal{E}_{\mathcal{K}}(I_{2,2}(C)) + \mathcal{E}_{\mathcal{K}}(I_{2,2}(C^*)) \le -\langle C, \mathcal{K}(C)\rangle_\rho - \langle C^*, \mathcal{K}(C^*)\rangle_\rho, \quad \forall C \in \mathcal{B}(\mathcal{H}'). \tag{3.29}$$

Again, we need a basic result for positive definiteness of block matrices.

Lemma 35. *Let $A \in M_n(\mathbb{C})$, then the block matrix $M = \begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix}$ is positive semidefinite.*

Proof of Lemma 35. It follows readily by polar decomposition of $A = U|A|$ where U is unitary:

$$M = \begin{pmatrix} |A| & |A|U^* \\ U|A| & U|A|U^* \end{pmatrix} = \begin{pmatrix} I & \\ & U \end{pmatrix} \begin{pmatrix} |A| & |A| \\ |A| & |A| \end{pmatrix} \begin{pmatrix} I & \\ & U^* \end{pmatrix} \geq 0. \tag{3.30}$$

□

Proof of Lemma 34. We denote $D := \Gamma_\rho^{\frac{1}{2}}(C)$. Then for $j \in \{0, 1\}$,

$$Y_j := \begin{pmatrix} |D| & (-1)^j D^* \\ (-1)^j D & |D^*| \end{pmatrix} \overset{\text{Lemma 35}}{\geq} 0. \tag{3.31}$$

It is easy to see that $\mathrm{id} \otimes \Gamma_\rho^{-\frac{1}{2}}$ is a congruent transformation of matrices (In fact, $\Gamma_\rho^{-\frac{1}{2}}$ is completely positive). Thus we have

$$Z_j := [\mathrm{id} \otimes \Gamma_\rho^{-\frac{1}{2}}](Y_j) \overset{\text{Recall } I_{2,2}(C) = \Gamma_\rho^{-\frac{1}{2}}(|D|)}{=} \begin{pmatrix} I_{2,2}(C) & (-1)^j C^* \\ (-1)^j C & I_{2,2}(C^*) \end{pmatrix} \geq 0. \tag{3.32}$$

On the other hand, the QMS $\Psi_t = e^{t\mathcal{K}}$ is completely positive, thus we have

$$[\mathrm{id} \otimes \Psi_t](Z_0) = \begin{pmatrix} \Psi_t(I_{2,2}(C)) & \Psi_t(C^*) \\ \Psi_t(C) & \Psi_t(I_{2,2}(C^*)) \end{pmatrix} \geq 0, \quad \forall t \geq 0. \tag{3.33}$$

Pushing together eq. (3.32) and eq. (3.33), we have

$$g(t) := \langle Z_1, [\mathrm{id} \otimes \Psi_t](Z_0) \rangle_{I \otimes \rho} = \mathrm{Tr}\Big((I \otimes \rho)^{\frac{1}{2}} Z_1 (I \otimes \rho)^{\frac{1}{2}} [\mathrm{id} \otimes \Psi_t](Z_0) \Big) \geq 0, \quad \forall t \geq 0. \tag{3.34}$$

Expanding the terms in the inner product, we find that $g(t)$ is given by

$$g(t) = \langle I_{2,2}(C), \Psi_t(I_{2,2}(C)) \rangle_\rho + \langle C^*, \Psi_t(C^*) \rangle_\rho - \langle C, \Psi_t(C) \rangle_\rho - \langle I_{2,2}(C^*), \Psi_t(I_{2,2}(C^*)) \rangle_\rho. \tag{3.35}$$

Differentiate on both sides, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) \bigg|_{t=0} = \langle I_{2,2}(C), \mathcal{K}(I_{2,2}(C)) \rangle_\rho + \langle C^*, \mathcal{K}(C^*) \rangle_\rho - \langle C, \mathcal{K}(C) \rangle_\rho - \langle I_{2,2}(C^*), \mathcal{K}(I_{2,2}(C^*)) \rangle_\rho \tag{3.36}$$

On the other hand,

$$g(0) = \left\| I_{2,2}(C) \right\|_{2,\rho}^2 - \|C^*\|_{2,\rho}^2 - \|C\|_{2,\rho}^2 + \left\| I_{2,2}(C^*) \right\|_{2,\rho}^2 \overset{\text{Recall } I_{2,2}(C) = \Gamma_\rho^{-\frac{1}{2}}(|\Gamma_\rho^{\frac{1}{2}}(C)|)}{=} 0. \tag{3.37}$$

Thus we have $g'(0) \geq 0$, which is the desired inequality. □

With Lemma 30 and Lemma 34 at hand, we are now at the position to prove the tensorization of the log-Sobolev constant for qubit generators.

Proof of Theorem 29. We denote $\alpha = \min\{\alpha_2(\mathcal{L}), \alpha_2(\mathcal{K})\}$. We claim that $\alpha_2(\mathcal{L} \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{K}) \leq \alpha$. In fact, for any $Y \in \mathcal{D}_+(\mathcal{H})$ and $Y' \in \mathcal{D}_+(\mathcal{H}')$, we have

$$\begin{aligned} \mathrm{Ent}_{2,\sigma\otimes\rho}(Y \otimes Y') &= \frac{1}{2} H(\Gamma_{\sigma\otimes\rho}^{\frac{1}{2}}(Y \otimes Y')^2 \| \sigma \otimes \rho) = \frac{1}{2} H(\Gamma_\sigma^{\frac{1}{2}}(Y) \otimes \Gamma_\rho^{\frac{1}{2}}(Y') \| \sigma \otimes \rho) \\ &= \frac{1}{2} H(\Gamma_\sigma^{\frac{1}{2}}(Y) \| \sigma) + H(\Gamma_\rho^{\frac{1}{2}}(Y') \| \rho) = \mathrm{Ent}_{2,\sigma}(Y) + \mathrm{Ent}_{2,\rho}(Y') \end{aligned} \tag{3.38}$$

Moreover, since $\mathcal{L}^*(\sigma) = 0$, $\mathcal{K}^*(\rho) = 0$ (see eq. (2.42)), for any $Y \in \mathcal{B}(\mathcal{H})$ and $Y' \in \mathcal{B}(\mathcal{H}')$,

$$\begin{aligned}\mathcal{E}_{\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{K}}(Y\otimes\widetilde{\rho}) &= -\langle Y,\mathcal{L}(Y)\rangle_\sigma\langle\widetilde{\rho},\widetilde{\rho}\rangle_\rho - \langle Y,Y\rangle_\sigma\langle\widetilde{\rho},\mathcal{K}(\widetilde{\rho})\rangle_\rho \\ &= -\langle Y,\mathcal{L}(Y)\rangle_\sigma = \mathcal{E}_{\mathcal{L}}(Y), \quad \text{where } \widetilde{\rho} = \frac{\rho}{\|\rho\|_{2,\rho}^2},\end{aligned} \tag{3.39}$$

$$\begin{aligned}\mathcal{E}_{\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{K}}(\widetilde{\sigma}\otimes Y') &= -\langle\widetilde{\sigma},\mathcal{L}(\widetilde{\sigma})\rangle_\sigma\langle Y',Y'\rangle_\rho - \langle\widetilde{\sigma},\widetilde{\sigma}\rangle_\sigma\langle Y',\mathcal{K}(Y')\rangle_\rho = -\langle Y',\mathcal{K}(Y')\rangle_\sigma \\ &= \mathcal{E}_{\mathcal{K}}(Y'), \quad \text{where } \widetilde{\sigma} = \frac{\sigma}{\|\sigma\|_{2,\sigma}^2},\end{aligned} \tag{3.40}$$

Thus we have

$$\alpha_2(\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{K}) \le \frac{\mathcal{E}_{\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{K}}(Y\otimes\widetilde{\rho})}{\mathrm{Ent}_{2,\sigma}(Y)+\mathrm{Ent}_{2,\rho}(\widetilde{\rho})} \le \frac{\mathcal{E}_{\mathcal{L}}(Y)}{\mathrm{Ent}_{2,\sigma}(Y)} \Rightarrow \alpha_2(\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{K}) \le \alpha_2(\mathcal{L}). \tag{3.41}$$

Likewise,

$$\alpha_2(\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{K}) \le \frac{\mathcal{E}_{\mathcal{K}}(Y')}{\mathrm{Ent}_{2,\rho}(Y')}(\forall Y'\in\mathcal{B}(\mathcal{H}')) \Rightarrow \alpha_2(\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{K}) \le \alpha. \tag{3.42}$$

To show the other direction, we need to prove that for any $X \in \mathbb{H}^{>0}_{\mathcal{H}\otimes\mathcal{H}'}$, we have

$$\alpha\,\mathrm{Ent}_{2,\sigma\otimes\rho}(X) \le \mathcal{E}_{\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\,\otimes\mathcal{K}}(X) \tag{3.43}$$

which implies the reverse inequality by taking the infimum over X. We denote $X = \left(\begin{smallmatrix} A & C \\ C^* & B\end{smallmatrix}\right)$. By the arbitrariness of X, we can W.L.O.G. assume that σ has the diagonal form $\sigma = \mathrm{diag}(\theta, 1-\theta)$ where $\theta \in (0,1)$. Then we can use Lemma 30 to obtain

$$\begin{aligned}\mathrm{Ent}_{2,\sigma\otimes\rho}(X) \le\, &\mathrm{Ent}_{2,\sigma}(M) + \theta\,\mathrm{Ent}_{2,\rho}(A) + (1-\theta)\,\mathrm{Ent}_{2,\rho}(B) \\ &+ \sqrt{\theta(1-\theta)}\,\mathrm{Ent}_{2,\rho}\left(I_{2,2}(C)\right) + \sqrt{\theta(1-\theta)}\,\mathrm{Ent}_{2,\rho}\left(I_{2,2}\left(C^*\right)\right).\end{aligned} \tag{3.44}$$

On the other hand, by the definition of the 2-log-Sobolev constant, we have

$$\alpha\,\mathrm{Ent}_{2,\sigma}(M) \le \alpha_2(\mathcal{L})\,\mathrm{Ent}_{2,\sigma}(M) \le \mathcal{E}_{\mathcal{L}}(M), \quad \alpha\,\mathrm{Ent}_{2,\rho}(Y) \le \alpha_2(\mathcal{K})\,\mathrm{Ent}_{2,\rho}(Y) \le \mathcal{E}_{\mathcal{K}}(Y). \tag{3.45}$$

Thus we can estimate $\mathrm{Ent}_{2,\sigma\otimes\rho}(X)$ further using Lemma 34 as

$$\begin{aligned}\alpha\,\mathrm{Ent}_{2,\sigma\otimes\rho}(X) \le\, &\mathcal{E}_{\mathcal{L}}(M) + \theta\mathcal{E}_{\mathcal{K}}(A) + (1-\theta)\mathcal{E}_{\mathcal{K}}(B) \\ &+ \sqrt{\theta(1-\theta)}\mathcal{E}_{\mathcal{K}}(I_{2,2}(C)) + \sqrt{\theta(1-\theta)}\mathcal{E}_{\mathcal{K}}(I_{2,2}\left(C^*\right)) \\ &\overset{\text{Lemma 34}}{\le} \mathcal{E}_{\mathcal{L}}(M) + \theta\mathcal{E}_{\mathcal{K}}(A) + (1-\theta)\mathcal{E}_{\mathcal{K}}(B) \\ &- \sqrt{\theta(1-\theta)}\langle C,\mathcal{K}(C)\rangle_\rho - \sqrt{\theta(1-\theta)}\langle C^*,\mathcal{K}(C^*)\rangle_\rho.\end{aligned} \tag{3.46}$$

On the other hand, the Dirichlet form $\mathcal{E}_{\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\,\otimes\mathcal{K}}(X)$ can be computed as follows:

$$\mathcal{E}_{\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\,\otimes\mathcal{K}}(X) = -\langle X,(\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\ \otimes\mathcal{K})(X)\rangle_{\sigma\otimes\rho} = -\langle X,\mathcal{L}\otimes\mathrm{id}(X)\rangle_{\sigma\otimes\rho} - \langle X,\mathrm{id}\otimes\mathcal{K}(X)\rangle_{\sigma\otimes\rho}. \tag{3.47}$$

We can estimate both terms separately. For the first term, we first compute the action of $\mathcal{L}\otimes\mathrm{id}$ on X:

$$(\mathcal{L}\otimes\mathrm{id})(X) = \mathcal{L}(E_{11})\otimes A + \mathcal{L}(E_{12})\otimes C + \mathcal{L}(E_{21})\otimes C^* + \mathcal{L}(E_{22})\otimes B. \tag{3.48}$$

Since $\mathcal{L}(X) = \mathrm{Tr}(\sigma X) - X$, we have $\mathcal{L}(E_{11}) = \theta I - E_{11} = \mathrm{diag}(\theta - 1, \theta)$, $\mathcal{L}(E_{12}) = -E_{12}$, $\mathcal{L}(E_{21}) = -E_{21}$ and $\mathcal{L}(E_{22}) = (1-\theta)I - E_{22} = \mathrm{diag}(1-\theta, -\theta)$. Thus we have

$$(\mathcal{L} \otimes \mathrm{id})(X) = \begin{pmatrix} -(1-\theta)A + (1-\theta)B & -C \\ -C^* & \theta A - \theta B \end{pmatrix} = \begin{pmatrix} (1-\theta)(B-A) & -C \\ -C^* & \theta(A-B) \end{pmatrix}. \tag{3.49}$$

Therefore,

$$\begin{aligned} -\langle X, \mathcal{L} \otimes \mathrm{id}(X) \rangle_{\sigma\otimes\rho} &= \left\langle \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}, \begin{pmatrix} (1-\theta)(A-B) & C \\ C^* & \theta(B-A) \end{pmatrix} \right\rangle_{\sigma\otimes\rho} \\ &\overset{\text{computing the diagonal blocks}}{=} \theta(1-\theta)\langle A, A-B\rangle_\rho + \theta(1-\theta)\langle B, B-A\rangle_\rho \\ &\quad + 2\sqrt{\theta(1-\theta)}\langle C, C^*\rangle_\rho \\ &= \theta(1-\theta)\|A\|_{2,\rho}^2 + \theta(1-\theta)\|B\|_{2,\rho}^2 - 2\theta(1-\theta)\langle A, B\rangle_\rho + 2\sqrt{\theta(1-\theta)}\|C\|_{2,\rho}^2 \\ &\overset{\text{Cauchy-Schwarz}}{\geq} \theta(1-\theta)\|A\|_{2,\rho}^2 + \theta(1-\theta)\|B\|_{2,\rho}^2 - 2\theta(1-\theta)\|A\|_{2,\rho}\|B\|_{2,\rho} \\ &\quad + 2\sqrt{\theta(1-\theta)}\|C\|_{2,\rho}^2 \\ &\overset{\text{by defn}}{=} -\langle M, \mathcal{L}(M)\rangle_\sigma = \mathcal{E}_{\mathcal{L}}(M), \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} -\langle X, \mathrm{id} \otimes \mathcal{K}(X) \rangle_{\sigma\otimes\rho} &= -\left\langle \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}, \begin{pmatrix} \mathcal{K}(A) & \mathcal{K}(C) \\ \mathcal{K}(C^*) & \mathcal{K}(B) \end{pmatrix} \right\rangle_{\sigma\otimes\rho} \\ &\overset{\text{computing diag. blocks}}{=} -\theta\langle A, \mathcal{K}(A)\rangle_\rho - (1-\theta)\langle B, \mathcal{K}(B)\rangle_\rho \\ &\quad - \sqrt{\theta(1-\theta)}[\langle C, \mathcal{K}(C)\rangle_\rho + \langle C^*, \mathcal{K}(C^*)\rangle_\rho] \\ &= \theta\mathcal{E}_{\mathcal{K}}(A) + (1-\theta)\mathcal{E}_{\mathcal{K}}(B) - \sqrt{\theta(1-\theta)}[\langle C, \mathcal{K}(C)\rangle_\rho + \langle C^*, \mathcal{K}(C^*)\rangle_\rho]. \end{aligned} \tag{3.51}$$

Therefore we have

$$\mathcal{E}_{\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\otimes\mathcal{K}}(X) \geq \mathcal{E}_{\mathcal{L}}(M) + \theta\mathcal{E}_{\mathcal{K}}(A) + (1-\theta)\mathcal{E}_{\mathcal{K}}(B) - \sqrt{\theta(1-\theta)}[\langle C, \mathcal{K}(C)\rangle_\rho + \langle C^*, \mathcal{K}(C^*)\rangle_\rho]. \tag{3.52}$$

Combined with eq. (3.46), we have

$$\alpha\, \mathrm{Ent}_{2,\sigma\otimes\rho}(X) \leq \mathcal{E}_{\mathcal{L}\otimes\mathrm{id}+\mathrm{id}\,\otimes\mathcal{K}}(X), \quad \forall X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}'). \tag{3.53}$$

□

Example 36. *We recall Remark 28, the log-Sobolev constant of generalized depolarizing channel is explicitly computed as*

$$\alpha_2(\mathcal{L}) = \frac{2(1-2\mu_{\min})(\sigma)}{\log(1/\mu_{\min}(\sigma) - 1)}. \tag{3.54}$$

Thus we can already get a result for the e.g. n-fold product channel of qubit ($\mathbb{C}^2$) depolarizing semigroup i.e. $\mathcal{K}_n = \sum_{i=1}^n \widehat{\mathcal{L}}_i$, we have

$$\alpha_2(\mathcal{K}_n) = \alpha_2(\mathcal{L}) = \frac{2(1-2\mu_{\min})(\sigma)}{\log(1/\mu_{\min}(\sigma) - 1)} \tag{3.55}$$

which has a better preconstant than the one in example 27 derived from the direct summation result Theorem 26 with mere knowledge of the system dimension and the spectral gap.

Even nicer, we can actually extend this result to general qubit generators other than the depolarizing channel, using the standard comparison of Dirichlet forms.

Corollary 37. *Let $\mathcal{H} = \mathbb{C}^2$ and $\sigma \in \mathcal{D}_+(\mathcal{H})$. $\mathcal{L}$ is a σ-symmetric primitive Lindbladian, then we have*

$$\alpha_2(\mathcal{K}_n) \geq \frac{2(1-2\mu_{\min}(\sigma))}{\log(1/\mu_{\min}(\sigma)-1)}\lambda(\mathcal{L}). \tag{3.56}$$

Here $\mathcal{K}_n := \sum_{i=1}^n \widehat{\mathcal{L}_i}$ is the n-fold product Lindbladian of the qubit generator $\mathcal{L}$, $\lambda(\mathcal{L})$ is the spectral gap of $\mathcal{L}$ and $\mu_{\min}(\sigma)$ denotes the minimal eigenvalue of σ.

Proof. We denote the generalied qubit depolarizing channel as $\mathcal{L}'(X) := \mathrm{Tr}(\sigma X)I - X$, $\alpha_2(\mathcal{L}') = \frac{2(1-2\mu_{\min})(\sigma)}{\log(1/\mu_{\min}(\sigma)-1)}$.. Let $X \in \mathbb{H}^{>0}_{\mathcal{H}^{\otimes n}}$ be any positive definite matrix on the product space. By Theorem 29, we have $\alpha_2(\sum_{i=1}^n \widehat{\mathcal{L}_i}) = \alpha_2(\mathcal{L}') = \frac{2(1-2\mu_{\min})(\sigma)}{\log(1/\mu_{\min}(\sigma)-1)}$. Thus we can write the following 2-LSI for the n-fold product of $\mathcal{L}'$:

$$\frac{2(1-2\mu_{\min})(\sigma)}{\log(1/\mu_{\min}(\sigma)-1)}\mathrm{Ent}_{2,\sigma^{\otimes n}}(X) \leq -\sum_{i=1}^n \langle X, \widehat{\mathcal{L}'}_i(X)\rangle_{\sigma^{\otimes n}} = \sum_{i=1}^n \mathcal{E}_{\widehat{\mathcal{L}'_i},\sigma^{\otimes n}}(X) \tag{3.57}$$

Let $\mathcal{W}_i \subset \mathcal{B}(\mathcal{H}^{\otimes n})$ be the subspace spanned by operators with tensor product form whose support does not contain the i-th qubit, i.e.

$$\mathcal{W}_i := \{A_i \otimes \cdots \otimes A_n : A_k \in \mathcal{B}(\mathcal{H})(1 \leq k \leq n), \quad A_i = I\}. \tag{3.58}$$

Note that $\mathcal{L}'(I) = 0$ and $\mathcal{L}'$ is primitive, we have $\mathcal{W}_i = \ker \widehat{\mathcal{L}'}_i$.

We note that for any tensor product operator $X_1 \otimes \cdots \otimes X_n$, $X_1 \otimes \cdots \otimes X_n \perp \mathcal{W}_i$ if and only if $\mathrm{Tr}(X_i \Gamma_{\sigma^{\otimes n}}(I)) = 0$ i.e. $\mathrm{Tr}(\sigma X_i) = 0$. Therefore, the action of the projection $P_{\mathcal{W}_i}$ onto $\mathcal{W}_i$ on tensor product operators is given by $P_{\mathcal{W}_i}(X_1 \otimes \cdots \otimes X_n) = \mathrm{Tr}(\sigma X_i) X_1 \otimes \cdots \otimes I_i \otimes \cdots \otimes X_n$ (one can verify that $P^2_{\mathcal{W}_i} = P_{\mathcal{W}_i}$, $\mathrm{Range}P_{\mathcal{W}_i} = \mathcal{W}_i$ and $\langle X, P_{\mathcal{W}_i}(X)\rangle_\sigma = \langle P_{\mathcal{W}_i}(X), X\rangle_\sigma$). That is, the action of $-\widehat{\mathcal{L}'_i}$ on tensor product operator is

$$-\widehat{\mathcal{L}'_i}(X_1 \otimes \cdots \otimes X_n) = (\mathrm{id} - P_{\mathcal{W}_i})(X_1 \otimes \cdots \otimes X_n) = P_{\mathcal{W}_i^\perp}(X_1 \otimes \cdots \otimes X_n). \tag{3.59}$$

On the other hand, since any operator $X \in \mathcal{B}(\mathcal{H}^{\otimes n})$ can be decomposed as a linear summation of tensor product operators, we actually have

$$\mathcal{E}_{\widehat{\mathcal{L}'_i},\sigma^{\otimes n}}(X) = -\langle X, \widehat{\mathcal{L}'_i}(X)\rangle_{\sigma^{\otimes n}} = \left\|P_{\mathcal{W}_i^\perp}(X)\right\|^2_{\sigma^{\otimes n}}. \tag{3.60}$$

On the other hand, since $\mathcal{L}$ itself is also primitive and $\mathcal{L}(I) = 0$, we have $\mathcal{W}_i = \ker \widehat{\mathcal{L}_i}$. Moreover, by the variational characterization of the spectral gap, we actually have

$$\lambda(\widehat{\mathcal{L}_i}) = \inf_{X \perp \ker \widehat{\mathcal{L}_i}} \frac{\mathcal{E}_{\widehat{\mathcal{L}_i},\sigma^{\otimes n}}(X)}{\|X\|^2_{\sigma^{\otimes n}}} = \inf_X \frac{\mathcal{E}_{\widehat{\mathcal{L}_i},\sigma^{\otimes n}}(X)}{\left\|P_{\mathcal{W}_i^\perp}(X)\right\|^2_{\sigma^{\otimes n}}} = \inf_X \frac{\mathcal{E}_{\widehat{\mathcal{L}_i},\sigma^{\otimes n}}(X)}{\mathcal{E}_{\widehat{\mathcal{L}'_i},\sigma^{\otimes n}}(X)} \tag{3.61}$$

Moreover, since $\widehat{\mathcal{L}_i}$ is just the tensor product of $\mathcal{L}$ with some identity maps, $\lambda(\widehat{\mathcal{L}_i}) = \lambda(\mathcal{L})$. Therefore

$$\lambda(\mathcal{L}) = \inf_X \frac{\mathcal{E}_{\widehat{\mathcal{L}_i},\sigma^{\otimes n}}(X)}{\mathcal{E}_{\widehat{\mathcal{L}'_i},\sigma^{\otimes n}}(X)}. \tag{3.62}$$

Therefore,

$$\begin{aligned}\frac{2(1-2\mu_{\min}(\sigma))}{\log(1/\mu_{\min}(\sigma)-1)}\lambda(\mathcal{L}) &\overset{\text{using eq. (3.57)}}{\le} \frac{\sum_{i=1}^n \mathcal{E}_{\widehat{\mathcal{L}_i'},\sigma^{\otimes n}}(X)}{\mathrm{Ent}_{2,\sigma^{\otimes n}}(X)}\lambda(\mathcal{L}) \\ &\overset{\text{using eq. (3.62)}}{\le} \frac{\sum_{i=1}^n \mathcal{E}_{\widehat{\mathcal{L}_i},\sigma^{\otimes n}}(X)}{\mathrm{Ent}_{2,\sigma^{\otimes n}}(X)} = \frac{\mathcal{E}_{\mathcal{K}_n}(X)}{\mathrm{Ent}_{2,\sigma^{\otimes n}}(X)}.\end{aligned} \tag{3.63}$$

Taking the infimum over X on both sides, we have $\frac{2(1-2\mu_{\min}(\sigma))}{\log(1/\mu_{\min}(\sigma)-1)}\lambda(\mathcal{L}) \le \alpha_2(\mathcal{K}_n)$ which is the desired result. □

In this section, we reviewed some of the general results for log-Sobolev inequalities on matrix algebra, especially the lower bound of $\alpha_2(\mathcal{L})$ and its tensorization for direct-sum Lindbladians and qubit generators. Interestingly, the entropy inequality Lemma 30 used for showing tensorization of qubit depolarizing semigroup *does not* hold for "qutrits" (i.e. $\mathbb{C}^3$, although it is much less important than qubit generators in quantum physics, see [Aud08]).

3.3 Modified log-Sobolev inequality and its quasi-tensorization

The main material of this section follows Section 10.1.1 of [Rou19].

Due to the above limitations, we aim at indentifying a characterization of the contraction behavior that is weaker than the hypercontractivity, yet which leads to similar estimation to the original hypercontractivity for the rate of convergence. The concept of strong data processing inequality (SDPI), as well as its continuous version known as modified log-Sobolev inequality (MLSI), is a natural candidate for this purpose.

Definition 38 (Strong data processing constant)**.** *Let Φ^* be a σ-DBC quantum Markov map where σ is the full-rank invariant state of Φ. We recall that the decoherence-free subalgebra $\mathcal{N}$ is also the multiplicative domain of Φ^*. The strong data processing constant is defined as*

$$s(\Phi) := \sup_{\rho\in\mathcal{D}(\mathcal{H})} \frac{H(\Phi(\rho)\|\Phi\circ E_{\mathcal{N}^*}(\rho))}{H(\rho\|E_{\mathcal{N}^*}(\rho))}. \tag{3.64}$$

In fact, the SDPI is weaker than the hypercontractivity and stronger than the spectral gap approach. We have the following result:

Proposition 39 (HC $\Longrightarrow$ SDPI $\Longrightarrow$ Gap)**.** *Same assumpations as Definition 38, then:*

(1) $\lambda(\Phi) \le s(\Phi)$;

(2) *If Φ^* is additionally assumed to be primitive and (p,q)-HC, then $s(\Phi) \le \frac{q(p-1)}{p(q-1)}$.*

Proof. The proof of (1) can be found in Theorem 4.1 of [GR22]. The proof of (2) can be found in Proposition 5.4 of [HRSF22]. □

Definition 40 (Modified log-Sobolev inequality)**.** *Let $\mathcal{L}$ be a σ-DBC Lindbladian w.r.t. the full-rank state σ. Let $\mathcal{F}$ be the fixed-point algebra of $\mathcal{P}_t = e^{t\mathcal{L}}$, then a* modified log-Sobolev inequality *(MLSI) is said to hold for $\mathcal{L}$ if there exists a constant α_1 such that*

$$\alpha_1 H(\rho\|E_{\mathcal{F}^*}(\rho)) \le \mathrm{EP}_{\mathcal{L}}(\rho), \quad \forall\rho\in\mathcal{D}(\mathcal{H}). \tag{3.65}$$

Here, $\mathrm{EP}_{\mathcal{L}}(\rho) := -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} H(\mathcal{P}_t^*(\rho)\|E_{\mathcal{F}^*}(\rho))$ *is called the* entropy production *of $\mathcal{L}$. The best constant α_1 statisfying MLSI is called the* MLSI constant *of $\mathcal{L}$ and denoted by $\alpha_1(\mathcal{L})$.*

Again, MLSI is weaker than LSI and we have in general the following strong-weak comparison.

Proposition 41 (LSI $\Longrightarrow$ MLSI $\Longrightarrow$ Gap)**.** *Same assumptions. We have $\alpha_1(\mathcal{L}) \le 2\lambda(\mathcal{L})$. If we additionally assume that $\mathcal{L}$ is primitive, then we have $\alpha_2(\mathcal{L}) \le \frac{1}{2}\alpha_1(\mathcal{L})$.*

Proof. The first claim is Theorem 3.3 in [GR22] and the second claim is Proposition 13 in [KT13]. □

We aim to construct the general theory trying to provide universal lower bounds for MLSI constant $\alpha_1(\mathcal{L})$. Before that, we would like to first focus on a warming-up example of the n-fold product Lindbladian $\mathcal{K}_n$ of generalied depolarizing channels. Basically, the MLSI tensorizes for $\mathcal{K}_n$. We will see the "strong subadditivity" strategy (see e.g. Proposition 4.3.9 of [Bha09]) we used here will be pivotal in the proof of the general argument that we will postpone to the next section.

Theorem 42 (Tensorization of Modified Log-Sobolev Inequality)**.** *Let $\sigma_i \in \mathcal{D}_+(\mathcal{H}_i)$ and $\mathcal{L}_i = \mathrm{Tr}(\sigma_i X)I - X$, where $\mathcal{H}_i$ is a finite dimensional Hilbert space. Denote $\widehat{\mathcal{L}_i}$ and $\mathcal{K}_n$ as in eq. (2.45), then we have $\alpha_1(\mathcal{K}_n) \ge 1$ independently of n.*

Proof. We denote $\mathcal{H} := \bigotimes_{i=1}^n \mathcal{H}_i$, $\sigma = \bigotimes_{i=1}^n \sigma_i$ and $\mathcal{P}_t := \exp(t\mathcal{K}_n)$. Since $\mathcal{L}_i^*(\sigma_i) = 0$, we have $\mathcal{P}_t^*(\sigma) = (\sigma)$ i.e. $E_{\mathcal{F}^*}(\sigma^*) = \sigma^*$. Recall that in eq. (3.59) we have already seen that the action $\widehat{\mathcal{L}_i}$. Using the same approach, we can see that the action of $\widehat{\mathcal{L}_i^*}$ is

$$\widehat{\mathcal{L}_i^*}(\tau_1 \otimes \cdots \otimes \tau_n) = \mathrm{Tr}(\tau_i)\tau_1 \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes \tau_n - \tau_1 \otimes \cdots \otimes \tau_n. \tag{3.66}$$

For brevity of notations, we write $\widehat{\mathcal{L}_i^*}(\tau) = \mathrm{Tr}_{\mathcal{H}_i}(\tau) \otimes \sigma_i - \tau$. [2] Then we can compute the entropy production of $\mathcal{K}_n$ as follows:

$$\begin{aligned}
\mathrm{EP}_{\mathcal{K}_n}(\rho) &= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Tr}(\mathcal{P}_t^*(\rho)\log[\mathcal{P}_t^*(\rho)] - \mathcal{P}_t^*(\rho)\log\sigma) \\
&= -\mathrm{Tr}(\mathcal{K}_n^*(\rho)(\log\rho - \log\sigma)) = -\sum_{i=1}^n \mathrm{Tr}\Big(\widehat{\mathcal{L}_i^*}(\rho)(\log\rho - \log\sigma)\Big) \\
&= \sum_{i=1}^n \mathrm{Tr}\Big[(\rho - \rho^{\sim i} \otimes \sigma_i)\cdot(\log\rho - \log\sigma)\Big] = \sum_{i=1}^n \Big[H(\rho\|\sigma) + H(\rho^{\sim i} \otimes \sigma_i\|\rho) - H(\rho^{\sim i} \otimes \sigma_i\|\sigma)\Big]
\end{aligned} \tag{3.67}$$

Here we denote $\rho^{\sim i} = \mathrm{Tr}_{\mathcal{H}_i}\rho$. We then only need to show that $H(\rho\|\sigma) \le \mathrm{EP}_{\mathcal{K}_n}(\rho)$ for any $\rho \in \mathcal{D}(\mathcal{H})$. Note that the relative entropy is non-negative, it suffices to show

$$H(\rho\|\sigma) \le \sum_{i=1}^n [H(\rho\|\sigma) - H(\rho^{\sim i} \otimes \sigma_i\|\sigma)]. \tag{3.68}$$

Moreover, since $\log(A \otimes B) = \log(A \otimes I) + \log(I \otimes B)$, the above is equivalent to

$$\begin{aligned}
-H(\rho) - \sum_{i=1}^n \mathrm{Tr}(\rho_i \log\sigma_i) &\le \sum_{i=1}^n \left[-H(\rho) - \sum_{j=1}^n \mathrm{Tr}\Big(\rho_j \log\sigma_j\Big) + H(\rho^{\sim i}) + \sum_{j\neq i} \mathrm{Tr}\Big(\rho_j \log\sigma_j\Big)\right] \\
&= \sum_{i=1}^n \Big[-H(\rho) - \mathrm{Tr}(\rho_i \log\sigma_i) + H(\rho^{\sim i})\Big], \quad \text{where } \rho_i = \mathrm{Tr}_{\mathcal{H}^{\sim i}}(\rho).
\end{aligned} \tag{3.69}$$

[2]The first term should be understood in the sense of isomorphism up to a permutation of the order of the tensor product i.e. $\mathrm{Tr}_{\mathcal{H}_i}(\tau) \in \mathcal{H}^{\sim i} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{i-1} \otimes \mathcal{H}_{i+1} \otimes \mathcal{H}_n$, $\sigma_i \in \mathcal{H}_i$ and the tensor product of the above two is in $\mathcal{H}^{\sim i} \otimes \mathcal{H}_i \cong \mathcal{H}$.

This equivalent to $H(\rho) \geq \sum_{i=1}^{n}[H(\rho) - H(\rho^{\sim i})]$, which follows immediately from the subadditivity of entropy once we use the chain rule

$$H(\rho) = H(\rho_1) + \sum_{i=2}^{n}[H(\rho) - H(\rho^{\sim i})]. \tag{3.70}$$

□

Remark 43. *Letting the σ_i to be identical in the above theorem, we obtain the promised tensorization-type result for the modified logarithmic Sobolev constant.*

3.4 Complete inequalities

The materials of this section are mainly from [BR22] and [GR22].

So far, we have developed general tensorization results for LSI and MLSI. However, they still possesses at least the following two limitations:

- The lower bound $\alpha_2(\mathcal{L})$ for unstructural generator (see the end of section 3.1) does not provide any asymptotic improvement over the gap bound.
- The lack of tensorization beyond specific cases like qubit generators in Theorem 29, or without resorting to the spectral approach such as Theorem 26, renders it difficult to adapt the theory to more general settings such as evolutions under correlated subsystems.

As stated previously, it is often helpful to recall the classical setting in order to see what is absent in the quantum case that prevents us to recover the much better properties of classical LSI. In fact, the tensorization of classical Markov semigroups holds mainly for two reasons: (1) the "chain rule" of the relative entropy, which still holds in non-commutative setting; (2) a key interpretation of the *conditional relative entropy* as **the average over the first subsystem** of **the second subsystem conditioned on the former**. That is, given two probability mass functions p_{XY} and q_{XY} over X, Y,

$$H(p_{XY}\|q_{XY}) = H(p_X\|q_X) + \mathbb{E}_{p_X}[H(p_{Y|X}\|q_{Y|X})]. \tag{3.71}$$

If we assume that we are given a product Markov Semigroup $e^{tL_X} \otimes e^{tL_Y}$ generated by $L_X + L_Y$ with an invariant probability measure with the form $q_{XY} = q_X \otimes q_Y$, then formally we have

$$\begin{aligned} H(p_{XY}\|q_{XY}) &= H(\rho_X\|q_X) + \mathbb{E}_{q_X}[H(\rho_{Y|X}\|q_{Y|X})] \\ &\leq \frac{1}{\alpha_2(L_X)}\mathcal{E}_{L_X}(\sqrt{p_X/q_X}) + \frac{1}{\alpha_2(L_Y)}\mathbb{E}_{p_X}\left[\mathcal{E}_{L_Y}\left(\sqrt{p_{Y|X}/q_{Y|X}}\right)\right] \\ &\leq \frac{1}{\min(\alpha_2(L_X), \alpha_2(L_Y))}\left\{\mathcal{E}_{L_X}(\sqrt{p_X/q_X}) + \mathbb{E}_{p_X}\left[\mathcal{E}_{L_Y}\left(\sqrt{p_{Y|X}/q_{Y|X}}\right)\right]\right\} \\ &\overset{\text{chain rules for Dirichlet forms}}{=} \frac{1}{\min(\alpha_2(L_X), \alpha_2(L_Y))}\mathcal{E}_{L_X+L_Y}(\sqrt{p_{XY}/q_{XY}}). \end{aligned} \tag{3.72}$$

The above "derivation" is a formal interpretation of why the tensorization of LSI holds for classical Markov semigroups. It is natural for us to attempt to transfer this to the SDPI/MLSI. Using the chain rule Proposition 6, we see that

$$H(\rho\|E_{\mathcal{N}^*}(\rho)) = H(\rho\|E_{\mathcal{N}_2^*}(\rho)) + H(E_{\mathcal{N}_2^*}(\rho)\|E_{\mathcal{N}^*}(\rho)), \tag{3.73}$$

where $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N} := \mathcal{N}_1 \cap \mathcal{N}_2$ are subalgebras of $\mathcal{B}(\mathcal{H})$. This is a generalization of eq. (3.72) to the quantum setting. However, the problem is the second relative entropy term of eq. (3.73) does not have a clear interpretation as the "conditional average" of the first subsystem over the second subsystem. To be more specific, we assume that the system has a tensor product form $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$. $\mathcal{F}_1$, $\mathcal{F}_2$ be the fixed-point algebras of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, $\mathcal{N}_1 := \mathcal{F}_1 \otimes \mathcal{B}(\mathcal{H}_2)$, $\mathcal{N}_2 := \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{F}_2$ and $\mathcal{N} := \mathcal{N}_1 \cap \mathcal{N}_2$. Then the conditional expectations onto $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}$ are given by

$$E_1 := E_{\mathcal{F}_1} \otimes \mathrm{id}_2, \quad E_2 := \mathrm{id}_1 \otimes E_{\mathcal{F}_2}, \quad E_{\mathcal{N}} = E_{\mathcal{F}_1} \otimes E_{\mathcal{F}_2}. \tag{3.74}$$

Since we cannot use the conditional expectation technique in eq. (3.72), we are forced to prove directly that, $H(\rho \| (\mathrm{id}_1 \otimes E_{\mathcal{F}_2^*})(\rho))$ is upper bounded e.g. by the entropy production $\mathrm{EP}_{\mathrm{id}_1 \otimes \mathcal{L}_2}(\rho)$, to recover our tensorization result. This observation is the generalization of the method we used for the tensorization of MLSI section 3.3. This technique allows us to first consider the weakened chain rule

$$H(\rho \| E_{\mathcal{N}^*}(\rho)) \leq H(\rho \| (E_{\mathcal{F}_1^*} \otimes \mathrm{id}_2)(\rho)) + E(\rho \| (\mathrm{id}_1 \otimes E_{\mathcal{F}_2^*})(\rho)) \tag{3.75}$$

where we use the monotocity of the first relative entropy with respect to the channel $E_{\mathcal{F}_2^*}$. Then the tensorization can be recovered upon assuming that $\mathcal{L}_1 \otimes \mathrm{id}_{M_k}$ and $\mathrm{id}_{M_k} \otimes \mathcal{L}_2$ satisfy the LSI or MLSI for reference system with arbitary dimension k.

The discussion above naturally to the introduction of *complete versions* of the log-Sobolev inequality and modified log-Sobolev inequality. We begin with the definition

Definition 44 (Complete log-Sobolev constant). *Let $\mathcal{L}$ be a σ-DBC Lindbladian w.r.t. the full rank state σ. We define the complete log-Sobolev constant $\alpha_{2,c}(\mathcal{L})$ as the infimum of all constants $\alpha_2(\mathcal{L} \otimes id_{M_k(\mathbb{C})})$, that is,*

$$\alpha_{2,c}(\mathcal{L}) = \inf_{k \in \mathbb{N}} \alpha_2(\mathcal{L} \otimes id_{M_k(\mathbb{C})}). \tag{3.76}$$

Here we recall

$$\alpha_2(\mathcal{K}) = \inf_X \frac{\mathcal{E}_{\mathcal{K}}(X)}{\mathrm{Ent}_{2,\sigma}(X)}. \tag{3.77}$$

With this definition at hand and in view of eq. (3.75), we can see that $\alpha_{2,c}(\mathcal{L})$ is a natural candidate for the tensorization of the log-Sobolev inequality. Unfortunately, we will prove that such constant will be trivially zero for all such generators.

Impossibility of complete LSI. The first focus of this section is to show the following theorem:

Theorem 45 (Impossibility of complete LSI). *Let $\mathcal{L}$ be a **non-primitive** generator of a QMS over $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a finite dimensional Hilbert space. Assume that the multiplicative domain $\mathcal{N}$ of $\mathcal{L}$ is non-trivial, i.e. $\mathcal{N} \neq \mathbb{C}I$ and $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$. Define*

$$\sigma_{\mathrm{Tr}} = E_{\mathcal{N}^*}\left(\frac{I_{\mathcal{H}}}{d_{\mathcal{H}}}\right), \tag{3.78}$$

then we have $\alpha_2(\mathcal{L}) = 0$.

Proof. By comparison of Dirichlet forms (see eq. (3.62), also Theorem 3.3 in [MHSFW16a]) in Corollary 37, it suffices to consider $\mathcal{L}' = E_{\mathcal{N}} - \mathrm{id}$ where $\mathcal{N}$ is any $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Specifically, for any $\mathcal{L}$ with $\mathcal{N}$ being its multiplicative domain and the same conditional expectation $E_{\mathcal{N}}$, we have

$$\lambda(\mathcal{L}) \leq \frac{\mathcal{E}_{\mathcal{L}}(X)}{\mathcal{E}_{\mathcal{L}'}(X)} \leq \frac{\left\| \mathcal{L} + \widehat{\mathcal{L}} \right\|_{L_2(\sigma_{\mathrm{Tr}})}}{2}. \tag{3.79}$$

Here we denote $\widehat{\mathcal{L}}$ as the adjoint of $\mathcal{L}$ w.r.t. the inner product $\langle\cdot,\cdot\rangle_{\sigma_{\mathrm{Tr}}} := \langle\cdot,\cdot\rangle_{\sigma_{\mathrm{Tr}},\frac{1}{2}}$. From which, we easily see that if $\alpha_2(\mathcal{L}') = 0$, then $\alpha_2(\mathcal{L}) = 0$. Then it suffices for us to construct a sequence $\{Z_k\}$ such that

$$\frac{\mathcal{E}_{\mathcal{L}'}(Z_k)}{\mathrm{Ent}_{2,\sigma_{\mathrm{Tr}}}(Z_k)} \to \infty. \tag{3.80}$$

More specifically, we shall construct a sequence $\{\rho_k\}$ of density matrices such that

$$\frac{\mathcal{E}_{\mathcal{L}'}(\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\rho_k^{\frac{1}{2}}\sigma_{\mathrm{Tr}}^{-\frac{1}{4}})}{H(\rho_k\|E_{\mathcal{N}^*}(\rho_k))} \to \infty. \tag{3.81}$$

This can be easily seen by taking $Z_k = \Gamma_{\sigma_{\mathrm{Tr}}}^{-\frac{1}{2}}(\rho_k^{\frac{1}{2}})$ in eq. (3.80).

We recall the classification of the structure of von Neumann alegbras on finite dimensional Hilbert spaces eq. (2.14).

$$\mathcal{N} = \bigoplus_{i=1}^{n} \mathcal{B}(\mathcal{H}_i)\otimes\mathbb{C}I_{\mathcal{K}_i}, \quad \mathcal{H} = \bigoplus_{i=1}^{n}\mathcal{H}_i\otimes\mathcal{K}_i. \tag{3.82}$$

Since $\mathcal{N}$ is non-trivial, there exists i such that $\dim\mathcal{H}_i > 1$ and $\dim\mathcal{K}_i > 1$, or $n > 1$.

We start with the first case. Without loss of generality, we assume that $\mathcal{H} = \mathcal{H}_A\otimes\mathcal{H}_B$ and $\mathcal{N} = \mathcal{B}(\mathcal{H}_A)\otimes\mathbb{C}I_B$, with $\dim\mathcal{H}_A = 2$ and $\dim\mathcal{H}_B := d_B > 1$. Otherwise, one can always recover the general case by adding zeros in the corresponding entries of ρ_k to make up the dimension. By the structure of conditional expectation eqs. (2.15) and (2.16), we have

$$E_{\mathcal{N}^*}(\omega) = \mathrm{Tr}_{\mathcal{H}_B}(\omega)\otimes\tau. \tag{3.83}$$

We define, in the eigenbasis of τ and any O.N. basis of $\mathcal{H}_A$, the following matrices,

$$\Delta := \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes\underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 1 & 0 & & \cdots & 0\\ 0 & & \ddots & & \vdots\\ \vdots & & & & \\ 0 & & \cdots & & 0\end{pmatrix}}_{d_B}, \quad \rho_{\mathcal{N},k} = \begin{pmatrix}\frac{1}{k} & \\ & 1-\frac{1}{k}\end{pmatrix}\otimes\tau. \tag{3.84}$$

It is easy to see that $E_{\mathcal{N}^*}(\Delta) = \mathrm{Tr}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 1 & 0 & & \cdots & 0\\ 0 & & \ddots & & \vdots\\ \vdots & & & & \\ 0 & & \cdots & & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes\tau = 0$. Next, we define

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\\ \vdots\\ 0\end{pmatrix}. \tag{3.85}$$

It is easy to verify that $\langle e_i, \Delta e_j\rangle = 0$. We define

$$\lambda_1 := k\langle e_1, \rho_{N,k}e_1\rangle, \quad \lambda_2 := \frac{k}{k-1}\langle e_2, \rho_{N,k}e_2\rangle. \tag{3.86}$$

Clearly, λ_1 and λ_2 do not depend on k. We now set for $\varepsilon \geq 0$, $\rho_{k,\varepsilon} := \rho_{\mathcal{N},k} + \varepsilon\Delta$. Thus we have $E_{\mathcal{N}^*}(\rho_{k,\varepsilon}) = \rho_{\mathcal{N},k}$. Since $\rho_{\mathcal{N},k}$ are full rank, $\rho_{k,\varepsilon}$ must be well-defined density matrices for ε small enough since Δ is traceless.

Next, we turn to the case where $n > 1$**.** Again by adding zero entries to the matrices defining ρ_k, we assume that $n = 2$. Denote by P_i the orthogonal projection on $\mathcal{H}_i \otimes \mathcal{K}_i$ and consider $\eta_i = \frac{I_{\mathcal{H}_i}}{\dim \mathcal{H}_i} \otimes \tau_i$ for τ_i defined in eqs. (2.15) and (2.16). We also denote by $e_i \in \mathcal{H}_i \otimes \mathcal{K}_i$ an eigenvector of η_i of associated eigenvalue $\lambda_i > 0$. We set

$$\Delta = e_1 e_2^* + e_2 e_1^*, \quad \rho_{\mathcal{N},k} = \frac{1}{k}\eta_1 + \left(1 - \frac{1}{k}\right)\eta_2. \tag{3.87}$$

Again, it is easy to verify that

$$E_{\mathcal{N}^*}(\Delta) = \mathrm{Tr}_{\mathcal{H}_1}(P_1 \Delta P_1) \otimes \tau_1 + \mathrm{Tr}_{\mathcal{H}_2}(P_2 \Delta P_2) \otimes \tau_2 = 0 + 0 = 0, \tag{3.88}$$

$$\langle e_i, \Delta e_j \rangle = 1 - \delta_{ij}. \tag{3.89}$$

As before, we define $\rho_{k,\varepsilon} := \rho_{\mathcal{N},k} + \varepsilon\Delta$.

So far, in both cases, we have $E_{\mathcal{N}^*}(\Delta) = 0$ and

$$\begin{aligned} &\Delta = e_1 e_2^* + e_2 e_1^*, \quad \left\langle e_i, \Delta e_j \right\rangle = 1 - \delta_{ij}, \\ &\lambda_1 := k \left\langle e_1, \rho_{\mathcal{N},k} e_1 \right\rangle, \quad \lambda_2 := \frac{k}{k-1} \left\langle e_2, \rho_{\mathcal{N},k} e_2 \right\rangle. \end{aligned} \tag{3.90}$$

Also, we can verify that in both cases, $\rho_{\mathcal{N},k}\Delta$ is traceless. For the first case it follows readily by direct calculation and the fact that τ is diagonal. For the second case, it suffices to note that $\mathrm{Tr}\left(\eta_1 e_2 e_1^*\right) = \mathrm{Tr}(e_2(\eta_1 e_1)^*) = \lambda_1 \mathrm{Tr}\left(e_2 e_1^*\right) = 0$ and similarly $\mathrm{Tr}\left(\eta_2 e_1 e_2^*\right) = 0$.

Next, we treat both cases simultaneously. For the purpose of estimating the quotient term in eq. (3.81), we compute the Taylor expansion of both the numerator and denominator. We recall the integral representations of the logarithm and square root functions:

$$\log X = \int_0^\infty \left(\frac{1}{\alpha + 1} - \frac{1}{X + \alpha}\right) \mathrm{d}\alpha, \tag{3.91}$$

$$X^{\frac{1}{2}} = \frac{\sin\left(\frac{1}{2}\pi\right)}{\pi} \int_0^\infty \frac{A}{1 + A\alpha} \alpha^{-\frac{1}{2}} \mathrm{d}\alpha = \frac{1}{\pi} \int_0^\infty \sqrt{t} \left(\frac{1}{t} - \frac{1}{A + t}\right) \mathrm{d}t. \tag{3.92}$$

we have

$$\begin{aligned} H(\rho_{k,\varepsilon} \| E_{\mathcal{N}^*}(\rho_{k,\varepsilon})) &= \mathrm{Tr}[(\rho_{\mathcal{N},k} + \varepsilon\Delta) \log(\rho_{\mathcal{N},k} + \varepsilon\Delta) - (\rho_{\mathcal{N},k} + \varepsilon\Delta) \log \rho_{\mathcal{N},k}] \\ &= \mathrm{Tr}\left[(\rho_{\mathcal{N},k} + \varepsilon\Delta) \int_0^\infty \left(\frac{1}{\alpha + 1} - \frac{1}{\rho_{\mathcal{N},k} + \varepsilon\Delta + \alpha}\right) \mathrm{d}\alpha\right] \\ &\quad - \mathrm{Tr}\left[(\rho_{\mathcal{N},k} + \varepsilon\Delta) \int_0^\infty \left(\frac{1}{\alpha + 1} - \frac{1}{\rho_{\mathcal{N},k} + \alpha}\right) \mathrm{d}\alpha\right] \\ &= \mathrm{Tr}\left[(\rho_{\mathcal{N},k} + \varepsilon\Delta) \int_0^\infty [(\rho_{\mathcal{N},k} + \alpha)^{-1} - (\rho_{\mathcal{N},k} + \varepsilon\Delta + \alpha)^{-1}] \mathrm{d}\alpha\right] \\ &= \mathrm{Tr}\left[(\rho_{\mathcal{N},k} + \varepsilon\Delta) \int_0^\infty (\rho_{\mathcal{N},k} + \alpha + \varepsilon\Delta)^{-1} \varepsilon\Delta (\rho_{\mathcal{N},k} + \alpha)^{-1} \mathrm{d}\alpha\right] \\ &= \mathrm{Tr}\left[(\rho_{\mathcal{N},k} + \varepsilon\Delta) \int_0^\infty (\rho_{\mathcal{N},k} + \alpha)^{-1} \varepsilon\Delta (\rho_{\mathcal{N},k} + \alpha)^{-1} - (\rho_{\mathcal{N},k} + \alpha)^{-1} \varepsilon\Delta (\rho_{\mathcal{N},k} + \alpha)^{-1}\right. \\ &\quad \left. \varepsilon\Delta (\rho_{\mathcal{N},k} + \alpha)^{-1} \mathrm{d}\alpha\right] + O(\varepsilon^3) \end{aligned} \tag{3.93}$$

Arranging the terms according to the order of ε, we have

$$\begin{aligned} H(\rho_{k,\varepsilon}\|\rho_{\mathcal{N},k}) &= \varepsilon\,\mathrm{Tr}\left[\rho_{\mathcal{N},k}\int_0^\infty(\rho_{\mathcal{N},k}+\alpha)^{-1}\Delta(\rho_{\mathcal{N},k}+\alpha)^{-1}\mathrm{d}\alpha\right] \\ &+\varepsilon^2\,\mathrm{Tr}\left[\int_0^\infty\Delta(\rho_{\mathcal{N},k}+\alpha)^{-1}\Delta(\rho_{\mathcal{N},k}+\alpha)^{-1}\mathrm{d}\alpha\right] \\ &+\varepsilon^2\,\mathrm{Tr}\left[\rho_{\mathcal{N},k}\int_0^\infty(\rho_{\mathcal{N},k}+\alpha)^{-1}\Delta(\rho_{\mathcal{N},k}+\alpha)^{-1}\Delta(\rho_{\mathcal{N},k}+\alpha)^{-1}\mathrm{d}\alpha\right]+O(\varepsilon^3). \end{aligned} \tag{3.94}$$

Because $\rho_{\mathcal{N},k}\Delta$ is traceless, the first and third terms vanish. We are left with

$$H(\rho_{k,\varepsilon}\|\rho_{\mathcal{N},k}) = \varepsilon^2\,\mathrm{Tr}\left[\int_0^\infty\Delta(\rho_{\mathcal{N},k}+\alpha)^{-1}\Delta(\rho_{\mathcal{N},k}+\alpha)^{-1}\mathrm{d}\alpha\right]+O(\varepsilon^3). \tag{3.95}$$

For the numerator, first we have

$$\rho_{k,\varepsilon}^{\frac{1}{2}} = \rho_{\mathcal{N},k}^{\frac{1}{2}} - \frac{\varepsilon}{\pi}\int_0^\infty\sqrt{t}\left(\frac{1}{t+\rho_{\mathcal{N},k}}\Delta\frac{1}{t+\rho_{\mathcal{N},k}}\right)\mathrm{d}t+O(\varepsilon^2). \tag{3.96}$$

$$\begin{aligned} \mathcal{E}_{\mathcal{L}'}(\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\rho_{k,\varepsilon}^{\frac{1}{2}}\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}) &= \langle\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\rho_{k,\varepsilon}^{\frac{1}{2}}\sigma_{\mathrm{Tr}}^{-\frac{1}{4}},\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\rho_{k,\varepsilon}^{\frac{1}{2}}\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\rangle_\sigma - \langle\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\rho_{k,\varepsilon}^{\frac{1}{2}}\sigma_{\mathrm{Tr}}^{-\frac{1}{4}},E_{\mathcal{N}}\left[\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\rho_{k,\varepsilon}^{\frac{1}{2}}\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\right]\rangle_\sigma \\ &= \pi^2\varepsilon^2\int_0^\infty\int_0^\infty\sqrt{ts}\,\mathrm{Tr}\left[\frac{1}{t+\rho_{\mathcal{N},k}}\Delta\frac{1}{t+\rho_{\mathcal{N},k}}\Delta\right]\mathrm{d}s\mathrm{d}t \\ &-\pi^2\varepsilon^2\int_0^\infty\int_0^\infty\sqrt{ts}\,\mathrm{Tr}\left[\frac{\sigma_{\mathrm{Tr}}^{\frac{1}{4}}}{t+\rho_{\mathcal{N},k}}\Delta\frac{\sigma_{\mathrm{Tr}}^{\frac{1}{4}}}{t+\rho_{\mathcal{N},k}}E_{\mathcal{N}}\left(\frac{\sigma_{\mathrm{Tr}}^{\frac{1}{4}}}{s+\rho_{\mathcal{N},k}}\Delta\frac{\sigma_{\mathrm{Tr}}^{\frac{1}{4}}}{s+\rho_{\mathcal{N},k}}\right)\right]+O(\varepsilon^3). \end{aligned} \tag{3.97}$$

Note that e_1, e_2 are eigenvectors of $\frac{\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}}{s+\rho_{\mathcal{N},k}}$, and $E_{\mathcal{N}}[e_1e_2^*] = E_{\mathcal{N}}[e_2e_1^*] = 0$. We have

$$H(\rho_{k,\varepsilon}\|\rho_{\mathcal{N},k}) = \varepsilon^2 g\left(\frac{1}{k}\lambda_1,\left(1-\frac{1}{k}\right)\lambda_2\right)\left|e_1^*\Delta e_2\right|^2+O(\varepsilon^3), \tag{3.98}$$

$$\mathcal{E}_{\mathcal{L}'}(\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\rho_{k,\varepsilon}^{\frac{1}{2}}\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}) = 2\pi^2\varepsilon^2 f\left(\frac{1}{k}\lambda_1,\left(1-\frac{1}{k}\right)\lambda_2\right)\left|e_1^*\Delta e_2\right|^2+O(\varepsilon^3). \tag{3.99}$$

Here we denote

$$f(x,y) := \begin{cases} \frac{(\sqrt{x}-\sqrt{y})^2}{(x-y)^2} & \text{if } x\neq y \\ \frac{1}{4x} & \text{else} \end{cases}, \quad g(x,y) := \begin{cases} \frac{\log(x)-\log(y)}{x-y} & \text{if } x\neq y \\ \frac{1}{x} & \text{else .} \end{cases} \tag{3.100}$$

Note that

$$\lim_{k\to\infty} f\left(\frac{1}{k}\lambda_1,\left(1-\frac{1}{k}\right)\lambda_2\right) = \frac{1}{\lambda_2}, \quad \lim_{k\to\infty} g\left(\frac{1}{k}\lambda_1,\left(1-\frac{1}{k}\right)\lambda_2\right) = +\infty. \tag{3.101}$$

Let $\varepsilon_k = \min\{\widetilde{\varepsilon}_k, \frac{1}{k}\}$ where $2\widetilde{\varepsilon}_k$ is defined to be the largest ε such that $\rho_{k,\varepsilon}$ is a well-defined density matrix. We set $\rho_k = \rho_{k,\varepsilon_k}$, then we have

$$\frac{\mathcal{E}_{\mathcal{L}'}(Z_k)}{\mathrm{Ent}_{2,\sigma_{\mathrm{Tr}}}(Z_k)} \to \infty \tag{3.102}$$

□

We recall that in the case of a primitive symmetric QMS, the log-Sobolev constant can always be lower upper bounded either in terms of the mixing time or the gap of the generator (see the end of section 3.1).

Next, we deal with another topic of this section: the complete strong data processing and modified logarithmic Sobolev inequalities. As we will see soon, the complete SDPI and MLSI can be effective in these important cases, in comparison with the failure of the complete LSI.

Definition 46 (Complete strong data processing constant)**.** *Let Φ^* be a σ-DBC quantum Markov map with σ being a full rank state. Let $E_{\mathcal{N}}$ be the conditional expectation on to the multiplicative domain $\mathcal{N}$ of Φ^*. The complete strong data processing constant $s_c(\Phi)$ is defined as*

$$s_c(\Phi) := \sup_{k\in\mathbb{N}} s(\Phi \otimes id_{M_k(\mathbb{C})}). \tag{3.103}$$

Definition 47 (Complete modified log-Sobolev constant)**.** *Let $\mathcal{L}$ be a σ-DBC Lindbladian w.r.t. the full rank state σ. The complete modified log-Sobolev constant $\alpha_c(\mathcal{L})$ is defined as the infimum of all constants $\alpha_1(\mathcal{L}\otimes id_{M_k(\mathbb{C})})$, that is,*

$$\alpha_c(\mathcal{L}) = \inf_{k\in\mathbb{N}} \alpha_1(\mathcal{L}\otimes id_{M_k(\mathbb{C})}). \tag{3.104}$$

Remark 48. *The complete SDPI and MLSI tensorize by design. Following eq. (3.75), we can easily see that*

$$\alpha_c(\mathcal{L}_1, \mathcal{L}_2) = \min\{\alpha_c(\mathcal{L}_1), \alpha_c(\mathcal{L}_2)\}. \tag{3.105}$$

Likewise, using the chain rule eq. (3.73), we have

$$s_c(\Phi_1 \otimes \Phi_2) = \max\{s_c(\Phi_1), s_c(\Phi_2)\}. \tag{3.106}$$

It remains to show that the complete SDPI and MLSI have non-trivial bounds. In fact, our strategy is to relate these constants to the complete version of *discrete and continuous return times*. We first give the statement of the main theorem.

Theorem 49.

$$s(\Phi) \le s_c(\Phi) \le 1 - \frac{1}{k_c(\Phi)}, \tag{3.107}$$

$$\frac{1}{2t_c(\mathcal{L})} \le \alpha_c(\mathcal{L}) \le \alpha_1(\mathcal{L}). \tag{3.108}$$

Here, the complete discrete and continuous return times are defined as

$$k_c(\Phi) = \inf\Big\{k \in \mathbb{N} : \frac{9}{10}E_{\mathcal{N}} \le_{cp} (\Phi^*)^{2k} \le_{cp} \frac{11}{10}E_{\mathcal{N}}\Big\}, \tag{3.109}$$

$$t_c(\mathcal{L}) = \inf\Big\{t > 0 : \frac{9}{10}E_{\mathcal{F}} \le_{cp} \mathcal{P}_t \le_{cp} \frac{11}{10}E_{\mathcal{F}}\Big\}, \tag{3.110}$$

where $\mathcal{N}$, $\mathcal{F}$ are the multiplicative domain and fixed-point algebra of Φ^* and $\mathcal{P}_t$ respectively.

Remark 50. *The generic return times are defined as*

$$\begin{aligned} k(\Phi) &:= \inf\left\{k \in \mathbb{N} \,\middle|\, \left\|\Phi^{2k}(X) - \mathrm{Tr}(\sigma X)I\right\|_\infty \le \frac{1}{10}\mathrm{Tr}(\sigma X) \quad \forall X \ge 0\right\}; \\ t(\mathcal{L}) &:= \inf\left\{t > 0 \,\middle|\, \|\mathcal{P}_t(X) - \mathrm{Tr}(\sigma X)I\|_\infty \le \frac{1}{10}\mathrm{Tr}(\sigma X) \quad \forall X \ge 0\right\}. \end{aligned} \tag{3.111}$$

It is easy to see from the definition that, for $X > 0$,

$$\frac{9}{10}\operatorname{Tr}(\sigma X)I \leq (\Phi^*)^{2k(\Phi)}(X), \mathcal{P}_{t(\mathcal{L})}(X) \leq \frac{11}{10}\operatorname{Tr}(\sigma X)I. \tag{3.112}$$

We will see later that these constants are closely related to the strong data processing and modified log-Sobolev constants.

For the proof of Theorem 49, we need the following two technical lemmas.

Lemma 51. *Let Ψ^* be a σ-DBC quantum Markov map w.r.t. the full rank state σ. Let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be the multiplicative domain algebra of Ψ^* and $E_{\mathcal{N}}$ be the conditional expectation onto $\mathcal{N}$.*

Assume further that $E_{\mathcal{N}} \circ \Psi^* = \Psi^* \circ E_{\mathcal{N}} = E_{\mathcal{N}}$, and that for some $0 < \varepsilon < \sqrt{2\log 2 - 1}$, we have

$$(1-\varepsilon)E_{\mathcal{N}} \leq_{cp} \Psi^* \leq_{cp} (1+\varepsilon)E_{\mathcal{N}}. \tag{3.113}$$

Then, for all $n \in \mathbb{N}$, and states $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^n)$, we have

$$H(\rho \| E_{\mathcal{N}^*}(\rho)) \leq \frac{1}{1 - \varepsilon^2(2\log 2 - 1)^{-1}} H(\rho \| \Psi^2(\rho)). \tag{3.114}$$

Proof of Lemma 51. Let $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^n)$ and $\rho_{\mathcal{N}} := E_{\mathcal{N}^*}(\rho)$. We have

$$H(\rho \| \rho_{\mathcal{N}}) - H(\rho \| \Psi^2(\rho)) = -H\left(\rho \middle\| \frac{A}{\operatorname{Tr} A}\right) + \log \operatorname{Tr} A \leq \log \operatorname{Tr} A. \tag{3.115}$$

Here $A = \exp\big(\log \Psi^2(\rho) - \log \rho_{\mathcal{N}} + \log \rho\big)$. The last inequality follows from the positivity of the relative entropy. By Golden-Thompson-Lieb inequality, we have

$$\log \operatorname{Tr} A \leq \log \int_0^\infty \operatorname{Tr}\left(\Psi^2(\rho) \frac{1}{\rho_{\mathcal{N}} + s} \rho \frac{1}{\rho_{\mathcal{N}} + s}\right) \mathrm{d}s =: \log \langle \Psi^2(\rho), \rho \rangle_{\rho_{\mathcal{N}}^{-1}}. \tag{3.116}$$

Here $\langle \cdot, \cdot \rangle_{\rho_{\mathcal{N}}^{-1}}$ is also called the quantum Fisher information. Then by the σ-symmetry of Ψ^* (see Lemma 15),

$$\begin{aligned} H(\rho \| \rho_{\mathcal{N}}) &\leq H(\rho \| \Psi^2(\rho)) + \langle \Psi^2(\rho), \rho \rangle_{\rho_{\mathcal{N}}^{-1}} \\ &= H(\rho \| \Psi^2(\rho)) + \langle \Psi(\rho), \Psi(\rho) \rangle_{\rho_{\mathcal{N}}^{-1}} \\ &\overset{(1)}{=} H(\rho \| \Psi^2(\rho)) + \int_0^\infty \operatorname{Tr}\left[(\Psi(\rho) - E_{\mathcal{N}*}(\rho)) \frac{1}{\rho_{\mathcal{N}} + s} (\Psi(\rho) - E_{\mathcal{N}*}(\rho)) \frac{1}{\rho_{\mathcal{N}} + s}\right] \mathrm{d}s. \end{aligned} \tag{3.117}$$

Here, (1) follows from $\log(x) \leq x - 1$ and the trace-preserving property of Ψ and $E_{\mathcal{N}^*}$. Now since $\Phi^* \geq_{cp} (1-\varepsilon)E_{\mathcal{N}^*}$, we can define $\widetilde{\Psi} := (\Psi - (1-\varepsilon)E_{\mathcal{N}^*})/\varepsilon$. Then we have $\widetilde{\Psi}$ is a quantum channel and $\Psi = (1-\varepsilon)E_{\mathcal{N}^*} + \varepsilon\widetilde{\Psi}$. Therefore,

$$\begin{aligned} H(\rho \| \rho_{\mathcal{N}}) &\leq H(\rho \| \Psi^2(\rho)) + \varepsilon^2 \int_0^\infty \operatorname{Tr}\left[\left(\widetilde{\Psi}(\rho) - E_{\mathcal{N}*}(\rho)\right) \frac{1}{\rho_{\mathcal{N}} + s} \left(\widetilde{\Psi}(\rho) - E_{\mathcal{N}*}(\rho)\right) \frac{1}{\rho_{\mathcal{N}} + s}\right] \mathrm{d}s \\ &= H(\rho \| \Psi^2(\rho)) + \varepsilon^2 \left\| (\widetilde{\Psi} - E_{\mathcal{N}^*})(\rho) \right\|^2_{\rho_{\mathcal{N}}^{-1}} \\ &\overset{\text{Lemma } 2}{\leq} H(\rho \| \Psi^2(\rho)) + \frac{\varepsilon^2}{k(2)} H(\widetilde{\Psi}(\rho) \| \rho_{\mathcal{N}}) \leq H(\rho \| \Psi^2(\rho)) + \frac{\varepsilon^2}{k(2)} H(\rho \| \rho_{\mathcal{N}}). \end{aligned} \tag{3.118}$$

Here we used the fact that

$$\widetilde{\Psi} \leq_{cp} (1+\varepsilon)E_{\mathcal{N}} \implies \widetilde{\Psi}(\rho) \leq \varepsilon^{-1}(1 + \varepsilon - (1-\varepsilon))\rho_{\mathcal{N}} = 2\rho_{\mathcal{N}}. \tag{3.119}$$

□

Our next technical can be seen as a strengthening of the so-called *quantum recoverability theorem* in the case of a GNS-symmetric quantum Markov map. In finite dimensional case, we provide a very simple proof.

Lemma 52 (Entropy difference theorem)**.** *Let Φ^* be a σ-DBC quantum Markov map w.r.t. the full rank state σ. Then for any two states ρ, ω,*

$$H(\rho\|\Phi^2(\omega)) - H(\rho\|\omega) \leq H_\Phi(\rho). \tag{3.120}$$

Here, $H_\Phi(\rho)$ is called the discrete entropy production defined as

$$H_\Phi(\rho) := H\left(\rho\|E_{\mathcal{N}^*}(\rho)\right) - H\left(\Phi(\rho)\|\Phi\circ E_{\mathcal{N}^*}(\rho)\right). \tag{3.121}$$

Here mcN is the multiplicative domain of Φ^* and $E_{\mathcal{N}}$ is the conditional expectation onto $\mathcal{N}$.

Remark 53. *By the chain rule eq. (3.73) and $\Phi|_{\mathcal{N}}$ is an involution (Lemma 15), one can easily see that for any fixed point σ of Φ,*

$$H_\Phi(\rho) = H\left(\rho\|\sigma\right) - H\left(\Phi(\rho)\|\sigma\right). \tag{3.122}$$

Proof of Lemma 52. We claim:

$$H\left(\rho\|\Phi^2(\omega)\right) - H(\rho\|\sigma) + H(\Phi(\rho)\|\sigma) \leq H(\Phi(\rho)\|\Phi(\omega)) \tag{3.123}$$

Then by $H(\Phi(\rho)\|\Phi(\omega)) \leq H(\rho\|\omega)$, the required inequality follows. By simple observations, one can see that the inequality holds if $\forall \omega \in \mathcal{D}(\mathcal{H})$, we have

$$\log\sigma - \log\Phi(\omega) \leq \Phi^*(\log\sigma - \log\omega). \tag{3.124}$$

Using $\frac{\mathrm{d}}{\mathrm{d}s}B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}}|_{s=0} = \log A - \log B$ and the fact that $\sigma^0\Phi(\omega)^0\sigma^0 = I = \Phi^*(\sigma^0\Phi(\omega)^0\sigma^0)$, it suffices to show

$$\Phi^*(\sigma^{-\frac{s}{2}}\omega^s\sigma^{-\frac{s}{2}}) \leq \sigma^{-\frac{s}{2}}\Phi(\omega)^s\sigma^{-\frac{s}{2}}, \quad \forall s \in [0,1). \tag{3.125}$$

Since we assumed that Φ^* is σ-DBC, by section 2.3 we have $\Phi\circ\Gamma_\sigma = \Gamma_\sigma\circ\Phi^*$. Therefore, it suffices to show

$$\Phi^*(\sigma^{-\frac{s}{2}}\omega^s\sigma^{-\frac{s}{2}}) \leq \sigma^{-\frac{s}{2}}\left(\sigma^{\frac{1}{2}}\Phi^*(\sigma^{-\frac{1}{2}}\omega\sigma^{-\frac{1}{2}})\sigma^{\frac{1}{2}}\right)^s\sigma^{-\frac{s}{2}}. \tag{3.126}$$

By Lemma 14, we write

$$\Phi^*(X) = \sum_k R_k X_k R_k^*, \quad \Delta_\sigma(R_k) = \eta_k R_k. \tag{3.127}$$

Moreover, we denote $\eta = \sum_k \eta_k e_k e_k^*$ and the Markov map Ψ^* with its Kraus operators being $R_k \otimes e_k^*$. We recall that

$$\sigma R_k = \eta_k R_k \sigma \tag{3.128}$$

in Lemma 14. Then, by operator Jensen inequality and the operator concavity of $x \mapsto x^s$, we have

$$\begin{aligned}
\Phi^*(\sigma^{-\frac{s}{2}}\omega^s\sigma^{-\frac{s}{2}}) &= \sum_k R_k\sigma^{-\frac{s}{2}}\omega^s\sigma^{-\frac{s}{2}}R_k^* \\
&\overset{\text{eq. (3.128)}}{=} \sigma^{-\frac{s}{2}}\left(\sum_k R_k\omega^s R_k^*\right)\eta_k^s\sigma^{-\frac{s}{2}} \overset{\text{by defn.}}{=} \sigma^{-\frac{s}{2}}\Psi^*((\omega\otimes\eta)^s)\sigma^{-\frac{s}{2}} \\
&\overset{\text{operator Jensen}}{\leq} \sigma^{-\frac{s}{2}}(\Psi^*(\omega\otimes\eta))^s\sigma^{-\frac{s}{2}} = \sigma^{-\frac{s}{2}}\left(\sigma^{\frac{1}{2}}\Phi^*(\sigma^{-\frac{1}{2}}\omega\sigma^{-\frac{1}{2}})\sigma^{\frac{1}{2}}\right)^s\sigma^{-\frac{s}{2}}.
\end{aligned} \tag{3.129}$$

Thus eq. (3.125) holds. ∎

Proof of Theorem 49. **Step 1.** We start with the proof of the first inequality. We consider the quantum Markov map $\Psi = (\Phi^*)^2$. By the definition of the complete return time, we have

$$\frac{9}{10} E_{\mathcal{N}} \leq_{cp} \Psi^{*2k_c(\Phi)} \leq_{cp} \frac{11}{10} E_{\mathcal{N}}. \tag{3.130}$$

By Lemma 51 applied to Ψ, we have for all $n \in \mathbb{N}$, and states $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^n)$

$$H(\rho \| E_{\mathcal{N}^*}(\rho)) \leq \frac{1}{1 - \frac{1}{100}(2\log 2 - 1)^{-1}} H(\rho \| \Psi^{*2k_c(\Phi)}(\rho)) \leq 2H(\rho \| \Phi^{4k_c}(\rho)). \tag{3.131}$$

By Lemma 52, we have

$$H(\rho \| \Phi^{4k_c}(\rho)) \leq H(\rho \| E_{\mathcal{N}^*}(\rho)) - H(\Phi(\rho) \| \Phi \circ E_{\mathcal{N}^*}(\rho)) + H(\rho \| \Phi^{4k_c - 2}(\rho)). \tag{3.132}$$

Applying Lemma 52 repeatedly, we have

$$H(\rho \| E_{\mathcal{N}^*}(\rho)) \leq 2H(\rho \| \Phi^{4k_c}(\rho)) \leq 4k_c \Big(H(\rho \| E_{\mathcal{N}^*}(\rho)) - H(\Phi(\rho) \| \Phi \circ E_{\mathcal{N}^*}(\rho)) \Big) \tag{3.133}$$

and thus

$$s_c(\Phi) \leq \sup_{n \in \mathbb{N}, \rho \in \mathcal{D}(\mathcal{H} \otimes M_n(\mathbb{C}))} \frac{H(\Phi(\rho) \| \Phi \circ E_{\mathcal{N}^*}(\rho))}{H(\rho \| E_{\mathcal{N}^*}(\rho))} \leq \frac{1}{4k_c(\Phi)}. \tag{3.134}$$

Step 2. We now prove the second inequality. By Lemma 51 applied to Ψ_{t_c}, we have that

$$H(\rho \| E_{\mathcal{F}^*}(\rho)) \leq \frac{1}{1 - \frac{1}{100}(2\log 2 - 1)^{-1}} H(\rho \| \mathcal{P}^2_{t_c}(\rho)) \leq 2H(\rho \| \mathcal{P}^2_{2t_c^*}(\rho)). \tag{3.135}$$

We divide the time $2t_c$ into m intervals of length $t_m = t_c/m$. By Lemma 52, we get

$$H(\rho \| \mathcal{P}_{2mt_m^*}) \leq H(\rho \| \mathcal{P}_{2(m-1)t_m^*}) + H_{\mathcal{P}_{t_m^*}}(\rho) \leq \cdots \leq mH_{\mathcal{P}_{t_m^*}}(\rho). \tag{3.136}$$

Thus,

$$H(\rho \| E_{\mathcal{F}^*}(\rho)) \leq 2m(H(\rho \| E_{\mathcal{N}^*}(\rho)) - H(\mathcal{P}_{t_m^*}(\rho) \| E_{\mathcal{N}^*}(\rho))) = 2t_c \frac{H(\rho \| E_{\mathcal{N}^*}(\rho)) - H(\mathcal{P}_{t_m^*}(\rho) \| E_{\mathcal{N}^*}(\rho))}{t_m}. \tag{3.137}$$

Taking the limit $m \to \infty$ (note that the quotient on the right hand side above converges to the entropy production $\mathrm{EP}_{\mathcal{L}}(\rho)$) and then the infimum $\inf_{n \in \mathbb{N}}$, we conclude that

$$\alpha_c(\mathcal{L}) \geq \frac{1}{2t_c(\mathcal{L})}. \tag{3.138}$$

□

The complement SDPI and MLSI Theorem 49 can be used to derive the bounds on the constants, by again relating the discrete and continuous return times to the spectral gap.

Corollary 54.

$$\begin{aligned} &\lambda(\Phi^*) \leq s(\Phi) \leq s_c(\Phi) \leq 1 - \frac{-\log(\lambda(\Phi^*))}{4\log(10C_{\mathrm{cb}}(E_{\mathcal{N}}))}; \\ &\frac{\lambda(\mathcal{L})}{2\log(10C_{\mathrm{cb}}(E_{\mathcal{N}}))} \leq \alpha_c(\mathcal{L}) \leq \alpha_1(\mathcal{L}) \leq 2\lambda(\mathcal{L}). \end{aligned} \tag{3.139}$$

We need the following technical lemma

Lemma 55. *Let $\mathcal{N}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$ and $E_{\mathcal{N}}$ is the trace-preserving conditional expectation onto $\mathcal{N}$. Let $\Phi^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital $\mathcal{N}$-bimodule map, i.e. $\Psi^*(X) = AXB$ for $A, B \in \mathcal{N}$ and any $X \in \mathcal{B}(\mathcal{H})$. Assume further that Ψ^* is self-adjoint w.r.t. the KMS inner product $\langle \cdot, \cdot \rangle_{\sigma, \frac{1}{2}}$ for any invertible invariant state $\sigma = E_{\mathcal{N}^*}(\sigma)$ and $E_{\mathcal{N}} \circ \Psi^* = \Psi^* \circ E_{\mathcal{N}} = E_{\mathcal{N}}$. Then, for $k > \frac{\log C_{cb}(E_{\mathcal{N}})}{-\log \lambda(\Psi^*)}$, we have*

$$(1 - \varepsilon)E_{\mathcal{N}} \leq_{cp} (\Phi^*)^k \leq_{cp} (1 + \varepsilon)E_{\mathcal{N}}, \quad \forall \varepsilon > 0, \tag{3.140}$$

for $\varepsilon = \lambda(\Psi^*)^{\frac{k}{2}} C_{cb}(E_{\mathcal{N}}) < 1$.

Proof. The proof relies on the properties of $\mathcal{N}$-bimodule maps. We refer to Lemma A.1 of [GR22]. □

Proof of Corollary 54. We start with the bounds in discrete time. Therefore the proof follows from an application of the bound in Lemma 55 for $\Psi = \Phi^2$. We now show the lower bound. Let

$$\omega_t := (1 - t)E_{\mathcal{N}^*}(\rho) + t\rho, \quad t \in [0, 1]. \tag{3.141}$$

By the complete SDPI, we have

$$H(\Phi(\omega_t) \| \Phi \circ E_{\mathcal{N}^*}(\rho)) \leq s(\Phi) H(\omega_t \| E_{\mathcal{N}^*}(\rho)). \tag{3.142}$$

That is because $E_{\mathcal{N}^*}^2 = E_{\mathcal{N}^*}$ is an idempotent and thus $E_{\mathcal{N}^*}(\omega_t) = E_{\mathcal{N}^*}(\rho)$. Consider

$$f(t) = s(\Phi) H(\omega_t \| E_{\mathcal{N}^*}(\rho)) - H(\Phi(\omega_t) \| \Phi \circ E_{\mathcal{N}^*}(\rho)), \tag{3.143}$$

then $f(0) = f'(0) = 0$ and $f''(0) = f''(0) = s(\Phi) \left\| \rho - E_{\mathcal{N}*}(\rho) \right\|^2_{E_{\mathcal{N}*}(\rho)^{-1}} - \left\| \Phi(\rho) - \Phi \circ E_{\mathcal{N}*}(\rho) \right\|^2_{\Phi \circ E_{\mathcal{N}*}(\rho)^{-1}}$. By $f(0) \geq 0$ on $[0, \varepsilon]$, we have $f''(0) \geq 0$. Therefore, we have

$$\left\| \Phi\left(\rho - E_{\mathcal{N}*}(\rho)\right) \right\|^2_{\Phi(\omega)^{-1}} \leq s(\Phi) \left\| \rho - E_{\mathcal{N}*}(\rho) \right\|^2_{E_{\mathcal{N}*}(\rho)^{-1}}. \tag{3.144}$$

This proves the lower bound. The bounds in the continuous case is shown similarly by taking $\Psi^* = \mathcal{P}_{t_c}$ and $k = 1$, and by noting $\lambda(\Psi^*) = e^{-2\lambda(\mathcal{L})t_c}$. □

4 Applications in Gibbs state preparation

The goal of this section is to understand the results in [BCG+23, BCG+24] using the established theories in the previous sections.

We consider the Lindblad dynamics of a quantum spin chain for preparing the Gibbs state on quantum devices by simulating this Markovian quantum dynamics (which is the standard idea of *dissipative state engineering*).

The basic setting is, we consider the spin chain $\Lambda = [[1, n]]$ with n sites. On each site lies a qudit system $\mathcal{H}_k = \mathbb{C}^d$ with $d \geq 2$. The total Hamiltonian is given by $\mathcal{H} = \bigotimes_{k \in \Lambda} \mathcal{H}_k$. The Hamiltonian H_Λ is assumed to be r-local, i.e. it is a sum of local Hamiltonians:

$$H_\Lambda = \sum_{A \subset \Lambda} h_A \otimes I_{A^c}, \quad \text{with } h_A = 0 \text{ whenever } \mathrm{diam}(A) > r. \tag{4.1}$$

The interacting strength is defined as $J := \max_{A\subset\Lambda} \|h_A\|_\infty$. Our goal is to estimate the mixing rate of some Lindblad dynamics whose fixed point is the Gibbs state $\sigma^\Lambda(\beta) := \exp(-\beta H_\Lambda)/\operatorname{Tr}\exp(-\beta H_\Lambda)$. The Lindblad generator considered here is the so-called *Davies generator*:

$$\mathcal{L}^D_{\Lambda^*}(\rho) = -\mathrm{i}[H_\Lambda, \rho] + \sum_{k\in\Lambda} \mathcal{L}^D_{k^*}(\rho), \tag{4.2}$$

for some local generator with Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) form:

$$\mathcal{L}^D_{k^*}(\rho) = \sum_{\omega,\alpha} \chi^{\beta,\omega}_{\alpha,k} \left(S^{\,\omega}_{\alpha,k} \rho S^{\,\omega *}_{\alpha,k} - \frac{1}{2}\{\rho, S^{\,\omega *}_{\alpha,k} S^{\,\omega}_{\alpha,k}\} \right). \tag{4.3}$$

The summation above runs over all index α of local jump operators $S_{\alpha,k}$ as well as the Bohr frequencies (all the possible differences of eigenvalues of H_Λ) ω. The coefficients $\chi^{\beta,\omega}_{\alpha,k}$ are the so-called *two-point correlation functions* which are assumed to satisfy the KMS condition:

$$\chi^{\beta,-\omega}_{\alpha,k} = e^{-\beta\omega} \chi^{\beta,\omega}_{\alpha,k}. \tag{4.4}$$

The operator $S^{\,\omega}_{\alpha,k}$ is the Fourier coefficients of $S_{\alpha,k}$, which means that they satisfies

$$e^{-\mathrm{i}tH_\Lambda} S_{\alpha,k} e^{\mathrm{i}tH_\Lambda} = \sum_\omega e^{\mathrm{i}t\omega} S^{\,\omega}_{\alpha,k}. \tag{4.5}$$

Note that it follows from the fact that H_Λ is a commuting Hamiltonian that the left hand side of eq. (4.5) is supported on an interval of diameter at most $r' + 2r$ centered at k, since all the terms h_A with support disjoint from the support of $S_{\alpha,k}$ cancel out. For the same reason, $S^{\,\omega}_{\alpha,k}$ is supported on a finite set centered at k. This implies $\mathcal{L}^D_{k^*}$ is a local generator, in the sense that the kernel of the adjoint $\mathcal{L}^D_k$ contains all operators that act trivally on $\{k' \in \Lambda : |k - k'| \le r\}$. In particular, both $\mathcal{L}^D_{k^*}$ and $\mathcal{L}^D_{A^*}$ are *non-primitive*.

The main result in [BCG+23, BCG+24] is

Theorem 56. *For any inverse temperature $\beta > 0$, we have $\alpha_1(\mathcal{L}^D_\Lambda) = \Omega(\log(n)^{-1})$.*

The key idea of the proof is to first reduce the problem to a quasi-local case on a subregion of Λ with size $\Theta(\log n)$ using the so-called *approximate tensorization* technique [CRF20]. Then the problem is reduced to showing $\alpha_c(\mathcal{L}_A) = \Omega(|A|)$ holds for any local subregions A whose size is $\Theta(\log n)$. For this, one directly makes use of Corollary 54 together with the fact that Davies semigroups on spin chains are gapped (see [KB16]).

Physically, this result suggests that we can prepare the quantum many-body Gibbs state for local spin systems efficiently on quantum computers, by simulating the corresponding Lindblad dynamics.

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