CSC265 Fall 2021 Homework Assignment 5

due Wednesday October 20, 2021

1. A family $\mathcal{H}$ of hash functions from a universe U to $\{0, \ldots, m-1\}$ is 3-universal if, for all distinct $x_1, x_2, x_3 \in U$ and all $y_1, y_2, y_3 \in \{0, \ldots, m-1\}$,

$$\text{Prob}_{h \in \mathcal{H}}[h(x_1) = y_1, h(x_2) = y_2, \text{ and } h(x_3) = y_3] = \frac{1}{m^3}.$$

   (a) Prove that any 3-universal family of hash functions is universal.

   (b) Prove that if $U = \{0, \ldots, m-1\}$ and m is prime, then the family $\mathcal{H} = \{h_{a,b,c} \mid a, b, c \in U\}$ is 3-universal, where $h_{a,b,c}(x) = ax^2 + bx + c \bmod m$.

2. In this question, you will construct a data structure that supports hashing with chaining using a hash function h chosen from a universal family of hash functions from a universe U to $[1..m]$ using only a table T of size m to store any set $S \subseteq U$ of size at most m. You may also use a constant number of additional words of memory.

   Each of the m slots of T contains a bit, b, indicating whether or not the slot is storing an element of S. If the bit is 1, the slot also contains one of the elements, v, of S and a value, w, in $\{0, \ldots, m\}$. If not, the slot also contains two values, v and w, each in $\{0, \ldots, m\}$.

   (a) Briefly describe your data structure.

   (b) Draw a picture of your data structure for $m = 5$, the hash function $h(x) = (x \bmod 5) + 1$ and the set $S = \{2, 7, 8\}$.

   (c) How should T be initialized, assuming that S is initially empty?

   (d) For any given $x \in U$, explain how to perform INSERT(x) in $O(1)$ worst-case expected time. You may assume as a precondition that $x \notin S$.

   (e) For any given $x \in U$, explain how to perform DELETE(x) in $O(1)$ worst-case expected time.

   (f) For any given $x \in U$, explain how to perform SEARCH(x) in $O(1)$ worst-case expected time.