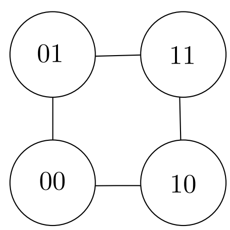
CSC265 Fall 2021 Homework Assignment 10

due Wednesday, December 8, 2021

Let $f : \mathbb{N}^+ \to \mathbb{N}^+$. A family of undirected graphs $\mathcal{F}$ has an *f-labelling scheme* if there is an *adjacency tester* $A : \{0,1\}^+ \times \{0,1\}^+ \to \{0,1\}$ such that, for all graphs $G = (V,E) \in \mathcal{F}$ with $n \geq 1$ vertices, there exists a 1-to-1 labelling function $\ell : V \to \{0,1\}^{f(n)}$ such that, for all $u, v \in V$,

$$A(\ell(u), \ell(v)) = \begin{cases} 1 & \text{if } \{u,v\} \in E \\ 0 & \text{if } \{u,v\} \notin E. \end{cases}$$

For example, consider the adjacency tester $A : \{0,1\}^+ \times \{0,1\}^+ \to \{0,1\}$ where $A(x,y) = 1$ if and only x and y are binary strings of the same length that differ in exactly one bit position.
Let V_d be the set of $n = 2^d$ vertices in $\mathbb{N}^d$ consisting of the vertices that can be reached from the origin by moving distance 1 in the positive direction along some subset of the dimensions.
Let E_d be the subsets $\{u,v\}$ of V_d such that the distance between u and v is exactly 1. Let $\mathcal{F}$ be the family of graphs $\{G_d = (V_d, E_d) \mid d \geq 1\}$. Then, for the labelling function $\ell : V_d \to \{0,1\}^d$ that maps each vertex in V_d to its d coordinates in $\mathbb{N}^d$, A is an adjacency tester for $\mathcal{F}$.
Here is a picture of G_2, with the labels on each node.



A labelling scheme is useful for storing a graph in a distributed system, with different parts of the graph stored in different places. Whenever a processor needs to determine whether two vertices are adjacent, it can do so by simply applying the adjacency tester to the labels of the two nodes.

1. Prove that the family of all undirected graphs has an f-labelling scheme, where $f(n) = n\lceil \log_2 n \rceil$.

2. Prove that the family of all undirected trees has an f-labelling scheme, where $f(n) = 2\lceil \log_2 n \rceil$.

3. Prove that if a family of undirected graphs $\mathcal{F}$ has an f-labelling scheme, then, for all positive integers n, there is a graph $G_n = (V_n, E_n)$ with $2^{f(n)}$ vertices such that every graph $G \in \mathcal{F}$ with n vertices is an induced subgraph of G_n.

4. Let $f : \mathbb{N}^+ \to \mathbb{N}^+$ be a nondecreasing function. Let $\mathcal{G}$ be a family of undirected graphs. Suppose that, for all positive integers n, there is an unidrected graph $G_n = (V_n, E_n)$ with $2^{f(n)}$ vertices such that every graph $G \in \mathcal{G}$ with n vertices is an induced subgraph of G_n. Prove that $\mathcal{G}$ has an f'-labelling scheme, where $f'(n) = f(n) + \lceil \log_2 n \rceil$.