Notes on Matrix Analysis with Applications in Quantum Theory

Haoen Li

Latest revision: June 6, 2025

This is a note on the matrix analysis course given by Prof. Jinsong Wu (BIMSA) in Spring 2025. It will be focused on some important matrix inequalities and their applications in quantum information theory.

Contents

Chapter 1

Eigenvalue Inequalities

1.1 The min-max inequality

Theorem 1.1.1. *$A \in M_n(\mathbb{C})$ is a Hermitian matrix, we denote $\lambda_k(A)$ as the k-th largest eigenvalue of A, then we have*

$$\lambda_k(A) = \max_{\dim V = k} \min_{\|x\|=1, x\in V} \langle Ax, x\rangle = \min_{\dim V = n-k+1} \max_{\|x\|=1, x\in V} \langle Ax, x\rangle. \tag{1.1.1}$$

Lemma 1. *Let V be a subspace of $\mathbb{C}^n$ with $\dim V = k$, then $\exists v \in V \cap \mathbb{S}(V)$ s.t. $\langle Av, v\rangle \leq \lambda_k$.*

Proof. Let v_i be the unit eigenvalue of $\lambda_i(A)$. We take $W = \mathrm{Span}(v_k, \cdots, v_n)$ then $\dim W = n - k + 1$. Note

$$\dim(V \cap W) = \dim(V) + \dim(W) - \dim(V + W) \geq \dim(V) + \dim(W) - n = 1, \tag{1.1.2}$$

then we have $V \cap W \neq \varnothing \Rightarrow \exists v \in V \cap W$ with $\|v\| = 1$. Since $v \in W$, we have $v = \sum_{j=k}^n a_j v_j$ with $\sum_{j=k}^n |a_j|^2 = \|v\|^2 = 1 \Rightarrow$

$$\langle Av, v\rangle = \left\langle \sum_{j=k}^n a_j \lambda_j v_j, \sum_{j=k}^n a_j v_j \right\rangle = \sum_{j=k}^n \lambda_j |a_j|^2 \leq \lambda_k \sum_{j=k}^n |a_j|^2 = \lambda_k. \tag{1.1.3}$$

□

Remark 1. *This inequality is also called* Poincaré inequality.

Proof of Theorem 1.1.1. We take any subspace V with dimension k of $\mathbb{C}^n$, by Poincaré's inequality, we have

$$\min_{x \in V \cap \mathbb{S}(V)} \langle Ax, x\rangle \leq \lambda_k. \tag{1.1.4}$$

By the arbitrariness of V we have

$$\max_{\dim V = k} \min_{x \in V \cap \mathbb{S}(V)} \langle Ax, x\rangle \leq \lambda_k. \tag{1.1.5}$$

□

Remark 2.

- $\lambda_2(A) = \min_{\mathrm{codim}=1} \max_{x \in \mathbb{S}(V) \cap V} \langle Ax, x\rangle$;

- $\lambda_k(A+B) \le \lambda_k(A) + \lambda_1(B)$;
- $|\lambda_k(A+B) - \lambda_k(A)| \le \|B\|$.

Theorem 1.1.2 (Poincaré separation theorem or Cauchy interlace theorem)**.** *A Hermitian, P a orthogonal projection in $M_n(\mathbb{C})$ s.t. $PAP = B$. Denote* $\mathrm{rank} P = m$*, and the eigenvalues*

$$A : \lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A); \tag{1.1.6}$$

$$B : \mu_1(B) \ge \mu_2(B) \ge \cdots \ge \mu_n(B). \tag{1.1.7}$$

Then for all $k \le m$, we have

$$\lambda_{n-m+k}(A) \le \mu_k(B) \le \lambda_k(A). \tag{1.1.8}$$

Proof. By the min-max theorem applied first on $\mathrm{Ran}P$ and then on $\mathbb{C}^n$, we have

$$\begin{aligned}\mu_k(B) &= \min_{\mathrm{dim}V=m-k+1, V\subset \mathrm{Ran}P} \max_{x\in V\cap\mathbb{S}(V)} \langle PAPx, x\rangle = \min_{\mathrm{dim}V=m-k+1, V\subset \mathrm{Ran}P} \max_{x\in V\cap\mathbb{S}(V)} \langle Ax, x\rangle \\ &\ge \min_{\dim V=m-k+1} \max_{x\in V\cap\mathbb{S}(V)} \langle Ax, x\rangle = \lambda_{n-m+k}.\end{aligned} \tag{1.1.9}$$

On the other hand,

$$\begin{aligned}\mu_k(B) &= \max_{\dim V=k, V\subset \mathrm{Ran}P} \min_{x\in V\cap\mathbb{S}(V)} \langle PAPx, x\rangle = \max_{\dim V=k, V\subset \mathrm{Ran}P} \min_{x\in V\cap\mathbb{S}(V)} \langle Ax, x\rangle \\ &\le \max_{\mathrm{dim}V=k} \min_{x\in V\cap\mathbb{S}} \langle Ax, x\rangle = \lambda_k.\end{aligned} \tag{1.1.10}$$

□

1.2 Reading: An application of the min-max inequality—Cheeger inequality

Theorem 1.2.1 (An application: Cheeger inequality)**.** *$G = (V, E)$ is a d-regular graph, $n = |V|$. Let A be the adjacent matrix of G.*

Consider $M = \frac{1}{d}A$ is a Hermitian matrix, then we observe

$$M\mathbf{1} = \mathbf{1}, \quad \mathbf{1} := (1, \cdots, 1)^T. \tag{1.2.1}$$

And 1 is the largest spectrum of M. What about $1 - \lambda_2$?

Cheeger:

$$h(G)^2/2 \le 1 - \lambda_2 \le \Phi(G) \le 2h(G). \tag{1.2.2}$$

Here,

$$\partial S := \{(x, y) \in E(G) : x \in S, y \in V \backslash S\}, \tag{1.2.3}$$

$$h(G) := \min\left\{\frac{|\partial S|}{d|S|} : S \subset V, 0 < |S| \le |V|/2\right\} \tag{1.2.4}$$

$$\Phi(G) := \min\left\{\frac{|\partial S|}{d|S| \cdot \frac{|V\backslash S|}{|V|}} : \varnothing \subsetneq S \subsetneq V\right\}. \tag{1.2.5}$$

Proof. Some important observations:

- In fact
$$\frac{1}{2}\min\{|S|,|V\backslash S|\} \le \frac{|S||V\backslash S|}{|V|} = \frac{|S||V\backslash S|}{|S|+|V\backslash S|} \le \min\{|S|,|V\backslash S|\}. \tag{1.2.6}$$
i.e. $h(G) \le \Phi(G) \le 2h(G)$. Therefore the two different "sparsities" of graphs are equivalent in some sense.

- $\lambda_2 = \max_{x\in\mathbb{S}^1\cap\{\mathbf{1}\}^\perp, x\in\mathbb{R}^n}\langle Mx, x\rangle$

- G is d-regular $\Rightarrow \sum_{j=1}^n M_{jk} = \sum_{k=1}^n M_{jk} = 1$.

- For $1-\lambda_2$, we have
$$\begin{aligned}
1-\lambda_2 &= \min_{x\perp\mathbf{1},\|x\|=1}\left(\langle x,x\rangle - \langle Mx,x\rangle\right)\\
&= \min_{x\perp\mathbf{1},\|x\|=1}\left(\sum_{j=1}^n x_j^2 - \sum_{j,k=1}^m M_{jk}x_jx_k\right)\\
&= \min_{x\perp\mathbf{1},\|x\|=1}\frac{1}{2}\sum_{j,k=1}^n M_{jk}(x_j-x_k)^2 \quad \text{(by the row-sum and col-sum are } 1)\\
&= \min_{x\neq\mathbf{1}}\frac{1}{2}\cdot\frac{\sum_{j,k=1}^n M_{jk}(x_j-x_k)^2}{\sum_{j,k=1}^n (x_j-\frac{1}{n}x_k)^2} \quad \text{(by the description } x\perp\mathbf{1} \iff x = \widetilde{x} - \frac{1}{n}\langle\widetilde{x},\mathbf{1}\rangle\mathbf{1})\\
&= \min_{x\neq\mathbf{1}}\frac{1}{2}\cdot\frac{n\sum_{j,k=1}^n M_{jk}(x_j-x_k)^2}{n\sum_{j=1}^n x_j^2 - \left(\sum_{j=1}^n x_j\right)^2}.
\end{aligned} \tag{1.2.7}$$

- On the other hand,
$$\begin{aligned}
\Phi(G) &= \min_S \frac{|\partial S|}{d|S|\frac{|V\backslash S|}{|V|}}\\
&= \min_{x\in\{0,1\}^n, x\neq 0,\mathbf{1}} \frac{\frac{1}{2}\sum_{j,k=1}^n dM_{jk}(x_j-x_k)^2}{\frac{d}{n}\left(\sum_{j=1}^n x_j\right)\left(n-\sum_{j=1}^n x_j\right)}\\
&= \min_{x\in\{0,1\}^n, x\neq 0,\mathbf{1}} \frac{1}{2}\cdot\frac{n\sum_{j,k=1}^n M_{jk}(x_j-x_k)^2}{n\sum_{j=1}^n x_j^2 - \left(\sum_{j=1}^n x_j\right)^2}.
\end{aligned} \tag{1.2.8}$$
Therefore
$$1-\lambda_2 \le \Phi(G). \tag{1.2.9}$$

- Now we turn to address another side. Let $\mathbf{v}\in\mathbb{R}^n$, $\mathbf{v}\neq 0$ s.t. $M\mathbf{v} = \lambda_2\mathbf{v}$. By Perron-Frobenius theorem, the components of $\mathbf{v}$ cannot be *all positive*. Thus, we can define
$$\mathbf{y}\neq 0, \quad \begin{cases} y_j = v_j, & v_j \ge 0,\\ y_j = 0, & v_j < 0.\end{cases} \tag{1.2.10}$$

Then we have

$$\langle M\mathbf{v}, \mathbf{y}\rangle = \lambda_2\langle \mathbf{v}, \mathbf{y}\rangle = \lambda_2(\langle \mathbf{y}, \mathbf{y}\rangle + \underbrace{\langle \mathbf{v} - \mathbf{y}, \mathbf{y}\rangle}_{=0}) = \lambda_2\langle \mathbf{y}, \mathbf{y}\rangle, \tag{1.2.11}$$

and

$$\begin{aligned} 1-\lambda_2 &= \frac{\langle \mathbf{y}, \mathbf{y}\rangle}{\langle \mathbf{y}, \mathbf{y}\rangle} - \frac{\langle M\mathbf{v}, \mathbf{y}\rangle}{\langle \mathbf{y}, \mathbf{y}\rangle} = \frac{\langle \mathbf{y}, \mathbf{y}\rangle - \langle M\mathbf{y}, \mathbf{y}\rangle}{\langle \mathbf{y}, \mathbf{y}\rangle} + \underbrace{\frac{\langle M(\mathbf{y}-\mathbf{v}), \mathbf{y}\rangle}{\langle \mathbf{y}, \mathbf{y}\rangle}}_{\geq 0} \geq \frac{\langle \mathbf{y}, \mathbf{y}\rangle - \langle M\mathbf{y}, \mathbf{y}\rangle}{\langle \mathbf{y}, \mathbf{y}\rangle} \\ &= \frac{\frac{1}{2}\sum_{j,k=1}^n M_{jk}(y_j - y_k)^2}{\langle \mathbf{y}, \mathbf{y}\rangle}. \end{aligned} \tag{1.2.12}$$

By the Cauchy-Schwarz Inequality, we have that

$$1-\lambda_2 \geq \frac{\left(\sum_{j,k=1}^n M_{jk}|y_j^2 - y_k^2|\right)^2}{2\|\mathbf{y}\|^2(\sum_{j,k=1}^n (y_j + y_k)^2)} \geq \frac{\left(\sum_{j,k=1}^n M_{jk}|y_j^2 - y_k^2|\right)^2}{4\|\mathbf{y}\|^2(\sum_{j,k=1}^n (y_j^2 + y_k^2))} \geq \frac{\left(\sum_{j,k=1}^n M_{jk}|y_j^2 - y_k^2|\right)^2}{8\|\mathbf{y}\|^4}. \tag{1.2.13}$$

W.L.O.G. let $y_1 \geq \cdots \geq y_n$, we take $t = \max\{k : y_k > 0\}$. Then we have

$$\begin{aligned} 1-\lambda_2 &\geq \frac{\left(2\sum_{j=1}^t \sum_{k=j+1}^n M_{jk}(y_j^2 - y_k^2)\right)^2}{8\|\mathbf{y}\|^4} \\ &= \frac{\left(\sum_{j=1}^t \sum_{k=j+1}^n \sum_{\ell=j}^{k-1} M_{jk}(y_\ell^2 - y_{\ell+1}^2)\right)^2}{2\|\mathbf{y}\|^4} \\ &= \frac{\left(\sum_{\ell=1}^t \left(\sum_{j=1}^{\ell} \sum_{k=\ell+1}^n M_{jk}\right)(y_\ell^2 - y_{\ell+1}^2)\right)^2}{2\|\mathbf{y}\|^4} \quad (j \leq \ell < k) \\ &= \frac{\left(\sum_{\ell=1}^t \frac{|\partial S_\ell|}{d}(y_\ell^2 - y_{\ell+1}^2)\right)^2}{2\|\mathbf{y}\|^4} \\ &\geq \frac{\left(\sum_{\ell=1}^t h(G)\ell(y_\ell^2 - y_{\ell+1}^2)\right)^2}{2\|\mathbf{y}\|^4} = \frac{h(G)^2\left(\sum_{\ell=1}^t y_\ell^2\right)^2}{2\|\mathbf{y}\|^4} = \frac{1}{2}h(G)^2. \end{aligned} \tag{1.2.14}$$

□

1.3 Singular value inequalities

Theorem 1.3.1. $A \in M_n(\mathbb{C})$, $|\lambda_j(A)| = \lim_{m\to\infty} \lambda_j(|A|^m)^{1/m}$

Remark 3. *Notations: $A \in M_n(\mathbb{C})$, recall the polar decomposition $A = U|A| = \widetilde{U}(V^*DV)$. Here V diagonalizes the matrix A^*A.*

Proof of Theorem 1.3.1. By Jordan decomposition $A = TJT^{-1}$, T nonsingular, J Jordan. In fact,

$$\begin{aligned}
\lambda_j(|A|^m)^2 &= \lambda_j(A^{*m}A^m) = \lambda_j(T^{-*}J^{m*}T^*TJ^mT^{-1}) \\
&= \max_{\dim V=j} \min_{\|x\|=1, x\in V} \langle T^{-*}J^{*m}T^*TJ^mT^{-1}x, x\rangle \\
&\leq \|T\|^2 \max_{\dim V=j} \min_{\|x\|=1, x\in V} \langle T^{-*}J^{*m}J^mT^{-1}x, x\rangle \\
&= \|T\|^2\lambda_j(T^{-*}J^{*m}J^mT^{-1}) = \|T\|^2\lambda_j(J^mT^{-1}T^{-*}J^{*m}) \quad (\lambda_j(A^*A) = \lambda_j(AA^*)) \\
&\leq \|T\|^2\|T^{-1}\|^2\lambda_j(J^m(J^m)^*) \quad \Rightarrow \quad \lambda_j(|A|^m) \leq \|T\|\|T^{-1}\|(\lambda_j(J^mJ^{*m}))^{1/2}
\end{aligned} \tag{1.3.1}$$

Since we have

$$J^mJ^{*m} = \mathrm{diag}(J_{n_1}^m(\mu_1)J_{n_1}^{*m}(\mu_1), \cdots, J_{n_k}^m(\mu_k)J_{n_k}^{*m}(\mu_k)), \quad \text{W.L.O.G.} \quad |\mu_1| \geq \cdots \geq |\mu_k|, \tag{1.3.2}$$

and

$$\lim_{m\to\infty}(J^mJ^{*m})^{1/2m} = \mathrm{diag}(|\mu_1|I_{n_1}, \cdots, |\mu_k|I_{n_k}), \tag{1.3.3}$$

we have

$$\limsup_{m\to\infty} \lambda_{n_j}(|A^m|)^{1/m} \leq |\mu_j|, \quad \text{i.e.} \quad \limsup_{m\to\infty} \lambda_j(|A|^m)^{1/m} \leq |\lambda_j(A)| \tag{1.3.4}$$

For the same reason, $\lim_{n\to\infty} \lambda_j(|A^m|)^{1/m} = |\lambda_j(A)|$. $\square$

Proposition 1. *Hermitian dilation*

$$A \mapsto B := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} & |A| \\ |A| & \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}^*. \tag{1.3.5}$$

It is easy to see that

$$\lambda_j(B) = \lambda_j(|A|) \quad \textit{for } 1 \leq j \leq n, \tag{1.3.6}$$

$$\lambda_j(B) = -\lambda_{n-j+1}(|A|), \quad \textit{for } n+1 \leq j \leq 2n. \tag{1.3.7}$$

Theorem 1.3.2. *For any matrices A, B, we have*

$$|\lambda_j(|A|) - \lambda(|B|)| = \left|\lambda_j\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} - \lambda_j\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}\right| \leq \left\|\begin{pmatrix} 0 & A-B \\ (A-B)^* & 0 \end{pmatrix}\right\| = \|A-B\|. \tag{1.3.8}$$

i.e.

$$\max_{1\leq j\leq n} |\sigma_j(A) - \sigma_j(B)| \leq \|A-B\|. \tag{1.3.9}$$

Proposition 2 (Schur-Horn Inequality)**.** *A Hermite, we introduce the notations:*

- $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$,
- $a_{i_1i_1} \geq \cdots \geq a_{i_ni_n}$ *are the diagonal elements of A ordered non-increasingly.*

Then we have $\{\lambda_j\}$ majorize $\{a_{i_ji_j}\}$

Proof. We define $B = \left(a_{i_j i_j}\right)_{1\le j\le k} = PAP^*$. By Cauchy interlace theorem, we have

$$\lambda_{n-k+j}(A) \le \lambda_j(B) \le \lambda_j(A) \quad 1 \le j \le k. \tag{1.3.10}$$

We take summation on both sides, yielding

$$\lambda_{n-k+1}(A) + \cdots + \lambda_n(A) \le \operatorname{Tr} B \le \lambda_1(A) + \cdots + \lambda_k(A). \tag{1.3.11}$$

□

Theorem 1.3.3 (von Neumann's trace theorem)**.** *A, B are Hermitian matrices, then we have*

$$\operatorname{Tr}(AB) \le \sum_{j=1}^{n} \lambda_j(A)\lambda_j(B), \tag{1.3.12}$$

$$\operatorname{Tr}(AB) \ge \sum_{j=1}^{n} \lambda_j(A)\lambda_{n-j+1}(B). \tag{1.3.13}$$

Proof. By $A = U^*DU$, we let $A = \operatorname{diag}(\lambda_i(A))$ ordered non-decreasingly, then

$$\operatorname{Tr} AB = \sum_{j=1}^{n} \lambda_j(A) b_{jj}. \tag{1.3.14}$$

By Proposition 2, we have

$$\sum_{j=1}^{k} b_{jj} \le \sum_{j=1}^{k} \lambda_j(B), \quad \forall 1 \le k \le n, \tag{1.3.15}$$

$$\sum_{j=1}^{k} b_{jj} \ge \sum_{j=1}^{k} \lambda_{n-k+j}(B), \quad \forall 1 \le k \le n. \tag{1.3.16}$$

Therefore, by the Abel formula, we have

$$\begin{aligned}
\operatorname{Tr} AB &= \sum_{j=1}^{n-1} (\lambda_j(A) - \lambda_{j+1}(A)) \sum_{k=1}^{j} b_{kk} + \lambda_n(A) \sum_{j=1}^{n} b_{jj} \\
&\le \sum_{j=1}^{n-1} (\lambda_j(A) - \lambda_{j+1}(A)) \sum_{k=1}^{j} \lambda_k(B) + \lambda_n(A) \sum_{j=1}^{n} \lambda_j(B) \\
&= \sum_{j=1}^{n} \lambda_j(A)\lambda_j(B).
\end{aligned} \tag{1.3.17}$$

□

Definition 1.3.4 (Majorization)**.** *Let x, y be two real vectors ordered non-increasingly. We say x majorizes y, if $\sum_{j=1}^{k} x_j \ge \sum_{j=1}^{k} y_j$ for all $1 \le k \le n$ and $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$.*

Theorem 1.3.5. *x majorizes $y \iff \exists$ a doubly-stochastic matrix S s.t. $y = Sx$.*
We say S is a doubly-stochastic matrix, if

$$S_{jk} \geq 0, \quad \sum_{j=1}^{n} S_{jk} = 1 = \sum_{k=1}^{n} S_{jk}, \quad \forall 1 \leq j, k \leq n. \tag{1.3.18}$$

To prove Theorem 1.3.5, we need the following lemma:

Lemma 2. *x majorizes $y \iff$ There exists an orthogonal matrix Q such that $[Q^T \mathrm{diag}(x) Q]_{ii} = y_i$.*

Proof of Lemma 2. The proof is based on the 2×2 case and then by induction. The complete proof can be found in the hand-written note. $\square$

Proof of Theorem 1.3.5. This is a very fast corollary of Lemma 2, since we have

$$y_j = \sum_{k=1}^{n} q_{kj}^2 x_k. \tag{1.3.19}$$

Let $S = (q_{kj}^2)_{j,k=1}^n$, then S is a real symmetric matrix and the row-summation of column-summation are both 1, therefore S is a doubly-stochastic matrix and $y = Sx$. $\square$

1.4 Exercise I

Exercise 1. *Show that $M_{n,m}(\mathbb{K}) \otimes M_{r,s}(\mathbb{K}) = M_{nr,ms}(\mathbb{K})$.*

Proof. It is easy to see that $M_{n,m}(\mathbb{K}) \otimes M_{r,s}(\mathbb{K}) \subset M_{nr,ms}(\mathbb{K})$. On the other hand, since we have the explicit description

$$M_{n,m}(\mathbb{K}) \otimes M_{r,s}(\mathbb{K}) = \mathrm{Span}\{E_{ij} \otimes E_{kl}, \quad 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq r, 1 \leq l \leq s\}. \tag{1.4.1}$$

We have $\dim M_{n,m}(\mathbb{K}) \otimes M_{r,s}(\mathbb{K}) = nrms = \dim M_{nr,ms}(\mathbb{K})$. Therefore $M_{n,m}(\mathbb{K}) \otimes M_{r,s}(\mathbb{K}) = M_{nr,ms}(\mathbb{K})$. $\square$

Exercise 2. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is a self-adjoint compact operator on a Hilbert space $\mathcal{H}$. Let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots$ be the list of all positive eigenvalues of A. Show that*

$$\begin{aligned} \lambda_k(A) &= \max_{\substack{\mathcal{V} \subset \mathcal{H}, \\ \dim \mathcal{V} = k}} \min_{\mathbf{x} \in \mathcal{V}, \|\mathbf{x}\| = 1} \langle A\mathbf{x}, \mathbf{x} \rangle \\ &= \min_{\mathcal{V} \subset \mathcal{H}, \mathrm{codim}\, \mathcal{V} = k-1} \max_{\mathbf{x} \in \mathcal{V}, \|\mathbf{x}\| = 1} \langle A\mathbf{x}, \mathbf{x} \rangle \end{aligned} \tag{1.4.2}$$

Proof. By the spectral theory of compact self-adjoint operators on the Hilbert space, we have

$$\mathcal{H} = \ker A \oplus \left(\bigoplus_{i=1}^{\infty} \mathrm{Span}(u_k) \right), \tag{1.4.3}$$

where $\{u_k\}$ are the eigenvectors corresponding to $\{\lambda_k(A)\}$. Let $\mathcal{S}_k = \overline{\mathrm{Span}\{u_k, u_{k+1}, \cdots\}}$, then $\mathcal{S}_k$ is a closed subspace with $\mathrm{codim} \mathcal{S}_k = k - 1$. We take $\mathcal{V}$ a k-dimensional subspace of $\mathcal{H}$, then we first consider

$$\pi : \mathcal{H} \to \mathcal{H}/\mathcal{S}_k, \quad \pi|_{\mathcal{V}} : \mathcal{V} \to \mathcal{H}/\mathcal{S}_k. \tag{1.4.4}$$

Since $\mathcal{V}$ is finite-dimensional, we have

$$k = \dim \mathcal{V} = \dim(\ker \pi|_{\mathcal{V}}) + \dim(\operatorname{Im} \pi|_{\mathcal{V}}) \leq \dim(\mathcal{V} \cap \mathcal{S}_k) + \dim(\mathcal{H}/\mathcal{S}_k) = \dim(\mathcal{V} \cap \mathcal{S}_k) + k - 1. \quad (1.4.5)$$

Therefore, $\mathcal{V} \cap \mathcal{S}_k \neq \{0\}$. We take $\mathbf{v} \in \mathcal{V} \cap \mathcal{S}_k$ with $\|\mathbf{v}\| = 1$, then by the construction of $\mathcal{S}_k$ we have

$$\langle A\mathbf{v}, \mathbf{v}\rangle \leq \lambda_k(A) \quad \Rightarrow \quad \inf_{\mathbf{x} \in \mathcal{V}, \|\mathbf{x}\|=1} \langle A\mathbf{x}, \mathbf{x}\rangle \leq \lambda_k(A). \tag{1.4.6}$$

Note that the unit ball in $\mathcal{H}$ is weak-compact, therefore $\{\mathbf{x} : \|\mathbf{x}\| = 1\}$ is compact, so we can write

$$\min_{\mathbf{x} \in \mathcal{V}, \|\mathbf{x}\|=1} \langle A\mathbf{x}, \mathbf{x}\rangle \leq \lambda_k(A). \tag{1.4.7}$$

By the arbitrariness of $\mathcal{V}$ we have

$$\sup_{\mathcal{V} \subset \mathcal{H}, \dim \mathcal{V}=k} \min_{\mathbf{x} \in \mathcal{V}, \|\mathbf{x}\|=1} \langle A\mathbf{x}, \mathbf{x}\rangle \leq \lambda_k(A). \tag{1.4.8}$$

Since the equality is achieved when taking $\mathcal{V} = \operatorname{Span}(u_1, \cdots, u_k)$, we actually have

$$\lambda_k(A) = \max_{\mathcal{V} \subset \mathcal{H}, \dim \mathcal{V}=k} \min_{\mathbf{x} \in \mathcal{V}, \|\mathbf{x}\|=1} \langle A\mathbf{x}, \mathbf{x}\rangle \leq \lambda_k(A). \tag{1.4.9}$$

The remaining part of the proposition can be proved analogously. $\square$

Exercise 3. *Suppose $A \in M_n(\mathbb{C})$ is Hermitian, show that*

$$\lambda_1(A) + \cdots + \lambda_k(A) = \sup_{P^*=P=P^2, \operatorname{rank} P=k} \operatorname{Tr}(AP), \tag{1.4.10}$$

$$\lambda_{n-k+1}(A) + \cdots + \lambda_n(A) = \inf_{P^*=P=P^2, \operatorname{rank} P=k} \operatorname{Tr}(AP). \tag{1.4.11}$$

Proof. We do this by intimating the proof of the min-max theorem. In fact, we assume P is the orthogonal projection to $\operatorname{Span}(\mathbf{v}_i)_{i=1}^k$ where $\{\mathbf{v}_i\}_{i=1}^k$ is the orthonormal basis. We denote the normalized eigenvectors of A by $\{\mathbf{u}_i\}_{i=1}^n$

$$\begin{aligned}\operatorname{Tr}(AP) = \operatorname{Tr}(PAP) = \sum_{i=1}^k \langle A\mathbf{v}_i, \mathbf{v}_i\rangle &= \sum_{i=1}^k \left\langle \sum_{j=1}^n \lambda_j(A)\mathbf{u}_j c_{ji}, \sum_{j=1}^n \mathbf{u}_j c_{ji} \right\rangle \\ = \sum_{i=1}^k \sum_{j=1}^n \lambda_j(A)|c_{ji}|^2 &= \sum_{j=1}^n \lambda_j(A) \left(\sum_{i=1}^k |c_{ji}|^2 \right)\end{aligned} \tag{1.4.12}$$

By the normalization condition, we have

$$\sum_{j=1}^n \sum_{i=1}^k |c_{ji}|^2 = \sum_{i=1}^k \|\mathbf{v}_i\|^2 = k, \quad 0 \leq \sum_{i=1}^k |c_{ji}|^2 \leq 1. \tag{1.4.13}$$

therefore,

$$\operatorname{Tr}(AP) \leq \lambda_1 + \cdots + \lambda_k. \tag{1.4.14}$$

By the arbitrariness of the orthogonal projection P, and by taking $P^\star = \sum_{i=1}^k \mathbf{u}_i \mathbf{u}_i^*$, we yield the first equality. The second equality follows by the same procedure.

$\square$

Proof using Cauchy interlace theorem. For any orthogonal projection P, by Cauchy interlace theorem, we have

$$\lambda_{n-\ell+1}(A) \leq \lambda_\ell(PAP) \leq \lambda_\ell(A). \tag{1.4.15}$$

Note that $\lambda_{k+1}(PAP) = \cdots = \lambda_n(PAP) = 0$, therefore $\mathrm{Tr}(PAP) = \sum_{\ell=1}^{k} \lambda_\ell(PAP)$. Taking the summation on both sides of eq. (1.4.15) yields the conclusion. □

Exercise 4 (Generalized Weyl inequality)**.** *Suppose $A, B \in M_n(\mathbb{C})$. Show that when $2 \leq j+k \leq n+1$, we have*

$$\lambda_{j+k-1}(|A+B|) \leq \lambda_j(|A|) + \lambda_k(|B|). \tag{1.4.16}$$

When $j+k \geq n+1$, we have

$$\lambda_{j+k-n}(|A+B|) \geq \lambda_j(|A|) + \lambda_k(|B|). \tag{1.4.17}$$

Proof. We consider $\widetilde{\bigstar} := \begin{pmatrix} & \bigstar \\ \bigstar^* & \end{pmatrix}$, by Weyl's inequality for Hermitian case, we have

$$\lambda_{j+k-1}(\widetilde{A+B}) \leq \lambda_j(\widetilde{A}) + \lambda_k(\widetilde{B}), \quad 2 \leq j+k \leq n+1, \tag{1.4.18}$$

$$\lambda_{j+k-n}(\widetilde{A+B}) \geq \lambda_j(\widetilde{A}) + \lambda_k(\widetilde{B}), \quad j+k \geq n+1. \tag{1.4.19}$$

Since we have

$$\lambda_\ell(\widetilde{\bigstar}) = \lambda_\ell(|\bigstar|), \quad 1 \leq \ell \leq n, \tag{1.4.20}$$

we conclude that when $2 \leq j+k \leq n+1$,

$$\lambda_{j+k-1}(|A+B|) \leq \lambda_j(|A|) + \lambda_k(|B|). \tag{1.4.21}$$

When $j+k \geq n+1$,

$$\lambda_{j+k-n}(|A+B|) \geq \lambda_j(|A|) + \lambda_k(|B|). \tag{1.4.22}$$

□

Exercise 5. *Suppose $A, B \in M_n(\mathbb{C})$, show that for $1/p + 1/q = 1$, $p, q > 0$,*

$$|\operatorname{Re}\mathrm{Tr}(AB)| \leq \left(\sum_{j=1}^{n} \lambda_j(|A|)^p\right)^{1/p} \left(\sum_{j=1}^{n} \lambda_j(|B|)^q\right)^{1/q}. \tag{1.4.23}$$

Proof. We first prove the vN trace theorem for the non-Hermitian scenario. In fact, we consider

$$\widetilde{\bigstar} := \begin{pmatrix} & \bigstar \\ \bigstar^* & \end{pmatrix}, \tag{1.4.24}$$

whose eigenvalues ordered non-increasingly are

$$\lambda_1(|\bigstar|) \geq \cdots \geq \lambda_n(|\bigstar|) \geq -\lambda_n(|\bigstar|) \geq \cdots \geq -\lambda_1(|\bigstar|). \tag{1.4.25}$$

In our case, we notice that

$$\widetilde{A}^* \widetilde{B} = \begin{pmatrix} & A^* \\ A & \end{pmatrix} \begin{pmatrix} & B \\ B^* & \end{pmatrix} = \begin{pmatrix} A^* B^* & \\ & AB \end{pmatrix}. \tag{1.4.26}$$

Therefore,

$$\operatorname{Tr}\left(\widetilde{A}^*\widetilde{B}\right) = 2\operatorname{Re}(\operatorname{Tr}(AB)). \tag{1.4.27}$$

By the vN trace theorem for Hermitian matrices, we have

$$\begin{aligned}|\operatorname{Re}(\operatorname{Tr}(AB))| \le \frac{1}{2}\left|\operatorname{Tr}\left(\widetilde{A}^*\widetilde{B}\right)\right| &\le \frac{1}{2}\left|2^{1/p+1/q}\left(\sum_{j=1}^n \lambda_j(|A|)^p\right)^{1/p}\left(\sum_{j=1}^n \lambda_j(|B|)^q\right)^{1/q}\right| \\ &= \left(\sum_{j=1}^n \lambda_j(|A|)^p\right)^{1/p}\left(\sum_{j=1}^n \lambda_j(|B|)^q\right)^{1/q}\end{aligned} \tag{1.4.28}$$

□

Exercise 6 ($*$)**.** *The same assumptions as in exercise 5, show that*

$$|\operatorname{Tr}(AB)| \le \left(\sum_{j=1}^n \lambda_j(|A|)^p\right)^{1/p}\left(\sum_{j=1}^n \lambda_j(|B|)^q\right)^{1/q}. \tag{1.4.29}$$

Proof. By the singular value decomposition, we may assume A is a non-negative diagonal matrix, then we have

$$\operatorname{Tr}(AB) = \sum_{j=1}^n \sigma_j(A)B_{jj} = \sum_{j=1}^n\sum_{\ell=1}^n \sigma_\ell(B)U_{j\ell}V_{j\ell}\sigma_j(A) \Rightarrow |\operatorname{Tr}(AB)| \le \sum_{1\le j,\ell\le n} \sigma_\ell(B)S_{j\ell}\sigma_j(A). \tag{1.4.30}$$

Here, $S_{j\ell} = |U_{j\ell}V_{j\ell}|$. Note that

$$\sum_{j=1}^n S_{j\ell} \le \left(\sum_{j=1}^n |U_{j\ell}|^2\right)^{1/2}\left(\sum_{j=1}^n |V_{j\ell}|^2\right)^{1/2} = 1, \tag{1.4.31}$$

$$\sum_{\ell=1}^n S_{j\ell} \le \left(\sum_{\ell=1}^n |U_{j\ell}|^2\right)^{1/2}\left(\sum_{\ell=1}^n |V_{j\ell}|^2\right)^{1/2} = 1, \tag{1.4.32}$$

therefore S is a sub-doubly stochastic matrix. There exists a doubly stochastic matrix Q s.t. $S \le Q$. By Bitkhoff-von Neumann theorem, we have

$$\exists \sum_{k=1}^N \alpha_k = 1, \quad Q = \sum_{k=1}^N \alpha_k P_k, \quad P_k \text{ permutation matrices.} \tag{1.4.33}$$

Therefore,

$$|\operatorname{Tr}(AB)| \le \sum_{k=1}^N \alpha_k \sum_{\ell=1}^n \sigma_\ell(B)\sigma_{\pi_k(\ell)}(A) \le \sum_{\ell=1}^n \sigma_\ell(B)\sigma_\ell(A). \tag{1.4.34}$$

Here, the equality holds if and only if $S = I$. Then the result follows by the Hölder inequality in the scalar case. □

Chapter 2

Operator Inequalities

2.1 Operator monotonicity and convexity

We recall some basic properties of positive operators:

Proposition 3. *For A, B Hermitian matrices, we define $A \geq B$ if and only $A - B$ is positive semidefinite. If $A \geq B$, we have*

- *$A + \lambda I = B + \lambda I$, for any $\lambda \in \mathbb{R}$;*
- *$S^*AS \geq S^*BS$, since the positive property is invariant under congruent transformations. Note that S does* not *have to be a square matrix.*

The question is, whether we have $A^\alpha \geq B^\alpha$? It is a very interesting problem and naturally leads to the concept of *operator monotonicity*. We begin with an example.

Proposition 4. *Let $0 \leq A \leq B$, then $A^{\frac{1}{2}} \leq B^{\frac{1}{2}}$. If A is invertible, then $B^{-1} \leq A^{-1}$.*

Proof. Let A be a invertible matrix, then B is also invertible. By $A \leq B$, we we have

$$B^{-1/2}AB^{-1/2} \leq \mathbf{1}. \tag{2.1.1}$$

(Note that there are also some similar techniques in numerical linear algebra)

To prove $B^{-1} \leq A^{-1}$, we only need to prove $A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \leq 1$ i.e. $\left\|A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right\| \leq 1$. To do this, we note that

$$\left\|A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right\| = r(A^{1/2}B^{-1}A^{1/2}) = r(B^{-1/2}AB^{-1/2}) = \left\|B^{-1/2}AB^{-1/2}\right\| \leq 1, \tag{2.1.2}$$

using the property of the spectral radius $r(AB) = r(BA)$ (follows from the *Sylvester determinant theorem*, $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$. Note that this property also holds for some infinite-dimensional cases by considering $(zI - AB)^{-1}$).

Inspired by this, we further consider $B^{-1/4}A^{1/2}B^{-1/4}$ and we want to estimate its spectral radius. In fact,

$$\left\|B^{-1/4}A^{1/2}B^{-1/4}\right\| = \left\|A^{1/2}B^{-1/2}\right\| \leq 1. \tag{2.1.3}$$

Now we have proved this proposition under the assumption that A is invertible. If A is not invertible, then for any $\varepsilon > 0$ we have $A + \varepsilon I$ is invertible since $A \geq 0$, and we have $A + \varepsilon I \leq B + \varepsilon I$, therefore $(A + \varepsilon I)^{1/2} \leq (B + \varepsilon I)^{1/2}$. Note that

$$\left\|(A+\varepsilon I)^{1/2} - A^{1/2}\right\| = \left|(\lambda_{\max}+\varepsilon)^{1/2} - \lambda_{\max}^{1/2}\right| \leq \varepsilon^{1/2}. \tag{2.1.4}$$

Then it follows that $A \leq B$ by taking $\varepsilon \to 0$. □

Definition 2.1.1. $A^0 := P_{\mathrm{Range}A}$*, then we have* $A^0 = \lim_{m\to\infty} |A|^{1/m}$.

Theorem 2.1.2. *Suppose* A, B *are Hermitian matrices, then*

- *For any* $\alpha \in [0,1]$*, we have* $A^\alpha \leq B^\alpha$.
- *If* A *is invertible, then for any* $\alpha \in [-1,0]$*, we have* $A^\alpha \geq B^\alpha$.

Proof. We fixed some A invertible and $B \geq A$. We prove by claiming that $I := \{\alpha \in [0,1] : A^\alpha \leq B^\alpha\}$ is a closed convex set. It holds trivially that $0, 1 \in I$ when $A > 0$.

For any $\alpha_1, \alpha_2 \in I$, we show using the same approach as in Proposition 4 that

$$\|C\| \leq 1, \quad C := B^{-\frac{1}{4}(\alpha_1+\alpha_2)} A^{\frac{1}{2}(\alpha_1+\alpha_2)} B^{-\frac{1}{4}(\alpha_1+\alpha_2)}. \tag{2.1.5}$$

Therefore we have $C \leq 1$ i.e. $A^{\frac{1}{2}(\alpha_1+\alpha_2)} \leq B^{\frac{1}{2}(\alpha_1+\alpha_2)}$. Thus we conclude that $I = [0,1]$.

If A is not invertible, we first consider $\alpha \in (0,1]$, $(A+\varepsilon I)^\alpha$ and $(B+\varepsilon I)^\alpha$. Taking $\varepsilon \to 0$, yields $A^\alpha \leq B^\alpha$. When $\alpha = 0$, we have

$$A^0 = \lim_{m\to\infty} A^{1/m} \leq \lim_{m\to\infty} B^{1/m} = B^0. \tag{2.1.6}$$

□

Another Proof. We provide a constructive and proof with more insights than just imitating Proposition 4.

In fact, we have

$$0 \leq A \leq B \Rightarrow B^{-1} \leq A^{-1} \Rightarrow B^{-1} + \lambda I \leq A^{-1} + \lambda I \Rightarrow (A^{-1}+\lambda I)^{-1} \leq (B^{-1}+\lambda I)^{-1}. \tag{2.1.7}$$

Consider

$$f_z(\lambda) = \frac{z}{1+\lambda z}. \tag{2.1.8}$$

Note that we have

$$t^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \frac{t}{1+\lambda t} \lambda^{-\alpha} \mathrm{d}\lambda \quad (\alpha \in (0,1)) \Rightarrow A^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty f_\lambda(A) \lambda^{-\alpha} \mathrm{d}\alpha. \tag{2.1.9}$$

Therefore $f_\lambda(A) \leq f_\lambda(B)$, i.e. $A^\alpha \leq B^\alpha$.

To see why this integral equality holds, we compute

$$\int_0^\infty \frac{u^{a-1}}{1+u} \mathrm{d}u = \int_{\mathbb{R}} \frac{e^{ax}}{1+e^x} \mathrm{d}x. \tag{2.1.10}$$

This is a textbook-example in complex analysis. Specifically, for the contour $[-R,R]\cup[R,2\pi\mathrm{i}+R]\cup[2\pi\mathrm{i}+R,2\pi\mathrm{i}-R]\cup[2\pi\mathrm{i}-R,-R]=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$. The only singularity inside the rectangular is $z=\pi\mathrm{i}$, with residue being

$$\lim_{z\to\pi\mathrm{i}}(z-\pi\mathrm{i})f(z)=\lim_{z\to\pi\mathrm{i}}e^{az}\frac{z-\pi\mathrm{i}}{e^z-e^{\pi\mathrm{i}}}=-e^{a\pi\mathrm{i}}. \tag{2.1.11}$$

Therefore

$$\int_\gamma f(z)\mathrm{d}z=-2\pi\mathrm{i}e^{a\pi\mathrm{i}}. \tag{2.1.12}$$

Note that

$$\left|\int_{\gamma_2} f\right|\le\int_0^{2\pi}\left|\frac{e^{a(R+\mathrm{i}t)}}{1+e^{R+\mathrm{i}t}}\right|\mathrm{d}t\le Ce^{(a-1)R}\to 0\quad(R\to\infty). \tag{2.1.13}$$

$$\left|\int_{\gamma_4} f\right|\le\int_0^{2\pi}\left|\frac{e^{a(-R+\mathrm{i}t)}}{1+e^{-R+\mathrm{i}t}}\right|\mathrm{d}t\le Ce^{-aR}\to 0\quad(R\to\infty). \tag{2.1.14}$$

$$\int_{\gamma_3} f=\int_R^{-R}\frac{e^{a(x+\mathrm{i}2\pi)}}{1+e^{(x+\mathrm{i}2\pi)}}\mathrm{d}x=-e^{2a\pi\mathrm{i}}\int_{-R}^R\frac{e^{ax}}{1+e^x}\mathrm{d}x. \tag{2.1.15}$$

Therefore we have

$$-2\pi\mathrm{i}e^{a\pi\mathrm{i}}=(1-e^{2a\pi\mathrm{i}})\int_{-\infty}^{+\infty}\frac{e^{ax}}{1+e^x}\mathrm{d}x\Rightarrow\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\mathrm{d}x=\frac{\pi}{\sin\pi a}. \tag{2.1.16}$$

□

Remark 4. *Using change of variables, we have*

$$t^\alpha=\frac{\sin(\alpha-1)\pi}{\pi}\int_0^\infty\left(\frac{t}{\lambda}+\frac{\lambda}{\lambda+t}-1\right)\lambda^\alpha\mathrm{d}\lambda,\quad\alpha\in(1,2), \tag{2.1.17}$$

$$t^\alpha=\frac{\sin(\alpha+1)\pi}{\pi}\int_0^\infty\frac{1}{\lambda+t}\lambda^\alpha\mathrm{d}\lambda,\quad\alpha\in(-1,0). \tag{2.1.18}$$

From which we can see that $A\le B\Rightarrow A^\alpha\ge B^\alpha$ for $\alpha\in(-1,0)$.

Definition 2.1.3 (Operator Monotonicity)**.** *Suppose $f:\mathrm{Dom}(f)\to\mathbb{R}$ is a function, where $\mathrm{Dom}(f)$ is an interval in $\mathbb{R}$. If for* any *$n\in\mathbb{N}$, $A\le B\in M_n(\mathbb{C})$ Hermitian and $\mathrm{Sp}(A),\mathrm{Sp}(B)\subset\mathrm{Dom}(f)$, we have $f(A)\le(<)f(B)$, then we say that f is (strictly)* operator monotone.

Remark 5. *$f(t)=t^\alpha$ is operator monotone for $\alpha\in[0,1]$ and is strictly operator monotone for $\alpha\in(0,1]$. $f(t)=-t^\alpha$ is strictly monotone on $(0,\infty)$ for $\alpha\in[-1,0)$.*

Example 1. *Can we prove the operator monotonicity for $\alpha>1$?*

$$A=\begin{pmatrix}\frac32 & \\ & \frac34\end{pmatrix}>\begin{pmatrix}\frac12 & \frac12\\ \frac12 & \frac12\end{pmatrix}=B. \tag{2.1.19}$$

Note that

$$B^\alpha=B=\begin{pmatrix}\frac12 & \frac12\\ \frac12 & \frac12\end{pmatrix}. \tag{2.1.20}$$

We compute

$$
\begin{aligned}
\det(A^\alpha - B^\alpha) &= \left[\left(\frac{3}{2}\right)^\alpha - \frac{1}{2}\right]\left[\left(\frac{3}{4}\right)^\alpha - \frac{1}{2}\right] - \frac{1}{4} = \left(\frac{9}{8}\right)^\alpha - \frac{1}{2}\left[\left(\frac{3}{2}\right)^\alpha + \left(\frac{3}{4}\right)^\alpha\right] \\
&= \left(\frac{3}{8}\right)^\alpha (2\cdot 3^\alpha - (4^\alpha + 2^\alpha)) < 0 \quad \textit{(by convexity)}
\end{aligned}
\tag{2.1.21}
$$

Therefore, although x^α is strictly monotone for $|\alpha| > 1$, but is not operator monotone.

The operator monotonicity is a very good property in quantum information theory, and usually cannot be satisfied.

Proposition 5. *$f(t) = \log t$ is operator on $(0,\infty)$. $f(t) = \frac{t-1}{\log t}$ operator on $[0,\infty)$.*

Proof. $\log t = \lim_{\alpha\to 0+} \frac{t^\alpha - 1}{\alpha}$, then by $\frac{t^\alpha-1}{\alpha}$ is operator monotone we conclude $\log t$ is also operator monotone. Moreover, $\frac{t-1}{\log t} = \int_0^1 t^\lambda \mathrm{d}\lambda$. □

Remark 6. *$t\log t$ is not operator monotone.*

$$
A\log A = \begin{pmatrix} \frac{3}{2}\log\frac{3}{2} & \\ & \frac{3}{4}\log\frac{3}{4} \end{pmatrix} \ngeq B\log B = 0, \quad \textit{since } B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \textit{ is a projection.} \tag{2.1.22}
$$

Remark 7. *In general, the function taking the form*

$$
f(t) = at + b + \int_0^\infty \frac{\lambda t}{t+\lambda}\mathrm{d}\mu(\lambda) \tag{2.1.23}
$$

is operator monotone on $[0,\infty)$. Here, $\mu(\lambda)$ is a positive Borel measure.

Proposition 6. *Let A, B be Hermitian matrices. Then we have, $(\frac{A+B}{2})^2 \leq \frac{A^2+B^2}{2}$.*

Proof.

$$
\left(\frac{A+B}{2}\right)^2 \leq \frac{A^2+B^2}{2} \Leftrightarrow A^2 - AB - BA + B^2 \geq 0 \Leftrightarrow (A-B)^2 \geq 0. \tag{2.1.24}
$$

□

Definition 2.1.4 (Operator convex)**.** *Suppose $f : \mathrm{Dom}(f) \subset \mathbb{R} \to \mathbb{R}$ is a function where Dom is an interval. We say that f is operator convex, if for any Hermitian matrices A, B with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset \mathrm{Dom}(f)$, we have*

$$
f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B), \quad \forall \lambda \in [0,1]. \tag{2.1.25}
$$

Theorem 2.1.5. *The following functions are operator convex:*

- *the function t^α on $(0,\infty)$, $\alpha \in [-1, 0)$;*
- *the function t^α on $[0,\infty)$, $\alpha \in [1, 2]$;*
- *the function $-t^\alpha$ on $[0,\infty)$, $\alpha \in [0, 1]$;*
- *the function t^2 on $(-\infty,\infty)$.*

Proof. **Step 1.** We prove that for $A, B > 0$, $\frac{A^{-1}+B^{-1}}{2} \geq \left(\frac{A+B}{2}\right)^{-1}$. We consider

$$A^{1/2}\frac{A^{-1}+B^{-1}}{2}A^{1/2} = \frac{1}{2}\left(I + A^{1/2}B^{-1}A^{1/2}\right), \tag{2.1.26}$$

$$A^{1/2}\left(\frac{A+B}{2}\right)^{-1}A^{1/2} = \left(\frac{I + A^{-1/2}BA^{-1/2}}{2}\right)^{-1}. \tag{2.1.27}$$

Denote $X = A^{1/2}B^{-1}X^{1/2} > 0$, then we only need to prove

$$\frac{1+X}{2} \geq \left(\frac{1+X^{-1}}{2}\right)^{-1} \quad \text{i.e.} \quad \left(\frac{1+X}{2}\right)^{-1} \leq \frac{1+X^{-1}}{2}. \tag{2.1.28}$$

This follows readily by the convexity of t^{-1} on $(0, \infty)$.

Step 2. By eq. (2.1.18) and the convexity of the function $t \mapsto \frac{1}{\lambda+t}$, we have $t^\alpha, \alpha \in (-1, 0)$ is operator convex. By eq. (2.1.9) and the convexity of the function $t \mapsto \frac{1}{\lambda+t}$, we have $-t^\alpha$, $\alpha \in (0, 1)$ is operator convex. By eq. (2.1.17) and the convexity of $t \mapsto \frac{t}{\lambda} + \frac{\lambda}{\lambda+t}$ for $\alpha \in (1, 2)$.

Step 3. For $\alpha = 0$, note that

$$P_{\mathrm{Range}(A)} = \lim_{m\to\infty} A^{1/m}. \tag{2.1.29}$$

Therefore $-t^0$ is operator convex (but not strictly operator convex, for example, consider $B = 2A$). ☐

Corollary 1. *The function $f(t) = \log t$ on $(0, \infty)$ is operator concave.*

The function $f(t) = t\log t$ on $[0, \infty)$ is operator convex.

The function $f(t) = \frac{t-1}{\log t}$ on $[0, \infty)$ is operator concave.

Proof.

$$\log t = \lim_{\alpha\to 0+} \alpha^{-1}(t^\alpha - 1), \quad t\log t = \lim_{\alpha\to 1+} \frac{t^\alpha - 1}{\alpha - 1}, \quad \frac{t-1}{\log t} = \int_0^1 t^\lambda \mathrm{d}\lambda. \tag{2.1.30}$$

☐

2.2 Non-commutative Jensen inequality

Is there any relation between *operator concavity* and *operator monotonicity*? This leads to a quite profound result: *non-commutative Jensen inequality*.

Proposition 7 (Sherman-Davis)**.** *Suppose $f : \mathrm{Dom}(f) \to \mathbb{R}$ is an operator convex function, then for any $n \in \mathbb{N}$ and $A \in M_n(\mathbb{C})$ Hermitian with $\mathrm{Sp}(A) \subset \mathrm{Dom}(f)$ and some projection $P \in M_n(\mathbb{C})$, we have*

$$Pf(PAP + s(1-P))P \leq Pf(A)P. \tag{2.2.1}$$

Here, $s \in \mathrm{Dom}(f)$

Remark 8. *Note that in the definition of operator convexity, we already assume that $\mathrm{Dom}(f)$ is an interval in $\mathbb{R}$, which ensures that $\mathrm{Sp}(PAP)\backslash\{0\} \subset \mathrm{Dom}(f)$ by the fact that $\mathrm{Sp}(A) \subset \mathrm{Dom}(f)$ and Cauchy interlace theorem.*

We add the term $s(1-P)$ because we likely have $0 \in \mathrm{Sp}(PAP)$ but $0 \notin \mathrm{Dom}(f)$. To deal with this and maximize the generality of our result, we add this term to ensure that $\mathrm{Sp}(PAP + s(1-P)) \subset \mathrm{Dom}(f)$, since we note that for any $x \in \ker P$,

$$(PAP + s(1-P))x = 0 + sx - 0 = sx, \quad s \in \mathrm{Dom}(f). \tag{2.2.2}$$

Proof. Consider $\widetilde{A} = \begin{pmatrix} A & \\ & A \end{pmatrix}$, $U = \begin{pmatrix} P & P-I \\ I-P & P \end{pmatrix}$ (*note that this is a very common strategy to construct a unitary matrix*), and $U_\perp = \begin{pmatrix} I-P & -P \\ P & I-P \end{pmatrix}$.

Since U and $U_\perp$ are unitary, therefore

$$U\widetilde{A}U^* = \begin{pmatrix} PAP + (I-P)A(I-P) & (P-I)AP + PA(I-P) \\ (I-P)AP + PA(P-I) & (I-P)A(I-P) + PAP \end{pmatrix}, \tag{2.2.3}$$

$$U_\perp\widetilde{A}U_\perp^* = \begin{pmatrix} PAP + (I-P)A(I-P) & (P-I)AP + PA(P-I) \\ (P-I)AP + PA(I-P) & (I-P)A(I-P) + PAP \end{pmatrix}, \tag{2.2.4}$$

and then

$$\frac{1}{2}(U\widetilde{A}U^* + U_\perp\widetilde{A}U_\perp^*) = \operatorname{diag}(PAP + (I-P)A(I-P), PAP + (I-P)A(I-P)). \tag{2.2.5}$$

By functional calculus and the convexity of f, we have

$$f\left[\frac{1}{2}(U\widetilde{A}U^* + U_\perp\widetilde{A}U_\perp^*)\right] \leq \frac{1}{2}\left[Uf(\widetilde{A})U^* + U_\perp f(\widetilde{A})U_\perp^*\right]. \tag{2.2.6}$$

We take the $(1,1)$-block

$$f(PAP + (I-P)A(I-P)) \leq Pf(A)P + (I-P)f(A)(I-P). \tag{2.2.7}$$

Hence we have

$$Pf(PAP + s(I-P))P \leq Pf(A)P. \tag{2.2.8}$$

Remark 9. *A very straightforward understanding is to consider the "matrix form" of $PAP + (I-P)A(I-P)$, which is "block-diagonalized" under the basis of $\operatorname{Range}P$ and its orthogonal complement. In this case we solely need to consider the $(1,1)$-block. The rest $(2,2)$-block is irrelevant to the inequality and we can write in a quite general form (replacing $(I-P)A(I-P)$ by $s(I-P)$), which is also well-defined because $s \in \operatorname{Dom}(f)$.*

□

Proposition 8. *Suppose $f : \operatorname{Dom}(f) \to \mathbb{R}$ is a function, where $\operatorname{Dom}(f)$ is an interval in $\mathbb{R}$. If for any $n \in \mathbb{N}$, Hermitian matrices $A \in M_n(\mathbb{C})$ with $\operatorname{Sp}(A) \subset \operatorname{Dom}(f)$ and projection P, the inequality $Pf(PAP + s(I-P))P \leq Pf(A)P$ holds for any $s \in \operatorname{Dom}(f)$, then f must be operator convex.*

Proof. We take $A, B \in M_n(\mathbb{C})$ Hermitian, $\lambda \in [0,1]$ along with

$$\widetilde{A} = \begin{pmatrix} A & \\ & B \end{pmatrix}, \quad P = \begin{pmatrix} I & \\ & \end{pmatrix}, \quad U = \begin{pmatrix} \sqrt{\lambda}I & -\sqrt{1-\lambda}I \\ \sqrt{1-\lambda}I & \sqrt{\lambda}I \end{pmatrix}. \tag{2.2.9}$$

We compute

$$U\widetilde{A}U^* = \begin{pmatrix} \lambda A + (1-\lambda)B & * \\ * & (1-\lambda)A + \lambda B \end{pmatrix}. \tag{2.2.10}$$

By our assumption, we have

$$Pf(PU\widetilde{A}U^*P + s(I-P))P \leq PUf(\widetilde{A})U^*P, \tag{2.2.11}$$

Note that

$$P(U\widetilde{A}U^*)P = \begin{pmatrix} \lambda A + (1-\lambda)B & \\ & \end{pmatrix}, \tag{2.2.12}$$

$$f(P(U\widetilde{A}U^*)P + s(I-P)) = \begin{pmatrix} f(\lambda A + (1-\lambda)B) & \\ & f(s)I \end{pmatrix}, \tag{2.2.13}$$

$$Pf(P(U\widetilde{A}U^*)P+s(I-P))P = \begin{pmatrix} f(\lambda A + (1-\lambda)B) & \\ & \end{pmatrix}, \quad PUf(\widetilde{A})U^*P = \begin{pmatrix} \lambda f(A) + (1-\lambda)f(B) & \\ & \end{pmatrix} \tag{2.2.14}$$

We read the $(1,1)$-block, yielding

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B). \tag{2.2.15}$$

Therefore f is operator convex. $\square$

Remark 10. *This proposition gives a much more simplified characterization of the operator convexity.*

In fact, we can extend the Sherman-Davis inequality to the case of partial isometries, which corresponds to a slightly different truncation approach compared to the case of projection.

Proposition 9 (Sherman-Davis)**.** *Suppose $f : \mathrm{Dom}(f) \to \mathbb{R}$ is an operator convex function, then for any $n \in \mathbb{N}$ and $A \in M_n(\mathbb{C})$ Hermitian with $\mathrm{Sp}(A) \subset \mathrm{Dom}(f)$ and some partial isometry $V \in M_n(\mathbb{C})$, we have*

$$VV^* f(VAV^* + s(1 - VV^*))VV^* \leq Vf(A)V^*. \tag{2.2.16}$$

Here, $s \in \mathrm{Dom}(f)$.

Remark 11. *We say V is a partial isometry iff VV^* and V^*V are both projections.*

Proof. Totally similar to Proposition 7. We take $A = \begin{pmatrix} A & \\ & A \end{pmatrix}$ and $U = \begin{pmatrix} V & V_\perp \\ 1-|V| & 1-|V_\perp| \end{pmatrix}$ ($V = Q|V|$ for some unitary Q) and $V_\perp$ is defined s.t. $VV^* + V_\perp V_\perp^* = I$. Then we do the same calculation. $\square$

Theorem 2.2.1 (Noncommutative Jensen Inequality)**.** *Suppose $f : \mathrm{Dom}(f) \to \mathbb{R}$ is an operator convex function on some interval in $\mathbb{R}$, $A_1, \cdots, A_m \in M_n(\mathbb{C})$ are Hermitian such that $\mathrm{Sp}(A_j) \subset \mathrm{Dom}(f)$, and $V_1, \cdots, V_m \in M_n(\mathbb{C})$ such that $\sum_{j=1}^m V_j^* V_j = I$, then*

$$f\left(\sum_{j=1}^m V_j^* A_j V_j\right) \leq \sum_{j=1}^m V_j^* f(A_j) V_j. \tag{2.2.17}$$

Remark 12. *Do not require each V_j to be a partial isometry. This is very useful in quantum information theory.*

Proof. We take

$$\widetilde{V} = \begin{pmatrix} V_1^* & \cdots & V_m^* \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix}. \tag{2.2.18}$$

Note that $\widetilde{V}$ is a partial isometry, therefore by Proposition 9, we have

$$\widetilde{V}\widetilde{V}^* f(\widetilde{V}A\widetilde{V}^* + s(1 - \widetilde{V}\widetilde{V}^*)) \leq \widetilde{V} f(A) \widetilde{V}^*. \tag{2.2.19}$$

Computing each side, we obtain

$$\begin{pmatrix} f(\sum_{j=1}^m V_j^* A_j V_j) & \\ & \end{pmatrix} \leq \begin{pmatrix} \sum_{j=1}^m V_j^* f(A_j) V_j & \\ & \end{pmatrix}, \tag{2.2.20}$$

i.e.

$$f\left(\sum_{j=1}^m V_j^* A_j V_j\right) \leq \sum_{j=1}^m V_j^* f(A_j) V_j. \tag{2.2.21}$$

□

Remark 13. *This theorem formally resembles the classical Jensen inequality. Moreover, by the characterization Proposition 8, the Jensen inequality implies operator convexity.*

By previous results, we use the additive term to deal with the case where $0 \in \mathrm{Dom}(f)$. Next we will explore another different way to understand the $0 \in \mathrm{Dom}(f)$ case.

Theorem 2.2.2. $0 \in \mathrm{Dom}(f)$ *is an interval, then TFAE:*

- f *is operator convex and* $f(0) \leq 0$*;*
- *For any* $n \in \mathbb{N}$*,* $X \in M_n(\mathbb{C})$ *with* $\|X\| \leq 1$ *and Hermitian matrix* $A \in M_n(\mathbb{C})$ *with* $\mathrm{Sp}(A) \subset \mathrm{Dom}(f)$*, the following inequality holds:*

$$f(X^*AX) \leq X^* f(A) X. \tag{2.2.22}$$

This inequality is sometimes also called noncommutative Jensen inequality.

Proof. $\Rightarrow$: Let

$$\widetilde{A} = \begin{pmatrix} A & \\ & 0 \end{pmatrix}, \quad U = \begin{pmatrix} X & (I - XX^*)^{1/2} \\ (I - X^*X)^{1/2} & -X^* \end{pmatrix}, \quad V = \begin{pmatrix} X & -(I - XX^*)^{1/2} \\ -(I - X^*X)^{1/2} & X^* \end{pmatrix} \tag{2.2.23}$$

Remark 14. *In quantum algorithm community, the construction of* U *is a very good example of "block-encoding" of* X*. The condition* $\|X\| \leq 1$ *is necessary.*

For simplicity, we let $Y = (I - XX^*)^{1/2}$. By operator convexity

$$f\left(\frac{U^*\widetilde{A}U + V^*\widetilde{A}V}{2}\right) \leq \frac{f(U^*\widetilde{A}U) + f(V^*\widetilde{A}V)}{2}. \tag{2.2.24}$$

The left hand side is $\mathrm{diag}(f(X^*AX), f(YAY))$. The right hand side is

$$\begin{aligned} \frac{1}{2}U^*\begin{pmatrix} f(A) & \\ & f(0)I \end{pmatrix}U + \frac{1}{2}V^*\begin{pmatrix} f(A) & \\ & f(0)I \end{pmatrix}V &\leq \frac{1}{2}U^*\begin{pmatrix} f(A) & \\ & 0 \end{pmatrix}U + \frac{1}{2}V^*\begin{pmatrix} f(A) & \\ & 0 \end{pmatrix}V \\ &= \begin{pmatrix} X^*f(A)X & \\ & Yf(A)Y \end{pmatrix}. \end{aligned} \tag{2.2.25}$$

We read the $(1,1)$-block, this implies that $f(X^*AX) \le X^*f(A)X$.

$\Leftarrow$: Suppose $A, B \in M_n(\mathbb{C})$ are Hermitian and $0 \le \lambda \le 1$. Let

$$\widetilde{A} = \begin{pmatrix} A & \\ & B \end{pmatrix}, \quad P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} \sqrt{\lambda} & -\sqrt{1-\lambda} \\ \sqrt{1-\lambda} & \sqrt{\lambda} \end{pmatrix}. \tag{2.2.26}$$

Then it is totally the same as Proposition 8. □

Corollary 2. *Same assumptions, TFAE*

- *f is operator convex and $f(0) \le 0$,*
- *For any $n \in \mathbb{N}$, Hermitian matrices A_j with $\mathrm{Sp}(A_j) \subset \mathrm{Dom}(f)$, and V_j s.t. $\sum_{j=1}^m V_j^* V_j \le 1$, we have*
$$f\left(\sum_{j=1}^m V_j^* A_j V_j\right) \le \sum_{j=1}^m V_j^* f(A_j) V_j. \tag{2.2.27}$$

Corollary 3. *Let $A \ge 0$, $X \in M_n(\mathbb{C})$, $\|X\| \le 1$, $\alpha \le [0,1]$, then*

$$X^* A^\alpha X \le (X^*AX)^\alpha. \tag{2.2.28}$$

Proof. t^α, $\alpha \in (0,1]$ is an operator concave function and $f(0) = 0$, then it follows from Theorem 2.2.2. Then let $\alpha \to 0$. □

Proposition 10. *$f : [0,b) \to \mathbb{R}$, $b > 0$, then TFAE:*

- *f is operator convex and $f(0) \le 0$;*
- *f_0 is operator convex and $f(0^+) \le f(0) \le 0$. Here*
$$f_0(t) = \begin{cases} f(t), & t \in (0,b), \\ f(0^+) = \lim_{t\to 0+} f(t), & t = 0. \end{cases} \tag{2.2.29}$$

Proof. $\Rightarrow$: It is obvious that f_0 is operator convex on $(0,b)$. By Löwner's theorem, $f \in C^2(0,b)$, therefore f_0 is operator convex on $[0,b)$. Moreover

$$f(0^+) = \lim_{t\to 0+} f(t) \le \limsup_{t\to 0^+} \frac{f(0)+f(2t)}{2} = \frac{f(0^+)+f(0)}{2} \Rightarrow f(0^+) \le f(0). \tag{2.2.30}$$

$\Leftarrow$: By the characterization of operator convexity in Proposition 8, we need to verify that $\forall P \in M_n(\mathbb{C})$ projection, $A \ge 0$, we have $f(PAP) \le Pf(A)P$. By the operator convexity of f_0, we have $f_0(PAP) \le Pf_0(A)P$. Note that

$$f(PAP) = f_0(PAP) + [f(0) - f_0(0)]P_{\ker(PAP)}, \tag{2.2.31}$$

$$f(A) = f_0(A) + [f(0) - f_0(0)]P_{\ker A}, \tag{2.2.32}$$

$$Pf(A)P = Pf_0(A)P + [f(0) - f_0(0)]PP_{\ker A}P. \tag{2.2.33}$$

We note that $f_0(PAP) = Pf_0(PAP)P$ and

$$P_{\ker(PAP)} = PP_{\ker(PAP)}P + (I-P)P_{\ker(PAP)}(I-P) \tag{2.2.34}$$

$$\begin{aligned}\Rightarrow [f(0)-f_0(0)]P_{\ker(PAP)} &= [f(0)-f_0(0)]PP_{\ker(PAP)}P + [f(0)-f_0(0)](I-P)P_{\ker(PAP)}(I-P) \\ &\le [f(0)-f_0(0)]PP_{\ker(PAP)}P + f(0)(I-P)P_{\ker(PAP)}(I-P) \\ &\le [f(0)-f_0(0)]PP_{\ker(PAP)}P + f(0)(I-P)I(I-P) \\ &= [f(0)-f_0(0)]PP_{\ker(PAP)}P + (I-P)f(0).\end{aligned} \tag{2.2.35}$$

Therefore we have

$$f(PAP) \le f_0(PAP)+[f(0)-f_0(0)]PP_{\ker(PAP)}P+(I-P)f(0) \le f_0(PAP)+[f(0)-f_0(0)]PP_{\ker(PAP)}P. \tag{2.2.36}$$

The last inequality follows from $f(0) \le 0$. Comparing with eq. (2.2.33) and noting $f_0(PAP) \le Pf_0(A)P$, it only remains to show that $[f(0)-f_0(0)]PP_{\ker(PAP)}P \le [f(0)-f_0(0)]PP_{\ker A}P$. Since $f_0(0) = f(0^+) \le f(0)$, we only need to verify $PP_{\ker(PAP)}P \le PP_{\ker A}P$. By Corollary 3 we have

$$P_{\mathrm{Range}(PAP)} \ge PP_{\mathrm{Range}(A)}P, \tag{2.2.37}$$

i.e.

$$P - PP_{\ker(PAP)}P \ge P - PP_{\ker A}P \Rightarrow PP_{\ker(PAP)}P \le PP_{\ker A}P. \tag{2.2.38}$$

□

Proposition 11. *Let $f:[0,b)\to\mathbb{R}$, $b>0$, then TFAE:*

- *f is operator concave and $f(0)\le 0$;*
- *$\frac{f(t)}{t}$ is operator monotonic on $(0,b)$.*

Proof. $\Rightarrow$: Let $0<A\le B$, $X:=B^{-1/2}A$, then by the operator convexity of f, we have

$$f(X^*BX) \le X^*f(B)X \Rightarrow f(A) \le X^*f(B)X = A^{1/2}B^{-1/2}f(B)B^{-1/2}A^{1/2}. \tag{2.2.39}$$

Therefore

$$A^{-1/2}f(A)A^{1/2} \le B^{-1/2}f(B)B^{1/2} \Rightarrow A^{-1}f(A) \le B^{-1}f(B). \tag{2.2.40}$$

$\Leftarrow$: $\frac{f(t)}{t}$ is operator monotonic $\Rightarrow$ f is continuous. We need to show that for $P\in M_n(\mathbb{C})$ a projection, $A\in M_n(\mathbb{C})$ Hermite,

$$f(PAP) \le Pf(A)P. \tag{2.2.41}$$

Take $P_\varepsilon = P+\varepsilon(I-P)$, $A_\varepsilon = A+\varepsilon I$. Then for $\varepsilon>0$ sufficiently small, $P_\varepsilon\le 1$, A_ε is invertible and therefore

$$A_\varepsilon^{1/2}P_\varepsilon A_\varepsilon^{1/2} \le A_\varepsilon. \tag{2.2.42}$$

We denote $h(t)=f(t)/t$, then by operator monotonicity, we have

$$h(A_\varepsilon^{1/2}P_\varepsilon A_\varepsilon^{1/2}) \le h(A_\varepsilon) \Rightarrow P_\varepsilon A_\varepsilon^{1/2}h(A_\varepsilon^{1/2}P_\varepsilon A_\varepsilon^{1/2})P_\varepsilon A_\varepsilon^{1/2} \le P_\varepsilon A_\varepsilon^{1/2}h(A_\varepsilon)P_\varepsilon A_\varepsilon^{1/2}. \tag{2.2.43}$$

By polar decomposition, $X=R|X|$, we have

$$Xh(X^*X)X^* = R|X|h(|X|^2)|X|R^* = Rh(|X|^2)|X|^2R^* = h(R|X|^2R^*)R|X|^2R^* = h(XX^*)XX^*. \tag{2.2.44}$$

$$\Rightarrow f(P_\varepsilon A_\varepsilon P_\varepsilon) \le P_\varepsilon f(A_\varepsilon)P_\varepsilon. \tag{2.2.45}$$

We take $\varepsilon\to 0$ and by the continuity of f, we conclude that $f_0(PAP)\le Pf_0(A)P$. Therefore f_0 is operator convex. By Proposition 10, f is operator convex. □

Remark 15. *From this we can easily see that $f(t) = f^{\alpha}(\alpha > 2)$ is not operator convex since otherwise, $\frac{f(t)}{t}$ is operator monotonic. But $t^{\alpha-1}$ is definitely not operator monotonic, which is a contradiction.*

Theorem 2.2.3. *$f : [0,\infty) \to [0,\infty)$, then TFAE:*

- *f is operator concave;*
- *f is operator monotonic.*

Lemma 3. *A, C are positive semidefinite matrices, A is invertible and $B \in M_n(\mathbb{C})$, then*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \iff C \geq B^* A^{-1} B. \tag{2.2.46}$$

Proof. We first note that

$$\begin{pmatrix} A^{\frac{1}{2}} & 0 \\ B^* A^{-\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & A^{-\frac{1}{2}} B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & B^* A^{-1} B \end{pmatrix} \geq 0. \tag{2.2.47}$$

$\Leftarrow$:

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & B^* A^{-1} B \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C - B^* A^{-1} B \end{pmatrix} \geq 0. \tag{2.2.48}$$

$\Rightarrow$:

$$\begin{aligned} 0 &\leq \left\langle \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} -A^{-1} B\xi \\ \xi \end{pmatrix}, \begin{pmatrix} -A^{-1} B\xi \\ \xi \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 0 \\ (C - B^* A^{-1} B)\xi \end{pmatrix}, \begin{pmatrix} -A^{-1} B\xi \\ \xi \end{pmatrix} \right\rangle = \langle (C - B^* A^{-1} B)\xi, \xi \rangle. \end{aligned} \tag{2.2.49}$$

□

Proof of Theorem 2.2.3. $\Rightarrow$: We take $0 \leq A \leq B$, $\lambda \in (0,1)$:

$$f(\lambda B) = f\left(\lambda A + (1-\lambda)\frac{1}{1-\lambda}(B - A)\right) \geq \lambda f(A) + (1-\lambda) f\left(\frac{\lambda}{1-\lambda}(B - A)\right). \tag{2.2.50}$$

This inequality follows from the operator concavity of f. Let $\lambda \to 0$, we have

$$f_0(B) \geq f_0(A). \tag{2.2.51}$$

Moreover, since $f_0(A) \leq f_0(B)$, $P_{\ker A} \geq P_{\ker B}$, $f(0) - f(0^+) \leq 0$, we have

$$f(A) = f_0(A) + [f(0) - f(0^+)] P_{\ker A} \leq f_0(B) + [f(0) - f(0^+)] P_{\ker B} = f(B). \tag{2.2.52}$$

Therefore,

$$f(A) \leq f(B). \tag{2.2.53}$$

$\Leftarrow$: We need to show that for A Hermitian, $X \in M_n(\mathbb{C})$, $\|X\| \leq 1$, we have $f(X^* A X) \leq X^* f(A) X$. We take

$$\widetilde{A} = \begin{pmatrix} A & \\ & 0 \end{pmatrix}, \quad U = \begin{pmatrix} X & (I - XX^*)^{1/2} \\ (I - X^* X)^{1/2} & -X^* \end{pmatrix}. \tag{2.2.54}$$

Then we have

$$U^*\widetilde{A}U = \begin{pmatrix} X^*AX & * \\ * & * \end{pmatrix}, \quad U^*f(\widetilde{A})U = \begin{pmatrix} X^*f(A)X & * \\ * & * \end{pmatrix}. \tag{2.2.55}$$

We take

$$\widetilde{B} = \begin{pmatrix} X^*AX + \varepsilon I & 0 \\ 0 & \gamma I \end{pmatrix}, \tag{2.2.56}$$

then when γ is large enough, we have $\widetilde{B} \geq U^*AU$ by Lemma 3. Then by the operator monoticity of f, we have

$$f(\widetilde{B}) \geq U^*f(\widetilde{A})U. \tag{2.2.57}$$

We read the $(1,1)$-block, we have

$$f(X^*AX + \varepsilon I) \geq X^*f(A)X. \tag{2.2.58}$$

We take $\varepsilon \to 0$, we have

$$f_0(X^*AX) \geq X^*f_0(A)X. \tag{2.2.59}$$

Therefore f_0 is operator concave and $f(0^+) \geq f(0) \geq 0$, and by Proposition 10, f is operator concave. □

Proposition 12. *$f : (0,\infty) \to (0,\infty)$, then TFAE:*

- *f is operator monotonic;*
- *$t/f(t)$ is operator monotonic;*
- *f is operator concave.*

Proof. Use Theorem 2.2.3 and Proposition 11 repeatly. □

2.3 Operator mean-value inequalities

We denote the set of $n \times n$ Hermitian matrices by $\mathbb{H}_n$. We denote the set of positive definite matrices by $\mathbb{H}_n^{>0}$.

Definition 2.3.1 (Harmonic mean)**.** *For $A, B \in M_n(\mathbb{C})$ positive definite (to ensure that it is well-defined), the harmonic mean of A and B is defined as*

$$M_{-1}(A,B) := \left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1}. \tag{2.3.1}$$

We can easily verify that:

$$M_{-1}(A,B) = 2B(B+A)^{-1}A = 2A(A+B)^{-1}B. \tag{2.3.2}$$

Theorem 2.3.2 (Ando's variational formula (for harmonic mean))**.** *Let $A, B > 0$, then*

$$\sup\left\{X \in \mathbb{H}_n : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \geq \frac{1}{2}\begin{pmatrix} X & X \\ X & X \end{pmatrix}\right\} = M_{-1}(A,B). \tag{2.3.3}$$

Proof. Let X s.t. $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \geq \frac{1}{2}\begin{pmatrix} X & X \\ X & X \end{pmatrix}$ (Note that $\mathrm{RHS} = \frac{1}{2}X \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$). We take

$$U = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad U^* \begin{pmatrix} A & \\ & B \end{pmatrix} U = \frac{1}{2}\begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix}. \tag{2.3.4}$$

Moreover

$$U^* \begin{pmatrix} X & X \\ X & X \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ 0 & 2X \end{pmatrix}. \tag{2.3.5}$$

Therefore we have

$$\begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \\ 0 & 2X \end{pmatrix}. \tag{2.3.6}$$

Recall. $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \iff C \geq B^* A^{-1} B.$

Hence

$$A + B - 2X \geq (A-B)(A+B)^{-1}(A-B). \tag{2.3.7}$$

We calculate the right hand side:

$$\begin{aligned} (A-B)(A+B)^{-1}(A-B) &= A(A+B)^{-1}A - B(A+B)^{-1}A - A(A+B)^{-1}B + B(A+B)^{-1}B \\ &= A(A+B)^{-1}A + B(A+B)^{-1}B - 2M_{-1}(A,B). \end{aligned} \tag{2.3.8}$$

And note that

$$A + B = (A+B)(A+B)^{-1}(A+B) = A(A+B)^{-1}A + B(A+B)^{-1}B + 2M_{-1}(A,B). \tag{2.3.9}$$

Therefore we have

$$A + B - 2X \geq A + B - 4M_{-1}(A,B), \tag{2.3.10}$$

i.e.

$$X \leq M_{-1}(A,B). \tag{2.3.11}$$

□

Another Proof. $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \geq \frac{1}{2}\begin{pmatrix} X & X \\ X & X \end{pmatrix}$ is equivalent to

$$\begin{pmatrix} I & \\ & I \end{pmatrix} \geq \frac{1}{2}\begin{pmatrix} A^{-1/2} & \\ & B^{1/2} \end{pmatrix}\begin{pmatrix} X & X \\ X & X \end{pmatrix}\begin{pmatrix} A^{-1/2} & \\ & B^{1/2} \end{pmatrix}. \tag{2.3.12}$$

We use the argument of spectral radius to "flip" the inequality, obtaining

$$\frac{1}{4}\begin{pmatrix} \sqrt{X} & \sqrt{X} \\ \sqrt{X} & \sqrt{X} \end{pmatrix}\begin{pmatrix} A^{-1} & \\ & B^{-1} \end{pmatrix}\begin{pmatrix} \sqrt{X} & \sqrt{X} \\ \sqrt{X} & \sqrt{X} \end{pmatrix} \leq \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{2.3.13}$$

We do the some calculations and obtain

$$\frac{1}{4}\begin{pmatrix} \sqrt{X}(A^{-1}+B^{-1})\sqrt{X} & \sqrt{X}(A^{-1}-B^{-1})\sqrt{X} \\ \sqrt{X}(A^{-1}-B^{-1})\sqrt{X} & \sqrt{X}(A^{-1}+B^{-1})\sqrt{X} \end{pmatrix} \leq \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{2.3.14}$$

We define $C := \frac{1}{2}\sqrt{X}(A^{-1}+B^{-1})\sqrt{X}$, then we have $C^*C \leq I \Rightarrow C \leq I$, i.e.

$$\frac{1}{2}\sqrt{X}(A^{-1}+B^{-1})\sqrt{X} \leq I \Rightarrow X \leq M_{-1}(A,B). \tag{2.3.15}$$

□

Drawing from the *another proof*, we can generalize the Ando's variational formula to

Proposition 13 (m-variables Ando's variational formula)**.** *Let* $A_1, \cdots, A_m > 0$*, then*

$$\sup\left\{X \in \mathbb{H}_n : \begin{pmatrix} A_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & A_m \end{pmatrix} \geq \frac{1}{m}\begin{pmatrix} X & X & \cdots & X \\ X & X & \cdots & X \\ \vdots & \vdots & \ddots & \vdots \\ X & X & \cdots & X \end{pmatrix}\right\} = \left(\frac{A_1^{-1} + \cdots + A_m^{-1}}{m}\right)^{-1}. \quad (2.3.16)$$

Definition 2.3.3 (Joint convexity)**.** $f : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}$*, if for* $\lambda \in [0,1]$*,* $x_1, y_1 \in \mathcal{X}_1$*,* $x_2, y_2 \in \mathcal{X}_2$*, we have*

$$f(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \leq \lambda f(x_1, x_2) + (1-\lambda)f(y_1, y_2), \quad (2.3.17)$$

then we say f *is joint convex.*

Proposition 14. *The mapping* $(A, B) \mapsto B^* A^{-1} B$ *is joint concave on* $\mathbb{H}_n^{>0} \times M_n(\mathbb{C})$*.*

Proof. We take $A_1, A_2 > 0$, $B_1, B_2 \in M_n(\mathbb{C})$, $\lambda \in [0,1]$. We have

$$\lambda\begin{pmatrix} A_1 & B_1 \\ B_1^* & B_1^* A_1^{-1} B_1 \end{pmatrix} + (1-\lambda)\begin{pmatrix} A_2 & B_2 \\ B_2^* & B_2^* A_2^{-1} B_2 \end{pmatrix} \geq 0. \quad (2.3.18)$$

Therefore

$$\lambda B_1^* A_1^{-1} B_1 + (1-\lambda) B_2^* A_2^{-1} B_2 \geq (\lambda_1 B_1^* + (1-\lambda)B_2^*)(\lambda A_1 + (1-\lambda)A_2)^{-1}(\lambda B_1 + (1-\lambda)B_2). \quad (2.3.19)$$

Therefore, the mapping is joint concave. □

Remark 16. *This is a very important and classical example of joint concavity. It is closely related to the (generalized) Lieb's concavity, as we may see later in section 3.7.*

Proposition 15. M_{-1} *is joint concave on* $\mathbb{H}_n^{>0} \times \mathbb{H}_n^{>0}$*.*

Proof. Take $A_1, A_2 > 0$, $B_1, B_2 > 0$, $\lambda \in [0,1]$. We have

$$\begin{aligned} &\frac{\lambda}{2}\left[M_{-1}(A_1, B_1) \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] + \frac{1-\lambda}{2}\left[M_{-1}(A_2, B_2) \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] \\ \overset{\text{(Ando's variational formula)}}{\leq} &\lambda\begin{pmatrix} A_1 & \\ & B_1 \end{pmatrix} + (1-\lambda)\begin{pmatrix} A_2 & \\ & B_2 \end{pmatrix} = \begin{pmatrix} \lambda A_1 + (1-\lambda)A_2 & \\ & \lambda B_1 + (1-\lambda)B_2 \end{pmatrix}. \end{aligned} \quad (2.3.20)$$

$$\Rightarrow M_{-1}(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda)B_2) \geq \lambda M_{-1}(A_1, B_1) + (1-\lambda)M_{-1}(A_2, B_2). \quad (2.3.21)$$

The last inequality follows from the Ando's variational formula Theorem 2.3.2 again (i.e. M_{-1} is the maximizer of the variational formula). Therefore, M_{-1} is joint concave. □

Proposition 16. M_{-1} *is operator monotone with respect to each component on* $\mathbb{H}_n^{>0} \times \mathbb{H}_n^{>0}$*.*

Proof. Let $A_1 \leq A_2$ be two positive definite matrices, then

$$\frac{1}{2}\begin{pmatrix} M_{-1}(A_1, B) & M_{-1}(A_1, B) \\ M_{-1}(A_1, B) & M_{-1}(A_1, B) \end{pmatrix} \leq \begin{pmatrix} A_1 & 0 \\ 0 & B \end{pmatrix} \leq \begin{pmatrix} A_2 & 0 \\ 0 & B \end{pmatrix}. \quad (2.3.22)$$

By the Ando's variational formula, we have

$$M_{-1}(A_2, B) \geq M_{-1}(A_1, B). \quad (2.3.23)$$

□

Remark 17. *We can see that the results for harmonic mean are very clean and elegant. In fact, the harmonic mean can be viewed as some "projection" from $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$ to $M_n(\mathbb{C})$ along the "direction" of* $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Next we will introduce the geometric mean. At first glance, the geometric mean is not as clean as the harmonic mean, but we will see later that this definition is reasonable and natural.

Definition 2.3.4 (Geometric mean)**.** *For $A, B \in M_n(\mathbb{C})$ positive definite, the geometric mean of A and B is defined as*

$$M_0(A, B) := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \tag{2.3.24}$$

Similar to the case of harmonic mean, we also have the Ando's variational formula for geometric mean.

Theorem 2.3.5 (Ando's variational formula (for geometric mean))**.** *Let $A, B > 0$, then*

$$\sup \left\{ X \in \mathbb{H}_n : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \geq - \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \right\} = M_0(A, B). \tag{2.3.25}$$

Proof. We take X s.t. $\left(\begin{smallmatrix} A & X \\ X & B \end{smallmatrix}\right) \geq 0$. Therefore we have

$$B \geq X^* A^{-1} X = XA^{-1}X. \tag{2.3.26}$$

Consider $A^{-1/2}BA^{-1/2}$, we have

$$A^{-1/2}BA^{-1/2} \geq (A^{-1/2}XA^{-1/2})(A^{-1/2}XA^{-1/2}). \tag{2.3.27}$$

Therefore, by the operator monotonicity of $t \mapsto t^{1/2}$, we have

$$A^{-1/2}XA^{-1/2} \leq (A^{-1/2}BA^{-1/2})^{1/2} \Rightarrow \text{The original equality holds.} \tag{2.3.28}$$

□

Remark 18. *We can also use the Ando's variational formula for geometric mean to show that M_0 is joint concave by putting M_0 on the off-diagonal block.*

Proposition 17.

- *M_0 is joint concave;*
- *(symmetric) $M_0(B, A) = M_0(A, B)$;*
- *For any invertible matrix D, $M_0(D^*AD, D^*BD) = D^*M_0(A, B)D$;*
- *M_0 is operator monotone with respect to each component.*
- *$M_{-1}(A, B) \leq M_0(A, B) \leq \frac{A+B}{2}$.*

Proof.

- By the Ando's variational formula, we have

$$\lambda \begin{pmatrix} A_1 & M_0(A_1, B_1) \\ M_0(A_1, B_1) & B_1 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} A_2 & M_0(A_2, B_2) \\ M_0(A_2, B_2) & B_2 \end{pmatrix} \geq 0. \tag{2.3.29}$$

This holds because the variational formula implies that each additive term is positive semidefinite. That is to say

$$\begin{pmatrix} \lambda A_1+(1-\lambda)A_2 & M_0(\lambda A_1+(1-\lambda)A_2,\lambda B_1+(1-\lambda)B_2) \\ M_0(\lambda A_1+(1-\lambda)A_2,\lambda B_1+(1-\lambda)B_2) & \lambda B_1+(1-\lambda)B_2 \end{pmatrix} \geq 0. \tag{2.3.30}$$

By the Ando's variational formula again (the geometric mean maximizes the 2 by 2 block matrix), we have

$$M_0(\lambda A_1+(1-\lambda)A_2,\lambda B_1+(1-\lambda)B_2) \geq \lambda M_0(A_1,B_1)+(1-\lambda)M_0(A_2,B_2). \tag{2.3.31}$$

- We consider $X=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then

$$X\begin{pmatrix} A & M_0(A,B) \\ M_0(A,B) & B \end{pmatrix}X=\begin{pmatrix} B & M_0(A,B) \\ M_0(A,B) & A \end{pmatrix} \geq 0. \tag{2.3.32}$$

By the Ando's variational formula, we have

$$M_0(B,A) \geq M_0(A,B) \overset{\text{symmetrically}}{\geq} M_0(B,A) \quad \Rightarrow \quad M_0(B,A)=M_0(A,B). \tag{2.3.33}$$

- We compute

$$\begin{pmatrix} D^* & \\ & D^* \end{pmatrix}\begin{pmatrix} A & M_0(A,B) \\ M_0(A,B) & B \end{pmatrix}\begin{pmatrix} D & \\ & D \end{pmatrix}=\begin{pmatrix} D^*AD & D^*M_0(A,B)D \\ D^*M_0(A,B)D & D^*BD \end{pmatrix} \geq 0. \tag{2.3.34}$$

Therefore

$$M_0(D^*AD,D^*BD) \geq D^*M_0(A,B)D. \tag{2.3.35}$$

Moreover, we use eq. (2.3.35) again

$$M_0(A,B)=M_0(D^{-*}(D^*AD)D^{-1},D^{-*}(D^*BD)D^{-1}) \geq D^{-*}M_0(D^*AD,D^*BD)D^{-1}. \tag{2.3.36}$$

Plugging it back to eq. (2.3.35), we obtain

$$\begin{aligned} M_0(D^*AD,D^*BD) &\geq D^*M_0(A,B)D \geq D^*D^{-*}M_0(D^*AD,D^*BD)D^{-1}D \\ &= M_0(D^*AD,D^*BD) \Rightarrow M_0(D^*AD,D^*BD)=D^*M_0(A,B)D. \end{aligned} \tag{2.3.37}$$

- Note that

$$-\begin{pmatrix} 0 & M_0(A_1,B) \\ M_0(A_1,B) & 0 \end{pmatrix} \leq \begin{pmatrix} A_1 & 0 \\ 0 & B \end{pmatrix} \leq \begin{pmatrix} A_2 & 0 \\ 0 & B \end{pmatrix}. \tag{2.3.38}$$

Therefore, by the maximization condition of the variational formula, we have

$$M_0(A_2,B) \geq M_0(A_1,B). \tag{2.3.39}$$

By symmetry, we have M_0 is operator monotone with respect to each component.

- We take $X = A^{-1/2}BA^{-1/2}$, then by

$$\left(\frac{1+t^{-1}}{2}\right) \le t^{1/2} \le \frac{1+t}{2} \Rightarrow \left(\frac{1+X^{-1}}{2}\right)^{-1} \le X^{1/2} \le \frac{1+X}{2}, \tag{2.3.40}$$

we conclude that the original inequality holds.

□

We next present a very profound application of the geometric mean—Ando's convexity.

Theorem 2.3.6 (Ando's convexity)**.** *$0 < p, r \le 1$, $p + r \ge 1$, then*

$$(A, B) \mapsto A^p \otimes B^r \tag{2.3.41}$$

is joint concave on $\mathbb{H}_n^{>0} \times \mathbb{H}_n^{>0}$.

Proof.

- We take $\Lambda = \{(p, r) : A^p \otimes B^r \text{ is joint concave}\}$. We have $(0, 0), (1, 0), (0, 1) \in \Lambda$.
- We next show that Λ is a convex set. We take $(p_1, r_1), (p_2, r_2) \in \Lambda$, we need to verify that $(p, r) := \left(\frac{p_1+p_2}{2}, \frac{r_1+r_2}{2}\right) \in \Lambda$. By the commutative of tensor product and the definition of geometric mean, we have

$$A^p \otimes B^r = A^{\frac{p_1+p_2}{2}} \otimes B^{\frac{r_1+r_2}{2}} = M_0(A^{p_1} \otimes B^{r_1}, A^{p_2} \otimes B^{r_2}). \tag{2.3.42}$$

- We take $A_1, A_2 > 0$, $B_1, B_2 > 0$, then by $(p_i, r_i) \in \Lambda$, we have

$$\left(\frac{A_1+A_2}{2}\right)^{p_i} \otimes \left(\frac{B_1+B_2}{2}\right)^{r_i} \ge \frac{1}{2}\left(A_1^{p_i} \otimes B_1^{r_i} + A_2^{p_i} \otimes B_2^{r_i}\right), \quad i = 1, 2 \tag{2.3.43}$$

Therefore,

$$\begin{aligned}
&\left(\frac{A_1+A_2}{2}\right)^{p} \otimes \left(\frac{B_1+B_2}{2}\right)^{r} \\
&= M_0\left(\left(\frac{A_1+A_2}{2}\right)^{p_1} \otimes \left(\frac{B_1+B_2}{2}\right)^{r_1}, \left(\frac{A_1+A_2}{2}\right)^{p_2} \otimes \left(\frac{B_1+B_2}{2}\right)^{r_2}\right) \\
&\overset{\text{monotonicity of } M_0 \text{ and eq. (2.3.42)}}{\ge} M_0\left(\frac{A_1^{p_1} \otimes B_1^{r_1} + A_2^{p_1} \otimes B_2^{r_1}}{2}, \frac{A_1^{p_2} \otimes B_1^{r_2} + A_2^{p_2} \otimes B_2^{r_2}}{2}\right) \\
&\overset{\text{joint concavity of } M_0}{\ge} \frac{1}{2}M_0(A_1^{p_1} \otimes B_1^{r_1}, A_1^{p_2} \otimes B_1^{r_2}) + \frac{1}{2}M_0(A_2^{p_1} \otimes B_2^{r_1}, A_2^{p_2} \otimes B_2^{r_2}) \\
&= \frac{1}{2}\left(A_1^p \otimes B_1^r + A_2^p \otimes B_2^r\right).
\end{aligned} \tag{2.3.44}$$

Thus $(p, r) \in \Lambda$.

□

Corollary 4. *$0 \le p_1, \cdots, p_m \le 1$, $\sum_{j=1}^m p_j \le 1$, then*

$$(A_1, \cdots, A_m) \mapsto A_1^{p_1} \otimes \cdots \otimes A_m^{p_m} \tag{2.3.45}$$

is joint concave on $\mathbb{H}_n^{>0} \times \cdots \times \mathbb{H}_n^{>0}$.

Next we explore the "geometric description" of the geometric mean.

Lemma 4.

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(A+tX)^{-1} = (-1)^n n!(A+tX)^{-\frac{1}{2}}[(A+tX)^{-\frac{1}{2}}X(A+tX)^{-\frac{1}{2}}]^n(A+tX)^{-\frac{1}{2}}. \tag{2.3.46}$$

Remark 19. *More generally, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}(A+X(t))^{-1} = -(A+X(t))^{-1}\frac{\mathrm{d}}{\mathrm{d}t}X(t)(A+X(t))^{-1}. \tag{2.3.47}$$

Theorem 2.3.7. *Let f be a continuous differentiable function, X is a Hermitian matrix, $\mathrm{Sp}(A) \subset \mathrm{Dom}(f)$, $A = \mathrm{diag}(t_1, \cdots, t_n)$, then $\frac{\mathrm{d}}{\mathrm{d}t}f(A+tX) = D \circ X$, here*

$$D_{jk} = \begin{cases} \frac{f(t_j)-f(t_k)}{t_j-t_k}, & j \neq k, \\ f'(t_j), & j = k. \end{cases} \tag{2.3.48}$$

Proof. Without loss of generality, we assume f is analytic. (Otherwise, we can use C^1 function to approximate f (we can always take a compact set since $\mathrm{Sp}(A)$ is bounded)). We have

$$f(A+tX) = \frac{1}{2\pi\mathrm{i}}\int_\gamma f(z)(zI-A-tX)^{-1}\mathrm{d}z. \tag{2.3.49}$$

Here we take γ efficiently large to enclose $\mathrm{Dom}(f)$, $\mathrm{Sp}(A)$ and $\mathrm{Sp}(A+tX)$. Then we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}f(A+tX) &= \frac{1}{2\pi\mathrm{i}}\int_\gamma f(z)(zI-A-tX)^{-1}X(zI-A-tX)^{-1}\mathrm{d}z \\ &= \left(\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(z)x_{jk}}{(z-t_j)(z-t_k)}\mathrm{d}z\right)_{1\le j,k\le n} = \left(\frac{f(t_j)-f(t_k)}{t_j-t_k}x_{jk}\right)_{1\le j,k\le n} \\ &= D\circ X. \end{aligned} \tag{2.3.50}$$

□

Example 2. $t \mapsto X(t)$ *a smooth path* $\subset \mathbb{H}_n^{>0}$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\log X(t) = \int_0^\infty (X(t)+\alpha I)^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}t}X(t)\right](X(t)+\alpha I)^{-1}\mathrm{d}\alpha. \tag{2.3.51}$$

Proof. We note that the following integral equality holds, then it follows readily by Remark 19.

$$\log t = \int_0^\infty \left(\frac{1}{\alpha+1} - \frac{1}{t+\alpha}\right)\mathrm{d}\alpha. \tag{2.3.52}$$

□

Example 3. $t \mapsto X(t)$ *a smooth path. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(X(t)) = \int_0^\infty \exp(\alpha X(t))\left[\frac{\mathrm{d}}{\mathrm{d}t}X(t)\right]\exp((1-\alpha)X(t))\mathrm{d}\alpha. \tag{2.3.53}$$

Proof.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}e^X &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{\infty}\frac{1}{k!}X^k \overset{\text{boundedness}}{=} \sum_{k=0}^{\infty}\frac{1}{k!}\frac{\mathrm{d}}{\mathrm{d}t}X^k \\
&= \sum_{k=0}^{\infty}\frac{1}{(1+k)!}\sum_{j=0}^{k}X^j\frac{\mathrm{d}X}{\mathrm{d}t}X^{k-j} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{j!(k-j)!}{(1+k)!}\cdot\frac{1}{j!(k-j)!}X^j\frac{\mathrm{d}X}{\mathrm{d}t}X^{k-j} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\int_0^1 s^j(1-s)^{k-j}\mathrm{d}s\right)\frac{1}{j!(k-j)!}X^j\frac{\mathrm{d}X}{\mathrm{d}t}X^{k-j} \\
&= \int_0^1\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{j!(k-j)!}X^j\frac{\mathrm{d}X}{\mathrm{d}t}X^{k-j}s^j(1-s)^{k-j}\mathrm{d}s \\
&= \int_0^1\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{s^j(1-s)^{k-j}}{j!(k-j)!}X^j\frac{\mathrm{d}X}{\mathrm{d}t}X^{k-j}\mathrm{d}s \\
&= \int_0^1\exp(sX)\frac{\mathrm{d}X}{\mathrm{d}t}\exp((1-s)X)\mathrm{d}s.
\end{aligned}
\tag{2.3.54}
$$

□

Remark 20. *The information encoded in the noncommutative exponential is much more than the commutative case.*

Example 4. *Let* $X > 0$, $\Phi_X : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $\Phi_X(A) = \int_0^\infty (X+\lambda)^{-1}A(X+\lambda)^{-1}\mathrm{d}\lambda$. *Then we can write down the inverse of* Φ_X *explicitly:*

$$
\Phi_X^{-1}(A) = \int_0^1 X^sAX^{1-s}\mathrm{d}s. \tag{2.3.55}
$$

Proof. We take A a self-adjoint matrix, then $X + tA = \exp(\log(X + tA))$

$$
\begin{aligned}
A &= \frac{\mathrm{d}}{\mathrm{d}t}(X+tA)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\exp(\log(X+tA))\bigg|_{t=0} \\
&= \int_0^1\exp(s\log(X+tA))\left[\frac{\mathrm{d}}{\mathrm{d}t}\log(X+tA)\right]\exp((1-s)\log(X+tA))\mathrm{d}s\bigg|_{t=0} \\
&= \int_0^1 X^s\left[\frac{\mathrm{d}}{\mathrm{d}t}\log(X+tA)\right]_{t=0}X^{1-s}\mathrm{d}s \\
&= \int_0^1 X^s\left[\int_0^\infty(X+\lambda)^{-1}A(X+\lambda)^{-1}\mathrm{d}\lambda\right]X^{1-s}\mathrm{d}s \\
&= \Phi_X^{-1}\Phi_X(A).
\end{aligned}
\tag{2.3.56}
$$

Similarly, $A = \frac{\mathrm{d}}{\mathrm{d}t}\log(\exp(X+tA))_{t=0} = \Phi_X\Phi_X^{-1}(A)$. □

With above results about matrix calculus, we are at the position to revisit the matrix geometric mean.

Example 5. *Define* $\|A\|_2 := \sqrt{\mathrm{Tr}(A^*A)}$ *the Hilbert-Schmidt norm. We consider the "generalized length of curve"*

$$\int_a^b \left\| [X(t)]^{-\frac{1}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t} X(t) \right) [X(t)]^{-\frac{1}{2}} \right\|_2 \mathrm{d}t. \tag{2.3.57}$$

We compute

$$\left\| [X(t)]^{-\frac{1}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t} X(t) \right) [X(t)]^{-\frac{1}{2}} \right\|_2^2 = \mathrm{Tr}\left(X' X^{-1} X' X^{-1} \right) = \mathrm{Tr}\big(Y' Y^{-1} Y' Y^{-1} \big). \tag{2.3.58}$$

Here, we take $Y(t) = D^* X(t) D$ *with* D *invertible. Inspired by this, we define:*

Definition 2.3.8 (Distance)**.** *Let* $A > 0$*,* $B > 0$*, define*

$$\delta(A, B) := \min \left\{ \int_0^1 \left\| [X(t)]^{-\frac{1}{2}} X'(t) [X(t)]^{-\frac{1}{2}} \right\|_2 \mathrm{d}t : X(0) = A,\ X(1) = B \right\}. \tag{2.3.59}$$

Here, we say $\left\| [X(t)]^{-\frac{1}{2}} X'(t) [X(t)]^{-\frac{1}{2}} \right\|_2$ *is the* speed *of* $X(t)$ *and we can denote*

$$\delta(X) = \int_0^1 \left\| [X(t)]^{-\frac{1}{2}} X'(t) [X(t)]^{-\frac{1}{2}} \right\|_2 \mathrm{d}t \tag{2.3.60}$$

for the simplicity of notation. Then the minimizer of $\delta(X)$ *i.e. the* X *such that* $\delta(X) = \delta(A, B)$ *is called the* geodesic *between* A *and* B*.*

Proposition 18. *If* D *is invertible, then* $\delta(D^*AD, D^*BD) = \delta(A, B)$*. In other words, the distance is invariant under the congruent action.*

Definition 2.3.9. *To facilitate the calculation of* $\delta(A, B)$*, we denote*

- $H(t) := H_X(t) = \log X(t)$*, then we have* $X(t) = \exp H(t)$*, Moreover, according to example 2, we have*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(t) &= \int_0^\infty (X(t) + \lambda I)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} X(t) (X(t) + \lambda I)^{-1} \mathrm{d}\lambda \\ &= \int_0^\infty \frac{X(t)^{\frac{1}{2}}}{\lambda + X(t)} \boldsymbol{X(t)^{-\frac{1}{2}} X'(t) X(t)^{-\frac{1}{2}}} \frac{X(t)^{\frac{1}{2}}}{\lambda + X(t)} \mathrm{d}\lambda. \end{aligned} \tag{2.3.61}$$

- *In order to calculate* $\left\| X(t)^{-\frac{1}{2}} \frac{\mathrm{d}X(t)}{\mathrm{d}t} X(t)^{-\frac{1}{2}} \right\|_2$*, we define*

$$\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad \Phi(A) = \int_0^\infty \frac{X(t)^{\frac{1}{2}}}{\lambda + X(t)} A \frac{X(t)^{\frac{1}{2}}}{\lambda + X(t)} \mathrm{d}\lambda. \tag{2.3.62}$$

Remark 21. *In fact* Φ *is a quantum channel since it can be viewed as a "continuous analog" of* $\sum_{j=1}^n V_j^\dagger (\cdot) V_j$*, where* $\sum_{j=1}^n V_j^\dagger V_j = I$*. See example 6 below.*

Example 6. *For* $s > 0$*,* $\int_0^\infty \frac{s}{(\lambda+s)^2} \mathrm{d}\lambda = 1$*.*

Proposition 19. $\mathrm{Tr}(\Phi(A)) = \mathrm{Tr}(A)$*,* $\Phi(A)^2 \leq \Phi(A^2)$*.*

Proof. example 6 $\Rightarrow \int_0^\infty \frac{X(t)}{\lambda+X(t)}\mathrm{d}\lambda = I$. Therefore,

$$\operatorname{Tr}(\Phi(A)) = \operatorname{Tr}\left(A\int_0^\infty \frac{X(t)^{\frac{1}{2}}}{\lambda+X(t)}\frac{X(t)^{\frac{1}{2}}}{\lambda+X(t)}\mathrm{d}\lambda\right) = \operatorname{Tr}(A\cdot I) = \operatorname{Tr}(A). \tag{2.3.63}$$

t^2 is operator convex, note that $\int_0^\infty \left(\frac{X(t)^{\frac{1}{2}}}{\lambda+X(t)}\right)^* \left(\frac{X(t)^{\frac{1}{2}}}{\lambda+X(t)}\right)\mathrm{d}\lambda = I$, then by the ("*continuous version of*") noncommutative Jensen's inequality, we have

$$\Phi(A)^2 \le \Phi(A^2). \tag{2.3.64}$$

□

Remark 22. *These properties can also be seen from the perspetive of quantum channel. That is to say, we can verify that Φ is 2-positive $\Rightarrow$ completely positive $\Rightarrow$ completely positive and trace-preserving (CPTP).*

Example 7 (A lower bound of $\delta(A,B)$)**.** *We try to calculate and bound the speed. In fact,*

$$\begin{aligned}\|H'(t)\|_2^2 = \operatorname{Tr}\left[\Phi\left(X^{-\frac{1}{2}}X'X^{-\frac{1}{2}}\right)^2\right] &\overset{\text{by Proposition 19}}{\le} \\ \le \operatorname{Tr}\left[\Phi\left((X^{-\frac{1}{2}}X'X^{-\frac{1}{2}})^2\right)\right] &\overset{\text{by trace preserving}}{\le} \operatorname{Tr}\left[(X^{-1}X'X^{-1}X'X^{-1})^2\right] = \delta(X)^2.\end{aligned} \tag{2.3.65}$$

Therefore,

$$\int_0^1 \|H'(t)\|_2\mathrm{d}t \le \delta(A,B). \tag{2.3.66}$$

By the triangular inequality of $\|\cdot\|_2$, we have

$$\left\|\int_0^1 H'(t)\mathrm{d}t\right\|_2 \le \int_0^1 \|H'(t)\|_2\mathrm{d}t \le \delta(A,B). \tag{2.3.67}$$

In other words,

$$\|\log B - \log A\|_2 = \|H(1)-H(0)\|_2 \le \delta(A,B). \tag{2.3.68}$$

We will see that the minimum can actually be achieved. We begin with the commutative or *classical* case.

Proposition 20. $A, B > 0$*,* $AB = BA$*, then there exists unique constant-speed geodesic from* A *to* B*.*

Proof. We take $X(t) = A^{1-t}B^t$, then by A and B commute we have $X'(t) = \log B - \log A$, therefore

$$\left\|X^{-\frac{1}{2}}X'X^{-\frac{1}{2}}\right\|_2 = \left\|A^{-\frac{1}{2}}(\log B - \log A)B^{-\frac{1}{2}}\right\|_2 = \|\log B - \log A\|_2. \tag{2.3.69}$$

That is to say $X(t)$ is a constant-speed geodesic from A to B. We assume there exists another constant-speed geodesic $Y(t)$, then we have

$$\int_0^1 \left\|Y^{-\frac{1}{2}}Y'Y^{-\frac{1}{2}}\right\|_2\mathrm{d}t = \|\log B - \log A\|_2. \tag{2.3.70}$$

Therefore the equality holds in eq. (2.3.67), i.e. for $H_Y(t) = \log Y(t)$ we have

$$\left\| \int_0^1 H_Y'(t)\mathrm{d}t \right\|_2 = \int_0^1 \|H_Y'(t)\|_2 \mathrm{d}t. \tag{2.3.71}$$

i.e.

$$\left(\sum_{j,k=1}^n \left| \int_0^1 h_{jk}(t)\mathrm{d}t \right|^2 \right)^{\frac{1}{2}} = \int_0^1 \left(\sum_{j,k=1}^n |h_{jk}(t)|^2 \right)^{\frac{1}{2}} \mathrm{d}t \tag{2.3.72}$$

Here we denote $H_Y'(t) = (h_{jk}(t))_{j,k=1}^n$.

By the equality condition of Cauchy-Schwarz inequality, we have $h_{jk}(t)$ is proportional to $h_{j,k}(s)$ for any $t, s \in [0,1]$, therefore $H_Y'(t)$ is a constant matrix, then

$$H_Y(t) = t\log B + (1-t)\log A, \quad (H_Y(0) = \log A,\ H_Y(1) = \log B). \tag{2.3.73}$$

Therefore $Y(t) = e^{H_Y(t)} = A^{1-t}B^t = X(t)$. □

Theorem 2.3.10. $A, B > 0$*, then there exists a unique constant-speed geodesic from* A *to* B*.*

$$X(t) \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} =: M_0^t(A,B). \tag{2.3.74}$$

$$\delta(X) = \left\| \log A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\|_2. \tag{2.3.75}$$

Proof. We consider $A \to B \mapsto I \to A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, then by I commutes with $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and the previous Proposition 20, we have

$$\delta(I, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) = \left\| \log A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - \log I \right\|_2 = \left\| \log A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\|_2. \tag{2.3.76}$$

Moreover, it is realized using the constant-speed geodesic $\widetilde{X}(t) = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t$. By the congruent-invariant property Proposition 18 we have

$$\delta(A,B) = \delta(A^{\frac{1}{2}}IA^{\frac{1}{2}}, A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}) = \left\| \log A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\|_2, \tag{2.3.77}$$

which is achieved by $X(t) = A^{\frac{1}{2}}\widetilde{X}(t)A^{\frac{1}{2}}$. By the uniqueness of $\widetilde{X}$ we have the uniqueness of X. □

Remark 23. *This in fact gives another characterization of the geometric mean. That is, we have* $M_0(A,B) = X(\frac{1}{2})$ *where* X *is the geodesic from* A *to* B*. In fact we can generlize this geometric mean to from* $M_0^{\frac{1}{2}}(A,B)$ *to* $M_0^{\alpha}(A,B)$*, see Definition 2.3.11.*

The generalization of the geometric mean to three and more variables case is quite difficult. In fact it has just been resolved in 2010s.

Definition 2.3.11. $M_0^{\alpha}(A,B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}$.

Proposition 21.

$$[(1-\alpha)A^{-1} + \alpha B^{-1}]^{-1} = M_0^{\alpha}(A,B) \le (1-\alpha)A + \alpha B. \tag{2.3.78}$$

Remark 24. *This is a very natural generalization of the equality* $M_{-1}(A,B) \le M_0(A,B) \le \frac{A+B}{2}$*.*

Proof. Let $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, recall the classical Young's inequality, we have

$$(1-\alpha)s + \alpha t \ge s^{1-\alpha}t^{\alpha}. \tag{2.3.79}$$

Thus we have

$$[(1-\alpha)I + \alpha X^{-1}]^{-1} = X^{\alpha} \le (1-\alpha)A + \alpha X. \tag{2.3.80}$$

□

Remark 25. *The phylosiphy is, we can usually reduce the problem to the commutative (classical) ase.*

2.4 The Schur product theorem

We define the Schur (or Hadamard) product of two matrices $A, B \in M_n(\mathbb{C})$ as

$$A \circ B = (a_{ij}b_{ij})_{i,j=1}^n, \tag{2.4.1}$$

which is the "entry-wise" product of two matrices.

The following lemma is vital in the proof of the Schur product theorem and many other results. Basically, it provides a way to reduce this problem to the "tensor product in an extended space", which allows us to better understand the essence of the Schur product principle.

Lemma 5. *Let $A, B \in M_n(\mathbb{C})$, $\{v_j\}_{j=1}^n$ being an O.N. basis of $\mathbb{C}^n$, then*

$$A \circ B = V^*(A \otimes B)V, \quad V : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}, \quad v_j \mapsto v_j \otimes v_j, \tag{2.4.2}$$

where V is a partial isometry. Intuitively, this amounts to consider

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}, \tag{2.4.3}$$

and we only take the (i, j)-entry of each block, i.e. truncate this block matrix using V.

Proof. $\langle V^*(A \otimes B)V v_j, v_k\rangle = \langle (A \otimes B)(v_j \otimes v_j), V v_k\rangle = \langle Av_j \otimes Bv_j, v_k \otimes v_K\rangle = \langle Av_j, v_k\rangle\langle Bv_j, v_k\rangle = a_{jk}b_{jk} = \langle (A \circ B)v_j, v_k\rangle$. □

Theorem 2.4.1 (Schur product theorem)**.** *$A, B \geq (>)0$, then $A \circ B \geq (>)0$.*

Proof.

$$A, B \geq 0 \Rightarrow A \otimes B \geq 0 \Rightarrow V^*(A \otimes B)V \geq 0 \Rightarrow A \circ B \geq 0. \tag{2.4.4}$$

$$A \geq aI, B \geq bI(a, b > 0) \Rightarrow A \otimes B \geq abI \Rightarrow V^*(A \otimes B)V \geq abV^*V \Rightarrow A \circ B \geq abI \circ I > 0. \tag{2.4.5}$$

□

Corollary 5. *$A \circ B$ is operator monotone with respect to each component.*

Proposition 22. *Let f be an analytic function with power series expansion*

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots \quad (c_0 \geq 0, c_1 \geq 0, \cdots) \tag{2.4.6}$$

with radius of convergence being R. If $A \geq 0$ and $|a_{jk}| < R$, then $[f(a_{jk})]_{j,k=1}^n \geq 0$.

Proof.

$$[f(a_{j,k})]_{j,k=1}^n = c_0(1)_{j,k=1}^n + c_1 A + c_2 A \circ A + \cdots + c_\ell A^{\circ\ell} + \cdots . \tag{2.4.7}$$

Note that each $A^{\circ\ell}(\ell \geq 1)$ is positive by Theorem 2.4.1, and $(1)_{j,k=1}^n$ is a projection thus is positive as well, we conclude that $[f(a_{jk})]_{j,k=1}^n \geq 0$. □

Proposition 23. *$A_1, A_2 > 0$, $B_1, B_2 \in M_n(\mathbb{C})$, then*

$$(B_1^* A_1^{-1} B_1) \circ (B_2^* A_2^{-1} B_2) \geq (B_1 \circ B_2)^*(A_1 \circ A_2)^{-1}(B_1 \circ B_2). \tag{2.4.8}$$

Proof. By Lemma 3 and Theorem 2.4.1 we have

$$\begin{pmatrix} A_i & B_i \\ B_i^* & B_i^* A_i^{-1} B_i \end{pmatrix} \geq 0 \Rightarrow \begin{pmatrix} A_1 & B_1 \\ B_1^* & B_1^* A_1^{-1} B_1 \end{pmatrix} \circ \begin{pmatrix} A_2 & B_2 \\ B_2^* & B_2^* A_2^{-1} B_2 \end{pmatrix} \geq 0 \tag{2.4.9}$$

i.e.

$$\begin{pmatrix} A_1 \circ A_2 & B_1 \circ B_2 \\ B_1^* \circ B_2^* & B_1^* A_1^{-1} B_1 \circ B_2^* A_2^{-1} B_2 \end{pmatrix} \geq 0. \tag{2.4.10}$$

Therefore

$$(B_1^* A_1^{-1} B_1) \circ (B_2^* A_2^{-1} B_2) \geq (B_1 \circ B_2)^* (A_1 \circ A_2)^{-1} (B_1 \circ B_2). \tag{2.4.11}$$

□

Corollary 6. $A_1^{-1} \circ A_2^{-1} \geq (A_1 \circ A_2)^{-1}$.

Proposition 24. $A_1, A_2, B_1, B_2 > 0$, *then*

- $M_{-1}(A_1, B_1) \circ M_{-1}(A_2, B_2) \leq 2M_{-1}(A_1 \circ A_2, B_1 \circ B_2)$;
- $M_0(A_1, B_1) \circ M_0(A_2, B_2) \leq M_0(A_1 \circ A_2, B_1 \circ B_2)$.

Proof. We only need to note that

$$\left[\frac{1}{2}\begin{pmatrix} M_{-1}(A_1, B_1) & M_{-1}(A_1, B_1) \\ M_{-1}(A_1, B_1) & M_{-1}(A_1, B_1) \end{pmatrix}\right] \circ \left[\frac{1}{2}\begin{pmatrix} M_{-1}(A_2, B_2) & M_{-1}(A_2, B_2) \\ M_{-1}(A_2, B_2) & M_{-1}(A_2, B_2) \end{pmatrix}\right] \leq \begin{pmatrix} A_1 \circ A_2 & \\ & B_1 \circ B_2 \end{pmatrix} \tag{2.4.12}$$

and

$$\begin{pmatrix} A_1 & M_0(A_1, B_1) \\ M_0(A_1, B_1) & B_1 \end{pmatrix} \circ \begin{pmatrix} A_2 & M_0(A_2, B_2) \\ M_0(A_2, B_2) & B_2 \end{pmatrix} \geq 0, \tag{2.4.13}$$

and then use the Ando's variational formula. □

Proposition 25 (Ando's concavity)**.** $p, r \in [0, 1]$, $p + r \leq 1$, *then* $(A, B) \mapsto A^p \circ B^r$ *is jointly concave on* $\mathbb{H}_n^{>0} \times \mathbb{H}_n^{>0}$.

Proof. $(A, B) \mapsto A^p \circ B^r = V^*(A^p \otimes B^r)V$. Then use the Ando's concavity for tensor product (Theorem 2.3.6). □

2.5 The absolute value of operators

We recall the results of polar decomposition

Proposition 26 (Polar decomposition)**.** $A \in M_n(\mathbb{C})$, *then there exists a unitary* V *and a positive semidefinite matrix* $|A|$, *s.t.* $A = V|A|$.

Remark 26. *The nonzero spectrum of* $|A|$ *equals to the sqaure root of the nonzero spectrum of* AA^* *or* A^*A.

Definition 2.5.1. *For* $A \in M_n(\mathbb{C})$, *we define* $\operatorname{Re} A = \frac{A+A^*}{2}$ *and* $\operatorname{Im} A = \frac{A-A^*}{2\mathrm{i}}$. *Note that* $\operatorname{Re} A$ *and* $\operatorname{Im} A$ *are both Hermitian.*

Example 8. *Let* $V = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $|V| = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\operatorname{Re} V = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $|V| \leq \operatorname{Re} V$.

Remark 27. *This example implies that* $|V| \geq \operatorname{Re} V$ *does not hold for operator case like in scalar case in general.*

Proposition 27. $A \in M_n(\mathbb{C})$*, then there exists a unitary matrix* W *such that*

$$\operatorname{Re} A \leq \frac{1}{2}(\operatorname{Re} A + \operatorname{Re} |A|) \leq W^*|A|W. \tag{2.5.1}$$

Proof.

$$\begin{aligned}
\lambda_j(\operatorname{Re} A) &= \max_{\dim V = j} \min_{x \in V \cap \mathbb{S}} \langle \operatorname{Re} Ax, x\rangle \\
&= \max \min \frac{1}{2}(\langle Ax, x\rangle + \overline{\langle Ax, x\rangle}) \\
&= \max \min \operatorname{Re}\langle U|A|x, x\rangle \\
&\leq \max \min \||A|x\| \\
&= \sqrt{\lambda_j(|A|^2)} \\
&= \lambda_j(|A|).
\end{aligned} \tag{2.5.2}$$

We decompose $\operatorname{Re} A = A_+ - A_-$ $(A_+ \geq 0,\ A_- \geq 0)$, $|\operatorname{Re} A| = A_+ + A_- \Rightarrow \frac{1}{2}(\operatorname{Re} A + |\operatorname{Re} A|) = A_+$. Therefore, $\operatorname{Re} A \leq A_+$. Moreover, A_+ takes only the positive part of the spectrum of $\operatorname{Re} A$. Therefore, we also have $\lambda_j(A_+) \leq \lambda_j(|A|)$. We diagonalize A_+ and $|A|$ and find

$$X^* A_+ X \leq Y^*|A|Y \Rightarrow A_+ \leq (XY^*)|A|(XY^*)^*. \tag{2.5.3}$$

We let $W = (XY^*)^*$, then by X, Y are both unitary we have W is unitary. Therefore we proved the right inequality. For the left inequality, it is trivial since $\operatorname{Re} A \leq A_+$. □

Theorem 2.5.2. $A, B \in M_n(\mathbb{C})$*, then there exists* U, V *unitary matrices, such that*

$$|A + B| \leq U|A|U^* + V|B|V^*. \tag{2.5.4}$$

Remark 28. *Note that this inequality is essentially different from the Weyl inequality for eigenvalues or sigular values, since the Weyl inequality describes the spectral (local) information while this inequality describes the operator (global) information. So it is hard to say which one is stronger.*

This theorem can be generalized to the von Neumann algebra case.

Proof. By polar decomposition, we have $A + B = W|A + B|$ for W unitary. Therefore

$$|A + B| = W^*A + W^*B. \tag{2.5.5}$$

We take the real part of both sides, then we have

$$|A + B| = \operatorname{Re} W^*A + \operatorname{Re} W^*B \leq U|W^*A|U^* + V|W^*B|V^* (U, V \text{ are unitary}) = U|A|U^* + V|B|V^*. \tag{2.5.6}$$

□

2.6 Exercise II

Exercise 7. *Suppose $A \in M_n(\mathbb{C})$ is Hermitian, $U \in M_n(\mathbb{C})$ is unitary and f is a function on $\mathrm{Sp}(A)$. Show that $f(UAU^*) = Uf(A)U^*$.*

Proof. We consider $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then we have

$$f(A) = \sum_{k=0}^{\infty} a_k A^k, \quad f(UAU^*) = \sum_{k=0}^{\infty} a_k UA^kU^* = U\sum_{k=0}^{\infty} a_k A^k U^* = Uf(A)U^*. \tag{2.6.1}$$

□

Exercise 8. *Suppose $A \in M_n(\mathbb{C})$ is and f is a function on $\mathrm{Sp}(A^*A)$. Show that*

$$Af(A^*A) = f(AA^*)A. \tag{2.6.2}$$

Proof. We consider the polar decomposition $A = U|A|$ with U being a unitary matrix. Then we have

$$A^*A = |A|U^*U|A| = |A|^2, \quad AA^* = U|A||A|U^* = U|A|^2U^*, \tag{2.6.3}$$

therefore

$$\mathrm{LHS} = U|A|f(|A|^2), \quad \mathrm{RHS} = f(U|A|^2U^*)U|A| = Uf(|A|^2)U^*U|A| = U|A|f(|A|^2) = \mathrm{LHS}. \tag{2.6.4}$$

□

Exercise 9. *Consider the α-log function f:*

$$f(t) := \frac{t^{1-\alpha} - 1}{1-\alpha}, \quad t \in (0,\infty), \quad \alpha > 0, \quad \alpha \neq 1. \tag{2.6.5}$$

Determine for which α, the function f is operator monotone or operator convex.

Proof. Since for $\alpha \in (1,2]$, we have $1-\alpha \in [-1,0)$ and for $\alpha \in [0,1)$, $1-\alpha \in (0,1]$, by Theorem 2.1.2, we have f is operator monotone for $\alpha \in (1,2]$ and $-f$ is operator monotone for $\alpha \in [0,1)$.

Similarly, by Theorem 2.1.5, we have f is operator convex for $\alpha \in [-1,0]$, $-f$ is operator convex for $\alpha \in [0,1)$, and f is operator convex for $\alpha \in (1,2]$. □

Exercise 10. *Show that the function $f(t) = \tan t$ is operator monotone on $(-\pi/2, \pi/2)$.*

Proof. By the polar expansion of $\tan t$ we have

$$\tan t = \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{(n-\frac{1}{2})\pi - t} - \frac{n\pi}{n^2\pi + 1} \right\}, \quad t \in (-\pi/2, \pi/2). \tag{2.6.6}$$

Then it follows by the operator monotonicity of t^{-1}. □

Exercise 11. *Show that*

$$f(t) = -t\log t + (t+1)\log(t+1), \quad t \in [0,\infty) \tag{2.6.7}$$

is operator monotone.

Proof. In fact,

$$f(t) = \int_0^1 [1 + \log(t + \alpha)] \mathrm{d}\alpha. \tag{2.6.8}$$

Since $\log t$ is operator monotone on $[0, \infty)$ and $t \in [0, \infty]$, we have $f(t)$ is also operator monotone. □

Exercise 12. *Show that* $f(t) = \sqrt{t^2 + 1}$ *is not operator monotone on* $[0, \infty)$.

Proof. We consider $A = \begin{pmatrix} \frac{3}{2} & \\ & \frac{3}{4} \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then we have

$$f(A) = \begin{pmatrix} \frac{\sqrt{13}}{2} & \\ & \frac{5}{4} \end{pmatrix}, \quad f(B) = \begin{pmatrix} \frac{2+\sqrt{2}}{2} & \frac{2-\sqrt{2}}{2} \\ \frac{2-\sqrt{2}}{2} & \frac{2+\sqrt{2}}{2} \end{pmatrix}. \tag{2.6.9}$$

We have $\det(f(A) - f(B)) < 0$, which means that f is *not* operator monotone. □

Exercise 13. *Suppose that* $A, B \in M_n(\mathbb{C})$ *are positive definite matrices, show that*

$$(A \log A + B \log B)(A + B)^{-1}(A \log A + B \log B) \leq A(\log A)^2 + B(\log B)^2. \tag{2.6.10}$$

Hint: Apply the noncommutative Jensen Inequality.

Proof. Since $A, B > 0$, we can define

$$V_A := A^{1/2}(A + B)^{-1/2}, \quad V_B := B^{1/2}(A + B)^{-1/2}. \tag{2.6.11}$$

Then we have

$$V_A^* V_A + V_B^* V_B = (A + B)^{-1/2} A (A + B)^{-1/2} + (A + B)^{-1/2} B (A + B)^{-1/2} = I. \tag{2.6.12}$$

Note that $f(t) = t^2$ is operator convex, we apply noncommutative Jensen inequality, obtaining

$$(V_A^* \log A V_A + V_B^* \log B V_B)^2 \leq V_A^* (\log A)^2 V_A + V_B^* (\log B)^2 V_B. \tag{2.6.13}$$

We compute

$$\begin{aligned} \text{LHS} &= [(A + B)^{-1/2}(A^{1/2}(\log A)A^{1/2} B^{1/2}(\log B)B^{1/2})(A + B)^{-1/2}]^2 \\ &= X^{-1}(A \log A + B \log B)(A + B)^{-1}(A \log A + B \log B)X^{-1} \end{aligned} \tag{2.6.14}$$

$$\begin{aligned} \text{RHS} &= (A + B)^{-1/2} A^{1/2} (\log A)^2 A^{1/2} (A + B)^{-1} + (A + B)^{-1/2} B^{1/2} (\log B)^2 B^{1/2} (A + B)^{-1} \\ &= X^{-1}(A(\log A)^2 + B(\log B)^2)X^{-1}. \end{aligned} \tag{2.6.15}$$

Here, we denote $X := (A + B)^{-1/2}$. It follows readily that

$$(A \log A + B \log B)(A + B)^{-1}(A \log A + B \log B) \leq A(\log A)^2 + B(\log B)^2. \tag{2.6.16}$$

□

Exercise 14. *Suppose that* $t_1, \cdots, t_n \in \mathbb{R}$. *Show that* $(\cos(t_j - t_k))_{1 \leq j,k \leq n}$ *is positive semidefinite.*

Proof. In fact this matrix is a Gram matrix with respect to the set $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ of vectors where

$$\mathbf{x}_i = \begin{pmatrix} \cos t_i \\ \sin t_i \end{pmatrix}. \tag{2.6.17}$$

Then we have

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \cos t_i \cos t_j + \sin t_i \sin t_j = \cos(t_i - t_j). \tag{2.6.18}$$

That is to say,

$$A := (\cos(t_i - t_j))_{1 \le i,j \le n} = (\langle \mathbf{x}_i, \mathbf{x}_j \rangle)_{1 \le i,j \le n}, \quad \text{i.e.} \quad A = (\mathbf{x}_i)^*_{1 \le i \le n} (\mathbf{x}_i)_{1 \le i \le n} \ge 0. \tag{2.6.19}$$

□

Exercise 15. *Suppose* $A \in M_n(\mathbb{C})$ *is a contraction i.e.* $\|A\| \le 1$*, show that for any* $n \in \mathbb{N}$*,*

$$\begin{pmatrix} I & A^* & \cdots & A^{*m} \\ A & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A^* \\ A^m & \cdots & A & I \end{pmatrix} \ge 0. \tag{2.6.20}$$

Proof. We assume further that A is normal, then we can let $\{u_j\}_{j=1}^n$ be a set of orthonormal basis of $\mathbb{C}^n$ that diagonalizes A:

$$A = \sum_{j=1}^n \lambda_j u_j u_j^*, \quad A^* = \sum_{j=1}^n \overline{\lambda_j} u_j u_j^*. \tag{2.6.21}$$

Since $\|A\| \le 1$, we have $|\lambda_j| \le 1$. We compute

$$\begin{pmatrix} I & A^* & \cdots & A^{*m} \\ A & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A^* \\ A^m & \cdots & A & I \end{pmatrix} = \sum_{j=1}^n M_n(\lambda_j) \otimes u_j u_j^*. \tag{2.6.22}$$

Here we denote

$$M_m(\lambda_j) = \begin{pmatrix} 1 & \overline{\lambda_j} & \cdots & \overline{\lambda_j}^m \\ \lambda_j & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{\lambda_j} \\ \lambda_j^m & \cdots & \lambda_j & 1 \end{pmatrix} \in M_m(\mathbb{C}). \tag{2.6.23}$$

Note that

$$\det M_m(\lambda_j) = (1 - |\lambda_j|^2)^{m-1} \ge 0, \quad \forall m \in \mathbb{N}. \tag{2.6.24}$$

Therefore, by Sylvester's criterion and $|\lambda_j| \le 1$, we have $M_n(\lambda_j)$ is positive semidefinite. It follows that $M_n(\lambda_j) \otimes u_j u_j^*$ and then the whole matrix is positive semidefinite.

In fact, for general case, let S be the matrix above and let

$$T = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ A & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & A & 0 \end{pmatrix}. \tag{2.6.25}$$

We have

$$S = I + T + \cdots + T^m + T^* + \cdots + T^{*m} = (I-T)^{-1} + (I-T^*)^{-1} - I. \tag{2.6.26}$$

Now we see that for any $v \in \mathbb{C}^{nm}$,

$$\begin{aligned}\langle Sv, v\rangle &= \langle (I-T)^{-1}v, v\rangle + \langle (I-T^*)^{-1}v, v\rangle - \langle v, v\rangle \\ &= \langle w, (1-T)w\rangle + \langle (1-T)w, w\rangle - \langle (1-T)w, (1-T)w\rangle \\ &= \|w\|^2 - \|Tw\|^2 \geq 0.\end{aligned} \tag{2.6.27}$$

Here we denote $w = (I-T)^{-1}v$. This indicates that $S \geq 0$. $\square$

Exercise 16. *Show that the map $A \mapsto A^{-1} \otimes A^{-1}$ is operator convex on $\mathbb{H}_n^{>0}$.*

Proof. Suppose that $A_1, A_2 > 0$, we need to show that

$$\left(\frac{A_1 + A_2}{2}\right)^{-1} \otimes \left(\frac{A_1 + A_2}{2}\right)^{-1} \leq \frac{A_1^{-1} \otimes A_1^{-1}}{2} + \frac{A_2^{-1} \otimes A_2^{-1}}{2}. \tag{2.6.28}$$

We multiply $A_1^{\frac{1}{2}} \otimes A_1^{\frac{1}{2}}$ on left and right sides of both sides of the inequality, we have

$$\left(\frac{I + A_1^{-\frac{1}{2}} A_2 A_1^{-\frac{1}{2}}}{2}\right)^{-1} \otimes \left(\frac{I + A_1^{-\frac{1}{2}} A_2 A_1^{-\frac{1}{2}}}{2}\right)^{-1} \leq \frac{I}{2} + \frac{A_1^{\frac{1}{2}} A_2^{-1} A_1^{\frac{1}{2}} \otimes A_1^{\frac{1}{2}} A_2^{-1} A_1^{\frac{1}{2}}}{2}. \tag{2.6.29}$$

Let $X = A_1^{\frac{1}{2}} A_2^{-1} A_1^{\frac{1}{2}}$ and let us consider the spectral decomposition of X, then it suffices to show that for $x, y > 0$, we have

$$\left(\frac{1 + x^{-1}}{2}\right)^{-1} \left(\frac{1 + y^{-1}}{2}\right) \leq \frac{1 + xy}{2}. \tag{2.6.30}$$

This is a simple inequality, we can compute directly. $\square$

Chapter 3

Trace Inequalities

3.1 The Hilbert-Schmidt inner product

Definition 3.1.1. *We define the (trace) p-functional as*

$$\|A\|_p := \operatorname{Tr}\left[(A^*A)^{p/2}\right]^{1/p} = \operatorname{Tr}(|A|^p)^{1/p}. \tag{3.1.1}$$

We note that in terms of singular values, we have in fact

$$\|A\|_p = \left(\sum_{j=1}^{n} \sigma_j(A)^p\right)^{1/p}, \quad \sigma_j(A) = \sqrt{\lambda_j(A^*A)}. \tag{3.1.2}$$

From this, we can easily see that

$$\|A\|_p \geq \|A\|, \quad \textit{and} \quad \lim_{p\to\infty} \|A\|_p = \|A\|. \tag{3.1.3}$$

Definition 3.1.2.

- *(Hilbert-Schmidt inner product) For $A, B \in M_n(\mathbb{C})$, we define the Hilbert-Schmidt inner product as ("mathematician's notation")*

$$\langle A, B\rangle = \operatorname{Tr}(B^*A). \tag{3.1.4}$$

- *Let ρ be a linear functional on $M_n(\mathbb{C})$, we say ρ is positive if $\rho(A) \geq 0$ for all $A \geq 0$.*
- *If a positive linear functional ρ satisfies $\rho(I) = 1$, we say ρ is a state.*
- *(density matrix) For any ρ a linear functional, by Riesz representation theorem applying to the H-S inner product space, there exists a unique $D_\rho \in M_n(\mathbb{C})$ such that $\rho(A) = \operatorname{Tr}(D_\rho A)$. If ρ is a state, we have $D_\rho \geq 0$, $\operatorname{Tr} D_\rho = 1$. In this case, we say D_ρ is the density matrix of ρ.*

Proposition 28.

$$|\operatorname{Tr}(BA)| \leq \|A\|\|B\|_1. \tag{3.1.5}$$

Proof.

$$\begin{aligned}|\operatorname{Tr}(BA)| &= |\operatorname{Tr}(V|B|A)| = |\operatorname{Tr}(|B|AV)| = \operatorname{Tr}\big(|B|AVe^{i\theta}\big)\\ &= \frac{1}{2}\left[|B|(AVe^{\mathrm{i}\theta} + (AVe^{\mathrm{i}\theta})^*)\right] \leq \operatorname{Tr}\Big(|B|^{\frac{1}{2}}\|A\||B|^{\frac{1}{2}}\Big)\\ &= \operatorname{Tr}(|B|)\|A\| = \|A\|\|B\|_1.\end{aligned} \tag{3.1.6}$$

□

Example 9 (Skew information).

- *ρ state, D_ρ density matrix, we define*

$$I(\rho, X) = \frac{1}{2}\operatorname{Tr}\Big([D_\rho^{\frac{1}{2}}, X]^*[D_\rho^{\frac{1}{2}}, X]\Big) = \operatorname{Tr}\bigg(\frac{X^*X + XX^*}{2}D_\rho\bigg) - \operatorname{Tr}\Big(X^*D_\rho^{\frac{1}{2}}XD_\rho^{\frac{1}{2}}\Big), \tag{3.1.7}$$

$$J(\rho, X) = \frac{1}{2}\operatorname{Tr}\Big(\{D_\rho^{\frac{1}{2}}, X\}^*\{D_\rho^{\frac{1}{2}}, X\}\Big) = \operatorname{Tr}\bigg(\frac{X^*X + XX^*}{2}D_\rho\bigg) + \operatorname{Tr}\Big(X^*D_\rho^{\frac{1}{2}}XD_\rho^{\frac{1}{2}}\Big). \tag{3.1.8}$$

We say that $I(\rho, X)$ is the skew information of the state ρ with respect to X. Sometimes we assume that X is self-adjoint, i.e. $X = X^*$. In this case, we can see that the first term is actually the variance of the observable X.

- *By Cauchy-Schwarz inequality, we have*

$$\operatorname{Tr}\Big([D_\rho^{\frac{1}{2}}, X]^*\{D_\rho^{\frac{1}{2}}, Y\}\Big) \le 2\sqrt{I(\rho, X)J(\rho, Y)}. \tag{3.1.9}$$

$$\begin{aligned}\text{L.H.S.} &= \Big|\operatorname{Tr}(X^*D_\rho Y) + \operatorname{Tr}\Big(X^*D_\rho^{\frac{1}{2}}YD_\rho^{\frac{1}{2}}\Big) - \operatorname{Tr}\Big(D_\rho^{\frac{1}{2}}X^*D_\rho^{\frac{1}{2}}Y\Big) - \operatorname{Tr}\Big(D_\rho^{\frac{1}{2}}X^*YD_\rho^{\frac{1}{2}}Y\Big)\Big| \\ &= |\operatorname{Tr}[(YX^* - X^*Y)D_\rho]| = |\rho[X^*, Y]|. \quad \Rightarrow I(\rho, X)J(\rho, Y) \ge \frac{1}{4}|\rho[X^*, Y]|^2.\end{aligned} \tag{3.1.10}$$

- *For the purpose of simplifying the notations, we denote*

$$C(\rho, X) = \operatorname{Tr}\Big(X^*D_\rho^{\frac{1}{2}}XD_\rho^{\frac{1}{2}}\Big). \tag{3.1.11}$$

Then we have

$$\begin{aligned}&\operatorname{Var}_\rho(X)\operatorname{Var}_\rho(Y) - C(\rho, X)C(\rho, Y) \\ &= \frac{1}{4}(I(\rho, X) + J(\rho, Y))(I(\rho, Y) + J(\rho, Y)) - \frac{1}{4}(I(\rho, X) - J(\rho, X))(I(\rho, X) - J(\rho, X)) \\ &= \frac{1}{2}I(\rho, X)J(\rho, Y) + \frac{1}{2}I(\rho, Y)J(\rho, X) \\ &\ge \frac{1}{4}|\rho[X^*, Y^*]|^2.\end{aligned} \tag{3.1.12}$$

We say this inequality is the Heisenberg uncertainty relation.

3.2 Trace monotonicity

In this section, we ask the following general question:

Can we derive the trace inequalities related to monotonic functions?

Quite intuitively, we have the following simple result:

Proposition 29. *Let $f : \operatorname{Dom}(f) \to \mathbb{R}$ be a non-decreasing function on some interval $\operatorname{Dom}(f)$. If A, B Hermitian matrices such that $A \le B$ and $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset \operatorname{Dom}(f)$, then $\operatorname{Tr} f(A) \le \operatorname{Tr} f(B)$.*

Proof. By $A \le B$ and the min-max theorem, we have $\lambda_j(A) \le \lambda_j(B)$ for all j and thus $f(\lambda_j(A)) \le f(\lambda_j(B))$ for all j. Therefore, we have

$$\operatorname{Tr} f(A) = \sum_{j=1}^{n} f(\lambda_j(A)) \le \sum_{j=1}^{n} f(\lambda_j(B)) = \operatorname{Tr} f(B). \tag{3.2.1}$$

□

In fact, we can derive a more refined result. We first examine the behavior of the differential of $\operatorname{Tr} f$ function.

Theorem 3.2.1. *Let f be a C^1 function, A is a Hermitian matrix with $\operatorname{Sp}(A) \subset \operatorname{Dom}(f)$ with $\operatorname{Dom}(f)$ being an open interval, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr} f(A+tX)\Big|_{t=0} = \operatorname{Tr}(f'(A)X). \tag{3.2.2}$$

Proof. Let $m \in \mathbb{N}$, $f(t) = t^m$, then we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr} f(A+tX) &= \sum_{j=1}^{n} \operatorname{Tr}\left[(A+tX)^{j-1} \frac{\mathrm{d}}{\mathrm{d}t}(A+tX)\right] \\ &= \operatorname{Tr}\big(X(A+tX)^{m-1} + \cdots + (A+tX)^{m-1}X\big) \\ &= \operatorname{Tr}\big[m(A+tX)^{m-1}X\big] \quad \text{(by the cyclic property of trace)} \\ &= \operatorname{Tr}(f'(A)X). \end{aligned} \tag{3.2.3}$$

Therefore, for $f \in \mathbb{C}[x]$, the conclusion holds. If f is C^1, the conclusion also holds since we can always approximate f by a polynomical function on a compact set. □

Remark 29. *In fact, if additionally assume that f is C^1, we can derive Proposition 29 using Theorem 3.2.1. In fact, we can define*

$$h(t) := f(tA + (1-t)B), \tag{3.2.4}$$

then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} h(t) = \operatorname{Tr}(f'(tA + (1-t)B)(A-B)) \le 0, \quad \forall t \in [0,1]. \tag{3.2.5}$$

This is because f is increasing and f' is a positive-valued function. Moreover, $A \le B \Rightarrow A - B \le 0$. Therefore $\frac{\mathrm{d}}{\mathrm{d}t} h(t) \le 0$. Thus we have

$$\operatorname{Tr}(f(A)) - \operatorname{Tr}(f(B)) = h(1) - h(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} h(t) \mathrm{d}t \le 0. \tag{3.2.6}$$

Remark 30. *Unfortunately, this result does not hold if we extend the trace to general state. That is to say, in general we do not have*

$$\operatorname{Tr}(X^* f(A) X) \le \operatorname{Tr}(X^* f(B) X), \quad \forall X \in M_n(\mathbb{C}). \tag{3.2.7}$$

Example 10. *If $0 \le A \le B$, then $\operatorname{Tr}(A^p) \le \operatorname{Tr}(B^p)$ $(p > 0)$.*

If $A \le B$, then $\operatorname{Tr}\big(e^A\big) \le \operatorname{Tr}\big(e^B\big)$.

We can see that the "trace monoticity" is easier to realize than the operator monoticity.

3.3 Jensen trace inequality

Recall the operator convexity. The operator convexity in fact implies C^2, which is much stronger than the generic convexity. It is natural for us to ask whether we can derive some trace ineqaulities for general convex functions. The answer is yes, and this is the Jensen trace inequality.

In fact, the trace inqualities related to convexity can be formulated in various ways. We begin with a simple result called *Peierls inequality*:

Proposition 30 (Peierls inequality)**.** *f is a convex function, A is Hermitian, $\mathrm{Sp}(A) \subset \mathrm{Dom}(f)$ and $\{v_j\}_{j=1}^n$ is a set of orthonormal basis of $\mathbb{C}^n$, then we have*

$$\sum_{j=1}^{n} f(\langle Av_j, v_j\rangle) \leq \mathrm{Tr}\, f(A). \tag{3.3.1}$$

Proof.

$$\text{RHS} = \sum_{j=1}^{n} \langle f(A)v_j, v_j\rangle. \tag{3.3.2}$$

We take the spectral decomposition of A, here P_k are orthogonal projection operators:

$$A = \sum_{k=1}^{m} \lambda_k P_k, \quad \sum_{k=1}^{m} P_k = I. \tag{3.3.3}$$

Therefore

$$\text{RHS} = \sum_{j=1}^{n}\sum_{k=1}^{m} f(\lambda_k)\|P_k v_j\|^2. \tag{3.3.4}$$

Note that $\sum_{k=1}^{m} \|P_k v_j\|^2 = \|v_j\|^2 = 1$, by the convexity of f, we have

$$\sum_{j=1}^{n}\sum_{k=1}^{m} f(\lambda_k)\|P_k v_j\|^2 \geq \sum_{j=1}^{n} f\left(\sum_{k=1}^{m} \lambda_k \|P_k v_j\|^2\right) = \sum_{j=1}^{n} f(\langle Av_j, v_j\rangle). \tag{3.3.5}$$

□

Unlike the case for monoticity, for the convexity we can examine the behavior of $\mathrm{Tr}(X^* f(A) X)$. In fact, we only need to modify a little bit the proof of Proposition 30 to obtain the following result:

Proposition 31. *Let f be a convex function, A is Hermitian, $\mathrm{Sp}(A) \subset \mathrm{Dom}(f)$ and $\{v_j\}_{j=1}^n$ is a set of orthonormal basis of $\mathbb{C}^n$. Assume $X \in M_n(\mathbb{C})$ and $\|X\| \leq 1$, then we have*

$$\sum_{j=1}^{n} \|Xv_j\|^2 f\left(\frac{\langle AXv_j, Xv_j\rangle}{\|Xv_j\|^2}\right) \leq \mathrm{Tr}(X^* f(A) X). \tag{3.3.6}$$

Proof.

$$\begin{aligned}\operatorname{Tr}[X^* f(A)X] &= \sum_{j=1}^{n}\sum_{k=1}^{m} f(\lambda_k)\|P_k X v_j\|^2 = \sum_{j=1}^{n}\|Xv_j\|^2 \sum_{k=1}^{m} f(\lambda_k)\frac{\|P_k X v_j\|^2}{\|Xv_j\|^2} \\ &\geq \sum_{j=1}^{n}\|Xv_j\|^2 f\left(\sum_{k=1}^{m}\lambda_k \frac{\|P_k X v_j\|^2}{\|Xv_j\|^2}\right) \\ &= \sum_{j=1}^{n}\|Xv_j\|^2 f\left(\frac{\langle AXv_j, Xv_j\rangle}{\|Xv_j\|^2}\right).\end{aligned} \tag{3.3.7}$$

□

We can also extend this result for general vector. The idea is to supplement the remaining part for $\|v\| \leq 1$.

Proposition 32. *f is a convex function, $0 \in \operatorname{Dom}(f)$, $f(0) \leq 0$. A Hermitian, $v \in \mathbb{C}^n$ with $\|v\| \leq 1$, then*

$$f(\langle Av, v\rangle) \leq \langle f(A)v, v\rangle. \tag{3.3.8}$$

Proof.

$$\begin{aligned}\langle f(A)v, v\rangle &= \sum_{j=1}^{n} f(\lambda_k)\|P_k v\|^2 + f(0)[1-\|v\|^2] - f(0)[1-\|v\|^2] \\ &\geq f\left(\sum_{k=1}^{m}\lambda_k\|P_k v\|^2\right) - f(0)[1-\|v\|^2] \\ &= f(\langle Av, v\rangle).\end{aligned} \tag{3.3.9}$$

□

We next derive three important results. These results can be viewed as the application of Proposition 30, Proposition 31 and Proposition 32 by taking trace.

Proposition 33. *f is a convex function, A, B Hermitian, $\lambda \in [0,1]$, then we have*

$$\operatorname{Tr} f(\lambda A + (1-\lambda)B) \leq \lambda \operatorname{Tr} f(A) + (1-\lambda)\operatorname{Tr} f(B). \tag{3.3.10}$$

Proof. We take v_j as the *eigenvectors* of $\lambda A + (1-\lambda)B$, then we have

$$\begin{aligned}\operatorname{Tr} f(\lambda A + (1-\lambda)B) &= \sum_{j=1}^{n}\langle f[(1-\lambda)B + \lambda A]v_j, v_j\rangle \\ &\overset{\text{spectral decomposition}}{=} \sum_{j=1}^{n} f(\langle \lambda A + (1-\lambda)Bv_j, v_j\rangle) \\ &\overset{\text{the convexity of } f}{\leq} \sum_{j=1}^{n}\lambda f(\langle Av_j, v_j\rangle) + (1-\lambda)f(\langle Bv_j, v_j\rangle) \\ &\overset{\text{Peierls (Proposition 30)}}{\leq} \lambda \operatorname{Tr} f(A) + (1-\lambda)\operatorname{Tr} f(B).\end{aligned} \tag{3.3.11}$$

□

Remark 31. *We use two different convexity: (1) the trivial convexity of f; (2) the Peierls convexity of f. In fact, Proposition 33 implies that the function $A \mapsto \operatorname{Tr} f(A)$ is operator convex if f is a convex function.*

Corollary 7.

$$\operatorname{Tr}\left(\frac{A+B}{2}\right)^p \le \frac{\operatorname{Tr} A^p + \operatorname{Tr} B^p}{2}, \tag{3.3.12}$$

$$\operatorname{Tr} e^{\frac{A+B}{2}} \le \frac{\operatorname{Tr} e^A + \operatorname{Tr} e^B}{2}. \tag{3.3.13}$$

Next we apply Proposition 31 to obtain the following *Jensen trace inequality*. It is formulated in the way similar to the operator Jensen inequality Theorem 2.2.1.

Theorem 3.3.1 (Jensen trace inequality). *Let f be a convex function, A_j are Hermitian, $V_j \in M_n(\mathbb{C})$, $\sum_{j=1}^m V_j^* V_j = 1$, then we have*

$$\operatorname{Tr} f\left(\sum_{j=1}^m V_j^* A_j V_j\right) \le \sum_{j=1}^m \operatorname{Tr}\big(V_j^* f(A_j) V_j\big). \tag{3.3.14}$$

Proof. The proof is also similar to Theorem 2.2.1. We take

$$X = \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ V_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ V_m & 0 & \cdots & 0 \end{pmatrix} \in M_{mn}(\mathbb{C}), \quad \widetilde{A} = \begin{pmatrix} A_1 - x_0 I & & \\ & \ddots & \\ & & A_m - x_0 I \end{pmatrix} \in M_{mn}(\mathbb{C}). \tag{3.3.15}$$

Let $\widetilde{f}(x) = f(x+x_0) - f(x_0)$ for $x_0 \in \operatorname{Dom}(f)$, then we have $\widetilde{f}$ is a convex function. We compute

$$X^* \widetilde{A} X = \begin{pmatrix} (\sum_{j=1}^m V_j^* A_j V_j)_{n\times n} & \\ & 0 \end{pmatrix} \in M_{mn}(\mathbb{C}). \tag{3.3.16}$$

We take the basis vectors with respect to the block matrix, i.e. $\{v_j\}_{j=1}^{mn}$, then we have

$$\begin{aligned} \|Xe_j\|^2 &= \left\langle \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ V_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ V_m & 0 & \cdots & 0 \end{pmatrix}^* \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ V_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ V_m & 0 & \cdots & 0 \end{pmatrix} e_j, e_j \right\rangle \\ &= \left\langle \begin{pmatrix} \sum_{j=1}^m V_j^* A_j V_j & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} e_j, e_j \right\rangle \\ &= \left\langle \begin{pmatrix} I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} e_j, e_j \right\rangle = \begin{cases} 1, & 1 \le j \le n, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{3.3.17}$$

Therefore

- $\mathrm{Tr}\Big(X^* f(\widetilde{A})X\Big) = \mathrm{Tr}\Big(\sum_{j=1}^m V_j^* \widetilde{f}(A_j - X_0)V_j\Big) = \mathrm{Tr}\Big(\sum_{j=1}^m V_j^* f(A_j)V_j - x_0\Big)$.
- $\sum_{k=1}^{mn} \|Xe_k\|^2 f\left(\frac{\langle X^*\widetilde{A}Xe_k, e_k\rangle}{\|Xe_k\|^2}\right) = \sum_{k=1}^n f(\langle X^*\widetilde{A}Xe_k, e_k\rangle) = \sum_{k=1}^n f(\langle \sum_{j=1}^m V_j^*(A_j - x_0 I)V_j e_k, e_k\rangle)$ $= \mathrm{Tr}\, f(\sum_{j=1}^m V_j^* A V_j - x_0)$.

□

Proposition 34. *f convex, $0 \in \mathrm{Dom}(f)$, $f(0) \le 0$, $\{A_j\}$ Hermitian, $V_1, \cdots, V_m \in M_n(\mathbb{C})$, $\sum_{j=1}^m V_j^* V_j \le I$, then we have*

$$\mathrm{Tr}\, f\left(\sum_{j=1}^m V_j^* A_j V_j\right) \le \sum_{j=1}^m \mathrm{Tr}\Big(V_j^* f(A_j) V_j\Big). \tag{3.3.18}$$

Proof. Let v_k be the eigenvectors of X^*AX for some $\|X\| \le 1$, then we have

$$\begin{aligned}\mathrm{Tr}\, f(X^*AX) &\overset{\text{spectral decomposition}}{=} \sum_{k=1}^n f(\langle X^*AXv_k, v_k\rangle) \\ &= \sum_{k=1}^n f(\langle AXv_k, Xv_k\rangle) \overset{\text{Proposition 32}}{\le} \sum_{k=1}^n \langle f(A)Xv_k, Xv_k\rangle = \mathrm{Tr}(X^* f(A) X).\end{aligned} \tag{3.3.19}$$

For general case, we can use the techniques of block matrices. □

In the next two propositions, we will see that if we additionally assume that f is increasing, we can derive even nicer results. The first one Proposition 35 in some sense achieves $f(X^*AX) \le X^* f(A) X$ up to a unitary matrix and avoids taking the trace. The second one Proposition 36 is a majorization result. It has a deep connection with the Gibbs state at different temperatures.

Proposition 35. *f is a convex and **increasing** function, $f(0) \le 0$ and $X \in M_n(\mathbb{C})$, then there exists U a unitary matrix, such that $f(X^*AX) \le U^* X^* f(A) X U$.*

Proof. Since we want to derive the inequality up to unitary matrix, we can estimate the eigenvalues of $X^* f(A) X$. In fact,

$$\begin{aligned}\lambda_k(X^* f(A) X) &= \min_{\dim V = k-1} \max_{\|v\|=1, v\in V} \langle X^* f(A) X v, v\rangle \\ &\overset{\text{Proposition 32}}{\ge} \min_{\dim V = k-1} \max_{\|v\|=1, v\in V} f(\langle AXv, Xv\rangle) \\ &\overset{\text{monoticity}}{=} \min_{\dim V = k-1} f(\max_{\|v\|=1, v\in V} \langle X^*AXv, v\rangle) \\ &\overset{\text{mean-max again}}{\ge} \min_{\dim V = k-1} f(\lambda_k(X^*AX)) \\ &= f(\lambda_k(X^*AX)) \overset{\text{monoticity again}}{=} \lambda_k(f(X^*AX)).\end{aligned} \tag{3.3.20}$$

Therefore, there exists a unitary matrix U such that $f(X^*AX) \le U^* X^* f(A) X U$. □

Proposition 36. *f is increasing and convex function, $f : [0, \infty) \to [0, \infty)$, $A \ge 0$ and $\mathrm{Tr}\, A = 1$, then we have*

$$\frac{f(A)}{\mathrm{Tr}\, f(A)} \succeq A. \tag{3.3.21}$$

Here the notation $\succeq$ denotes majorization.

Proof. Note that for $0 \le a \le b \neq 0$ we have

$$f(b)a - f(a)b = f(b)a - f\left(0 \cdot \left(1 - \frac{a}{b}\right) + b \cdot \frac{a}{b}\right) b \ge f(b)a - \frac{a}{b} \cdot f(b)b = 0. \tag{3.3.22}$$

Therefore, we have

$$f(\lambda_j)\lambda_k \ge \lambda_j f(\lambda_k), \quad \text{for} \quad j \le k. \tag{3.3.23}$$

Therefore

$$[f(\lambda_1) + \cdots + f(\lambda_l)](\lambda_{l+1} + \cdots + \lambda_n) \ge (\lambda_1 + \cdots + \lambda_l)[f(\lambda_{l+1}) + \cdots + f(\lambda_n)]. \tag{3.3.24}$$

We add $(\lambda_1 + \cdots + \lambda_l)[f(\lambda_{l+1}) + \cdots + f(\lambda_n)]$ to both sides, we have

$$[f(\lambda_1) + \cdots + f(\lambda_l)] \operatorname{Tr} A \ge (\lambda_1 + \cdots + \lambda_l) \operatorname{Tr} f(A). \tag{3.3.25}$$

Therefore

$$\frac{f(\lambda_1) + \cdots + f(\lambda_l)}{\operatorname{Tr} f(A)} \ge \lambda_1 + \cdots + \lambda_l, \quad \forall 1 \le l \le n. \tag{3.3.26}$$

Therefore $\frac{f(A)}{\operatorname{Tr} f(A)}$ majorizes A. □

Remark 32. *For two Gibbs states with different temperatures, we have*

$$\frac{e^{-\beta H}}{\operatorname{Tr}(e^{-\beta H})} \preceq \frac{e^{-\beta' H}}{\operatorname{Tr}(e^{-\beta' H})}, \quad \forall \beta \le \beta'. \tag{3.3.27}$$

This is because $t^{\beta'/\beta}$ is convex for $\beta'/\beta \ge 1$.

3.4 Klein inequality and relative entropy

Theorem 3.4.1. *Let f be a C^1 and convex function, A, B are Hermitian matrices, then*

$$\operatorname{Tr}((A-B)f'(B)) \le \operatorname{Tr}(f(A) - f(B)) \le \operatorname{Tr}((A-B)f'(A)). \tag{3.4.1}$$

Proof. Let $h(t) = \operatorname{Tr} f(A + t(B-A))$, then we have $h(0) = \operatorname{Tr} f(A)$, $h(1) = \operatorname{Tr} f(B)$.

By the Jensen trace inequality we have $A \mapsto \operatorname{Tr} f(A)$ is a convex map, therefore $h(t)$ is a convex function. Thus

$$h(t) \le th(1) + (1-t)h(0) \Rightarrow \frac{h(1) - h(0)}{1 - 0} \ge \frac{h(t) - h(0)}{t - 0}, \quad \forall t \in [0,1]. \tag{3.4.2}$$

Therefore, by taking the limit $t \to 0$, we have

$$h'(0) \le h(1) - h(0) \Rightarrow \operatorname{Tr}[f'(A)(B-A)] \le \operatorname{Tr}(f(B) - f(A)). \tag{3.4.3}$$

□

Definition 3.4.2 (Relative entropy)**.** *A, B are two density matrices, $\mathrm{P}_{Range}(B) \le \mathrm{P}_{Range}(A)$, then we define the relative entropy as*

$$H(B||A) = \operatorname{Tr}(B \log B - B \log A). \tag{3.4.4}$$

We define the entropy of a density matrix A as

$$H(A) = -\operatorname{Tr}(A \log A). \tag{3.4.5}$$

Note that 0 is just a first-order pole of $\log x$ at 0. Therefore $H(A)$ is well-defined even if A is singular. Also, we know that $t \log t$ is an operator convex function, therefore $(\cdot)$ is an operator concave mapping.

From the definition of relative entropy, we can easily find that

Proposition 37. $H(B||A) \geq 0$.

Proof. $f(t) = t \log t$ is convex, therefore $\mathrm{Tr}(A \log A - B \log B) \leq \mathrm{Tr}((A - B) \log A)$. □

In fact, we can show a stronger result. In fact, the Klein inequality is a first-order Taylor expansion of f. We can in fact use the information of the second order derivative of $f(x) = x \log x$ to derive the following result, which is closely related to the trace distance.

In fact, we have the following result:

Theorem 3.4.3. *f_k, g_k are functions on (a, b), $\lambda_k \in \mathbb{R}$. If $\sum_{k=1}^{m} \lambda_k f_k(x) g_k(y) \geq 0$ for any $x, y \in (a, b)$. Then for A, B Hermtian with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset (a, b)$, we have*

$$\sum_{k=1}^{m} \lambda_k \mathrm{Tr}(f_k(A) g_k(B)) \geq 0. \tag{3.4.6}$$

Proof. We apply the spectral decomposition of A and B:

$$A = \sum_{j=1}^{m'} a_j P_j, \quad B = \sum_{k=1}^{m''} b_k Q_k, \quad \sum_{j=1}^{m'} P_j = I, \quad \sum_{l=1}^{m''} Q_l = I. \tag{3.4.7}$$

Therefore, by functional calculus via spectral decomposition, we have

$$\sum_{k=1}^{m} \lambda_k \mathrm{Tr}(f_k(A) g_k(B)) = \sum_{kjl} \lambda_k f_k(a_j) g_k(b_l) \mathrm{Tr}(P_j Q_l) = \sum_{j,l} \left(\sum_{k} \lambda_k f_k(a_j) g_k(b_l) \right) \mathrm{Tr}(P_j Q_l) \geq 0. \tag{3.4.8}$$

□

Example 11. *For $f(x) = x \log x$, we have*

$$-f(x) + f(y) + (x - y) f'(y) = -\frac{1}{2}(x - y) f''(x_0) \geq -\frac{1}{2}(x - y)^2, \quad x, y \in (0, 1). \tag{3.4.9}$$

Here we use that $f'(x) = 1 + \log x$ and $f''(x) = \frac{1}{x} > 1$ for $x \in (0, 1)$.

Therefore, by Theorem 3.4.3, we have

$$\mathrm{Tr}(B \log B - B \log A) \geq \frac{1}{2} \mathrm{Tr}\big((A - B)^2\big). \tag{3.4.10}$$

Therefore

$$H(B||A) \geq \frac{1}{2} \mathrm{Tr}\big((A - B)^2\big) = \frac{1}{2} \|B - A\|_2^2. \tag{3.4.11}$$

Remark 33. *If $H(B||A) = 0$, we can easily see that $A = B$. However, this is not that obvious if we only know Proposition 37.*

Example 12 (Lieb's convexity)**.** *We claim that*

$$A, B \mapsto \mathrm{Tr}(X^* A^p X B^r) \tag{3.4.12}$$

is jointly concave for $0 \le p, r \le 1$, $p + r \le 1$.

We recall the Ando's concavity, we have $(A, B) \mapsto A^p \otimes B^r$ *is jointly concave.*

We take

$$E = \sum_{j,k=1}^{n} E_{jk} \otimes E_{jk}. \tag{3.4.13}$$

Then we can verify That

$$\mathrm{Tr}(X^* A^p X B^r) = \frac{1}{n} \mathrm{Tr}\big(E(X^* \otimes I)(A^p \otimes (B^T)^q)(X \otimes I)E\big). \tag{3.4.14}$$

By Ando's concavity, we have $A, B \mapsto \mathrm{Tr}(X^* A^p X B^r)$ *is jointly concave.*

Theorem 3.4.4. *The map* $(A, B) \mapsto H(B||A)$ *is jointly convex on the set*

$$\{(A, B) : P_{\mathit{Range}}(B) \le P_{\mathit{Range}}(A), A, B \textit{ are density matrices}\}. \tag{3.4.15}$$

Proof. Let

$$f(p) = \mathrm{Tr}\left((\lambda A_1 + (1-\lambda)A_2)^p \left(\lambda B_1 + (1-\lambda)B_2\right)^{1-p}\right) - \lambda \,\mathrm{Tr}\left(A_1^p B_1^{1-p}\right) - (1-\lambda)\,\mathrm{Tr}\left(A_2^p B_2^{1-p}\right). \tag{3.4.16}$$

Here, $A_1, A_2, B_1, B_2 \in \mathbb{H}_n^{>0}$ and $p, \lambda \in [0, 1]$.

By Lieb's concavity, we have $f(p) \ge 0$. Therefore $f'(0) \ge 0$ since $f(0) = 0$. This implies that

$$\begin{aligned}&\mathrm{Tr}\left((\lambda B_1 + (1-\lambda)B_2)\log\left(\lambda A_1 + (1-\lambda)A_2\right)\right) - \mathrm{Tr}\left((\lambda B_1 + (1-\lambda)B_2)\log\left(\lambda B_1 + (1-\lambda)B_2\right)\right)\\ \ge&\lambda\left(\mathrm{Tr}\left(B_1 \log A_1\right) - \mathrm{Tr}\left(B_1 \log B_1\right)\right) + (1-\lambda)\left(\mathrm{Tr}\left(B_2 \log A_2\right) - \mathrm{Tr}\left(B_2 \log B_2\right)\right)\end{aligned} \tag{3.4.17}$$

Here we note that Tr ensures the commutative property when taking derivatives, thus $f'(0)$ can be computed using the similar way as in the classical case.

This indicates that $H(B||A)$ is jointly convex on invertible density matrices. For general case, we can approximate $H(B||A)$ by invertible density matrices to see that it is still jointly convex. □

3.5 Peierls-Bogoliubov inequality and Gibbs variational principle

Next we want to explore the variational formula for relative entropy. In fact, these results are usually called (quantum) Gibbs variational principle. To prove these formulas, we first present the Peierls-Bogoliubov inequality

Theorem 3.5.1 (Peierls-Bogoliubov)**.** A, B *are Hermitian,* $\lambda \in [0, 1]$*, then we have*

$$\log \mathrm{Tr}\, e^{\lambda A + (1-\lambda)B} \le \lambda \log \mathrm{Tr}\, e^{A} + (1-\lambda) \log \mathrm{Tr}\, e^{B}. \tag{3.5.1}$$

Proof. We use the similar technique like our proof of $\operatorname{Tr} f$ is convex (see Proposition 33). We take v_j as the normal eigenvectors of $\lambda A + (1-\lambda)B$, then by spectral decomposition, we have

$$\log \sum_j \langle e^{\lambda A+(1-\lambda)B} v_j, v_j\rangle = \log\left(\sum_j e^{\langle[\lambda A+(1-\lambda B)]v_j,v_j\rangle}\right) = \log \sum_j e^{\lambda\langle Av_j,v_j\rangle+(1-\lambda)\langle Bv_j,v_j\rangle}. \tag{3.5.2}$$

We should note that at this point we cannot directly apply the Peierls inequality Proposition 30 because the λ term and $(1-\lambda)$ term are not yet separated. However, we can use the convexity of $\log\sum_j e^{x_j}$ to separate them. In fact, we can define

$$\psi : \mathbb{R}^n \to \mathbb{R}, \quad \psi(\mathbf{x}) = \log \sum_{j=1}^n e^{x_j}, \quad x_j \in \mathbb{R}. \tag{3.5.3}$$

We can see that ψ is a convex function by calculating the Hessian matrix. In fact, through a tedious but straightforward calculation, we can show that the Hessian matrix of ψ is given by

$$H_\psi(\mathbf{x}) = (h_{jk})_{j,k=1}^n, \quad h_{jk} = t_j\delta_{jk} - t_jt_k, \quad t_j = \frac{e^{x_j}}{\sum_{l=1}^n e^{x_l}}. \tag{3.5.4}$$

We can verify that $H_\psi(\mathbf{x})$ is positive semidefinite. Therefore, we can apply the convexity of ψ to separate the λ and $(1-\lambda)$ terms. We take $\mathbf{x} = (\langle Av_j, v_j\rangle)_{j=1}^n$, $\mathbf{y} = (\langle Bv_j, v_j\rangle)_{j=1}^n$, then

$$\begin{aligned} \log \sum_j e^{\lambda\langle Av_j,v_j\rangle+(1-\lambda)\langle Bv_j,v_j\rangle} &= \psi(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \\ &\le \lambda\psi(\mathbf{x}) + (1-\lambda)\psi(\mathbf{y}) \\ &= \lambda \log\sum_j e^{\langle Av_j,v_j\rangle} + (1-\lambda)\log\sum_j e^{\langle Bv_j,v_j\rangle} \\ &\overset{\text{Peierls Proposition 30}}{\le} \lambda \log \operatorname{Tr} e^A + (1-\lambda)\log\operatorname{Tr} e^B. \end{aligned} \tag{3.5.5}$$

□

Remark 34. *In this proof, we use to different convexity just like Proposition 33. The first one is the trivial convexity of* $\log\sum_j e^{x_j}$*, and the second one is the Peierls convexity.*

We can also state Peierls-Bogoliubov inequality using logarithmic convexity, i.e.

$$\operatorname{Tr}\left(e^{\lambda A+(1-\lambda B)}\right) \le \operatorname{Tr}\left(e^A\right)^{1-\lambda}\operatorname{Tr}\left(e^B\right)^{\lambda}. \tag{3.5.6}$$

Next we want to derive the variational formula for relative entropy. We first give a quick corollary of Theorem 3.5.1, which actually finds a lower bound of $\log e^{\operatorname{Tr} A} - \log e^{\operatorname{Tr} B}$.

Proposition 38. *Let A, B be Hermitian matrices, Then*

$$\log\frac{\operatorname{Tr} e^{A+B}}{\operatorname{Tr} e^B} \ge \frac{\operatorname{Tr}\left(e^B A\right)}{\operatorname{Tr} e^B}. \tag{3.5.7}$$

Proof. Let

$$h(t) = \log\operatorname{Tr} e^{tA+(1-t)B}, \quad t\in[0,1]. \tag{3.5.8}$$

Then $h(1) = \log \operatorname{Tr} e^A$ and $h(0) = \log \operatorname{Tr} e^B$. By Theorem 3.5.1, we have $A \mapsto \log \operatorname{Tr} e^A$ is operator convex. Therefore, we have $h(t)$ is convex. Thus we have

$$\frac{h(1) - h(0)}{1 - 0} \geq \frac{h(t) - h(0)}{t - 0}, \quad \forall t \in [0, 1]. \tag{3.5.9}$$

$$\text{LHS} = \log \operatorname{Tr} e^A - \log \operatorname{Tr} e^B. \tag{3.5.10}$$

To calculate the right-hand side, we only need to calculate the derivative of $h(t)$. In fact we should again take the advantage of the trace in the sense that the part inside the trace is commutative. Therefore, we have

$$h'(t) = \frac{\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr}\left(e^{tA+(1-t)B}\right)}{\operatorname{Tr} e^{tA+(1-t)B}} = \frac{\operatorname{Tr}\left(e^{tA+(1-t)B}(A - B)\right)}{\operatorname{Tr} e^{tA+(1-t)B}}. \tag{3.5.11}$$

Therefore

$$\lim_{t \to 0+} \text{RHS} = \lim_{t \to 0+} \frac{\operatorname{Tr}\left(e^B(A - B)\right)}{\operatorname{Tr} e^B}. \tag{3.5.12}$$

Therefore we have

$$\log \operatorname{Tr} e^A - \log \operatorname{Tr} e^B \geq \frac{\operatorname{Tr}\left(e^B(A - B)\right)}{\operatorname{Tr} e^B}. \tag{3.5.13}$$

Replacing A by $A + B$, we yield the desired result. □

Theorem 3.5.2 (Gibbs variational principle). *Let X be a Hermitian matrix, then*

$$\log \operatorname{Tr} e^X = \sup_{D \text{ density matrix}} \{\operatorname{Tr}(XD) + H(D)\}. \tag{3.5.14}$$

Proof. We take $B = \log D$, $A = X$ for any density matrix D and Hermitian matrix X, then by Proposition 38 we have

$$\log \operatorname{Tr} e^X \geq \frac{\operatorname{Tr}(D(X - \log D))}{\operatorname{Tr} e^{\log D}} = \operatorname{Tr}(XD) - \operatorname{Tr}(D \log D) = \operatorname{Tr}(XD) + H(D). \tag{3.5.15}$$

To see that the supremum is achieved, we can take $D = e^X / \operatorname{Tr} e^X$. Then we have

$$\begin{aligned} \operatorname{Tr}(XD) + H(D) &= \operatorname{Tr}\left(X \frac{e^X}{\operatorname{Tr} e^X}\right) + H\left(\frac{e^X}{\operatorname{Tr} e^X}\right) = \operatorname{Tr}\left(X \frac{e^X}{\operatorname{Tr} e^X}\right) - \operatorname{Tr}\left(\frac{e^X}{\operatorname{Tr} e^X} \log \frac{e^X}{\operatorname{Tr} e^X}\right) \\ &= \operatorname{Tr}\left(X \frac{e^X}{\operatorname{Tr} e^X}\right) - \operatorname{Tr}\left(X \frac{e^X}{\operatorname{Tr} e^X}\right) + \operatorname{Tr}\left(\frac{e^X}{\operatorname{Tr} e^X} \log \operatorname{Tr} e^X\right) = \log \operatorname{Tr} e^X. \end{aligned} \tag{3.5.16}$$

Thus the equality holds when we take D as the Gibbs state. □

Theorem 3.5.3 (Gibbs variational principle for entropy).

$$H(D) = \inf_{x \in \mathbb{H}^n} \left[\log \operatorname{Tr} e^X - \operatorname{Tr}(DX)\right] \tag{3.5.17}$$

Proof. For any $X \in \mathbb{H}^n$, we have

$$H(D) \leq \log \operatorname{Tr} e^X - \operatorname{Tr} DX. \tag{3.5.18}$$

If we take $X = \log D$, then

$$\log \operatorname{Tr} e^{\log D} - \operatorname{Tr}(D \log D) = -\operatorname{Tr}(D \log D). \tag{3.5.19}$$

□

Theorem 3.5.4 (Gibbs variational principle for relative entropy)**.**

$$\log \operatorname{Tr} e^{X+\log A} = \sup\{\operatorname{Tr} DX - H(D||A)\}. \tag{3.5.20}$$

$$H(D||A) = \inf_{X\in\mathbb{H}^n} \left[\log \operatorname{Tr} e^{X+\log A} - \operatorname{Tr} DX\right]. \tag{3.5.21}$$

Remark 35. *From the results above, we can see that* $\log \operatorname{Tr} e^X$ *(*$\log \operatorname{Tr} e^{X+\log A}$*) is in fact the Legendre transform of* $H(D)$ *(*$H(D||A)$*)*

3.6 Trace Hölder inequalities and Minkowski inequalities

Theorem 3.6.1. $1 \le p, q \le \infty$*,* $1/p + 1/q = 1$*,* $A, B \in M_n(\mathbb{C})$*, then we have*

$$|\operatorname{Tr}(AB)| \le \|A\|_p \|B\|_q. \tag{3.6.1}$$

When $1 < p, q < \infty$*, the equality is achieved iff* $V_A = V_B^*$ *and* $\frac{|A|^p}{\|A\|_p^p} = \frac{|B^*|^q}{\|B\|_q^q}$*. Here* V_A, V_B *are the polar part of* A, B *respectively.*

Remark 36. *The result is trivial if one of* p, q *is* 1*. We assume that* $q = 1$*. In this case,* $p = \infty$*. Then*

$$|\operatorname{Tr}(AB)| \le \operatorname{Tr}(|AB|) \le \|A\| \|B\|_1, \tag{3.6.2}$$

where the first inequality follows from the Cauchy-Schwarz inequality

$$|\operatorname{Tr}(AB)| = \operatorname{Tr}\Big(V_{AB}|AB|^{1/2}|AB|^{1/2}\Big) \le \operatorname{Tr}(|AB|) \tag{3.6.3}$$

and the second inequality is Proposition 28.

First Proof. exercise 6 □

Second Proof.

Lemma 6 (Hadamard three line theorem)**.** *Suppose* $f(z)$ *is an analytic function in the strip* $\Omega = \{z \in \mathbb{C} : a \le \operatorname{Re} z \le b\}$*, then we have* $M(x) := \sup_{y\in\mathbb{R}} |f(x + \mathrm{i}y)|$ *is a logarithmic convex function in* $[a, b]$*. That is,*

$$M(ta + (1-t)b) \le M(a)^t M(b)^{1-t}, \quad \forall t \in [0, 1]. \tag{3.6.4}$$

Proof of Lemma 6. W.L.O.G. we assume that $M(a) = M(b) = 1$. Apply maximum principle to $F_n(z) = f(z)e^{z^2/n}e^{-1/n}$, we have $|F_n(z)| \le 1$ on Ω. Letting $n \to \infty$ we have $|f(z)| \le 1$ on Ω. □

Without loss of generality, we assume that $\|A\|_p = \|B\|_q = 1$. We construct the following analytic function on the stripe $0 < \operatorname{Re} z < 1$ that continuously extends to the boundary:

$$f(z) = \operatorname{Tr}\Big(V_A|A|^{pz}|B^*|^{q(1-z)}V_B\Big) \tag{3.6.5}$$

Let $z = 1/p$ and apply Lemma 6, we have

$$|\operatorname{Tr}(AB)| = |f(1/p)| \le 1. \tag{3.6.6}$$

Suppose that $|\operatorname{Tr}(AB)| = 1$, then $|f(1/p)| = 1$. By maximum modulus principle, we have $|f(1/2)| = 1$. By the equality of Cauchy-Schwarz inequality we have $V_A|A|^{p/2} = V_B^*|B^*|^{q/2}$. Therefore, we have $\frac{|A|^p}{\|A\|_p^p} = \frac{|B^*|^q}{\|B\|_q^q}$ and $V_A = V_B^*$. We can verify that if these conditions hold, then we indeed have $|\operatorname{Tr}(AB)| = \operatorname{Tr}(V_A|A||B^*|V_B) = \operatorname{Tr}(A|A|^pV_A^*) = \operatorname{Tr}(|A|^p) = \|A\|_p^{1/p} = 1$. □

Third Proof. We apply the rank-1 decomposition.

We take $A = V_A|A| = V_A \sum_{j=1}^n s_j^A P_j^A$ with each P_j^A being a minimal projection. We absorb the polar part into P_j^A by defining $V_j^A := V_A P_j^A$. Then V_j^A is a rank-1 partial isometry. We can also do the same for B. Then we have

$$|\operatorname{Tr}(AB)| = \left| \sum_{j,k=1}^n \operatorname{Tr}\left(s_j^A s_k^B V_j^A V_k^B\right) \right| \leq \sum_{j,k=1}^n s_j^A s_j^B \left|\operatorname{Tr}\left(V_j^A V_k^B\right)\right|. \tag{3.6.7}$$

Next we estimate $\left|\operatorname{Tr}\left(V_j^A V_k^B\right)\right|$. By Proposition 28

$$\left|\operatorname{Tr}\left(V_j^A V_k^B\right)\right| \leq \min\{\|V_k^B\|\|V_j^A\|_1, \|V_j^A\|\|V_k^B\|_1\} = \min\{\operatorname{Tr}\left(|V_j^A|\right), \operatorname{Tr}\left(|V_j^B|\right)\} \leq \operatorname{Tr}\left(|V_j^A|\right)^{1/p} \operatorname{Tr}\left(|V_k^B|\right)^{1/q}. \tag{3.6.8}$$

By classical Hölder inequality, we have

$$\begin{aligned} |\operatorname{Tr}(AB)| &\leq \sum_{j,k=1}^n s_j^A s_k^B \operatorname{Tr}\left(|V_j^A|\right)^{1/p} \operatorname{Tr}\left(|V_k^B|\right)^{1/q} = \sum_{j=1}^n \left\{(s_j^A)^p \operatorname{Tr}\left(|V_j^A|\right)\right\}^{1/p} \left\{(s_k^B)^q \operatorname{Tr}\left(|V_k^B|\right)\right\}^{1/q} \\ &= \left(\sum_{j=1}^n \operatorname{Tr}\left(s_j^A\right)^p P_j^A\right)^{1/p} \left(\sum_{k=1}^n \operatorname{Tr}\left(s_k^B\right)^q P_k^B\right)^{1/q} = (\operatorname{Tr}|A|^p)^{1/p} (\operatorname{Tr}|B|^q)^{1/q} = \|A\|_p \|B\|_q. \end{aligned} \tag{3.6.9}$$

□

Remark 37. *From both proof above we can see that the polar decomposition is vital to the proof.*

Proposition 39.

$$|\operatorname{Tr}(A_1 \cdots A_m)| \leq \|A_1\|_{p_1} \cdots \|A_m\|_{p_m}, \quad \frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1, \quad 1 \leq p_i \leq \infty. \tag{3.6.10}$$

Proposition 40 (The variational formula for p-functional)**.**

$$\|A\|_p = \sup\{\operatorname{Tr}(AX) : \|X\|_q = 1\}, \quad 1 \leq p \leq \infty. \tag{3.6.11}$$

Proof. On the one hand,

$$|\operatorname{Tr} AX| \leq \|A\|_p \|X\|_q = \|A\|_p. \tag{3.6.12}$$

On the other hand, if we take

$$X = \frac{|A|^{p-1}}{\|A\|_p^p} |V|_A^*, \quad \operatorname{Tr}(AX) = \|A\|_p. \tag{3.6.13}$$

□

Theorem 3.6.2. $1 \leq r, p, q \leq \infty$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$*, then* $\|AB\|_r \leq \|A\|_p \|B\|_q$*.*

Proof. By the variational formula, we have (r' denotes the Hölder conjugate of r)

$$\|AB\|_r = \sup\{|\operatorname{Tr}(ABX)| : \|X\|_{r'} = 1\} \overset{\text{Proposition 39}}{\leq} \sup\{\|A\|_p \|B\|_q \|X\|_{r'} : \|X\|_{r'} = 1\} = \|A\|_p \|B\|_q. \tag{3.6.14}$$

□

In fact, in the classical case, the Hölder inequality also holds for $0 < p < 1$. However we should point out that the proof of the trace Hölder inequality for $0 < p < 1$ would be *much* more difficult than the case $p \geq 1$. In fact, we will use the Minkowski inequality to prove the Hölder inequality for $0 < p < 1$. We first state the Minkowski inequality for $1 \leq p \leq \infty$. This also implies that the p-functional we have defined previously is actually a norm and $(M_n(\mathbb{C}), \|\cdot\|_p)$ is a normed vector space when $1 \leq p \leq \infty$.

Theorem 3.6.3 (Minkowski inequality)**.**

$$\|A+B\|_p \leq \|A\|_p + \|B\|_p, \quad 1 \leq p \leq \infty. \tag{3.6.15}$$

Proof. We can use the variational formula to show that. In fact, we have

$$\|A+B\|_p = \sup\{|\mathrm{Tr}(A+B)X| : \|X\|_q = 1\} \leq \sup_{\|X\|_q=1}\{|\mathrm{Tr}(AX)|\} + \sup_{\|X\|_q=1}\{|\mathrm{Tr}(BX)|\} = \|A\|_p + \|B\|_p. \tag{3.6.16}$$

□

For $0 < p < 1$, the p-functional is actually not a norm. However, we can still show that the p-functional induces a metric on $\mathbb{H}_n^{\geq 0}$. In fact we have

Proposition 41 (Minkowski inequality for $0 < p < 1$)**.** $A, B \geq 0$, $0 < p < 1$, *then*

$$\|A+B\|_p^p \leq \|A\|_p^p + \|B\|_p^p, \quad 0 < p < 1. \tag{3.6.17}$$

Proof. We define

$$T = A^{1/2}(A+B)^{-1/2}, \quad S = B^{1/2}(A+B)^{-1/2}. \tag{3.6.18}$$

Here we define $(A+B)^{-1/2}$ to be

$$(A+B)^{-1/2} := (A+B)^{-1/2}|_{\mathrm{Range}(A+B)} P_{\mathrm{Range}(A+B)}. \tag{3.6.19}$$

Therefore

$$T^*T + S^*S = A^{1/2}(A+B)^{-1/2} + B^{1/2}(A+B)^{-1/2} = (A+B)^{-1/2}(A+B)(A+B)^{-1/2} = P_{\mathrm{Range}(A+B)}. \tag{3.6.20}$$

□

Therefore

$$\begin{aligned} \mathrm{Tr}((A+B)^p) &= \mathrm{Tr}\big((A+B)^{p/2}(T^*T+S^*S)(A+B)^{p/2}\big) = \mathrm{Tr}(T(A+B)^pT^*) + \mathrm{Tr}(S(A+B)^pS^*) \\ &\overset{T^*T+S^*S\leq 1,\ x^p \text{ concave, Proposition 34}}{\leq} \mathrm{Tr}[(T(A+B)T^*)^p] + \mathrm{Tr}[(S(A+B)S^*)^p] \\ &\overset{\text{monoticity}}{\leq} \mathrm{Tr}(A^p) + \mathrm{Tr}(B^p). \end{aligned} \tag{3.6.21}$$

Theorem 3.6.4 (Minkowski inequality)**.** *Suppose* $A, B \in M_n(\mathbb{C})$*, then we have* $0 < p < 1$*,*

$$\|A+B\|_p^p \leq \|A\|_p^p + \|B\|_p^p. \tag{3.6.22}$$

Proof. Recall Theorem 2.5.2

$$|A+B| \le U|A|U^* + V|A|V^*, \tag{3.6.23}$$

then by the operator monoticity of $x \mapsto x^p$ $(0<p<1)$ we have

$$|A+B|^p \le (U|A|U^* + V|B|V^*)^p. \tag{3.6.24}$$

Since $U|A|U^*, V|B|V^* \ge 0$, by Proposition 41 we have

$$\operatorname{Tr}|A+B|^p \le \operatorname{Tr}(U|A|U^*)^p + \operatorname{Tr}(V|B|V^*)^p = \operatorname{Tr}(|A|^p) + \operatorname{Tr}(|B|^p). \tag{3.6.25}$$

□

Now, we are at the place to prove the most general Hölder's inequality for general matrices $A, B \in M_n(\mathbb{C})$. This is a very strong theorem.

Theorem 3.6.5 (Hölder's inequality)**.** *Suppose that* $0 \le p,q,r \le \infty$ *such that* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, $A, B \in M_n(\mathbb{C})$, *then we have*

$$\|AB\|_r \le \|A\|_p \|B\|_q. \tag{3.6.26}$$

Proof. **Step 1.** For $r \ge 1$, we have already proved this in Theorem 3.6.2.

Step 2. If $0<r<1$, $\max\{p,q\} \ge 2$. Without loss of generality, we assume $q \ge 2$. Then we have

$$\|AB\|_r^r = \operatorname{Tr}|AB|^r = \operatorname{Tr}(B^*A^*AB)^{r/2} = \operatorname{Tr}\big(B^*|A|^2B\big)^{r/2} = \||A|B\|_r^r. \tag{3.6.27}$$

Therefore, without loss of generality, we can assume that $A \ge 0$, then we can apply spectral decomposition to A

$$A = \sum_{j=1}^n s_j P_j, \quad s_j \ge 0, \quad P_j \text{ minimal projection.} \tag{3.6.28}$$

By Minkowski inequality, we have

$$\|AB\|_r^r = \operatorname{Tr}(|AB|)^r = \sum_{j=1}^n s_j^r \|P_jB\|^r. \tag{3.6.29}$$

Not that $\|P_kB\|_r \le \|P_kP_kB\|_r = \|P_k|P_kB|\|_r$ and P_k commutes with $|P_kB| = (P_kBB^*P_k)^{\frac{1}{2}}$, we can use the classical Hölder inequality to obtain

$$\|P_kB\|_r^r \le \|P_k\|_p^p \|P_kB\|_q^q. \tag{3.6.30}$$

Therefore, we have

$$\|AB\|_r^r \le \sum_{k=1}^n \|P_k\|_p^r \|P_kB\|_q^r s_k^r \overset{1=\frac{r}{p}+\frac{r}{q}}{\le} \left[\sum_{k=1}^n s_k^p \operatorname{Tr}(P_k)\right]^{\frac{r}{p}} \left[\sum_{k=1}^n \|P_kB\|_q^q\right]^{\frac{r}{q}} = \|A\|_p^r \left[\sum_{k=1}^n \|P_kB\|_q^q\right]^{\frac{r}{q}} \tag{3.6.31}$$

Moreover, by $q \ge 2$, we have $t \mapsto t^{q/2}$ is a convex function, therefore by Jensen trace inequality Theorem 3.3.1, we have

$$\sum_{k=1}^n \|P_kB\|_q^q = \sum_{k=1}^n \operatorname{Tr}(P_kB^*BP_k)^{q/2} \overset{\text{Jensen trace inequality}}{\le} \sum_{k=1}^n \operatorname{Tr}\big(P_k(B^*B)^{q/2}P_k\big) = \operatorname{Tr}\big((B^*B)^{q/2}\big) = \|B\|_q^q. \tag{3.6.32}$$

Therefore,

$$\|AB\|_r^r \le \|A\|_p^r \|B\|_q^r. \tag{3.6.33}$$

Step 3. For the last step, $0 < r < 1$, $\max\{p, q\} < 2$. We take $l \in \mathbb{N}$ such that $lp \ge 2$, then by the result of **Step2** we have

$$\|AB\|_r = \left\| A^{\frac{1}{l}} A^{\frac{l-1}{l}} B \right\|_r \le \left\| A^{\frac{1}{l}} \right\|_{lp} \left\| A^{\frac{l-1}{l}} B \right\|_{r_1}. \tag{3.6.34}$$

Here r_1 is the Hölder conjugate of lp. If $r_1 \ge 1$, then we take $p_1 = \frac{lp}{l-1}$. By Hölder's inequality for $r \ge 1$ Theorem 3.6.2, we have

$$\left\| A^{\frac{l-1}{l}} B \right\|_{r_1} \le \left\| A^{\frac{l-1}{l}} \right\|_{p_1} \|B\|_q. \tag{3.6.35}$$

therefore, by the equality of Theorem 3.6.1

$$\|AB\|_r \le \left\| A^{\frac{1}{l}} \right\|_{lp} \left\| A^{\frac{l-1}{l}} \right\|_{p_1} \|B\|_q = \|A\|_p \|B\|_q. \tag{3.6.36}$$

If r_1, we repeat the above procedure, $\left\| A^{\frac{l-1}{l}} B \right\|_{r_1} \le \|A\|_p^l \left\| A^{\frac{l-2}{l}} B \right\|_{r_2}$. After at most l times, finally we can see the inequality holds. □

Proposition 42. *Suppose that* $0 \le p_1, \cdots, p_m, r \le \infty$ *with* $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{r}$, *then we have*

$$\|A_1 \cdots A_m\|_r \le \|A_1\|_{p_1} \cdots \|A_m\|_{p_m}, \quad A_i \in M_n(\mathbb{C}). \tag{3.6.37}$$

Proof. It follows from Theorem 3.6.5. □

Proposition 43 (Reverse Hölder inequality)**.** *Suppose that* $0 < p, q < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $A, B > 0$, *then we have*

$$\|AB\|_1 \ge \|A\|_p \|B\|_q. \tag{3.6.38}$$

Proof.

$$\|A\|_p = \left\| ABB^{-1} \right\|_p \overset{\text{Theorem 3.6.5}}{\le} \|AB\|_1 \left\| B^{-1} \right\|_q. \tag{3.6.39}$$

□

Proposition 44 (Reverse Minkowski inequality)**.** *Suppose that* $0 < p < 1$, *then we have*

$$\|A + B\|_p^p \ge \|A\|_p^p + \|B\|_p^p. \quad 0 < p < 1, \quad A, B \ge 0. \tag{3.6.40}$$

Proof. Let A, B invertible, then we have

$$\begin{aligned} \operatorname{Tr}((A+B)^p) &= \operatorname{Tr}\big((A+B)^{p-1}(A+B)\big) = \operatorname{Tr}\big((A+B)^{p-1}A\big) + \operatorname{Tr}\big((A+B)^{p-1}B\big) \\ &\overset{\text{Reverse Hölder Proposition 43}}{\ge} \|A\|_p \|A+B\|_p^{p-1} + \|B\|_p \|A+B\|_p^{p-1}. \end{aligned} \tag{3.6.41}$$

□

Next we present two applications of the Hölder's inequality.

Proposition 45. $A, B \ge 0$, $0 \le \alpha \le 1$, *then we have*

$$\operatorname{Tr}\big(A^{1-\alpha} B^{\alpha}\big) \ge \operatorname{Tr} \frac{A+B}{2} - \operatorname{Tr}|A - B|. \tag{3.6.42}$$

Proof. We take $X = (A-B)_+$, then $A - B \le X$ i.e. $A \le B + X$. Now we have

$$\begin{aligned}
\operatorname{Tr}(A) - \operatorname{Tr}\left(B^\alpha A^{1-\alpha}\right) &= \operatorname{Tr}\left(\left(A^\alpha - B^\alpha\right) A^{1-\alpha}\right) \\
&\overset{\text{operator monoticity}}{\le} \operatorname{Tr}\left(\left((X+B)^\alpha - B^\alpha\right) A^{1-\alpha}\right) \\
&\overset{\text{monoticity and } (X+B)^\alpha - B^\alpha \ge 0}{\le} \operatorname{Tr}\left(\left((X+B)^\alpha - B^\alpha\right)(X+B)^{1-\alpha}\right) \\
&= \operatorname{Tr}(X+B) - \operatorname{Tr}\left(B^\alpha (B+X)^{1-\alpha}\right) \\
&\le \operatorname{Tr}(X) + \operatorname{Tr}(B) - \operatorname{Tr}(B) = \operatorname{Tr}(X) \\
&\le \operatorname{Tr}(|A-B|)
\end{aligned} \tag{3.6.43}$$

Therefore $\operatorname{Tr}(A^{1-\alpha}B^\alpha) \ge \operatorname{Tr}(A) - \operatorname{Tr}(|A-B|)$, similarly $\operatorname{Tr}(A^{1-\alpha}B^\alpha) \ge \operatorname{Tr}(B) - \operatorname{Tr}(|B-A|)$. Therefore we have

$$\operatorname{Tr}\left(A^{1-\alpha}B^\alpha\right) \ge \frac{\operatorname{Tr}(A) + \operatorname{Tr}(B)}{2} - \operatorname{Tr}(|A-B|). \tag{3.6.44}$$

□

Remark 38. *If* $\operatorname{Tr} A = \operatorname{Tr} B = 1$ *(density matrices), then we have* $\operatorname{Tr}(A^{1-\alpha}B^\alpha) \ge 1 - \operatorname{Tr}(|A-B|)$ *i.e.* $\|A-B\|_1 \ge 1 - \operatorname{Tr}(A^{1-\alpha}B^\alpha)$.

Theorem 3.6.6 (Weyl's inequality)**.** $A \in M_n(\mathbb{C})$*, then*

$$\sum_{j=1}^n |\lambda_j(A)|^k \le \sum_{j=1}^n \lambda_j(|A|)^k. \tag{3.6.45}$$

Proof. Recall Theorem 1.3.1, we have

$$|\lambda_j(A)| = \lim_{m\to\infty} (\lambda_j(|A|^m))^{1/m}. \tag{3.6.46}$$

For any $\varepsilon > 0$, $\exists m_0$ s.t. $|\lambda_j(A)| \le (1+\varepsilon)\lambda_j(|A^m|)^{1/m}(m \ge m_0)$. Therefore

$$\begin{aligned}
\sum_{j=1}^n |\lambda_j(A)|^k &\le (1+\varepsilon)^k \sum_{j=1}^n \lambda_j(|A^m|)^{k/m} = (1+\varepsilon)^k \sum_{j=1}^n \|A^m\|_{k/m}^{1/m} \\
&\overset{\text{Hölder}}{\le} (1+\varepsilon)^k \Big(\underbrace{\|A\|_k^k \cdots \|A\|_k^k}_{m}\Big)^{\frac{1}{m}} = (1+\varepsilon)^k \|A\|_k^k
\end{aligned} \tag{3.6.47}$$

□

Remark 39. *This global Weyl's inequality will be useful for Golden-Thompson inequality we will discuss later.*

3.7 Trace joint convexity

The main goal of this section is to show several joint convexity results that are strongly related to the Lieb concavity (see example 12). To do this, we first recall some essential results.

Proposition 46. *The mapping $(A, X) \to X^* A^{-1} X$ is jointly concave on $\mathbb{H}_n^{>0} \times M_n(\mathbb{C})$.*

Proof. See Proposition 14. $\square$

Proposition 47 (Ando's concavity)**.** *$(A, B) \mapsto A^p \otimes B^r$ is jointly concave on $\mathbb{H}_n^{>0} \times H_n^{>0}$ for $0 \le p, r \le 1$ and $p + r \le 1$.*

Remark 40. *Can we say for more generalized p and r?*

Proposition 48. *The mapping $(A, B) \mapsto A^p \otimes B^r$ is jointly convex on $\mathbb{H}_n^{>0} \times H_n^{>0}$ for $1 \le p \le 2$, $-1 \le r \le 0$, and $p + r \ge 1$.*

Proof. From the conditions we see that $0 \le 2 - p - r \le 1$, $0 \le 2 - p, -r \le 1$. By Ando's concavity Proposition 47 we have

$$(A, B) \mapsto A^{2-p} \otimes B^{-r} \text{ is jointly concave on } \mathbb{H}_n^{>0} \times \mathbb{H}_n^{>0}. \tag{3.7.1}$$

Moreover, we notice that

- $(A^{2-p} \otimes B^{-r})^{-1} = A^{-2} A^p \otimes B^r$;
- We consider
$$(A \otimes I)(A^{-2} A^p \otimes B^r)(A \otimes I) = A^p \otimes B^r. \tag{3.7.2}$$

Thus, by the operator convexity of $t \mapsto t^{-1}$ and the joint concavity of $(A, X) \mapsto X^* A^{-1} X$ Proposition 46, we have

$$(A, B) \mapsto (A \otimes I)^* [(A^{2-p} \otimes B^{-r})^{-1}](A \otimes I) = A^p \otimes B^r \tag{3.7.3}$$

is jointly convex. $\square$

Remark 41. *And symmetrically, by $-1 \le p \le 0$, $1 \le r \le 2$ and $0 \le p + r \le 1$, the mapping is also jointly convexity.*

Proposition 49. *The mapping $(A, B) \mapsto A^p \otimes B^r$ is jointly concave on $\mathbb{H}_n^{>0} \times H_n^{>0}$ for $-1 \le p, r \le 0$, and $-1 \le p + r \le 0$.*

Proof. It follows readily from the convexity of $A \mapsto A^{-1}$. $\square$

Remark 42. *The only essence is the Ando's concavity. But it does not hold for $p + r \ge 1$. A quick explanation is by considering the algebraic homomorphism $A^p \otimes A^r \mapsto A^{p+r}$. So when $p + r \ge 1$, the concavity is changed to convexity when $p + r \ge 1$.*

Recall the skew information

$$I(\rho, X) := \frac{1}{2} \operatorname{Tr}\Big([\rho^{\frac{1}{2}}, X]^* [\rho^{\frac{1}{2}}, X] \Big), \quad \text{where } \rho \text{ is a density matrix.} \tag{3.7.4}$$

Remark 43 (Remark of history)**.** *Wigner-Yanase-Dyson (1973) conjecture: to study the convesity of*

$$I_s(\rho, X) = \frac{1}{2} \operatorname{Tr}\big([\rho^s, X]^* [\rho^{1-s}, X] \big), \quad 0 \le s \le 1. \tag{3.7.5}$$

Lieb (1976) gave the Lieb's concavity

Theorem 3.7.1 (Lieb, 1976)**.** *$(A, B) \mapsto \mathrm{Tr}(X^* A^p X B^r)$ is jointly concave if $0 \leq p + r \leq 1$ and $0 \leq p, r \leq 1$.*

Strategy of proof. Consider the analytic function

$$f(z) = \mathrm{Tr}\big(X^* A^z X B^{s-z}\big) \tag{3.7.6}$$

on a strip. Then we use the Hadamard's three line theorem to discuss the maximum of $f(z)$ on the boundary of the strip. □

In fact, Lieb's concavity can also be generalized like the case of Ando's concavity. We can also consider the joint convexity of the mapping $(A, B) \mapsto \mathrm{Tr}(X^* A^p X B^r)$ for $p + r \geq 1$.

Proposition 50. *The mapping $(A, B) \mapsto \mathrm{Tr}(X^* A^p X B^r)$ is jointly convex on $\mathbb{H}_n^{>0} \times \mathbb{H}_n^{>0}$ for*

- $1 \leq p \leq 2$, $-1 \leq r \leq 0$, $p + r \geq 1$;
- $-1 \leq p \leq 0$, $1 \leq r \leq 2$, $p + r \geq 1$;
- $-1 \leq p \leq 0$, $-1 \leq r \leq 0$, $-1 \leq p + r \leq 0$.

Proof. We take $E = \sum_{j,k=1}^n E_{jk} \otimes E_{jk}$, then

$$\mathrm{Tr}(X^* A^p X B^r) = \frac{1}{n} (\mathrm{Tr} \otimes \mathrm{Tr}) \left[E (X^* \otimes I)(A^p \otimes B^r)(X \otimes I) E \right] \tag{3.7.7}$$

then the results follows from the generalized Ando's convexity Proposition 48 and Proposition 49. □

Another natural question is whether we can also consider the joint convexity of the mapping

$$(A, B) \mapsto \mathrm{Tr}\big(X^* A^{-p} X B^{-r}\big), \quad 0 \leq p, r \leq 1, \quad p + r \leq 1. \tag{3.7.8}$$

The original proof is given by Lieb, which seems a little bit complicated but quite provocative. In fact he considered the extended problem on $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ rather than the tensor product space.

Proof. Let $\lambda \in [0, 1]$ and let

$$A = \lambda A_1 + (1 - \lambda) A_2, \quad B = \lambda B_1 + (1 - \lambda) B_2. \tag{3.7.9}$$

Let $X_1, X_2, X_1', X_2' \in M_n(\mathbb{C})$, then we would like to show

$$\langle \lambda X_1' + (1-\lambda) X_2', A^{-p} (\lambda X_1 + (1-\lambda) X_2) B^{-r} \rangle \leq \lambda \langle X_1', A^{-p} X_1 B^{-r} \rangle + (1-\lambda) \langle X_2', A^{-p} X_2 B^{-r} \rangle. \tag{3.7.10}$$

Here $\langle \cdot, \cdot \rangle$ is the Hilbert-Schmidt inner product on $M_n(\mathbb{C})$. Since $A, B \in \mathbb{H}_n^{>0}$, we can verify that $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ defined as follows are both inner products on $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$:

$$\begin{aligned} \langle X_1' \oplus X_2', X_1 \oplus X_2 \rangle_1 &:= \langle \lambda X_1' + (1-\lambda) X_2', A^{-p} (\lambda X_1 + (1-\lambda) X_2) B^{-r} \rangle, \\ \langle X_1' \oplus X_2', X_1 \oplus X_2 \rangle_2 &:= \langle X_1', A^{-p} X_1 B^{-r} \rangle + (1-\lambda) \langle X_2', A^{-p} X_2 B^{-r} \rangle. \end{aligned} \tag{3.7.11}$$

By Riesz's representation theorem, there exists a linear operator T on $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ such that

$$\langle X_1' \oplus X_2', X_1 \oplus X_2 \rangle_1 = \langle X_1' \oplus X_2', T(X_1 \oplus X_2) \rangle_2. \tag{3.7.12}$$

Let a be an eigenvalue of T with eigenvector $Y_1 \oplus Y_2$. Then we have

$$\langle X_1' \oplus X_2', Y_1 \oplus Y_2 \rangle_2 = a\lambda \langle X_1', A_1^{-p} Y_1 B_1^{-r} \rangle + a(1-\lambda)\langle X_2', A_2^{-p} Y_2 B_2^{-r} \rangle. \tag{3.7.13}$$

$$\langle X_1' \oplus X_2', Y_1 \oplus Y_2 \rangle_1 = \lambda \langle X_1', A^{-p}(\lambda X_1 + (1-\lambda)X_2)B^{-r} \rangle + (1-\lambda)\langle X_2', A^{-p}(\lambda X_1 + (1-\lambda)X_2)B^{-r} \rangle. \tag{3.7.14}$$

We denote $Y := A^{-p}(\lambda X_1 + (1-\lambda)X_2)B^{-r}$, then

$$\lambda \langle X_1', Y \rangle + (1-\lambda)\langle X_2', Y \rangle = a\lambda \langle X_1', A_1^{-p} Y_1 B_1^{-r} \rangle + a(1-\lambda)\langle X_2', A_2^{-p} Y_2 B_2^{-r} \rangle \tag{3.7.15}$$

By the arbitrariness of X_1', X_2', we can see that

$$aA_1^{-p} Y_1 B_1^{-r} = Y = aA_2^{-p} Y_2 B_2^{-r}. \tag{3.7.16}$$

Therefore, by Lieb's concavity example 12 we have

$$\begin{aligned}
\operatorname{Tr}(Y^*(\lambda Y_1 + (1-\lambda)Y_2)) &\overset{\text{definition of } Y}{=} \operatorname{Tr}(Y^* A^p Y B^r) \\
&\overset{\text{Lieb concavity}}{\geq} \lambda \operatorname{Tr}(Y^* A_1^p Y B_1^r) + (1-\lambda)\operatorname{Tr}(Y^* A_2^p Y B_2^r) \\
&= a\lambda \operatorname{Tr}(Y^* Y_1) + a(1-\lambda)\operatorname{Tr}(Y^* Y_2) \\
&= a \operatorname{Tr}(Y^*(\lambda Y_1 + (1-\lambda)Y_2)) \Rightarrow a \leq 1.
\end{aligned} \tag{3.7.17}$$

Therefore $T \leq 1$, thus

$$\langle X_1' \oplus X_2', X_1 \oplus X_2 \rangle_1 \leq \langle X_1' \oplus X_2', X_1 \oplus X_2 \rangle_2 \tag{3.7.18}$$

which is what we want to show. □

Another proof.

$$(X^* \otimes I)(A^p \otimes (B^T)^r)^{-1}(X \otimes I) \tag{3.7.19}$$

is joint convex since $t \mapsto t^{-1}$ is decreasing on $(0, \infty)$ together with example 12 and Proposition 46. □

Remark 44. *The tensor product proof is only single line. But we should remark again that Lieb's original proof is sometimes the only viable approach.*

Theorem 3.7.2. $(D, X) \mapsto \int_0^\infty \operatorname{Tr}\big(X^* \frac{1}{s+D} X \frac{1}{s+D}\big)\mathrm{d}s$ *is jointly convex.*

Remark 45. *Recall that*

$$\Phi_D(X) := \int_0^\infty \frac{1}{s+D} X \frac{1}{s+D} \mathrm{d}s, \tag{3.7.20}$$

then

$$\Phi_D^{-1}(X) = \int_0^1 D^s X D^{1-s} \mathrm{d}s. \tag{3.7.21}$$

Proof. We follow the proof of Lieb. We define

$$D = \lambda D_1 + (1-\lambda)D_2. \tag{3.7.22}$$

Define the conjugate bilinear form

$$\langle X_1' \oplus X_2', X_1 \oplus X_2 \rangle_1 := \langle \lambda X_1' + (1-\lambda)X_2', \Phi_D(\lambda X_1 + (1-\lambda)X_2) \rangle, \tag{3.7.23}$$

$$\langle X_1' \oplus X_2', X_1 \oplus X_2 \rangle_2 := \lambda \langle X_1', \Phi_{D_1}[X_1] \rangle + (1-\lambda) \langle X_2', \Phi_{D_2}[X_2] \rangle. \tag{3.7.24}$$

Repeat the previous proof, we have

$$a\Phi_{D_1}(Y_1) = \Phi_D(\lambda Y_1 + (1-\lambda)Y_2) =: Y = a\Phi_{D_2}(Y_2). \tag{3.7.25}$$

By the convexity of Φ_D^{-1} (since $\Phi_D^{-1}(X) = \int_0^1 D^s X D^{1-s} \mathrm{d}s$, by noncommutative Jensen inequality), we have

$$\begin{aligned} \operatorname{Tr}(Y^*[\lambda Y_1 + (1-\lambda)Y_2]) &= \operatorname{Tr}\big(Y^* \Phi_D^{-1}(Y)\big) \\ &\overset{\text{Lieb's concavity}}{\geq} \lambda \operatorname{Tr}\big(Y^*, \Phi_{D_1}^{-1}(Y_1)\big) + (1-\lambda) \operatorname{Tr}\big(Y^* \Phi_{D_2}^{-1}(Y_2)\big) \\ &= a \operatorname{Tr}(Y^*[\lambda Y_1 + (1-\lambda)Y_2]) \Rightarrow a \leq 1 \Rightarrow T \leq 1. \end{aligned} \tag{3.7.26}$$

□

Remark 46. *The phylosiphy is that: the concavity of the inverse gives the* joint *convexity of the original mapping.*

Remark 47 (Remark of history)**.** *WYD conjecture: we study the convexity of* $\operatorname{Tr}(\rho^s X^* \rho^{1-s} X)$ *by studying the convexity of* $\operatorname{Tr}(A^p X^* B X)$*. The generalized WYD conjecture for* $\operatorname{Tr}\big([A^{\frac{p}{2}} X^* B^r X A^{\frac{p}{2}}]^s\big)$ *was also resolved very recently by Zhang 2019.*

3.8 Golden-Thompson inequality

Lemma 7 (Lie-Trotter formula)**.** *Let* $A, B \in M_n(\mathbb{C})$*, then*

$$e^{A+B} = \lim_{m \to \infty} \left(e^{\frac{A}{m}} e^{\frac{B}{m}} \right)^m. \tag{3.8.1}$$

Remark 48. *For analytic function* $e^A = I + A + \frac{1}{2}A^2 + \cdots$ *can be defined using power-series, which means that the definition can in general be the whole domain of convergence, instead of simply the spectrum like in the usual case of Hermitian matrices.*

Proof. Let $X_m = e^{A/m} e^{B/m}$, $Y_m = e^{\frac{A+B}{m}}$.

$$\|X_m^m - Y_m^m\| \leq m \|X_m - Y_m\| \{\max(\|X_m\|, \|Y_m\|)\}^{m-1} (\text{by factorization of } A^m - B^m) \tag{3.8.2}$$

Moreover,

$$\|X_m\| \leq e^{\frac{\|A\|+\|B\|}{m}}, \quad \|Y_m\| \leq e^{\frac{\|A\|+\|B\|}{m}}, \tag{3.8.3}$$

thus

$$\|X_m^m - Y_m^m\| \leq m \|X_m - Y_m\| e^{\|A\|+\|B\|}. \tag{3.8.4}$$

We compute

$$X_m = \left(I + \frac{A}{m} + \frac{A^2}{2m^2} + o\left(\frac{1}{m^2}\right) \right) \left(I + \frac{B}{m} + \frac{B^2}{2m^2} + o\left(\frac{1}{m^2}\right) \right), \tag{3.8.5}$$

$$Y_m = \left(I + \frac{A+B}{m} + \frac{(A+B)^2}{2m^2} + o\left(\frac{1}{m^2}\right) \right). \tag{3.8.6}$$

$$X_m - Y_m = \frac{AB - BA}{2m^2} + o\left(\frac{1}{m^2}\right). \tag{3.8.7}$$

Thus

$$\|X_m^m - Y_m^m\| \lesssim m \cdot \frac{1}{m^2} \exp(\|A\| + \|B\|) \in o\left(\frac{1}{m}\right). \tag{3.8.8}$$

Let $m \to \infty$, we have

$$\lim_{m\to\infty} \|X_m^m - Y_m^m\| = 0. \tag{3.8.9}$$

$$\Rightarrow \lim_{m\to\infty} \left[e^{\frac{A}{m}} e^{\frac{B}{m}}\right]^m = e^{A+B}. \tag{3.8.10}$$

□

Theorem 3.8.1 (Golden-Thompson inequality)**.** *Suppose that $A, B \in M_n(\mathbb{C})$, then we have*

$$\left|\operatorname{Tr}\left(e^{A+B}\right)\right| \le \operatorname{Tr}\left(e^{\operatorname{Re} A} e^{\operatorname{Re} B}\right). \tag{3.8.11}$$

Proof. For $r \in \mathbb{N}$, by Weyl's inequality Theorem 3.6.6

$$\left|\operatorname{Tr}(AB)^{2r}\right| = \sum_{j=1}^n |\lambda_j(AB)|^{2r} \le \sum_{j=1}^n \lambda_j(|AB|)^{2r} = \operatorname{Tr}|AB|^{2r}. \tag{3.8.12}$$

$$\operatorname{Tr}|AB|^{2r} = \operatorname{Tr}((B^*A^*AB)^r) = \operatorname{Tr}((A^*ABB^*)^r) = \operatorname{Tr}\left([|A|^2|B^*|^2]^r\right). \tag{3.8.13}$$

Let $m = 2^k$, we have

$$\left|\operatorname{Tr}(AB)^{2^k}\right| \le \operatorname{Tr}\left([|A|^2|B^*|^2]^{2^{k-1}}\right) \le \operatorname{Tr}\left([|A|^4|B^*|^4]^{2^{k-2}}\right) \le \cdots \le \operatorname{Tr}\left([|A|^{2^k}|B^*|^{2^k}]\right). \tag{3.8.14}$$

We apply the change of variable $A \mapsto e^{2^{-k}A}$, $B \mapsto e^{2^{-k}B^*}$ to get the result. We have

$$\left|\operatorname{Tr}\left(e^{A+B}\right)\right| = \lim_{k\to\infty}\left|\operatorname{Tr}\left(e^{2^{-k}A}e^{2^{-k}B}\right)^{2^k}\right| \le \lim_{k\to\infty} \operatorname{Tr}\left((e^{-2^kA^*}e^{2^{-k}A})^{2^{k-1}}(e^{-2^kB^*}e^{2^{-k}B})^{2^{k-1}}\right) = \operatorname{Tr}\left(e^{\operatorname{Re} A}e^{\operatorname{Re} B}\right). \tag{3.8.15}$$

□

Remark 49. *For A, B Hermitian, we have*

$$\operatorname{Tr}\left(e^{A+B}\right) = \operatorname{Tr}\left(e^A e^B\right) \tag{3.8.16}$$

But in general we do not have

$$\left|\operatorname{Tr}\left(e^{A+B+C}\right)\right| = \operatorname{Tr}\left(e^A e^B e^C\right). \tag{3.8.17}$$

A more generlized result is Golden-Thompson-Lieb inequality.

Lemma 8. *$A \mapsto \operatorname{Tr}\left(e^{X+\log A}\right)$ is concave, where X is a density matrix.*

Proof. By Gibbs variational formula, we have

$$\log \operatorname{Tr}\left(e^{X+\log A}\right) = \sup_{\widetilde{A}}\{\operatorname{Tr}\left(X\widetilde{A}\right) - H(\widetilde{A}||A)\} \tag{3.8.18}$$

The supremum is achieved when $\widetilde{A} = \frac{e^{X+\log A}}{\operatorname{Tr}\left(e^{X+\log A}\right)}$. For $A = \lambda A_1 + (1-\lambda)A_2$ ($\lambda \in [0,1]$), we have

$$\begin{aligned}
&\lambda \log \operatorname{Tr}\left(e^{X+\log A_1}\right) + (1-\lambda)\log \operatorname{Tr}\left(e^{X+\log A_2}\right)\\
=&\lambda \operatorname{Tr}\left(X\widetilde{A}_1\right) - \lambda H(\widetilde{A}_1||A_1) + (1-\lambda)\operatorname{Tr}\left(X\widetilde{A}_2\right) - (1-\lambda)H(\widetilde{A}_2||A_2).
\end{aligned} \tag{3.8.19}$$

Since $-H(B||A)$ is jointly concave for $P_{\text{Range}(B)} \geq P_{\text{Range}(A)}$ by Theorem 3.4.4, we have

$$\begin{aligned}
&\lambda \log \operatorname{Tr}\left(e^{X+\log A_1}\right) + (1-\lambda)\log \operatorname{Tr}\left(e^{X+\log A_2}\right)\\
\leq&\operatorname{Tr}\left(X(\lambda\widetilde{A}_1 + (1-\lambda)\widetilde{A}_2)\right) - H(\lambda\widetilde{A}_1 + (1-\lambda)\widetilde{A}_2|| \underbrace{\lambda A_1 + (1-\lambda)A_2}_{=A})\\
\overset{\text{Gibbs variational formula again}}{\leq}& \sup_{\widetilde{A}}\{\operatorname{Tr}\left(X\widetilde{A}\right) - H(\widetilde{A}||A)\} = \log \operatorname{Tr}\left(e^{X+\log A}\right).
\end{aligned} \tag{3.8.20}$$

Since $t \mapsto e^{-t}$ is decreasing and concave, we have

$$A \mapsto -\operatorname{Tr}\left(e^{X+\log A}\right) \tag{3.8.21}$$

is convex. Thus $A \mapsto \operatorname{Tr}\left(e^{X+\log A}\right)$ is concave. □

Theorem 3.8.2 (Golden-Thompson-Lieb inequality)**.** *Suppose that A, B, C are Hermitian matrices, then*

$$\operatorname{Tr}\left(e^{A+B+C}\right) \leq \int_0^\infty \operatorname{Tr}\left(e^A \frac{1}{\lambda + e^{-C}} e^B \frac{1}{\lambda + e^{-C}}\right) \mathrm{d}\lambda. \tag{3.8.22}$$

Proof. We denote

$$h(t) = \operatorname{Tr}\left(e^{X+\log(D+tY)}\right) \tag{3.8.23}$$

where Y is a Hermitian matrix. By Lemma 8 we have $h(t)$ is concave. Therefore

$$h(1) - h(0) \leq h'(0). \tag{3.8.24}$$

Moreover,

$$h'(0) = \operatorname{Tr}\left(e^{X+\log D}\int_0^\infty \frac{1}{\lambda + D} Y \frac{1}{\lambda + D}\mathrm{d}\lambda\right). \tag{3.8.25}$$

Lemma 9. *Let Ω be a convex cone, f is a convex function on Ω and f is homogeneous of order 1 i.e. $f(\lambda x) = \lambda f(x)$ for $\lambda > 0$ and $x \in \Omega$. Let $y \in \Omega$ and $\lim_{t\to 0+} \frac{f(x+ty)-f(x)}{t}$ exists, then $f(y) \geq \lim_{t\to 0+} \frac{f(x+ty)-f(x)}{t}$.*

Proof. By the convexity of f, we have $h(t) := f(x+ty)$ is convex and thus $h'(0) \leq h(1) - h(0)$, therefore

$$f(x+y) - f(x) \geq \lim_{t\to 0+} \frac{f(x+ty)-f(x)}{t}. \tag{3.8.26}$$

By the convexity and homogeneity of f, we have

$$\frac{1}{2}f(x+y)=f\left(\frac{x+y}{2}\right)\leq\frac{1}{2}[f(x)+f(y)]\Rightarrow f(x+y)-f(x)\leq f(x)+f(y)-f(x)=f(y). \quad (3.8.27)$$

Thus we have

$$f(y)\geq\lim_{t\to 0+}\frac{f(x+ty)-f(x)}{t}. \quad (3.8.28)$$

□

Note that in our case, $h(t)$ is also a homogeneous function of order 1 and $-h(t)$ is convex, then we have

$$h'(0)\geq h(1)-h(0)=\operatorname{Tr}\exp(X+\log(D+Y))-\operatorname{Tr}\exp(X+\log D)\geq\operatorname{Tr}\left(e^{X+\log Y}\right). \quad (3.8.29)$$

Thus we take $D=e^{-C}$, $X=A+C$, $Y=e^{B}$, then we have

$$\operatorname{Tr}\left(e^{A+B+C}\right)\leq\int_0^\infty\operatorname{Tr}\left(e^{A}\frac{1}{\lambda+e^{-C}}e^{B}\frac{1}{\lambda+e^{-C}}\right)\mathrm{d}\lambda. \quad (3.8.30)$$

□

Next we give another version of Golden-Thompson theorem in terms of the so-called weak majorization. The statement is

Theorem 3.8.3 (Informal)**.**

$$e^{A+B}\prec_w e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}. \quad (3.8.31)$$

This is called Weyl's majorization theorem. We will focus on addressing this in the next section.

3.9 Weyl majorization theorem

Theorem 3.9.1 (Karamata's inequality)**.** *$x,y\in\mathbb{R}^n$ are vector ordered non-increasingly, then $y\prec x$ $\iff$ for any convex function f we have $\sum_{j=1}^n f(y_j)\leq\sum_{j=1}^n f(x_j)$.*

Proof of Theorem 3.9.1. $\Rightarrow$ We recall that $y\prec x$ iff there exists a doubly stochastic matrix S such that $y=Sx$. We take f to be a convex function, then we have

$$\sum_{j=1}^n f(y_j)=\sum_{j=1}^n f\left(\sum_{k=1}^n S_{jk}x_k\right)\overset{f\text{ convex}}{\leq}\sum_{j=1}^n\sum_{k=1}^n S_{jk}f(x_k)=\sum_{k=1}^n f(x_k)\left(\sum_{j=1}^n S_{jk}\right)=\sum_{k=1}^n f(x_k). \quad (3.9.1)$$

$\Leftarrow$ We consider $f(x)=|x-t|$ convex, then we have

$$\sum_{j=1}^n|y_j-t|\leq\sum_{j=1}^n|x_j-t|. \quad (3.9.2)$$

We note that

$$|y_j-r|=2(y_j-r)_+-(y_j-r). \quad (3.9.3)$$

Here t_+ is the positive part of t i.e. $t_+ = \max(t, 0)$. Then we have

$$\sum_{j=1}^{n} |y_j - r| = 2\sum_{j=1}^{n}(y_j - r)_+ - \sum_{j=1}^{n}(y_j - r) = 2\sum_{j=1}^{n}(y_j - r)_+ - \sum_{j=1}^{n} y_j + nr. \tag{3.9.4}$$

We take r large enough ($r \geq x_1$), then

$$(y_j - r)_+ = (x_j - r)_+ = 0, \quad \forall j = 1, \cdots, n. \tag{3.9.5}$$

Therefore

$$-\sum_{j=1}^{n} y_j + nr \leq -\sum_{j=1}^{n} x_j + nr \Rightarrow \sum_{j=1}^{n} y_j \geq \sum_{j=1}^{n} x_j. \tag{3.9.6}$$

Likewise, we can take r small enough ($r \leq \min\{x_j, y_j : j = 1, \cdots, n\}$), then we have

$$\sum_{j=1}^{n} y_j \leq \sum_{j=1}^{n} x_j. \tag{3.9.7}$$

Combining the two inequalities, we have $\sum_{j=1}^n y_j = \sum_{j=1}^n x_j$. Thus we have

$$\begin{aligned}\sum_{j=1}^{n}(y_j - r)_+ &= \frac{1}{2}\left(\sum_{j=1}^{n}|y_j - r| + \sum_{j=1}^{n}(y_j - r)\right)\\ &= \frac{1}{2}\left(\sum_{j=1}^{n}|y_j - r| + \sum_{j=1}^{n}(x_j - r)\right) \leq \frac{1}{2}\left(\sum_{j=1}^{n}|x_j - r| + \sum_{j=1}^{n}(x_j - r)\right) = \sum_{j=1}^{n}(x_j - r)_+.\end{aligned} \tag{3.9.8}$$

Next, we take $x_{k-1} \leq r \leq x_k$, then we have

$$\sum_{j=1}^{k} y_j - kr \leq \sum_{j=1}^{k}(y_j - r)_+ \leq \sum_{j=1}^{n}(y_j - r)_+ \leq \sum_{j=1}^{n}(x_j - r)_+ \overset{\text{by construction}}{=} \sum_{j=1}^{k} x_j - kr. \tag{3.9.9}$$

Eliminating kr from both sides, we have

$$\sum_{j=1}^{k} y_j \leq \sum_{j=1}^{k} x_j. \tag{3.9.10}$$

By the arbitrariness of k, we have $\sum_{j=1}^k y_j \leq \sum_{j=1}^k x_j$ for any $1 \leq k \leq n$, that is $y \prec x$ together with $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$. $\square$

We give the definition of weak majorization. In a nutshell, the difference between weak majorization and strong majorization is that the former does not require the equality condition $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$.

Definition 3.9.2 (Weak majorization)**.** *Let x, y be two real vectors ordered non-increasingly. We say that y weakly majorizes x, denoted by $y \prec_w x$, if $\sum_{j=1}^k y_j \leq \sum_{j=1}^k x_j$ for any $1 \leq k \leq n$.*

The following equivalence is a very important characterization of weak majorization. When we encounter weak majorization, we will use this equivalence to deal with it.

Proposition 51. *Let $x, y \in \mathbb{R}^n$, then $y \prec_w x$ iff there exists $\widetilde{x} \in \mathbb{R}^n$ such that*

$$y \leq \widetilde{x} \quad \text{and} \quad \widetilde{x} \prec x. \tag{3.9.11}$$

Proof. $\Rightarrow$ Let $y \prec_w x$. We prove by induction. For $n = 1$, $y_1 \leq x_1$, we only need to take $\widetilde{x}_1 = x_1$. Next we assume that the statement holds for any $1 \leq k \leq n-1$. We take

$$a = \min_{1 \leq k \leq n} \left\{ \sum_{j=1}^{k} x_j - \sum_{j=1}^{k} y_j \right\} \geq 0. \tag{3.9.12}$$

We consider

$$\begin{pmatrix} y_1 + a \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{3.9.13}$$

Then we observe that $y_1 + a \geq y_2 \geq \cdots \geq y_n$, and by construction and $y \prec_w x$, we still have $\begin{pmatrix} y_1 + a \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \prec_w \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

By the minimum is achieved, we have $\exists k_0$, such that $\sum_{j=1}^{k_0} y_j + a = \sum_{j=1}^{k_0} x_j$. That is,

$$\begin{pmatrix} y_1 + a \\ y_2 \\ \vdots \\ y_{k_0} \end{pmatrix} \prec \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k_0} \end{pmatrix}. \tag{3.9.14}$$

If $k_0 = n$, we can just take $\widetilde{x} = \begin{pmatrix} y_1 + a \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$. If $k_0 \neq n$, then we use the induction hypothesis to the remaining part $\begin{pmatrix} y_{k_0+1} \\ \vdots \\ y_n \end{pmatrix}$ to get the required $\widetilde{x}$. $\square$

Definition 3.9.3 (Doubly substochastic matrix)**.** *$S \in M_n(\mathbb{R})$ is called doubly substochastic matrix if S is a non-negative matrix and $\sum_{j=1}^{n} S_{jk} \leq 1,\ \sum_{k=1}^{n} S_{jk} \leq 1$ for any $1 \leq k \leq n$.*

Proposition 52. *$x, y \geq 0$, then $y \prec_w x$ iff $\exists$ a doubly substochastic matrix S such that $y = Sx$.*

Proof. $\Rightarrow$ By $y \prec_w x$ and Proposition 51, there exists $\widetilde{x}$ such that $y \leq \widetilde{x} \prec x$. Therefore, there exists a doubly stochastic matrix S_0 such that $\widetilde{x} = S_0 x$. Since $x, y \geq 0$, we have $\widetilde{x}_j \neq 0$, thus we can take $a_j = \frac{y_j}{\widetilde{x}_j}, 0 \leq a_j \leq 1$ and $S = \mathrm{diag}(a_1, \cdots, a_n) S_0$. Then we have $y = Sx$ and S is doubly substochastic.
$\Leftarrow$ is by iteratively adjust S to obtain a doubly stochastic matrix S_0. We omit the details here. $\square$

Theorem 3.9.4 (Karamata's inequality)**.** *Let $x, y \in \mathbb{R}^n$, $y \prec_w x$, then for any convex increasing function f we have*

$$\sum_{j=1}^{n} f(y_j) \leq \sum_{j=1}^{n} f(x_j). \tag{3.9.15}$$

Proof. $\Rightarrow$ By Proposition 51, we have $y \leq \widetilde{x} \prec x$. By the convexity of f and Karamata's inequality, we have

$$\sum_{j=1}^{n} f(\widetilde{x}_j) \leq \sum_{j=1}^{n} f(x_j). \tag{3.9.16}$$

Then by the monotonicity of f, we have

$$\sum_{j=1}^{n} f(y_j) \leq \sum_{j=1}^{n} f(\widetilde{x}_j) \leq \sum_{j=1}^{n} f(x_j). \tag{3.9.17}$$

$\Leftarrow$ We take $f(t) = (t - r)_+$, by f is convex and increasing, we have

$$\sum_{j=1}^{n} (y_j - r)_+ \leq \sum_{j=1}^{n} (x_j - r)_+. \tag{3.9.18}$$

We take $x_{k-1} \leq r \leq x_k$, then we have

$$\sum_{j=1}^{k} y_j - kr \leq \sum_{j=1}^{k} (y_j - r)_+ \leq \sum_{j=1}^{n} (y_j - r)_+ \leq \sum_{j=1}^{n} (x_j - r)_+ = \sum_{j=1}^{k} x_j - kr. \tag{3.9.19}$$

The last equality follows from the construction of r. Therefore we have

$$\sum_{j=1}^{k} y_j \leq \sum_{j=1}^{k} x_j, \quad \forall 1 \leq k \leq n. \tag{3.9.20}$$

$\square$

We next define the logarithmic majorization.

Definition 3.9.5 (Logarithmic majorization)**.** *Let $x, y \geq 0 \in \mathbb{R}^n$ ordered non-increasingly, we say that y weakly logarithmically majorizes x, denoted by $y \prec_{w \log} x$, if*

$$\prod_{j=1}^{k} y_j \leq \prod_{j=1}^{k} x_j, \quad \forall 1 \leq k \leq n. \tag{3.9.21}$$

If we additionally require $\sum_{j=1}^{n} y_j = \sum_{j=1}^{n} x_j$, we say that y logarithmically majorizes x, denoted by $y \prec_{\log} x$ (this conception is less frequently used).

Remark 50. *If $x, y > 0$, then $y \prec_{w \log} x$ iff $\log y \prec_w \log x$. Here $\log x, \log y \in \mathbb{R}^n$ is calculated component-wise.*

Proposition 53. *Let $x, y > 0$ and $y \prec_{w\log} x$, then for any function $f : [0,\infty) \to \mathbb{R}$ with $t \mapsto f(e^t)$ convex, we have*

$$f(y) \prec_w f(x). \tag{3.9.22}$$

Here, $f(x)$ and $f(y)$ are calculated component-wise.

Proof. By imitating the proof of Theorem 3.9.4. □

We will see that the Weyl majorization theorem represents a quite important phenomenon of log-majorization. To prove this result, our strategy is to realise the product of eigenvalues via a constructive way. That is to consider the eigenvalue of the operators acting on an antisymmetric tensor product space.

Definition 3.9.6. *$v_1, \cdots, v_k \in \mathbb{C}^n$, we define*

$$v_1 \wedge \cdots \wedge v_k := \frac{1}{\sqrt{k!}} \sum_{\sigma \in \mathfrak{S}_k} \mathit{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in \bigwedge^k \mathbb{C}^n. \tag{3.9.23}$$

Here, $\bigwedge^k \mathbb{C}^n = \mathit{Span}\{k\text{-order antisymmetric tensors}\}$, $\mathit{dim}(\bigwedge^k \mathbb{C}^n) = \binom{n}{k}$. Moreover,

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_j, w_\ell \rangle)_{j,\ell=1}^k. \tag{3.9.24}$$

We can define the orthogonal projection operator $P_\wedge : \bigotimes^k \mathbb{C}^n \to \bigwedge^k \mathbb{C}^n$ as follows:

$$P_\wedge(v_1 \otimes \cdots \otimes v_k) = \frac{1}{\sqrt{k!}} v_1 \wedge \cdots \wedge v_k, \quad P_\wedge^2 = P_\wedge = P_\wedge^*. \tag{3.9.25}$$

We define an operator $A^{\wedge k}$ by its action:

$$A^{\wedge k}(v_1 \wedge \cdots \wedge v_k) = (Av_1) \wedge \cdots \wedge (Av_k). \tag{3.9.26}$$

Then it is easy to verify that

$$P_\wedge A^{\wedge k} P_\wedge = A^{\wedge k} P_\wedge : \bigotimes^k \mathbb{C}^n \to \bigwedge^k \mathbb{C}^n, \tag{3.9.27}$$

$$(A^{\wedge k})^* = (A^*)^{\wedge k}, \quad A^{\wedge k} B^{\wedge k} = (AB)^{\wedge k}, \quad \left|A^{\wedge k}\right| = |A|^{\wedge k}. \tag{3.9.28}$$

Lemma 10. *Let $A \in M_n(\mathbb{C})$, then we have*

$$\left\|A^{\wedge k}\right\| = \prod_{j=1}^k \lambda_j(|A|), \quad \forall 1 \le k \le n. \tag{3.9.29}$$

Remark 51. *This naturally gives rise to the logarithmic majorization between the eigenvalues of A and $|A|$. In some sense, $\left\|A^{\wedge k}\right\|$ is the multiplicative version of trace.*

Proof. Let v_j be the eigenvector of $|A|$ corresponding to $\lambda_j(|A|)$, then we have

$$\left|A^{\wedge k}\right|(v_{j_1} \wedge \cdots \wedge v_{j_k}) = \prod_{i=1}^k \lambda_{j_i}(|A|) v_{j_1} \wedge \cdots \wedge v_{j_k}. \tag{3.9.30}$$

Thus, under the basis $\{v_{j_1} \wedge \cdots \wedge v_{j_k}\}$, $|A|^{\wedge k}$ is diagonalized and

$$\left\| |A|^{\wedge k} \right\| = \left\| \text{diag}\left(\prod_{i=1}^{k} \lambda_{j_i}(|A|) \right) \right\| \Rightarrow \sup \prod_{i=1}^{k} \lambda_{j_i}(|A|) = \prod_{j=1}^{k} \lambda_j(|A|). \tag{3.9.31}$$

Thus we have

$$\left\| A^{\wedge k} \right\| = \left\| |A^{\wedge k}| \right\| = \left\| |A|^{\wedge k} \right\| = \prod_{j=1}^{k} \lambda_j(|A|). \tag{3.9.32}$$

$\square$

Theorem 3.9.7 (Weyl's majorization theorem)**.** *Let $A \in M_n(\mathbb{C})$, then we have*

$$\prod_{j=1}^{k} |\lambda_j(A)| \leq \prod_{j=1}^{k} \lambda_j(|A|), \quad \forall 1 \leq k \leq n. \tag{3.9.33}$$

Proof. Let λ be an eigenvalue of A with algebraic multiplicity m_λ, then we have there exists a "cyclic basis" $\{x_1, \cdots, x_{m_\lambda}\}$ such that $Ax_j - \lambda x_j \in \text{Span}(x_1, \cdots, x_{j-1})$ (One may understand this by thinking of the Jordan form). In particular, we have a linearly indepedent set $\{v_1, \cdots, v_n\}$ such that

$$Av_j - \lambda_j(A)v_j \in \text{Span}(v_1, \cdots, v_{j-1}), \quad \forall 1 \leq j \leq n. \tag{3.9.34}$$

Note that the antisymmetric tensor vanishes whenever two of the vectors coincide, thus $w := Av_j - \lambda_j(A)v_j$ does not contribute to the action of $A^{\wedge k}$ on the antisymmetric tensor product space. That is,

$$A^{\wedge k}(v_1 \wedge \cdots \wedge v_k) = (Av_1) \wedge \cdots \wedge (Av_k) = \prod_{j=1}^{k} \lambda_j(A)(v_1 \wedge \cdots \wedge v_k). \tag{3.9.35}$$

Thus, by Lemma 10, we have

$$\prod_{j=1}^{k} |\lambda_j(A)| \leq \left\| A^{\wedge k} \right\| = \prod_{j=1}^{k} \lambda_j(|A|). \tag{3.9.36}$$

$\square$

Next, we can immediately apply the results of log-majorization to the Weyl majorization theorem. This will give us the generalization of the Weyl inequality Theorem 3.6.6 and the stronger Golden-Thompson inequality.

Proposition 54. *Let $A \in M_n(\mathbb{C})$, f is a increasing function on $[0, \infty)$ with $t \mapsto f(e^t)$ convex, then we have*

$$\begin{pmatrix} f(|\lambda_1(A)|) \\ \vdots \\ f(|\lambda_n(A)|) \end{pmatrix} \prec_w \begin{pmatrix} f(\lambda_1(|A|)) \\ \vdots \\ f(\lambda_n(|A|)) \end{pmatrix}. \tag{3.9.37}$$

Corollary 8. *We take $f(t) = t^\alpha$ for $\alpha \geq 1$, then we have*

$$\sum_{j=1}^{k} |\lambda_j(A)|^\alpha \leq \sum_{j=1}^{k} \lambda_j(|A|)^\alpha, \quad \forall 1 \leq k \leq n. \tag{3.9.38}$$

Remark 52. *This is a much stronger and more essential result than the original Weyl inequality Theorem 3.6.6. However, this is only directly applicable in the finite-dimensional case.*

Theorem 3.9.8 (Golden-Thompson inequality, formal version of Theorem 3.8.3)**.** *Let $A, B \in M_n(\mathbb{C})$, f is increasing and $f(e^t)$ is convex, then*

$$f(e^{A+B}) \prec_w f\left(e^{\frac{B+B^*}{4}} e^{\frac{A+A^*}{2}} e^{\frac{B+B^*}{4}}\right). \tag{3.9.39}$$

Remark 53. *By taking $f = Id$, we can obtain the original trace version of the Golden-Thompson inequality.*

3.10 Araki-Lieb-Thirring inequality

We extend the weak majorization results of Weyl to a "parametrized case". That is, we consider the trace inequality of the following form

$$\mathrm{Tr}\Big(B^{\frac{1}{2}} A B^{\frac{1}{2}}\Big) \leq \mathrm{Tr}\Big((B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\frac{1}{s}}\Big). \tag{3.10.1}$$

Theorem 3.10.1 (Furata inequality)**.** *Let $A, B \geq 0$, $0 \leq s \leq 1$, then we have*

$$\|A^s B^s\| \leq \|AB\|^s. \tag{3.10.2}$$

Proof. We denote

$$\Lambda = \{s : \|A^s B^s\| \leq \|AB\|^s\}, \tag{3.10.3}$$

we need to show that Λ is a convex set. It is easy to see that $0, 1 \in \Lambda$. We take $s, t \in \Lambda$, then we want to show that $\frac{s+t}{2} \in \Lambda$ i.e. $\left\|A^{\frac{s+t}{2}} B^{\frac{s+t}{2}}\right\|^2 \leq \|AB\|^{s+t}$. Therefore,

$$\begin{aligned} \text{LHS} &= \left\|B^{\frac{s+t}{2}} A^{\frac{s+t}{2}} B^{\frac{s+t}{2}}\right\| = r(B^{\frac{s+t}{2}} A^{s+t} B^{\frac{s+t}{2}}) = r(A^{s+t} B^{s+t}) \\ &= r(B^s A^s A^t B^t) \leq \|B^s A^s\| \left\|B^t A^t\right\| = \|AB\|^{s+t} = \text{RHS}. \end{aligned} \tag{3.10.4}$$

Therefore, Λ is a convex set i.e. $\Lambda = [0, 1]$. □

To prove the Araki-Lieb-Thirring inequality, we first need to prove the following key lemma. Then the Araki-Lieb-Thirring inequality follows readily via connection bewteen weak logarithmic majorization and the weak majorization (see Proposition 53).

Lemma 11. *$A, B \geq 0$, $s \geq 1$, then*

$$B^{\frac{1}{2}} A B^{\frac{1}{2}} \prec_{w \log} (B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\frac{1}{s}}. \tag{3.10.5}$$

The main idea of the proof of this lemma is again using the structure of wedge product operators (like in the proof of Weyl's majorization theorem).

Proof. We let $Y_s = B^{\frac{s}{2}} A^s B^{\frac{s}{2}}$, then we need to show that

$$\prod_{j=1}^{k} \lambda_j(B^{\frac{1}{2}} A B^{\frac{1}{2}}) \le \prod_{j=1}^{k} \lambda_j(Y_s^{\frac{1}{s}}), \quad \forall 1 \le k \le n. \tag{3.10.6}$$

Note that $\lambda(Y_s^{\frac{1}{s}})^s = \lambda(Y_s)$, we consider the product of distinct eigenvalues of Y_s. We consider the k-wedge product of Y_s

$$\begin{aligned}\prod_{j=1}^{k} \lambda_j(Y_s) &= \left\|Y_s^{\wedge k}\right\| = \left\|\left[(A^{\frac{s}{2}} B^{\frac{s}{2}})^* (A^{\frac{s}{2}} B^{\frac{s}{2}})\right]^{\wedge k}\right\| = \left\|\left[(A^{\frac{s}{2}} B^{\frac{s}{2}})^{\wedge k}\right]^* \left[(A^{\frac{s}{2}} B^{\frac{s}{2}})^{\wedge k}\right]\right\| \\ &= \left\|(A^{\frac{s}{2}} B^{\frac{s}{2}})^{\wedge k}\right\|^2 = \left\|(A^{\frac{s}{2}})^{\wedge k} (B^{\frac{s}{2}})^{\wedge k}\right\|^2 = \left\|(A^{\wedge k})^{\frac{s}{2}} (B^{\wedge k})^{\frac{s}{2}}\right\|^2 \\ &\overset{\text{Furata inequality}}{\ge} \left\|(A^{\wedge k})^{\frac{s}{2}} (B^{\wedge k})^{\frac{s}{2}}\right\|^2 = \left\|(A^{\wedge k})^{\frac{1}{2}} (B^{\wedge k})^{\frac{1}{2}}\right\|^{2s} = \left\|Y_1^{\wedge k}\right\|^s \Rightarrow \left\|Y_s^{\wedge k}\right\|^{\frac{1}{s}} \ge \left\|Y_1^{\wedge k}\right\|.\end{aligned} \tag{3.10.7}$$

Therefore,

$$\prod_{j=1}^{k} (\lambda_j(Y_s))^{\frac{1}{s}} \ge \prod_{j=1}^{k} \lambda_j(Y_1), \quad \forall k. \tag{3.10.8}$$

We note that $Y_1 = B^{\frac{1}{2}} A B^{\frac{1}{2}}$, thus we have

$$B^{\frac{1}{2}} A B^{\frac{1}{2}} \prec_{w \log} \left(B^{\frac{s}{2}} A^s B^{\frac{s}{2}}\right)^{\frac{1}{s}}. \tag{3.10.9}$$

□

Theorem 3.10.2 (Araki-Lieb-Thirring inequality)**.** *Let $A, B \ge 0$, f is increasing and $f(e^t)$ is convex, then we have*

$$f((B^{\frac{1}{2}} A B^{\frac{1}{2}})^s) \prec_w f(B^{\frac{s}{2}} A^s B^{\frac{s}{2}}). \tag{3.10.10}$$

Remark 54. *The more "popular" version of the Araki-Lieb-Thirring inequality is the trace version:*

$$\operatorname{Tr}\left(B^{\frac{1}{2}} A B^{\frac{1}{2}}\right) \le \operatorname{Tr}\left((B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\frac{1}{s}}\right). \tag{3.10.11}$$

As a remark of history, the original version of the Araki-Lieb-Thirring inequality (Lieb-Thirring inequality) relies on a different proof technique, which is based on the properties of analytic functions. The proof based on the construction of wedge product operators in fact provides more structural information.

Corollary 9. *Let $\alpha > 0$, $s \ge 1$, then we have*

$$(B^{\frac{1}{2}} A B^{\frac{1}{2}})^{\alpha s} \prec_w (B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\alpha}. \tag{3.10.12}$$

In particular,

$$\operatorname{Tr}\left[(B^{\frac{1}{2}} A B^{\frac{1}{2}})^{\alpha s}\right] \le \operatorname{Tr}\left[(B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\alpha}\right]. \tag{3.10.13}$$

That is, $s \mapsto \operatorname{Tr}\left(B^{\frac{s}{2}} A^s B^{\frac{s}{2}}\right)^{\frac{1}{s}}$ is increasing.

As an application, we show (1) a generalized Golden-Thompson inequality; (2) an entropic inequality.

Theorem 3.10.3 (Generalized Golden-Thompson inequality). *Let A, B be Hermitian matrices, then $s \mapsto \operatorname{Tr}\Big[(e^{\frac{sB}{2}}e^{sA}e^{\frac{sB}{2}})\Big]$ is increasing. In particular,*

$$\operatorname{Tr} e^{A+B} \leq \operatorname{Tr}\Big[(e^{\frac{sB}{2}}e^{sA}e^{\frac{sB}{2}})^{\frac{1}{s}}\Big]. \tag{3.10.14}$$

Remark 55. *In particular, if we take $s \to 0$, then we get the Lie-Trotter formula.*

Theorem 3.10.4. *Let $A, B > 0$, $s > 0$, then*

$$\frac{1}{s}\operatorname{Tr}\big(A\log B^{\frac{s}{2}}AB^{\frac{s}{2}}\big) \leq \operatorname{Tr}(A\log A + A\log B). \tag{3.10.15}$$

Or equivalently, we have

$$\frac{1}{s}\operatorname{Tr}\big(A\log B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}}\big) \leq H(A\|B). \tag{3.10.16}$$

Proof. Without loss of generality, we let $\operatorname{Tr} A = 1$. By the Gibbs variational formula, we have

$$H(A\|e^D) \geq \operatorname{Tr}(AX) - \log\operatorname{Tr} e^{X+D} \overset{\text{Golden-Thompson}}{\geq} \operatorname{Tr}(AX) - \log\operatorname{Tr}\Big((e^{\frac{sD}{2}}e^{sX}e^{\frac{sD}{2}})^{\frac{1}{s}}\Big) \tag{3.10.17}$$

Let $X = \frac{1}{s}\log\big(e^{-sD/2}e^{sX}e^{-sD/2}\big)$. Then we have

$$H(A\|e^D) \geq \frac{1}{s}\operatorname{Tr}\big(A\log\big(e^{-sD/2}e^{sX}e^{-sD/2}\big)\big). \tag{3.10.18}$$

Then the required result follows by taking $D = -\log B$. $\square$

3.11 The convexity of some entropy functionals

The material in this section is mainly based on the work of Carlen and Lieb [CL08]. Specifically, we care about the following types of functionals:

$$(A_1, \cdots, A_m) \mapsto \left\|\left(\sum_{j=1}^m A_j^p\right)^{\frac{1}{p}}\right\|_q; \tag{3.11.1}$$

and

$$\Gamma_{p,q}(A) := \operatorname{Tr}\Big((B^*A^pB)^{\frac{q}{p}}\Big). \tag{3.11.2}$$

Lemma 12. *If $1 \leq p \leq 2$, and $q \geq p$, $\Gamma_{p,q}$ is convex function.*

Proof. Since $A \mapsto A^p$ is operator convex, so is $A \mapsto B^*A^pB$. Moreover, if we let $r := q/p \geq 1$, then by the variational formula of r-norm, we have

$$\|B^*A^pB\|_r = \sup_{\|Y\|_{r'}\leq 1, Y\geq 0} \operatorname{Tr}(B^*A^pBY) \tag{3.11.3}$$

Thus, as the supremum of a family of convex functions, $\Gamma_{p,q}$ is convex. $\square$

Lemma 13 (Young's inequality). *If $\alpha+\beta=1$, then $\alpha x^{\frac{1}{\alpha}}+\beta y^{\frac{1}{\beta}}\geq x^\alpha y^\beta$ for any $x,y\geq 0$. Moreover, if $r>1$, $a,b>0$, then we have*

$$\frac{1}{r}a^r+\frac{r-1}{r}b^r\geq ab^{r-1}. \tag{3.11.4}$$

Proposition 55. *For $r>1$, we have the following key observations:*

$$\operatorname{Tr}\Big((A^*A)^{\frac{1}{r}}\Big)=\frac{1}{r}\inf\{\operatorname{Tr}\big(A^*X^{1-r}A\big)+(r-1)\operatorname{Tr}X:X>0\}. \tag{3.11.5}$$

Similarly, for $r<1$,

$$\operatorname{Tr}\Big((A^*A)^{\frac{1}{r}}\Big)=\frac{1}{r}\sup\{\operatorname{Tr}\big(A^*X^{1-r}A\big)+(r-1)\operatorname{Tr}X:X>0\}. \tag{3.11.6}$$

Proof. By eq. (3.11.4) and the previous result Theorem 3.4.3 for Klein inequality, we have

$$\operatorname{Tr}\Big((AA^*)^{\frac{1}{r}}\Big)\leq\frac{1}{r}\operatorname{Tr}\big(A^*X^{1-r}A\big)+\frac{r-1}{r}\operatorname{Tr}X. \tag{3.11.7}$$

□

Proposition 56.

$$\Gamma_{p,q}(A)=\frac{q}{p}\inf_X\left\{\operatorname{Tr}\Big(A^{\frac{p}{2}}BX^{1-\frac{p}{q}}B^*A^{\frac{p}{2}}\Big)+\left(\frac{p}{q}-1\right)X:X>0\right\}\quad(p>q); \tag{3.11.8}$$

$$\Gamma_{p,q}(A)=\frac{q}{p}\sup_X\left\{\operatorname{Tr}\Big(A^{\frac{p}{2}}BX^{1-\frac{p}{q}}B^*A^{\frac{p}{2}}\Big)+\left(\frac{p}{q}-1\right)X:X>0\right\}\quad(p<q). \tag{3.11.9}$$

Proof. It follows readily from Proposition 55. □

Lemma 14. *Let $f(x,y)$ be a jointly convex/concave function on $I_1\times I_2$, then $g(x)=\inf_{y\in I_2}f(x,y)$ is convex/concave on I_1.*

Proof. Take $x_1,x_2\in I_1$ and $\lambda\in(0,1)$. For any $\varepsilon>0$, there exists $y_1,y_2\in I_2$ such that $f(x_1,y_1)\leq g(x_1)+\varepsilon$, $f(x_2,y_2)\leq g(x_2)+\varepsilon$. Then we have

$$\begin{aligned}g(\lambda x_1+(1-\lambda)x_2)&\overset{\text{by defn.}}{\leq}f(\lambda x_1+(1-\lambda)x_2,\lambda y_1+(1-\lambda)y_2)\\&\overset{\text{by joint convexity}}{\leq}\lambda f(x_1,y_1)+(1-\lambda)f(x_2,y_2)\\&\leq\lambda g(x_1)+(1-\lambda)g(x_2)+\varepsilon.\end{aligned} \tag{3.11.10}$$

Taking $\varepsilon\to 0$ on both sides yields the desired result readily. □

Remark 56. *This means that the convexity or concavity is somewhat "stable" under taking infimum or supremum. That is why the variational formula technique as we have established in Proposition 55 and Proposition 56 is useful in proving the convexity of these entropy functionals. In the next reading section section 3.12, we will revisit this strategy again.*

Theorem 3.11.1. (1) *If $1\leq p\leq 2$, $q\geq 1$, then $\Gamma_{p,q}$ is convex;*

(2) *If $0<p\leq q\leq 1$, then $\Gamma_{p,q}$ is concave;*

(3) *If $p > 2$ and $p \neq 1$, then $\Gamma_{p,q}$ is neither convex nor concave.*

Proof. **For (1)**, by Lieb's concavity Proposition 50 (see also example 12), $(A, X) \mapsto \mathrm{Tr}\Big(BX^{1-\frac{p}{q}}B^*A^p\Big)$ is jointly convex for $1 \le p \le 2$, $-1 \le 1 - \frac{p}{q} \le 0$ and $p + 1 - \frac{p}{q} \ge 1$. By Lemma 14, we have $\Gamma_{p,q}(A)$ is convex.

For (2), if $0 < p \le q \le 1$, $0 < 1 - \frac{p}{q} \le 1$, wev have $0 \le 1 - \frac{p}{q} + p \le 1$. By Lieb's concavity, we have $(A, X) \mapsto \mathrm{Tr}\Big(BX^{1-\frac{p}{q}}B^*A^p\Big)$ is jointly concave. By Lemma 14, we have $\Gamma_{p,q}(A)$ is concave. If $0 < q \le p \le 1$, then the result is trivial by $A \mapsto A^p$ is concave and $A \mapsto A^{\frac{q}{p}}$ is concave. **For (3)**, the proof is based on Taylor expansion. We let

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} tX & \\ & Y \end{pmatrix}. \tag{3.11.11}$$

Then direct calculation shows that

$$\Gamma_{p,q}(A) = \mathrm{Tr}\Big((t^pX^p + Y^p)^{\frac{q}{p}}\Big). \tag{3.11.12}$$

By Taylor expansion, we have

$$\Gamma_{p,q}(A) = \mathrm{Tr}(Y^q) + \frac{q}{p}\mathrm{Tr}\big(Y^{q-p}X^p\big)t^p + \mathcal{O}(t^{2p}). \tag{3.11.13}$$

Note that we have taken the advantage of expanding w.r.t. t^p and the commutativity of the trace. We replace X with A_1, A_2 and $\frac{A_1+A_2}{2}$, we find

$$\begin{aligned} &\frac{1}{2}\Gamma_{p,q}(A_1) + \frac{1}{2}\Gamma_{p,q}(A_2) - \Gamma_{p,q}\left(\frac{A_1 + A_2}{2}\right) \\ &= \frac{q}{p}\left(\frac{1}{2}\mathrm{Tr}\big(Y^{q-p}A_1^p\big) + \frac{1}{2}\mathrm{Tr}\big(Y^{q-p}A_2^p\big) - \mathrm{Tr}\left[Y^{q-p}\left(\frac{A_1+A_2}{2}\right)^p\right]t^p\right) + \mathcal{O}(t^{2p}). \end{aligned} \tag{3.11.14}$$

That is,

$$\begin{aligned} &\frac{1}{2}\mathrm{Tr}\Big[(t^pA_1^p + Y^p)^{\frac{q}{p}}\Big] + \frac{1}{2}\mathrm{Tr}\Big[(t^pA_2^p + Y^p)^{\frac{q}{p}}\Big] - \mathrm{Tr}\left[\left(t^p\frac{A_1 + A_2}{2}^p + Y^p\right)^{\frac{q}{p}}\right] \\ &= \frac{q}{p}\left(\frac{1}{2}\mathrm{Tr}\big(Y^{q-p}A_1^p\big) + \frac{1}{2}\mathrm{Tr}\big(Y^{q-p}A_2^p\big) - \mathrm{Tr}\left[Y^{q-p}\left(\frac{A_1+A_2}{2}\right)^p\right]t^p\right) + \mathcal{O}(t^{2p}). \end{aligned} \tag{3.11.15}$$

Since $A \mapsto A^p$ is *not* operator convex, $p > 2$, thus there exists $v \in \mathbb{C}^n$, $\|v\| = 1$, such that

$$\frac{1}{2}\langle A_1^p v, v\rangle + \frac{1}{2}\langle A_2^p v, v\rangle - \langle\left(\frac{A_1 + A_2}{2}\right)^p v, v\rangle < 0. \tag{3.11.16}$$

However, take Y to be a projection to $\mathbb{C}v$, then the right hand side of eq. (3.11.15) is nothing other than the inner product terms in eq. (3.11.16). This shows that the left hand side of eq. (3.11.15) is negative, which implies that $\Gamma_{p,q}$ cannot be convex for such p and q. Similarly we can show that $\Gamma_{p,q}$ cannot be concave.

∎

Lemma 15. *$\Gamma_{p,q}$ and $\Gamma_{p,q}^{\frac{1}{q}}$ have the same convexity properties.*

Proof. The proof is based on the fact that, when f is homogeneous with degree 1, then f is convex if and only if $\{x \in \mathrm{Dom}(f) : f(x) \leq 1\}$ is a convex set. □

Corollary 10. $(A_1, \cdots, A_m) \mapsto \left\| \left(\sum_{j=1}^m A_j^p \right)^{\frac{1}{p}} \right\|_q$ *is jointly convex for $1 \leq p \leq 2$, $q \geq 1$; and is jointly concave for $0 < p, q \leq 1$.*

Proof. We consider $\mathrm{diag}(A_1, \cdots, A_m)$ and use Theorem 3.11.1. □

Remark 57 (Remark of history). *Many lines of investigation of such types of functional inequalities can be traced back to, again, the WYK hypothesis, which drew attention to the concavity of $A \mapsto \mathrm{Tr}(A^p X A^{1-p} X^*)$. The physical motivation is: some states are easier to measure than others; if a density matrix ρ commutes with a conserved quantity (say the energy) then it is easy to measure, and otherwise not. Thus, while the von Neuman entropy of any pure state ρ is zero, some pure states have a higher information content than others – namely those that are not functions of the conserved quantities, such as the Wigner–Yanase skew information $I(\rho) = \mathrm{Tr}\, X^2 \rho - \mathrm{Tr}\, \rho^{\frac{1}{2}} K \rho^{\frac{1}{2}} K$.*

3.12 Reading: How far can we go with Lieb's concavity and Ando's convexity?

We consider a very important family of entropy functionals: the $\alpha - z$ Rényi entropy. We define the $\alpha - z$ Rényi entropy as

$$H_{\alpha,z}(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}} \right)^z, \quad |\alpha| > 1, z > 0. \tag{3.12.1}$$

In particular, we define

- $z = 1$: α-Rényi entropy $H_\alpha(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}(\sigma^{1-\alpha} \rho^\alpha)$;
- $z = \alpha$: sandwiched α-Rényi entropy $\widetilde{H}_\alpha(\rho \| \sigma) = \frac{1}{\alpha-1} \mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{\alpha}{\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha$.

In fact there are many other types of entropy functionals with quite complicated connections among them (see fig. 3.1).

We are interested in the "monotonicity" or data-processing inequality of the relative entropy functional. That is, for any quantum channel Φ, whether we have

$$H_{\alpha,z}(\Phi(\rho) \| \Phi(\sigma)) \leq H_{\alpha,z}(\rho \| \sigma), \quad \forall \rho, \sigma \in \mathcal{D}(\mathcal{H}) \text{ being density matrices.} \tag{3.12.2}$$

The standard argument shows that it is essentially equivalent to some Lieb/Ando-type convexity. Specifically,

Example 13. *Set*

$$\Psi(A, B) = \mathrm{Tr} \left(B^{\frac{q}{2}} A^p B^{\frac{q}{2}} \right)^{\frac{1}{p+q}}, \quad A, B \in \mathbb{H}_n^{>0}, \tag{3.12.3}$$

with

$$p := \frac{\alpha}{z}, \quad q := \frac{1-\alpha}{z} \tag{3.12.4}$$

Then the DPI holds for the $\alpha - z$ Rényi entropy $H_{\alpha,z}$ if and only if one of the following holds

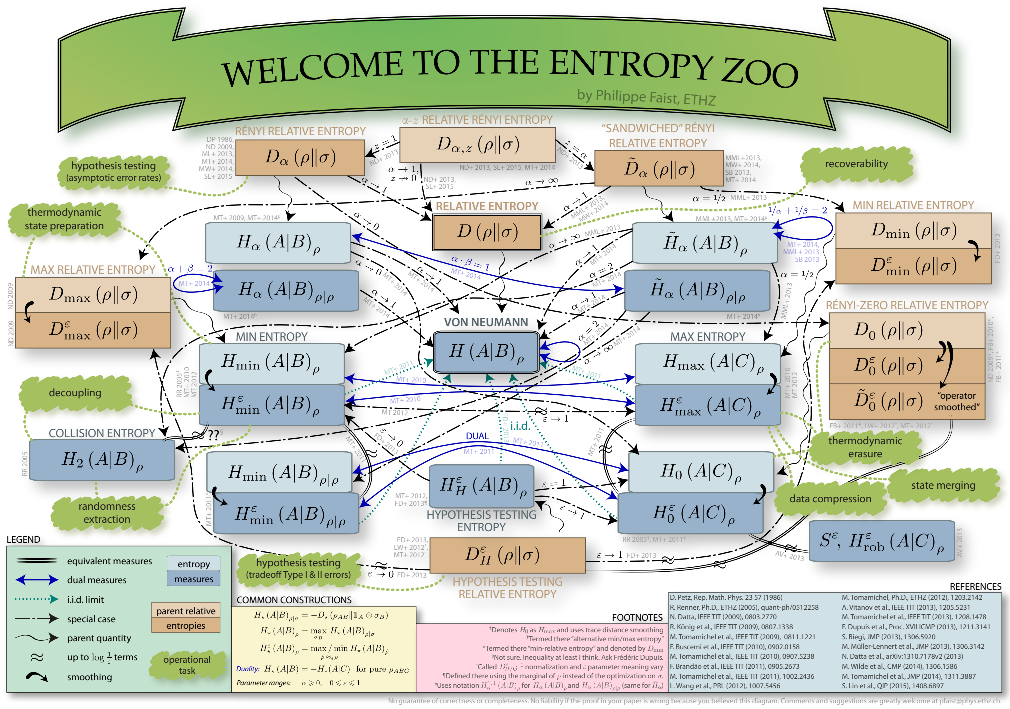

Figure 3.1: *Welcome to the entropy zoo by Philippe Faist (ETH-Zürich)*

- $\alpha < 1$ *and* Ψ *is jointly concave;*
- $\alpha > 1$ *and* Ψ *is jointly convex.*

This is one of the very important motivation of the study of Lieb/Ando type convexity properties. Before [Zha20], the known results for data-processing inequalities are summarized in fig. 3.2

The Carlen-Frank-Lieb conjecture is to ask whether we have the necessary and sufficient conditions.

$$\Psi_{p,q,s}(A,B) = \operatorname{Tr}\left(B^{\frac{q}{2}} X^* A^p X B^{\frac{q}{2}}\right)^s = \operatorname{Tr}\left|A^{\frac{p}{2}} X B^{\frac{q}{2}}\right|^{2s}, \tag{3.12.5}$$

is jointly convex/concave for all $X \in \mathbb{H}_n^{>0}$. The main result in [Zha20] is

- $\Psi_{p,q,s}$ is jointly concave iff $0 \le s \le \frac{1}{p+q}$, $0 \le p, q \le 1$;
- $\Psi_{p,q,s}$ is jointly convex iff $s \ge \frac{1}{p+q}$, $1 \le p \le 2$, $-1 \le q \le 0$ or $s \ge \frac{1}{p+q}$, $1 \le q \le 2$, $-1 \le p \le 0$ or $s \ge 0$, $-1 \le p, q \le 0$.

The building blocks are very fundamental:

1. **A special case of Lieb concavity** For any $0 < p < 1$, $\Psi_{p,1-p,1}(A,B) = \operatorname{Tr}(X^* A^p X B^{1-p})$ is jointly concave if $X \in \mathbb{H}_n^{>0}$;
2. **A special case of Ando convexity** For any $-1 < p < 0$, $\Psi_{p,1-p,1}(A,B)$ is jointly convex if $X \in \mathbb{H}_n^{>0}$;
3. **A quite well-known fact** For any $t > 0$, $A \mapsto \operatorname{Tr} A^{-t}$ is convex.

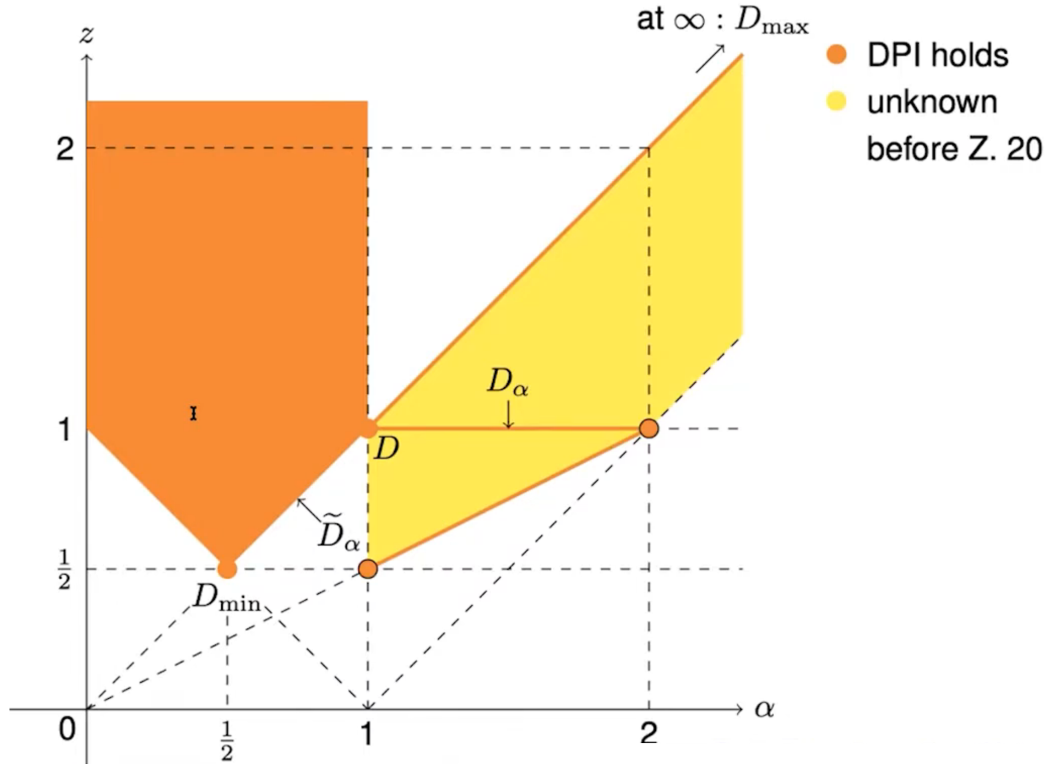

Figure 3.2: Known results of DPI for $H_{\alpha,z}$ before [Zha20]

Besides the building blocks listed above, the key technique to prove the results is actually the variational formula approach in section 3.11 just like the proof of the results of Carlen and Lieb [CL08]. We begin with a "toy example" for the purpose of warm-up.

Example 14. *Let $\psi_{\alpha,\beta}(x,y) = x^\alpha y^\beta$, show that it is jointly concave for $0 < \alpha, \beta < 1$ and $\alpha + \beta \le 1$.*

Lemma 16. *If $f(\cdot, y)$ is concave for each y, then $\min_y f(\cdot, y)$ is concave.*

Lemma 17. *By Young's inequality, we have*

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.12.6}$$

By taking $(a,b) \mapsto (ac, bc^{-1})$, we have for any $c > 0$,

$$ab = ac \cdot bc^{-1} \le \frac{1}{p}(ac)^p + \frac{1}{q}(bc^{-1})^q, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.12.7}$$

Then we have the variational formula for the product ab

$$ab = \min_{c>0} \left\{ \frac{1}{p}(ac)^p + \frac{1}{q}(bc^{-1})^q \right\}. \tag{3.12.8}$$

This gives readily

$$x^\alpha y^\beta = \min_{z>0} \left\{ \frac{\alpha}{\alpha+\beta}(xz^{\frac{1}{\alpha}})^{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(yz^{-\frac{1}{\beta}})^{\alpha+\beta} \right\}, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1 \tag{3.12.9}$$

for $0 < \alpha, \beta < 1$ and $\alpha + \beta \le 1$. This shows that $\psi_{\alpha,\beta}$ is jointly concave.

Note that the p-functional also admits the Hölder inequality, thus the non-commutative version also holds. Thus we have

Lemma 18. *For $X, Y \in \mathbb{H}_n^{>0}$, $r_i > 0$, $\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}$, we have*

$$\operatorname{Tr}|XY|^{r_0} := \min_{Z\in\mathbb{H}_n^{>0}} \left\{ \frac{r_0}{r_1} \operatorname{Tr}|XZ|^{r_1} + \frac{r_0}{r_2} \operatorname{Tr}\left|YZ^{-1}\right|^{r_2} \right\}, \tag{3.12.10}$$

$$\operatorname{Tr}|XY|^{r_1} := \max_{Z\in\mathbb{H}_n^{>0}} \left\{ \frac{r_1}{r_0} \operatorname{Tr}|XZ|^{r_0} - \frac{r_1}{r_2} \operatorname{Tr}\left|Y^{-1}Z\right|^{r_2} \right\}. \tag{3.12.11}$$

Next we show how we can reduce the $p, q > 0$ case to the $p = 0$ or $q = 0$ case. This can be easily done by the following variational treatment:

$$\begin{aligned} \Psi_{p,q,s}(A, B) &= \min_{Z\in\mathbb{H}_n^{>0}} \left\{ \frac{p}{p+q} \operatorname{Tr}\left|A^{\frac{p}{2}}KZ\right|^{\frac{2\lambda}{p}} + \frac{q}{p+q} \operatorname{Tr}\left|Z^{-1}B^{\frac{q}{2}}\right|^{\frac{2\lambda}{q}} \right\} \\ &= \min_{Z\in\mathbb{H}_n^{>0}} \left\{ \frac{p}{p+q} \underbrace{\operatorname{Tr}(Z^*K^*AKZ)^{\frac{\lambda}{p}}}_{q=0 \text{ case}} + \frac{q}{p+q} \underbrace{\operatorname{Tr}\left(Z^{-1}B^qZ^{*-1}\right)^{\frac{\lambda}{q}}}_{p=0 \text{ case}} \right\} \end{aligned} \tag{3.12.12}$$

The $p = 0$ or $q = 0$ case is nothing but the special case of Lieb concavity (building block 1) abd the special case of Ando convexity (Note that they also in some sense follow from some types of variational formulae).

3.13 Exercise III

Exercise 17. *Suppose that $f : \operatorname{Dom}(f) \to \mathbb{R}$ is an increasing convex function with $f(0) \le 0$ and $A_1, \cdots, A_m \in \mathbb{H}_n$ with $\operatorname{Sp}(A_j) \subset \operatorname{Dom}(f)$ for $j = 1, \cdots, m$. Suppose that $V_1, \cdots, V_m \in M_n(\mathbb{C})$ with $\sum_{j=1}^m V_j^* V_j = I$, then we have $\exists$ a unitary matrix $U \in M_n(\mathbb{C})$ such that*

$$f\left(\sum_{j=1}^m V_j^* A_j V_j\right) \le U^* \left(\sum_{j=1}^n V_j^* f(A_j) V_j\right) U. \tag{3.13.1}$$

Proof. We let

$$\widetilde{X} = \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ V_m & 0 & \cdots & 0 \end{pmatrix} \in M_{mn}(\mathbb{C}), \quad \widetilde{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix} \in M_{mn}(\mathbb{C}). \tag{3.13.2}$$

Then by Proposition 35 we have

$$\lambda_k(f(\widetilde{X}^*\widetilde{A}\widetilde{X})) \le \lambda_k(\widetilde{X}^* f(\widetilde{A})\widetilde{X}), \quad \forall 1 \le k \le mn. \tag{3.13.3}$$

It is easy for us to calculate that

$$f(\widetilde{X}^*\widetilde{A}\widetilde{X}) = \begin{pmatrix} f\left(\sum_{j=1}^m V_j^* A_j V_j\right) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \widetilde{X}^* f(\widetilde{A})\widetilde{X} = \begin{pmatrix} \sum_{j=1}^m V_j^* f(A_j) V_j & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.13.4}$$

Therefore we have

$$\lambda_k \left[f\left(\sum_{j=1}^{m} V_j^* A_j V_j \right) \right] \leq \lambda_k \left[\sum_{j=1}^{m} V_j^* f(A_j) V_j \right], \quad \forall 1 \leq k \leq n. \tag{3.13.5}$$

Therefore, there exists some unitary matrix $U \in M_n(\mathbb{C})$ such that

$$f\left(\sum_{j=1}^{m} V_j^* A_j V_j \right) \leq U^* \left(\sum_{j=1}^{m} V_j^* f(A_j) V_j \right) U. \tag{3.13.6}$$

□

Exercise 18 (Hadamard inequality)**.** *Suppose that $A = (a_{jk})_{j,k=1}^n \in M_n(\mathbb{R})$ is a positive semi-definite matrix, then we have*

$$\det(A) \leq \prod_{j=1}^{n} a_{jj}, \quad \forall 1 \leq j \leq n. \tag{3.13.7}$$

Proof. Let $A = LL^*$ where L is a lower-triangular matrix, then

$$a_{jj} = \sum_{k=1}^{n} |L_{jk}|^2 \geq |L_{jj}|^2, \quad \forall 1 \leq j \leq n. \tag{3.13.8}$$

On the other hand

$$\det(A) = \det(LL^*) = |\det(L)|^2 = \left(\prod_{j=1}^{n} |L_{jj}| \right)^2 = \prod_{j=1}^{n} |L_{jj}|^2 \leq \prod_{j=1}^{n} a_{jj} \tag{3.13.9}$$

which is the desired result. □

Exercise 19. *Suppose $A, B \in M_n(\mathbb{C})$ are Hermitian matrices with $A \prec B$, f is an convex function, then*

$$\operatorname{Tr} f(A) \leq \operatorname{Tr} f(B). \tag{3.13.10}$$

Proof. Since $\lambda_j(A)$ is majorized by $\lambda_j(B)$ and f is convex, by the Karamata's inequality we have

$$\sum_{j=1}^{n} f(\lambda_j(A)) \leq \sum_{j=1}^{n} f(\lambda_j(B)). \tag{3.13.11}$$

That is $\operatorname{Tr} f(A) \leq \operatorname{Tr} f(B)$. □

Exercise 20. *$A, B \in \mathbb{H}_n^{>0}$, $t > 0$, then*

$$\frac{1}{t} \operatorname{Tr}\big(B - B^{1-t} A^t\big) \leq H(B||A) \leq \frac{1}{t} \operatorname{Tr}\big(B^{1+t} A^{-t} - B\big). \tag{3.13.12}$$

Proof. Let $g(t) = \operatorname{Tr}(B^{1+t} A^{-t} - B)$, then we have $g'(t) = \operatorname{Tr}(B^{1+t} A^{-t} \log B - B^{1+t} A^{-t} \log A)$, $g''(t) = \operatorname{Tr}(B^{1+t} A^{-t} (\log B - \log A)^2)$. Since $A, B > 0$, $(\log B - \log A)^2 \geq 0$, we have $g''(t) \geq 0$. Therefore,

$$g(t) - g(0) = t g'(0) + \frac{1}{2} g''(t_0) t^2 \geq t g'(0). \tag{3.13.13}$$

Note that $g'(0) = \lim_{t\to 0+} \frac{\operatorname{Tr}\left(B^{1+t}A^{-t}-B\right)}{t} = \operatorname{Tr}(B\log B - B\log A)$, therefore we have

$$\operatorname{Tr}(B\log B - B\log A) \le \frac{g(t)-g(0)}{t} = \frac{1}{t}\operatorname{Tr}\left(B^{1+t}A^{-t} - B\right). \tag{3.13.14}$$

Similarly, let $\widetilde{g}(t) = \operatorname{Tr}(B - B^{1-t}A^t)$, we have $\widetilde{g}'(t) = \operatorname{Tr}(B^{1-t}A^t\log B - B^{1-t}A^t\log A)$, $\widetilde{g}''(t) = -\operatorname{Tr}(B^{1-t}A^t(\log A - \log B)^2)$. Since $A, B > 0$, $(\log A - \log B)^2 \le 0$, we have $\widetilde{g}''(t) \le 0$. Therefore,

$$\widetilde{g}(t) - \widetilde{g}(0) \le t\widetilde{g}'(0). \tag{3.13.15}$$

Therefore,

$$\operatorname{Tr}(B\log B - B\log A) \ge \frac{\widetilde{g}(t)-\widetilde{g}(0)}{t} = \frac{1}{t}\operatorname{Tr}\left(B - B^{1-t}A^t\right). \tag{3.13.16}$$

□

Exercise 21. *Suppose that $A, B \in M_n(\mathbb{C})$, show that*

$$e^{A+B} = \lim_{m\to\infty}\left(e^{\frac{A}{2m}}e^{\frac{B}{m}}e^{\frac{A}{2m}}\right)^m. \tag{3.13.17}$$

Proof. Let $X_m = e^{\frac{A}{2m}}e^{\frac{B}{m}}e^{\frac{A}{2m}}$, $Y_m = e^{\frac{A+B}{m}}$. Then we have

$$\|X_m^m - Y_m^m\| \le m\|X_m - Y_m\|\max(\|X_m\|, \|Y_m\|)^{m-1} \quad \text{(by factorization of } A^m - B^m\text{)}. \tag{3.13.18}$$

We calculate

- $\|X_m\| \le e^{\frac{\|A\|}{2m}}e^{\frac{\|B\|}{m}}e^{\frac{\|A\|}{2m}} = e^{\frac{\|A\|+\|B\|}{2}}$; $\|Y_m\| \le e^{\frac{\|A+B\|}{m}} \le e^{\frac{\|A\|+\|B\|}{m}}$.
-
$$\begin{aligned}
X_m &= \left(I + \frac{A}{2m} + \frac{A^2}{8m^2} + o\left(\frac{1}{m^2}\right)\right)\left(I + \frac{B}{m} + \frac{B^2}{2m^2} + o\left(\frac{1}{m^2}\right)\right) \\
&\quad \left(I + \frac{A}{2m} + \frac{A^2}{8m^2} + o\left(\frac{1}{m^2}\right)\right) \\
&= I + \frac{A+B}{m} + \frac{AB+BA}{2m^2} + \frac{A^2}{4m^2} + \frac{A^2}{4m^2} + \frac{B^2}{2m^2} + o\left(\frac{1}{m^2}\right) \\
&= I + \frac{A+B}{m} + \frac{A^2+B^2+AB+BA}{2m^2} + o\left(\frac{1}{m^2}\right) \\
&= I + \frac{A+B}{m} + \frac{(A+B)^2}{2m^2} + o\left(\frac{1}{m^2}\right).
\end{aligned} \tag{3.13.19}$$

  Moreover
$$Y_m = I + \frac{A+B}{m} + \frac{(A+B)^2}{2m^2} + o\left(\frac{1}{m^2}\right) \tag{3.13.20}$$
- Thus we have
$$\|X^m - Y^m\| \le m\cdot o\left(\frac{1}{m^2}\right)\exp(\|A\|+\|B\|) \lesssim \frac{1}{m^2}. \tag{3.13.21}$$

Then, the required result follows readily by taking $m\to\infty$. □

Exercise 22. *Suppose* $A_1, \cdots, A_k \in M_n(\mathbb{C})$*, show that*

$$e^{A_1+\cdots+A_k} = \lim_{m\to\infty}\left(e^{\frac{A_1}{m}}e^{\frac{A_2}{m}}\cdots e^{\frac{A_k}{m}}\right)^m. \tag{3.13.22}$$

Proof. We denote

$$X_m = e^{\frac{A_1}{m}}e^{\frac{A_2}{m}}\cdots e^{\frac{A_k}{m}}, \quad Y_m = e^{\frac{A_1+\cdots+A_k}{m}}. \tag{3.13.23}$$

Then we have

$$\|X_m^m - Y_m^m\|_2 \le m\|X_m - Y_m\|_2 \max(\|X_m\|_2, \|Y_m\|_2)^{m-1} \text{(by factorization of } A^m - B^m\text{)}. \tag{3.13.24}$$

Note that

$$\begin{aligned} \|X_m\|_2 &\le e^{\frac{\|A_1\|+\cdots+\|A_k\|}{m}}, \\ \|Y_m\|_2 &\le e^{\frac{\|A_1+\cdots+A_k\|}{m}} \le e^{\frac{\|A_1\|+\cdots+\|A_k\|}{m}}. \end{aligned} \tag{3.13.25}$$

Thus we have

$$\max(\|X_m\|_2, \|Y_m\|_2)^{m-1} \le e^{\frac{\|A_1\|+\cdots+\|A_k\|}{m}(m-1)} \le e^{\|A_1\|+\cdots+\|A_k\|}. \tag{3.13.26}$$

Moreover

$$\begin{aligned} X_m &= \left(I + \frac{A_1}{m} + \frac{A_1^2}{2m^2} + o\left(\frac{1}{m^2}\right)\right)\cdots\left(I + \frac{A_k}{m} + \frac{A_k^2}{2m^2} + o\left(\frac{1}{m^2}\right)\right) \\ &= I + \frac{A_1+\cdots+A_k}{m} + \sum_{j=1}^k \frac{A_j^2}{2m^2} + \sum_{1\le j<l\le k}\frac{A_jA_l}{m^2} + o\left(\frac{1}{m^2}\right) \\ &= I + \frac{A_1+\cdots+A_k}{m} + \frac{(A_1+\cdots+A_k)^2}{2m^2} + \sum_{1\le j<l\le k}\frac{[A_j, A_l]}{2m^2} + o\left(\frac{1}{m^2}\right). \end{aligned} \tag{3.13.27}$$

$$Y_m = I + \frac{A_1+\cdots+A_k}{m} + \frac{(A_1+\cdots+A_k)^2}{2m^2} + o\left(\frac{1}{m^2}\right). \tag{3.13.28}$$

Thus we have

$$\|X_m - Y_m\|_2 \le me^{\|A_1\|+\cdots+\|A_k\|}\cdot\left(\frac{1}{2m^2}\sum_{1\le j<l\le k}\|[A_j, A_l]\|_2 + o\left(\frac{1}{m^2}\right)\right) \lesssim \frac{1}{m}. \tag{3.13.29}$$

Thus the required result follows by taking $m \to \infty$. $\square$

Exercise 23. *Suppose that* $A \in M_n(\mathbb{C})$*, show that* $\det e^A = e^{\operatorname{Tr} A}$.

Proof. We can show this by direct calculation for Jordan blocks. In fact, we can give another stronger result

Lemma 19. *Let* $A : I \to M_n(\mathbb{C})$ *be a operator valued continuous function and assume that the operator valued* C^1 *function* $\Phi(t)$ *solves the following equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}X = A(t)X, \tag{3.13.30}$$

then $\det\Phi(t)$ *solves*

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\Phi = \operatorname{Tr} A(t)\det\Phi, \quad \forall t \in I. \tag{3.13.31}$$

Proof of the lemma. Without loss of generality, we assume that $\Phi(t)^{-1}$ exists for any $t \in I$, since otherwise there exists t_1 and $c_1, \cdots, c_n \in \mathbb{C}$ such that $\psi(t) := c_1\Phi_1(t) + \cdots + c_n\Phi_n(t)$ satisfies $\psi(t_1) = 0$. Note that $\psi : I \to \mathbb{C}^n$ solves the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x = A(t)x, \tag{3.13.32}$$

thus by the uniqueness of the solution we have $\psi(t) \equiv 0$ on I. Thus we have $\Phi(t)$ is not invertible for any $t \in I$ thus $\det \Phi \equiv 0$, which proves the conclusion. In the rest of our proof, we assume that $\Phi(t)$ is invertible for any $t \in I$. Let $\Phi^*(t)$ be the adjugate matrix of $\Phi(t)$, then we have

$$\Phi^*(t) = \det \Phi(t)\Phi(t)^{-1}, \quad \forall t \in I. \tag{3.13.33}$$

Here the adjugate matrix is defined as the matrix whose (j, i)-entry is the (i, j)-cofactor of $\Phi(t)$. By the expansion of the determinant, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\det\Phi &= \lim_{\epsilon\to 0}\frac{\det\Phi(t+\epsilon) - \det\Phi(t)}{\epsilon} \\
&= \sum_{\sigma\in\mathfrak{S}_n}(-1)^{\mathrm{sgn}(\sigma)}\lim_{\epsilon\to 0}\frac{\prod_{k=1}^n \Phi_{k,\sigma(k)}(t+\varepsilon) - \prod_{k=1}^n \Phi_{k,\sigma(k)}(t)}{\epsilon} \\
&= \sum_{\sigma\in\mathfrak{S}_n}(-1)^{\mathrm{sgn}(\sigma)}\sum_{k=1}^n \Phi'_{k,\sigma(k)}(t)\prod_{j\neq k}\Phi_{j,\sigma(j)}(t) \\
&= \sum_{k,\ell}\Phi'_{k,\ell}(t)\Phi^*_{k,\ell}(t) = \sum_{k,\ell}\det\Phi(t)\Phi'_{k,\ell}(t)[\Phi^{-1}(t)]_{\ell,k} = \det\Phi(t)\operatorname{Tr}\big(\Phi'(t)\Phi^{-1}(t)\big) \\
&= \det\Phi(t)\operatorname{Tr}\big(A(t)\Phi(t)\Phi(t)^{-1}\big) = \det\Phi(t)\operatorname{Tr}A(t).
\end{aligned} \tag{3.13.34}$$

□

In particular, we take $\Phi_i(t) = e^{tA}e_i$ for $A \in M_n(\mathbb{C})$ and then $\Phi(t) = e^{tA}I = e^{tA}$, thus we have $\frac{\mathrm{d}}{\mathrm{d}t}\det e^{tA} = \operatorname{Tr} A \det e^{tA}$. Then we know that both

$$\varphi_1(t) = \det e^{tA} \quad \text{and} \quad \varphi_2(t) = e^{t\operatorname{Tr} A} \tag{3.13.35}$$

are solutions to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x = \operatorname{Tr}(A)x, \\ x(0) = 1, \end{cases} \tag{3.13.36}$$

which implies that $\varphi_1(t) = \varphi_2(t)$ for any $t \in I$. Thus we have $\det e^{tA} = e^{t\operatorname{Tr} A}$ for any $t \in I$. In particular, we have $\det e^A = e^{\operatorname{Tr} A}$. □

Exercise 24. *Suppose that $X \in M_n(\mathbb{C})$ is Hermitian and $\beta > 0$, show that*

$$\operatorname{Tr}(DX) - \frac{1}{\beta}H(D) \geq -\frac{1}{\beta}\log\operatorname{Tr}\big(e^{-\beta X}\big). \tag{3.13.37}$$

The equality holds iff D is the Gibbs state i.e. $D = \frac{e^{-\beta X}}{\operatorname{Tr}\left(e^{-\beta X}\right)}$.

Proof. By the Gibbs variational formula Theorem 3.5.2, we have

$$\begin{aligned}\log \operatorname{Tr} e^{-\beta X} &= \sup_{\widetilde{D} \text{ density matrix}} \{\operatorname{Tr}\left(-\beta X \widetilde{D}\right) - H(\widetilde{D})\} = \sup_{\widetilde{D} \text{ density matrix}} \{-\beta \operatorname{Tr}\left(X \widetilde{D}\right) - H(\widetilde{D})\} \\ &\geq -\beta \operatorname{Tr}(XD) - H(D)\end{aligned} \tag{3.13.38}$$

for density matrix D. Moreover, the equality holds if and only if

$$D = \frac{e^{-\beta X}}{\operatorname{Tr}(e^{-\beta X})}. \tag{3.13.39}$$

□

Exercise 25. *Suppose $A, B \in M_n(\mathbb{C})$, show that*

$$\exp\begin{pmatrix} A & B \\ & A \end{pmatrix} = \begin{pmatrix} \exp(A) & \int_0^1 e^{tA} B e^{(1-t)A} \mathrm{d}t \\ & \exp(A) \end{pmatrix} \tag{3.13.40}$$

Proof. We claim

$$\begin{pmatrix} A & B \\ & A \end{pmatrix}^m = \begin{pmatrix} A^m & C_m \\ & A_m \end{pmatrix}, \quad C_m = \sum_{j=0}^{m-1} A^j B A^{m-1-j}. \tag{3.13.41}$$

We can show this by induction. In fact, it follows readily from

$$\begin{pmatrix} A^m & C_m \\ & A_m \end{pmatrix} \begin{pmatrix} A & B \\ & A \end{pmatrix} = \begin{pmatrix} A^{m+1} & A^m B + C_m A \\ & A_m A \end{pmatrix} = \begin{pmatrix} A^{m+1} & C_{m+1} \\ & A_{m+1} \end{pmatrix}. \tag{3.13.42}$$

We denote $M = \begin{pmatrix} A & B \\ & A \end{pmatrix}$, then we have

$$\exp M = \sum_{m=0}^{\infty} \frac{M^m}{m!} = \sum_{m=0}^{\infty} \frac{1}{m!} \begin{pmatrix} A^m & C_m \\ & A_m \end{pmatrix} = \begin{pmatrix} \sum_{m=0}^{\infty} \frac{A^m}{m!} & \sum_{m=0}^{\infty} \frac{C_m}{m!} \\ & \sum_{m=0}^{\infty} \frac{A_m}{m!} \end{pmatrix} = \begin{pmatrix} e^A & \int_0^1 e^{tA} B e^{(1-t)A} \mathrm{d}t \\ & e^A \end{pmatrix}. \tag{3.13.43}$$

That is because

$$\begin{aligned}\int_0^1 e^{tA} B e^{(1-t)A} \mathrm{d}t &= \sum_{m,l=0}^{\infty} \left(\int_0^1 t^m (1-t)^l \mathrm{d}t\right) \frac{A^m B A^l}{m! l!} \\ &= \sum_{m,l=0}^{\infty} \frac{\Gamma(m+1)\Gamma(l+1)}{\Gamma(m+l+2)} \frac{A^m B A^l}{\Gamma(m+1)\Gamma(l+1)} \\ &= \sum_{k=0}^{\infty} \sum_{m+l=k} \frac{A^m B A^l}{(k+1)!} \\ &= (\text{change the dummy variable}) \sum_{m=0}^{\infty} \sum_{j=0}^{m-1} \frac{A^j B A^{m-j}}{m!} = \sum_{m=0}^{\infty} \frac{C^m}{m!}.\end{aligned} \tag{3.13.44}$$

□

Exercise 26. *Suppose that $A, B \in M_n(\mathbb{C})$ are Hermitian matrices, show that*

$$\left|\mathrm{Tr}\left(e^{A+\mathrm{i}B}\right)\right| \leq \mathrm{Tr}\left(e^{A}\right). \tag{3.13.45}$$

Proof. By Golden-Thompson inequality, we have

$$\left|\mathrm{Tr}\, e^{A+\mathrm{i}B}\right| \leq \mathrm{Tr}\left(e^{\frac{A+A^\dagger}{2}} e^{\frac{\mathrm{i}B+(\mathrm{i}B)^\dagger}{2}}\right) = \mathrm{Tr}\left(e^{A} e^{0}\right) \leq \mathrm{Tr}\, e^{A}. \tag{3.13.46}$$

□

Exercise 27. *Suppose that $A, B \in M_n(\mathbb{C})$. Show that*

$$\left\|e^{A+B} - (e^{A/m} e^{B/m})^m\right\|_2 \leq \frac{1}{2m} \|[A, B]\|_2 \exp(\|A\|_2 + \|B\|_2). \tag{3.13.47}$$

Proof. We denote

$$X_m = e^{\frac{A}{m}} e^{\frac{B}{m}}, \quad Y_m = e^{\frac{A+B}{m}}. \tag{3.13.48}$$

From the proof of Lemma 7, we have

$$X_m - Y_m = \frac{[A, B]}{2m^2} + o\left(\frac{1}{m^2}\right). \tag{3.13.49}$$

$$\|X_m^m - Y_m^m\| \leq m \|X_m - Y_m\| \exp(\|A\| + \|B\|) \tag{3.13.50}$$

We plug eq. (3.13.49) into eq. (3.13.50), we have

$$\left\|e^{A+B} - (e^{A/m} e^{B/m})^m\right\| = \|X_m^m - Y_m^m\| \leq \frac{1}{2m} \|[A, B]\|_2 \exp(\|A\| + \|B\|). \tag{3.13.51}$$

□

Chapter 4

Completely Positive Maps

4.1 Overview

Definition 4.1.1. *Let $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map, we say:*

- *Φ is bounded, if $\sup_{\|A\|=1} \|\Phi(A)\| < \infty$. We denote $\|\Phi\| = \sup_{\|A\|=1} \|\Phi(A)\|$.*
- *Φ is positive, if $\Phi(A) \geq 0$ for any $A \geq 0$.*
- *Φ is unital, if $\Phi(I) = I$.*
- *Φ is trace-preserving, if $\operatorname{Tr} \Phi(A) = \operatorname{Tr} A$ for any $A \in M_m(\mathbb{C})$.*
- *Φ is contractive, if $\|\Phi(A)\| \leq \|A\|$ for any $A \in M_m(\mathbb{C})$ i.e. $\|\Phi\| \leq 1$.*

Example 15. $\Phi(A) = \operatorname{Tr}(A)I$ *is positive.*

Example 16. $X \in M_{m,n}(\mathbb{C})$, $\Phi(A) = X^*AX$ *is positive.* $\|\Phi\| = \|X\|^2$.
Φ is unital $\iff X^*X = I$; Φ is trace-preserving $\iff XX^* = I$.

Example 17. $\Phi(A) = A^T$ *is positive,* $\|\Phi\| = 1$.

Proposition 57. $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ *is positive, then*

$$\|\Phi\| \leq 2\|\Phi(I)\|. \tag{4.1.1}$$

Remark 58. *In fact if we assume that Φ is positive, then we have $\|\Phi\| = \|\Phi(I)\|$, but the proof is very complicated, as we will see in the discusssion of von Neuman inequality (4.8.4).*

Proof. In fact, we have for any A, $A = \operatorname{Re} A + \mathrm{i} \operatorname{Im} A$ with $\operatorname{Re} A, \operatorname{Im} A \geq 0$, thus we have

$$\Phi(A) = \Phi(\operatorname{Re} A) + \mathrm{i}\Phi(\operatorname{Im} A), \quad \Phi(\operatorname{Re} A), \Phi(\operatorname{Im} A) \geq 0 \text{ by } \Phi \text{ is positive.} \tag{4.1.2}$$

Thus we have

$$\Phi(A)^* = \Phi(\operatorname{Re} A) - \mathrm{i}\Phi(\operatorname{Im} A) = \Phi(A^*). \tag{4.1.3}$$

Let A be a Hermitian matrix, then we have

$$-\|A\|I \leq A \leq \|A\|I, \tag{4.1.4}$$

which means that $\Phi(I)$ can be viewed as the largest support that can be achieved by Φ. Thus we have

$$\|\Phi(A)\| \leq \|A\|\|\Phi(I)\|. \tag{4.1.5}$$

Then we assume that A is an arbitrary matrix, then we have

$$\|\Phi(A)\| = \left\|\Phi\left(\frac{A+A^*}{2}\right) + \Phi\left(\frac{\mathrm{i}A - \mathrm{i}A^*}{2\mathrm{i}}\right)\right\| \leq 2\|A\|\|\Phi(I)\| \Rightarrow \|\Phi\| \leq 2\|\Phi(I)\|. \tag{4.1.6}$$

□

Definition 4.1.2. $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$. $\ell \in \mathbb{N}$, $id_\ell \otimes \Phi : M_\ell(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_\ell(\mathbb{C}) \otimes M_n(\mathbb{C})$ *is defined as (using the block matrix form)*

$$(id_\ell \otimes \Phi)((A_{j,k})_{j,k=1}^{\ell}) = (\Phi(A_{j,k}))_{j,k=1}^{\ell}. \tag{4.1.7}$$

This is because, we actually have

$$(A_{j,k})_{j,k=1}^{\ell} = \sum_{j,k=1}^{\ell} E_{j,k} \otimes A_{j,k}, \tag{4.1.8}$$

thus,

$$(id_\ell \otimes \Phi)((A_{j,k})_{j,k=1}^{\ell}) = \sum_{j,k=1}^{\ell} E_{j,k} \otimes \Phi(A_{j,k}) = (\Phi(A_{j,k}))_{j,k=1}^{\ell}. \tag{4.1.9}$$

We define:

- *If $id_\ell \otimes \Phi$ is positive, then we say Φ is ℓ-positive.*
- *If $id_\ell \otimes \Phi$ is positive for any $\ell \in \mathbb{N}$, then we say Φ is completely positive.*
- *If $\sup_{\ell\in\mathbb{N}} \|id_\ell \otimes \Phi\| < \infty$, then we say Φ is completely bounded. We denote $\|\Phi\|_{cb} = \sup_{\ell\in\mathbb{N}} \|id_\ell \otimes \Phi\|$. It is easy to see that $\|\Phi\|_{cb} \geq \|\Phi\|$.*
- *If $\|\Phi\|_{cb} \leq 1$, we say that Φ is completely contractive.*

Example 18.

- *$\Phi(A) = \mathrm{Tr}(A)I$ is completely positive.*
- *$\Phi(A) = X^*AX$ is completely positive.*
- *If Ψ, Φ are completely positive, then $\Psi + \Phi, a\Psi\ (a > 0), \Psi \circ \Phi$ is completely positive.*
- *$\Phi(A) = A^T$ is **not** completely positive. In fact,*

$$(id_2 \otimes \Phi)\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} E_{11}^T & E_{12}^T \\ E_{21}^T & E_{22}^T \end{pmatrix} = \begin{pmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}. \not\geq 0. \tag{4.1.10}$$

However, $\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} 1&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&1 \end{pmatrix}$ itself is a rank-1 projection matrix, thus it is positive. Therefore, Φ is not 2-positive.

Proposition 58. *$\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is completely positive, then $\|\Phi\|_{cb} = \|\Phi\| = \|\Phi(I)\|$.*

Proof. We have $\|\Phi(I)\| \leq \|\Phi\| \leq \|\Phi\|_{cb}$ holds by definition. We only need to show the reverse inequality $\|\Phi\|_{cb} \leq \|\Phi\|$.

We take $A \in M_\ell(\mathbb{C}) \otimes M_n(\mathbb{C})$ and $\|A\| \leq 1$, then we have

$$M = \begin{pmatrix} I & A \\ A^* & I \end{pmatrix} \geq \begin{pmatrix} I & A \\ A^* & A^*A \end{pmatrix} = \begin{pmatrix} I & 0 \\ A^* & 0 \end{pmatrix}\begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix} \geq 0. \tag{4.1.11}$$

Thus we have

$$(\mathrm{id}_{2\ell} \otimes \Phi)(M) = [\mathrm{id}_2 \otimes (\mathrm{id}_\ell \otimes \Phi)]\begin{pmatrix} I & A \\ A^* & I \end{pmatrix} = \begin{pmatrix} (\mathrm{id}_\ell \otimes \Phi)(I) & (\mathrm{id}_\ell \otimes \Phi)(A) \\ (\mathrm{id}_\ell \otimes \Phi)(A^*) & (\mathrm{id}_\ell \otimes \Phi)(I) \end{pmatrix} \geq 0. \tag{4.1.12}$$

Thus by Lemma 3, we have

$$[(\mathrm{id}_\ell \otimes \Phi)(A)]^*[\varepsilon + (\mathrm{id}_\ell \otimes \Phi)(I)]^{-1}[(\mathrm{id}_\ell \otimes \Phi)(A)] \leq (\mathrm{id}_\ell \otimes \Phi)(I) + \varepsilon. \tag{4.1.13}$$

That is,

$$\left\|[\varepsilon + (\mathrm{id}_\ell \otimes \Phi)(I)]^{-1/2}[(\mathrm{id}_\ell \otimes \Phi)(A)][\varepsilon + (\mathrm{id}_\ell \otimes \Phi)(I)]^{-1/2}\right\| \leq 1. \tag{4.1.14}$$

Thus we have

$$\|(\mathrm{id}_\ell \otimes \Phi)(A)\| \leq \left\|[\varepsilon + (\mathrm{id}_\ell \otimes \Phi)(I)]^{1/2}\right\|^2 = \|I_\ell \otimes \Phi(I) + \varepsilon\| = \|\Phi(I)\| + \varepsilon. \tag{4.1.15}$$

By taking $\varepsilon \to 0$ we have $\|(\mathrm{id}_\ell \otimes \Phi)(A)\| \leq \|\Phi(I)\|$ for any ℓ and $\|A\| \leq 1$. Thus we have $\|\mathrm{id}_\ell \otimes \Phi\| \leq \|\Phi(I)\|$ for any $\ell \in \mathbb{N}$. Therefore, we have $\|\Phi\|_{cb} = \sup_{\ell \in \mathbb{N}} \|\mathrm{id}_\ell \otimes \Phi\| \leq \|\Phi(I)\|$. □

Lemma 20. *Let $A \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, then $A \geq 0$ if and only if A is a summation of "rank-1 block matrices" $(B_j^* B_k)_{j,k=1}^n$.*

Proof. $\Leftarrow$: obvious. $\Rightarrow$: Since $A \geq 0$, we have $A = X^*X$ for some $X \in M_{n^2}(\mathbb{C})$. Then we expand this blockwisely. □

Proposition 59. *$A \in M_n(\mathbb{C})$, then TFAE:*

(1) $A \geq 0$.

(2) *$X \mapsto X \circ A$ is positive.*

(3) *$X \mapsto X \circ A$ is completely positive.*

Proof. $(1) \implies (2)$ follows readily from the Schur product theorem Theorem 2.4.1; $(2) \implies (1)$ follows by taking $X = (1)_{1 \leq j,k \leq n}$ and $A = \Phi_A(X) \geq 0$.

For (1) or $(2) \implies (3)$, we recall that

$$\Phi_A(X) = V^*(X \otimes A)V, \quad V : e_i \otimes e_i \mapsto e_i. \tag{4.1.16}$$

Since we have $V^*(\cdot)V$ is a completely positive map, thus we have Φ_A is completely positive. □

4.2 Characteriation of completely positive and k-positive maps

Definition 4.2.1 (Choi matrix theorem)**.** *$\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map, we define the Choi matrix of Φ as*

$$C_\Phi = \sum_{i,j=1}^{n} E_{i,j} \otimes \Phi(E_{i,j}) = (id_n \otimes \Phi)(\underbrace{E}_{:=\sum_{j,k=1}^{m} E_{jk}\otimes E_{jk}}) \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}). \tag{4.2.1}$$

Remark 59. *$C_\Phi = (id_n \otimes \Phi)(E)$. We note that $E^2 = \frac{1}{m}E$ thus $E \geq 0$ (E is some multiple of a projection). In some literatures, E is called the Jones projection.*

Therefore, if Φ is completely positive, then $C_\Phi \geq 0$.

Remark 60. *We have the following very useful identity:*

$$\Phi(A) = (\mathrm{Tr} \otimes id_n)(C_\Phi(A^T \otimes I)). \tag{4.2.2}$$

Caution: the trace is taken on the first factor. More generally, we have

$$B\Phi(A)C = (\mathrm{Tr} \otimes \mathcal{M}_B)(C_\Phi(A^T \otimes C)). \tag{4.2.3}$$

This can be understood as

$$\Phi(A) = \sum_{i,j=1}^{n} \mathrm{Tr}\big(E_{i,j}A^T\big)\Phi(E_{i,j}) = \sum_{i,j=1}^{n} a_{ij}\Phi(E_{ij}), \tag{4.2.4}$$

$$(\mathrm{Tr} \otimes \mathcal{M}_B)(C_\Phi(A^T \otimes C)) = \sum_{i,j=1}^{n} \mathrm{Tr}\big(E_{i,j}A^T\big)\mathcal{M}_B(\Phi(E_{i,j})C) = \sum_{i,j=1}^{n} a_{ij}B\Phi(E_{ij})C = B\Phi(A)C. \tag{4.2.5}$$

This means that all the information of Φ is contained in C_Φ.

Remark 61. *Two important examples of choi matrices are*

$$id : A \mapsto A, \quad C_{id} = \sum_{i,j=1}^{n} E_{i,j} \otimes E_{i,j} = E. \tag{4.2.6}$$

$$\mathrm{Tr} : A \mapsto (\mathrm{Tr}\, A)I, \quad C_{\mathrm{Tr}} = I \otimes I = I. \tag{4.2.7}$$

Next we will use the Choi matrix representation to deduce a quite important characterization of completely positive maps on finite-dimensional matrix algebras.

Theorem 4.2.2. *$\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map, then TFAE:*

(1) *Φ is completely positive.*

(2) ***(Kraus decomposition or Choi decomposition)*** *$\Phi(A) = \sum_{j=1}^{r} X_j^* A X_j$ for some $X_j \in M_{m,n}(\mathbb{C})$ and $r \leq mn$.*

(3) *$C_\Phi \geq 0$.*

(4) ***(CP in f.d. case is equiv. to $m \wedge n$-positive)*** *$id_{m\wedge n} \otimes \Phi$ is positive where $m \wedge n = \min(m, n)$.*

Remark 62. *The main difficulties lie in two aspects. Our strategies are: (1) show $(3) \implies (2)$ using the "rank-1 decomposition" of C_Φ to "open up" its structure and then using standard algebraic arguments; (2) to show that $(m\wedge n) \Rightarrow$ is sufficient to show $\langle C_\Phi v, v\rangle \geq 0$ for any $v \in \mathbb{C}^m \otimes \mathbb{C}^n$ by direct computation.*

Proof. The proof is totally algebraic construction.

$(1) \implies (3)$: $C_\Phi = (\mathrm{id} \otimes \Phi)(E) \geq 0$ follows readily from $E \geq 0$ and Φ is completely positive.

$(3) \implies (2)$: By $C_\Phi \geq 0$ and the approach of the previous lemma Lemma 20, we have the "straightening" of C_Φ as:

$$C_\Phi = \sum_{t=1}^{r} Y_t^* Y_t, \quad Y_t \in M_{1,mn}(\mathbb{C}). \tag{4.2.8}$$

We write

$$Y_t = (v_{1,t}, \cdots, v_{m,t}), \quad \text{where each } v_{j,t} \text{ is an } n\text{-dimensional row vector.} \tag{4.2.9}$$

We define $X_t = \begin{pmatrix} v_{1,t} \\ \vdots \\ v_{m,t} \end{pmatrix} \in M_{m,n}(\mathbb{C})$, then we have:

- One the one hand, $C_\Phi = \sum_{j,k=1}^{m} E_{j,k} \otimes \Phi(E_{j,k}) = \sum_{t=1}^{r} Y_t^* Y_t$.
- On the other hand, we compute $\sum_{j,k=1}^{m} E_{j,k} \otimes X_t^* E_{jk} X_t = (X_t^* E_{jk} X_t)_{j,k=1}^{m} = (X_t^* e_j e_k^* X_k)_{j,k=1}^{m} = (v_{jt}^* v_{kt})_{j,k=1}^{m} = Y_t^* Y_t$.

Taking summation over t, we have

$$\sum_{j,k=1}^{m} E_{j,k} \otimes \Phi(E_{j,k}) = C_\Phi = \sum_{t=1}^{r} Y_t^* Y_t = \sum_{j,k=1}^{m} E_{j,k} \otimes \left(\sum_{t=1}^{r} X_t^* E_{jk} X_t\right). \tag{4.2.10}$$

Therefore we have that the action of Φ on E_{jk} is given by

$$\Phi(E_{jk}) = \sum_{t=1}^{r} X_t^* E_{jk} X_t. \tag{4.2.11}$$

That is,

$$\Phi(A) = \sum_{t=1}^{r} X_t^* A X_t, \quad \forall A \in M_m(\mathbb{C}). \tag{4.2.12}$$

$(2) \implies (1)$ is obvious by the previous example.

$(4) \implies (3)$: We take arbitrary $v \in \mathbb{C}^m \otimes \mathbb{C}^n$, we need to show that $\langle C_\Phi v, v\rangle \geq 0$.

In fact, we have

$$v = \sum_{j=1}^{m}\sum_{k=1}^{n} v_{jk} e_j \otimes e_k = \sum_{k=1}^{n}\left(\sum_{j=1}^{m} v_{jk} e_j\right) \otimes e_k = \sum_{j=1}^{m} e_j \otimes \left(\sum_{k=1}^{n} v_{jk} e_k\right). \tag{4.2.13}$$

Therefore, we can always write

$$v = \sum_{j=1}^{r} x_j \otimes y_j, \quad x_j \in \mathbb{C}^m, y_j \in \mathbb{C}^n, \quad r \leq \min(n, m) = m \wedge n. \tag{4.2.14}$$

We take $v_1, \cdots, v_r$ to be the O.N. basis of $\mathbb{C}^r$ and let Tr be the trace on $M_m(\mathbb{C})$ or $M_n(\mathbb{C})$, then we have

$$\begin{aligned}\langle C_\Phi v, v\rangle &= (\mathrm{Tr}\otimes\mathrm{Tr})(C_\Phi vv^*)\\ &= \sum_{j,k=1}^r (\mathrm{Tr}\otimes\mathrm{Tr})(C_\Phi x_jx_k^*\otimes y_jy_k^*)\\ &\overset{\text{by the previous remark}}{=} \sum_{j,k=1}^r \mathrm{Tr}[\Phi(x_k^*x_j)(y_jy_k^*)]\\ &\overset{\text{dilate again}}{=} \sum_{j,k=1}^r (\mathrm{Tr}\otimes\mathrm{Tr})[(v_kv_j^*)\otimes\Phi(x_kx_j^*)(v_jv_k^*\otimes y_jy_k^*)]\\ &= (\mathrm{Tr}\otimes\mathrm{Tr})\underbrace{\left(\sum_{j,k=1}^r (\mathrm{id}\otimes\Phi)(v_kv_j^*\otimes x_kx_j^*)\right)}_{\geq 0 \text{ by (4)}}\underbrace{\left(\sum_{j,k=1}^r v_jv_k^*\otimes y_jy_k^*\right)}_{\geq 0}\geq 0.\end{aligned} \tag{4.2.15}$$

$(1) \implies (4)$ follows readily by definition. $\square$

Another very important characterization is the so-called Stinespring dilation theorem. It can be seen as a GNS construction in terms of operator algebras.

Theorem 4.2.3 (Stinespring dilation theorem)**.** $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ *is a completely positive map, then there exists a finite dimensionl Hilbert space* $\mathcal{H}$*. Then there exists a unital* $*$*-homomorphism* $\pi : M_m(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ *and a bounded operator* $V : \mathbb{C}^n \to \mathcal{H}$ *such that* $\|\Phi\|_{cb} = \|V\|^2$ *and for any* $A \in M_m(\mathbb{C})$*, we have*

$$\Phi(A) = V^*\pi(A)V. \tag{4.2.16}$$

Proof. This is in fact the standard procedure of GNS construction. We denote $\mathcal{H}_0 = M_m(\mathbb{C})\otimes\mathbb{C}^n$ and we define

$$\langle A\otimes x, B\otimes y\rangle_0 := \langle\Phi(B^*A)x, y\rangle. \tag{4.2.17}$$

This is a Hermitian bilinear form on $M_m(\mathbb{C})\otimes\mathbb{C}^n$. Since Φ is completely positive, we know that the bilinear form is positive semidefinite, thus the Cauchy-Schwarz inequality holds.

We next deal with the null space. We define

$$\mathcal{N} = \{x\in\mathcal{H}_0 : \langle x, x\rangle_0 = 0\}\subset\mathcal{H}_0. \tag{4.2.18}$$

Then $\mathcal{H} := \mathcal{H}_0/\mathcal{N}$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_0$ since it becomes now strictly positive. We define

$$\pi : M_m(\mathbb{C}) \to \mathcal{B}(\mathcal{H}), \quad \pi(A)(B\otimes x+\mathcal{N}) := AB\otimes x+\mathcal{N}. \tag{4.2.19}$$

It is easy to verify that π is in fact a $*$-homomorphism and $\|\pi(A)\|\leq\|A\|$.

In fact, Φ "acts like" an identity map on $\mathbb{C}^n$, thus we define

$$V : \mathbb{C}^n \to \mathcal{H}, \quad Vx := I\otimes x+\mathcal{N}. \tag{4.2.20}$$

Then we have

$$(V^*\pi(A)V)(x) = (V^*\pi(A))(I\otimes x+\mathcal{N}) = V^*(A\otimes x+\mathcal{N}) = \Phi(A)x. \tag{4.2.21}$$

Here, the last equality is because

$$\langle \Phi(A)x, y\rangle = \langle A\otimes x, I\otimes y\rangle_0 = \langle A\otimes x+\mathcal{N}, Vy\rangle_0 = \langle V^*(A\otimes x+\mathcal{N}), y\rangle. \tag{4.2.22}$$

We can also compute

$$\|Vx\|^2 = \langle \Phi(I)x, x\rangle \leq \|\Phi(I)\|\|x\|^2 \overset{\text{Proposition 58}}{=} \|\Phi\|_{cb}\|x\|^2. \tag{4.2.23}$$

Thus we have $\|V\|^2 = \|\Phi\|_{cb}$ since the equality of the Cauchy-Schwarz inequality can be achieved. □

Remark 63. *This is an abstract construction and does not rely on the structure of the matrix algebra. Thus this theorem itself is also true for general von-Neumann algebras and even C^*-algebras.*

Corollary 11. *Let $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a completely positive map, then there exists $X_j \in M_{m,n}(\mathbb{C})$ such that*

$$\Phi(A) = \sum_{j=1}^{r} X_j^* A X_j, \quad \forall A \in M_m(\mathbb{C}). \tag{4.2.24}$$

Remark 64. *This corollary implies that the Choi matrix decomposition can also be derived from the Stinespring dilation theorem.*

Proof. By the above theorem, we have $\pi : M_n(\mathbb{C}) \to \mathcal{B}(\mathcal{H}) \cong M_\ell(\mathbb{C})$, $\Phi(A) = V^*\pi(A)V$. But we note that the $\mathcal{N}$ above is trivial in this case, thus π is in fact a $*$-isomorphism. Therefore, we have ℓ must be some multiple of n. Thus π must be:

$$\pi : A \mapsto \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}, \quad V : \mathbb{C}^n \to \mathbb{C}^\ell \in M_{\ell,n}(\mathbb{C}). \tag{4.2.25}$$

We write $V = (X_1, \cdots, X_r)$, then we have

$$\Phi(A) = \begin{pmatrix} X_1^* \\ \vdots \\ X_r^* \end{pmatrix} \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} \begin{pmatrix} X_1 & \cdots & X_r \end{pmatrix} = \sum_{j=1}^{r} X_j^* A X_j. \tag{4.2.26}$$

□

Example 19. *Two sets of Kraus operators $\{X_j\}_{j=1}^r$ and $\{\widetilde{X}_\ell\}_{\ell=1}^s$ represent the same complemently positive map, if and only if they are related by a unitary transform, i.e.*

$$(\widetilde{X}_1, \cdots, \widetilde{X}_\ell, \cdots, \widetilde{X}_s) = (X_1, \cdots, X_j, \cdots, X_r)U. \tag{4.2.27}$$

Here U is unitary and the smaller set is padded with zeros.

Proof. We note that

$$\begin{aligned} \widetilde{\Phi}(A) &= \sum_{\ell=1}^{s} \widetilde{X}_\ell^* A \widetilde{X}_\ell = \sum_{j,k=1}^{r} \overline{U_{j\ell}} \sum_{\ell=1}^{s} \left(X_j^* A X_k\right) U_{k\ell} = \sum_{j,k=1}^{r} \left(\sum_{\ell=1}^{s} U_{k\ell} U_{\ell j}^*\right) X_j^* A X_k = \sum_{j,k=1}^{r} \delta_{jk} X_j^* A X_k \\ &= \sum_{j=1}^{r} X_j^* A X_j = \Phi(A). \end{aligned} \tag{4.2.28}$$

The above derivation holds if and only if $UU^* = I$ i.e. U is unitary. □

Remark 65. *In physics literatures, the invariance of the Kraus representation under unitary transformations is called the "gauge invariance" of the quantum channel.*

Next we will give a quite different characterization of k-positive maps $\iff (P\otimes I)C_\Phi(P\otimes I)\geq 0$ for any projection P with rank k.

Proposition 60. $\Phi: M_n(\mathbb{C})\to M_n(\mathbb{C})$ *is a linear map, then* Φ *is* k*-positive* $\iff (P\otimes I)C_\Phi(P\otimes I)\geq 0$ *for any projection* P *with rank* k*.*

Proof. We take $v_1,\cdots,v_k$ be the orthonormal basis of $P\mathbb{C}^n$. For any $v\in\mathbb{C}^m\otimes\mathbb{C}^n$,

$$\langle (P\otimes I)C_\Phi(P\otimes I)v,v\rangle\geq 0 \iff \left\langle C_\Phi\sum_{j=1}^k v_j\otimes x_j,\sum_{j=1}^k v_j\otimes x_j\right\rangle\geq 0,\forall x_j\in\mathbb{C}^n. \tag{4.2.29}$$

This is equivalent to

$$\sum_{j,\ell=1}^k(\mathrm{Tr}\otimes\mathrm{Tr})(C_\Phi v_jv_\ell^*\otimes x_jx_\ell)=\sum_{j,\ell=1}^k(\mathrm{id}\otimes\mathrm{Tr})(\Phi)=\sum_{j,\ell=1}^k\mathrm{Tr}(\Phi(v_\ell^*v_j)x_jx_\ell^*)\geq 0. \tag{4.2.30}$$

This is equivalent to

$$x^*\left[(\mathrm{id}_k\otimes\Phi)(v_\ell v_j^*)\right]x\geq 0. \tag{4.2.31}$$

Thus it is further equivalent to Φ is k-positive. $\square$

Remark 66. *We can also see from above that the Choi matrix representation is indepedent of the choice of the basis. Another way to understand this is by direct inspection: For* $U=(u_1,\cdots,u_n)$ *unitary, if we have*

$$C_\Phi=(id_n\otimes\Phi)[(U\otimes U)E(U\otimes U)^*]=(id_n\otimes\Phi)\left(\sum_{i,j=1}^n u_iu_j^*\otimes u_iu_j^*\right), \tag{4.2.32}$$

then we have

$$(\mathrm{Tr}\otimes id)(C_\Phi(A^T\otimes I))=\sum_{i,j=1}^n\underbrace{\mathrm{Tr}\big(u_iu_j^*A^T\big)}_{=u_i^*Au_j}\Phi(u_iu_j^*)=\Phi(A) \tag{4.2.33}$$

which also recovers the action of the original map Φ *on* A*.*

We can readily see how the above proposition can be useful to characterize the k-positivity.

Theorem 4.2.4. $1\leq k\leq n$, $A\mapsto(1-t)\frac{1}{n}\mathrm{Tr}(A)I+tA$, $t\in\mathbb{R}$ *is* k*-positive* $\iff t\in[-\frac{1}{nk-1},1]$.

Proof. Let $\Phi_t(A):(1-t)\frac{1}{n}\mathrm{Tr}(A)I+tA$. Then we can compute the Choi matrix as

$$C_{\Phi_t}=(1-t)\frac{1}{n}I+tE. \tag{4.2.34}$$

By the characterization Proposition 60 above, we have Φ_t is k-positive $\iff (P\otimes I)C_{\Phi_t}(P\otimes I)\geq 0$ for any projection P with rank k. That is,

$$(1-t)\frac{1}{n}(P\otimes I)+t(P\otimes I)E(P\otimes I)\geq 0. \tag{4.2.35}$$

Note that E is independent of the choice of basis. Then without loss of generality, we can take the basis corresponding to the projection P, then under this basis we actually have the following very simple form:

$$(P \otimes I)E(P \otimes I) = \begin{pmatrix} \underbrace{\begin{matrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{matrix}}_{k \times k} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.2.36}$$

Thus we have that $\frac{1}{k}(P \otimes I)E(P \otimes I)$ is a rank-1 projection matrix. Therefore, we have eq. (4.2.35) holds if and only if

$$(1-t)\frac{1}{n} + kt \geq 0, \quad 1 - t \geq 0, \tag{4.2.37}$$

and hence

$$t \in \left[-\frac{1}{nk-1}, 1 \right]. \tag{4.2.38}$$

$\square$

Remark 67. *Inspired by the idea above, we actually have:*

- Φ_t *is positive if and only if* $t \in [-\frac{1}{n-1}, 1]$ *by taking* $k = 1$.
- Φ_t *is completely positive if and only if* $t \in [-\frac{1}{n^2-1}, 1]$ *by taking* $k = n$ *(recall that from (4) in Theorem 4.2.2, we know that* $\Phi_t : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ *is completely positive* $\iff$ *it is* $(m \wedge n)$*-positive).*

For the last part of this section, we will give a simple convexity inequality regarding the completely positive maps and their adjoint maps.

Definition 4.2.5. *We denote* Φ^* *as the adjoint map of* $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ *with respect to the Hilbert-Schmidt inner product.*

Proposition 61. *It is clear that*

- Φ *is* k*-positve* $\iff$ Φ^* *is* k*-positive.*
- Φ *is completely positive* $\iff$ Φ^* *is completely positive.*
- Φ *is unital* $\iff$ Φ^* *is trace-preserving.*

Remark 68. *Caution! In general, we de not have* $\|\Phi\|_{cb} = \|\Phi^*\|_{cb}$. *To see this, if* Φ *is unital, then* $\|\Phi\|_{cb} = \|\Phi(I)\| = \|I\| = 1$, *but* $\|\Phi^*\|_{cb}$ *can be numbers other than* 1.

The example above in Theorem 4.2.4 is both unital and trace-preserving.

Theorem 4.2.6 (Majorization inequality)**.** *Let* $A \in \mathbb{H}_n$, $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *be a positive (we only need* 1*-positive here), trace-preserving and unital map. Then we have* $\Phi(A) \prec A$.

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be eigenvalues of $\Phi(A)$, P_k is the orthogonal projection onto the eigenspace spanned by the first k eigenvectors. Then we have

$$\sum_{j=1}^{k} \lambda_j = \operatorname{Tr}(P_k \Phi(A)) = \operatorname{Tr}(\Phi^*(P_k)A) \overset{\Phi^* \text{ is TP and unital}}{\leq} \sup_{0 \leq T \leq 1, \operatorname{Tr} T = k} \operatorname{Tr}(TA) \overset{\text{like min-max theorem}}{=} \sum_{j=1}^{k} \lambda_j(A). \tag{4.2.39}$$

By trace-preserving we have $\operatorname{Tr}(\Phi(A)) = \operatorname{Tr}(A)$, thus we have $\Phi(A) \prec A$. $\square$

Corollary 12. *$A \in \mathbb{H}_n$, $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital and trace-preserving map. f is a convex function, then we have*

$$\operatorname{Tr} f(\Phi(A)) \le \operatorname{Tr} f(A). \tag{4.2.40}$$

Proof. Recall that $A \prec B$ if and only if $\operatorname{Tr} f(A) \le \operatorname{Tr} f(B)$ for any convex function f. □

Corollary 13. *Φ is a unital and trace-preserving positive map, then $\|\Phi(A)\|_p \le \|A\|_p$.*

Corollary 14. *$B \ge 0$, $\Phi(A) := \int_0^\infty \frac{B^{\frac{1}{2}}}{\lambda+B} A \frac{B^{\frac{1}{2}}}{\lambda+B} d\lambda$ which is trace-preserving and unital. If $A \in \mathbb{H}_n$, then $\Phi(A) \prec A$.*

4.3 Conditional expectations

Definition 4.3.1. *Let $\mathfrak{A}$ be a unital (which means $I \in \mathfrak{A}$) $*$-subalgebra of $M_n(\mathbb{C})$. We say $E_{\mathfrak{A}}$ is the conditional expectation onto $\mathfrak{A}$ (or given $\mathfrak{A}$), if*

- *$E_{\mathfrak{A}} : M_n(\mathbb{C}) \to \mathfrak{A}$ is a positive map;*
- *$E_{\mathfrak{A}}$ is unital i.e. $E_{\mathfrak{A}}(I) = I$;*
- *$E_{\mathfrak{A}}(B_1 A B_2) = B_1 E_{\mathfrak{A}}(A) B_2$ for any $B_1, B_2 \in \mathfrak{A}$ and $A \in M_n(\mathbb{C})$. In particular, $E_{\mathfrak{A}}(A) = A$ for $A \in \mathfrak{A}$.*

We say a conditional expectation $E_{\mathfrak{A}}$ is a trace-preserving conditional expectation (TPCE) if it also satisfies

$$\operatorname{Tr}(E_{\mathfrak{A}}(A)) = \operatorname{Tr}(A), \quad \forall A \in M_n(\mathbb{C}). \tag{4.3.1}$$

Remark 69.

- *It is easy to see that $E_{\mathfrak{A}}^2(A) = E_{\mathfrak{A}}(\underbrace{E_{\mathfrak{A}}(A)}_{\in \mathfrak{A}})$ for any $A \in M_n(\mathbb{C})$, i.e. $E_{\mathfrak{A}}^2 = E_{\mathfrak{A}}$.*
- *By noting that*
$$E_{\mathfrak{A}}\left[(A - E_{\mathfrak{A}})^*(A - E_{\mathfrak{A}})\right] \ge 0 \tag{4.3.2}$$
we see that $E_{\mathfrak{A}}(A^*A) \ge E_{\mathfrak{A}}(A^*)E_{\mathfrak{A}}(A)$.
- *If $E_{\mathfrak{A}}$ is the **trace-preserving** conditional expectation, then we have $\operatorname{Tr}(A^*A) \operatorname{Tr} E_{\mathfrak{A}}(A^*A) \ge \operatorname{Tr}[E_{\mathfrak{A}}(A^*)E_{\mathfrak{A}}(A)]$, i.e. $\|E_{\mathfrak{A}}(A)\|_2 \le \|A\|_2$. In other words, $E_{\mathfrak{A}}$ can be viewed as an orthogonal projection: $(M_n(\mathbb{C}), \|\cdot\|_2) \to (\mathfrak{A}, \|\cdot\|_2)$. By the uniqueness of orthogonal projection, we know that the* trace-preserving *conditional expectation is unique.*
- *In general, if we have a faithful state σ, we can also use the trace to define the weighted inner product and the corresponding Hilbert space $L^2(\sigma)$, then the mapping $F_{\mathfrak{A}} : L^2(M_n(\mathbb{C}), \sigma) \to L^2(\mathfrak{A}, \sigma)$ is an orthogonal projection as well as a conditional expectation.*

Theorem 4.3.2 (von Neumann's double commutant theorem)**.** *Let $\mathfrak{A}$ be a unital $*$-subalgebra of $M_n(\mathbb{C})$. We define the commutant of $\mathfrak{A}$ as*

$$\mathfrak{A}' = \{X \in M_n(\mathbb{C}) : AX = XA, \forall A \in \mathfrak{A}\}. \tag{4.3.3}$$

Then $\mathfrak{A}'$ is also a unital $*$-subalgebra of $M_n(\mathbb{C})$. Likewise, we define the double commutant of $\mathfrak{A}$ as

$$\mathfrak{A}'' = (\mathfrak{A}')' = \{X \in M_n(\mathbb{C}) : AX = XA, \forall A \in \mathfrak{A}'\}. \tag{4.3.4}$$

We have $\mathfrak{A}'' = \mathfrak{A}$.

Proof. exercise 28. $\square$

Definition 4.3.3. • *Let $\mathcal{U}$ be a group consist of unitary matrices in $\mathfrak{A}$. We say $\mathcal{U}$ is the generating unitary group of $\mathfrak{A}$ if $\mathfrak{A} = Span\{U \in \mathcal{U}\}$.*

- $X = \sum_{j=1}^{n} E_{j,j+1}$, $Z = \operatorname{diag}(1, \omega, \cdots, \omega^{n-1})$ *where* $\omega = e^{\frac{2\pi i}{n}}$. *Note that X and Z are both unitary matrices.*

- *We consider the* finite set

$$\mathcal{U} = \{\omega^{\ell} X^j Z^k : j, k, l = 1, 2, \cdots, n\}. \tag{4.3.5}$$

Obviously this is in fact a finite unitary group. Moreover, since $\{X^j\}_{1\leq j\leq n}$ already linearly generates $M_n(\mathbb{C})$, this is in fact a finite and generating unitary group of $M_n(\mathbb{C})$.

- *Note that any finite dimensional $*$-algebra is a direct sum of matrix algebras $M_n(\mathbb{C})$, thus the finite, generating unitary group of $\mathfrak{A}$ always exists.*

The next theorem gives a very important description of the TPCE. Briefly speaking, in finite dimension, the TPCE is a convex combination of conjugations using unitary matrices in $\mathfrak{A}'$.

Theorem 4.3.4. *Let $\mathfrak{A}$ be a $*$-subalgebra of $M_n(\mathbb{C})$ with $E_{\mathfrak{A}}$ being the TPCE. Then there exists $U_1, \cdots, U_m \in \mathfrak{A}'$ being unitary matrices in $\mathfrak{A}'$, such that*

$$E_{\mathfrak{A}}(A) = \frac{1}{m}\sum_{j=1}^{m} U_j A U_j^*, \quad \forall A \in M_n(\mathbb{C}). \tag{4.3.6}$$

Proof. Let $\mathcal{U}$ being the generating unitary group of $\mathfrak{A}'$ with $|\mathcal{U}| < \infty$. We define

$$E : M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad E(A) = \frac{1}{|\mathcal{U}|}\sum_{U\in\mathcal{U}} UAU^*. \tag{4.3.7}$$

For any $Q \in \mathcal{U} \subset \mathfrak{A}'$, we have

$$\begin{aligned} QE(A) &= \frac{1}{|\mathcal{U}|}\sum_{U\in\mathcal{U}} QUAU^* \overset{U\mapsto QU \text{ is a bijection (left-translation) on } \mathcal{U}}{=} \frac{1}{|\mathcal{U}|}\sum_{U\in\mathcal{U}} UA(Q^{-1}U)^* \\ &= \frac{1}{|\mathcal{U}|}\sum_{U\in\mathcal{U}} UAU^*Q = E(A)Q. \end{aligned} \tag{4.3.8}$$

Thus $E(A)$ commutes with $\mathcal{U}$ and thus commutes with the whole $\mathfrak{A}'$, i.e. $E(A) \in (\mathfrak{A}'') \overset{\text{Theorem 4.3.2}}{=} \mathfrak{A}$. Therefore $E : M_n(\mathbb{C}) \to \mathfrak{U}$. It is clear that E is positive. Moreover, $E(I) = I$ and for $B_1, B_2 \in \mathfrak{A}$,

$$E(B_1AB_2) = \frac{1}{|\mathcal{U}|}\sum_{U\in\mathcal{U}} UB_1AB_2U^* \overset{B_1, B_2 \in \mathfrak{A}}{=} \frac{1}{|\mathcal{U}|}\sum_{U\in\mathcal{U}} B_1UAU^*B_2 = B_1E(A)B_2. \tag{4.3.9}$$

Thus $E = E_{\mathfrak{A}}$ is a conditional expectation given $\mathfrak{A}$. We can take $m = |\mathcal{U}|$. In fact, it is easy to verify that E is trace-preserving and thus $E_{\mathfrak{A}}$ is the unique TPCE. $\square$

Corollary 15 (Finite dimensional TPCEs are CPTP)**.** *By the above characterization Theorem 4.3.4 of the TPCE, we actually have $E_{\mathfrak{A}}$ is a CPTP and unital map. In particular, we have $\|E_{\mathfrak{A}}\|_{cb} = \|E_{\mathfrak{A}}\| = \|E_{\mathfrak{A}}(I)\| = 1$.*

Example 20. *Sometimes we care about the adjoint of the conditional expectation. Since the conditional expectation itself is unital, the adjoint of the conditional expectation is a positive and trace-preserving. Physically speaking, this corresponds to the* Schrödinger picture *acting on the density matrix.*

With respect to the Hilbert-Schmidt inner product, we have

$$\mathrm{Tr}(E_{\mathfrak{A}}^*(A)^*B) = \mathrm{Tr}(A^*E_{\mathfrak{A}}(B)). \tag{4.3.10}$$

- ***The adjoint of $E_{\mathfrak{A}}$ is also an idempotent map.*** *This is because*

$$\mathrm{Tr}\big(E_{\mathfrak{A}}^{*2}(A)^*B\big) = \mathrm{Tr}(E_{\mathfrak{A}}^*(A)^*E_{\mathfrak{A}}(B)) = \mathrm{Tr}\big(A^*E_{\mathfrak{A}}^2(B)\big) = \mathrm{Tr}(A^*E_{\mathfrak{A}}(B)) = \mathrm{Tr}(E_{\mathfrak{A}}^*(A)^*B) \tag{4.3.11}$$

  for any $A, B \in M_n(\mathbb{C})$. Thus we have $E_{\mathfrak{A}}^{*2} = E_{\mathfrak{A}}^*$.

- ***The adjoint of $E_{\mathfrak{A}}$ is a bimodular map.*** *For any $C_1, C_2 \in \mathfrak{A}$, we have*

$$\begin{aligned}\mathrm{Tr}(E_{\mathfrak{A}}^*(C_1AC_2)^*B) &= \mathrm{Tr}((C_1AC_2)^*E_{\mathfrak{A}}(B)) \\ &= \mathrm{Tr}(A^*C_1^*E_{\mathfrak{A}}(B)C_2^*) \overset{E \text{ is a bimodular map, } \mathfrak{A} \text{ is a } *\text{-subalgebra}}{=} \mathrm{Tr}(A^*E_{\mathfrak{A}}(C_1^*BC_2^*)) \\ &= \mathrm{Tr}(E_{\mathfrak{A}}^*(A^*)C_1^*BC_2^*) = \mathrm{Tr}(C_2^*E_{\mathfrak{A}}^*(A)^*C_1^*B) = \mathrm{Tr}([C_1E_{\mathfrak{A}}^*(A)C_2]^*B)\end{aligned} \tag{4.3.12}$$

  for any $A, B \in M_n(\mathbb{C})$. Thus we have the bimodular property of the adjoint of the conditional expectation

$$E_{\mathfrak{A}}^*(C_1AC_2) = C_1E_{\mathfrak{A}}^*(A)C_2, \quad \forall A \in M_n(\mathbb{C}), C_1, C_2 \in \mathfrak{A}. \tag{4.3.13}$$

- ***In general, the adjoint of the conditional expectation is not a conditional expectation.*** *This is because the adjoint of the conditional expectation is unital if and only if the conditional expectation is a trace-preserving i.e. itself is a TPCE. We have Proposition 62.*

Proposition 62. *Let $\mathfrak{A}$ be a $*$-subalgebra of $M_n(\mathbb{C})$. The TPCE $E_{\mathfrak{A}}$ is a self-adjoint map w.r.t. the Hilbert-Schmidt inner product.*

Proof. By example 20, we know that $E_{\mathfrak{A}}^*$ is a unital, positive, idempotent and bimodular map. We also know that $E_{\mathfrak{A}}^* : M_n(\mathbb{C}) \to \mathfrak{A}$ is a trace-preserving map since $E_{\mathfrak{A}}$ itself is unital. Thus we have $E_{|mfA}^*$ is a TPCE onto $\mathfrak{A}$. By the uniqueness of the TPCE, we know that $E_{\mathfrak{A}}^* = E_{\mathfrak{A}}$, i.e. $E_{\mathfrak{A}}$ is self-adjoint w.r.t. the Hilbert-Schmidt inner product. □

Example 21. *In fact, with the help of the adjoint of conditional expectations, we are able to say more about the "weight" of the L^2 space corresponding to the orthogonal projection as discussed in Remark 69.*

We define the following positive linear functional

$$\rho(X) := \mathrm{Tr}(XE_{\mathfrak{A}}^*(I)), \quad \forall X \in M_n(\mathbb{C}). \tag{4.3.14}$$

Then $E_{\mathfrak{A}}$ is a D_ρ-preserving conditional expectation. That is because

$$\begin{aligned}\rho(E_{\mathfrak{A}}(A)) &= \mathrm{Tr}(E_{\mathfrak{A}}(A)E_{\mathfrak{A}}^*(I)) = \mathrm{Tr}\big(AE_{\mathfrak{A}}^{*2}(I)\big) \\ &\overset{E_{\mathfrak{A}}^* \text{ is an idempotent}}{=} \mathrm{Tr}(AE_{\mathfrak{A}}^*(I)) = \rho(A), \quad \forall A \in M_n(\mathbb{C}).\end{aligned} \tag{4.3.15}$$

Thus $E_{\mathfrak{A}}$ is a D_ρ-preserving conditional expectation.

Next we discuss two most fundamental examples of conditional expectations. The first is the so-called *pinching map* and the second is the so-called *partial trace*.

Definition 4.3.5 (Pinching maps)**.** *Let $\{P_j\}_{j=1}^m$ be a family of orthogonal projections on $\mathbb{C}^n$ such that $\sum_{j=1}^m P_j = I$ i.e. a set of unital decomposition. We define the pinching map as*

$$E : M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad E(A) = \sum_{j=1}^m P_j A P_j. \tag{4.3.16}$$

Then, it is easy to verify that E is in fact the conditional expectation from $M_n(\mathbb{C})$ onto the unital $*$-subalgebra

$$\mathfrak{A} = \bigoplus_{j=1}^m P_j M_n(\mathbb{C}) P_j. \tag{4.3.17}$$

Intuitively, the pinching map is equivalent to taking the "block diagonal" of the matrix A with respect to the orthogonal projections $\{P_j\}_{j=1}^m$.

Definition 4.3.6 (Partial trace)**.** *Consider $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$, then $\mathrm{Tr} \otimes id$ is a linear map from $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ to $M_n(\mathbb{C}) \cong \mathbb{C}I \otimes M_n(\mathbb{C}) \overset{*\text{-subalgebra}}{\subset} M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$. We define the partial trace as*

$$\mathrm{Tr}_1(A) := (\mathrm{Tr} \otimes id)(A), \quad A \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C}). \tag{4.3.18}$$

Note that

$$(\mathrm{Tr} \otimes id)(A \otimes B) = \mathrm{Tr}(A) B = \mathrm{Tr}(A) I \otimes B. \tag{4.3.19}$$

Therefore, we have $\frac{1}{m} \mathrm{Tr}_1 = \frac{1}{m} \mathrm{Tr} \otimes id$ is a conditional expectation.

Similarly, we can define the partial trace $\mathrm{Tr}_2(A) = (id \otimes \mathrm{Tr})(A)$ from $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ to $M_m(\mathbb{C}) \cong M_m(\mathbb{C}) \otimes \mathbb{C}I \overset{*\text{-subalgebra}}{\subset} M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$. Then $\frac{1}{n} id \otimes \mathrm{Tr}$ is a conditional expectation.

Remark 70. *From the structural perspetive, since we have for any finite-dimensional unital $*$-subalgebra, we can write*

$$\mathfrak{A} \cong \bigoplus_{j=1}^m M_{n_j}(\mathbb{C}) \otimes \mathbb{C} I_{n_j'}, \quad \sum_{j=1}^m n_j n_j' = n. \tag{4.3.20}$$

Thus any conditional expectation can be viewed as a combination of the pinching map and the partial trace. More explicitly, we can write

$$E_{\mathfrak{A}}(A) = \bigoplus_{j=1}^m (id_{n_j} \otimes \mathrm{Tr})(P_j A P_j) \otimes I_{n_j'} \in M_n(\mathbb{C}), \quad A \in M_n(\mathbb{C}). \tag{4.3.21}$$

4.4 Schwarz inequalities

In this section, we will discuss the convexity inequalities related to positive maps. Basically, we will apply the operator and trace Jensen inequalities to the positive maps.

We will first give an important technical lemma with respect to the structural properties of positive maps onto commutative $*$-subalgebras.

Lemma 21. *Suppose $\mathfrak{A}$ is a commutative $*$-subalgebra of $M_m(\mathbb{C})$ and $\Phi : \mathfrak{A} \to M_n(\mathbb{C})$ is a positive map. Then Φ is a completely positive map.*

Remark 71. *Basically, we have that positive $\implies$ CP for commutative $*$-algebras.*

Proof. Note that $\mathfrak{A} \cong \bigoplus_{j=1}^{\ell} \mathbb{C} I_{n_j}$, we may denote the family minimal projections as $\{P_j\}_{j=1}^{\ell}$. Suppose $A \in M_k(\mathbb{C}) \otimes \mathfrak{A}$, by the structure of $\mathfrak{A}$, we may write $A = \bigoplus_{j=1}^{\ell} A_j \otimes P_j$. Since this operator is "block-diagonalized" with respect to certian basis, we have $A \geq 0 \implies$ each $A_j \geq 0$. Thus $\Phi(A_j) \geq 0$ for each j and thus $(\mathrm{id}_k \otimes \Phi)(A) = \sum_{j=1}^{\ell} A_j \otimes \Phi(P_j) \geq 0$ and hence Φ is completely positive. □

We first deal with the case of general matrices before moving on to the case of Hermitian matrices.

Proposition 63. *Suppose that $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a* 2-positive *map. Then for any $A \in M_m(\mathbb{C})$,*

$$\Phi(A)^*\Phi(A) \leq \|\Phi\| \Phi(A^*A). \tag{4.4.1}$$

Moreover, $\|\Phi\| = \|\Phi(I)\|$.

Proof. By the fact that Φ is 2-positive, we have:

$$\begin{pmatrix} I & A \\ A^* & A^*A \end{pmatrix} = \begin{pmatrix} I & \\ A^* & \end{pmatrix} \begin{pmatrix} I & A \\ & \end{pmatrix} \geq 0 \Rightarrow \begin{pmatrix} \Phi(I) + \varepsilon & \Phi(A) \\ \Phi(A)^* & \Phi(A^*A) + \varepsilon \end{pmatrix} \geq 0. \tag{4.4.2}$$

By the Schur complement lemma Lemma 3 we have

$$\Phi(A^*A) + \varepsilon \geq \Phi(A)^*[\Phi(I) + \varepsilon]^{-1}\Phi(A) \geq 0 \implies \Phi(A^*A) + \varepsilon \geq \Phi(A)^* \|\Phi(I) + \varepsilon\|^{-1} \Phi(A) \geq 0. \tag{4.4.3}$$

This indicates that

$$\|\Phi(I)\| \Phi(A^*A) \geq \Phi(A)^*\Phi(A) \tag{4.4.4}$$

by taking limit $\varepsilon \to 0$. Note again that $\Phi(A^*A) \leq \|A\|^2 \Phi(I)$, we have that $\|\Phi(A)\| \leq \|A\| \|\Phi(I)\|$. Thus we have $\|\Phi(I)\| \geq \|\Phi\|$ i.e. $\|\Phi\| = \|\Phi(I)\|$. □

Remark 72. *For non-Hermitian case, we see that the "square" is essential. Thus we need to assume that Φ is 2-positive to "open up" the structure.*

Theorem 4.4.1 (Choi-Schwarz inequality for positive maps)**.** *Suppose that $\mathfrak{A} \subset M_m(\mathbb{C})$ is a $*$-subalgebra, $\Phi : \mathfrak{A} \to M_n(\mathbb{C})$ is a positive map with $\|\Phi\| \leq 1$ and $f : \mathrm{Dom}(f) \subset \mathbb{R} \to \mathbb{R}$ is an operator convex function with $f(0) \leq 0$ and $Sp(A) \subset \mathrm{Dom}(f)$. Then we have*

$$f(\Phi(A)) \leq \Phi(f(A)), \quad \forall A \in \mathfrak{A} \cap \mathbb{H}_m. \tag{4.4.5}$$

Lemma 22 (Choi-Schwarz inequality for CP maps)**.** *Suppose that $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a completely positive map. Let f be an operator convex function with $f(0) \leq 0$ and $Sp(A) \subset \mathrm{Dom}(f)$. For convenience we assume further that $\|\Phi\| \leq 1$. Then we have the Schwarz inequality eq. (4.4.5) holds for any $A \in \mathbb{H}_m$.*

Proof of Lemma 22. By Stinespring dilation theorem Theorem 4.2.3, we have

$$\Phi(A) = V^*\pi(A)V, \quad \pi : M_m(\mathbb{C}) \to \mathcal{B}(\mathcal{H}) \text{ } *\text{-homomorphism}, \quad V : \mathbb{C}^n \to \mathcal{H} \text{ bounded linear operator.} \tag{4.4.6}$$

Moreover, $\|\Phi\|_{cb} = \|V\|^2$ and $\|\Phi\| \leq 1$ implies that $V^*V \leq 1$. By operator Jesnsen inequality Theorem 2.2.2, we have

$$f(V^*\pi(A)V) \leq V^* f(\pi(A))V = V^*\pi(f(A))V = \Phi(f(A)). \tag{4.4.7}$$

□

Corollary 16. *Lemma 22 holds for any unital $*$-subalgebra $\mathfrak{A}$ of $M_m(\mathbb{C})$.*

Proof of Corollary 16. We take the TPCE $E_{\mathfrak{A}}$ from $M_m(\mathbb{C})$ to $\mathfrak{A}$, then $\Phi \circ E_{\mathfrak{A}}$ is a completely positive map from $M_m(\mathbb{C})$ to $M_n(\mathbb{C})$ by "the fact that f.d. TPCEs are CPTP" (see Corollary 15). $\square$

Proof of Theorem 4.4.1. Let $C^*(A)$ be the unital $*$-subalgebra of $\mathfrak{A}$ generated by A and I, then it is clear that $C^*(A)$ is commutative. By Lemma 21, we have $\Phi|_{C^*(A)} : C^*(A) \to M_n(\mathbb{C})$ and $E_{C^*(A)} : \mathfrak{A} \to C^*(A)$ are completely positive maps. Thus we have $\Phi|_{C^*(A)} \circ E_{C^*(A)}$ is a completely positive map. Then the result follows from Corollary 16. $\square$

Remark 73. *The proof strongly depends on the structural properties of positive maps on commutative $*$-subalgebras, which significantly strengthens the result in the sense that we do not need to assume Φ is completely positive but only 1-positive (in the case of Hermitian matrices).*

Corollary 17 (Kadison-Schwarz inequality)**.** *$\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a positive map with $\|\Phi\| \le 1$, $A \in \mathbb{H}_m$, then we have $\Phi(A)^2 \le \Phi(A^2)$.*

Corollary 18. *$\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital positive map with $\|\Phi\| \le 1$, then we have $\Phi(A)^{-1} \le \Phi(A^{-1})$ for any $A \ge 0$.*

Proof. $t \mapsto t^{-1}$ is operator convex on $(0, \infty)$, thus $t \mapsto f(t) := \frac{1}{t+\varepsilon} - \frac{1}{\varepsilon}$ is operator convex on $(0, \infty)$ for any $\varepsilon > 0$ and we have $f(0) = 0$. By Choi-Schwarz inequality Theorem 4.4.1, we have

$$(\Phi(A) + \varepsilon)^{-1} - \varepsilon^{-1} \le \Phi((A + \varepsilon)^{-1} + \varepsilon^{-1}) - \varepsilon^{-1}. \tag{4.4.8}$$

Thus we have

$$(\Phi(A) + \varepsilon)^{-1} \le \Phi((A + \varepsilon)^{-1}) + \varepsilon^{-1}\Phi(I) - \varepsilon^{-1} \overset{\Phi \text{ is unital}}{=} \Phi((A + \varepsilon)^{-1}). \tag{4.4.9}$$

Taking limit $\varepsilon \to 0$, we have $\Phi(A)^{-1} \le \Phi(A^{-1})$. $\square$

Comparing this result with Proposition 63, we have a natural question: can we generalize the Choi-Schwarz type inequality to the case of non-Hermitian matrices? The answer is yes, however, as we may expect, we need to assume that the map is 2-positive. Moreover, since our strategy is to use the "Hermitian-dilation" of the non-Hermitian matrices, we need to assume that the operator convex function is even.

Theorem 4.4.2. *Let $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a 2-positive map with $\|\Phi\| \le 1$. f is an operator convex function with $f(0) \le 0$ and $Sp(|A|) \subset \mathrm{Dom}(f) = (-a, a)$ $(a > 0)$. Assume that f is an even function, then we have*

$$f(|\Phi(A)|) \le \Phi(f(|A|)) \tag{4.4.10}$$

Proof. We have $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ is Hermitian and $\mathrm{id}_2 \otimes \Phi$ is positive, thus by Choi-Schwarz inequality Theorem 4.4.1 we have

$$f\begin{pmatrix} 0 & \Phi(A) \\ \Phi(A)^* & 0 \end{pmatrix} \le (\mathrm{id}_2 \otimes \Phi)\left[f\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\right]. \tag{4.4.11}$$

Since f is even, we have the following functional calculus

$$
\begin{aligned}
f\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} &= f\left(\left|\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\right|\right) = f\left(\left|\begin{pmatrix} 0 & U|A| \\ |A|U^* & 0 \end{pmatrix}\right|\right) \\
&= f\left(\left|\begin{pmatrix} 0 & U|A| \\ |A|U^* & 0 \end{pmatrix}\right|\right) = f\left[\left|\begin{pmatrix} 0 & U \\ I & 0 \end{pmatrix}\begin{pmatrix} |A| & 0 \\ 0 & |A| \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}^*\right|\right] = f\begin{pmatrix} |A| & 0 \\ 0 & |A| \end{pmatrix} \\
&= \begin{pmatrix} f(|A|) & 0 \\ 0 & f(|A|) \end{pmatrix}.
\end{aligned}
\tag{4.4.12}
$$

Then eq. (4.4.11) becomes

$$
\begin{pmatrix} f(|\Phi(A)|) & 0 \\ 0 & f(|\Phi(A)|) \end{pmatrix} \leq (\mathrm{id}_2 \otimes \Phi)\begin{pmatrix} f(|A|) & 0 \\ 0 & f(|A|) \end{pmatrix} = \begin{pmatrix} \Phi(f(|A|)) & 0 \\ 0 & \Phi(f(|A|)) \end{pmatrix}. \tag{4.4.13}
$$

Thus we have

$$
f(|\Phi(A)|) \leq \Phi(f(|A|)). \tag{4.4.14}
$$

$\square$

Remark 74. *Take $f(t) = t^2$, we have $|\Phi(A)|^2 \leq \Phi(|A|^2)$. Thus we have $\Phi(A)^*\Phi(A) \leq \Phi(A^*A)$ for $\|\Phi\| \leq 1$ and 2-positive. This is just what we have proved in the original version of Schwarz inequality Proposition 63.*

We can also apply the Jensen trace inqualities Theorem 3.3.1 and Proposition 34 to positive maps to obtain the so-called "trace Schwarz inequality".

Lemma 23. *$\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ a completely positive map with $\|\Phi\| \leq 1$. $f : \mathrm{Dom}(f) \to \mathbb{R}$ a convex (no need to be operator convex) function with $f(0) \leq 0$. Then we have*

$$
\mathrm{Tr}\, f(\Phi(A)) \leq \mathrm{Tr}\, \Phi(f(A)), \quad \forall A \in \mathbb{H}_m. \tag{4.4.15}
$$

Proof. It follows from the Stinespring dilation theorem Theorem 4.2.3 and Jensen trace inequality Proposition 34 in the similar way as the proof of Lemma 22. $\square$

Then by imitating the proof of Theorem 4.4.1, we can easily obtain

Theorem 4.4.3. *$\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a positive (no need to be completely positive) map with $\|\Phi\| \leq 1$. $f : \mathrm{Dom}(f) \to \mathbb{R}$ is an operator convex function with $f(0) \leq 0$ and $Sp(A) \subset \mathrm{Dom}(f)$. Then we have that for any $A \in \mathbb{H}_m$ with $sp(A) \subset \mathrm{Dom}(f)$,*

$$
\mathrm{Tr}\, f(\Phi(A)) \leq \mathrm{Tr}\, \Phi(f(A)). \tag{4.4.16}
$$

4.5 Strong subadditivity of entropy functionals

Example 22. *Let D be a density matrix in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \otimes M_l(\mathbb{C})$ (a tripartite system), we consider*

$$
\begin{aligned}
D_{12} &= \mathrm{Tr}_3(D) \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C}), \quad \mathrm{Tr}\, D_{12} = 1, \\
D_{23} &= \mathrm{Tr}_1(D) \in M_m(\mathbb{C}) \otimes M_l(\mathbb{C}), \quad \mathrm{Tr}\, D_{23} = 1,
\end{aligned}
\tag{4.5.1}
$$

as well as

$$D_2 = \mathrm{Tr}_{13}(D) = (\mathrm{Tr} \otimes id \otimes \mathrm{Tr})(D) \in M_m(\mathbb{C}), \quad \mathrm{Tr}\, D_2 = 1. \tag{4.5.2}$$

That is, D_{12}, D_{23} *and* D_2 *are still density matrices in the corresponding matrix algebras. We consider the entropy functional*

$$H(D) = -\,\mathrm{Tr}(D \log D). \tag{4.5.3}$$

We ask: what is the relation between the entropies of these density matrices?

The answer is given by the strong subadditivity of the entropy functional, which states that

$$H(D) + H(D_2) \leq H(D_{12}) + H(D_{23}). \tag{4.5.4}$$

Example 23. *In this example, we will give some techniques for computing the partial trace. Let* $D \in \bigotimes_{i=1}^k M_{n_i}(\mathbb{C})$ *and* $\mathbb{A} \subset \{1, \cdots, k\}$*. We denote*

$$D_{\mathbb{A}} = \mathrm{Tr}_{\mathbb{A}^c}(D) = \mathrm{Tr}_{\{1,\cdots,k\}\setminus\mathbb{A}}(D) \in \bigotimes_{i\in\mathbb{A}} M_{n_i}(\mathbb{C}). \tag{4.5.5}$$

Then we have

$$\mathrm{Tr}(Df(D_{\mathbb{A}})) = \mathrm{Tr}(D_{\mathbb{A}} f(D_{\mathbb{A}})). \tag{4.5.6}$$

That is because,

$$\mathrm{Tr}((A \otimes B)\,\mathrm{Tr}_1 C) = \mathrm{Tr}((A \otimes B)(I \otimes \mathrm{Tr}_1 C)) = \mathrm{Tr}\, A\, \mathrm{Tr}(B\, \mathrm{Tr}_1 C) = \mathrm{Tr}(\mathrm{Tr}_1(A \otimes B)\,\mathrm{Tr}_1 C). \tag{4.5.7}$$

Thus in general, we have

$$\mathrm{Tr}(D\,\mathrm{Tr}_1 C) = \mathrm{Tr}(\mathrm{Tr}_1 D\,\mathrm{Tr}_1 C), \quad \textit{where } \mathrm{Tr}_1 C \textit{ on LHS should be understood as } I \otimes \mathrm{Tr}_1 C. \tag{4.5.8}$$

For general conditional expectations, we can do the same calculations. For simplicity, we consider again the partical trace which we now write as E_{12}*, denoting the conditional expectation that keep the first two factors. A concrete example is*

$$\begin{aligned}
\mathrm{Tr}(D_{12} f(D_2) D_{23} g(D_2)) &= \mathrm{Tr}(E_{12}(D_{12}) f(D_2) D_{23} g(D_2)) \\
&\overset{E_{12} \text{ is self-adjoint by Proposition 62}}{=} \mathrm{Tr}(D_{12} E_{12}[f(D_2) D_{23} g(D_2)]) \\
&\overset{E_{12} \text{ is a bimodular map}}{=} \mathrm{Tr}(D_{12} f(D_2) E_{12}[D_{23}] g(D_2)) \\
&= \mathrm{Tr}(D_{12} f(D_2) D_2 g(D_2)) = \mathrm{Tr}(D_{12} E_2[f(D_2) D_2 g(D_2)]) \\
&\overset{E_2 \text{ is self-adjoint}}{=} \mathrm{Tr}(E_2(D_{12}) f(D_2) D_2 g(D_2)) = \mathrm{Tr}(D_2 f(D_2) D_2 g(D_2))
\end{aligned} \tag{4.5.9}$$

It is easy to see that similar calculations can also be generalized to abstract conditional expectations.

Theorem 4.5.1 (Strong subadditivity of entropy functionals)**.** *Let* $D \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \otimes M_l(\mathbb{C})$ *be a density matrix, then we have*

$$H(D) + H(D_2) \leq H(D_{12}) + H(D_{23}). \tag{4.5.10}$$

Proof. Note that

$$
\begin{aligned}
&H(D_{12}) + H(D_{23}) - H(D) - H(D_2) \\
&= -\operatorname{Tr}(D_{12}\log D_{12}) - \operatorname{Tr}(D_{23}\log D_{23}) + \operatorname{Tr}(D\log D) + \operatorname{Tr}(D_2\log D_2) \\
&\overset{\text{by example 23}}{=} -\operatorname{Tr}(D\log D_{12} - D\log D_{23} + D\log D + D\log D_2) \\
&= \operatorname{Tr}(D\log D - D[\log D_{12} + \log D_{23} - \log D_2]) \\
&\overset{X := e^{\log D_{12}+\log D_{23}-\log D_2}}{=} \operatorname{Tr}(D\log D - D\log X) = H\left(D\|\frac{X}{\operatorname{Tr} X}\right) - \log\operatorname{Tr} X \\
&\overset{\text{by the non-negativity of relative entropy}}{\geq} -\log\operatorname{Tr} X.
\end{aligned}
\tag{4.5.11}
$$

Thus it remains to verify that $\operatorname{Tr} X \leq 1$. This follows readily from the Golden-Thompson-Lieb inequality Theorem 3.8.2

$$
\begin{aligned}
\operatorname{Tr}(X) &\overset{\text{Theorem 3.8.2}}{\leq} \int_0^\infty \operatorname{Tr}\big(D_{12}(t+D_2)^{-1}D_{23}(t+D_2)^{-1}\big)\mathrm{d}t \\
&\overset{\text{example 23}}{=} \int_0^\infty \operatorname{Tr}\big(D_2(t+D_2)^{-1}D_2(t+D_2)^{-1}\big)\mathrm{d}t \\
&= \operatorname{Tr} D_2 = 1.
\end{aligned}
\tag{4.5.12}
$$

□

Corollary 19 (Subadditivity of entropy). *$D \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ is a density matrix, then we have*

$$H(D) \leq H(D_1) + H(D_2). \tag{4.5.13}$$

Proof. We view D as a matrix in $M_n(\mathbb{C}) \otimes M_r(\mathbb{C}) \otimes M_m(\mathbb{C})$. Then $D_{12} = D_1$, $D_{23} = D_2$ and $D_2 = I$. Then it follows from Theorem 4.5.1 that

$$H(D) \leq H(D_{12}) + H(D_{23}) - H(D_2) = H(D_1) + H(D_2) - H(I) = H(D_1) + H(D_2). \tag{4.5.14}$$

□

4.6 Generalized data processing inequality

In this section, we consider the invertible density matrix $D \in \mathbb{H}_n^{>0}$.

We define the left action and right action of D on $M_n(\mathbb{C})$ as follows:

$$\mathbf{L}_D : M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad X \mapsto DX, \quad \mathbf{R}_D : M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad X \mapsto XD. \tag{4.6.1}$$

Definition 4.6.1 (Quasi-entropy). *Let $D_1, D_2 \in \mathbb{H}_n^{>0}$ be two invertible density matrices, $f : [0,\infty) \to [0,\infty)$ be a operator monotone function with $f(0) \geq 0$, we define the* quasi-entropy *as*

$$
\begin{aligned}
H_f^A(D_1\|D_2) &:= \operatorname{Tr}\Big(D_2^{\frac{1}{2}}A^* f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})(AD_2^{\frac{1}{2}})\Big) \\
&= \langle f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})(AD_2^{\frac{1}{2}}), AD_2^{\frac{1}{2}}\rangle \\
&= \operatorname{Tr}\big(A^* f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})(\mathbf{R}_{D_2}(A))\big)
\end{aligned}
\tag{4.6.2}
$$

Remark 75. *Without ambiguity, we can also write*

$$\operatorname{Tr}\left(D_2^{\frac{1}{2}} A^* f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})(AD_2^{\frac{1}{2}})\right) \operatorname{Tr}\left(D_2^{\frac{1}{2}} A^* f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})(A) D_2^{\frac{1}{2}}\right). \tag{4.6.3}$$

That is because, for polynomial f, we obviously have

$$f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})(A)D_2^{\frac{1}{2}} = f(\mathbf{L}_{D_1}\mathbf{R}_{D_2^{-1}})(A)D_2^{\frac{1}{2}} = f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})(AD_2^{\frac{1}{2}}). \tag{4.6.4}$$

Definition 4.6.2 (Schwarz mapping)**.** *We say Φ is a Schwarz mapping, if it satisfies*

$$\Phi(A^*)\Phi(A) \leq \Phi(A^*A), \quad \forall A \in M_n(\mathbb{C}). \tag{4.6.5}$$

Example 24. *By Proposition 63 or Theorem 4.4.2, we have that any 2-positive map with $\|\Phi\| \leq 1$ is a Schwarz mapping.*

If Φ is a CPTP map, then Φ^* is a CP and unital map with $\|\Phi^*\|_{cb} = \|\Phi^*\| = \|\Phi^*(I)\| = 1$, thus Φ^* is a unital Schwarz mapping.

Theorem 4.6.3 (Generalized data processing inequality)**.** *Let $D_1, D_2 \in \mathbb{H}_n^{>0}$ be two invertible density matrices, $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a unital Schwarz mapping and $f : [0,\infty) \to [0,\infty)$ be an operator monotone function with $f(0) \geq 0$. Then we have*

$$H_f^A(\Phi^*(D_1)\|\Phi^*(D_2)) \geq H_f^{\Phi(A)}(D_1\|D_2). \tag{4.6.6}$$

Proof. Without loss of generality, we assume that $f(0) = 0$ since

$$H_{f+\lambda}^A(D_1\|D_2) = H_f^A(D_1\|D_2) + \lambda \operatorname{Tr}(A^*AD_2). \tag{4.6.7}$$

We assume further that $\Phi^*(D_2) > 0$. We define a linear map

$$\mathbf{V} : M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad X\Phi^*(D_2)^{\frac{1}{2}} \mapsto \Phi(X)D_2^{\frac{1}{2}}, \quad X \in M_n(\mathbb{C}). \tag{4.6.8}$$

This map is well-defined since $\Phi^*(D_2)^{\frac{1}{2}}$ is invertible. We have

$$\begin{aligned} \left\|\Phi(X)D_2^{\frac{1}{2}}\right\|^2 &= \operatorname{Tr}(D_2\Phi(X)^*\Phi(X)) \overset{\Phi \text{ is a Schwarz mapping}}{\leq} \operatorname{Tr}(D_2\Phi(X^*X)) = \operatorname{Tr}(\Phi^*(D_2)(X^*X)) \\ &= \left\|X\Phi^*(D_2)^{\frac{1}{2}}\right\|^2, \quad \forall X \in M_n(\mathbb{C}) \Rightarrow \|\mathbf{V}\| \leq 1. \end{aligned} \tag{4.6.9}$$

Thus we can estimate

$$\begin{aligned} \langle \mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1}\mathbf{V}X\Phi^*(D_2)^{\frac{1}{2}}, \mathbf{V}X\Phi^*(D_2)^{\frac{1}{2}}\rangle &= \langle \mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1}\Phi(X)D_2^{\frac{1}{2}}, \Phi(X)D_2^{\frac{1}{2}}\rangle \\ &= \operatorname{Tr}(D_1\Phi(X)\Phi(X^*)) \\ &\overset{\text{Schwarz}}{\leq} \operatorname{Tr}(D_1\Phi(X^*X)) \\ &= \operatorname{Tr}(\Phi^*(D_1)X^*X) \\ &= \langle \mathbf{L}_{\Phi^*(D_1)}\mathbf{R}_{\Phi^*(D_2)}^{-1}X\Phi^*(D_2)^{\frac{1}{2}}, X\Phi^*(D_2)^{\frac{1}{2}}\rangle. \end{aligned} \tag{4.6.10}$$

Thus we have

$$\mathbf{V}^*\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1}\mathbf{V} \leq \mathbf{L}_{\Phi^*(D_1)}\mathbf{R}_{\Phi^*(D_2)}^{-1}. \tag{4.6.11}$$

By the operator monotonicity of f, we have

$$f(\mathbf{V}^*\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1}\mathbf{V}) \le f(\mathbf{L}_{\Phi^*(D_1)}\mathbf{R}_{\Phi^*(D_2)}^{-1}). \tag{4.6.12}$$

By the equivalence of operator convexity and operator monotonicity Theorem 2.2.3, we have that f is operator convex, thus

$$\mathbf{V}^* f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})\mathbf{V} \le f(\mathbf{L}_{\Phi^*(D_1)}\mathbf{R}_{\Phi^*(D_2)}^{-1}). \tag{4.6.13}$$

By the operator monotonicity of the mapping $\mathrm{Tr}\Big(\Phi^*(D_2)^{\frac{1}{2}}A^*(\cdot)A\Phi^*(D_2)^{\frac{1}{2}}\Big)$, we have

$$\mathrm{Tr}\Big(\Phi^*(D_2)^{\frac{1}{2}}A^*\mathbf{V}^*f(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})\mathbf{V}A\Phi^*(D_2)^{\frac{1}{2}}\Big) \le \mathrm{Tr}\Big(\Phi^*(D_2)^{\frac{1}{2}}A^*f(\mathbf{L}_{\Phi^*(D_1)}\mathbf{R}_{\Phi^*(D_2)}^{-1})A\Phi^*(D_2)^{\frac{1}{2}}\Big) \tag{4.6.14}$$

It follows readily that the generalized data processing inequality holds.

For $\Phi^*(D_2) \ge 0$, we only need to consider $\Phi^*(D_2) + \varepsilon$ ($\varepsilon > 0$) and we have

$$H_f^A(\Phi^*(D_1)\|\Phi^*(D_2)+\varepsilon) \ge H_f^{\Phi(A)}(D_1\|D_2+\varepsilon). \tag{4.6.15}$$

Then we take $\varepsilon \to 0$.

□

Corollary 20. *If Φ^* is a quantum channel (CPTP map), then we have*

$$H_f^A(\Phi(D_1)\|\Phi(D_2)) \le H_f^{\Phi^*(A)}(D_1\|D_2). \tag{4.6.16}$$

Proof. Recall example 24, we have Φ^* is a unital Schwarz mapping, thus we can apply Theorem 4.6.3 to obtain the result. □

Corollary 21. *Let $D_1, D_2 \in \mathbb{H}_n^{>0}$ be two invertible density matrices, $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a CPTP map, then*

$$H(\Phi(D_1)\|\Phi(D_2)) \le H(D_1\|D_2). \tag{4.6.17}$$

Proof. Take $f(t) = t^\alpha$ for $\alpha \in (0,1)$, then by Theorem 4.6.3, we have

$$\mathrm{Tr}\Big(A^*(\mathbf{L}_{\Phi(D_1)}\mathbf{R}_{\Phi(D_2)}^{-1})^\alpha\mathbf{R}_{\Phi(D_2)}(A)\Big) \ge \mathrm{Tr}\big(\Phi^*(A)(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1})^\alpha\Phi(A)\big). \tag{4.6.18}$$

Note that $\log t = \lim_{\alpha\to 0}\frac{t^\alpha - 1}{\alpha}$, thus we have

$$\begin{aligned}&\mathrm{Tr}\Big(A^*\log\Big(\mathbf{L}_{\Phi(D_1)}\mathbf{R}_{\Phi(D_2)}^{-1}\Big) + \mathrm{Tr}(A^*A\Phi(D_2))\mathbf{R}_{\Phi(D_2)}(A)\Big)\\ &\ge \mathrm{Tr}\big(\Phi^*(A)\log\big(\mathbf{L}_{D_1}\mathbf{R}_{D_2}^{-1}\big)\Phi(A)\big) + \mathrm{Tr}(\Phi(A)^*\Phi(A)\Phi(D_2)).\end{aligned} \tag{4.6.19}$$

We take $A = I$, then

$$\mathrm{Tr}[(\log\Phi(D_1) - \log\Phi(D_2))\Phi(D_2)] \ge \mathrm{Tr}[(\log D_1 - \log D_2)D_2]. \tag{4.6.20}$$

□

4.7 Quantum Perron-Frobenius theorem

In this section we consider the so-called *strictly positive* and *positive irreducible* matrices. Basically, we will study their spectral properties. Next we will see that we can actually put these analysis into a more abstract framework of *strictly positive maps* and *positive irreducible maps*.

Definition 4.7.1. *We say a real matrix $A \in M_n(\mathbb{R})$ is* positive or non-negative *(*strictly positive*) if $a_{ij} \geq (>)\, 0$ for all $1 \leq i, j \leq n$, denoted as $A \trianglerighteq (\triangleright)\, 0$.*

We denote $|A|_+ := (|a_{ij}|)_{1\leq i,j\leq n}$.

Lemma 24. *$A, B \in M_n(\mathbb{C})$ and $B \trianglerighteq 0$, $|A|_+ \trianglelefteq B$, then $r(A) \leq r(|A|_+) \leq r(B)$.*

Here, $r(A)$ is a spectral radius of A and can be explicitly computed as

$$r(A) = \lim_{m\to\infty} \|A^m\|^{\frac{1}{m}} = \lim_{m\to\infty} \|A^m\|_2^{\frac{1}{m}}. \tag{4.7.1}$$

Proof. We observe that

$$|A^m|_+ \trianglelefteq |A|_+^m \trianglelefteq B^m. \tag{4.7.2}$$

Since we want to use the information of the entries, we use the formula corresponding to the 2-norm

$$\|A^m\|_2 \leq \left\| |A^m|_+ \right\|_2 \leq \left\| |A|_+^m \right\|_2 \leq \|B^m\|_2. \tag{4.7.3}$$

Thus we have

$$r(A) \leq r(|A|_+^m) \leq r(B). \tag{4.7.4}$$

□

Remark 76. *This lemma can be viewed as an* analogue *of the spectral radius bound given by the order of positive definite matrices. That is, for Hermitian matrices A, B, if $|A| \leq B$, then we also have*

$$r(A) \leq r(B). \tag{4.7.5}$$

Lemma 25 (Row-sum bounds of spectral radii)**.** *A is a positive matrix, then we have*

$$\min_{1\leq j\leq n} \sum_{k=1}^{n} a_{jk}\ (\textit{the smallest row-sum.}) \leq r(A) \leq \max_{1\leq j\leq n} \sum_{k=1}^{n} a_{jk}\ (\textit{the largest row-sum.}). \tag{4.7.6}$$

Note that $r(A) = r(A^T)$, we have further

$$\min_{1\leq k\leq n} \sum_{j=1}^{n} a_{jk}\ (\textit{the smallest column-sum.}) \leq r(A) \leq \max_{1\leq k\leq n} \sum_{j=1}^{n} a_{jk}\ (\textit{the largest column-sum.}). \tag{4.7.7}$$

Proof. Let $\alpha = \min_{1\leq j\leq n} \sum_{k=1}^{n} a_{jk} \geq 0$.

If $\alpha = 0$, we take $B = 0$. If $\alpha > 0$, we take $B = (b_{jk})_{1\leq j,k\leq n} = \left(\frac{a_{jk}}{\sum_{\ell=1}^{n} a_{j\ell}}\alpha\right)$. Then $A \trianglerighteq B$ holds obviously (since $\alpha \geq \sum_{\ell=1}^{n} a_{j\ell}$). By the previous lemma Lemma 24, we have $r(B) \leq r(A)$.

Note that after this arrangement, we have

$$B\mathbf{1} = \left(\sum_{k=1}^{n} b_{jk}\right)_{1\leq j\leq n} = (\alpha)_{1\leq j\leq n} = \alpha\mathbf{1} \implies \alpha \in \mathrm{Sp}(B). \tag{4.7.8}$$

Thus we have

$$\alpha \leq r(B) \leq r(A) \implies \min_{1\leq j\leq n} \sum_{k=1}^{n} a_{jk} \leq r(A). \tag{4.7.9}$$

correspondingly, for $\beta := \max_{1\leq j\leq n} \sum_{k=1}^{n} a_{jk} \geq 0$. We take $C := (c_{jk})_{1\leq j,k\leq n} = \left(\frac{a_{jk}}{\sum_{\ell=1}^{n} a_{j\ell}}\beta\right)_{1\leq j,k\leq n}$. Then $A \trianglerighteq C$ holds obviously. By the previous lemma Lemma 24, we have $r(C) \leq r(A)$.

Let λ be in the spectrum of C such that $|\lambda| = r(C)$. Let the eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$ corresponding to λ, then we have

$$\left|\sum_{k=1}^{n} c_{jk} x_k\right| = |\lambda x_j| \implies r(C)|x_j| \leq \left(\max_{1\leq k\leq n} |x_k|\right) \sum_{k=1}^{n} c_{jk} = \max_{1\leq k\leq n} |x_k| \cdot \beta. \tag{4.7.10}$$

Thus

$$r(C) \leq \beta \cdot \frac{\max_{1\leq k\leq n} |x_k|}{|x_j|} \quad \forall 1 \leq j \leq n \implies r(C) \leq \beta. \tag{4.7.11}$$

Thus we have

$$r(A) \leq r(C) \leq \beta = \max_{1\leq j\leq n} \sum_{k=1}^{n} a_{jk}. \tag{4.7.12}$$

□

It follows readily that we have the following proposition.

Proposition 64. *Let A be a positive matrix, then we have*

(1) If there exists a strictly positive vector $x \in \mathbb{C}^n$, $\alpha > 0$, such that $Ax \geq (>)\, \alpha x$, then $r(A) \geq (>)\, \alpha$;

(2) If there exists a strictly positive vector $x \in \mathbb{C}^n$, $\alpha > 0$, such that $Ax = \alpha x$, then $r(A) = \alpha$;

(3) If there exists a strictly positive vector $x \in \mathbb{C}^n$, $\alpha > 0$, such that $Ax \leq (<)\, \alpha x$, then $r(A) \leq (<)\, \alpha$.

Remark 77. *By A positive, x positive, we have $Ax \geq 0$.*

Remark 78. *We only prove (1) and (3). (2) follows from (1) and (3).*

Proof. Since $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is strictly positive, we have $X = \operatorname{diag}(x_1, \cdots, x_n)$ is invertible. Thus we have

$$r(A) = r(X^{-1}AX) \overset{\text{Lemma 25}}{\geq} \min_{1\leq j\leq n} a_{jk} x_j^{-1} x_k \overset{Ax \geq \alpha x}{\geq} \min_{1\leq j\leq n} \alpha x_j x_j^{-1} = \alpha. \tag{4.7.13}$$

If $Ax \geq \alpha x$ holds strictly, then the second inequality is strict, thus we have $r(A) > \alpha$. Likewise we have $Ax \leq (<)\, \alpha x$ implies $r(A) \leq (<)\, \alpha$. □

With the above analysis, we are now at the position to give a structural characterization of positive matrices, particularly, their largest eigenvalue and the corresponding eigenvector.

First, we will show that the largest eigenvalue must be the spectral radius of the positive matrix.

Proposition 65. *Let A be a positive matrix, then we have $r(A) \in Sp(A)$. Furthermore, there exists two positive (non-negative) vectors x, y such that*

$$Ax = r(A)x, \quad y^T A^T = r(A)y^T. \tag{4.7.14}$$

Furthermore, if A is strictly positive, then x, y can be chosen to be strictly positive *and* unique. *In this case, the geometric multiplicity of the eigenvalue $r(A)$ is* 1.

Proof. **We begin with the case of strictly positive matrices.** Let $\lambda \in \mathrm{Sp}(A)$ such that $|\lambda| = r(A)$ and the corresponding eigenvector be v. Then we have

$$r(A)|v| = |\lambda v| = |Av| \overset{\text{triangular inequality}}{\leq} |A|_+|v| = A|v|. \tag{4.7.15}$$

Here $|x|$ denotes taking the absolute value of the vector x entrywise. We denote

$$w = A|v| - r(A)|v| \geq 0. \tag{4.7.16}$$

Case 1. If $w = 0$, then $A|v| = |\lambda||v|$. Note that A is strictly positive, $|v| \geq 0$, $|v| \neq 0$, we have $A|v| > 0$ is strictly positive. Thus we take $x = A|v| > 0$, then we have

$$Ax = A(A|v|) = |\lambda|A(|v|) = r(A)x. \tag{4.7.17}$$

Case 2. If $w > 0$, then Aw is also strictly positive, thus

$$0 < Aw = A(A|v|) - |\lambda|(A|v|) =: Az - |\lambda|z. \tag{4.7.18}$$

By Proposition 64, $|\lambda|$ is strictly upper-bounded by $r(A)$ i.e. $r(A) > |\lambda|$. But we know $|\lambda| = r(A)$, thus we have $r(A) > r(A)$, which is a contradiction.

Thus we have $A|v| = r(A)|v| = x$.

For the uniqueness, let x' be **a real vector** such that $Ax' = r(A)x'$, then we can define

$$t_* = \min\{t > 0 : tx - x' \geq 0\} > 0. \tag{4.7.19}$$

Then, $t_*x - x' \geq 0$, and there is at least one entry is 0 in $t_*x - x'$. If all the entries are zero i.e. $t_*x - x' = 0$, then x and x' are proportional. Otherwise, by the strict positivity of A again, we have $A(t_*x - x')$ is strictly positive, thus we have

$$0 < A(t_*x - x') = r(A)(t_*x - x'), \tag{4.7.20}$$

which means that $t_*x - x'$ is strictly positive, which contradicts the fact that $t_*x - x' \geq 0$ and at least one entry is zero. Thus we have the uniqueness. Note that we prove the uniqueness the eigenvector of $r(A)$ as a real vector and not only as a strictly positive vector. Thus we can conclude that the geometric multiplicity of the eigenvalue $r(A)$ is 1.

For the case that A is non-negative, we only need to consider $A + \varepsilon\mathbf{1}$, where $\mathbf{1} = (1)_{1\leq j,k\leq n}$. Then by the previous case there exists $x_\varepsilon \in \mathbb{S}^{n-1}$ such that $(A + \varepsilon\mathbf{1})x_\varepsilon = r_\varepsilon(A)x_\varepsilon$.

- First, we have

$$\lim_{\varepsilon} r_\varepsilon(A) = \lim_{\varepsilon\to 0}\lim_{m\to\infty} \|(A + \varepsilon\mathbf{1})^m\|_2^{\frac{1}{m}} = \lim_{m\to\infty}\lim_{\varepsilon\to 0} \|(A + \varepsilon\mathbf{1})^m\|_2^{\frac{1}{m}} = r(A). \tag{4.7.21}$$

- Then, by the compactness of the unit sphere $\mathbb{S}^{n-1}$, we can take a convergent subsequence and then take the limit to conclude that there exists some non-negative vector $x \in \mathbb{S}^{n-1}$ such that $Ax = r(A)x$.

$\square$

We will show next that the algebraic multiplicity of the eigenvalue $r(A)$ is also 1 for strictly positive matrix A, which rules out the possibility of the existence of nontrivial Jordan blocks (generalized eigenspaces) corresponding to the eigenvalue $r(A)$.

Proposition 66 ($r(A)$ is a simple eigenvalue)**.** *Let A be a strictly positive matrix, then $r(A)$ is a simple eigenvalue of A .*

Proof. We have shown that $r(A)$ is at least a semisimple eigenvalue of A in Proposition 65. If it is not simple, then there exists $w \neq 0$ and w, x being linearly indepedent, such that

$$Aw = r(A)w + x. \tag{4.7.22}$$

Taking the complex conjugate on both sides, we have

$$A\overline{w} = r(A)\overline{w} + x. \tag{4.7.23}$$

Thus W.L.O.G. we can assume that w is real. Moreover, since x is strictly positive, for t sufficiently large, we have $wt + x > 0$ and

$$A(wt + x) = r(A)(wt + x) + x. \tag{4.7.24}$$

Thus again, W.L.O.G. we can assume that w itself is already strictly positive. However, since $x > 0$, we have

$$Aw > r(A)w \overset{\text{Proposition 64}}{\Longrightarrow} r(A) > r(A), \tag{4.7.25}$$

which is a contraction. $\square$

Remark 79. *The above proposition is structural, which means that*

$$A \sim \begin{pmatrix} r(A) & \\ & B \end{pmatrix}. \tag{4.7.26}$$

Next, we will show that there is an essential separation between the eigenvalue $r(A)$ and the rest of the spectrum of A.

Proposition 67. *Let A be a strictly positive matrix, $\lambda \in Sp(A)$ and $\lambda \neq r(A)$, then we have $|\lambda| < r(A)$.*

Proof. Assume $|\lambda| = r(A)$, then

$$r(A)|v| = |\lambda v| = |Av| \overset{\text{triangular inequality}}{\leq} A|v|. \tag{4.7.27}$$

If $A|v| > r(A)|v|$, then by Proposition 64 we have $r(A) > r(A)$, which is a contradiction. Thus we have $A|v| = r(A)|v| = |Av|$. By the equality condition of the triangular inequality, we have $v = e^{i\theta}|v|$ for some $\theta \in \mathbb{R}$ which means that $|v|$ is also an eigenvector of A corresponding to the eigenvalue λ. However we have shown that $|v|$ is an eigenvector of A corresponding to the eigenvalue $r(A)$, thus $r(A) = \lambda$ which is a contradiction with the assumption that $\lambda \neq r(A)$.

$\square$

Proposition 68. *A is strictly positive, $x, y > 0$, $Ax = r(A)x$, $y^T A^T = r(A)y^T$, and we assume that x and y are normalized such that $y^T x = 1$, then we have*

$$\lim_{m\to\infty}\left(\frac{A}{r(A)}\right)^m = xy^T. \tag{4.7.28}$$

Proof. The proof is by direct calculation. W.L.O.G. $r(A) = 1$, then we have that $\exists\, S$ invertible such that $SAS^{-1} = \begin{pmatrix} 1 & \\ & B \end{pmatrix}$, and

$$\lim_{m\to\infty} SA^mS^{-1} = \begin{pmatrix} 1 & \\ & \lim_{m\to\infty} B^m \end{pmatrix} \overset{r(B)<1}{=} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}. \tag{4.7.29}$$

Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then we have $SAS^{-1}v_1 = v_1$. By Proposition 65, $S^{-1}v_1 = ax$ for some $a \in \mathbb{C}$. Likewise we have $v_1^T S = by^T$ for some $b \in \mathbb{C}$. On the other hand,

$$1 = v_1^T v_1 = baS^{-1}\underbrace{y^T x}_{=1} S = ba \implies \lim_{m\to\infty} A^m = (ab)^{-1}S^{-1}v_1v_1^T S = (ab)^{-1}xy^T = xy^T. \tag{4.7.30}$$

□

Remark 80. *This is a structural result which gives an explicit characterization of the limit behavior of the strictly positive matrix A.*

Next, we will extend the above results from the case of strictly positive matrices to the case of irreducible matrices.

Definition 4.7.2 (Irreducible matrix)**.** *A matrix $A \in M_n(\mathbb{C})$ is called* reducible *if there exists a permutation matrix V such that*

$$VAV^{-1} = \begin{pmatrix} B & C \\ 0_{(n-r)\times r} & D \end{pmatrix}. \tag{4.7.31}$$

We say that A is irreducible *if it is not reducible.*

Proposition 69. *If A is positive (non-negative), then A is irreducible if and only if $(I + A)^{n-1}$ is strictly positive (i.e. $(I + A)^{-1} \rhd 0$)*

Proof. exercise 30. □

Theorem 4.7.3 (Perron-Frobenius theorem for matrices)**.** *Let A be a positive (non-negative) irreducible matrix, then we have the following properties:*

(1) $r(A) > 0$*;*

(2) $r(A)$ *is a simple eigenvalue of A;*

(3) There exists a unique (up to a positive scalar) strictly positive eigenvector x such that $Ax = r(A)x$, which is called the right Perron vector of A;

(4) There exists a unique (up to a positive scalar) strictly positive left eigenvector y such that $y^T A = r(A)y^T$, which is called the left Perron vector of A.

Proof. Using the characterization of irreducible matrices Proposition 69, we can apply the previous results on strictly positive matrices to obtain the result. □

Next, we will consider the P-F theorem for the case of positive and strictly positive maps. We will first give the definition of strictly positive maps.

Definition 4.7.4. *Let $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. If for any $A \in M_m(\mathbb{C}), A \geq 0$ and $A \neq 0$, we have $\Phi(A) > 0$, then we say that Φ is a* strictly positive map, *denoted as $\Phi > 0$.*

Definition 4.7.5. *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a positive map. Let $P \in M_n(\mathbb{C})$ be an orthogonal projection, we say P reduces Φ if $PM_n(\mathbb{C})P$ is an "invariant subalgebra" with respect to Φ, i.e.*

$$\Phi(PM_n(\mathbb{C})P) \subset PM_n(\mathbb{C})P. \tag{4.7.32}$$

If there does not exist any nontrivial orthogonal projection $P \in M_n(\mathbb{C})$ that reduces Φ, then we say that Φ is irreducible.

We have the following simple characterizations of reduction of positive maps.

Proposition 70. *P reduces Φ iff (1) $\Phi(P) \leq aP$ for some $a > 0$ iff (2) $I - P$ reduces Φ^*.*

Proof. (1) " $\Longrightarrow$ " is trivial. " $\Longleftarrow$ ", if $\Phi(P) \leq aP$, then

$$\Phi(PXP) \overset{\text{positive maps preserve the order}}{\leq} \|X\|\Phi(P) \leq a\|X\|P \in PM_n(\mathbb{C})P. \tag{4.7.33}$$

(2) " $\Longrightarrow$ " follows by direct calculations:

$$\operatorname{Tr}(\Phi^*(I-P)P) = \operatorname{Tr}((I-P)\Phi(P)(I-P)) \leq a\operatorname{Tr}((I-P)P) = 0 \Rightarrow P\Phi^*(I-P)P = 0 \tag{4.7.34}$$

$$\Rightarrow \Phi^*(I-P)^{\frac{1}{2}}P = 0 \Rightarrow \Phi^*(I-P)P = 0 \Rightarrow \Phi^*(I-P) \text{ annihilates } P\mathbb{C}^n. \tag{4.7.35}$$

Therefore, Range$[\Phi^*(I-P)] \subset (I-P)\mathbb{C}^n$, thus there must exist some $b > 0$ such that $\Phi^*(I-P) \leq b(I-P)$, which means that $I - P$ reduces Φ^*.

" $\Longleftarrow$ " is symmetric to the above argument. □

To discuss the P-F theorem for positive maps, we first define the concept of the spectrum of a linear map on matrix algebras. Then, we will give a non-trivial and abstract definition of the *irreducibility* of positive maps.

Definition 4.7.6. *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. If there exists $\Psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $\Phi \circ \Psi = \Psi \circ \Phi = I_{M_n(\mathbb{C})}$, then we say that Φ is invertible and denote the inverse as $\Psi = \Phi^{-1}$.*

We define the spectrum of Φ as

$$Sp(\Phi) := \{\alpha \in \mathbb{C} : \alpha id - \Phi \text{ is not invertible}\}. \tag{4.7.36}$$

We also define the spectral radius of Φ as

$$r(\Phi) := \max\{|\alpha| : \alpha \in Sp(\Phi)\}. \tag{4.7.37}$$

Interestingly, we have a very similar characterization of the irreducibility of positive maps as the irreducibility of positive matrices.

Proposition 71. *Suppose $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a positive map, then Φ is irreducible if and only if $(id + \Phi)^{n-1} > 0$.*

Proof. Suppose that Φ is irreducible and $0 \neq A \geq 0$. Let $B = A + \Phi(A)$. We will show that $P_{\text{Range}((\text{id}+\Phi)^{n-1}(A))} = I$.

We first note that $\ker B \subset \ker A$. This is because if $Bv = 0$, then we have

$$v^* A v + v^* \Phi(A) v = 0. \tag{4.7.38}$$

However, $\Phi(A) \geq 0$ and $A \geq 0$, thus we have $v^*Av = 0$ and $v^*\Phi(A)v = 0$. Thus we have $A^{\frac{1}{2}}v = 0$ and $\Phi(A)^{\frac{1}{2}}v = 0$, which means that $v \in \ker A$. Since A and B are Hermitian, we have Range$A = \ker(A^*)^\perp = \ker A \subset \ker B^\perp = \text{Range}(B^*) = \text{Range}B$. Therefore, $P_{\text{Range}(A)} \leq P_{\text{Range}(B)}$.

Case 1. If $P_{\text{Range}(B)} = P_{\text{Range}(A)}$, then $P_{\text{Range}(\Phi(A))} \leq P_{\text{Range}(B)} = P_{\text{Range}(A)}$. Hence Range$(\Phi(A)) \subset$ Range(A) i.e. $P_{\text{Range}(A)}$ reduces Φ. The irreducibility of Φ implies that $P_{\text{Range}(A)} = I$ which is equivalent to A being invertible. Thus $\ker B = \ker A = 0$, which means that $B = (\text{id} + \Phi)(A) > 0$. Now we have proved that $0 \neq X \geq 0 \implies (\text{id} + \Phi)(X) > 0$, thus we have $(\text{id} + \Phi)^{n-1}(X) > 0$ i.e. $(\text{id} + \Phi)^{n-1} > 0$.

Case 2. If $P_{\text{Range}(B)} > P_{\text{Range}(A)}$ i.e. Range$(A) \subsetneq$ Range$(\text{id} + \Phi)(A)$. If Range$(\text{id} + \Phi)^{k-1}(A) =$ Range$(\text{id} + \Phi)^k(A)$ for some $k - 1 = 1, \cdots, n-1$, then it reduces to Case 1. If Range$(\text{id}+\Phi)^{k-1}(A) \subsetneq$ Range$(\text{id} + \Phi)^k(A)$ for any $k = 1, \cdots, n-1$, we have that $P_{\text{Range}((\text{id}+\Phi)^{n-1}(A))} = I$ by counting the rank of the projections. This means that $(\text{id} + \Phi)^{n-1}(A)$ is invertible i.e. $(\text{id} + \Phi)^{n-1}(A) > 0$, which means that $(\text{id} + A)^{n-1} > 0$. □

Remark 81. *For the matrix version, we use some techniques from the graph theory to prove this characterization as we do in exercise 30. For this version, it is more straightforward to use the abstract properties of positive maps. However, the graph theory approach may provide us more information in the positive matrix case.*

With this property at hand, we can now give the P-F theorem for positive maps.

Theorem 4.7.7 (Perron-Frobenius theorem for positive maps)**.** *Suppose Φ: $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is an irreducible positive map. Then $r(\Phi) \in Sp(\Phi)$ and there exists a unique positive definite matrix $A \in M_n(\mathbb{C})$ up to scalar such that $\Phi(A) = r(\Phi)A$. Moreover, $r(\Phi)$ is an algebraically simple eigenvalue of Φ.*

Proof. We define the resolvent operator on $\text{Sp}(\Phi)^c \overset{\text{open}}{\subset} \mathbb{C}$ as

$$\phi(z) = (z\text{id} - \Phi)^{-1}. \tag{4.7.39}$$

Then ϕ is a holomorphic function on $\text{Sp}(\Phi)^c$ with Taylor-series expansion

$$\phi(z) = \sum_{k=1}^{\infty} \frac{\Phi^{k-1}}{z^k}. \tag{4.7.40}$$

This series converges absolutely on $\{|z| > r(\Phi)\}$.

We claim that $r(\Phi)$ is a singularity. Otherwise, $\phi(r(\Phi))$ exists and thus $\lim_{z \to r(\Phi)} \text{Tr}(B\phi(z)(A))$ exists for $A, B \geq 0$. Hence

$$|\text{Tr}(B\phi(z)(A))| \leq \text{Tr}(B\phi(|z|)(A)), \quad \forall |z| > r(\Phi). \tag{4.7.41}$$

By the Banach-Steinhaus theorem, we have $\phi(z)$ exists for $|z| = r(\Phi)$. This implies that ϕ is analytic on $\{|z| \geq r(\Phi)\}$ which is a contradiction to the fact that $r(\Phi)$ is the spectral radius. Thus $r(\Phi)$ is a singularity of $\phi(z)$ and consequently $r(\Phi) \in \mathrm{Sp}(\Phi)$.

By taking the Laurent series expansion of $\phi(z)$ at $r(\Phi)$, we have

$$\phi(z) = \sum_{k=-\ell}^{\infty} (z - r(\Phi))^k \Phi_k, \quad \text{where } \Phi_k = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{\phi(z)}{(z - r(\Phi))^{k+1}} \mathrm{d}z. \tag{4.7.42}$$

Here, Γ is chosen to be a smooth curve in the neighborhood of $r(\Phi)$ which encloses $r(\Phi)$. Note that

$$\Phi_{-\ell} = \lim_{z \to r(\Phi)} (z - r(\Phi))^{\ell} \phi(z) = \lim_{\mathbb{R} \ni \alpha \to r(\Phi)+} \underbrace{(\alpha - r(\Phi))^{\ell}}_{\geq 0} \underbrace{\phi(\alpha)}_{\geq 0} \tag{4.7.43}$$

is a positive linear map.

For any $k, j \geq -\ell$, by direct computations, we have

$$\begin{aligned} \Phi_k \Phi_j &= -\frac{1}{4\pi^2} \oint_{\Gamma_k} \oint_{\Gamma_j} \frac{\phi(z_1)\phi(z_2)}{(z_1 - r(\Phi))^{k+1}(z_2 - r(\Phi))^{j+1}} \mathrm{d}z_1 \mathrm{d}z_2 \\ &\overset{\text{resolvent formula}}{=} \frac{1}{4\pi^2} \oint_{\Gamma_k} \oint_{\Gamma_j} \frac{\phi(z_1) - \phi(z_2)}{(z_1 - z_2)(z_1 - r(\Phi))^{k+1}(z_2 - r(\Phi))^{j+1}} \mathrm{d}z_1 \mathrm{d}z_2 \\ &= \begin{cases} -\Phi_{j+1+k}, & k, j \geq 0, \\ 0, & k < 0, j \geq 0, \text{ or } j < 0, k \geq 0 \\ \Phi_{j+1+k}, & k, j < 0. \end{cases} \end{aligned} \tag{4.7.44}$$

Here, W.L.O.G. we assume that Γ_k is contained in Γ_j. This implies that

$$\phi(z)\Phi_{-\ell} = \Phi_{-\ell}\phi(z) = \sum_{k=-\ell}^{\infty} \Phi_{-\ell}\Phi_k (z - r(\Phi))^k = \sum_{k'=1}^{\ell} \frac{\Phi_{-k'-\ell+1}}{(z - r(\Phi))^{k'}} = (z - r(\Phi))^{-1}\Phi_{-\ell}, \tag{4.7.45}$$

which is because $\Phi_{-\ell'} = 0$ for any $\ell' > \ell$ since we note that $r(\Phi)$ is an r-order pole. This means that

$$(z - r(\Phi))\Phi_{-\ell}\phi(z) = \Phi_{-\ell} = (z - r(\Phi))\phi(z)\Phi_{-\ell}. \tag{4.7.46}$$

By eq. (4.7.40) and comparing the coefficient of z^{-2} on both sides, we have

$$\Phi_{-\ell}\Phi - r(\Phi)\Phi_{-\ell} = 0 = \Phi\Phi_{-\ell} - r(\Phi)\Phi_{-\ell}. \tag{4.7.47}$$

Therefore,

$$\Phi\Phi_{-\ell} = \Phi_{-\ell}\Phi = r(\Phi)\Phi_{-\ell}. \tag{4.7.48}$$

For any $A \geq 0$, we have that

$$\Phi(\Phi_{-\ell}(A)) = r(\Phi)\Phi_{-\ell}(A) \quad \text{where } \Phi_{-\ell}(A) \geq 0 \text{ by the previous observation eq. (4.7.43).} \tag{4.7.49}$$

If Φ is irreducible, then by Proposition 71, we have $(\mathrm{id} + \Phi)^{n-1}(\Phi_{-\ell}(A)) = (r(\Phi) + 1)^{n-1}\Phi_{-\ell}(A) > 0$ i.e. $\Phi_{-\ell}(A) > 0$ if $\Phi_{-\ell}(A) \neq 0$. However, when $\ell \geq 2$, $\Phi_{-\ell}^2 = \Phi_{-\ell}\Phi_{-\ell} = \Phi_{-2\ell+1} = 0$ by eq. (4.7.44), which leads to a contradiction. Thus we can only have $\ell = 1$ and $\Phi_{-1}^2 = \Phi_{-1-1+1} = \Phi_{-1}$. From this we can also see that $\Phi_{-1}(A) > 0$ is a "Perron vector" since

$$\Phi(\Phi_{-1}(A)) = r(\Phi)\Phi_{-1}(A). \tag{4.7.50}$$

Now we are at the position to prove the uniqueness of "Perron vector". Similar to the case of positive matrices, we will actually show the uniqueness in the regime of any matrices and do not restrict to the regime of only positive definite matrices. We suppose $A \in \mathbb{H}_n^{>0}$, $A' \in \mathbb{H}_n$ (A' is not a multiple of A), such that $\Phi(A) = r(\Phi)A$ and $\Phi(A') = r(\Phi)A'$. Then we let $t_* = \min\{t : tA + A' \geq 0\}$. Since A' is not a multiple of A, we have $t_*A + A' \neq 0$, thus by the irreducibility of Φ we have

$$0 < (\mathrm{id} + \Phi)^{n-1}(t_*A + A') = (r(\Phi) + 1)^{n-1}(t_*A + A') \Rightarrow t_*A + A' > 0. \tag{4.7.51}$$

which is a contraction to the minimality of t_*. Thus we have the uniqueness. Suppose $B \in M_n(\mathbb{C})$ is a general matrix (not required to be Hermitian) such that $\Phi(B) = r(\phi)B$, then by considering $B + B^*$ and $B - B^*$ we know that B is also a multiple of A, which proves that the geometric multiplicity is 1.

We already know that $r(\Phi)$ is semisimple. Suppose that $r(\Phi)$ is not an algebraically simple eigenvalue, then there exists a matrix B such that $\Phi(B) = r(\Phi)B + A$ and A, B are linearly indepedent. Note that A is a multiple of the Perron vector $\Phi_{-1}(\widetilde{A})$, thus $\Phi_{-1}(A) = A$. Therefore,

$$\Phi_{-1}(\Phi(B)) = r(\Phi)\Phi_{-1}(B) + \Phi_{-1}(A) = r(\Phi)\Phi_{-1}(B) + A. \tag{4.7.52}$$

On the other hand,

$$\Phi_{-1}(\Phi(B)) = \Phi_{-1}(r(\Phi)B) = r(\Phi)\Phi_{-1}(B), \tag{4.7.53}$$

thus $A = 0$, which is a contradiction. We see that $r(\Phi)$ is an algebraically simple eigenvalue of Φ. This completes the proof. □

Proposition 72. *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an irreducible positive map, then*

(1) If $A > 0$ such that $\Phi(A) \geq \alpha A$ for some $\alpha > 0$, then $\alpha \leq r(\Phi)$;

(2) If $A \geq 0$ such that $\Phi(A) \leq \alpha A$ for some $\alpha > 0$, then $\alpha \geq r(\Phi)$;

(3) If $A \geq 0$ such that $\Phi(A) = \alpha A$ for some $\alpha > 0$, then $\alpha = r(\Phi)$.

Proof. Note that Φ_{-1} is a positive map, thus

$$\Phi_{-1}(\Phi(A)) \geq \alpha\Phi_{-1}(A) \Rightarrow r(\Phi)\Phi_{-1}(A) \geq \alpha\Phi_{-1}(A). \tag{4.7.54}$$

Since $\Phi_{-1}(A)$ is positive definite, we have $r(\Phi) \geq \alpha$. □

4.8 von Neumann inequality

Theorem 4.8.1 (Dilation theorem)**.** *Let $A \in M_n(\mathbb{C})$, $\|A\| \leq 1$, then $\exists$ a Hilbert space $\mathcal{H}$ and a unitary operator U on $\mathbb{C}^n \oplus \mathcal{H}$ such that for any $m \in \mathbb{N}$, we have*

$$A^m = PU^mP, \quad \textit{where } P : \mathbb{C}^n \oplus \mathcal{H} \to \mathbb{C}^n \textit{ is the orthogonal projection.} \tag{4.8.1}$$

Remark 82. *This is a totally structural result, which means that any operator can be put into a "corner" of a very large unitary operator, and the polynomial operation on the operator can be preserved.*

Remark 83. *If an operator V is isometric satisfying*

$$V^*V = I, \quad I - VV^* = P, \tag{4.8.2}$$

then

$$U = \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix} \tag{4.8.3}$$

is a unitary operator, which can be verified by straightforward calculations

$$U^*U = \begin{pmatrix} V^* & \\ P & V \end{pmatrix}\begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4.8.4}$$

Moreover, we have

$$U^m = \begin{pmatrix} V^m & * \\ 0 & V^{*m} \end{pmatrix}. \tag{4.8.5}$$

Proof. Let

$$\mathcal{H}_0 := \bigoplus_{j\geq 1} \mathbb{C}^n := \left\{ (v_j)_{j\geq 1} : \sum_{j=1}^{\infty} \|v_j\|^2 < \infty, \quad v_j \in \mathbb{C}^n \right\} = \ell^2(\mathbb{N}; \mathbb{C}^n). \tag{4.8.6}$$

We define $V : \mathcal{H}_0 \to \mathcal{H}_0$ as

$$V(v_j)_{j\geq 1} = (Av_1, \sqrt{1 - A^*A}v_1, v_2, v_3, \cdots). \tag{4.8.7}$$

Then we have

$$\begin{aligned} \|V(v_j)_{j\geq 1}\|^2 &= \|Av_1\|^2 + \left\|\sqrt{1 - A^*A}v_1\right\|^2 + \sum_{j=2}^{\infty} \|v_j\|^2 \\ &= \langle A^*Av_1, v_1\rangle + \langle (1 - A^*A)v_1, v_1\rangle + \sum_{j=2}^{\infty} \|v_j\|^2 \\ &= \|v_1\|^2 + \sum_{j=2}^{\infty} \|v_j\|^2 \\ &= \|(v_j)_{j\geq 1}\|^2. \end{aligned} \tag{4.8.8}$$

Therefore, we have V is an isometry, i.e. $V^*V = I_{\mathcal{H}_0}$. Moreover, by the action of V we have

$$V^m(v_j)_{j\geq 1} = (A^m v_1, \cdots, \cdots). \tag{4.8.9}$$

We define

$$U = \begin{pmatrix} V & I - VV^* \\ & V^* \end{pmatrix} = \begin{pmatrix} \underbrace{A^m}_{\in M_{\mathbb{C}}(\mathbb{C})} & * & * \\ * & * & \\ * & & * \end{pmatrix} \in M_2(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H}_0) \cong \mathcal{B}(\mathbb{C}^n \oplus \mathcal{H}) \text{ for some } \mathcal{H} \tag{4.8.10}$$

which is a unitary operator, then we have

$$U^m = \begin{pmatrix} V^m & * \\ & V^{*m} \end{pmatrix} \implies PU^mP = \begin{pmatrix} A^m & 0 \\ 0 & 0 \end{pmatrix} \overset{\text{with a bit abuse of notations}}{=} A^m. \tag{4.8.11}$$

□

Theorem 4.8.2 (von Neumann inequality)**.** *Let $A \in M_n(\mathbb{C})$, $\|A\| \leq 1$, then for any polynomial $p \in \mathbb{C}[z]$, we have*

$$\|p(A)\| \leq \max_{|z|\leq 1} |p(z)|. \tag{4.8.12}$$

Proof. By the dilation theorem, we have $\exists$ a Hilbert space $\mathcal{H}$ and a unitary operator $U : \mathbb{C}^n \oplus \mathcal{H} \to \mathbb{C}^n \oplus \mathcal{H}$ such that for any $m \in \mathbb{N}$, we have

$$A^m = PU^m P, \quad \text{where } P : \mathbb{C}^n \oplus \mathcal{H} \to \mathbb{C}^n \text{ is the orthogonal projection.} \tag{4.8.13}$$

Then we have

$$\|p(A)\| = \|Pp(U)P\| \leq \|p(U)\| = \sup_{\lambda \in \mathrm{Sp}(U)} |f(\lambda)| \leq \sup_{\lambda \in \mathbb{S}^1} |f(\lambda)| \overset{\text{by the maximum theorem for hol. functions}}{=} \max_{|z|\leq 1} |p(z)|. \tag{4.8.14}$$

□

We will see next that this theorem can be generalized into a more generalized form, which is stated for positive maps. We will first give a highly non-trivial characterization of the positive map acting on some commutative $*$-algebra.

Lemma 26. *Let $f(z) = \sum_{j=-m}^{m} a_j z^j$ is strictly positive on $\mathbb{S}^1$, then there exists a polynomial $p \in \mathbb{C}[z]$ such that $f = |p|^2$.*

Proof. Since f is real-valued on $\mathbb{S}^1$, we see that $a_j = \overline{a_{-j}}$ for any j and $a_0 \in \mathbb{R}$. W.L.O.G. we assume that $a_m \neq 0$ then $a_{-m} \neq 0$. We denote

$$g(z) := z^m f(z) \tag{4.8.15}$$

then $g(z)$ is a polynomial and can be extended to the whole complex plane $\mathbb{C}$. Moreover, it satisfies

$$g(z) z^{-2m} = \overline{g(1/\overline{z})}. \tag{4.8.16}$$

Therefore, with respect to the distribution of the zeros of g, we claim that, if $\alpha_1, \cdots, \alpha_m$ are the zeros of g, then $1/\overline{\alpha_1}, \cdots, 1/\overline{\alpha_m}$ are also the zeros of g. Let

$$g_1(z) = (z - \alpha_1) \cdots (z - \alpha_m), \quad g_2(z) = (z - 1/\overline{\alpha_1}) \cdots (z - 1/\overline{\alpha_m}). \tag{4.8.17}$$

Then we have

$$g(z) = a_m g_1(z) g_2(z). \tag{4.8.18}$$

Moreover, by direct calulations, we have

$$\overline{g_2(z)} = (\overline{z} - 1/\alpha_1) \cdots (\overline{z} - 1/\alpha_m) = \overline{z}^m \frac{(-1)^m}{\alpha_1 \cdots \alpha_m} g_1(1/\overline{z}). \tag{4.8.19}$$

For any $z \in \mathbb{S}^1$, since $f(z) > 0$ and $|z^{-m}| = 1$, we have

$$\begin{aligned} f(z) = |f(z)| = \left|z^{-m} g(z)\right| = |g(z)| = |a_m||g_1(z)||\overline{z}^m| \left|\frac{(-1)^m}{\alpha_1 \cdots \alpha_m}\right| |g_1(1/\overline{z})| \\ = \frac{|a_m|}{|\alpha_1 \cdots \alpha_m|} |g_1(z)||g_1(1/\overline{z})|. \end{aligned} \tag{4.8.20}$$

Note that $z \in \mathbb{S}^1$, thus we have $\overline{z}z = |z|^2 = 1$ and then $1/\overline{z} = z$,

$$f(z) = \frac{|a_m|}{|\alpha_1 \cdots \alpha_m|} |g_1(z)|^2 \tag{4.8.21}$$

Let $p(z) = \sqrt{\frac{|a_m|}{|\alpha_1 \cdots \alpha_m|}} g_1(z)$, then we have $p(z)$ is a polynomial and $f(z) = |p(z)|^2$ for any $z \in \mathbb{S}^1$. $\square$

Theorem 4.8.3. $A \in M_n(\mathbb{C})$, $\|A\| \leq 1$. *Let* $\Phi : C(\mathbb{S}^1) \to M_n(\mathbb{C})$ *is a linear map defined by its action on polynomials*

$$\Phi(f_1 + \overline{f_2}) = f_1(A) + f_2(A)^*. \tag{4.8.22}$$

Here, $f_1, f_2 \in \mathcal{P}(\mathbb{S}^1)$ *are polynomials on* $\mathbb{S}^1$*. Then* Φ *must be a positive map and* $\|\Phi\| \leq \|\Phi(\mathbf{1})\| \leq 1$.

Proof. By Stone-Weierstrass theorem, Φ is defined on the dense subset $\mathcal{P}(\mathbb{S}^1) + \overline{\mathcal{P}}(\mathbb{S}^1)$ of $C(\mathbb{S}^1)$, where

$$\overline{\mathcal{P}}(\mathbb{S}^1) := \{\overline{p} : p \in \mathcal{P}(\mathbb{S}^1)\} \tag{4.8.23}$$

Specifically, we define

$$\Phi_0 : \mathcal{P}(\mathbb{S}^1) + \overline{\mathcal{P}}(\mathbb{S}^1) \to M_n(\mathbb{C}), \quad \Phi_0(f + \overline{g}) = f(A) + g(A)^*. \tag{4.8.24}$$

We take $f \geq 0$, $f \in \mathcal{P}(\mathbb{S}^1) + \overline{\mathcal{P}}(\mathbb{S}^1)$ and consider $f + \varepsilon\mathbf{1} > 0$ for $\varepsilon > 0$ small. Then we have $f + \varepsilon\mathbf{1} \in \mathcal{P}(\mathbb{S}^1) + \overline{\mathcal{P}}(\mathbb{S}^1)$. If we can show that $\Phi_0(f + \varepsilon\mathbf{1}) \geq 0$, then we have $\Phi_0(f) \geq 0$ by the continuity (f.d. and linear) of Φ_0. Thus W.L.O.G. we assume that f is already strictly positive, and we want to show $\Phi_0(f) \geq 0$. By the previous Lemma 26 characterizing strictly positive elements in $\mathcal{P}(\mathbb{S}^1) + \overline{\mathcal{P}}(\mathbb{S}^1)$, we have $f = |p|^2$ for some $p(z) = \sum_{j=0}^m a_j z^j \in \mathbb{C}[z]$. Then we have

$$f(z) = \sum_{j=0}^m \sum_{k=0}^m a_j \overline{a_k} z^{j-k}, \quad z \in \mathbb{S}^1. \tag{4.8.25}$$

Thus we have

$$\Phi_0(f) = \sum_{j=0}^m \sum_{k=0}^m a_j \overline{a_k} A_{j-k}, \quad \text{where} \quad A_j := \begin{cases} A^j, & j > 0, \\ A^{*(-j)}, & j < 0, \\ I, & j = 0. \end{cases} \tag{4.8.26}$$

We claim that $\Phi_0(f)$ is positive semidefinite. To see this, we calculate

$$\langle \Phi_0(f)v, v\rangle = \left\langle \begin{pmatrix} I & A^* & \cdots & A^{*m} \\ A & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A^* \\ A^m & \cdots & A & I \end{pmatrix} \begin{pmatrix} \overline{a_0}v \\ \vdots \\ \overline{a_m}v \end{pmatrix}, \begin{pmatrix} \overline{a_0}v \\ \vdots \\ \overline{a_m}v \end{pmatrix} \right\rangle. \tag{4.8.27}$$

By exercise 15, the matrix $\begin{pmatrix} I & A^* & \cdots & A^{*m} \\ A & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A^* \\ A^m & \cdots & A & I \end{pmatrix}$ is positive semidefinite, thus we have $\langle \Phi_0(f)v, v\rangle \geq 0$ for any $v \in \mathbb{C}^n$. Therefore, we have $\Phi_0(f) \geq 0$. Thus Φ_0 is a positive map on $\mathcal{P}(\mathbb{S}^1) + \overline{\mathcal{P}}(\mathbb{S}^1)$. By the density of $\mathcal{P}(\mathbb{S}^1) + \overline{\mathcal{P}}(\mathbb{S}^1)$, we can uniquely and continuously extend Φ_0 to a positive map Φ on $C(\mathbb{S}^1)$, whose positivity is preserved under continuity.

Next we will show that $\|\Phi\| \leq 1$. This follows readily by the generalized Proposition 73 below. $\square$

Proposition 73. *Ω is a compact Hausdorff space and $C(\Omega)$ is the Banach space of continuous functions on Ω endowed with the sup-norm. Let $\Phi : C(\Omega) \to M_n(\mathbb{C})$ be a positive map. Then we have $\|\Phi\| = \|\Phi(\mathbf{1})\|$.*

Proof. Let $\Phi(\mathbf{1}) \le I$ (W.L.O.G.). Then for any $f \in C(\Omega)$ with $\|f\| \le 1$.

For any $\varepsilon > 0$, there exist $x_1, \cdots, x_m \in \Omega$ and a finite open covering $\{U_j\}_{j=1}^m$ of Ω such that

$$|f(x) - f(x_j)| < \varepsilon, \quad \forall x \in U_j, \quad j = 1, \cdots, m \tag{4.8.28}$$

using the compactness of Ω. We take $\{g_j\}$ the unital decomposition of $\mathbf{1}$ w.r.t. $\{U_j\}$, i.e. supp$g_j \subset U_j$, $g_j \ge 0$ and $\sum_{j=1}^m g_j = \mathbf{1}$. We let $g(x) = \sum_{j=1}^m f(x_j)g_j(x)$. Then we have $\|f - g\| \le \varepsilon$. Thus we have $\|\Phi(f) - \Phi(g)\| \le \|\Phi\|\|f - g\| \le \varepsilon\|\Phi\|$.

On the other hand, we have

$$\begin{aligned}
\|\Phi(g)\| &= \left\| \sum_{j=1}^m f(x_j)\Phi(g_j) \right\| \\
&= \left\| \begin{pmatrix} \Phi(g_1)^{\frac{1}{2}} & \cdots & \Phi(g_m)^{\frac{1}{2}} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \underbrace{\begin{pmatrix} f(x_1) & & \\ & \ddots & \\ & & f(x_m) \end{pmatrix}}_{\le 1} \begin{pmatrix} \Phi(g_1)^{\frac{1}{2}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \Phi(g_m)^{\frac{1}{2}} & \cdots & 0 \end{pmatrix} \right\| \\
&\le \left\| \sum_{j=1}^m \Phi(g_j) \right\| = \|\Phi(\mathbf{1})\| \le 1.
\end{aligned} \tag{4.8.29}$$

Thus we have

$$\|\Phi(f)\| \le \|\Phi(g)\| + \varepsilon\|\Phi\| \le \|\Phi(\mathbf{1})\| + \varepsilon\|\Phi\|. \tag{4.8.30}$$

Taking $\varepsilon \to 0$, we have $\|\Phi(f)\| \le \|\Phi(\mathbf{1})\|$ for any $f \in C(\Omega)$ with $\|f\| \le 1$. Thus we have $\|\Phi\| \le \|\Phi(\mathbf{1})\| \le \|\Phi\|$ and thus $\|\Phi\| = \|\Phi(\mathbf{1})\|$. In particular, we have $\|\Phi\| \le 1$. ☐

Corollary 22 (von Neumann inequality)**.**

Proof. p polynomial, we take $\Phi(p) := p(A)$, then $\Phi(\mathbf{1}) = I$. By the above proposition, we have $\|\Phi\| = 1$. Thus

$$\|p(A)\| = \|\Phi(A)\| \le \|p\| = \sup_{|z| \le 1} |p(z)|. \tag{4.8.31}$$

☐

Theorem 4.8.4 (The norm of positive maps)**.** *If $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a positive map, then $\|\Phi\| = \|\Phi(I)\|$.*

Proof. Let $\|A\| \le 1$, Ψ be the map defined on $C(\mathbb{S}^1)$. By Theorem 4.8.3, we have that Ψ is a positive map and thus

$$\|\Phi(A)\| = \|\Phi(\Psi(z))\| \le \|\Phi\Psi\|\|z\| \le \|\Phi\Psi\| = \|\Phi\Psi(\mathbf{1})\| = \|\Phi(I)\| \tag{4.8.32}$$

Thus we have

$$\|\Phi\| \le \|\Phi(I)\| \le \|\Phi\| \implies \|\Phi\| = \|\Phi(I)\|. \tag{4.8.33}$$

☐

4.9 Exercise IV

Exercise 28. *Prove Theorem 4.3.2.*

Proof. **Step 1.** We claim that for any $*$-subalgebra $\mathfrak{A}$, and any $B \in \mathfrak{A}''$, $v \in \mathbb{C}^n$, there exists $A \in \mathfrak{A}$ such that $Bv = Av$.

If this has already been established, we apply this to the $*$-subalgebra

$$\mathfrak{M} := \left\{ \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} = A \otimes I_n : A \in \mathfrak{A} \right\} \subset M_{n^2}(\mathbb{C}), \tag{4.9.1}$$

then for any $\widetilde{B} \in \mathfrak{M}''$, $\exists A \in \mathfrak{A}$ such that $\widetilde{B}v = (A \otimes I_n)v$ for $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^{n^2}$ where $\{v_1, \cdots, v_n\}$ is a basis of $\mathbb{C}^n$. Now we compute the structure of $\mathfrak{M}' = \left\{ \begin{pmatrix} X & & \\ & \ddots & \\ & & X \end{pmatrix} = X \otimes I_n : X \in \mathfrak{A}' \right\}$, thus we have $\mathfrak{M}'' = \left\{ \begin{pmatrix} X & & \\ & \ddots & \\ & & X \end{pmatrix} = X \otimes I_n : X \in \mathfrak{A}'' \right\}$. W.L.O.G. we let $\widetilde{B} = \begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix} = B \otimes I_n$ for $B \in \mathfrak{A}''$, then we have

$$\widetilde{B}v = (A \otimes I_n)v \Rightarrow Bv_i = Av_i, \quad \text{for } i = 1, \cdots, n. \tag{4.9.2}$$

Thus the actions of B and A coincide on the basis thus $B = A$. We have proved that $\mathfrak{A}'' \subset \mathfrak{A}$. But $\mathfrak{A} \subset \mathfrak{A}''$ holds trivially, thus we have $\mathfrak{A}'' = \mathfrak{A}$.

Step 2. It suffices to prove the claim above. We fix any vector $v \in \mathbb{C}^n$, let $V = \{Av : A \in \mathfrak{A}\} = \mathfrak{A}v$ the subspace of $\mathbb{C}^n$ and consider P_V, the orthogonal projection onto the subspace of V.

We claim that $P_V \in \mathfrak{A}'$. That is because V is $\mathfrak{A}$-invariant, thus we have $P_V A P_V = A P_V$ for any $A \in \mathfrak{A}$. By taking adjoint we have $P_V A^* P_V = P_V A^*$ for any $A \in \mathfrak{A}$. By $\mathfrak{A}$ is a $*$-subalgebra, we have $P_V A P_V = P_V A$ for any $A \in \mathfrak{A}$. Together with $P_V A P_V = P_V A$ we have $A P_V = P_V A P_V = P_V A$ for any $A \in \mathfrak{A}$. Thus we have $P_V \in \mathfrak{A}'$.

Therefore, $\forall B \in \mathfrak{A}''$, $BP_V = P_V B \Rightarrow V$ is $\mathfrak{A}''$-invariant. In particular, $\underbrace{B}_{\in \mathfrak{A}''} \underbrace{v}_{\in V \text{ since } \mathfrak{A} \text{ is unital}} \in V$ (by V is $\mathfrak{A}''$-invariant). That is, $\exists A \in \mathfrak{A}$ such that $Bv = Av$ by the definition of V. $\square$

Exercise 29 (The closedness of inverse on $*$-subalgebras)**.** *Let $\mathfrak{A}$ be a unital $*$-subalgebra of $M_n(\mathbb{C})$. Let $A \in \mathfrak{A}$ be a matrix that is invertible in $M_n(\mathbb{C})$. We will show that $A^{-1} \in \mathfrak{A}$ i.e. $A|_{\mathfrak{A}}$ is invertible in $\mathfrak{A}$.*

(1) *If $A \in \mathfrak{A}$ is a Hermitian matrix with spectral decomposition $A = \sum_{j=1}^m \lambda_j P_j$ for $\lambda_j \in \mathbb{R}, \lambda_j \neq \lambda_i$ $(j \neq i)$. Then each P_j belongs to $\mathfrak{A}$. Moreover, for general $A \in \mathfrak{A}$ (no need to be Hermitian), each $A \in \mathfrak{A}$ can be written as a linear combination of* at most 4 *unitaries with each of them belongs to $\mathfrak{A}$.*

(2) *Let $B \in \mathbb{H}_n^{>0} \cap \mathfrak{A}$ be a positive definite matrix in $\mathfrak{A}$, then $B^{-1} \in \mathfrak{A}$.*

(3) *Prove the main result.*

Proof. (1) Note that

$$P_j = \prod_{i \in \{1, \cdots, n\} \setminus \{j\}} \frac{1}{\lambda_i - \lambda_j} (\lambda_i I - A). \tag{4.9.3}$$

Thus we have $P_j \in \mathfrak{A}$ since $\mathfrak{A}$ is a subalgebra. If A is contractive, then $\lambda_j \in [-1,1]$ and we can write $\lambda_j = \cos\theta_j$ for some $\theta_j \in [0,\pi]$. Then we have

$$A = \sum_{j=1}^{m} \cos\theta_j P_j = \sum_{j=1}^{m} \frac{e^{\mathrm{i}\theta_j} + e^{-\mathrm{i}\theta_j}}{2} P_j = \frac{1}{2}\left[\underbrace{\sum_{j=1}^{m}(e^{\mathrm{i}\theta_j}P_j)}_{\text{unitary}} + \underbrace{\sum_{j=1}^{m}(e^{-\mathrm{i}\theta_j}P_j)}_{\text{unitary}}\right]. \tag{4.9.4}$$

Therefore, each Hermitian matrix can be written as a linear combination of 2 unitary matrices

$$A = \frac{1}{2\|A\|}(U_1 + U_2) \tag{4.9.5}$$

according to eq. (4.9.4) where $U_1, U_2 \in \mathfrak{A}$ are unitaries. For general $A \in \mathfrak{A}$, we write

$$A = \frac{1}{2}\underbrace{(A + A^*)}_{\text{self-adjoint}} + \frac{1}{2\mathrm{i}}\underbrace{[\mathrm{i}(A - A^*)]}_{\text{self-adjoint}}. \tag{4.9.6}$$

It follows readily that each $A \in \mathfrak{A}$ can be written as a linear combination of at most 4 unitaries that come from $\mathfrak{A}$.

(2) $\left\|I - \|B\|^{-1}B\right\| < 1$, thus we have

$$(\|B\|^{-1}B)^{-1} = \sum_{n=0}^{\infty}\underbrace{(I - \|B\|^{-1}B)}_{\in\mathfrak{A}}{}^{n} \overset{\text{any f.d. subalgebra is closed}}{\in} \|B\|B^{-1} \in \mathfrak{A} \Rightarrow B^{-1} \in \mathfrak{A}. \tag{4.9.7}$$

(3) For any $\varepsilon > 0$, $A \in \mathfrak{A}$, we have $(A^*A)^{\frac{1}{2}} \in \mathfrak{A}$ and $A(A^*A + \varepsilon I)^{-\frac{1}{2}} \in \mathfrak{A}$. That is, if $A = U|A|$ is the polar decomposition of A, then both U and $|A|$ belong to $\mathfrak{A}$.

Let $A \in M_n(\mathbb{C})$ be a invertible matrix with polar decomposition $A = U|A|$. Then we have $A^{-1} = |A|^{-1}U^*$. Since $A \in \mathfrak{A}$, we have $U, |A| \in \mathfrak{A}$. By (2), we have $|A|^{-1} \in \mathfrak{A}$. By $\mathfrak{A}$ is a $*$-subalgebra, we have $U^* \in \mathfrak{A}$. Thus we have $A^{-1} = |A|^{-1}U^* \in \mathfrak{A}$.

$\square$

Exercise 30. *If A is positive (non-negative), then A is irreducible if and only if $(I + A)^{n-1}$ is strictly positive (i.e. $(I + A)^{-1} \rhd 0$)*

(Hint: Consider $\widetilde{A} = I + A$, then $(\widetilde{A}^p)_{jk} \neq 0$ if and only if there exists a path connecting v_j to v_k in the directed graph $\Gamma = (v_1, \cdots, v_n; \widetilde{A})$, where v_j is the j-th vertex of Γ and $\widetilde{A}_{jk} > 0$ means there is a directed edge from v_j to v_k. Consider $\Omega = \{v \in \Gamma :\ \nexists$ a path from v to $v_{k_0}\}$. Then show that there is no path between Ω and Ω^c.)

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