The 3x+1 Problem

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1 Introduction

The 3x+1 problem, also known as the Collatz Conjecture, is one of the unsolved mathematical problems in the world. Let C: $\mathbb{Z} \to \mathbb{Z}$ be the function defined by

$$C(x) = \begin{cases} 3x + 1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}$$

The conjecture states that if we plug in any positive integer for x in function C, the function will iterate to 1 eventually. So we will try to find a heuristic of what might be true to prove the Collatz conjecture: why x is limited to positive integers, what about negative integers and 0, and why function C iterates to 1 when we plug in any positive integer for x.

Theorem 1. *The function C is iterating to 1 for some positive integers as x.*

In Section 2 we will prove that if x is an positive integer in function C, then function C will iterate to 1 since negative integers and 0 will not iterate to 1. Therefore the value of x of $C(x)$ is $[1, \infty)$. In Section 3 we will prove that if we randomly choose an integer from 1 to n, the probability of $C(x)$ being even approaches $2/3$ as n goes to ∞. In Section 4 we will prove that the equation of the function for even numbers, is an equation that produces an smaller integer after each iteration.

2 Possible Values of x

This section proves that only if x is an positive integer in function C, the function can iterate to 1. We will do that by finding out if x is an negative integer in function C, the function will iterate to -1 or negative integers, and if x is 0, the function will iterate to 0. Below is a list of facts we will need for the proof:

1. 0 is an even number.

2. An integer is either zero, negative or positive.

Lemma 1. *If we start the function with a negative integer, the function will always produce negative integers.*

Proof. Assume $C(x)$ iterates to 1 is true.

Then $C(x)$ will always produce positive integers.

Then

$$C(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}$$

$C(x)$ iterates to 1, produces positive integers.

Let x be a negative number.

Then $x = -y$, for some $y > 0$

Then

$$C(-y) = \begin{cases} -3y+1 & \text{if } -y \text{ is odd} \\ -y/2 & \text{if } -y \text{ is even} \end{cases}$$

$-3y+1$ will always be negative because for some $y > 0$, $-3 * y$ will always be smaller or equal to -3 and adding 1 will not make it become positive.

$-y/2$ will always be negative because a negative number over a positive number will get a negative number.

Because $-3y+1$ and $-y/2$ will always be negative numbers.

Therefore negative x will produce negative integers or iterate to -1 in some cases. □

Lemma 2. *If we start the function with* 0*, the function will always produce* 0*.*

Proof. Let $x = 0$ in function C.

Then $x = 0$ is even by Fact 1.

Let y be one of the equations from function C.

If x is even, we use the equation of $y = x/2$ from function C.

Then $y = 0/2 = 0$, it goes back to even integer.

The equation iterates to 0 since x will always be 0 and an even integer.

Therefore $x = 0$ will produce 0 in function C. □

Theorem 2. *If function C iterates to* 1*, then x must be positive, not negative and* 0*.*

Proof. By Lemma 1,we conclude that negative x in function C makes the function iterates to -1 or negative integers.

By Lemma 2, we conclude that $x = 0$ in function C makes the function iterates to 0.

x can only be positive, negative or 0 by Fact 2.

If x cannot be negative or 0 because they do not iterate to 1.

Then we know $C(x)$ iterates to 1 is true for x is an positive integer.

Then the only possible values on which the Collatz conjecture holds is the positive integers. □

3 The function C has more chance of producing an even number

From Section 2 we know that x can only be positive integers, so in Section 3 we will prove that if x is an positive integer, randomly chosen from 1 to n, the probability of $C(x)$ being even approaches $2/3$ as n goes to ∞.

Lemma 3. *If x is an odd positive integer, then it will become an even integer after an iteration of function C.*

Proof. Assume x is an odd positive integer.
Then $x = 2m + 1$.
Let y be one of the equations from the function.
Since x is odd we use the equation of $y = 3x + 1$ from function C.
Then $y = 3(2m + 1) + 1 = 6m + 1 + 1 = 6m + 2$ is even.
Therefore y is an even number. □

Lemma 4. *If x is an even positive integer, then it will become either an even or odd integer after an iteration of function C.*

Proof. Assume x is an even positive integer.
Then $x = 2m$.
Let y be one of the equations from the function.
Since x is even we use the equation of $y = x/2$ from function C.
Then $y = 2m/2 = m$.
Therefore y can be an even number or an odd number. □

Theorem 3. *If x is an positive integer, it has a greater chance of becoming an even integer after an iteration of function C.*

Proof. By Lemma 3, we conclude that if x is an odd positive integer, it will produce an even integer.

By Lemma 4, we conclude that if x is an even positive integer, it will produce an odd or even integer.

Therefore, we say that if x is an positive integer chosen randomly from 1 to n, as n goes to ∞, the probability of $C(x)$ being even approaches to 2/3. □

4 The even function of $C(x)$ is approaching to a smaller integer

In section 3 we proved that function C has a greater probability of producing an even number, and the even function of $C(x)$ is $f(x) = x/2$. So this section proves that the even function of $C(x)$, $f(x) = x/2$, is producing an smaller integer after each iteration, in other words, is approaching to 1.

Theorem 4. *If x is an even positive integer, then function C will produce a smaller integer.*

Proof. Assume x is an even positive integer.
Let $f(x)$ be one of the equations from the function.
Since x is even we use $f(x) = x/2$ from function C.
We take the limit of x to the ∞ for **even numbers only**.

$$\lim_{x\to\infty} f(x) = \infty/2 \cdots \cdots 2/2 = 1.$$
$$\infty/2 > 2/2.$$

Therefore the even function of $C(x)$, $f(x) = x/2$, is producing a smaller integer and approaching to 1. $\square$

Example 1. Assume Theorem 4 is true, 2^n will always be an even positive integer for $n \geq 1$, and will eventually iterates to 1 with theorem 4, then we can say that:

$$\frac{2^1}{2} + \frac{2^2}{2} + \frac{2^3}{2} + \cdots + \frac{2^n}{2} = 1 * n$$

(Each fraction divides 2 until it gets to 1)

We will prove this by induction.

Proof. For the base case, $n = 1$, the left hand side is $\frac{2^1}{2} = 1$, and the right hand side is $1 * 1 = 1$ as well. So the equation holds for the base case.

For the induction step, let $k \geq 1$ be an arbitrary natural number and assume $\frac{2^1}{2} + \frac{2^2}{2} + \frac{2^3}{2} + \cdots + \frac{2^k}{2} = 1 * k$. Then we have:

$$\begin{aligned}
\frac{2^1}{2} + \frac{2^2}{2} + \frac{2^3}{2} + \cdots \frac{2^{k+1}}{2} + &= (\frac{2^1}{2} + \frac{2^2}{2} + \frac{2^3}{2} + \cdots \frac{2^k}{2}) + \frac{2^{k+1}}{2} && (1)\\
&= (1 * k) + \frac{2^{k+1}}{2} && (2)\\
&= (1 * k) + (\frac{2^k}{2} * \frac{2}{2}) && (3)\\
&= (1 * k) + (1 * 1) && (4)\\
&= (1 * k) + 1 && (5)\\
&= 1 * (k + 1). && (6)
\end{aligned}$$

This proves that the equation holds for $n = k + 1$, and the induction is complete. $\square$

5 Conclusion and further questions

In conclusion, we have discovered a heuristic of what it might be true for the proof of Collatz conjecture. When x is 0, the function iterates to 0 and when x is a negative integer, the function iterates to -1 or negative integers. So x can only be positive integers for function iterates to 1. Also that function C has a greater probability of producing an even number and the even function is approaching to 1. Therefore, these might provides evidence to solve for the proof of the Collatz conjecture.

Conjecture 1. *If x is an negative integer, function C will sometimes iterate to -1 or to an random negative integer. For example, when $x = -3$, the function will iterate to -1, and when $x = -5$, the function will iterate to -5 and back to where it started.*

Conjecture 2. *If we continue the function after reaching to 1, the function will always end with $4, 2, 1$ as the results of last three iterations.*