# Lecture 8

ARMA Models

2/13/2018

# AR models

# AR(p) models

We can generalize from an AR(1) to an AR(p) model by simply adding additional autoregressive terms to the model.

$$
\begin{aligned}
AR(p): \quad y_t &= \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \cdots + \phi_p\, y_{t-p} + w_t \\
&= \delta + w_t + \sum_{i=1}^{p} \phi_i\, y_{t-i}
\end{aligned}
$$

. . .

What are the properities of $AR(p)$,

1. Expected value?

2. Autocovariance / autocorrelation?

3. Stationarity conditions?

# Lag operator

The lag operator is convenience notation for writing out AR (and other) time series models.

We define the lag operator $L$ as follows,

$$L\,y_t = y_{t-1}$$

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We define the lag operator $L$ as follows,

$$L\,y_t = y_{t-1}$$

this can be generalized where,

$$\begin{aligned} L^2 y_t &= L\,(L\,y_t) \\ &= L\,y_{t-1} \\ &= y_{t-2} \end{aligned}$$

therefore,

$$L^k\,y_t = y_{t-k}$$

# Lag polynomial

Lets rewrite the $AR(p)$ model using the lag operator,

$$y_t = \delta + \phi_1 \, y_{t-1} + \phi_2 \, y_{t-2} + \cdots + \phi_p \, y_{t-p} + w_t$$
$$y_t = \delta + \phi_1 \, L \, y_t + \phi_2 \, L^2 \, y_t + \cdots + \phi_p \, L^p \, y_t + w_t$$

# Lag polynomial

Lets rewrite the $AR(p)$ model using the lag operator,

$$
\begin{aligned}
y_t &= \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \cdots + \phi_p\, y_{t-p} + w_t \\
y_t &= \delta + \phi_1\, L\, y_t + \phi_2\, L^2\, y_t + \cdots + \phi_p\, L^p\, y_t + w_t
\end{aligned}
$$

$$
\begin{aligned}
y_t - \phi_1\, L\, y_t - \phi_2\, L^2\, y_t - \cdots - \phi_p\, L^p\, y_t &= \delta + w_t \\
(1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_p\, L^p)\, y_t &= \delta + w_t
\end{aligned}
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$$

$$
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(1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_p\, L^p)\, y_t &= \delta + w_t
\end{aligned}
$$

This polynomial of the lags

$$\phi_p(L) = (1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_p\, L^p)$$

is called the characteristic polynomial of the AR process.

# Stationarity of $AR(p)$ processes

**Claim**: An $AR(p)$ process is stationary if the roots of the characteristic polynomial lay *outside* the complex unit circle

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**Claim**: An $AR(p)$ process is stationary if the roots of the characteristic polynomial lay *outside* the complex unit circle

If we define $\lambda = 1/L$ then we can rewrite the characteristic polynomial as

$$(\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \dots - \phi_{p-1} \lambda - \phi_p)$$

then as a corollary of our claim the $AR(p)$ process is stationary if the roots of this new polynomial are *inside* the complex unit circle (i.e. $|\lambda| < 1$).

# Example AR(1)

AR(1) $y_t = \delta + \phi_1 L y_t + w_t$

$$(1-\phi_1 L)\, y_t = \delta + w_t$$

Find roots at

$$(1-\phi_1 L) = 0$$

$$L = 1/\phi_1$$

stationarity

$$|L| > 1$$

$$|1/\phi_1| > 1$$

$$1 > |\phi_1|$$

$$-1 < \phi_1 < 1$$

# Example AR(2)

AR(2) $y_t = \delta + \phi L y_t + \phi_2 L^2 y_t + w_t$

$(1-\phi L + \phi_2 L^2) y_t = \delta + w_t$

Characteristic polynomial

$(1-\phi_1 L + \phi_2 L^2) = 0$

$(\lambda - \phi_1 \lambda - \phi_2) = 0$

解 $\lambda = \dfrac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$

What if complex root?

if $\phi_1^2 + 4\phi_2 < 0$

$\dfrac{\phi_1}{2} \pm \dfrac{\sqrt{-(\phi_1^2+4\phi_2)}}{2} i$

Next page

How to get to next step?

Stationarity $\|L\| > 1$

分情况讨论 $\|\lambda\| < 1$

$\boxed{\phi_1^2 + 4\phi_2 > 0}$

$\dfrac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1 \Rightarrow \sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$

$\Rightarrow \phi_1 + \phi_2 < 1$ A

& . $\dfrac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} > -1$

$-\sqrt{\phi_1^2 + 4\phi_2} > -2 - \phi_1$

$\Rightarrow \phi_2 - \phi_1 < 1$ A

$$\sqrt{\left(\frac{\phi_1}{2}\right)^2 + \left(\frac{\sqrt{-(\phi_1^2+4\phi_2)}}{2}\, i\right)\left(-\frac{\sqrt{-(\phi_1^2-4\phi_2)}}{2}\, i\right)} < 1$$

$$\sqrt{\frac{\phi_1^2}{4} + \frac{-(\phi_1^2+4\phi_2)}{4}} = \sqrt{-\phi_2} < 1$$

$$\boxed{-1 < \phi_2 < 1}$$

# AR(2) Stationarity Conditions

$-1<\phi_2<1$

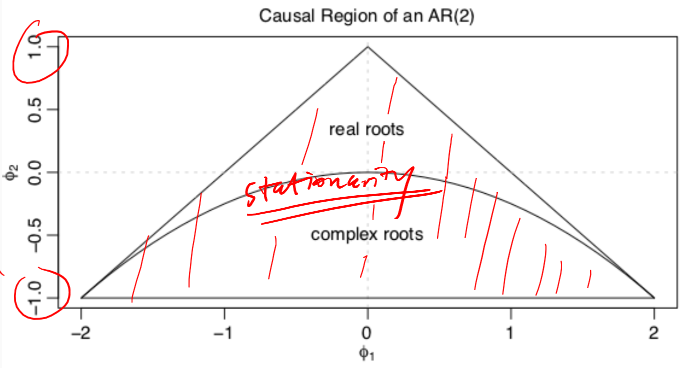


*Fig. 3.3. Causal region for an AR(2) in terms of the parameters.*

From Shumway&Stofer4thed.

complex is real root. Slight difference.

# Proof Sketch

supplementary proof. not be tested.

We can rewrite the $AR(p)$ model into an $AR(1)$ form using matrix notation

$$y_t = \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \cdots + \phi_p\, y_{t-p} + w_t$$
$$\boldsymbol{\xi}_t = \boldsymbol{\delta} + \mathbf{F}\, \boldsymbol{\xi}_{t-1} + \mathbf{w}_t$$

where

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \delta + w_t + \sum_{i=1}^{p} \phi_i\, y_{t-i} \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}$$

# Proof sketch (cont.)

So just like the original $AR(1)$ we can expand out the autoregressive equation

$$\begin{aligned}
\boldsymbol{\xi}_t &= \boldsymbol{\delta} + \mathbf{w}_t + \mathbf{F}\,\boldsymbol{\xi}_{t-1} \\
&= \boldsymbol{\delta} + \mathbf{w}_t + \mathbf{F}\left(\boldsymbol{\delta} + \mathbf{w}_{t-1}\right) + \mathbf{F}^2\left(\boldsymbol{\delta} + \mathbf{w}_{t-2}\right) + \cdots \\
&\qquad\qquad + \mathbf{F}^{t-1}\left(\boldsymbol{\delta} + \mathbf{w}_1\right) + \mathbf{F}^t\left(\boldsymbol{\delta} + \mathbf{w}_0\right) \\
&= (\sum_{i=0}^{t} F^i)\boldsymbol{\delta} + \sum_{i=0}^{t} F^i\, w_{t-i}
\end{aligned}$$

and therefore we need $\lim_{t\to\infty} F^t \to 0$.

# Proof sketch (cont.)

We can find the eigen decomposition such that $\mathbf{F} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ where the columns of $\mathbf{Q}$ are the eigenvectors of $\mathbf{F}$ and $\mathbf{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues.

A useful property of the eigen decomposition is that

eigen vector.

$$\mathbf{F}^i = \mathbf{Q}\mathbf{\Lambda}^i\mathbf{Q}^{-1}$$

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$$\mathbf{F}^i = \mathbf{Q}\boldsymbol{\Lambda}^i\mathbf{Q}^{-1}$$

Using this property we can rewrite our equation from the previous slide as

$$\begin{aligned}\boldsymbol{\xi}_t &= (\sum_{i=0}^{t} F^i)\boldsymbol{\delta} + \sum_{i=0}^{t} F^i\, w_{t-i} \\ &= (\sum_{i=0}^{t} \mathbf{Q}\boldsymbol{\Lambda}^i\mathbf{Q}^{-1})\boldsymbol{\delta} + \sum_{i=0}^{t} \mathbf{Q}\boldsymbol{\Lambda}^i\mathbf{Q}^{-1}\, w_{t-i}\end{aligned}$$

# Proof sketch (cont.)

$$\mathbf{\Lambda}^i = \begin{bmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^i \end{bmatrix}$$

Therefore,

$$\lim_{t \to \infty} F^t \to 0$$

when

$$\lim_{t \to \infty} \Lambda^t \to 0$$

which requires that

$$|\lambda_i| < 1 \quad \text{for all } i$$

## Proof sketch (cont.)

Eigenvalues are defined such that for $\boldsymbol{\lambda}$,

$$\det(\mathbf{F} - \boldsymbol{\lambda}\,\mathbf{I}) = 0$$

based on our definition of $\mathbf{F}$ our eigenvalues will therefore be the roots of

$$\lambda^p - \phi_1\,\lambda^{p-1} - \phi_2\,\lambda^{p-2} - \cdots - \phi_{p_1}\,\lambda^1 - \phi_p = 0$$

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which if we multiply by $1/\lambda^p$ where $L = 1/\lambda$ gives

$$1 - \phi_1\,L - \phi_2\,L^2 - \cdots - \phi_{p_1}\,L^{p-1} - \phi_p\,L^p = 0$$

For a *stationary* $AR(2)$ process where $w_t$ has $E(w_t) = 0$ and $Var(w_t) = \sigma_w^2$

Weak stationary

1. $E[y_t^2] < \infty$
2. $E[y_t] = \mu \Rightarrow E[y_t] = E(y_{t-h})$
3. $\gamma(h) = Cov(y_t, y_{t-h}) = f(h)$

$\Rightarrow \gamma(0) = Var(y_t) = c = Var(y_{t-h})$

For AR(2):

1. $E(y_t)$ –
2. $\gamma(0) = Var(y_t)$ –
3. $\gamma(h)$ 4. $\rho(h)$

① $y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$

$E(y_t) = \delta + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2}) + E(w_t)$

$E(y_t) = \delta + \phi_1 E(y_t) + \phi_2 E(y_t)$

$(1-\phi_1-\phi_2)E(y_t) = \delta$

$E(y_t) = \delta / (1-\phi_1-\phi_2)$

② $\tilde{y}_t = y_t - E(y_t)$

$= \phi_1 \tilde{y}_{t-1} + \phi_2 \tilde{y}_{t-2} + w_t$

$\gamma(0) = Var(Y_t) = Var(\tilde{Y}_t)$

$= \phi_1 Var(\tilde{Y}_{t-1}) + \phi_2 Var(\tilde{Y}_{t-2}) + \phi_1\phi_2 Cov(\tilde{Y}_{t-1}, \tilde{Y}_{t-2})$

$+ \sigma_w^2$

$= \phi_1 \gamma(0) + \phi_2 \gamma(0) + \phi_1\phi_2 \gamma(1) + \sigma_w^2$ plug in

$= \phi_1 \gamma(0) + \phi_2 \gamma(0) + \phi_1\phi_2 \cdot \frac{\phi_1 \gamma(0)}{1-\phi_2} + \sigma_w^2$

$= \frac{(1-\phi_2)\sigma_w^2}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1\phi_2)}$

$\gamma(h) = E[(Y_t - \mu)(Y_{t-h} - \mu)]$

$= E(\tilde{Y}_t \tilde{Y}_{t-h})$

$\tilde{Y}_t \tilde{Y}_{t-h} = \phi_1 \tilde{Y}_{t-1} \tilde{Y}_{t-h} + \phi_2 (\tilde{Y}_{t-2} \tilde{Y}_{t-h})$

$+ w_t \tilde{Y}_{t-h}$

$E(\tilde{Y}_t \tilde{Y}_{t-h}) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + 0$

$\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 (h-2)$

$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1)$

$(1-\phi_2)\gamma(1) = \phi_1 \gamma(0) \Rightarrow \boxed{\gamma(1) = \frac{\phi_1 \gamma(0)}{1-\phi_2}}$

# Properties of $AR(p)$

For a *stationary* $AR(p)$ process where $w_t$ has $E(w_t) = 0$ and $Var(w_t) = \sigma_w^2$

$$E(Y_t) = \frac{\delta}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

$$\gamma(0) = \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + \sigma_w^2$$
$$\gamma(h) = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} + \cdots + \phi_p\gamma_{j-p}$$

$$\rho(h) = \phi_1\rho_{j-1} + \phi_2\rho_{j-2} + \cdots + \phi_p\rho_{j-p}$$

# Moving Average (MA) Processes

A moving average process is similar to an AR process, except that the autoregression is on the error term.

$$MA(1): \qquad y_t = \delta + w_t + \theta\, w_{t-1}$$

Properties:

$$E(y_t) = \delta + E(w_t) + \theta E(w_{t-1})$$
$$= \delta$$

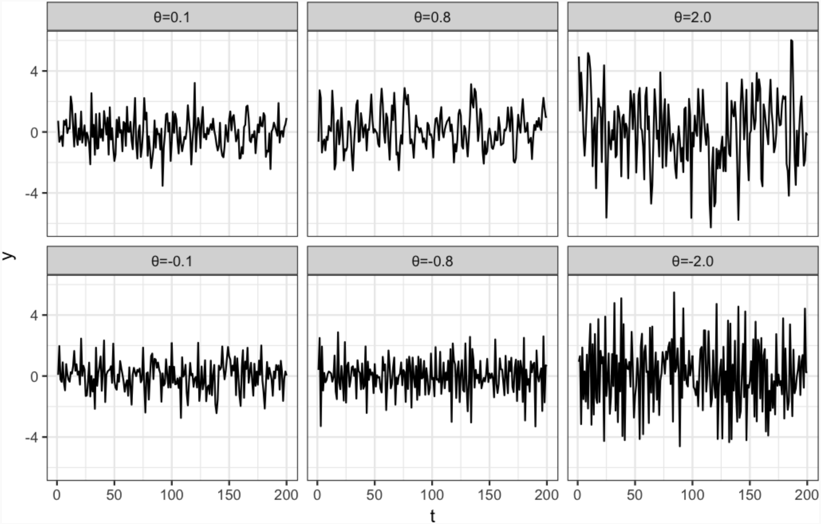
$$Var(y_t) = Var(w_t) + \theta^2 Var(w_{t-1})$$
$$= \sigma_w^2 + \theta^2 \sigma_w^2$$
$$= (1+\theta^2)\sigma_w^2$$

$$\gamma(h) = E[(y_t - E(y_t))(y_{t-h} - E(y_{t-h}))]$$
$$= E[\theta w_{t-1} w_{t-h}] + E[\theta w_t \theta w_{t-h-1}]$$

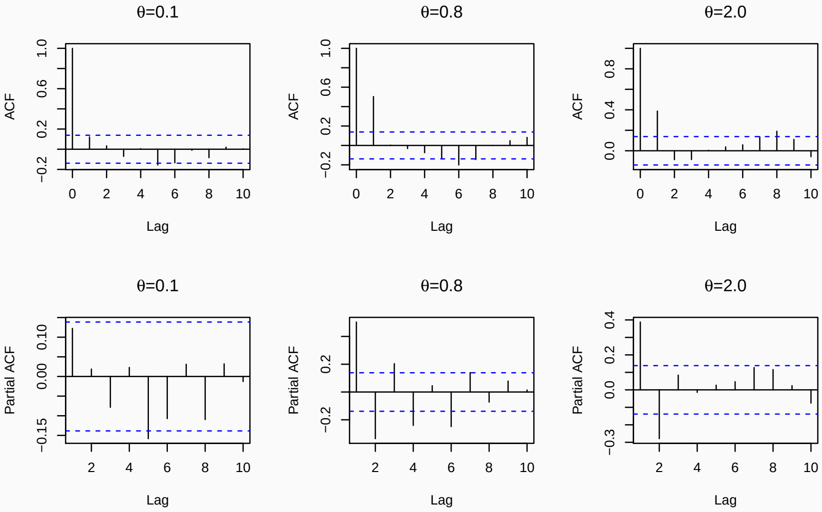
$$= \begin{cases} (1+\theta^2)\sigma_w^2 & \text{if } h=0 \\ \theta\sigma_w^2 & \text{if } h=1 \\ 0 & \text{otherwise} \end{cases}$$

(4) $$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

$$= \begin{cases} 1 & \text{if } h=0 \\ \theta/(1+\theta^2) & \text{if } h=1 \\ 0 & \text{otherwise} \end{cases}$$


θ=0.1
θ=0.8
θ=2.0
θ=-0.1
θ=-0.8
θ=-2.0
y
t
4
0
-4
0
50
100
150
200

# ACF



$$MA(q): \qquad y_t = \delta + w_t + \theta_1\, w_{t-1} + \theta_2\, w_{t-2} + \cdots + \theta_q\, w_{t-q}$$

Properties:

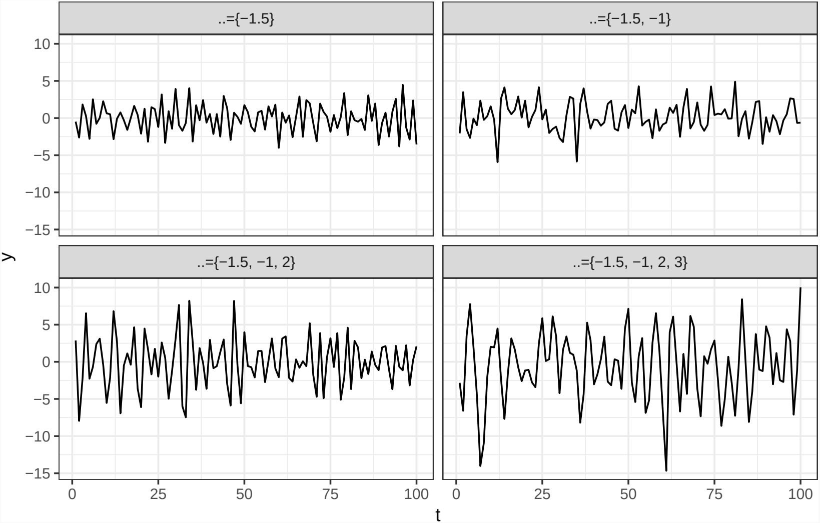
$$E(y_t) = \delta$$

$Var(y_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_w^2$

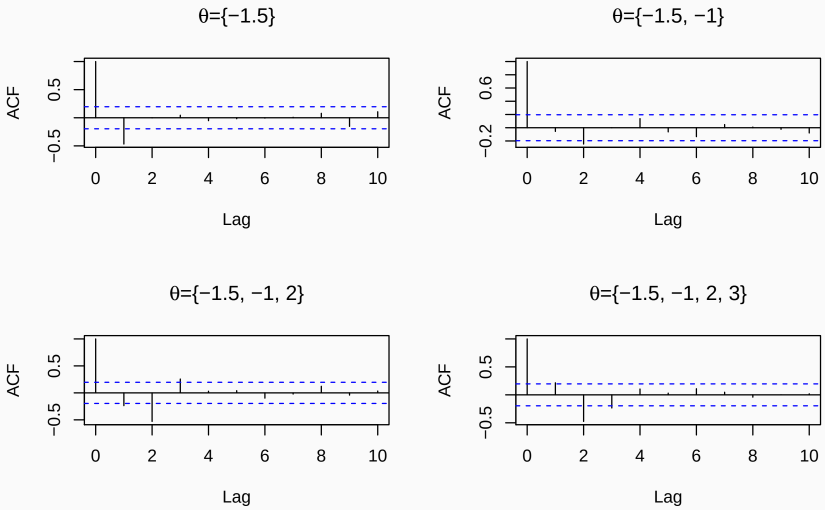
$$\gamma(0) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\, \sigma_w^2$$

$$\gamma(h) = \begin{cases} -\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q+k}\theta_q & \text{if } |k| \in \{1, \ldots, q\} \\ 0 & \text{otherwise} \end{cases}$$

# Example series



# ACF



# ARMA Model

# ARMA Model

An ARMA model is a composite of AR and MA processes,

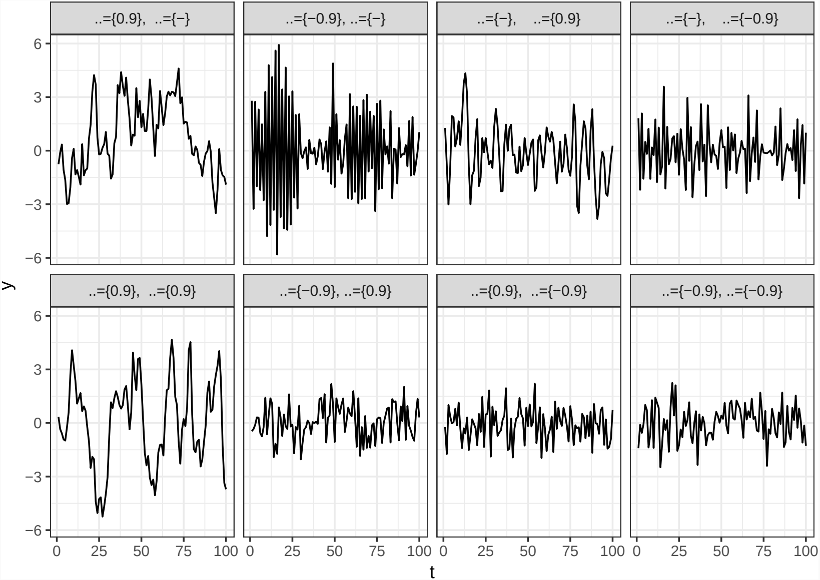
$ARMA(p, q)$:

$$y_t = \delta + \phi_1 \, y_{t-1} + \cdots \phi_p \, y_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t_q}$$

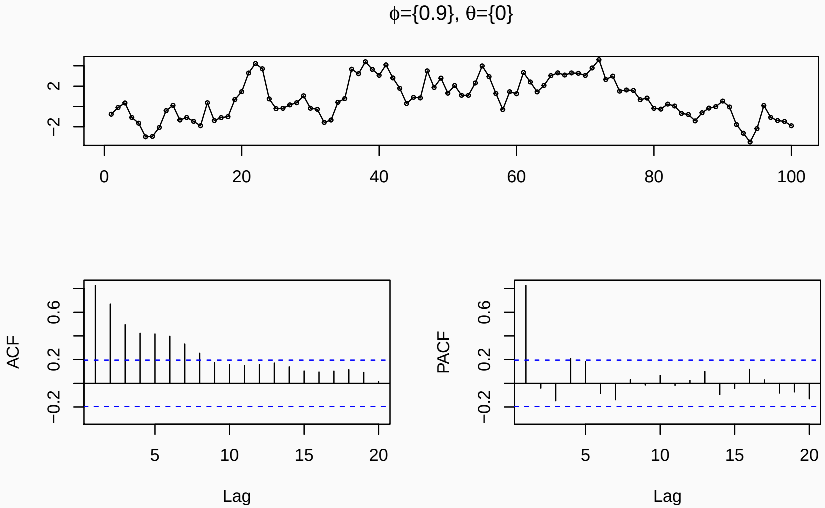
$$\phi_p(L) y_t = \delta + \theta_q(L) w_t$$

Since all $MA$ processes are stationary, we only need to examine the $AR$ aspect to determine stationarity (roots of $\phi_p(L)$ lie outside the complex unit circle).

# Time series



# $\phi = 0.9, \theta = 0$



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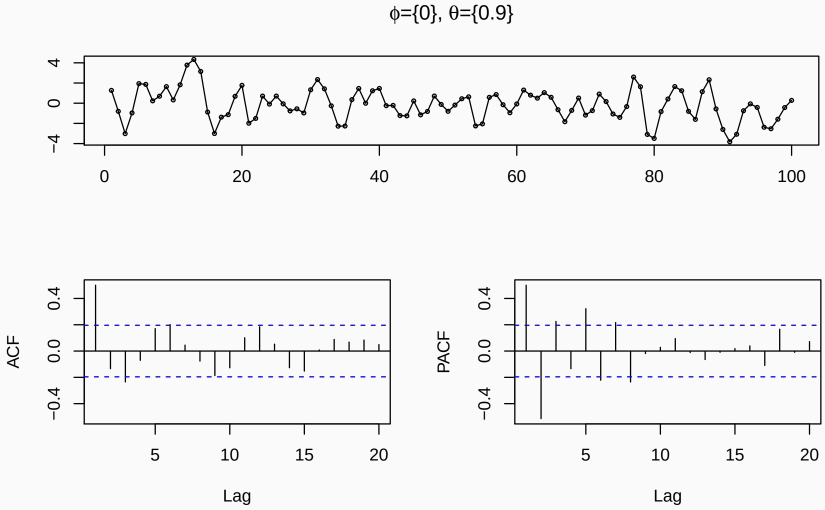

# $\phi = 0.9, \theta = 0.9$