ORF 474: High Frequency Trading
Notes 5a

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March 2, 2020

The model of Kyle (1985) considers the strategic interplay between a dealer who sets the trade price, and an informed trader who knows the final value, in the presence of uninformed traders. Here information is transmitted by the *size* of the orders submitted by the informed trader relative to the size of the orders submitted by the noise traders. This is in contrast to to the model of Glosten and Milgrom (1985), where the dealer eventually deduces the true final value by observing the relative fraction of buys and sells; here the dealer observes a bias in the sign and size of the submitted order or orders.

1 Single period

There is a single security, whose final value is v. This is believed by the market makers to be normal with expected value p_0 and variance Σ_0^2. The informed traders know v.

Noise, or uninformed, traders send market orders in a total volume u, chosen randomly with mean 0 and variance σ_u^2, independently from v. If $u > 0$ then they are buying; if $u < 0$ then they are selling.

The market proceeds with the following steps:

0. The market chooses the value of $v \sim \mathcal{N}(p_0, \Sigma_0^2)$
1. Two choices are made simultaneously and independently:
   (a) The uninformed traders choose their trade sign and size $u \sim \mathcal{N}(0, \sigma_u^2)$
   (b) The informed traders choose their trade sign and size x. They do not know the uninformed trade size u. The only information they have is the true value v. So their trade size must be a function $x = X(v)$.

   The total trade size submitted to the market is $y = u + x$.
2. The market maker sets the price p. She sees the total trade size $y = u + x$, but not the individual values of u and x. So her price must be a function $p = P(u + x)$.

Everyone in the market (that is, the market maker and the informed traders, since the noise traders do not incorporate any information) knows the structure of the model, and knows the values of p_0, Σ_0, and σ_u. Furthermore, the informed traders and the market makers both believe that the other will make strategically optimal decisions, and each uses the other's rule in making their own decision.

The profits of the participants are

$$\begin{aligned} \text{Informed traders:} \quad & (v-p)x \\ \text{Noise traders:} \quad & (v-p)u \\ \text{Market maker:} \quad & (p-v)(u+x) \end{aligned}$$

which add to zero. The equilibrium conditions are

1. The informed traders choose their rule $x = X(v)$ to maximise their expected profit:

$$\max_x \mathbb{E}\big(\pi(x)\big), \qquad \pi = (v-p)x.$$

2. The market makers choose their rule $p = P(y)$ to have zero expected profit:

$$p \;=\; \mathbb{E}\big(v \,|\, u+x\big), \qquad y = u+x.$$

The intuition is that the trade size $u+x$ observed by the market makers does not have mean zero: it is biased away from zero by the difference between v and p_0. They use this bias to adjust their price to reduce the losses to the informed traders. We shall assume—it can be shown that this is the optimum—that both rules have linear form:

$$x \;=\; \beta(v-p_0)$$

and

$$p \;=\; p_0 \;+\; \lambda(u+x).$$

The coefficient in the latter expression is the famous "Kyle's lambda," which measures market impact, the change in price per unit trade size.

The calculation has two steps:

- To determine β knowing λ, write the informed traders' profit:

$$\begin{aligned} \pi \;&=\; (v-p)x \\ &=\; \big(v-p_0-\lambda(u+x)\big)x, \end{aligned}$$

  so

$$E(\pi) \;=\; (v-p_0)x \;-\; \lambda x^2$$

  since $\mathbb{E}(u) = 0$. Maximizing this over x gives

$$x \;=\; \beta(v-p_0) \qquad \text{with} \qquad \beta = \frac{1}{2\lambda}. \tag{1}$$

- To determine λ knowing β, write the market makers' zero-profit condition:

$$p \;=\; \mathbb{E}(\,v \mid y\,) \;=\; p_0 \;+\; \frac{\mathrm{Cov}(v,y)}{\mathrm{Var}(y)}\Big(\,y \;-\; \mathbb{E}(y)\,\Big),$$

  where the latter is the projection theorem for correlated normal variables. Since

$$y \;=\; u + x \;=\; u \;+\; \beta(v - p_0),$$

  we have

$$\mathbb{E}(y) = 0, \qquad \mathrm{Var}(y) = \sigma_u^2 + \beta^2\Sigma_0^2, \qquad \text{and} \quad \mathrm{Cov}(v,y) = \beta\Sigma_0^2,$$

  so

$$p \;=\; \mathbb{E}(\,v \mid y\,) \;=\; p_0 \;+\; \lambda y, \qquad \text{with} \quad \lambda = \frac{\beta\Sigma_0^2}{\sigma_u^2 + \beta^2\Sigma_0^2}. \tag{2}$$

It only remains to solve (1) and (2) for both β and λ. But

$$2\beta \;=\; \frac{1}{\lambda} \;=\; \frac{\sigma_u^2}{\beta\Sigma_0^2} \;+\; \beta,$$

from which

$$\beta = \frac{\sigma_u}{\Sigma_0} \qquad \text{and} \qquad \lambda = \frac{\Sigma_0}{2\sigma_u}.$$

Both the informed traders and the market makers choose their strategies in terms of their own partial information, knowing also their counterparty's strategic choice.

As noted above, the coefficient λ is a measure of market impact. From the informed traders' point of view, in order to trade a quantity x, they must accept a price that is worse than the no-trade price p_0 by λx.

- A *liquid* market is one in which λ is small, so that the informed traders can execute large quantities with small price changes. This requires either Σ_0 small, so that the the market makers do not have much motivation to adjust their prices in response to the information, or σ_u large, so that the noise traders trade with large volume. With large noise volume, the informed traders can more easily hide among them.

- An *illiquid* market is one in which λ is large, so that small trades cause large price changes. This requires either Σ_0 large, so that the information revealed by their trades has large value to the market maker, or σ_u small, so that there are few noise traders to hide among.

2 Multi-period

Given the opportunity to trade across a finite time interval, how do the informed traders act so as to maximise their total profit? How do they spread their trading across time?

Trading takes place at N discrete times $\{t_n\}_{n=0}^N$ between $t_0 = 0$ and $t_N = 1$, with $t_0 < t_1 < \cdots < t_{N-1} < t_N$. We denote $\Delta t_n = t_n - t_{n-1}$ for $n = 1, \dots, N$. The case $N = 1$ will reduce to the single-period case.

Before trading begins at $t = 0$, the value of the asset is fixed by $v \sim \mathcal{N}(p_0, \Sigma_0)$. Going from time $n-1$ to n, trading happens in three steps:

1. The uninformed traders choose a trade sign and size $\Delta u_n \sim \mathcal{N}(\,0,\ \sigma_u^2 \Delta t_n\,)$. Thus their cumulative position $u_n = \sum_{j=1}^n \Delta u_j$ (so $\Delta u_n = u_n - u_{n-1}$ for $n = 1, \dots, N$ and $u_0 = 0$) follows a Brownian motion with variance σ_u^2 per unit time. Note that knowledge of the cumulative positions $u_1, \dots, u_n$ is equivalent to knowledge of the increments $\Delta u_1, \dots, \Delta u_n$.

2. The informed traders choose a trade sign and size $\Delta x_n = X_n(p_1, \dots, p_{n-1}, v)$. Their trade may depend on the true value v, and on the prices at steps before n, but they do not know the uninformed trade at the current step, nor the price p_n that will be in effect for this step. Their cumulative position is $x_n = \sum_{j=1}^n \Delta x_j$, so $\Delta x_n = x_n - x_{n-1}$. From knowledge of the historical prices $p_1, \dots, p_{n-1}$, of the market makers' pricing rule, and of their own trades $x_1, \dots, x_{n-1}$, the informed traders can determine the historical uninformed trades $u_1, \dots, u_{n-1}$ if they need them (though they do not).

3. The market makers set the price $p_n = P_n(x_1 + u_1, \dots, x_n + u_n)$. They know the combined trade sizes at each step up to and including the current step, but they do not know how much is informed and how much is uninformed.

At step n, denote

$$\pi_n \;=\; \sum_{k=n}^{N} (v - p_k) x_k$$

the informed traders's total profit for all trades from the current one to the end (at step k they trade x_k at price p_k, and at the end liquidate at price v).

The informed traders solve the optimization problem

$$\max_{X_n} \mathbb{E}(\,\pi_n \mid p_1, \dots, p_{n-1}, v\,)$$

for each n, and the market makers apply the zero-profit condition

$$p_n \;=\; \mathbb{E}(\,v \mid x_1 + u_1, \dots, x_n + u_n\,).$$

Kyle shows that the solution has the linear form

$$\begin{aligned} p_n &= p_{n-1} \;+\; \lambda_n(\,\Delta x_n \;+\; \Delta u_n\,) \\ \Delta x_n &= \beta_n\,(v - p_{n-1})\,\Delta t_n \\ \mathbb{E}(\,\pi_n \mid p_1, \dots, p_{n-1}, v\,) &= \alpha_{n-1}(v - p_{n-1})^2 \;+\; \delta_{n-1}. \end{aligned}$$

The expected value of the informed traders' future profit is quadratic in the mispricing at the previous step. Further, denote

$$\Sigma_n \;=\; \mathrm{Var}(\, v \mid x_1 + u_1, \dots, x_n + u_n \,)$$

the variance of the market makers' knowledge of the true value after trade n.

Kyle shows that the coeffcients α_n and δ_n solve the recurrence relations

$$\alpha_{n-1} \;=\; \frac{1}{4\lambda_n(1-\alpha_n\lambda_n)}, \qquad\qquad \delta_{n-1} \;=\; \delta_n \;+\; \alpha_n\,\lambda_n^2\,\sigma_u^2\,\Delta t_n$$

with $\alpha_N = 0$ and $\delta_N = 0$. The coefficients β_n and λ_n are determined at step n by

$$\beta_n\,\Delta t_n \;=\; \frac{1-2\alpha_n\lambda_n}{2\lambda_n(1-\alpha_n\lambda_n)}, \qquad\qquad \lambda_n \;=\; \frac{\beta_n\Sigma_n}{\sigma_u^2}.$$

The variance Σ_n is determined by the forward relation

$$\Sigma_n \;=\; (1-\beta_n\,\lambda_n\,\Delta t_n)\,\Sigma_{n-1}$$

To solve these, assume a value for Σ_N. Then the above relations can be solved for α_k, δ_k, β_k, λ_k, and Σ_k for $k = N-1, N-2, \dots, 1, 0$. If the resulting value of Σ_0 does not agree with the specified value, then adjust Σ_N and repeat. Kyle shows that there is a unique value of Σ_N that reproduces the correct Σ_0.

The resulting solution has Σ_n decreasing as n increases from zero, as the market maker gets information from the informed trades. The final value Σ_N is small but still slightly positive.

3 Continuous-time limit

The continuous limit takes N large, and all Δt_n small (see Back (1992) for a more general and rigorous discussion). The cumulative position of the uninformed traders approaches a Brownian motion

$$u(t) \;=\; \sigma_u\,B(t).$$

The cumulative informed trade and the price approach continuous functions $x(t)$,and $p(t)$ that solve the differential equations

$$\begin{aligned} dx(t) \;&=\; \beta(t)\,\big(v-p(t)\big)\,dt & x(0) \;&=\; 0 \\ dp(t) \;&=\; \lambda(t)\,\big(dx(t)+du(t)\big) & p(0) \;&=\; p_0. \end{aligned}$$

The incremental profit of the informed traders solves

$$d\pi(t) \;=\; \big(v-p(t)\big)\,dx(t)$$

The liquidity coefficient is constant in time:

$$\lambda(t) \;=\; \frac{\sqrt{\Sigma_0}}{\sigma_u}$$

The other coefficients have the forms

$$\beta(t) = \frac{\sigma_u}{\sqrt{\Sigma_0}}\,\frac{1}{1-t}, \qquad \Sigma(t) = (1-t)\Sigma_0,$$
$$\alpha(t) = \frac{\sigma_u}{2\sqrt{\Sigma_0}}, \qquad \text{and} \qquad \delta(t) = \frac{\sigma_u\sqrt{\Sigma_0}}{2}\,(1-t).$$

Combining the above, the price solves the stochastic differential equation (SDE)

$$dp(t) \;=\; \frac{v-p(t)}{1-t}\,dt \;+\; \sqrt{\Sigma_0}\,dB(t)$$

The first term in this expression tells us that the price $p(t)$ always converges to the true value v as $t \to 1$, despite the continual fluctuation introduced by the uninformed traders.

For a given true value v, the expectation of the total profit of the informed traders is

$$\mathbb{E}_u\big(\,\pi(0)\mid v\,\big) \;=\; \frac{\sigma_u\sqrt{\Sigma_0}}{2}\left(\frac{(v-p_0)^2}{\Sigma_0}+1\right).$$

The expectation is taken over the random trajectories $u(t)$ of the uninformed traders. In contrast to the single-period case, they have positive expected profit even if $v = p_0$, that is, the true value exactly matches its consensus expectation. When the price fluctuates away from its initial value (also the true value) the informed traders know that it will revert back to the true value, and they can put on trades to profit from this reversion.

Taking expectation over the true value as well, the unconditional expected profit is

$$\mathbb{E}_{u,v}\big(\,\pi(0)\,\big) \;=\; \sigma_u\,\sqrt{\Sigma_0},$$

exactly twice its value in the single-period model.

References

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