CSC265 Homework 10 | Fall 2021
Name: Haojun Qiu | Student Number: 1006900622
Who I discussed with: Darcy Wang, Runshi Yang

Q1

Lemma 1: Function $f(n) = n\lceil \log_2 n \rceil$ is injective its domain $\mathbb{N}^+$.

Proof: We know that $g(n) = n$ is strictly increasing on N^+, and $h(n) = \lceil \log_2 n \rceil$ is non-decreasing on $\mathbb{N}^+$. Therefore, for all $n \in \mathbb{N}^+$, $f(n) = g(n)h(n)$ is strictly increasing, thus injective. ■

The lemma 1 tells if there is some $n \in \mathbb{N}^+$ such that $n\lceil \log_2 n \rceil$, it will be the only one, and can be determined somehow.

Proof:

- Let $\mathcal{F}$ be the family of all undirected graph.
- Take $A : \{0,1\}^+ \times \{0,1\}^+ \to \{0,1\}$ such that $A(x, y) = 1$ if x, y are binary strings such that
  (1) there is $k \in \mathbb{N}^+$ such that $|x| = |y| = k\lceil \log_2 k \rceil$; and
  (2) first $\lceil \log_2 k \rceil$ bits of x and y are, respectively, binary representation of some integers i and j in $\{0, \ldots, k-1\}$; and
  (3) one of the latter $(k-1)$ contiguous blocks of substring of length $\lceil \log_2 k \rceil$ of x is the binary representation of j, and one of the latter $(k-1)$ blocks of substring of length k of y is the binary representation of i.

  and $A(x, y) = 0$ otherwise. Note that A can determine unique $k \in \mathbb{N}$ given $|x|$ if such k exists, by lemma 1.
- Let $G = (V, E) \in \mathcal{F}$ with $n \geq 1$ vertices. Say, $V = \{v_0, \ldots, v_{n-1}\}$.
- Take $l : V \to \{0,1\}^{n\lceil \log_2 n \rceil}$ that maps $v_i \in V$ to a binary string $l(v_i)$ of length $n\lceil \log_2 n \rceil$ where
  - first $\lceil \log_2 n \rceil$ bits of $l(v_i)$ are binary representation of i,
  - the rest $(n-1)\lceil \log_2 n \rceil$ bits of $l(v_i)$ is divided into $(n-1)$ contiguous blocks of subtrings of length $\lceil \log_2 n \rceil$, such that each neighbour v_j of $v_i \in V$ takes up a block of substring of length $\lceil \log_2 n \rceil$ by the binary representation of j, the rest of blocks are each filled with binary representation of i.

  Notice that by the definition of l, for any distinct nodes $v_i, v_j \in V$, binary representation of i and j are not equal (since $i \neq j$) hence $l(v_i)$ and $l(v_j)$ differs in first $\lceil \log_2 n \rceil$ bits thus not equal. Therefore, l is injective.
- If $\{v_i, v_j\} \in E$, then $l(v_i), l(v_j)$ has length $n\lceil \log_2 n \rceil$, where first $\lceil \log_2 n \rceil$ bits of $l(v_i)$ and $l(v_j)$ are binary representation of i and j in $\{0, \ldots, n-1\}$, respectively. Furthermore, one of $l(v_i)$'s $(n-1)$ blocks of substring length $\lceil \log_2 n \rceil$ is the binary representation of j, and one of $l(v_i)$'s $(n-1)$ blocks of substring length $\lceil \log_2 n \rceil$ is the binary representation of i. Therefore, all three conditions satisfied, we have $A(l(v_i), l(v_j)) = 1$.

  If $\{v_i, v_j\} \notin E$, the first two conditions of $A(l(v_i), l(v_j))$ are satisfied as above. However, since v_i and v_j are not neighbour, $l(v_i)$ has no block (blocks are as described above) of subtring containing binary representation of j, and vice versa, hence the third condition fails. So, $A(l(v_i), l(v_j)) = 0$, as wanted. ■

Q2

Proof:

- Let $\mathcal{F}$ be the family of all undirected tree.
- Take $A : \{0,1\}^+ \times \{0,1\}^+ \to \{0,1\}$ such that $A(x, y) = 1$ if x, y are binary strings such that
  (1) not both x and y are all 0's; and

(2) there is $k \in \mathbb{N}^+$ such that $|x| = |y| = 2k$ (i.e., can be splitted into two halves of same length) so that first k bits of x matches (equals to) last k bits of y **OR** last k bits of x matches first k bits of y.

and $A(x, y) = 0$ otherwise.

- Let $G = (V, E) \in \mathcal{F}$ with $n \geq 1$ vertices. Say, $V = \{v_0, \ldots, v_{n-1}\}$. And, without loss, say v_0 is the root of G (any node can be root).
- Then, since G is a free tree (undirected, acyclic, and connected graph), by Theorem B.2 (2) of CLRS, we know that for all $v_i \in V - \{V_0\}$, there is a unique simple path from v_0 to $v_i = \langle 0, \ldots, v_p, v_i \rangle$, and in particular, there is an v_p precedes v_i on the path (v_p can be v_0) and such $\{v_p, v_i\} \in E$.
- Take $l : V \to \{0,1\}^{2\lceil \log_2 n \rceil}$ that maps $v_i \in V$ to a binary string $l(v_i)$ of length $2\lceil \log_2 n \rceil$ where
  - first $\lceil \log_2 n \rceil$ bits of $l(v_i)$ are binary representation of i,
  - the last $\lceil \log_2 n \rceil$ bits of $l(v_i)$ is
    the binary representation of 0 (i.e. $\lceil \log_2 n \rceil$ bits of 0's) if $v_i = v_0$;
    the binary representation of p (from v_p as defined above) ("parent of v_i") if $v_i \neq v_0$.

  Notice that by the definition of l, for any distinct nodes $v_i, v_j \in V$, binary representation of i and j are not equal (since $i \neq j$) hence $l(v_i)$ and $l(v_j)$ differs in first $\lceil \log_2 n \rceil$ bits thus not equal. Therefore, l is injective.
- If $\{v_i, v_j\} \in E$, then $l(v_i), l(v_j)$ has length $2\lceil \log_2 n \rceil$. Also, (v_0, v_0) is not an edge in E (self-loop not allowed in undirected graph/tree) so not both v_i and v_j are v_0, it follows that not both $l(v_i)$ and $l(v_j)$ are 0's. Also, it is either the case that there is a simple path from v_0 to v_j containing v_i preceding v_j, $\langle 0, \ldots, v_i, v_j \rangle$, **OR**, there is a simple path from v_0 to v_i containing v_j preceding v_i, $\langle 0, \ldots, v_j, v_i \rangle$. That is, either the last $\lceil \log_2 n \rceil$ bits of $l(v_j)$ equals to the first $\lceil \log_2 n \rceil$ bits of $l(v_i)$ **OR** the last $\lceil \log_2 n \rceil$ bits of $l(v_i)$ equals to the first $\lceil \log_2 n \rceil$ bits of $l(v_j)$. Hence, $A(l(v_j), l(v_i)) = 1$.

  If $\{u, v\} \notin E$, then we know that they are not neighbor of each other in any unique simple path in the graph, we by definition know that both $l(v_j)$ does not contain the binary representation of i on its last $\lceil \log_2 n \rceil$ bits and $l(v_i)$ does not contain the binary representation of j on its last $\lceil \log_2 n \rceil$ bits. So, $A(l(v_i), A(v_j)) = 0$. ■

Q3

Proof:

- Assume a family of undirected graohs F has an f-labelling scheme. Then, by definition, there is an adjacency tester $A : \{0,1\}^+ \times \{0,1\}^+ \to \{0,1\}$ such that for all graphs $G = (V, E) \in \mathcal{F}$ with $n \geq 1$ vertices, there exists a 1-to-1 labelling function $l : V \to \{0,1\}^{f(n)}$ such that for all $u, v \in V$,

$$A(l(u), l(v)) = \begin{cases} 1 & \text{if } \{u, v\} \in E \\ 0 & \text{if } \{u, v\} \notin E. \end{cases}$$

- Let n be a positive integer.
- Take $V_n = \{v_0, v_1, \ldots, v_{2^{f(n)}-1}\}$ to be a set of $2^{f(n)}$ vertices.

  Then, take a bijection labelling function $l_n : V_n \to \{0,1\}^{f(n)}$ by $l_n(v_i)$ to be binary representation of i of length $f(n)$ for all $v_i \in V_n$.

  Define $E_n = \{\{u, v\} \subseteq V_n \mid A(l_n(u), l_n(v)) = 1\}$.

  Take $G_n = (V_n, E_n)$.
- Let $G = (V, E) \in \mathcal{F}$ with n vertices. Then, we know that there exists a 1-to-1 labelling function $l : V \to \{0,1\}^{f(n)}$ such that for all $u, v \in V$,

$$A(l(u), l(v)) = \begin{cases} 1 & \text{if } \{u, v\} \in E \\ 0 & \text{if } \{u, v\} \notin E. \end{cases}$$

- Take $G' = (V', E')$, where
  - $V' = \{v_i \in V_n \mid l_n(v_i) = l(v) \text{ for some } v \in V\}$
  - $E' = \{\{u, v\} \in E_n \mid u, v \in V'\}$.

  Then, notice that $V' \subseteq V_n$ and G' is a subgraph of G_n induced by V', by construction.

- It remains to show that G and G' is isomorphic.

  Since l_n is invertible, we know that there is a inverse of l_n, called it $l_n^{-1} : \{0,1\}^{f(n)} \to V$. Since image$(l) \subseteq \{0,1\}^{f(n)}$, it is valid to define restriction of l_n^{-1} on image(l), call it, $l_n^{-1}|_{img(l)}$.

  Take $F : V \to V'$ by $F(v) = l_n^{-1}|_{img(l)}(l(v))$ for all $v \in V$. Note that F bijective by construction, as l is injective and $l_n^{-1}|_{img(l)}$ is bijective with domain being image of l.

  Let $\{u, v\} \in E$. Then, $A(l(u), l(v)) = 1$. And, $A\Big(l_n\big(l_n^{-1}|_{img(l)}(l(u))\big), l_n\big(l_n^{-1}|_{img(l)}(l(v))\big)\Big) = 1$, then $\{(l_n^{-1}|_{img(l)} \circ l)(u), (l_n^{-1}|_{img(l)} \circ l)(u)\} = \{(l_n^{-1} \circ l)(u), (l_n^{-1} \circ l)(u)\} \in E_n$. And, note that $(l_n^{-1} \circ l)(u) \in V'$ since $(l_n^{-1} \circ l)(u) \in V_n$ and $l_n\big((l_n^{-1} \circ l)(u)\big) = l(u)$. Similarly, $(l_n^{-1} \circ l)(v) \in V'$. So, by definition of E', $\{(l_n^{-1}|_{img(l)} \circ l)(u), (l_n^{-1}|_{img(l)} \circ l)(u)\} = \{F(u), F(v)\} \in E'$.

  Let $\{F(u), F(v)\} \in E'$. Then, since our map F is bijection from V to V', we have that $u, v \in V$. Expanding F, we get $\{(l_n^{-1}|_{img(l)} \circ l)(u), l_n^{-1}|_{img(l)} \circ l)(v)\} \in E' \subseteq E_n$. Hence, by definition of E_n, we have that $A\Big(l_n\big(l_n^{-1}|_{img(l)} \circ l)(u)\big), l_n\big(l_n^{-1}|_{img(l)} \circ l)(v)\big)\Big) = 1$, that is $A(l(u), l(v)) = 1$ and $u, v \in V$. This implies $\{u, v\} \in E$.

  We have show that $\{u, v\} \in E$ iff $\{F(u), F(v)\} \in E'$

Q4

Lemma 2: For all $n \in \mathbb{N}^+$, there is a bijection l_n with domain V_n and codomain $\{0,1\}^{f(n)}$.

Proof: This is straightforward. Let $n \in \mathbb{N}^+$. By assumption, we know that the existed V_n that forms G_n has size $2^{f(n)}$ ($|V| = 2^{f(n)}$). And $|\{0,1\}^{f(n)}| = 2^{f(n)}$. Hence, a bijection between V and $\{0,1\}^{f(n)}$ must exist. ■

The lemma 2 says once we are given an postive integer n, we are able to corresponds an existed $G_n = (V_n, E_n)$ given in assumption, and knows there is a bijection between V_n and $\{0,1\}^{f(n)}$, mapping vertex $v \in V_n$ to a binary string of length $f(n)$.

For this question, for all positive integer n, fix an l_n from existed one(s) given in lemma globally, called in standard labelling with n. So, for all $n \in \mathbb{N}^+$, when referring to l_n, we refer to the standard labelling with n we just fixed.

Lemma 3: For all $n_1, n_2 \in \mathbb{N}^+$, if $f'(n_1) = f'(n_2)$, then $\lceil \log_2 n_1 \rceil = \lceil \log_2 n_2 \rceil$.

Proof: Let $n_1, n_2 \in \mathbb{N}^+$. Without loss of generality, say $n_1 \leq n_2$. Assume $f'(n_1) = f'(n_2)$, then $f(n_1) + \lceil \log_2 n_1 \rceil = f'(n_2) + \lceil \log_2 n_2 \rceil$. Since $\lceil \log_2 n \rceil$ is non-decreasing, we have $\lceil \log_2 n_1 \rceil \leq \lceil \log_2 n_2 \rceil$. Assume for a contradiction that $\lceil \log_2 n_1 \rceil \neq \lceil \log_2 n_2 \rceil$, then $\lceil \log_2 n_1 \rceil < \lceil \log_2 n_2 \rceil$. Then, $f(n_1) > f(n_2)$, however since $n_1 \leq n_2$, this is a contradiction to f is an non-decreasing function. Therefore, $\lceil \log_2 n_1 \rceil = \lceil \log_2 n_2 \rceil$, as wanted. ■

Proof of Q4:

- Take $A : \{0,1\}^+ \times \{0,1\}^+ \to \{0,1\}$ such that $A(x, y) = 1$ if x, y are binary strings such that
  (1) there is an $k \in \mathbb{N}^+$ such that $|x| = |y| = \lceil \log_2 k \rceil + f(k)$; and
  (2) first $\lceil \log_2 k \rceil$ bits of x and y matches and is the binary representation of k; and
  (3) the last $f(k)$ bits of x, say a, and the last $f(k)$ bits of y, say b, satisfies that $(a, b) = (l_k(u), l_k(v))$ (where l_k is the standard labelling function with k) for some $u, v \in V_k$ such that $\{u, v\} \in E_k$ (equivalently, $\{l_k^{-1}(a), l_k^{-1}(b)\} \in E_k$)

  $A(x, y) = 0$ otherwise. Note that if such k exists, given $|x|$, we can determine and compute the unique $\lceil \log_2 k \rceil$ by lemma 3, and by the second step checking, A can directly computes what k exactly is.

- Let $G = (V, E) \in \mathcal{G}$ with $n \geq 1$ vertices.
- Then, we know that there is an undirected graph $G_n = (V_n, E_n)$ with $2^{f(n)}$ vertices such that G is an induced subgraph of G_n. That is, $V \subseteq V_n$, and $E = \{\{u, v\} \in E_n | u, v \in V\}$ (so $E \subseteq E_n$).
- Take $l : V \to \{0, 1\}^{f'(n)}$ that maps $v \in V$ to a binary string $l(v)$ of length $f'(n) = f(n) + \lceil \log_2 n \rceil$, where
  - the first $\lceil \log_2 n \rceil$ bits is the binary representation of n
  - the last $f(n)$ bits is $l_n(v)$ (this is valid mapping since $v \in V \subseteq V_n$).

  Notice that l is 1-to-1 since l_n is bijective.
- Assume that $\{u, v\} \in E$, then condition 1 and 2 are satisfied trivially by labelling $l(u), l(v)$. Furthermore, we know that $u, v \in V \subseteq V_n$ and $\{u, v\} \in E_n$. And, the last $f(n)$ bits of $l(u)$ and $l(v)$ are respectively $l_n(u)$ and $l_n(v)$. So, $A(l(u), l(v)) = 1$.

  Assume that $\{u, v\} \notin E$. Since there are different numbers $k_1, \ldots, k_j$ satisfies first condition, one of which is n by the labelling function codomain. Furthermore, by Lemma 3, the labelling function can determine unique $\lceil \log_2 n \rceil = \lceil \log_2 k \rceil$ for $k \in \{k_1, \ldots, k_j\} - \{k_n\}$. And, after reading first $\lceil \log_2 n \rceil$ bits of $l(u)$, it finds out n. However, since it must be that $\{u, v\} \notin E_n$ ($E \subseteq E_n$), by the definition of l and l_n we have that $\{l_n^{-1}(a), l_n^{-1}(b)\} = \{u, v\} \notin E_n$, where a and b are the last $f(n)$ bits of $l(u)$ and $l(v)$, respectively. Then $l(u)$ and $l(v)$ does not satisfies (3). Hence, $A(l(u), l(v)) = 0$.