ORF 474: High Frequency Trading
Notes 4a

Robert Almgren

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1 Information flow

Papers by Bagehot (1971) and Glosten and Milgrom (1985).

We assume that the asset has a "true value" v, which is unknown (to most people) until some future date T. At that future time, all positions will be closed out at that price by some magical external agent, so this is the benchmark price for all trades.

In this model there is no ϵ_t representing the price change due to news arrivals between the $(t-1)$st and the tth trade; information comes into the market only through the orders submitted by the external traders. The information set Ω_{t-1} is the dealer's information just after the $(t-1)$st trade, which is also the information just before the tth trade. The market maker must set bid and ask prices b_t, a_t using information Ω_{t-1}, so that the prices still look reasonable in Ω_t, the information after the trade happens.

Let $\mu_{t-1} = \mathbb{E}(\, v \mid \Omega_{t-1}\,)$, the dealer's expected value of the true price just before the tth trade. If the trade carries no information, then $\Omega_t = \Omega_{t-1}$ and $\mu_{t-1} = \mu_t = \mathbb{E}(\, v \mid \Omega_t\,)$, the dealer's expected value of the true price just *after* the tth trade. If the dealer buys at a price $p_t > \mu_t$, or sells at a price $p_t < \mu_t$, then he expects to lose money: he will not be willing to do that trade. If the dealer buys at a price $p_t < \mu_t$ or sells at a price $p_t > \mu_t$, then he has a positive expected profit: competition will eliminate those prices. The only possible trade price that yields a zero expected profit is $p_t = \mu_t$. Therefore the dealer will set quotes $b_t = a_t = p_t$ and the spread $S = 0$. The price will not change from one trade to the next. (In the Roll model above, new information arrived between trades $t-1$ and t, which caused the quotes at t to be different than the quotes at $t-1$.) Information carried by the trades is the essential ingredient that leads to a positive spread.

Now we assume that some traders have information about the future value v. Traders who have this information will submit orders using it, and the market maker's task is to extract this information. The information available to the market maker just after the tth trade is the information before, combined with the direction of the tth trade, or $\Omega_t = \Omega_{t-1} \cap \{d_t\}$. Since the market maker sets bid and ask prices independently, she can set those prices to be *ex post* accurate in the two possible cases.

For a specific model, suppose that the final asset value v has two possible values v_{L} and v_{H}, with $v_{\mathrm{L}} < v_{\mathrm{H}}$. Let θ_{t-1} denote the dealer's estimate, before the tth trade, of the probability that $v = v_{\mathrm{H}}$. The dealer's estimate of the true value, *before* the tth trade, is

$$\mu_{t-1} \;=\; \mathbb{E}(\, v \mid \Omega_{t-1} \,) \;=\; \theta_{t-1} v_{\mathrm{H}} \;+\; (1-\theta_{t-1}) v_{\mathrm{L}}.$$

The dealer sets prices so that, whether the tth trade is a buy or a sell, *after* that trade,

$$p_t \;=\; \mu_t \;=\; \mathbb{E}(\, v \mid \Omega_t \,) \;=\; \theta_t v_{\mathrm{H}} \;+\; (1-\theta_t) v_{\mathrm{L}}.$$

That is, there is no expected profit following the trade.

Since the dealer sets bid and ask prices separately, representing the prices at which a sell happens or a buy happens respectively, it is enough to set these prices as

$$\begin{aligned} a_t \;&=\; \mathbb{E}(\, v \mid \Omega_{t-1},\, d_t = +1 \,) \quad \text{and} \\ b_t \;&=\; \mathbb{E}(\, v \mid \Omega_{t-1},\, d_t = -1 \,). \end{aligned}$$

The dealer will also update her value of the probability θ_t, based on previous beliefs θ_{t-1} and the new information d_t. Let us assume for now that

$$v_{\mathrm{L}} \;<\; b_t \;<\; a_t \;<\; v_{\mathrm{H}},$$

which can be shown to always be true in the model below.

The dealer knows that the external traders are of two types:

- A fraction π of the traders are "informed:" they know with certainty which of the two possible final values $v_{\mathrm{L}}, v_{\mathrm{H}}$ is true. They use this information to decide whether to send a buy or a sell order.

- The remaining fraction $1-\pi$ are "uninformed," "liquidity," or "noise" traders, who trade for reasons unconnected with information about the asset. Uninformed traders submit buy and sell orders randomly with equal probability.

Here is how each step of trading works:

1. The dealer sets bid and ask prices b_t, a_t.

2. The market selects randomly whether this trader is informed (probability π) or uninformed ($1-\pi$). These choices are uncorrelated from one trade to the next.

3. If this trader is uninformed, then he sends a buy or sell order randomly with equal probability, and uncorrelated with everything else.

4. If this trader is informed, then she sends a buy order if $v = v_{\mathrm{H}}$, and a sell order if $v = v_{\mathrm{L}}$. Since $a_t < v_{\mathrm{H}}$ and $b_t > v_{\mathrm{L}}$, this guarantees her a profit at the end of trading.

5. The dealer updates her estimates of the probability θ_t.

The trade directions will not be equally distributed. If $v = v_{\mathrm{H}}$, then the liquidity traders always submit buys, so $d = +1$ will be much more common, and conversely if $v = v_{\mathrm{L}}$. The dealer's task is to extract this information from the stream of market orders, in the presence of noise introduced by the liquidity traders.

The dealer knows the two possible final values v_{L} and v_{H}, and the fraction π of informed traders. The randomness comes from the initial selection of whether $v = v_{\mathrm{L}}$ or $v = v_{\mathrm{H}}$, and at each step whether the trader is informed or uninformed, and if uninformed, whether he sends a buy or sell order.

Table 1 shows the 8 possible events that can happen on the tth trade, with probabilities from the point of view of the dealer just before the trade. The dealer wants to compute the probabilities of $v = v_{\mathrm{H}}$ and $v = v_{\mathrm{L}}$, conditional on observing either a buy $d = +1$ or a sell $d = -1$. Using Bayes' rule

$$P(\,A \mid B\,) \;=\; \frac{P(\,A \cap B\,)}{P(B)},$$

we have

$$\begin{aligned}
\theta_t^+ \;&\equiv\; P(\,v = v_{\mathrm{H}} \mid d_t = +1\,) \;=\; \frac{\frac{1}{2}(1+\pi)\theta_{t-1}}{\pi\theta_{t-1} + \frac{1}{2}(1-\pi)} \\
1 - \theta_t^+ \;&\equiv\; P(\,v = v_{\mathrm{L}} \mid d_t = +1\,) \;=\; \frac{\frac{1}{2}(1-\pi)(1-\theta_{t-1})}{\pi\theta_{t-1} + \frac{1}{2}(1-\pi)} \\
\theta_t^- \;&\equiv\; P(\,v = v_{\mathrm{H}} \mid d_t = -1\,) \;=\; \frac{\frac{1}{2}(1-\pi)\theta_{t-1}}{\pi(1-\theta_{t-1}) + \frac{1}{2}(1-\pi)} \\
1 - \theta_t^- \;&\equiv\; P(\,v = v_{\mathrm{L}} \mid d_t = -1\,) \;=\; \frac{\frac{1}{2}(1+\pi)(1-\theta_{t-1})}{\pi(1-\theta_{t-1}) + \frac{1}{2}(1-\pi)}
\end{aligned}$$

Odds ratios The formulas are a bit simpler if expressed in terms of *odds ratios*. If θ is a probability, then the associated odds ratio is $r = \theta/(1-\theta)$, between 0 and $+\infty$. The inverse is $\theta = 1/(1+r)$. For example, if an event has a $2/5$ probability, then its odds ratio is $2/3$ or $2 : 3$ ("two-to-three").

In these terms, the update formulas for the dealer's probability that $v = v_{\mathrm{H}}$ are

$$\begin{aligned}
\frac{\theta_t^+}{1-\theta_t^+} &= \frac{1+\pi}{1-\pi}\,\frac{\theta_{t-1}}{1-\theta_{t-1}} \\
\frac{\theta_t^-}{1-\theta_t^-} &= \frac{1-\pi}{1+\pi}\,\frac{\theta_{t-1}}{1-\theta_{t-1}}
\end{aligned}$$

Thus the dealer changes her estimate of the probability such that the odds ratio increases by a factor of $(1+\pi)/(1-\pi) > 1$ each time an external trader buys at the ask, and decreases by a factor of $(1-\pi)/(1+\pi) < 1$ each time an external trader sells at the bid.

If $\pi = 0$ then the probabilities do not change from one trade to the next. If $\pi = 1$ then as soon as one trade is observed, one odds ratio becomes zero and the other infinite.

$v = v_{\mathrm{H}}$	θ_{t-1}	Informed	π	Buy	1	$\pi\theta_{t-1}$
				Sell	0	0
		Uninformed	$1-\pi$	Buy	$\frac{1}{2}$	$\frac{1}{2}(1-\pi)\theta_{t-1}$
				Sell	$\frac{1}{2}$	$\frac{1}{2}(1-\pi)\theta_{t-1}$
$v = v_{\mathrm{L}}$	$1-\theta_{t-1}$	Informed	π	Buy	0	0
				Sell	1	$\pi(1-\theta_{t-1})$
		Uninformed	$1-\pi$	Buy	$\frac{1}{2}$	$\frac{1}{2}(1-\pi)(1-\theta_{t-1})$
				Sell	$\frac{1}{2}$	$\frac{1}{2}(1-\pi)(1-\theta_{t-1})$

Table 1: Possible events on the tth trade, with probabilities as measured by the dealer just before the trade. The values after each event are the probability of that event, given events to the left. The final column is the net probability of that combination.

Quotes The market maker then sets quotes

$$
\begin{aligned}
a_t &= \mu_t^+ \equiv \mathbb{E}(\, v \mid \Omega_{t-1}, d_t = +1\,) = \theta_t^+ v_{\mathrm{H}} + (1-\theta_t^+)v_{\mathrm{L}} = \mu_{t-1} + s_t^a \\
b_t &= \mu_t^- \equiv \mathbb{E}(\, v \mid \Omega_{t-1}, d_t = -1\,) = \theta_t^- v_{\mathrm{H}} + (1-\theta_t^-)v_{\mathrm{L}} = \mu_{t-1} - s_t^b
\end{aligned}
$$

where the two-sided spreads are

$$
\begin{aligned}
s_t^a &= \frac{\pi\theta_{t-1}(1-\theta_{t-1})}{\pi\theta_{t-1} + \frac{1}{2}(1-\pi)}\,(v_{\mathrm{H}} - v_{\mathrm{L}}) \\
s_t^b &= \frac{\pi\theta_{t-1}(1-\theta_{t-1})}{\pi(1-\theta_{t-1}) + \frac{1}{2}(1-\pi)}\,(v_{\mathrm{H}} - v_{\mathrm{L}})\,.
\end{aligned}
$$

The total spread is $S = s_t^a + s_t^b$. The spreads are proportional to $v_{\mathrm{H}} - v_{\mathrm{H}}$, that is, to the uncertainty in the final value. The final time T does not appear so may be as long as we like; we only assume that as some future point all positions are cleared at price v.

If $\pi = 1$ then $\theta_t^+ = 1$ and $\theta_t^- = 0$. Since every trader is informed, the market maker immediately knows the true value after the first trade. She sets quotes $a_t = v_{\mathrm{H}}$ and $b_t = v_{\mathrm{L}}$, since those are the only possible future values. If $\pi = 0$, then $S = 0$.

As as simple case, suppose that at the start of trading, the market maker believes that both outcomes are equally likely: $\theta_0 = \frac{1}{2}$. The bid and ask spreads for the first trade are

$$
s_1^a = s_1^b = \frac{\pi}{2}(v_{\mathrm{H}} - vL)
$$

and the total spread is

$$S = s_1^b + s_1^a = \pi(v_{\mathrm{H}} - vL)$$

As t increases, the market makers get better and better information about which of the final values is the true one. The bid and ask prices converge to the true one, and the spread converges to zero. This is because the market makers are not willing to sell at prices $a_t < v$, nor buy at prices $b_t > v$, because they would lose money. They would love to sell at prices $a_t > v$ or buy at prices $b_t < v$, but competitive pressures make it impossible to sustain such prices in the market. Therefore $b_t \to v$ and $a_t \to v$ (Figure 1).

From the market makers' point of view, trade prices are *martingales:* $\mathbb{E}(p_{t+k}|\Omega_t) = p_t$. From the informed traders' point of view, trade prices drift steadily toward the true value.

The initial bid-ask spread is determined by the uncertainty in the final value (more uncertainty means wider spread), and by the probability of informed traders π. The larger the value of π, the wider the market makers need to make the spread to compensate their information deficit. In the extreme case $\pi = 1$ they need to set $b_1 = v_{\mathrm{L}}$ and $a_1 = v_{\mathrm{H}}$ since they know that all traders will be informed starting with the very first one.

To track the money flow, we recall that at some final time $t = T$, all positions are liquidated at the value v. Then each time a trade occurs, we can record a profit or loss by comparing the trade price to v, even though that value is not known to most market participants. For example, if you buy a share at $p^{(1)}$ and later sell it at $p^{(2)}$, then you would say you have a profit of $p^{(2)} - p^{(1)}$. I would say that you have a profit of $(p^{(2)} - v) - (p^{(1)} - v)$, which is the same. Then the average flow is

$$\text{Uninformed} \quad \longrightarrow \quad \text{Market makers} \quad \longrightarrow \quad \text{Informed.}$$

The market makers make no net profit on average. Informed traders use their information to make profit, which comes from the uninformed traders who pay the spread.

Finally a general comment: Any time that a model has a probability, your responsibility as a trader is to game the probability. For example, if you read that the price has equal probability to go up or down, then you should try to get extra information to predict which direction it is more likely to go.

In this case, the key information is whether a particular trade comes from an informed or an uniformed trader. Therefore as a market maker, you should put a lot of effort into discovering whether your clients have knowledge or not, so you can continue to make profit from the uniformed traders while not losing too much to informed. So you would like to do things like trade against retail flow (individual traders) who are unlikely to have high-frequency information, or set up restricted trading venues where only low frequency traders are allowed in.

Conversely, if you are an informed trader (or believe yourself to be informed), then you will want to work hard to appear uninformed, so you can continue to extract profits from the other market participants before they recognize you. So you may not want to jump too quickly on favorable price moves, and perhaps you may occasionally want to put in trades in the wrong direction.

Trading is all about information: trying to extract it from other market participants, and concealing your own information.

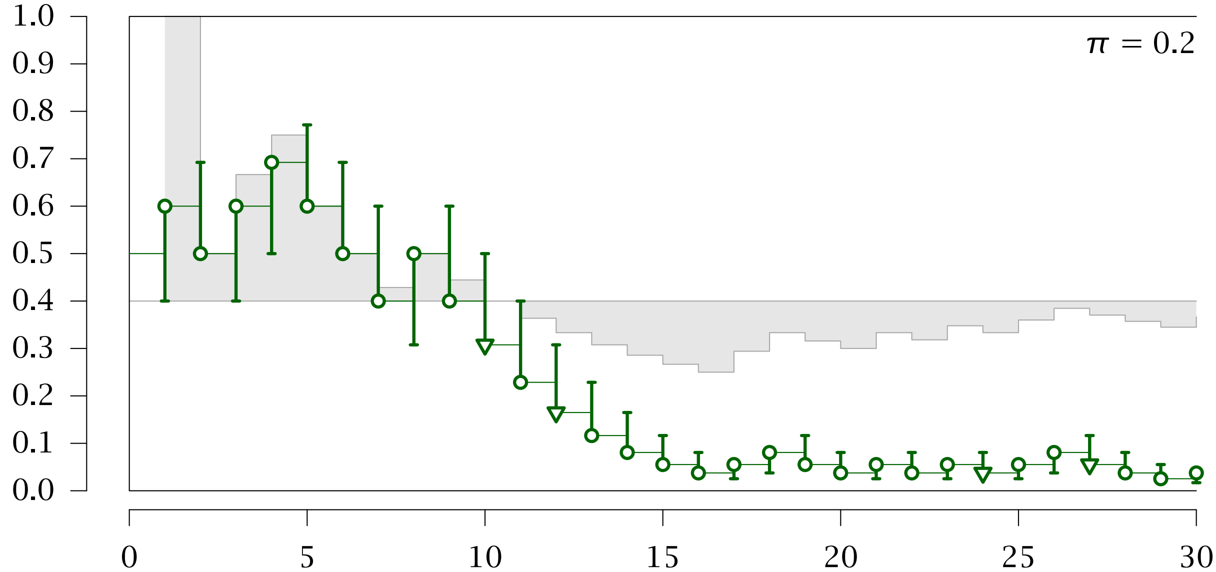

Figure 1: Horizontal axis is steps; vertical axis is price. The market maker knows that the possible true values are $v_L = 0$ and $v_H = 1$, and that the probability of informed trading is $\pi = 0.2$. Vertical bars are the bid-ask spread set by the market maker. Circles are uninformed trades, triangles are informed trades; the distinction between them is not known by the market maker. In this case, the true value is $v = 0$, so the informed traders always send sell orders as indicated by the downward-pointing triangles. The gray region is the cumulative fraction of buy orders, which converges to $(1 + \pi)/2$ if $v = v_H$ and to $(1 - \pi)/2$ if $v = v_L$; here 0.4. The horizontal segments indicate that the previous trade value p_t is used as the best estimate for the next midpoint μ_t. In this realization, the market makers initially adjust their prices upwards because 3 of the first 4 trades happen to be buys from the uninformed traders. Even though very few of the trades are informed, eventually the imbalance is detected by the market makers and their quote prices converge to the true value $v = 0$.

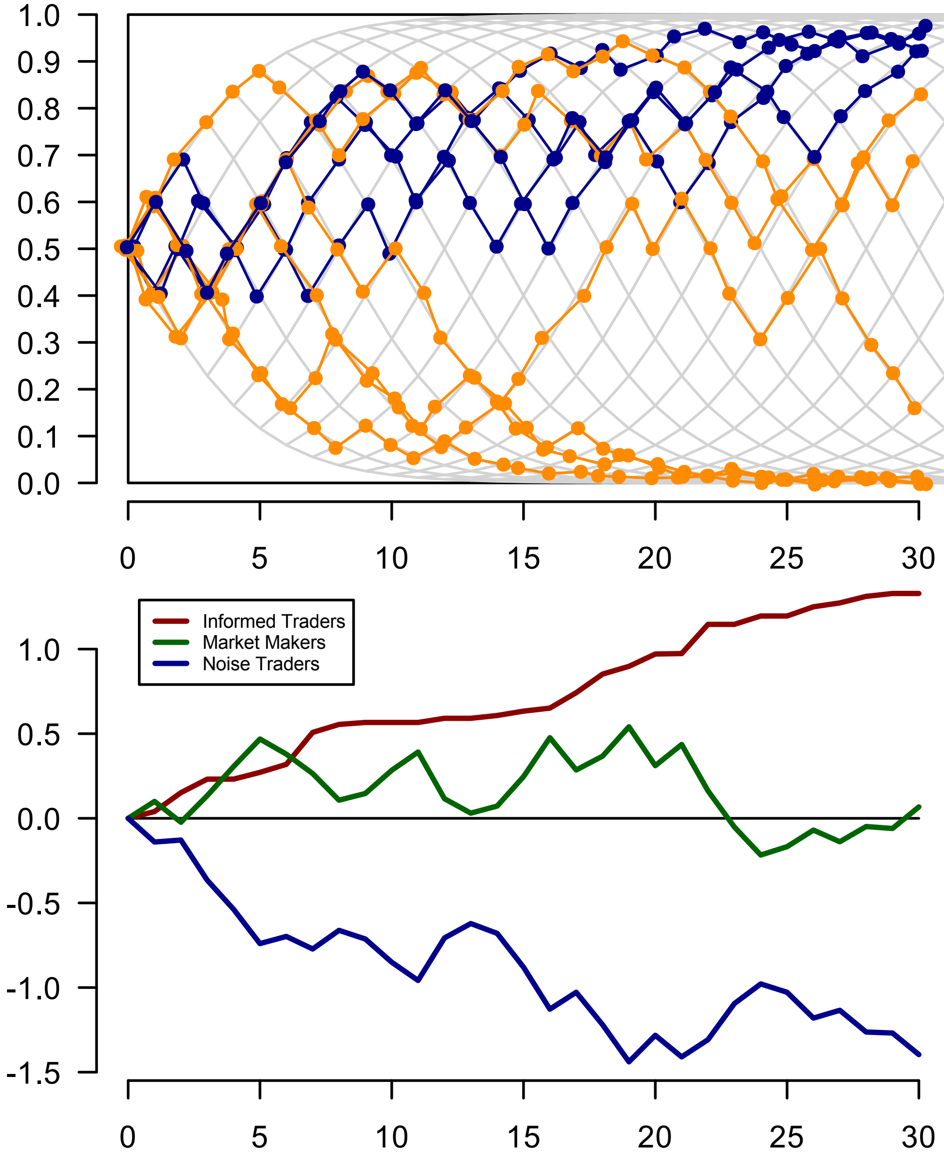

Figure 2: 20 independent realizations of Glosten-Milgrom model, with $\pi = 0.2$. Top panel is three separate variables, all of which are the same: the dealer's probability θ_t that $v = v_{\mathrm{H}}$, the dealer's expected value of v (since $v_{\mathrm{L}} = 0$ and $v_{\mathrm{H}} = 1$), and the bid and ask quotes. The paths eventually converge to $v = v_{\mathrm{L}}$ (orange) or $v = v_{\mathrm{H}}$ (blue). Bottom picture is the cumulative net profits of the three types of participants, averaged across the realizations. On aera

Profit and loss Suppose that $v = v_H$. The informed traders consistently buy at $a_t < v$, so they make a profit from the trend, or drift, as the price moves in the direction that they know it will. The uninformed traders are randomly buying at a_t or selling at b_t. On average, the uninformed traders neither gain nor lose consistently from the trend, but they consistently lose money on the spread. Thus, to extend the picture that was in the previous notes:

$$\text{Uninformed} \xrightarrow[\text{spread}]{} \text{Market makers} \xrightarrow[\text{trend}]{} \text{Informed.}$$

Competition drives the spread to be such that the two cash flows are equal on average.

Why don't the uninformed traders quit the market, and why doesn't competition among the informed traders drive their profit to zero? A quick answer would be that those effects are not included in this model. More fully, the uninformed traders presumably think they are making profit in some other way, like long-term investment returns, so they are willing to pay the spread. And indeed there might be competition among informed traders, to get to the market quickly and profit from their information before the market makers realize what is going on and adjust their quotes.

Critique The model is very stylized, and does a good job of capturing what it is intended to do, but still one may point out two limitations:

1. Two final values is an oversimplification. The model can be extended in various ways to incorporate distributions for the beliefs of the informed traders and the market makers, depending on how much analytical complexity one is willing to address.

2. More seriously, the model does not incorporate the continual inflow of information, represented in the Roll model by the innovation ϵ_t, nor the partial information that some market participants have about this evolving state. That is much harder to incorporate into a reasonable model.

References

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