Junior Balkan Mathematical Olympiad 2025

Abstract:

This document shares Junior Balkan Mathematical Olympiad 2025 (JBMO 2025)'s problems and Solutions. After showing such solutions, I will make some ~~meaningful~~ remark by explain the motivation and thought of **most of these problems**, which can build up your experience, with a hindsight to observe these problems.

Problems:

1. For all positive reals a, b, c prove that

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{2abc(a+b+c)^2}{ab+bc+ca}.$$

Proposed by Hakan Karakuş, Türkiye

2. Determine all numbers of the form

$$20252025\ldots2025$$

(consisting of one or more consecutive blocks of 2025*) that are perfect squares of positive integers.*

Proposed by Ognjen Tešić, Serbia

3. Let ABC *be a right-angled triangle with* $\angle A = 90^\circ$, $AD \perp BC$ *at* D, *and let* E *be the midpoint of* DC. *The circumcircle of* ABD *intersects* AE *again at point* F. *Let* X *be the intersection of the lines* AB *and* DF. *Prove that* $XD = XC$.

Proposed by Dren Neziri, Albania

4. Let n *be a positive integer. The integers from* 1 *to* n *are written in the cells of an* $n \times n$ *table (one integer per cell) so that each of them appears exactly once in each row and exactly once in each column. Denote by* r_i *the number of pairs* (a, b) *of numbers in the* i^{th} *row* ($1 \leq i \leq n$)*, such that , but* a *is written to the left of* b *(not necessarily next to it). Denote by* c_j *the number of pairs* (a, b) *of numbers in the* j^{th} *column* ($1 \leq j \leq n$)*, such that* $a \geq b$*, but* a *is written above* b*(not necessarily next to it). Determine the largest possible value of the sum*

$$\sum_{i=1}^{n} r_i + c_i$$

$Note$: *In the* $n \times n$ *table we label the rows* 1 *to* n *from top to bottom, and we label the columns* 1 *to* n *from left to right.*

Proposed by Boris Mihov, Bulgaria

1. For all positive reals a, b, c prove that

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{2abc(a+b+c)^2}{ab+bc+ca}.$$

Proposed by Hakan Karakuş, Türkiye

Proof:

By AM-GM inequality we have $(x^2+yz)^2 \geq (2\sqrt{x^2yz})^2 \geq 4x^2yz.\,(x,y,z\ >0)$*, so*

$$LHS \geq \sum_{cyc}\frac{4a^2bc}{b+c}=4abc\cdot\sum_{cyc}\frac{a}{b+c}.$$

Therefore, it's suffice to prove

$$\sum_{cyc}\frac{a}{b+c}\ \geq\frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

But Titu Lemma inequality said that

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{a^2}{ab+ac}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Thus Thanks Titu Lemma and we're done.

Remark:

It's easy to connect with AM-GM while look at $(a^2+bc)^2$*, and after doing AM-GM the expression is still "uninjured" (even it's "elegant", you should know why, right?) since the square "surrounds".*

Furthermore, the RHS is really burden, so it's necessary to find some similar elements in LHS (such as $4abc$*, you know, the proof have offset it).*

~~*Actually, I don't have any thought on this Algebra Problem.*~~

2. Determine all numbers of the form

$$20252025\ldots2025$$

(consisting of one or more consecutive blocks of 2025*) that are perfect squares of positive integers.*

Proposed by Ognjen Tešić, Serbia

Solution: (For convenience letting PS denote perfect square)

The answer is $2025(=45^2)$. *We will proof that it's the only answer.*

Actually all numbers of the form $20252025\ldots2025$ *can be written as* $2025\cdot(10^{4(k-1)}+10^{4(k-2)}+\ldots+1)$*(consisting k consecutive blocks of 2025), and* 2025 *is a PS, so* $20252025\ldots2025$ *is a PS* $\Leftrightarrow$ $10^{4(k-1)}+10^{4(k-2)}+\ldots+1$ *is a PS.*

Claim. $S=10^{4(k-1)}+10^{4(k-2)}+\ldots+1$ *is a PS* $\Leftrightarrow$ *k = 1.*

Proof.

Necessity is trivial.

When $k\geq 2$*, Suppose that* S *is a PS and assume that* $S=a^2$*, then*

$$a^2=\frac{10^{4k}-1}{10^4-1}$$

$$(10^{2k}+1)(10^k+1)(10^k-1)=11\times 101\times(3a)^2$$

Via Euclidean algorithm $10^{2k}+1, 10^k+1, 10^k-1$ *are pairwise coprime (See for yourself!!!)*

.However $11,\ 101$ *are prime, hence one of* $10^{2k}+1, 10^k+1, 10^k-1$ *must be a PS.*

But(!!!), mod 3 give that $10^{2k}+1, 10^k+1\ \equiv 2(\mathrm{mod}\ 3)$ *not a PS, mod 4 give that* 10^k-

Remark:

Venture to guess that 2025 is the only answer.

What would happen if the form change to $361361\ldots361$*?*

It's similar to the Claim. When $k=1$*,* 361 *is a PS. Now discuss* $k\geq 2$.

We suppose $S=10^{3(k-1)}+10^{3(k-2)}+\ldots+1$ *is a PS and let* $S=a^2$*, then*

$$a^2=\frac{10^{3k}-1}{10^3-1}$$

$$(10^k-1)(10^{2k}+10^k+1)=3^3\cdot 37\cdot a^2$$

Via Euclidean algorithm $(10^k-1, 10^{2k}+10^k+1)=3$. *By* $10^{2k}+10^k+1\ \equiv 3(\mathrm{mod}\ 9)$ *we know it only take a "3". As* $37, q^2\equiv 1(\mathrm{mod}4)$*(here q is arbitrary factor of a),* $10^{2k}+10^k+1\equiv 3(\mathrm{mod}\ 4)$.

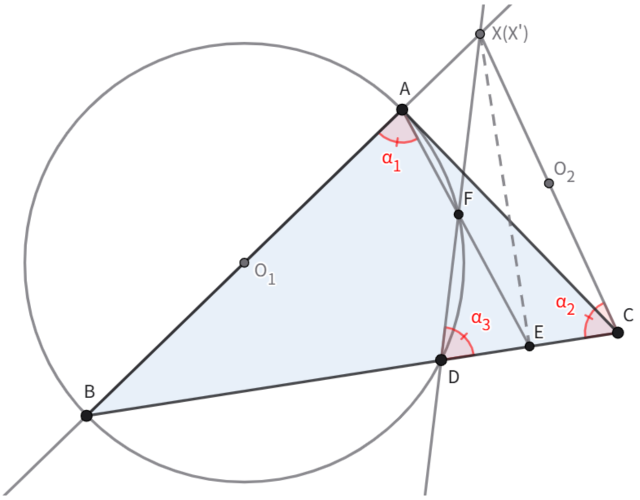
However in other hand $10^{2k}+10^k+1\equiv 0+0+1\equiv 1(\mathrm{mod}\ 4)$*, contradiction.*

3. Let ABC be a right-angled triangle with $\angle A = 90°$, $AD \perp BC$ at D, and let E be the midpoint of DC. The circumcircle of ABD intersects AE again at point F. Let X be the intersection of the lines AB and DF. Prove that $XD = XC$.

Proposed by Dren Neziri, Albania

Proof:

Let X' denote the intersection of BA and DC's perpendicular bisector(As shown in diagram below).



Then we have $\angle X'EC = 90°$, and $XAEC$ is inscribed in the circle with diameter XC.
Quickly given that $\angle FDC = \angle FDE = \angle BAE = \angle X'CE = \angle X'CD = \angle X'DC$, hence X', F, D collinear. Finally X's definition said that $X = X'$.

Remark:
When you're stuck on a problem, Phantom Point is indeed a good choice.

4. Let n be a positive integer. The integers from 1 to n are written in the cells of an $n \times n$ table (one integer per cell) so that each of them appears exactly once in each row and exactly once in each column. Denote by r_i the number of pairs (a, b) of numbers in the i^{th} row ($1 \leq i \leq n$), such that , but a is written to the left of b (not necessarily next to it). Denote by c_j the number of pairs (a, b) of numbers in the j^{th} column ($1 \leq j \leq n$), such that $a \geq b$, but a is written above b(not necessarily next to it). Determine the largest possible value of the sum

$$\sum_{i=1}^{n} r_i + c_i$$

$Note$: *In the $n \times n$ table we label the rows 1 to n from top to bottom, and we label the columns 1 to n from left to right.*

Proposed by Boris Mihov, Bulgaria

“光是題目就佔了頁面的一大半”

Solution:

The answer is $\frac{(n-1)n(2n-1)}{3}$.

Let's decide the upper boundary:

Claim. $\sum_{i=1}^{n} r_i \leq \frac{(n-1)n(2n-1)}{6}$.

Proof.

For any $1 \leq i \leq n$, every "i" are written in different row in such table, hence in every row there exist a "i".

Let (i, k) denote the cell which lie on row k and "i" written on. Consider the size of i and the number of cell which behind k, we know that the contribution of (i, k) in $\sum_{i=1}^{n} r_i$ is at most min(n-k, i-1). Therefore,

$$\sum_{i=1}^{n} r_i \leq \sum_{i=1}^{n}\sum_{k=1}^{n} m\,in(n-k, i-1) \leq \sum_{i=1}^{n} \left(\sum_{k=1}^{n-i} i-1 + \sum_{k=n-i+1}^{n} n-k\right)$$

$$= \sum_{i=1}^{n} (n-i)(i-1) + (1+2+\ldots+(i-1))$$

$$= \sum_{i=1}^{n} (n-i)(i-1) + \frac{(i-1)i}{2} = \sum_{i=1}^{n} \frac{(i-1)(2n-i)}{2}$$

$$= \sum_{i=1}^{n} \frac{-i^2 + (2n+1)i - 2n}{2}$$

$$= \frac{-(1^2+2^2+\ldots+n^2) + (2n+1)(\frac{(n+1)n}{2}) - 2n^2}{2}$$

$$= \frac{\frac{(2n+1)(n+1)n}{2} - \frac{(2n+1)(n+1)n}{6} - 2n^2}{2}$$

$$=\frac{\frac{(2n+1)(n+1)n}{3}-2n^2}{2}=\frac{2(1^2+2^2+\ldots+n^2)-2n^2}{2}=(1^2+2^2+\ldots+(n-1)^2)$$

$$=\frac{(n-1)n(2n-1)}{6}.$$

Similarly, $\sum_{i=1}^{n} c_i \le \frac{(n-1)n(2n-1)}{6}$. *Hence* $\sum_{i=1}^{n} r_i + c_i \le \frac{(n-1)n(2n-1)}{3}$.

Construction:

1	*n*	*n-1*	*...*	*2*
n	*n-1*	*...*	*2*	*1*
n-1	*...*	*2*	*1*	*n*
...	*2*	*1*	*n*	*n-1*
2	*1*	*n*	*n-1*	*...*

For all the (i, k), the number of cell which behind (or below) it are always small than it. Thus all the contribution of all the (i, k) can achieve to min(n-k, i-1), we're done.