PRINCETON UNIVERSITY

ORF 474: High Frequency Trading
Spring 2020
Robert Almgren

Lecture 12a

April 27, 2020

Market Impact and Trade Optimization

- Last week: Impact of a single order
  - Micro impact: single market order
  - Macro impact: single parent order
  - No dependence on how order is executed
  - Not useful for optimization of execution strategies
- This week: Impact of a trade trajectory
  - how order is executed through time
  - necessary to determine optimal trajectories

Final problems

Choose one of (details to follow)

1. Multidimensional cointegration
2. Technical indicators from daily data
3. Optimal trajectories with Bayesian drift

Outline

- Definition of a trade trajectory
- Models for market impact

  Permanent, temporary, and other
- Optimization

  static optimization: calculus of variations

  dynamic optimization (description)
- Additional topics

  calibration to real data

  other benchmarks (VWAP, etc)

1. Optimal trading
Portfolio transition

Current portfolio → Desired portfolio

$$P_0 \xrightarrow[\text{Trades}]{} P_1$$

Challenges:

- Cannot do whole trade instantly
  finite liquidity
- Your trading will move market against you
  information leakage, front-running
- Market may move while you are doing trade
  volatility

Examples

- Put on a position (probably have alpha)
  - P_0=flat, P_1=long or short
- Liquidate a position (may not have alpha)
  - P_0=long or short, P_1=flat
- Transition
  - P_0, P_1 both large nonzero baskets
- Option hedging, time-varying alpha
  - $P_1(t)$ depends on time

Questions to answer

- How should you execute trades?

  what is your metric for "good" execution?

  execute quickly or slowly?

  time-dependent strategy or pre-computed?

  in what proportions if multi-asset portfolio?

- How much will the trade cost?

  how much will this affect returns?

  what is the capacity of your strategy?

  should you even do the trade?

Example #1: trade execution

Must buy 20,000 shares by end of day
- How fast? On what time horizon?
- Buy fast then slow down? Exact trajectory?

Inputs:
- Properties of stock: average daily volume, volatility, etc
- Transaction cost model
- Risk aversion
- Anticipated alpha horizon

Example #2: option hedging

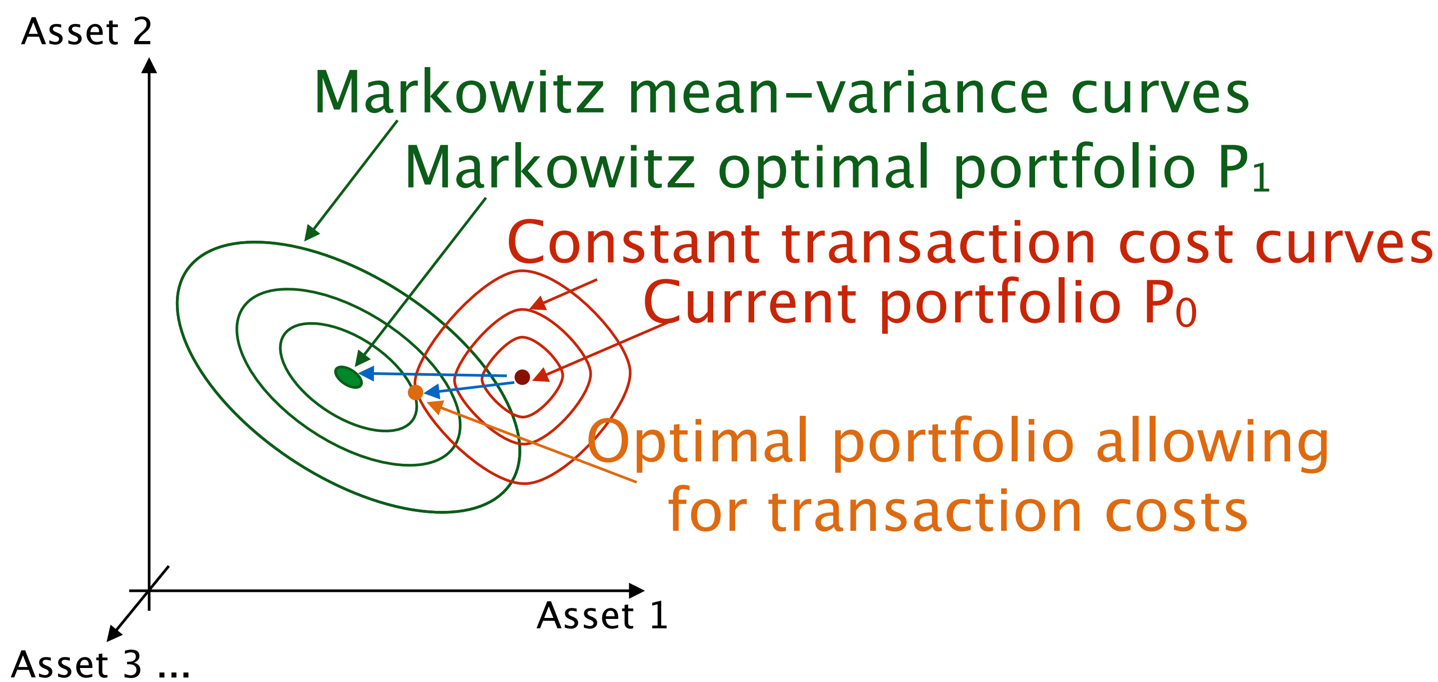
You are sell side dealer that sold large OTC option
Hedge by trading underlying
Size is large enough to affect market

- How precisely should you hedge?
  (Black–Scholes has infinite trade activity)
- What should be your exact strategy?
- Do you want to be more accurate near close?
- What will be visible effects on market?

Inputs:

- Properties of stock: average daily volume, volatility, etc
- Properties of option: delta, gamma, etc
- Transaction cost model
- Risk aversion
- Overnight risk

Example #3: portfolio with trade costs



Inputs:

- Expected return vector and covariance matrix (for Markowitz)
- Current portfolio
- Transaction cost model
- Intended holding period (to balance transaction cost against alpha)

2. Optimization Formulation

- Order execution is separate from investment
- Goal: best final average execution price
- What is a good price?
- Evaluate relative to benchmark
  - benchmark defines an "ideal" trade
  - different benchmarks give different strategies

Optimal trading

Impact model + Objective function $\xrightarrow{\text{Mathematics}}$ Trading Strategy + Optimal cost

How does my trading affect the market?
How do I measure quality of an execution

Calculus for simple models
Calculus of variations for trajectories
Dynamic optimization in general

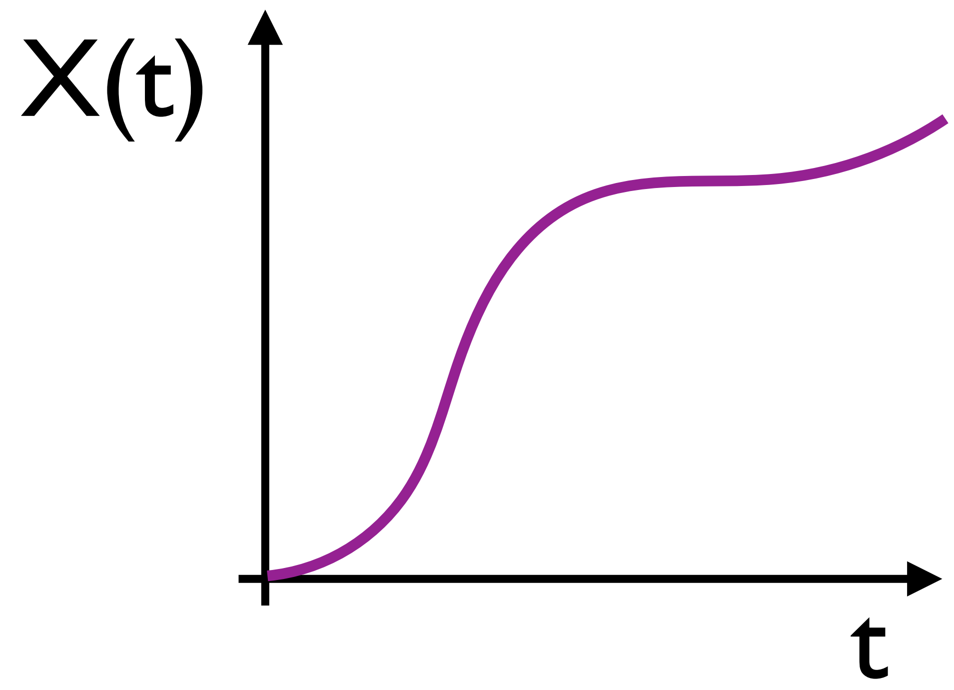
What is the optimal solution?
How good is it?

Main idea of Arrival price optimization

fast *vs* slow

- Reasons to trade quickly
  - P_1 is your benchmark
    - incur risk relative to benchmark until trade is complete
    - do not get alpha until trade is complete
- Reasons to trade slowly
  - market impact
  - large trades cannot be executed quickly

Trade trajectory

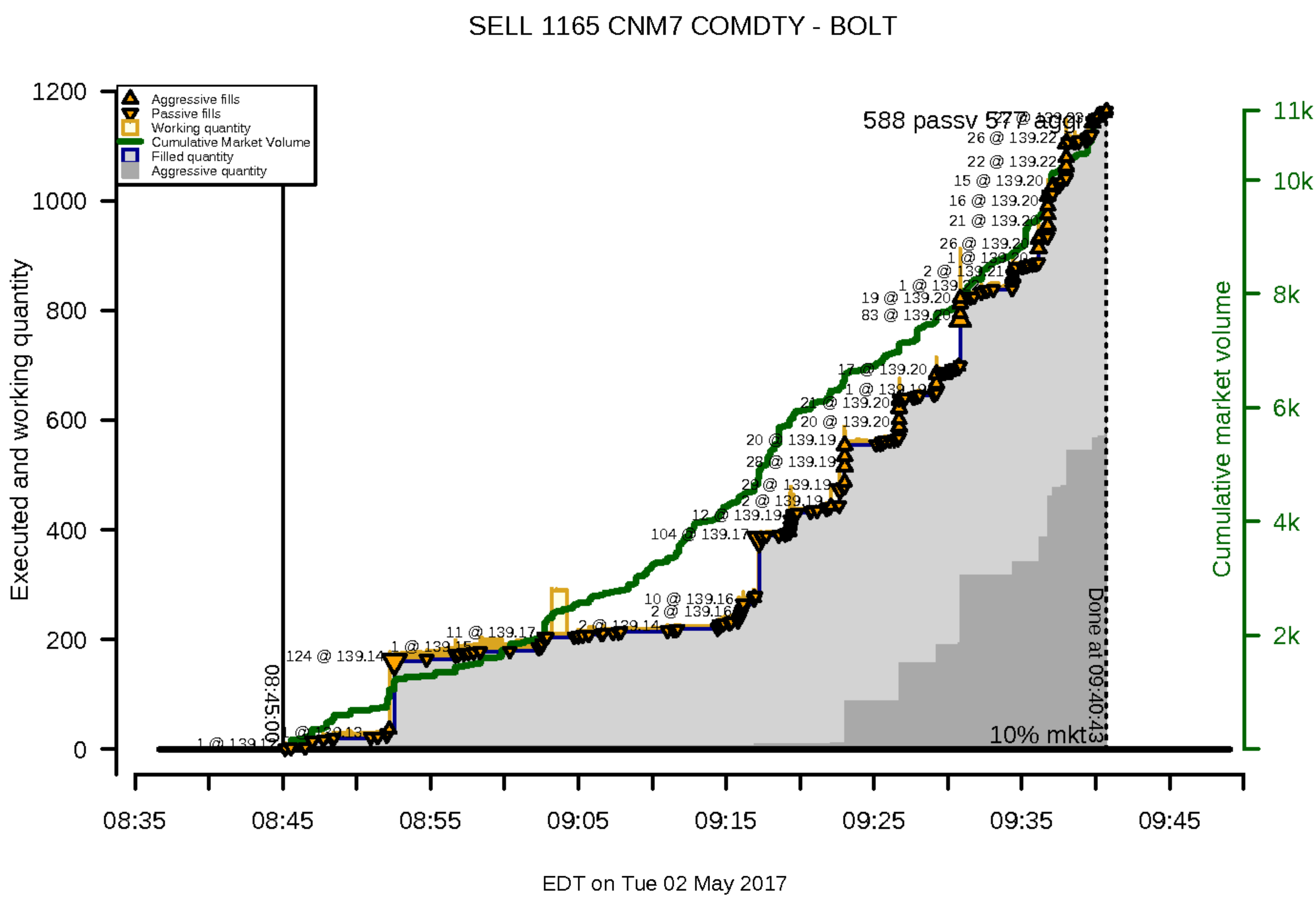


$$X_t = X_0 + \int_0^T \theta_s \, ds$$

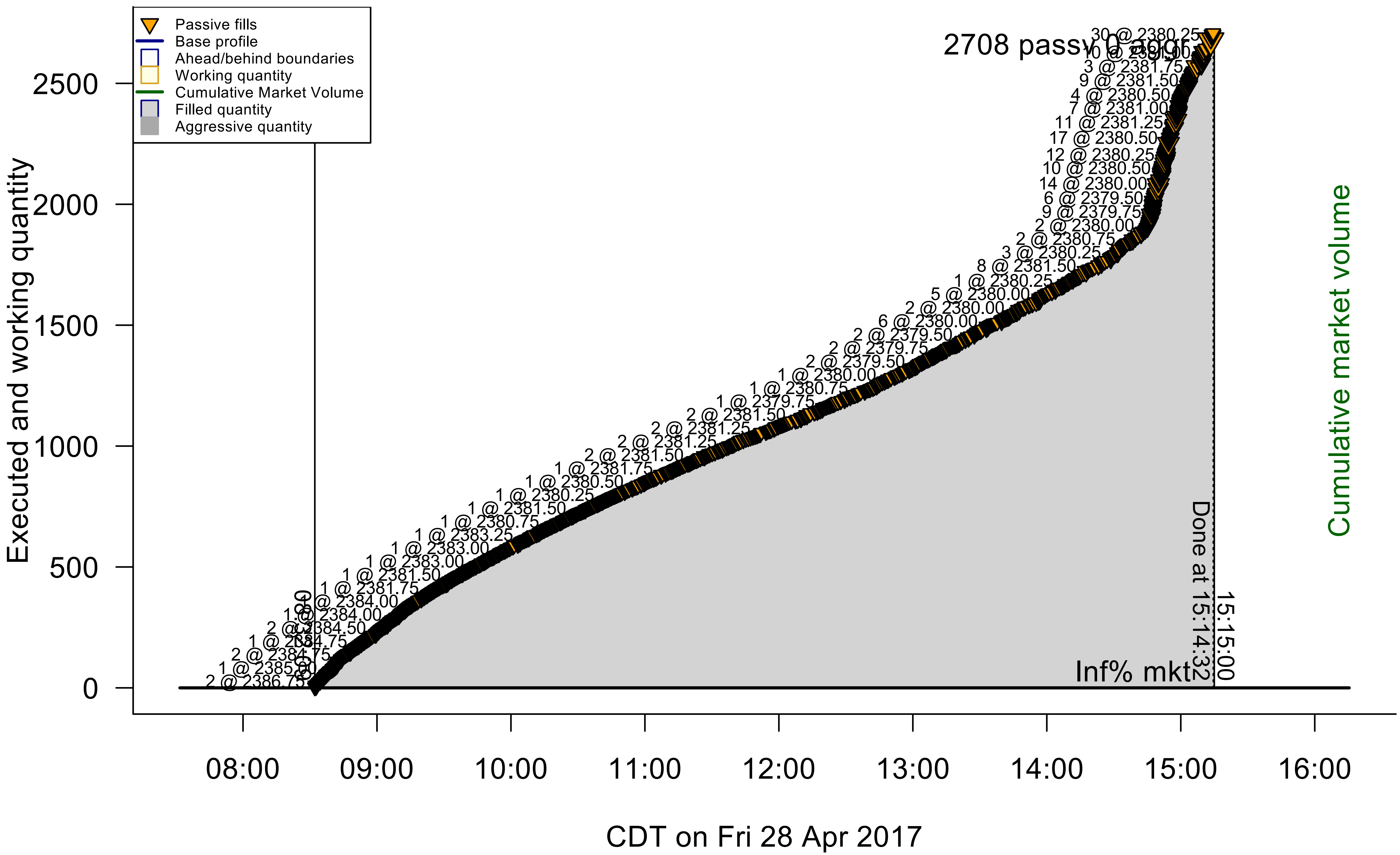
θ_t = instantaneous *rate* of trading
(pretend that trajectory is smooth)

Specify a model for public price evolution and execution price for any trajectory X_t, $\boldsymbol{\theta}_t$

This averages over all the details of the microstructure.
Detailed orders are placed by "micro-algorithm"


SELL 1165 CNM7 COMDTY - BOLT
Aggressive fills
Passive fills
Working quantity
Cumulative Market Volume
Filled quantity
Aggressive quantity
Executed and working quantity
Cumulative market volume
EDT on Tue 02 May 2017
588 passv 577 aggr
Done at 09:40:43
08:45:00
10% mkt

SELL 2708 ESM7 STROBE



Market impact models for trading

- Two types of market impact (both active, both important):
- Permanent
    - due to information transmission
    - affects public market price
- Temporary
    - due to finite instantaneous liquidity
    - “private” execution price not reflected in market
- Many richer structures are possible

The implementation shortfall: Paper versus reality

André F. Perold

Journal of Portfolio Management; Spring 1988; pg. 4

Reality involves the cost of trading and *the cost of not trading*.

Temporary impact: liquidity

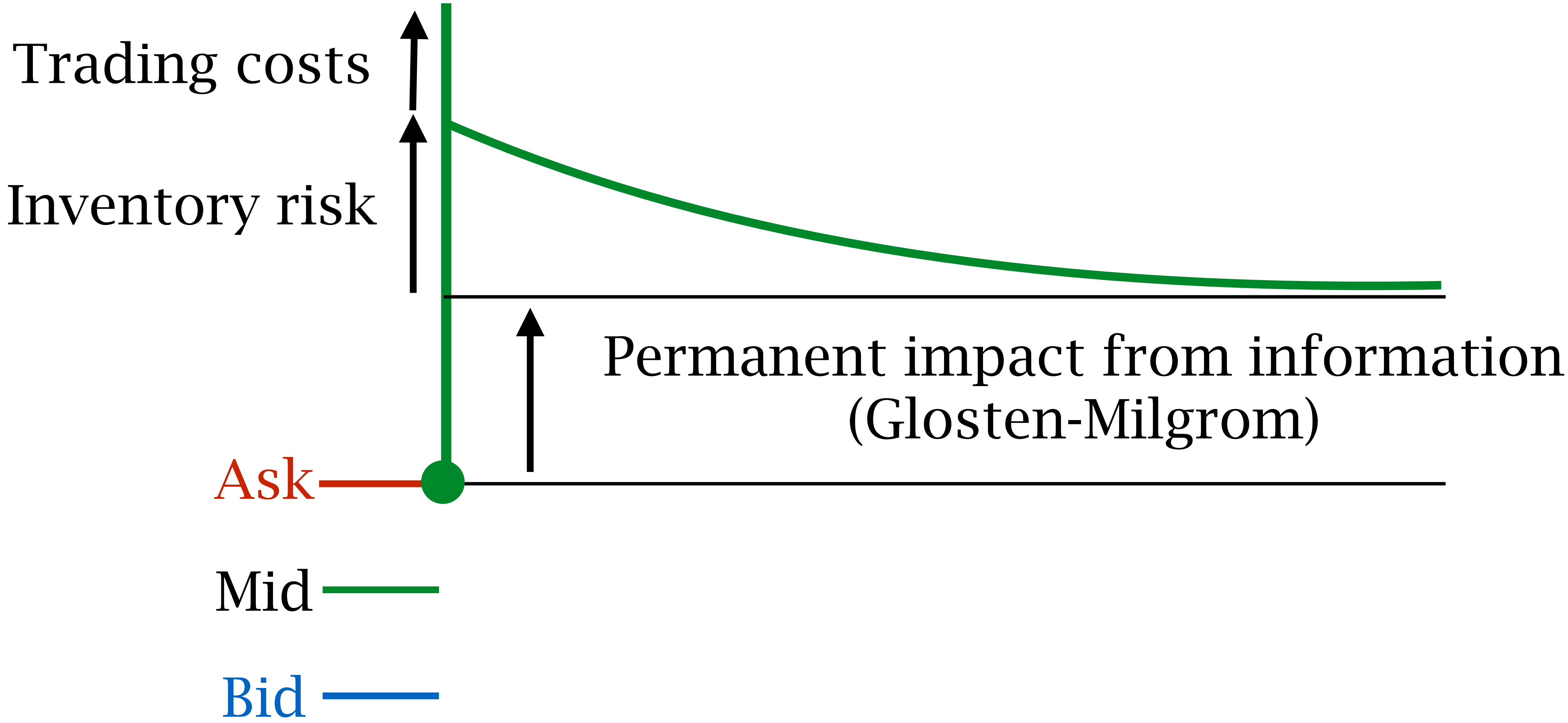
Permanent impact: information

Execution cost also measures price impact. For the purposes of this discussion, let us define price impact to be the difference between the price you could have transacted at on paper (the average of the bid and ask at the time of the decision to trade) and the price you actually transacted at, whether immediately following the decision to trade or later. For example, if you buy at the ask (or sell at the bid) prevailing at the time of the decision to trade, your price impact will be half the bid–ask spread.

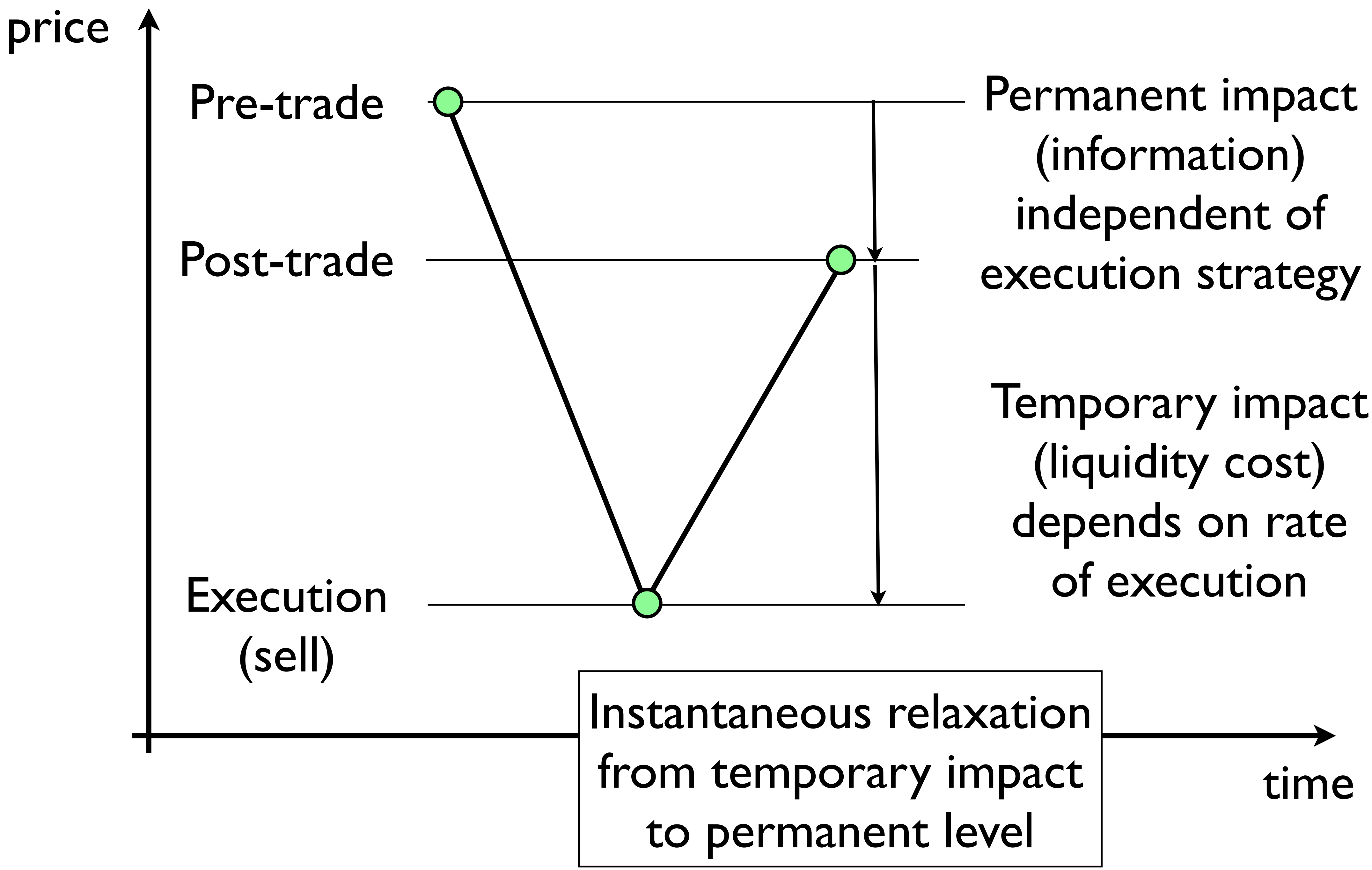
Price impact may occur because you have to move the market temporarily away from its current price in order to induce someone to supply the liquidity you are seeking. From time to time, there may be negative price impact, because you are able to take advantage of someone on the other side who needs the liquidity more than you do. When the price impact is purely a liquidity effect, the price of the stock will usually return to the level it was at before you traded.

Price impact may occur also because the market suspects you know something. Think of the block trader who has to find the other side of the trade for you. If you often show up with "soiled merchandise," he is going to go out of business if he always accommodates you at current prices and bags his clients on your behalf. More likely, he will adjust the price somewhat. The smarter he thinks you are, the bigger the adjustment. Once you have traded, the price may not return to its previous level because the cat is now out of the bag. In that case, part of the price impact will be permanent.

Spread models (micro level)



Temporary vs. permanent market impact



More general cost models #1

Quantitative Finance, Vol. 10, No. 7, August–September 2010, 749–759

No-dynamic-arbitrage and market impact

JIM GATHERAL*

Bank of America Merrill Lynch and the Courant Institute, New York University, New York, NY 10080, USA

(*Received 31 October 2008; in final form 25 September 2009*)

Starting from a no-dynamic-arbitrage principle that imposes that trading costs should be non-negative on average and a simple model for the evolution of market prices, we demonstrate a relationship between the shape of the market impact function describing the average response of the market price to traded quantity and the function that describes the decay of market impact. In particular, we show that the widely assumed exponential decay of market impact is compatible only with linear market impact. We derive various inequalities relating the typical shape of the observed market impact function to the decay of market impact, noting that, empirically, these inequalities are typically close to being equalities.

In the following we assume that the stock price S_t at time t is given by

$$S_t = S_0 + \int_0^t f(\dot{x}_s) G(t-s) \mathrm{d}s + \int_0^t \sigma \, \mathrm{d}Z_s, \qquad (1)$$

where $\dot{x}_s$ is our rate of trading in dollars at time $s < t$, $f(\dot{x}_s)$ represents the impact of trading at time s and $G(t-s)$ is a decay factor.

$$G(t-s) = \frac{1}{(t-s)^{\gamma}}, \quad 0 < \gamma < 1. \qquad \boldsymbol{f(v) \propto v^{\delta}.}$$

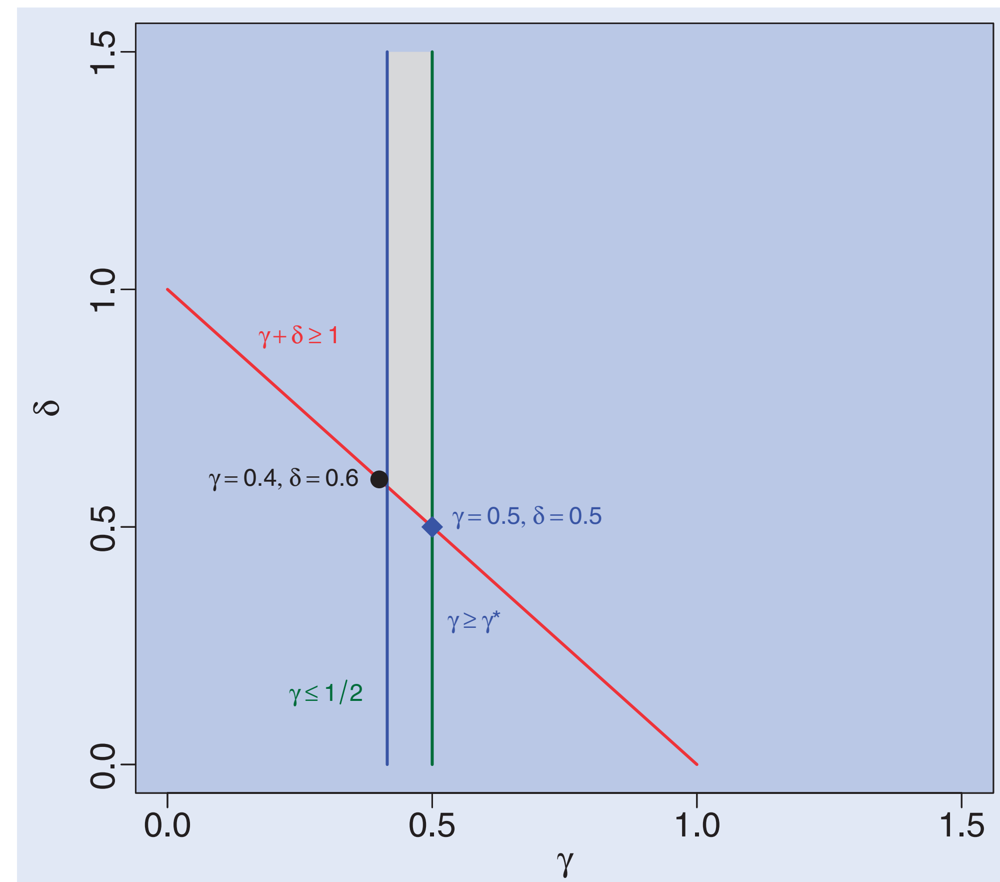


Figure 3. Combinations to the right of the red line satisfy $\gamma+\delta \geq 1$, to the right of the blue line $\gamma \geq \gamma^*$, to the left of the green line $\gamma \leq 1/2$ and in the shaded intersection, the allowable values of γ and δ consistent with the stylized facts of market impact. The black dot represents the empirical estimates $\gamma \approx 0.4$ and $\delta \approx 0.6$ and the blue diamond, the values $\gamma = 0.5$ and $\delta = 0.5$ consistent with the square-root formula.

More general cost models #2

Journal of Financial Markets 16 (2013) 1–32

Optimal trading strategy and supply/demand dynamics ☆

Anna A. Obizhaeva[a,1], Jiang Wang[b,c,d,*]

The ask price at any time t is

$$A_t = V_t + s/2 + \sum_{i=0}^{n(t)} x_{t_i} \kappa e^{-\rho(t-t_i)},$$

Block trades at start and end

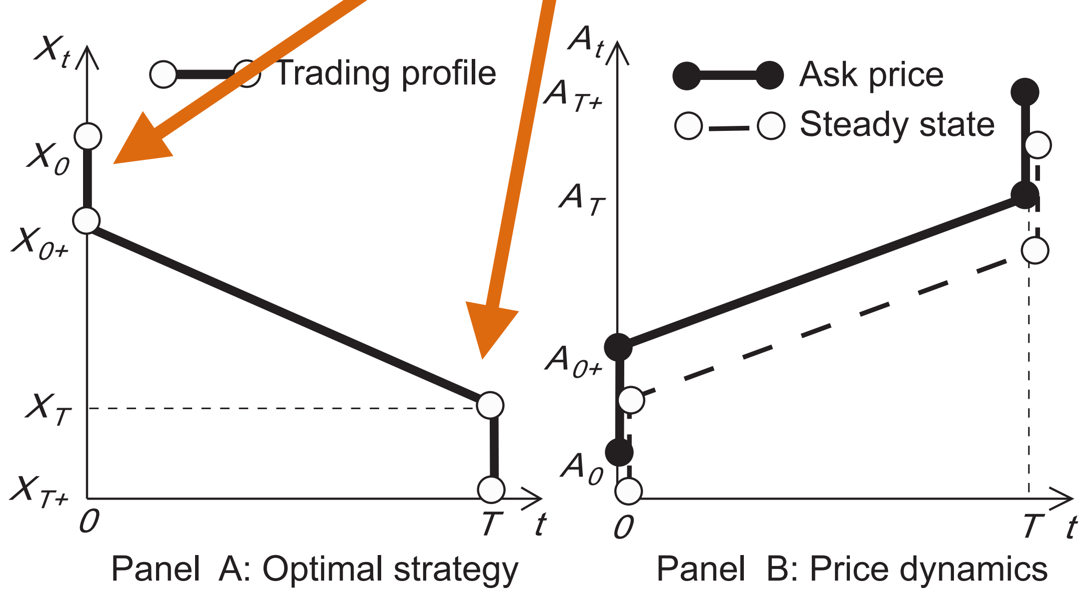
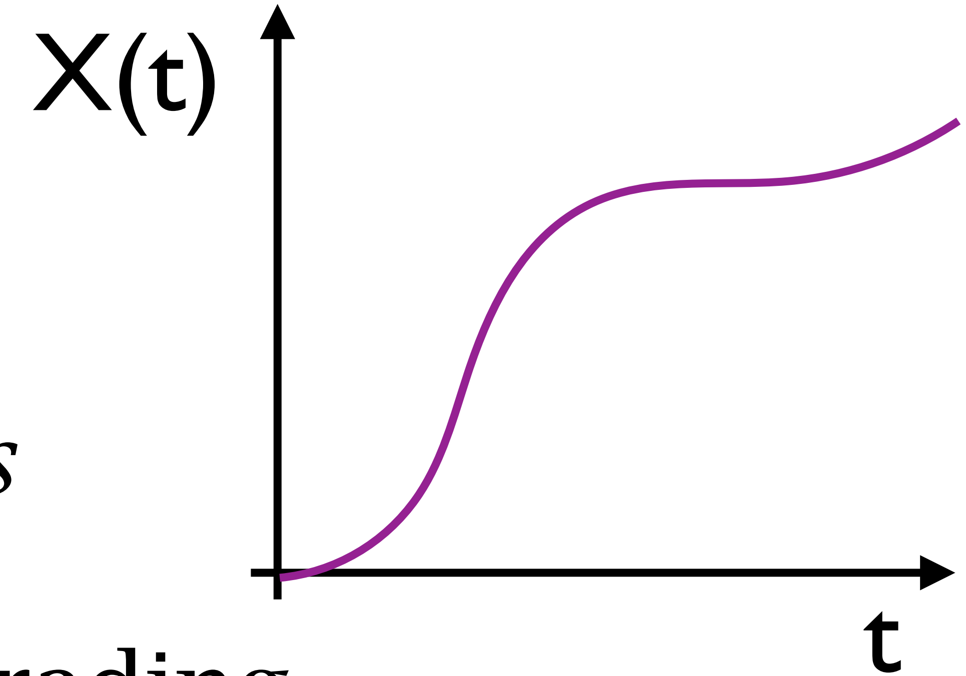


Fig. 3. Profiles of the optimal execution strategy and ask price. Panel A plots the profile of optimal execution policy as described by X_t, the number of units left to be traded at time t. Panel B plots the profile of realized ask price A_t. A discrete trade occurs at time 0 and moves the ask price up. After the initial trade, continuous trades are executed as a constant fraction of newly incoming sell orders to keep the deviation of the ask price A_t from its steady state $V_t + s/2$, shown with dashed line in Panel B, at a constant level. A discrete trade occurs at the last moment T to complete the order.

Executions consume liquidity, which replenishes exponentially in time

Permanent impact



$$X_t = X_0 + \int_0^T \theta_s \, ds$$

θ_t = instantaneous *rate* of trading

$$dP_t = \sigma \, dW_t + G(\theta_t) \, dt$$

G must be linear to avoid round-trip arbitrage (Huberman & Stanzl)

$$G(\theta) = \nu \, \theta$$

$$\boxed{P_t = P_0 + \sigma W_t + \nu (X_t - X_0)}$$

(independent of path)

Cost to execute net X shares = $\frac{1}{2} \nu X^2$

Temporary impact

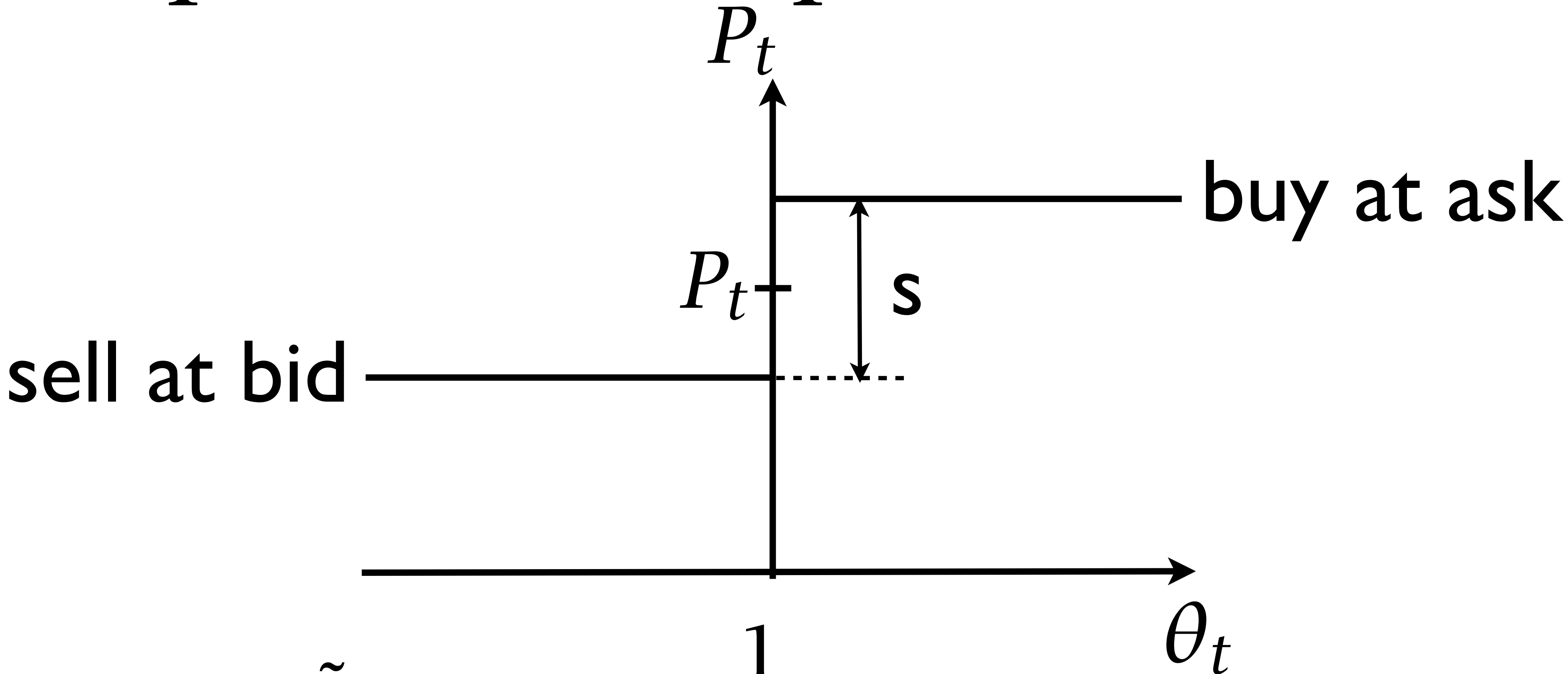
We trade at $\tilde{P}_t \neq P_t$

$\tilde{P}_t$ depends on instantaneous trade rate θ_t

$$\tilde{P}_t = P_t + H(\theta_t)$$

Require finite instantaneous trade rate
(no block trades, no market orders)

Example: bid-ask spread



$$\tilde{P}_t \;=\; P_t \;+\; \frac{1}{2}\, s \,\mathrm{sgn}(\theta_t)$$

"Linear" model: cost to trade $\theta_t \, \Delta t$ shares

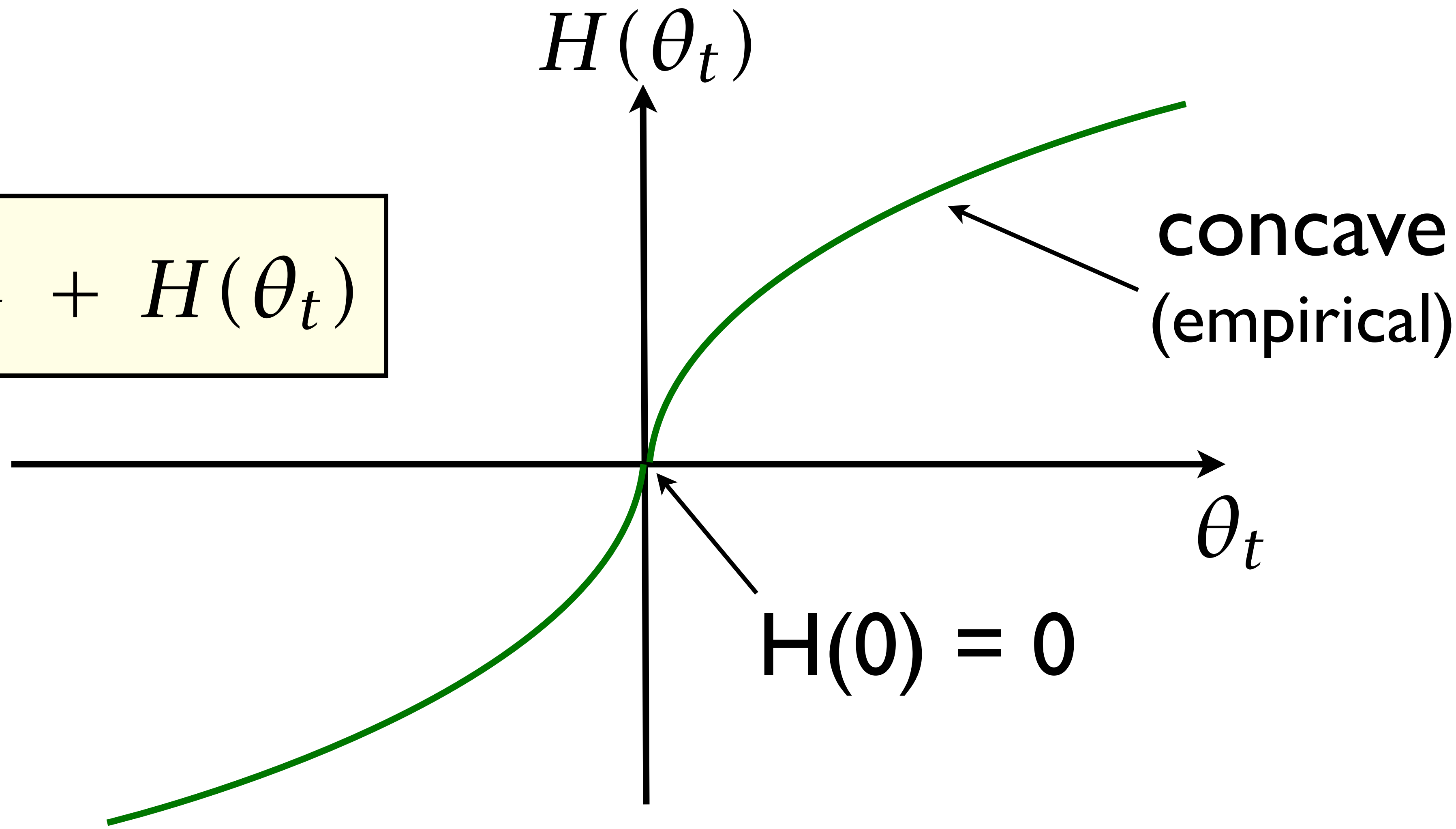
$$\frac{1}{2}\, s\,\mathrm{sgn}(\theta_t) \cdot \theta_t \,\Delta t \;=\; \frac{1}{2}\, s\,|\theta_t|\,\Delta t$$

Critique of linear cost model

- independent of trade size
  - not suitable for large traders
- in practice, effective execution near midpoint
  - spread cost not consistent with modern cost models
  - liquidity takers act as liquidity providers

Temporary cost model

$$\tilde{P}_t = P_t + H(\theta_t)$$



Empirical results: $H(\theta) = \eta\theta^{1/2}$

Summary: how trades affect prices

Trade trajectory

$$X_t = X_0 + \int_0^T \theta_s \, ds$$

Permanent impact affects public market price P_t

$$dP_t = \sigma \, dW_t + G(\theta_t) \, dt$$

$$G(\theta) = \nu \, \theta$$

$$P_t = P_0 + \sigma \, W_t + \nu (X_t - X_0)$$

"Private" execution price

$$\tilde{P}_t = P_t + H(\theta_t)$$

Determine optimal trajectory for some objective function
Specific example: Must purchase X shares by time T

Issue #1: Static vs dynamic

Is θ_t
- fixed in advance at $t = 0$, or
- adjusted in response to price motions?

$$\theta_t, X_t \text{ must be measurable in } \mathcal{F}_t\text{, filtration of } W_t$$

Example: if hedging an option, then clearly trade strategy θ_t must be specified as *rule* in terms of price.

Answer (for optimal execution):
Adaptivity is important, but not the most important
Return to this later.

Cost to acquire X shares on a particular trajectory

Temporary impact varies in time because trade rate varies

$$
\begin{aligned}
Z &= \int_0^T \tilde{P}(t)\,\theta(t)\,dt \\
&= \int_0^T \Big[\overset{\text{Initial price}}{P_0} + \overset{\text{Volatility}}{\sigma W(t)} + \overset{\text{Permanent impact}}{\nu x(t)} + H\big(\theta(t)\big)\Big]\,\theta(t)\,dt \\
&= \underbrace{P_0 X \;+\; \nu \int_0^T x(t)\,dx(t) \;+\; \int_0^T H\big(\theta(t)\big)\,\theta(t)\,dt}_{\substack{\text{Nonrandom} \\ \text{(if strategy is non-adaptive)}}} \;+\; \underbrace{\sigma \int_0^T W(t)\,dx(t)}_{\text{Normal, mean zero}}
\end{aligned}
$$

$$
\int_0^T x(t)\,dx(t) = \frac{1}{2}X^2
$$

$$
\int_0^T W(t)\,dx(t) \;=\; \int_0^T \big(X - x(t)\big)\,dW(t) \;-\; \underbrace{\Big[\big(X - x(t)\big)W(t)\Big]_{t=0}^{T}}_{=\,0}
$$

$$
\mathrm{Var} \;=\; \int_0^T \big(X - x(t)\big)^2\,dt
$$

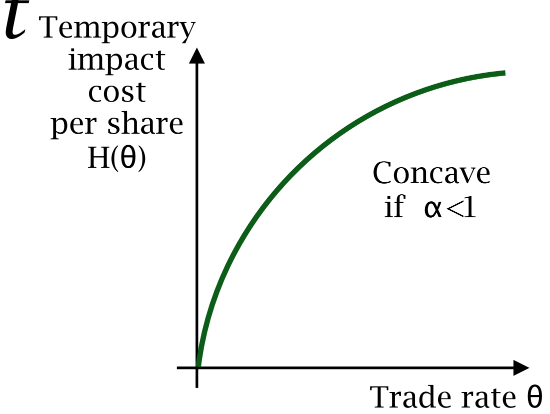
Cost of trading

$$C = Z - X P_0$$ Normal random variable

$$\mathbb{E}(C) = \frac{1}{2}\nu X^2 + \int_0^T H(\theta(t))\,\theta(t)\,dt$$

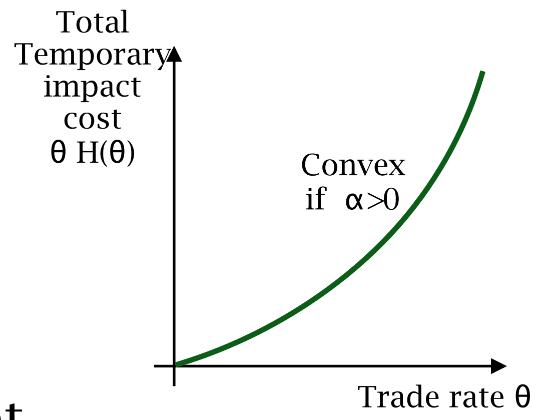
$$\mathrm{Var}(C) = \sigma^2 \int_0^T (X - x(t))^2\,dt$$

With $H(\theta) = \eta\,\theta^\alpha$ typically $0<\alpha\leq 1$



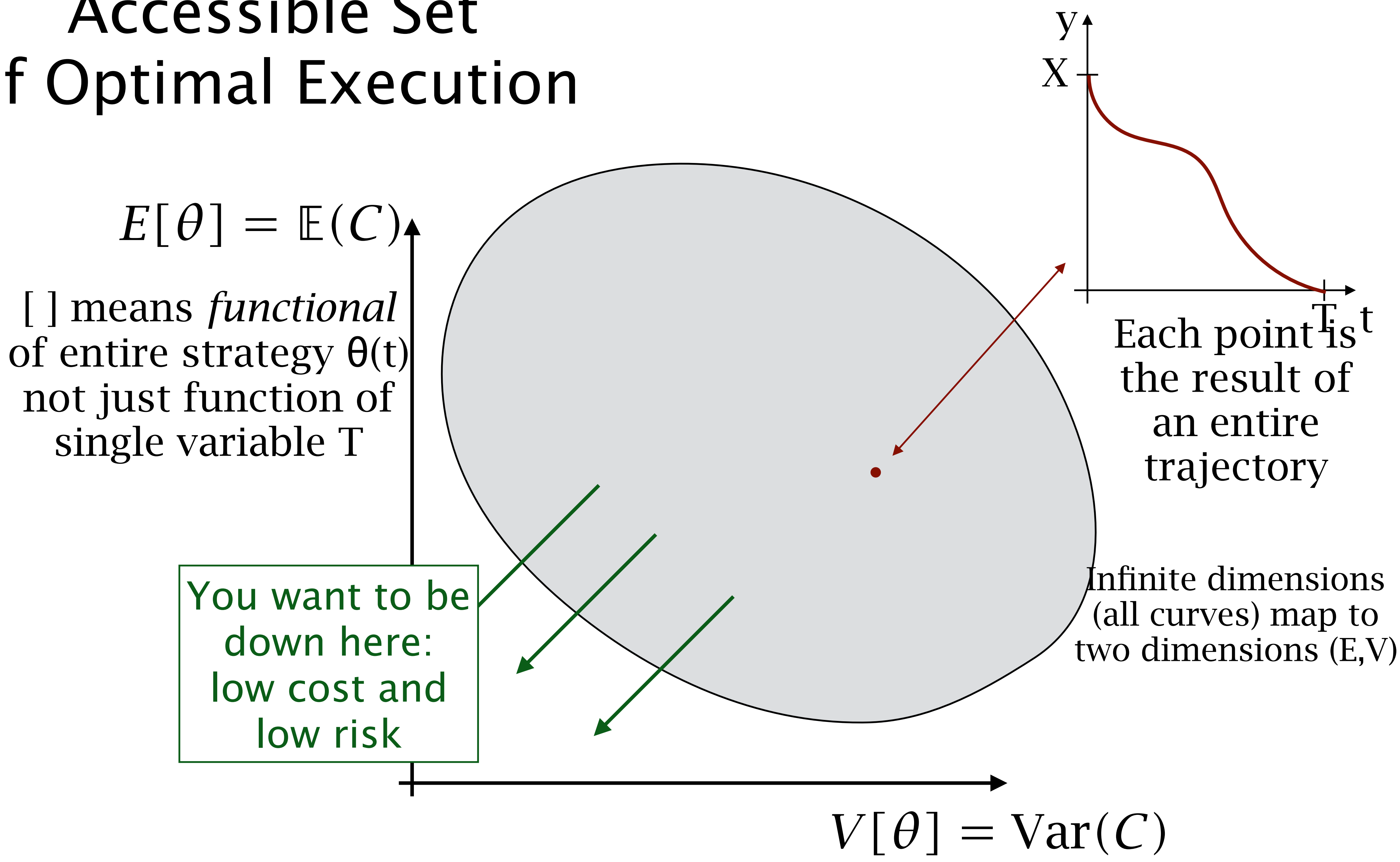
$$E[\theta] = \mathbb{E}(C) = \frac{1}{2}\nu X^2 + \eta \int_0^T \theta(t)^{1+\alpha}\,dt$$

$$V[\theta] = \mathrm{Var}(C) = \sigma^2 \int_0^T y(t)^2\,dt$$



$y(t) = X - x(t)$, deviation from target

Accessible Set of Optimal Execution



- How do you find the "best" strategies?
- Boundary of the accessible set on the small-E, small-V side

$$\min_{\{x(t) \,|\, x(0)=0,\ x(T)=X\}} \Big(E[\theta] + \lambda V[\theta] \Big) \qquad \theta(t) = x'(t)$$

$$E + \lambda V \;=\; \eta \int_0^T \theta(t)^{1+\alpha}\,dt \;+\; \lambda\sigma^2 \int_0^T y(t)^2\,dt$$

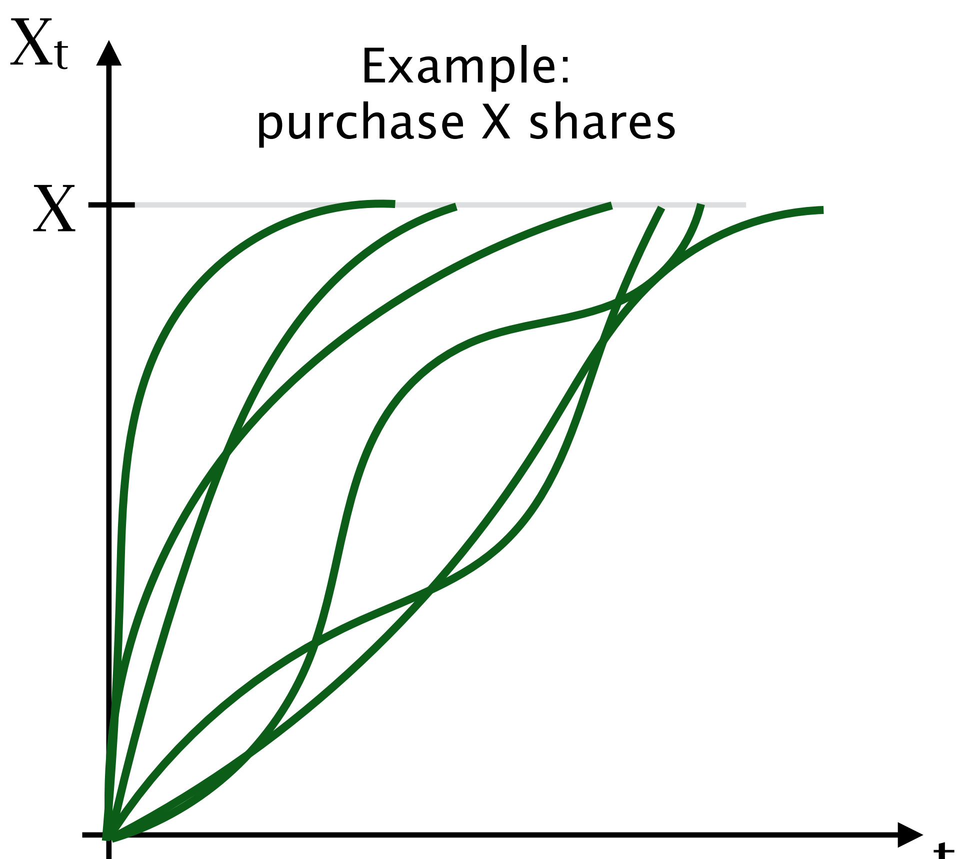
$y(t) = X - x(t)$
Distance from target

Minimize this
by taking θ constant
TWAP on whole interval

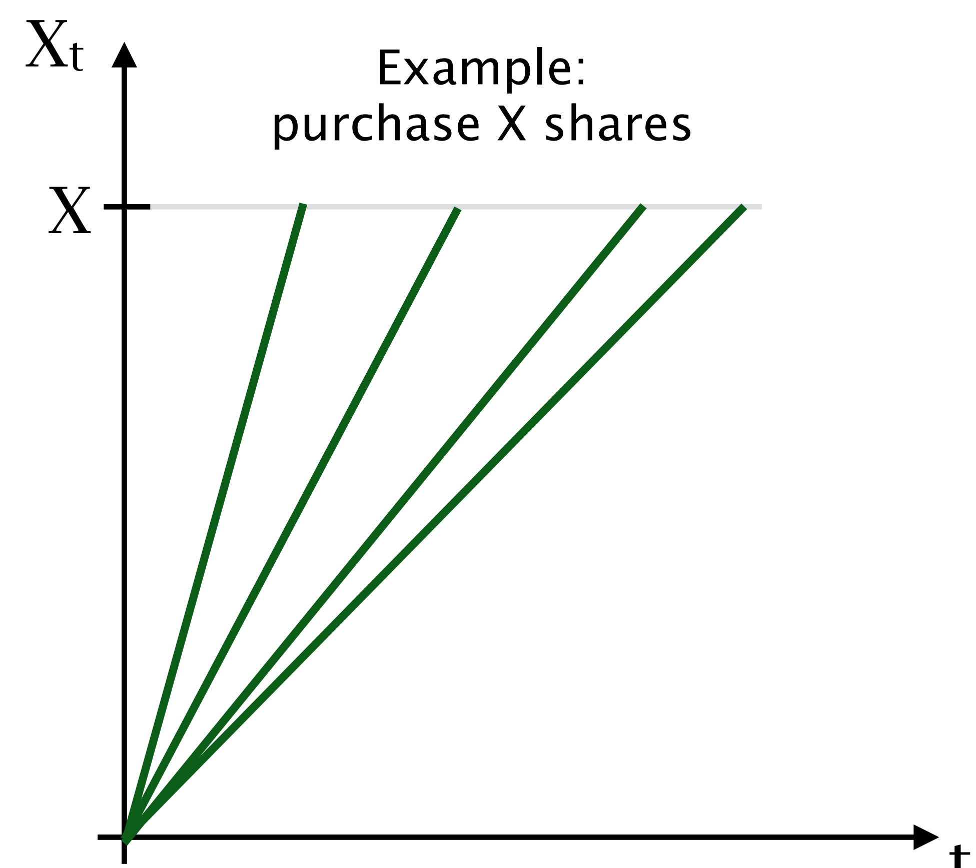
Minimize this
by taking y to 0 (x to X)
as rapidly as possible

Issue #2: General strategy vs restricted family

Trajectory optimization



Optimize over all possible trajectories (Calculus of variations for infinite-dimensional optimization)



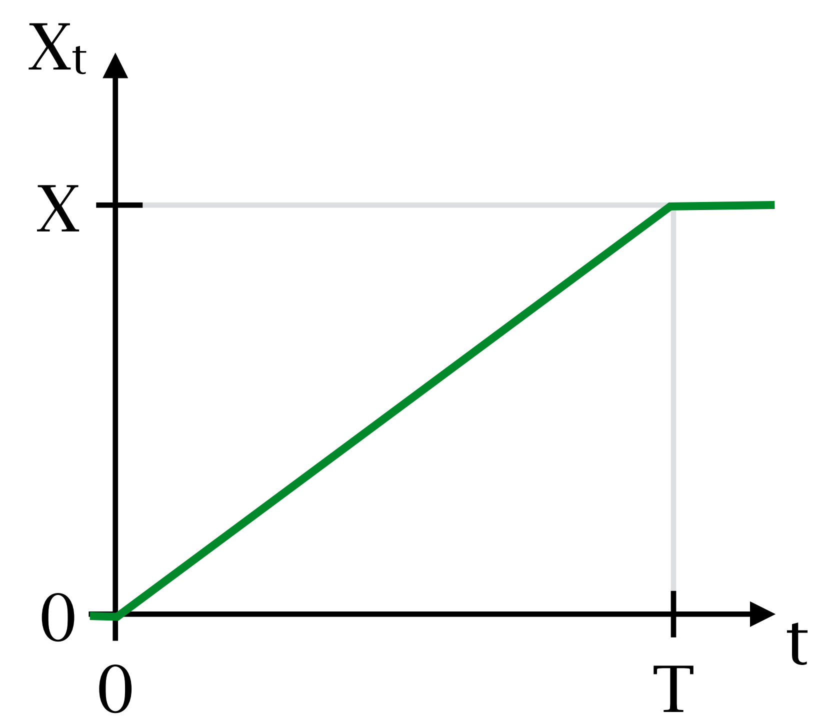
Optimize over specified family (finite-dimensional optimization)

Example with restricted family

- Must purchase X shares (specified)
- Strategy is to buy at constant rate v for time T=X/v

  optimize over restricted class of trajectories
- T arbitrary to be determined

$$X_t = \begin{cases} 0, & t \le 0 \\ Xt/T, & 0 \le t \le T \\ X, & t \ge T \end{cases}$$



Drift from permanent impact

$$P_t = P_0 + \sigma W_t + \begin{cases} 0, & t \le 0 \\ \nu X t / T, & 0 \le t \le T \\ \nu X, & t \ge T \end{cases}$$

Permanent offset from permanent impact

Private trade price

$$\begin{aligned} \tilde{P}_t &= P_t + H\left(\frac{X}{T}\right) \\ &= P_0 + \sigma W_t + \frac{\nu X t}{T} + H\left(\frac{X}{T}\right) \end{aligned}$$

Cost to acquire X shares

$$Z = \int_0^T \tilde{P}_t \underbrace{\frac{X}{T}\,dt}_{\text{Quantity purchased in time } dt}$$

Cost to acquire X shares

Temporary impact is constant in time because trade rate is constant

$$Z = \int_0^T \tilde{P}_t \, \frac{X}{T} \, dt$$

Initial price

Volatility

Permanent impact

$$= \int_0^T \left(P_0 + \sigma W_t + \frac{\nu X t}{T} + H\left(\frac{X}{T}\right)\right) \frac{X}{T} \, dt$$

$$= X P_0 + \frac{1}{2} \nu X^2 + X H\left(\frac{X}{T}\right) + \frac{\sigma X}{T} \int_0^T W_t \, dt$$

$$\int_0^T t \, dt = \frac{1}{2} T^2$$

What you expected to pay

Nonrandom

Normal, mean zero

$$\int_0^T W_t \, dt = \int_0^T (T-t) \, dW_t - \underbrace{\Big[(T-t) W_t \Big]_{t=0}^T}_{=\,0}$$

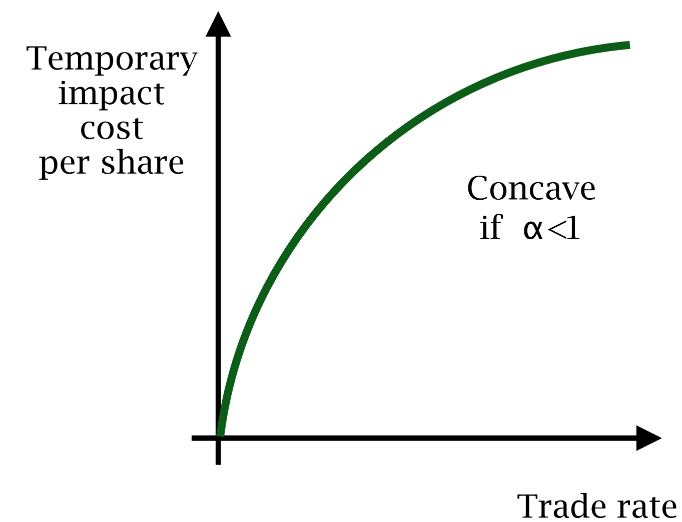
$$\mathrm{Var} = \int_0^T (T-t)^2 \, dt = \frac{1}{3} T^3$$

Cost of trading

$$C = Z - X P_0$$ Normal random variable

$$\mathbb{E}(C) = \frac{1}{2}\nu X^2 + X H\left(\frac{X}{T}\right)$$

$$\mathrm{Var}(C) = \frac{1}{3}\sigma^2 X^2 T$$



If $H(\theta) = \eta\,\theta^\alpha$ (typically $0 < \alpha \le 1$) then

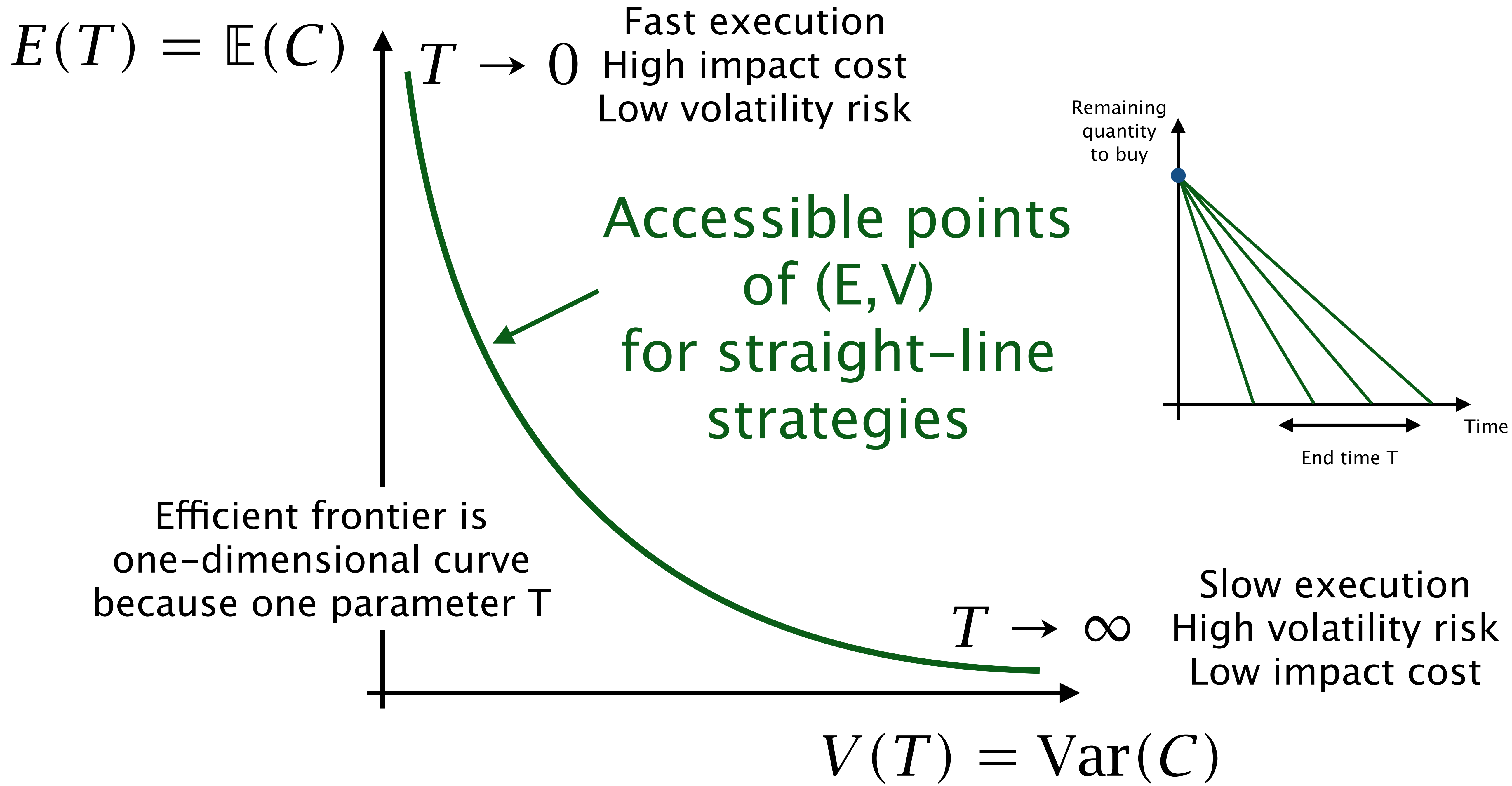
$$E(T) = \mathbb{E}(C) = \frac{1}{2}\nu X^2 + \eta\,\frac{X^{\alpha+1}}{T^\alpha}$$

Decreasing in T: slower trading is cheaper

$$V(T) = \mathrm{Var}(C) = \frac{1}{3}\sigma^2 X^2 T$$

Increasing in T: slower trading is more risky

Efficient Frontier of Optimal Execution



Optimal T depends on risk aversion
All points are reasonable for some aversion level