PDE.2.3 Nonhomogeneous boundary conditions

1. Suppose $\mathcal{L}$ is a linear partial differential operator, $u(z;t)$ is the solution of the initial-boundary value problem

$$\begin{aligned} \mathcal{L}u &= r(z;t) && (z,t) \in (0,L) \times (0,\infty), \\ u(0;t) &= f_1(t) && t \in (0,\infty), \\ u(L;t) &= f_2(t) && t \in (0,\infty), \\ u(z;0) &= u_0(z) && z \in (0,L), \end{aligned}$$

that $w_1(z;t)$ is the solution of the problem

$$\begin{aligned} \mathcal{L}w &= 0 && (z,t) \in (0,L) \times (0,\infty), \\ w(0;t) &= f_1(t) && t \in (0,\infty), \\ w(L;t) &= 0 && t \in (0,\infty), \\ w(z;0) &= 0 && z \in (0,L), \end{aligned}$$

that $w_2(z;t)$ is the solution of the problem

$$\begin{aligned} \mathcal{L}w &= 0 && (z,t) \in (0,L) \times (0,\infty), \\ w(0;t) &= 0 && t \in (0,\infty), \\ w(L;t) &= f_2(t) && t \in (0,\infty), \\ w(z;0) &= 0 && z \in (0,L), \end{aligned}$$

and that $w_s(z;t)$ is the solution of the problem

$$\begin{aligned} \mathcal{L}w &= r(z;t) && (z,t) \in (0,L) \times (0,\infty), \\ w(0;t) &= 0 && t \in (0,\infty), \\ w(L;t) &= 0 && t \in (0,\infty), \\ w(z;0) &= 0 && z \in (0,L). \end{aligned}$$

Define $v(z;t) = u(z;t) - w_1(z;t) - w_2(z;t) - w_s(z;t)$. Formulate an initial-boundary value problem of which v is the solution.

2. Solve the problem

$$\begin{aligned} u_t &= K u_{zz} + az && (z,t) \in (0,L) \times (0,\infty), \\ u(0;t) &= 0 && t \in (0,\infty), \\ u(L;t) &= 0 && t \in (0,\infty), \\ u(z;0) &= 0 && z \in (0,L). \end{aligned}$$

3. Consider a metallic rod of length L and thermal diffusivity K, insulated along its length and insulated at its ends. Suppose that at initial time $t = 0$, the temperature at the left end of the rod is $20 + 2L$, the temperature at the right end of the rod is $20 - 2L$, and the initial temperature profile is linear. Find an expression for the temperature $u(z;t)$ at all points in the rod and all times $t > 0$.