# ORF 474: High Frequency Trading
# Notes 4b

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Paper by Glosten and Milgrom (1985). Exposition and examples indebted to Bouchaud et al. (2018).

Now we discuss a somewhat more general formulation of the Glosten-Milgrom model. As before, let

$$v \;=\; \text{true value of the asset, which will be revealed at some future time } T.$$

This means that all positions will be converted to cash at time $T$, at values $v$. We take the position of a market maker who must post a bid $b$ and an ask $a$, before time $T$ without knowing the true value. Also, let

$$\hat{v} \;=\; \text{the belief of some trader who comes to trade with us.}$$

When a trader arrives at the market, he will buy from us at our ask price $a$ if his particular belief $\hat{v}$ has $\hat{v} > a$, and will sell to us at our bid $b$ if his particular belief $\hat{v}$ has $\hat{v} < b$. If $b < \hat{v} < a$ then the trader does not trade, and another trader is chosen, until a trade occurs. There is a distribution of these beliefs among the trading population. When someone trades with us we do not know what his belief is, only that he has submitted a trade of a particular sign.

To determine our quoting strategy, we need two distributions. Let

$$G(v) \;=\; \text{our prior distribution for the value of } v,$$

and

$$P(\,\hat{v} \mid v\,) \;=\; \text{how we believe the external traders form their beliefs.}$$

**Example** We shall consider only the structure

$$P(\,\hat{v} \mid v\,) \;=\; \pi\,\delta(\hat{v} - v) \;+\; (1-\pi)\,F(\hat{v})$$

where $\delta$ is the Dirac mass (a point concentration at zero) and $0 < \pi < 1$. Thus, a fraction $\pi$ of all traders are informed, with believed value $\hat{v}$ precisely equal to the true value $v$. The remaining fraction $1 - \pi$ are uninformed, and have a distribution $F(\cdot)$ of beliefs.

**Example** The previous Glosten-Milgrom example is obtained by setting

$$G(v) \;=\; \theta\,\delta(v - v_{\mathrm{H}}) \;+\; (1-\theta)\,\delta(v - v_{\mathrm{L}})$$

(the true value can be $v_{\mathrm{L}}$ or $v_{\mathrm{H}}$, and we assign probability $\theta$ to the case $v = v_{\mathrm{H}}$), and

$$F(\hat{v}) \;=\; \frac{1}{2}\Big(\,\delta(\hat{v} - v_{\mathrm{L}}) \;+\; \delta(\hat{v} - v_{\mathrm{H}})\,\Big)$$

(half of the uniformed traders believe the value is high and will always buy, and half of them believe the true value is low and will always sell).

The new thing here is the continuous distribution $F(\cdot)$ of uninformed beliefs, and the continuous distribution $G(\cdot)$ of our prior beliefs. We will be able to see different types of behavior depending on the relative structure of these two distributions.

As before, we (the market maker) set bid and ask prices by the "no regret" conditions

$$a \;=\; \mathbb{E}(\,v \mid \hat{v} > a\,) \;=\; \int v\,G_+(v)\,dv$$
$$b \;=\; \mathbb{E}(\,v \mid \hat{v} < b\,) \;=\; \int v\,G_-(v)\,dv.$$

where $G_+(v)$ and $G_-(v)$ are the updated versions of our estimates of $G(v)$, conditional on the next trade being a buy or a sell respectively. We know that if a trader buys, then that trader believes $\hat{v} > a$, and if a trader sells, then that trader believes $\hat{v} < b$. Those observations give us information about the distribution of the underlying true value $v$.

In these expressions, it is is crucial that $v$ appears in the first part, and $\hat{v}$ in the second. Indeed, $\mathbb{E}(\,v \mid v > a\,) > a$ (if $\{v > a\}$ has nonzero probability), and hence these conditions would be impossible to satisfy. In fact, it is not obvious at all that values $a$ and $b$ can be found that meet these conditions, and we will show examples where the market fails, and no values $a$ and $b$ are adequate.

Our tool for extracting information about the distribution of $v$ is Bayes' theorem

$$P(\,A \mid B\,) \;=\; \frac{P(\,B \mid A\,)\,P(A)}{P(B)}.$$

Thus,

$$G_+(v) \;=\; P(\,v \mid \hat{v} > a\,) \;=\; \frac{P(\,\hat{v} > a \mid v\,)\,G(v)}{\int P(\,\hat{v} > a \mid v\,)\,G(v)\,dv}$$
$$G_-(v) \;=\; P(\,v \mid \hat{v} < b\,) \;=\; \frac{P(\,\hat{v} < b \mid v\,)\,G(v)}{\int P(\,\hat{v} < b \mid v\,)\,G(v)\,dv}.$$

Hence, $a$ and $b$ are determined by the conditions

$$a \;=\; \frac{\int v\,P(\hat{v} > a \mid v\,)\,G(v)\,dv}{\int P(\hat{v} > a \mid v\,)\,G(v)\,dv}$$
$$b \;=\; \frac{\int v\,P(\hat{v} < b \mid v\,)\,G(v)\,dv}{\int P(\hat{v} < b \mid v\,)\,G(v)\,dv}.$$

For a specific example in which we can compute the integrals explicitly, let us set

$$F(\hat{v}) \;=\; \frac{1}{2\rho}\exp\left(-\frac{|\hat{v}-v_0|}{\rho}\right), \qquad G(v) \;=\; \frac{1}{2\sigma}\exp\left(-\frac{|v-v_0|}{\sigma}\right).$$

Thus, $\rho$ represents the uncertainty in the beliefs of the uninformed traders, and $\sigma$ is the uncertainty in our own intial belief. Both are centered on the same reference value $v_0$. More or less, we may interpret $\sigma$ as the daily volatility: the market's uncertainty about the true value of the asset. Then, after some algebra, we obtain

$$\begin{aligned} a &= v_0 + \frac{s}{2} \\ b &= v_0 - \frac{s}{2} \end{aligned}$$

where the spread $s$ satisfies the implicit condition

$$\tilde{s}e^{\mu\tilde{s}} \;=\; \frac{2\pi}{1-\pi} \qquad \text{with} \quad \tilde{s} = \frac{s}{\sigma}, \quad \mu = \frac{\rho-\sigma}{\rho}.$$

In this expression, by defining $\tilde{s} = s/\sigma$ we measure the spread as a nondimensional multiple of volatility, and we have identified the single nondimensional parameter $\mu$. Our question is whether there exists a solution $\tilde{s}$ to this implicit relation. See Figure 1.

There are two cases to consider.

1. Suppose $\rho \ge \sigma$: the uncertainty in our belief is *less* than the uncertainty of the informed traders. Then $\mu \ge 0$, and $f(\tilde{s}) = \tilde{s}\exp(\mu\tilde{s})$ increases monotonically from $f(0) = 0$ to $f(\tilde{s}) \to \infty$ as $\tilde{s} \to \infty$. Therefore there is a unique value $\tilde{s} > 0$ at which $f(\tilde{s})$ equals the positive constant $2\pi/(1-\pi)$.

   (a) For any fixed $\mu$, we have $\tilde{s} \to 0$ as $\pi \to 0$, and $\tilde{s} \to \infty$ as $\pi \to 1$.

   (b) For any fixed $\pi$, we have $\tilde{s} \to 0$ as $\mu \to \infty$, and $\tilde{s} \to 2\pi/(1-\pi)$ as $\mu \to 0$.

2. Suppose $\rho < \sigma$: the uncertainty in our belief is *greater* than the uncertainty of the informed traders. Then $\mu < 0$, and $f(\tilde{s}) = \tilde{s}\exp(\mu\tilde{s})$ increases from $f(0) = 0$ to a finite maximum value $f_{\max} = 1/e|\mu|$ at $\tilde{s} = 1/|\mu| = (\sigma-\rho)/\rho$, before decreasing back to $f(\tilde{s}) \to 0$ as $\tilde{s} \to \infty$.

   (a) If $2\pi/(1-\pi) < 1/\mu$, that is, $\pi < 1/(1+(2/e)\mu)$, then there are two solutions for $\tilde{s}$. I am not sure of the economic significance of these two solutions.

   (b) If $2\pi/(1-\pi) > 1/\mu$, that is, $\pi > 1/(1+(2/e)\mu)$, then there is *no* solution $\tilde{s}$. The market has broken down.

In Case 1, when $\tilde{s}$ is small, we have the asymptotic expression

$$\frac{s}{2} \;\sim\; \pi\sigma \;+\; \mathcal{O}(\pi^2), \qquad \pi, s \to 0.$$

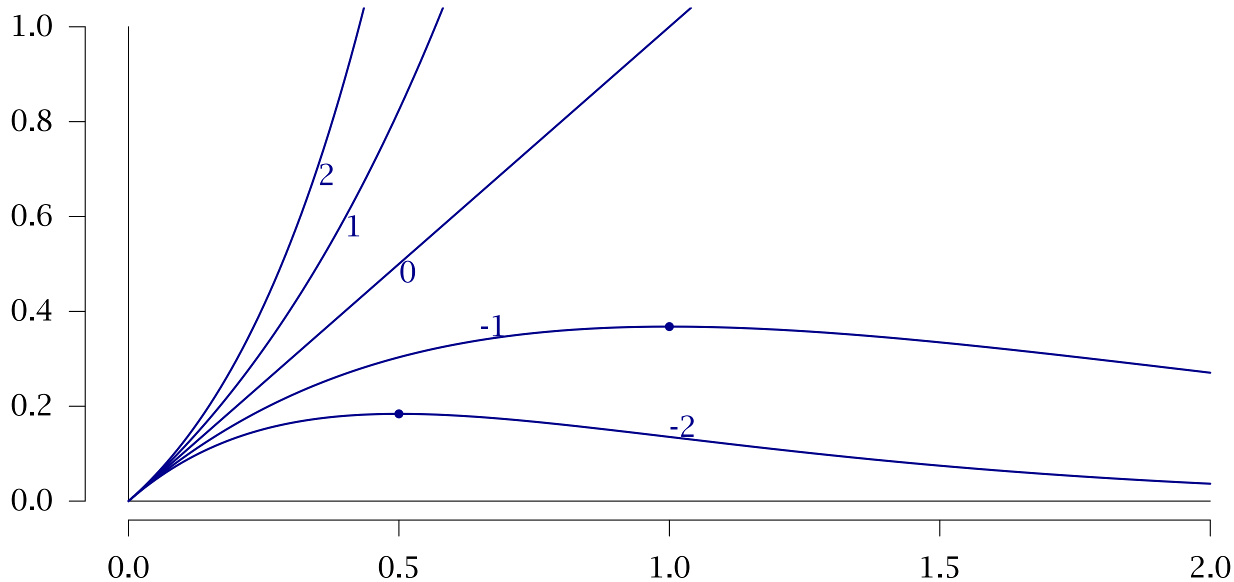


Figure 1: The function $f(s) = s\exp(\mu s)$ for different values of $\mu$ as labelled. For $\mu \geq 0$, $f$ increases monotonically from $f(0) = 0$ to $f(s) \to \infty$ as $s \to \infty$, so the relation $f(s) = Y$ has a unique solution for all $Y \geq 0$. For $\mu < 0$, it attains a finite maximum of $f_{\max} = 1/e|\mu|$ at $s = 1/|\mu|$, so the relation $f(s) = Y$ has a solution only when $Y$ is not too large.

The gain per trade with an uninformed trader is $s/2$. This must balance the loss per trade with an informed trader. Our expected loss on each trade with an informed trader is approximately $\sigma$, since that is how much the price can vary away from its central value $v_0$. This is multiplied by the probability $\pi$ that a trader is informed.

In Case 2, as the market maker increases the spread $s$ to try to capture more profit, the number of informed traders who are willing to trade decreases. But the number of uninformed traders who are willing to trade decreases even faster, since the density of their beliefs decays as an exponential with a larger coefficient. So you simply cannot get enough uninformed traders to balance the informed.

# References

Bouchaud, J.-P., J. Bonart, J. Donier, and M. Gould (2018). *Trades, Quotes, and Prices: Financial Markets Under the Microscope.* Cambridge University Press.

Glosten, L. R. and P. R. Milgrom (1985). Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *J. Financial Econ. 14*(1), 71–100.